Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Lehrstuhl für
INFORMATIK I
Algorithmen & Komplexität

Institut für Informatik

Exact Algorithms

Sommer Term 2020

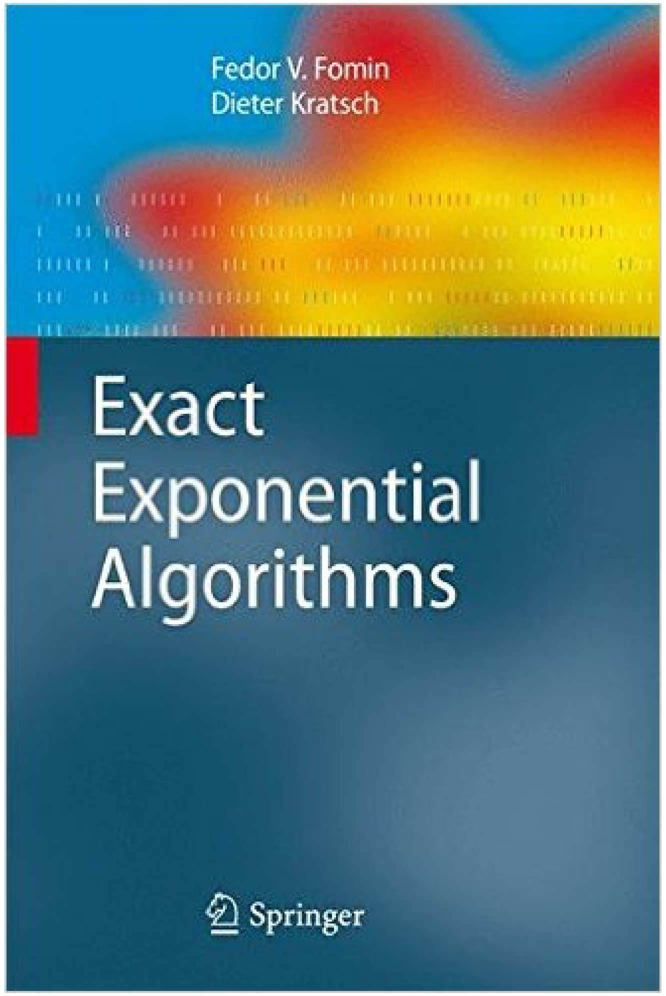
Lecture 1. Introduction & Two Examples

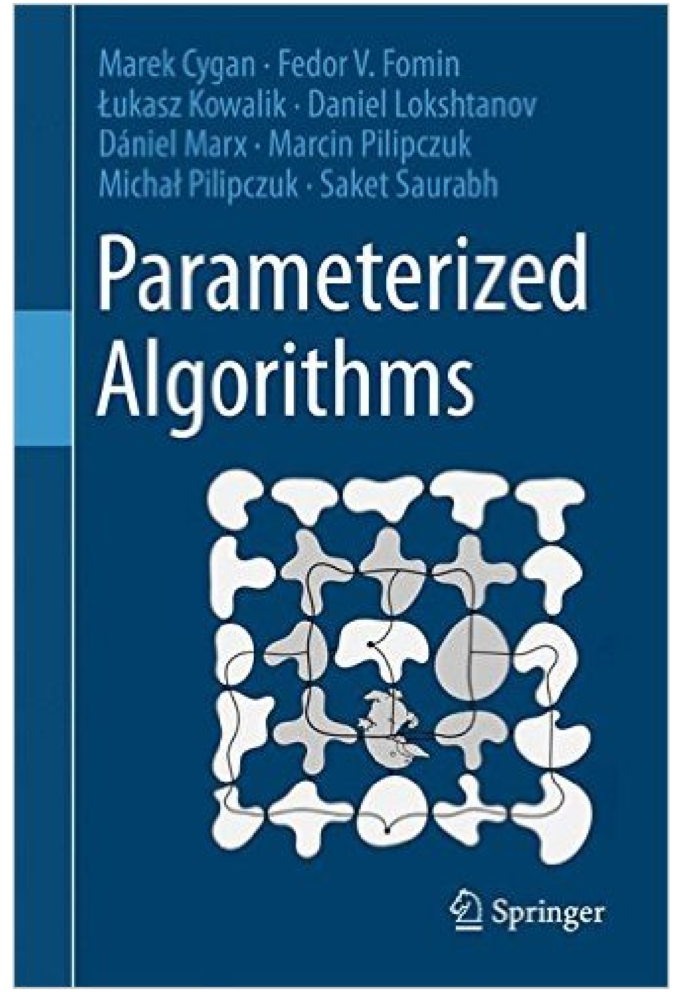
(slides by J. Spoerhase, Th. van Dijk, S. Chaplick, and A. Wolff)

Alexander Wolff

Lehrstuhl für Informatik I

Textbooks





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Marek Cygan et al.:
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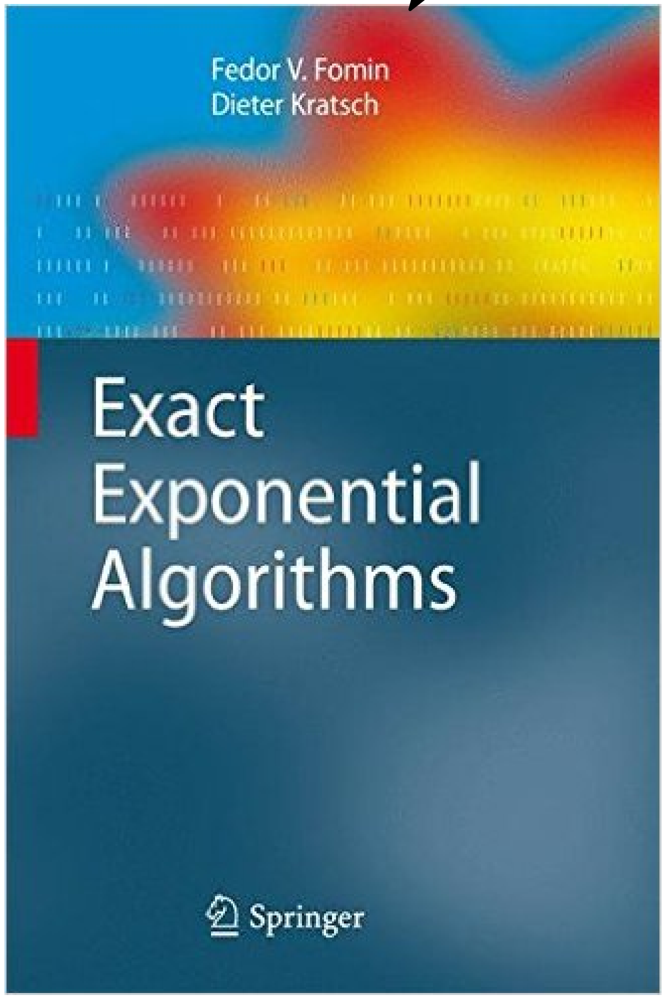
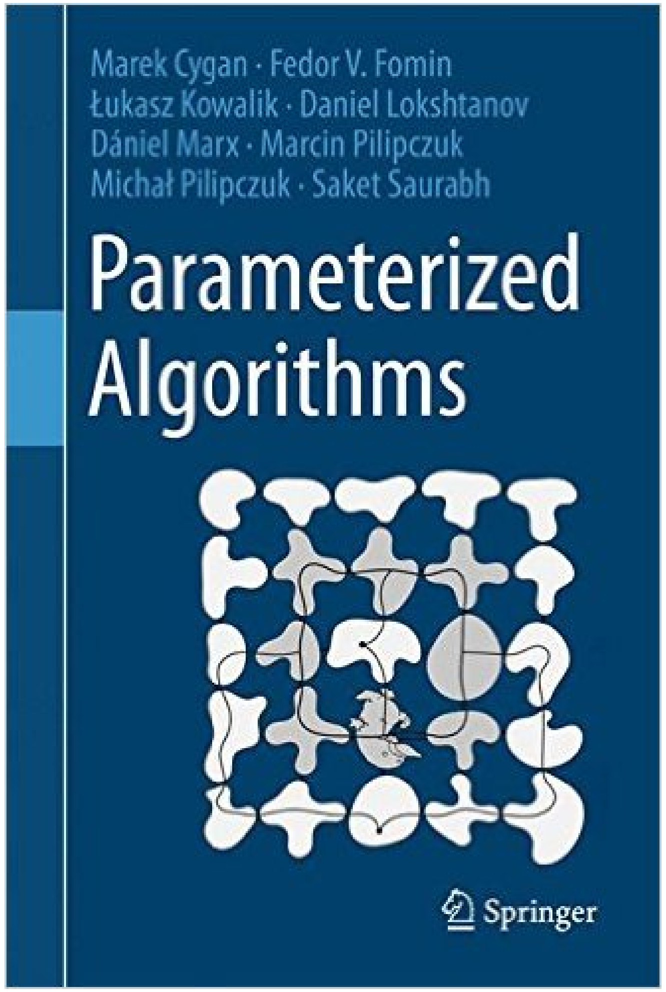
This Lecture: Chapter 1





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Motivation

Efficient vs. inefficient algorithms

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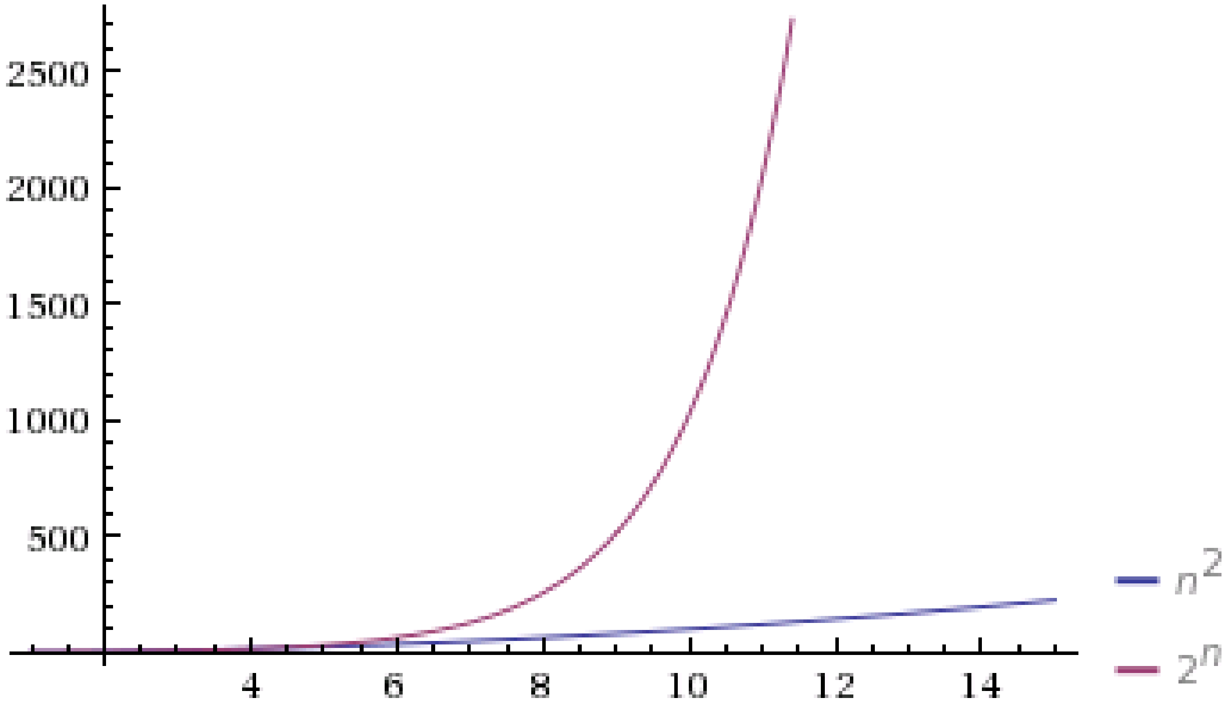
$\rightsquigarrow$ polynomial vs. super-polynomial algorithms



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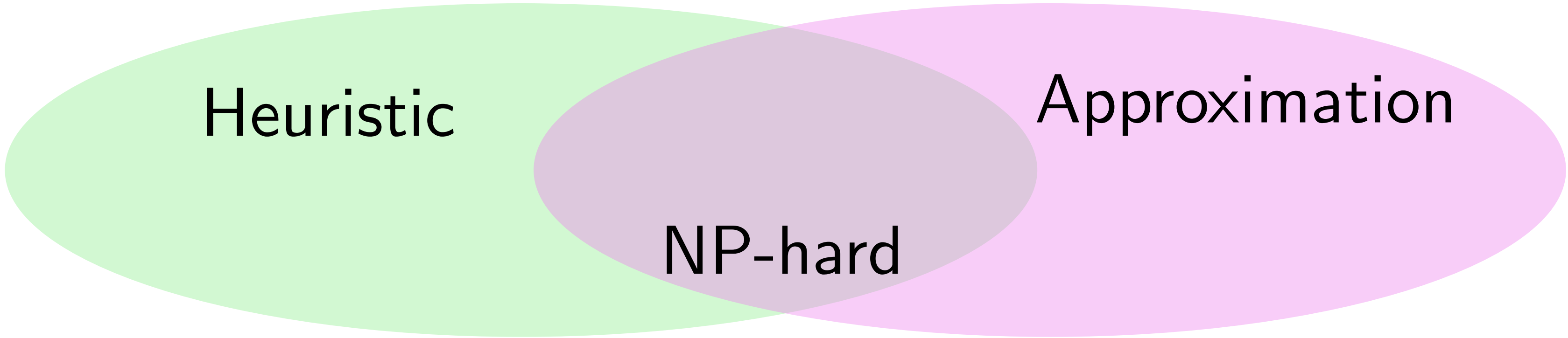
  - heuristics (simulated annealing, tabu search)
  - approximation algorithms (Christofides' algorithm)



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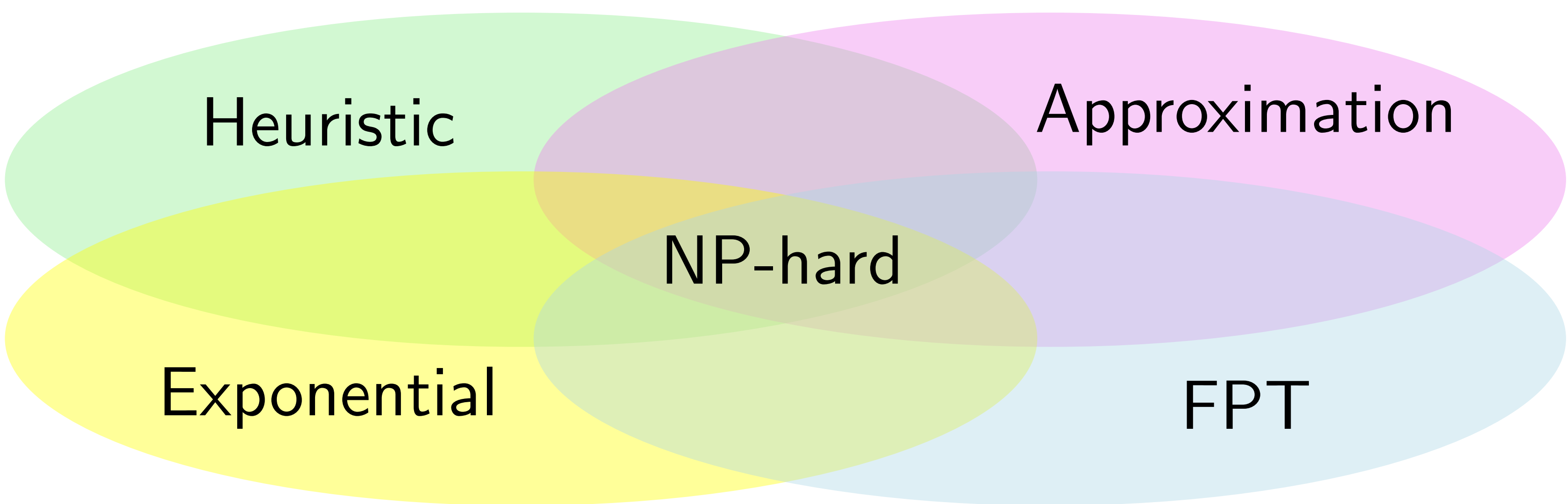

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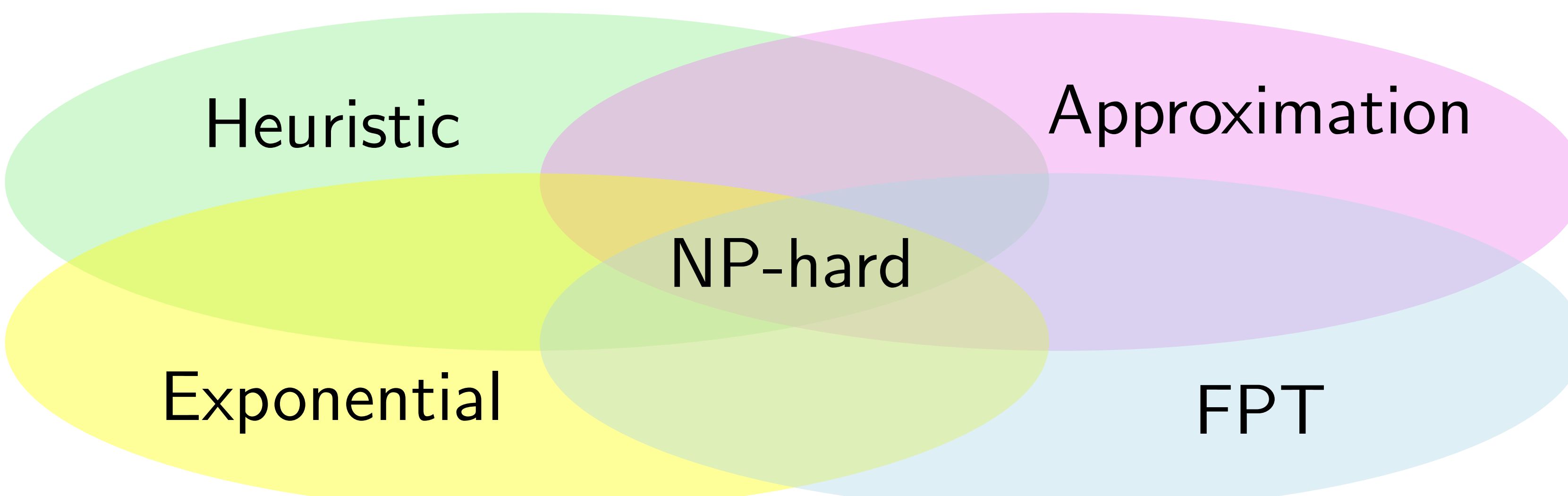
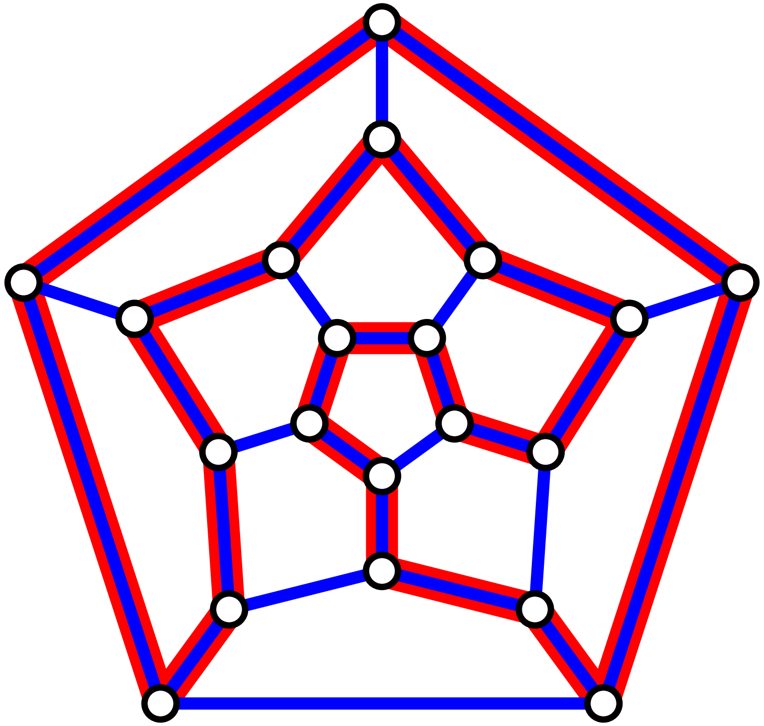


Motivation: Exact Exponential Algorithms

- Can be “fast” for **medium-sized** instances:



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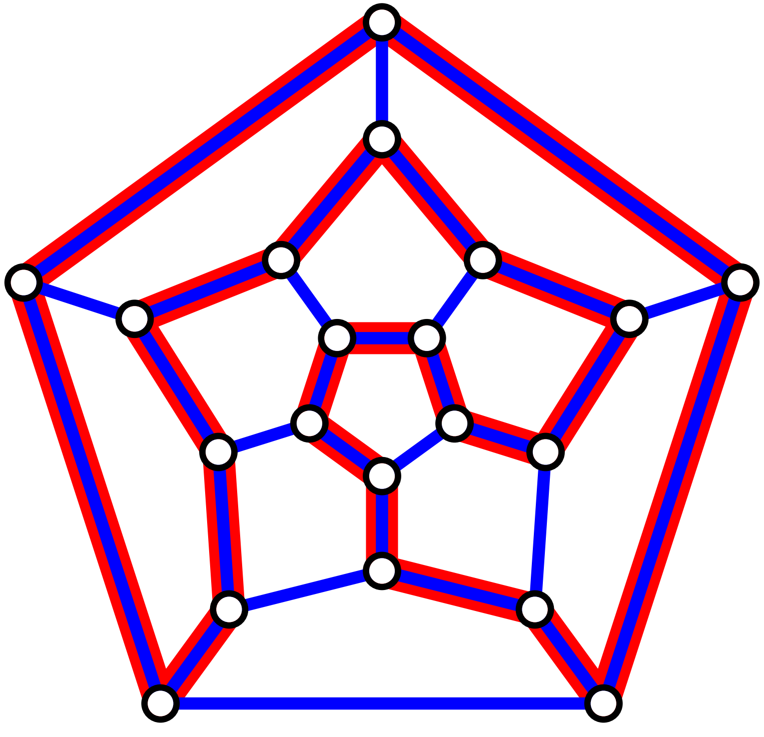

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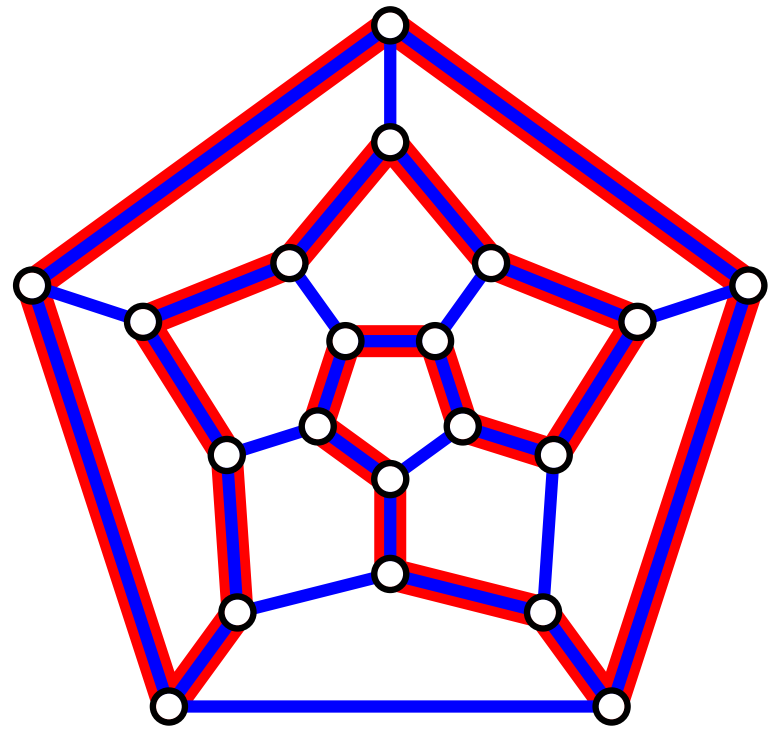

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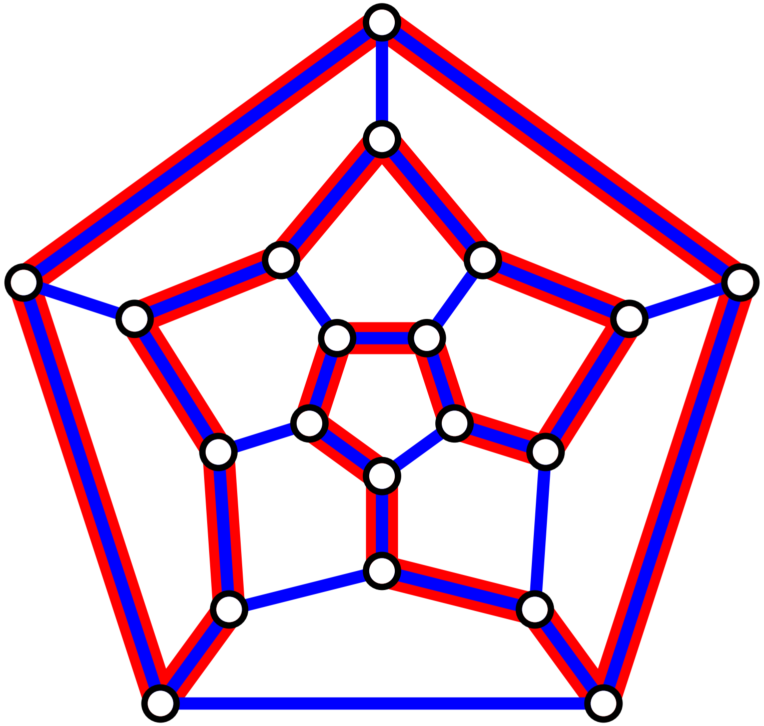

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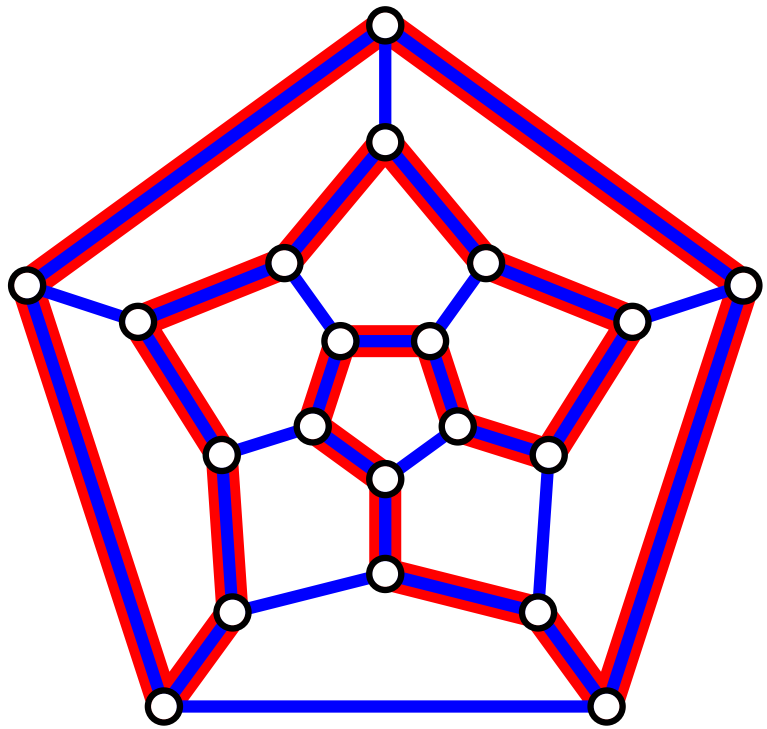
- Theoretical interest!



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$f \in O^*(g) \;\Leftrightarrow\; \exists$ polynomial p: $f \in O(g \cdot p)$

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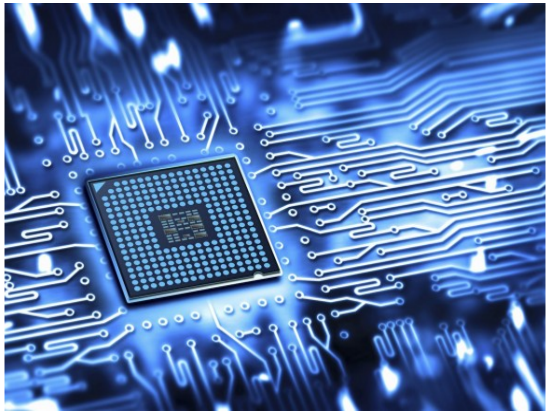

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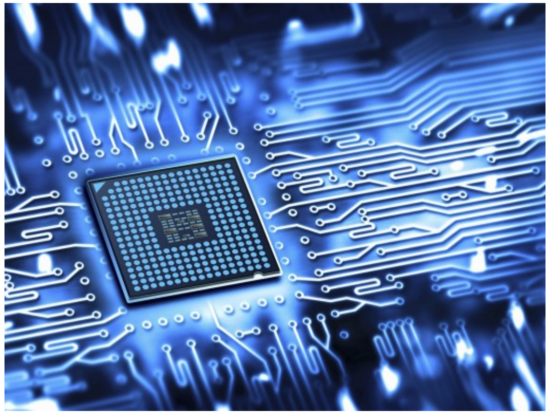

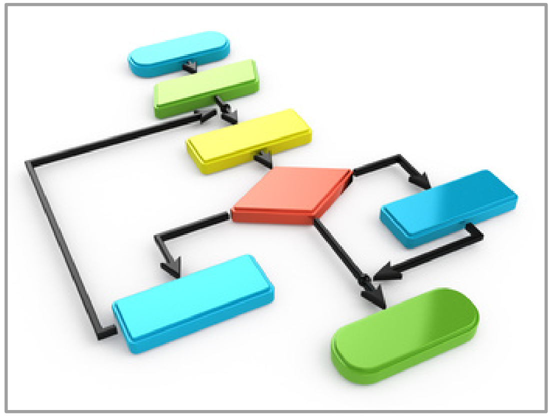

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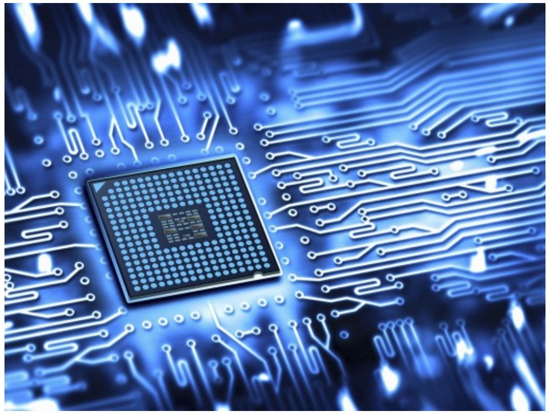

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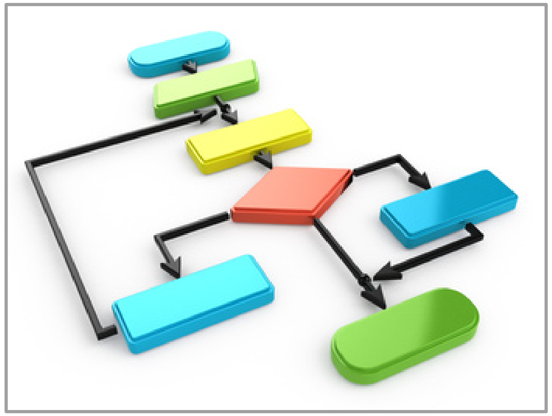

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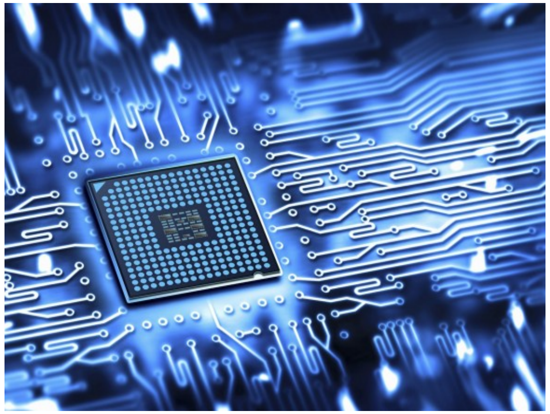

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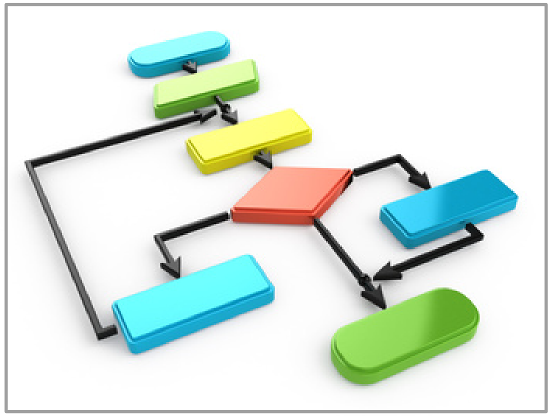

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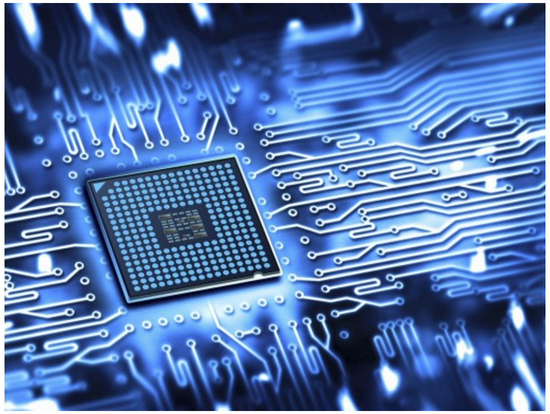

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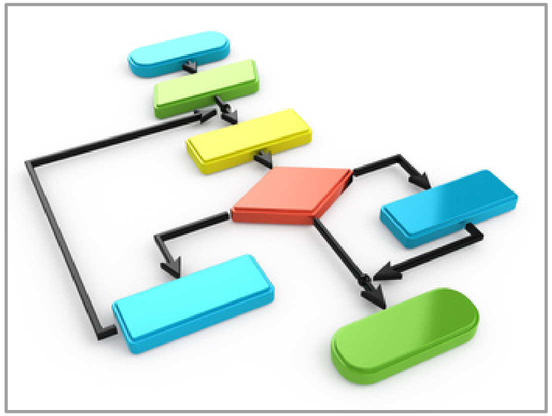

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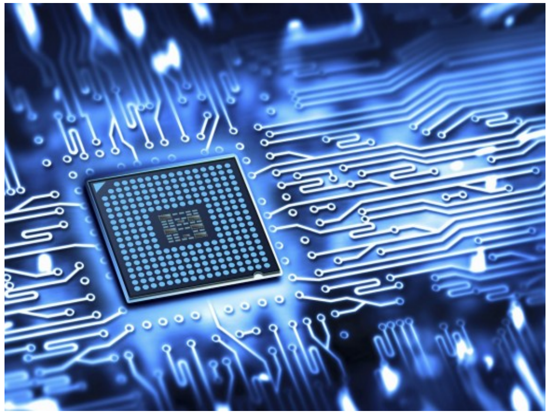

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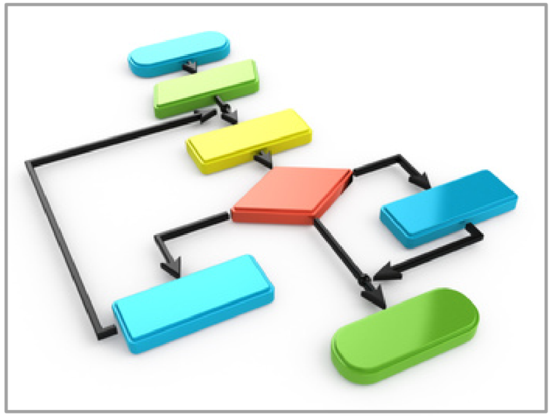

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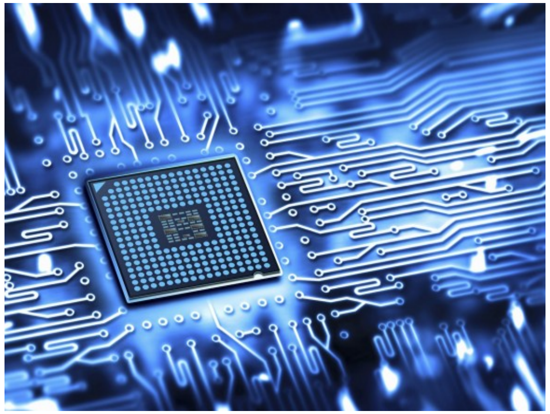

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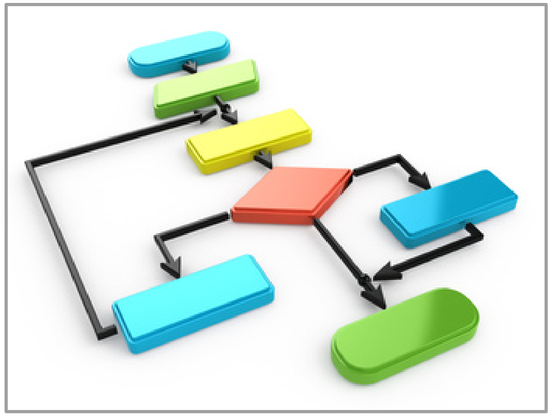

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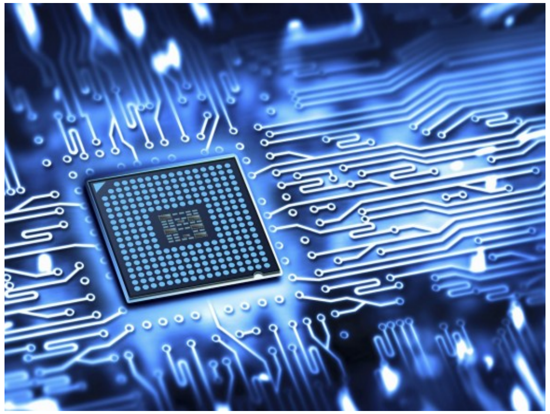

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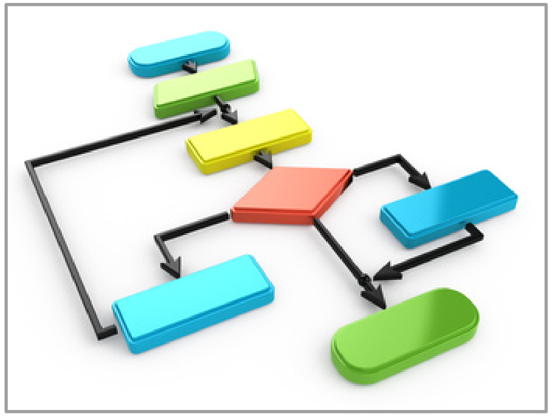

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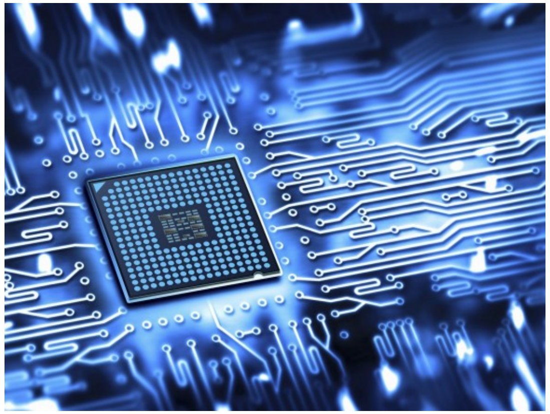

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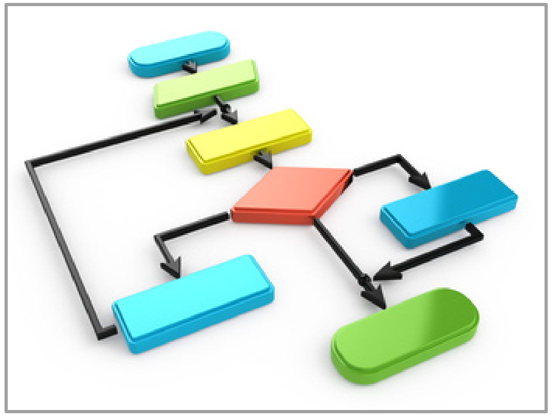

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Traveling Salesperson Problem (TSP)

Input: Complete directed graph $G = (V, E)$ with n vertices and edge weights $c \colon E \to \mathbb{Q}_{\geq 0}$

Output: A Hamiltonian cycle $C = (v_1, \ldots, v_n, v_{n+1} = v_1)$ of G, of minimum weight $\sum_{i=1}^{n} c(v_i, v_{i+1})$.

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Runtime: $\Theta(n! \cdot n) = n \cdot 2^{\Theta(n \log n)}$

Bellman–Held–Karp Algorithm

Richard M. Karp

Richard E. Bellman

Bellman–Held–Karp Algorithm

Technique: Dynamic Programming!

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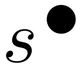

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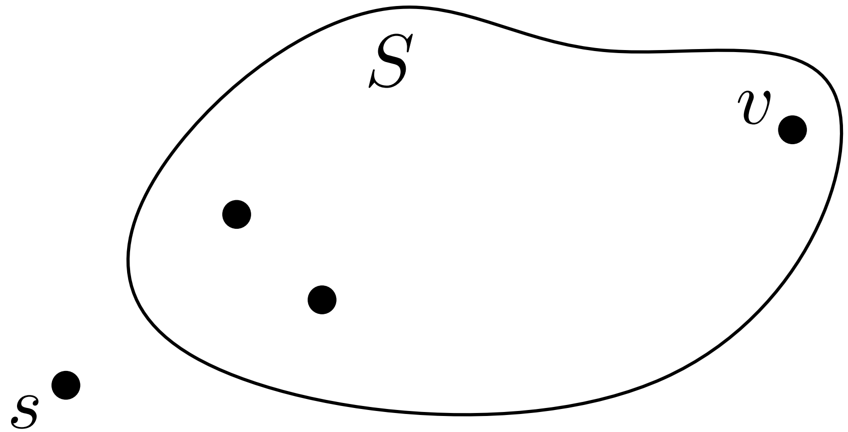

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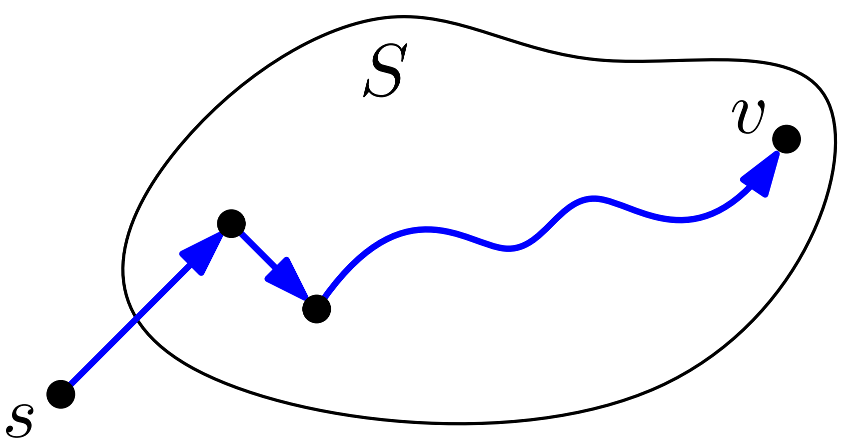
OPT$[S, v]$:= length of the shortest s–v path that visits precisely the vertices of $S \cup \{s\}$.



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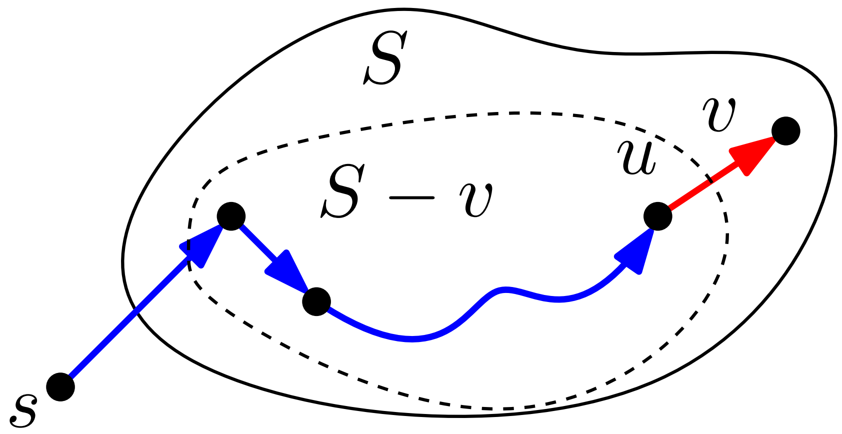

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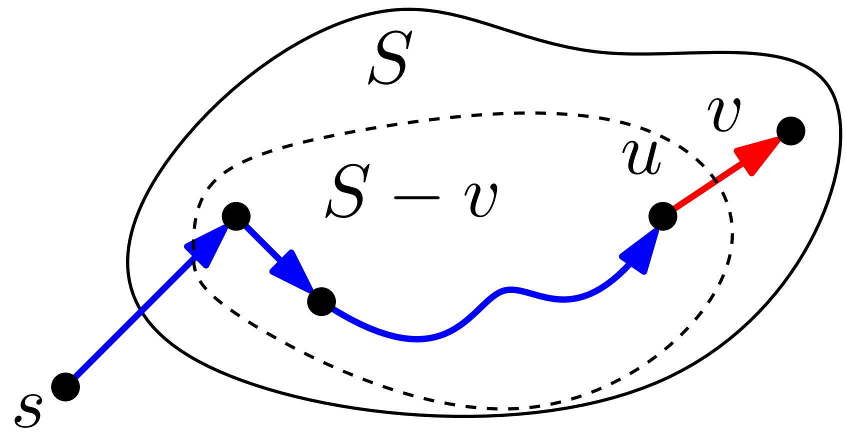

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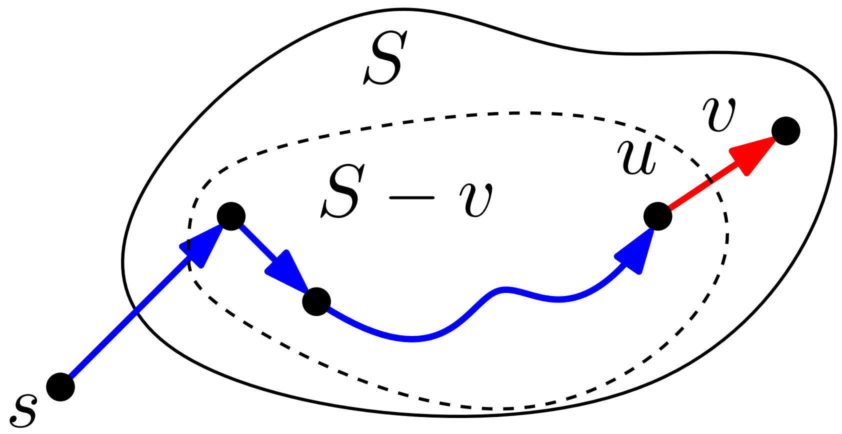

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When $|S| \geq 2$, we compute $\mathrm{OPT}[S, v]$ recursively:

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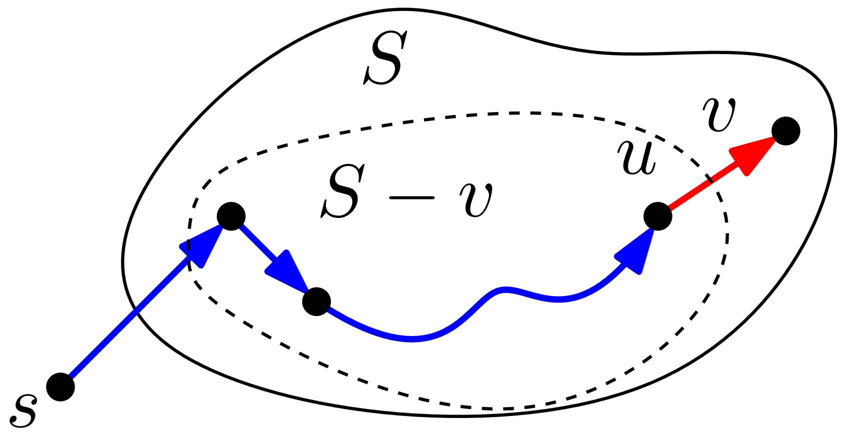

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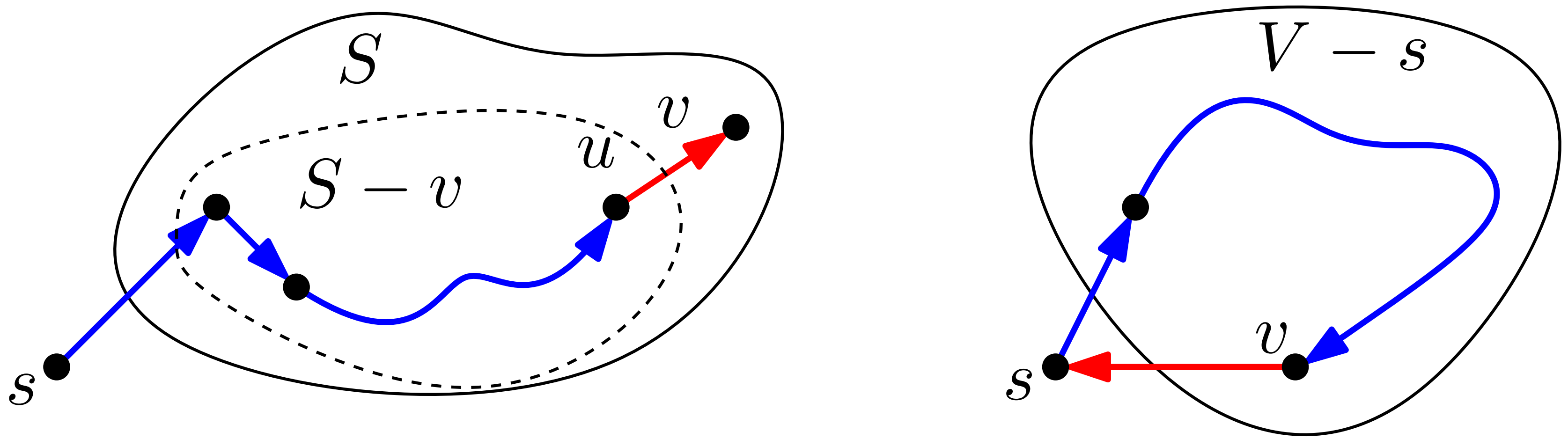

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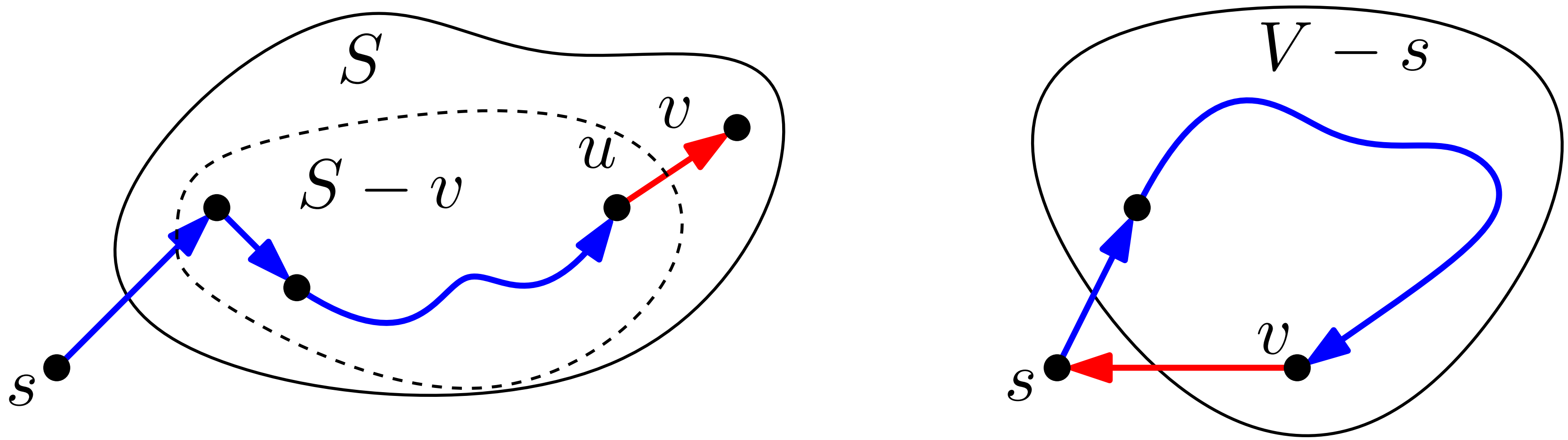

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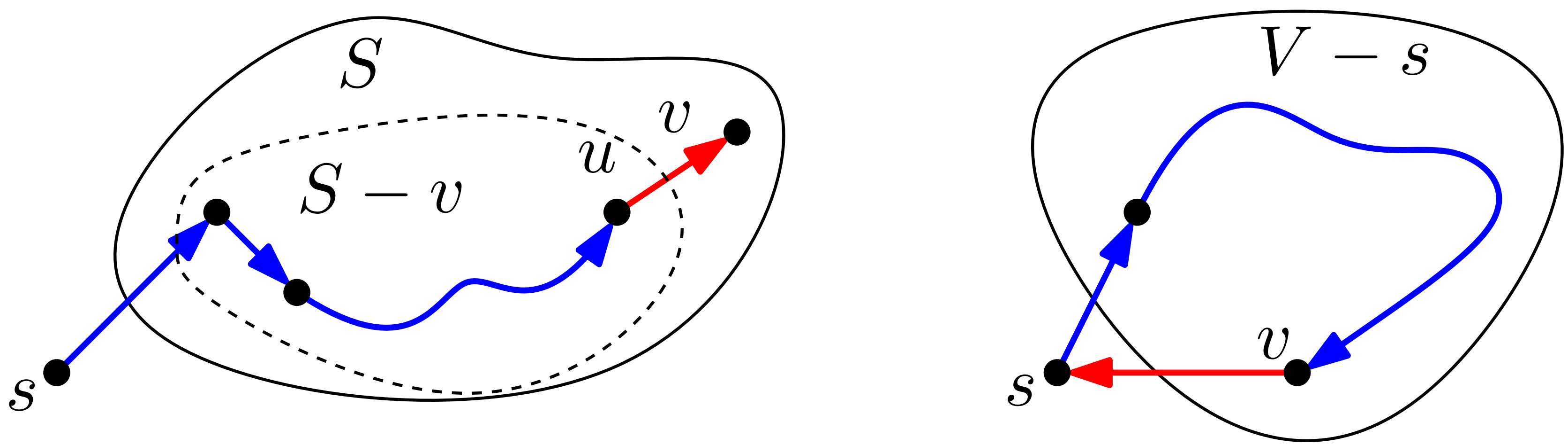

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$$\text{OPT} = \min\{\, \text{OPT}[V - s, v] + c(v, s) \mid v \in V - s \,\} \qquad \square$$

Pseudocode for the Dynamic Program

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Algorithm Bellmann–Held–Karp(G, c)
  foreach v ∈ V − s do
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In total: $O(\quad) = O^*(\quad)$.

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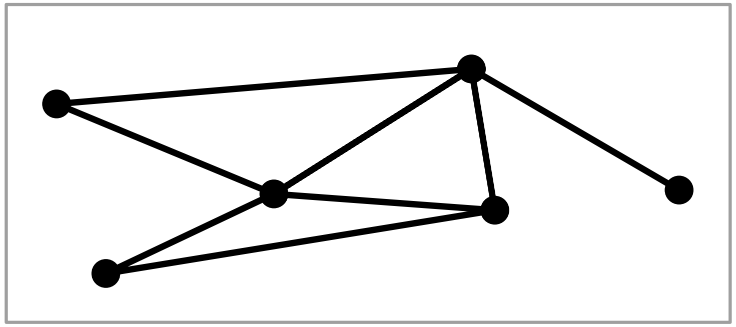
A shortest tour can be produced by backtracking the DP table (as usual). Compare: $O^*(2^n)$ with $2^{O(n \log n)}$ for Brute-Force!

Maximum Independent Set (MIS)

Input: Graph $G = (V, E)$ with n vertices.

Output: Maximum size *independent* set, i.e., a largest set $U \subseteq V$, such that no pair of vertices in U are adjacent in G.



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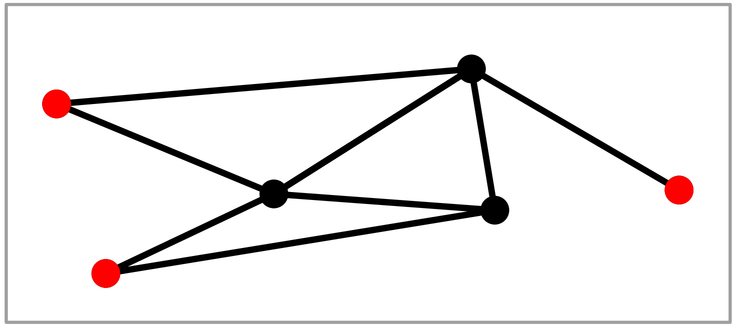

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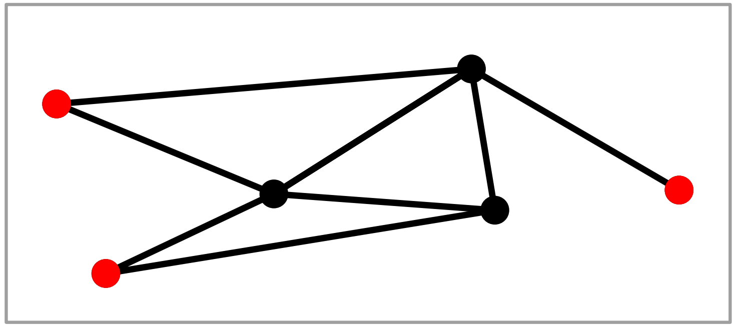

Brute Force?

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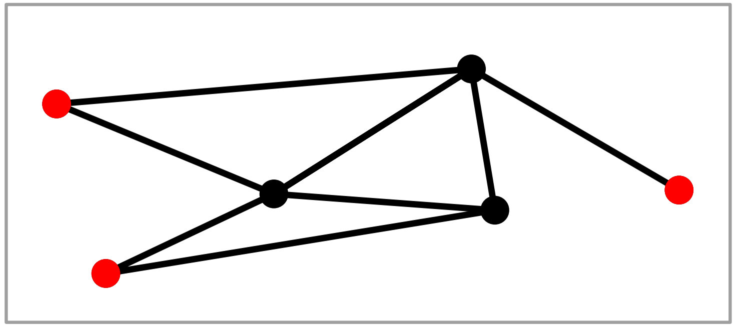

Brute Force? Try all subsets of $V \Rightarrow$ runtime $O(2^n \cdot n)$.

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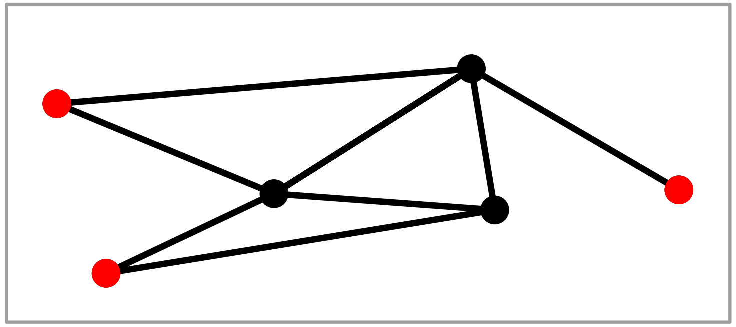

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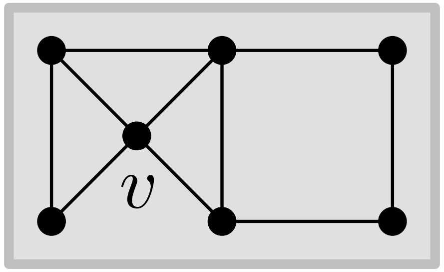
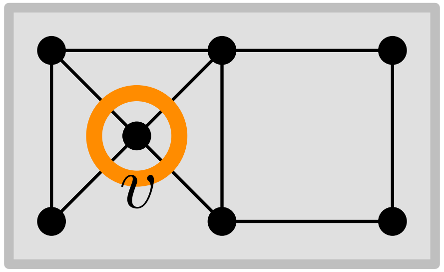
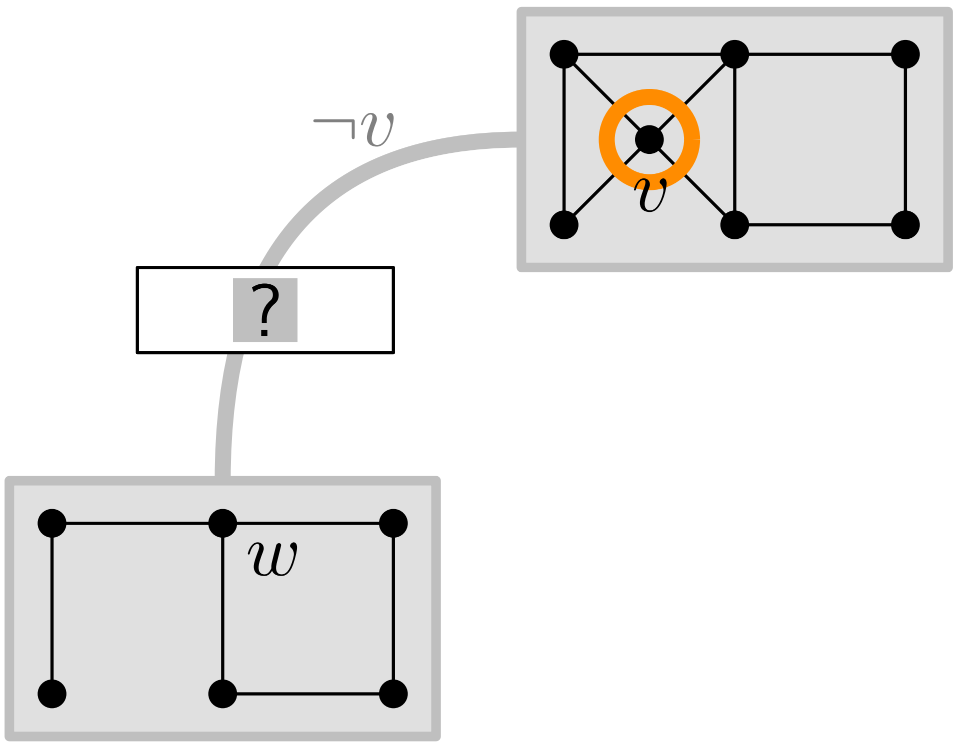
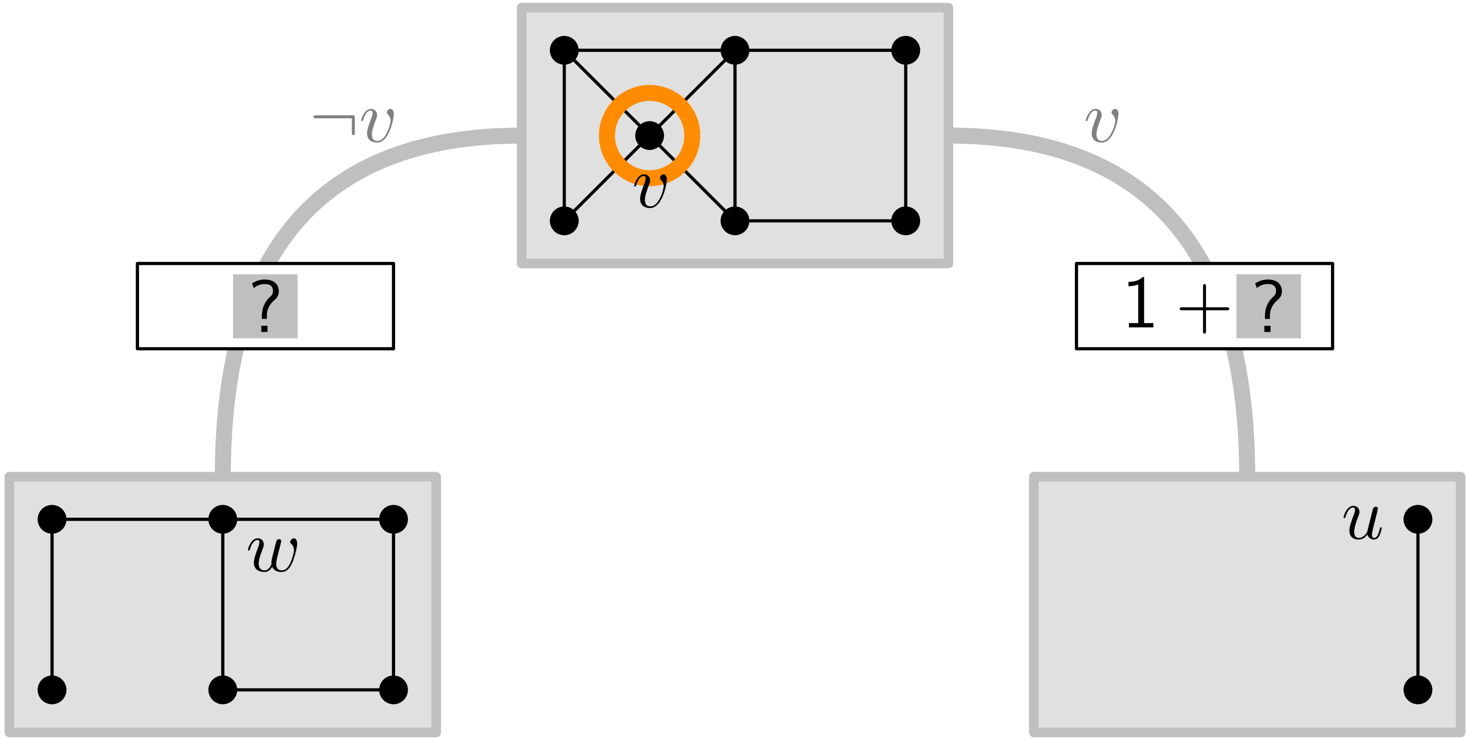
Algorithm NaiveMIS(graph $G = (V, E)$)
  if $V = \emptyset$ **then**
    return 0
  $v \leftarrow$ arbitrary vertex in $V(G)$
  return max$\{1 +$ NaiveMIS$(G - N(v) - \{v\})$, NaiveMIS$(G - \{v\})\}$


v


v



$\neg v$
v
?
w



$\neg v$
v
v
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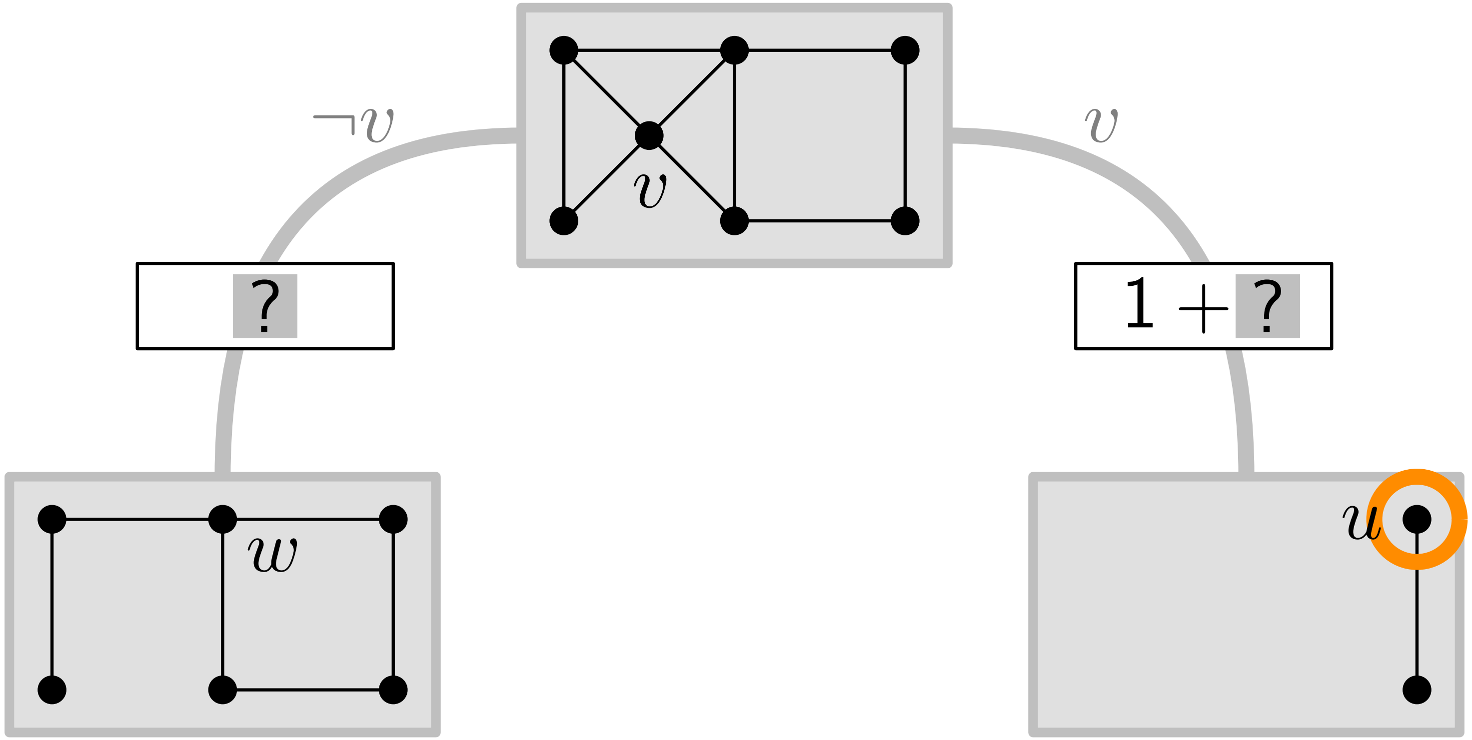
$1 + ?$
w
u



¬v
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v
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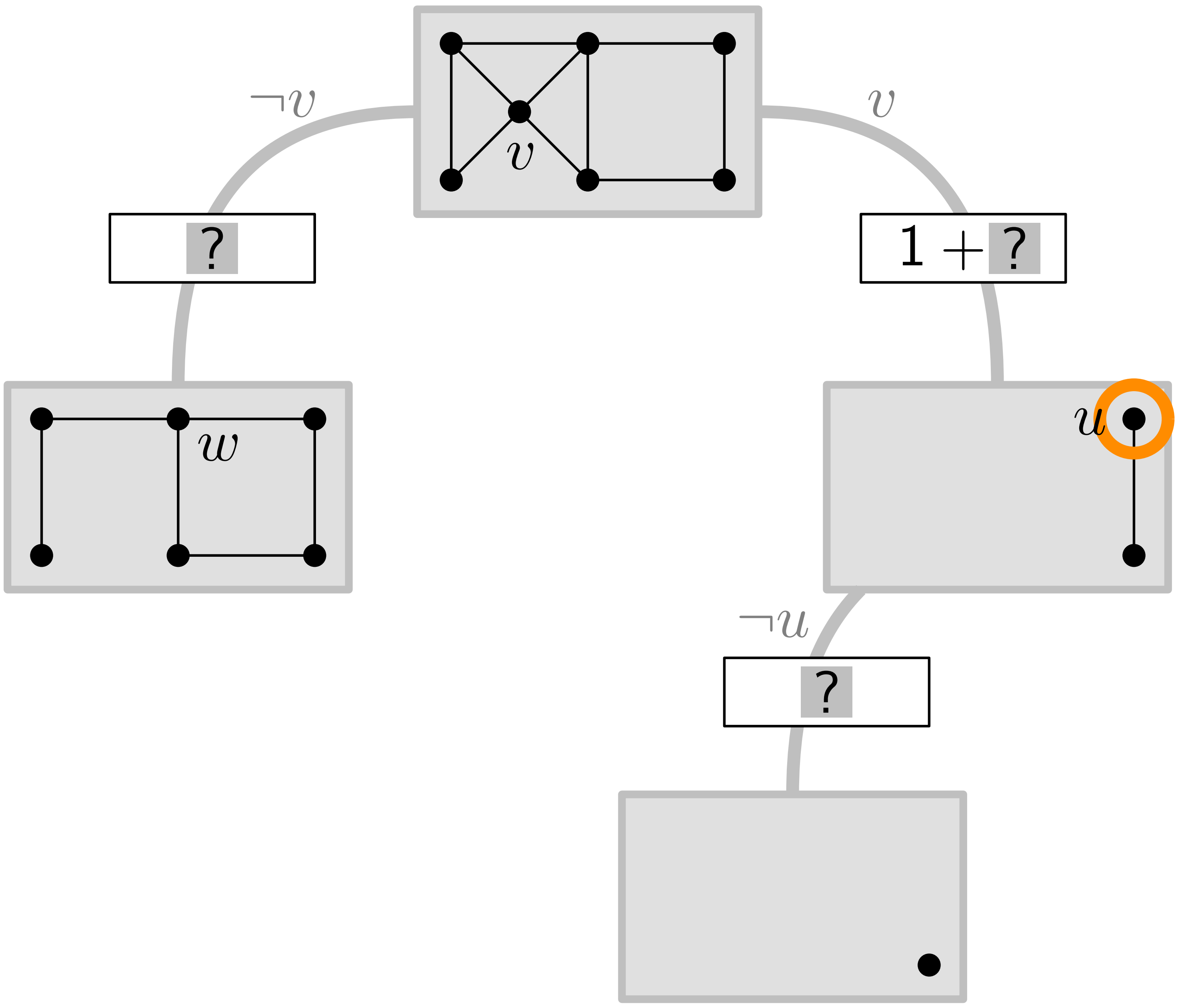
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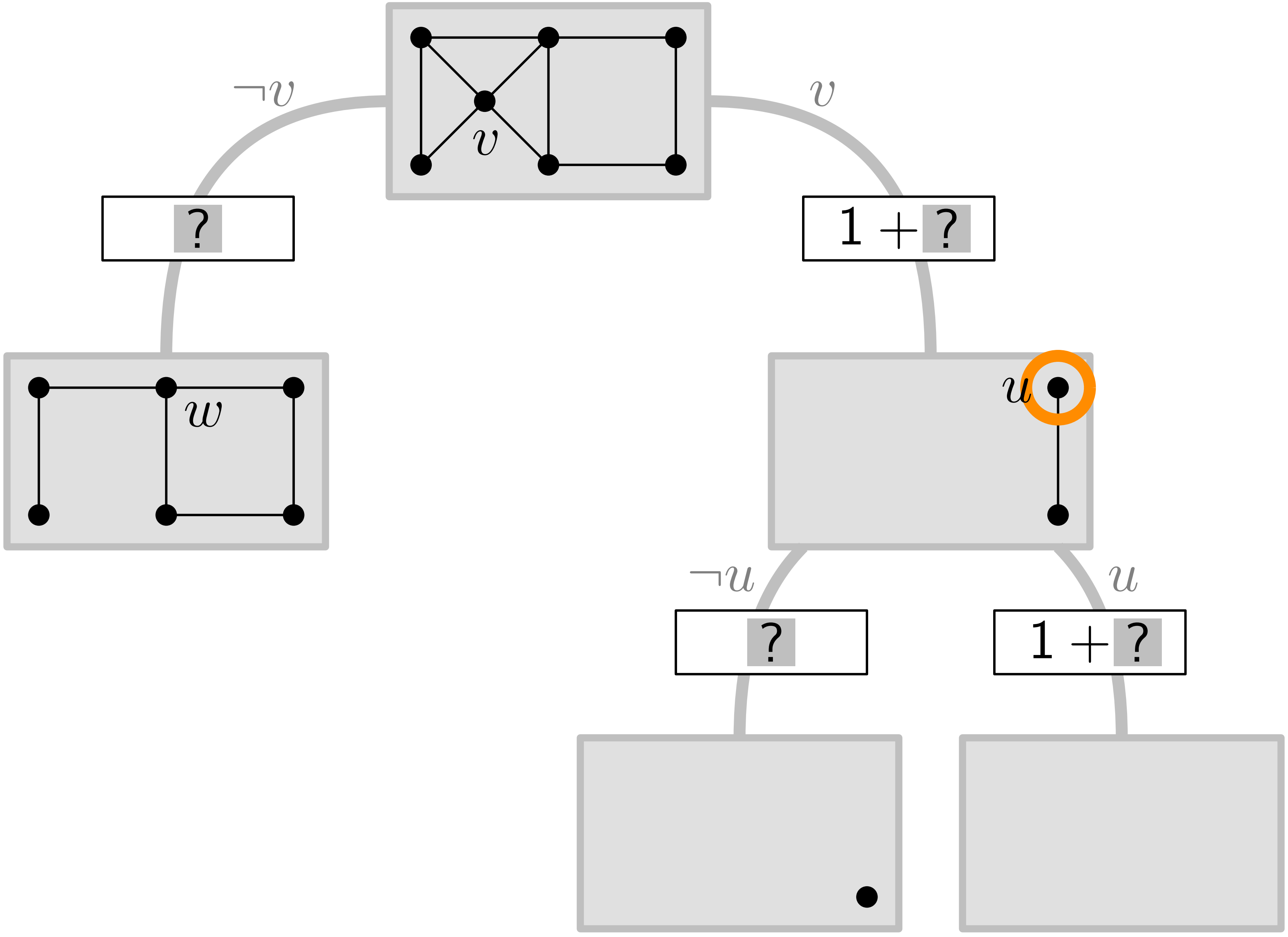

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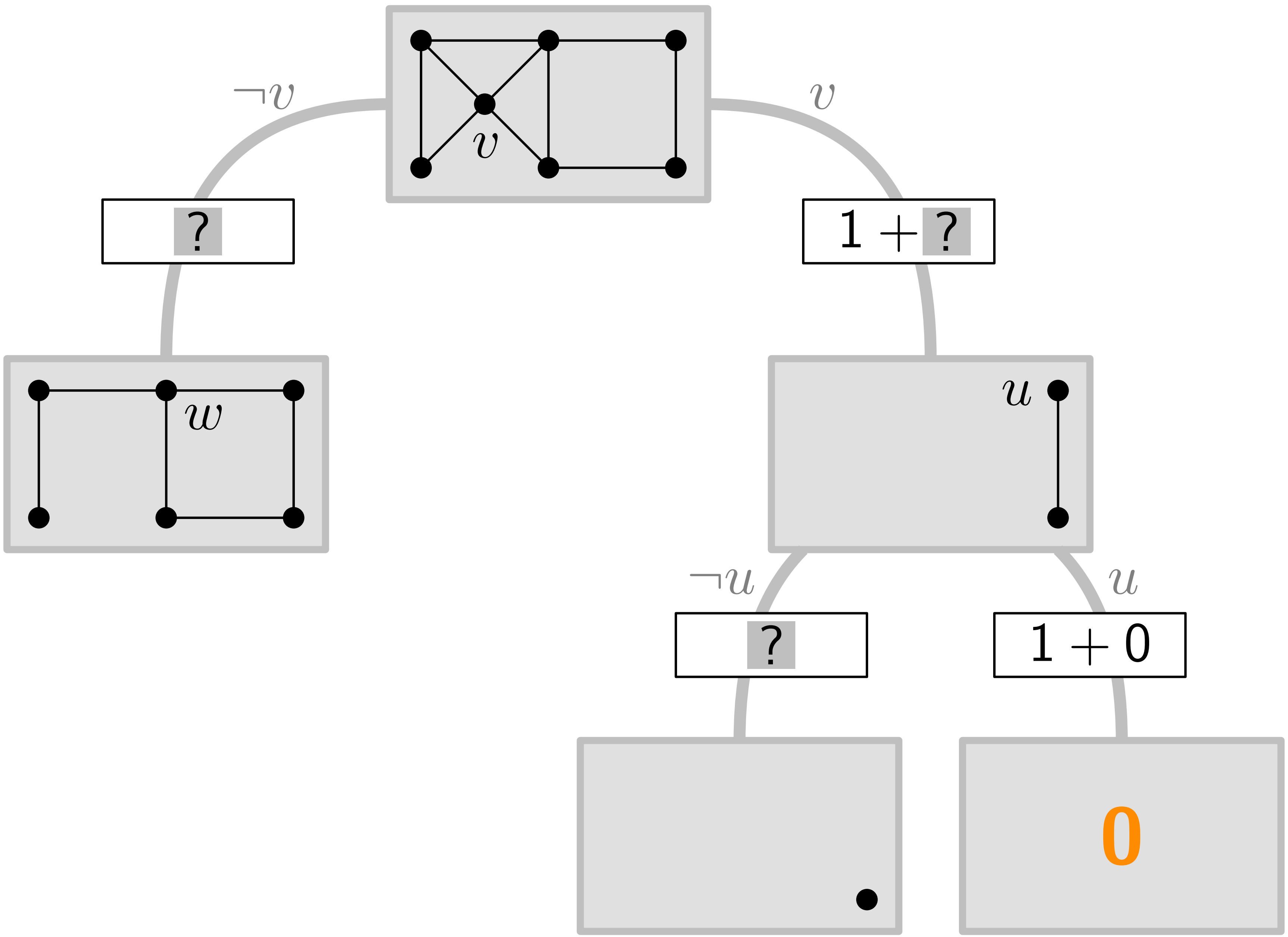
¬v
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?
1 + 0
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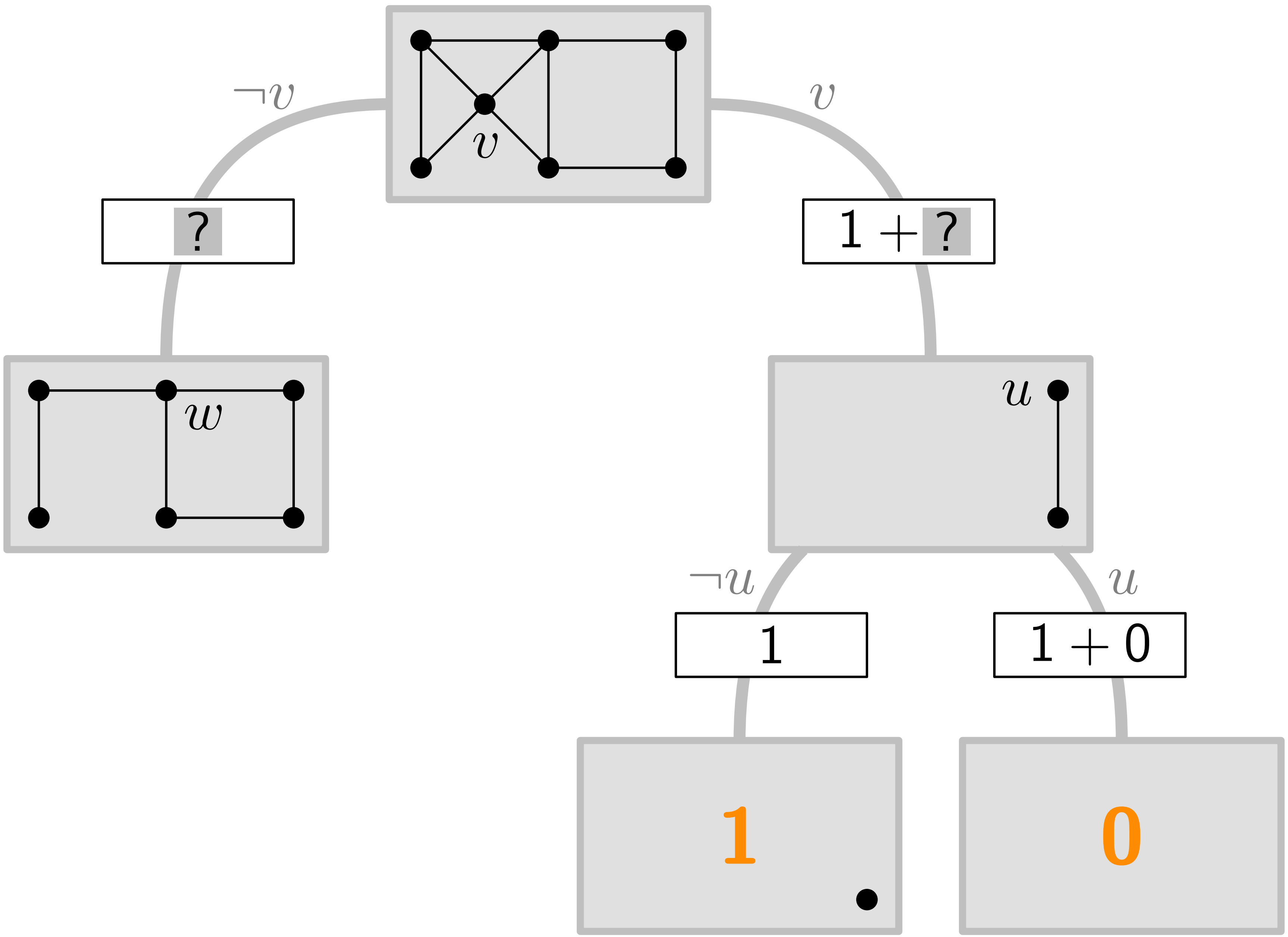
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u
1
1 + 0
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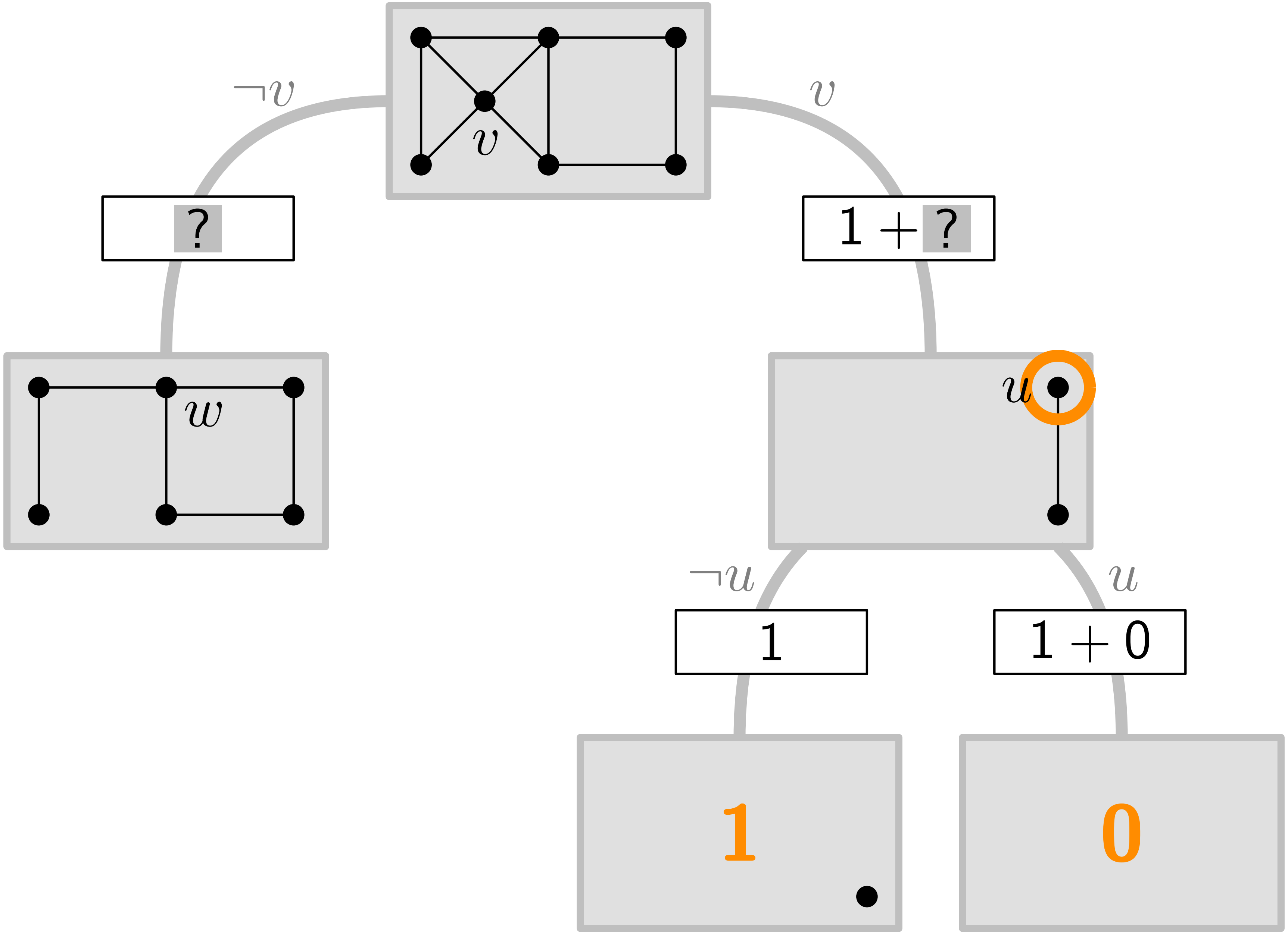

$\neg v$
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$1 + ?$
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$\neg u$
u
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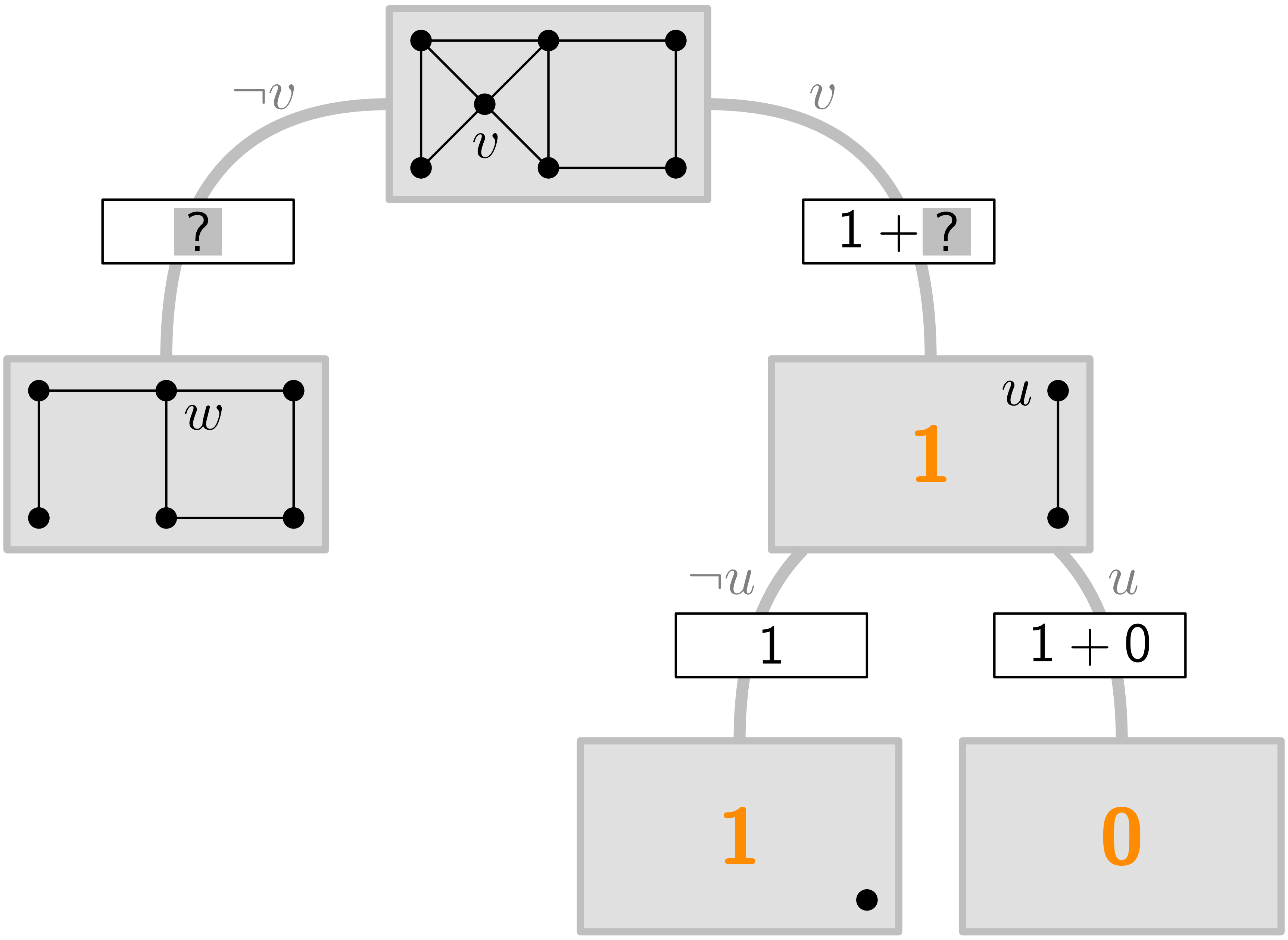
$1 + 0$
1
0



¬v
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v
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1 + ?
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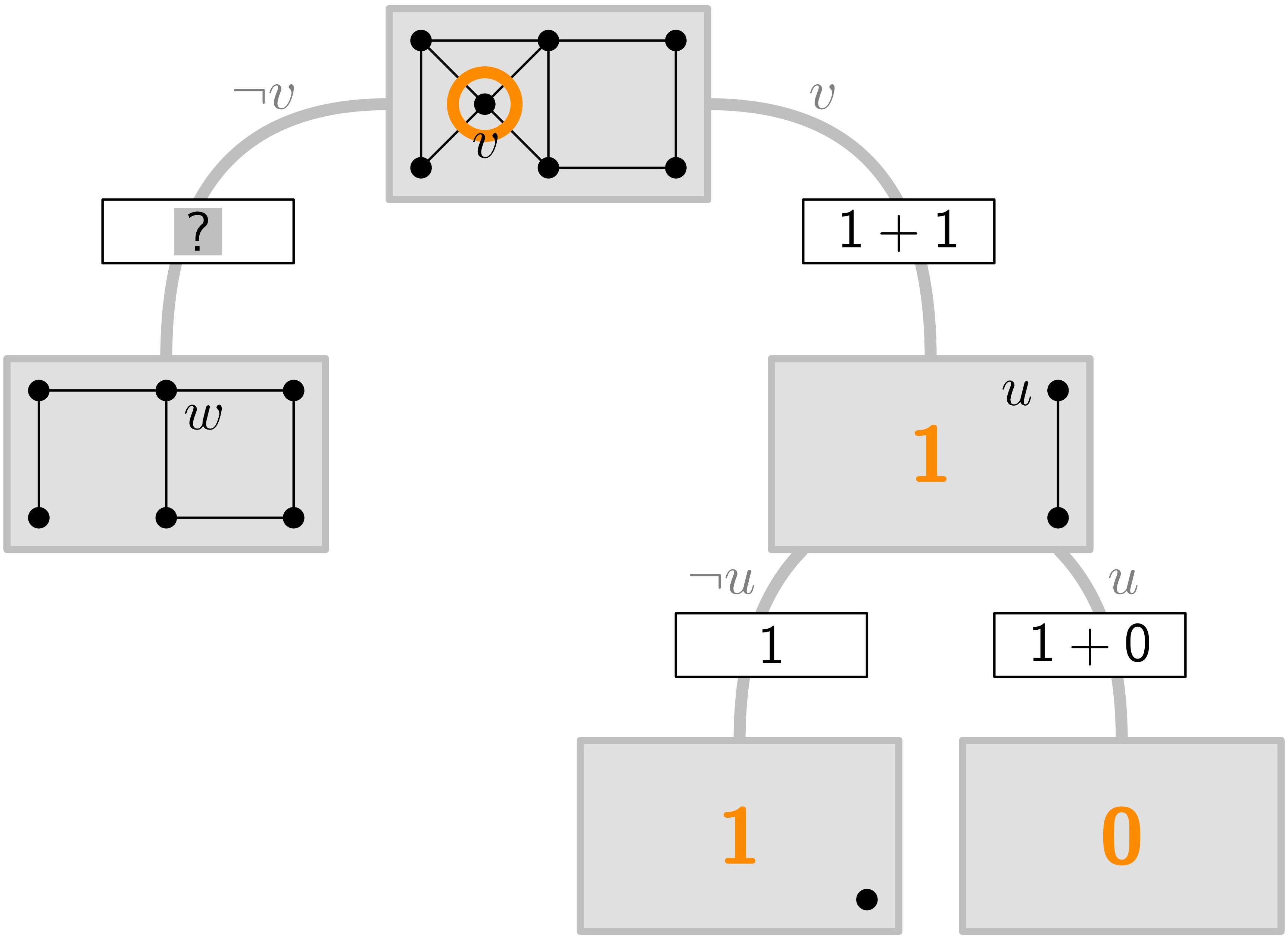
¬u
u
1
1 + 0
1
0



$\neg v$
v
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$1+1$
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$\neg u$
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$1+0$
1
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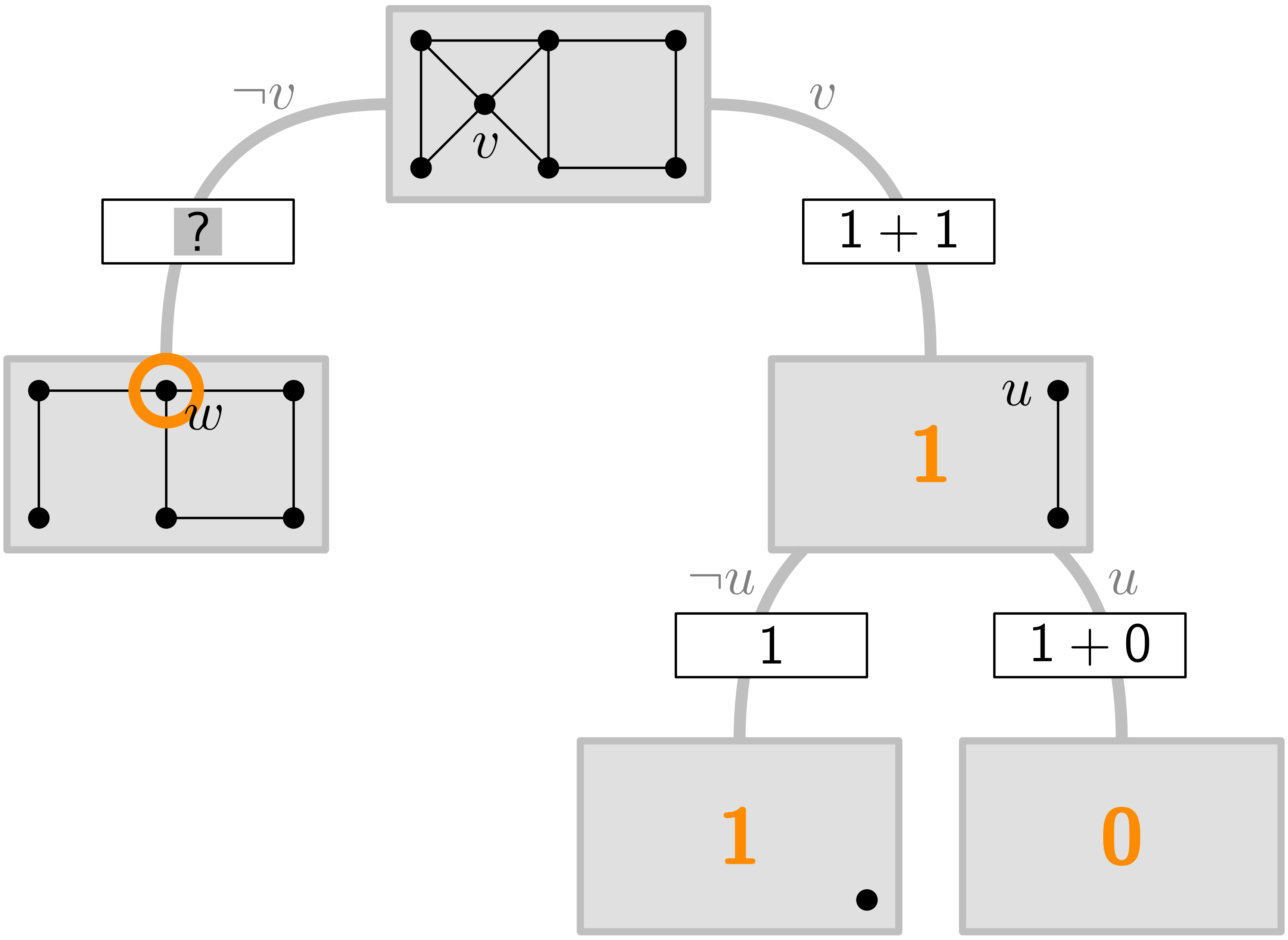

¬v
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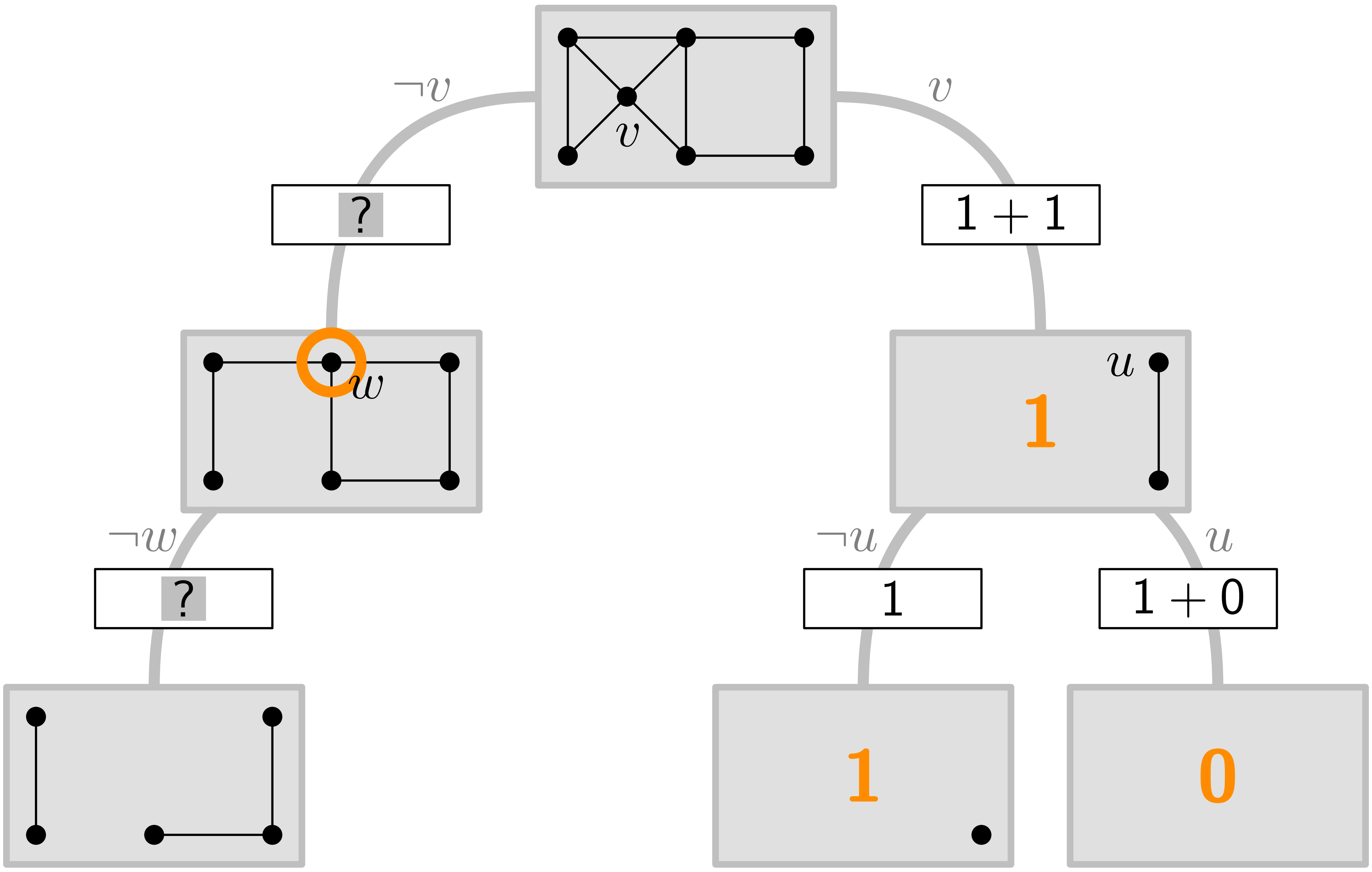

$\neg v$
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$1 + 1$
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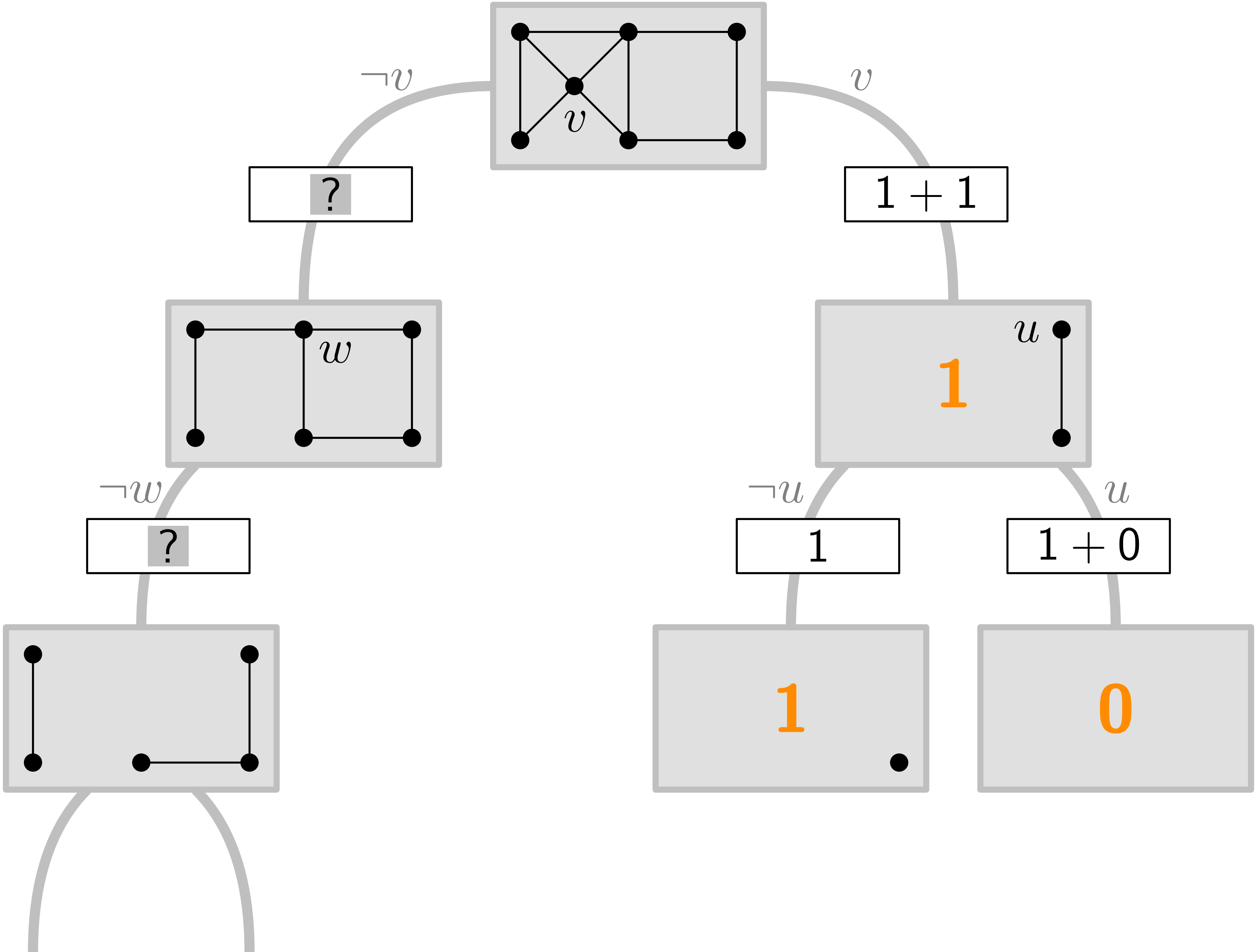
1
$\neg w$
$\neg u$
u
?
1
$1 + 0$
1
0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1 + 1
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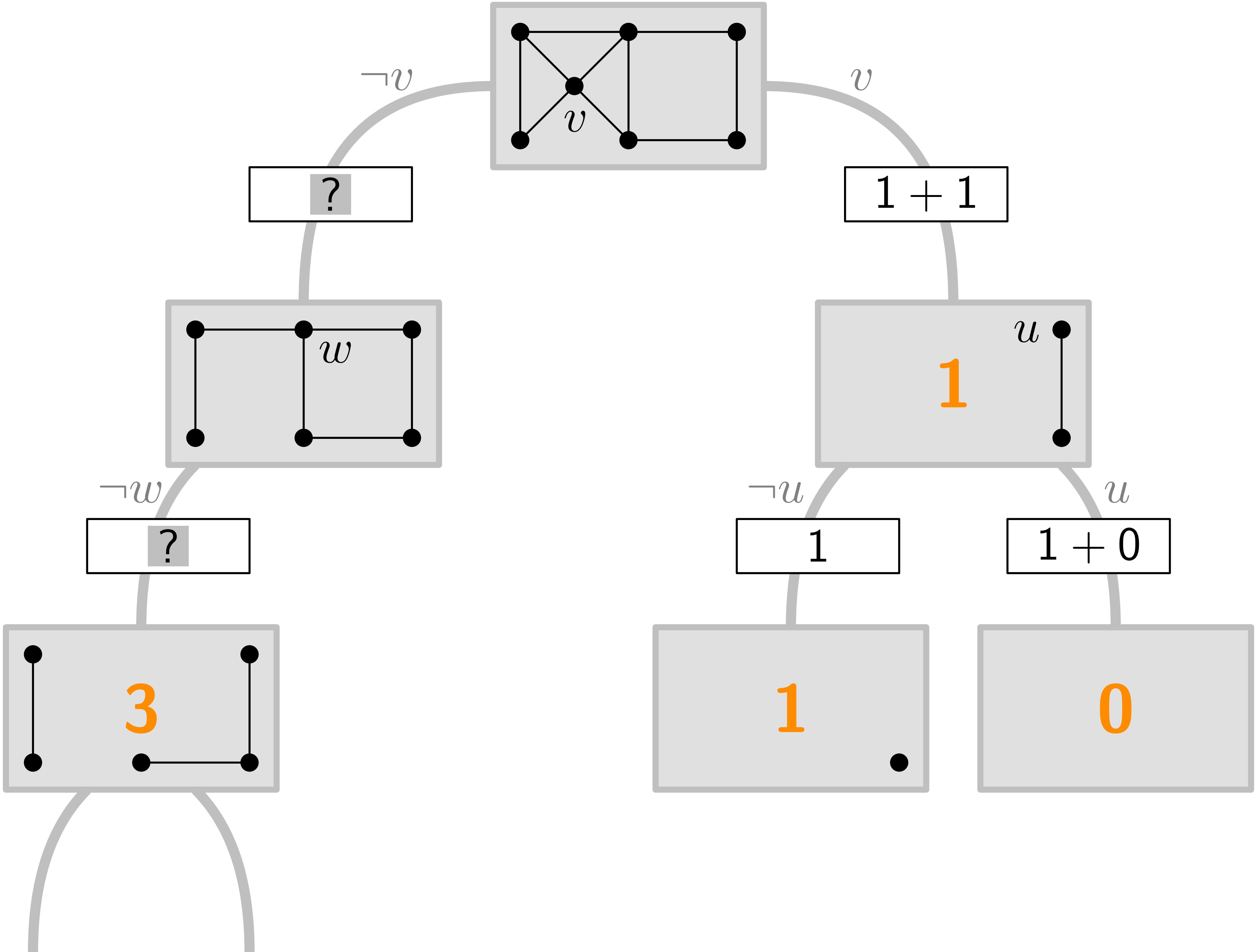
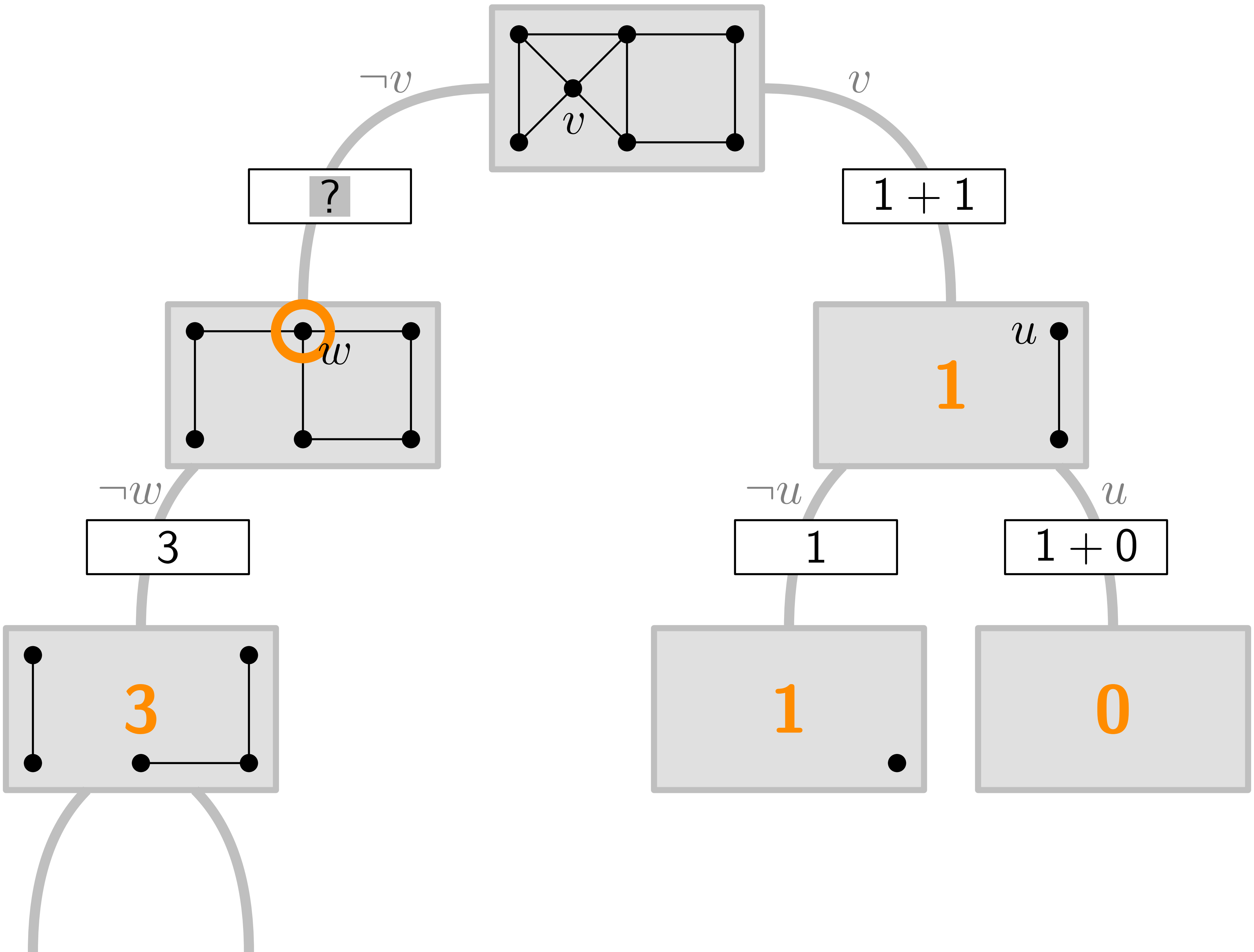
¬u
u
?
1
1 + 0
1
0



$\neg v$
v
v
?
$1 + 1$
w
u
1
$\neg w$
$\neg u$
u
?
1
$1 + 0$
3
1
0



$\neg v$
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$1+1$
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$\neg w$
$\neg u$
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$1+0$
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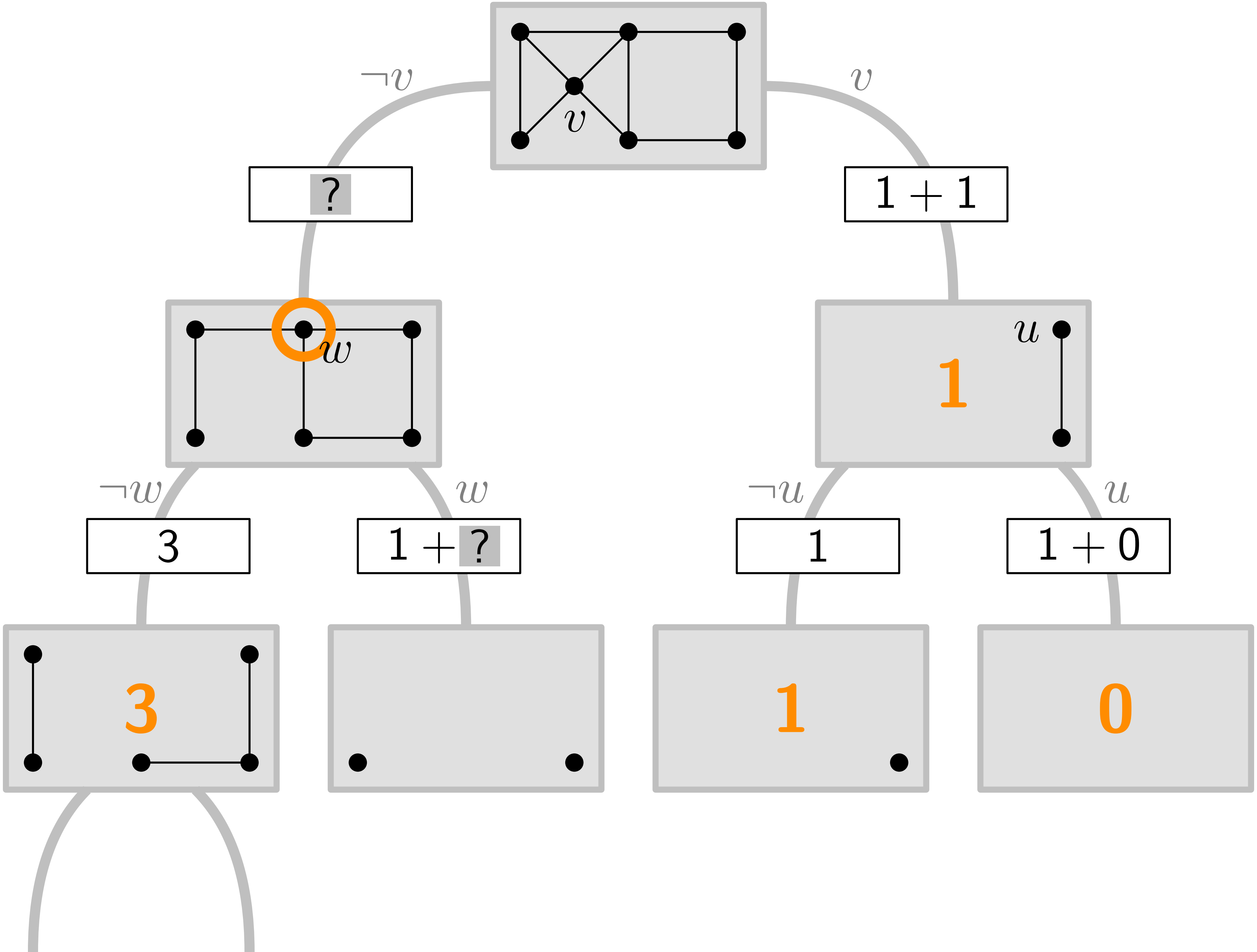
0



$\neg v$
v
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$1+1$
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$\neg w$
w
$\neg u$
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3
$1+?$
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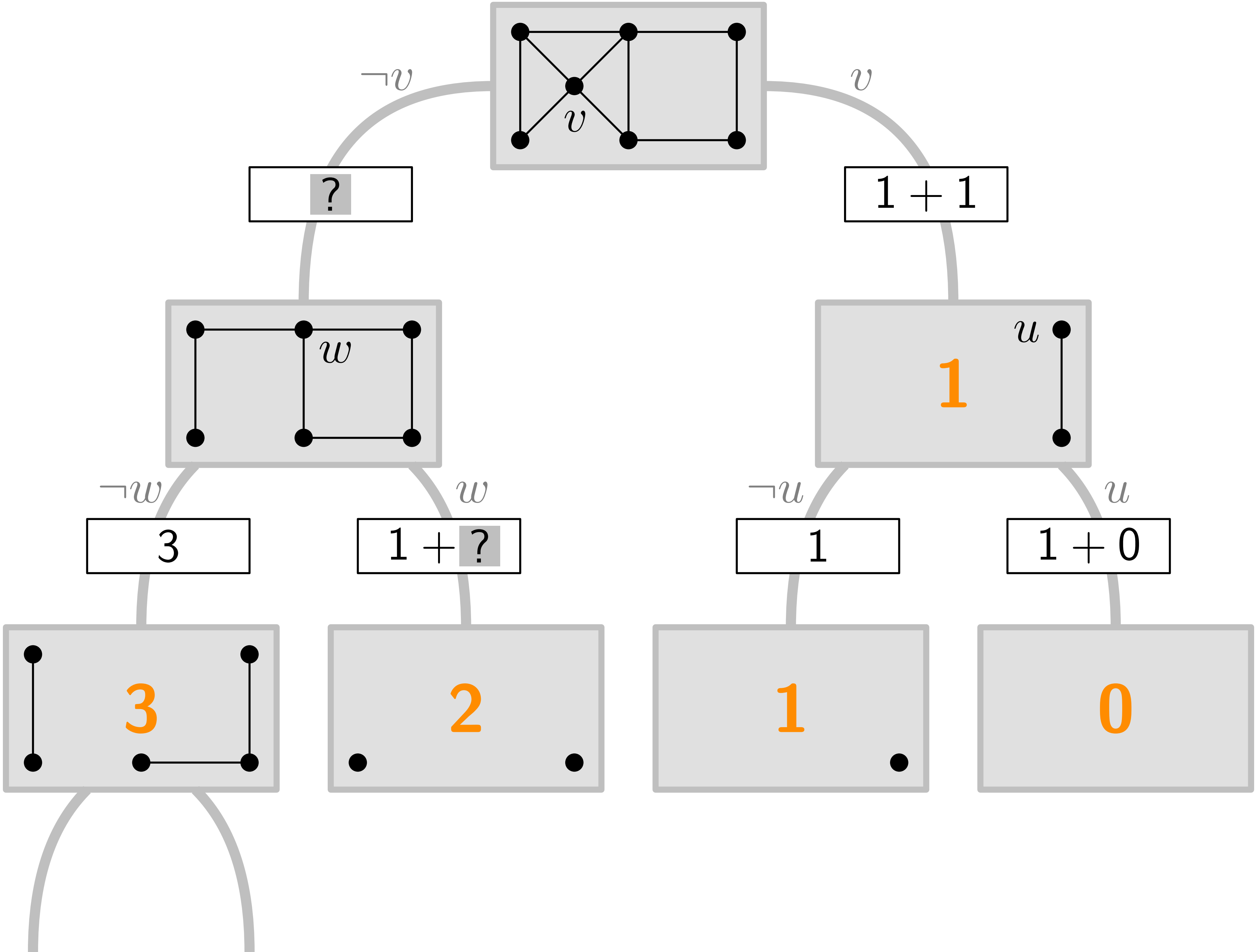
$1+0$
3
1
0



v
$\neg v$
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?
$1+1$
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$\neg w$
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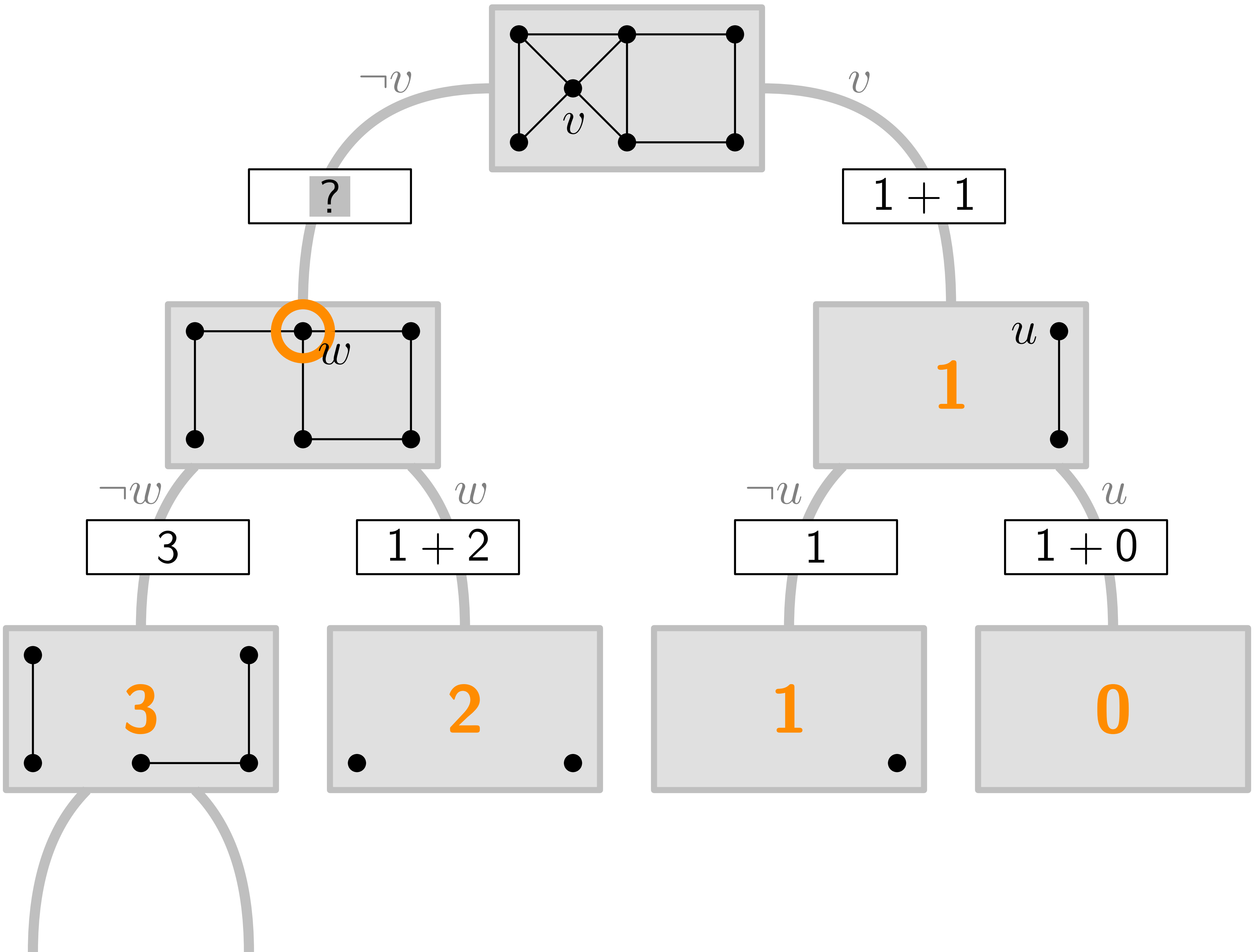
$\neg u$
u
3
$1+?$
1
$1+0$
3
2
1
0



¬v
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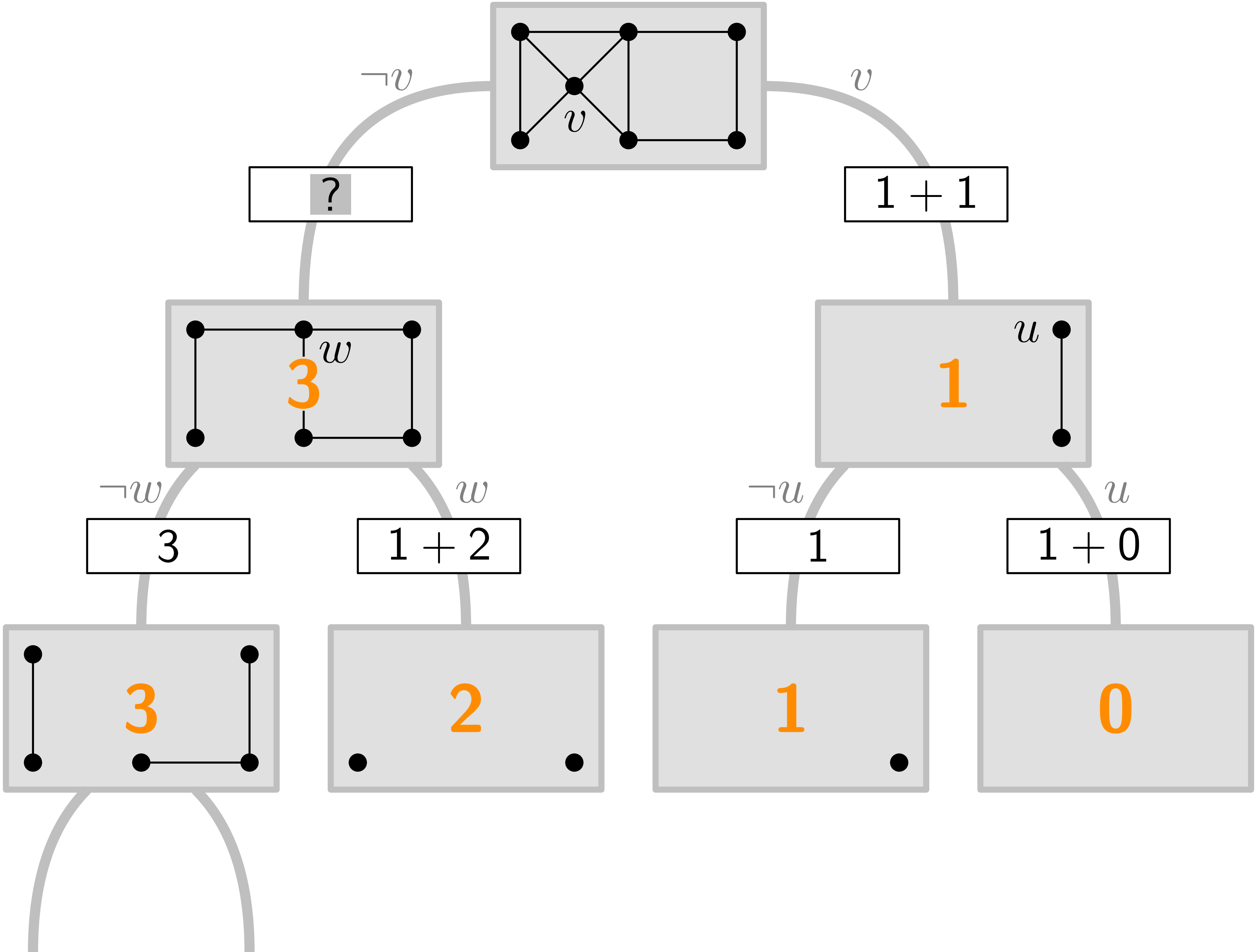

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$1+2$
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3
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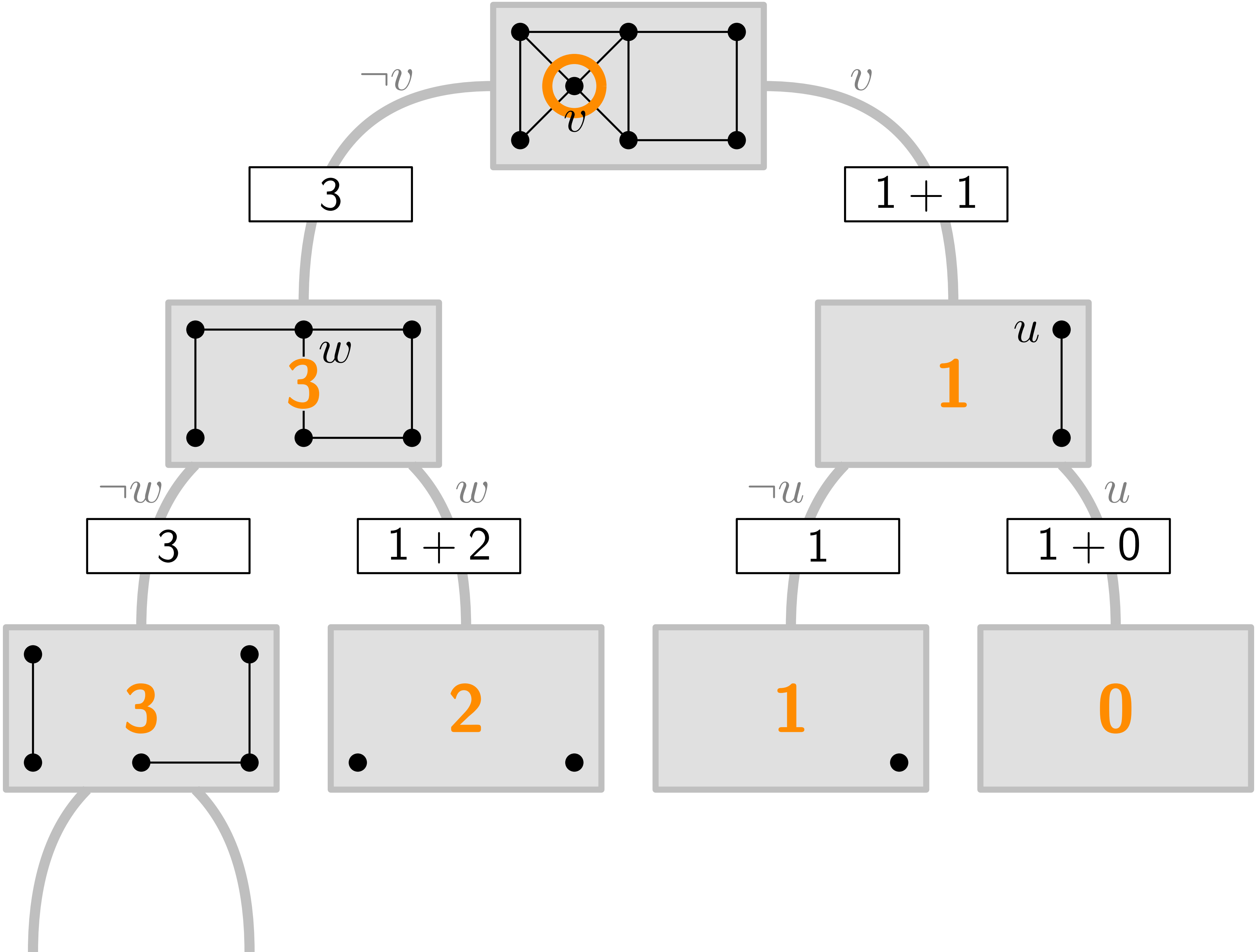
¬v
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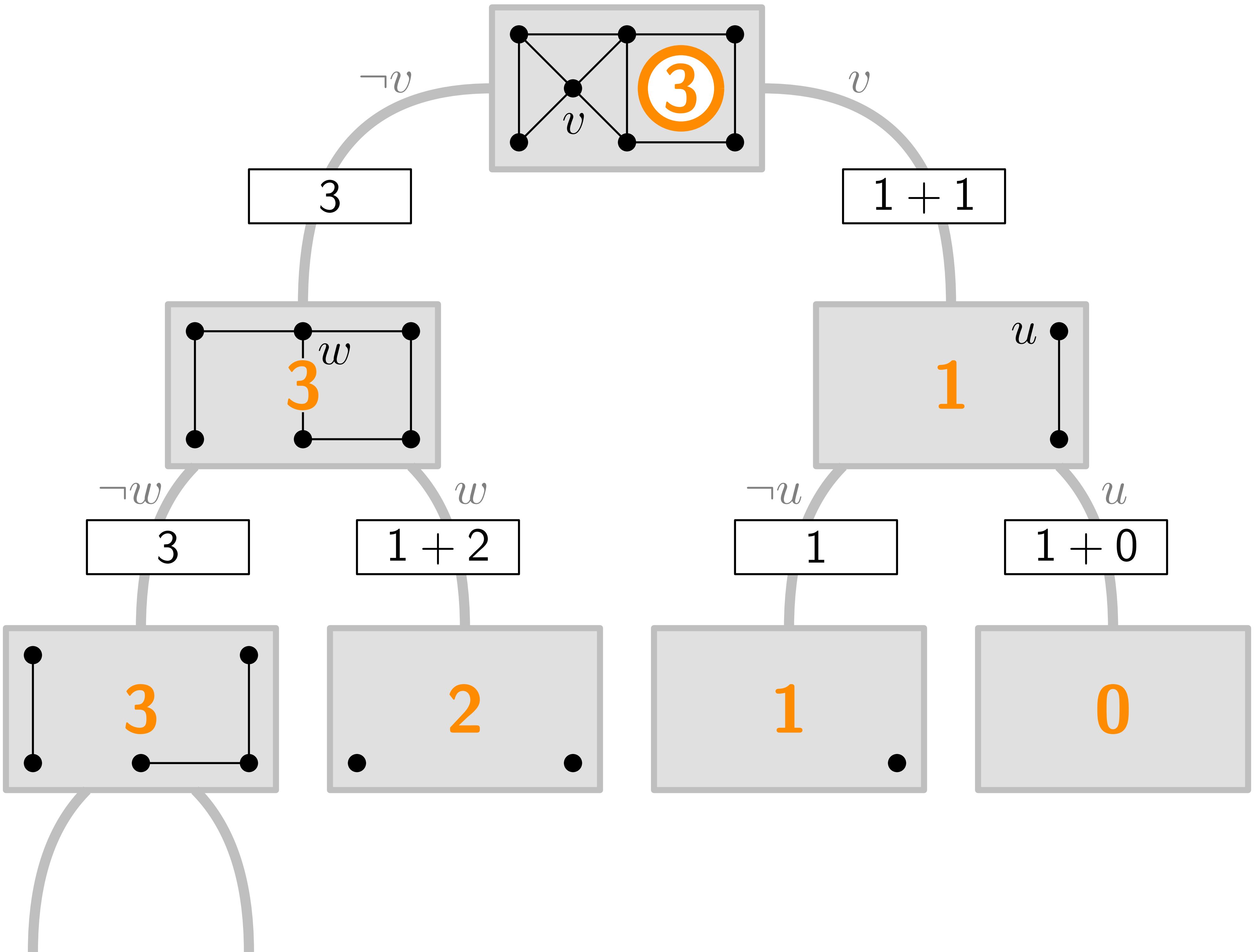

v
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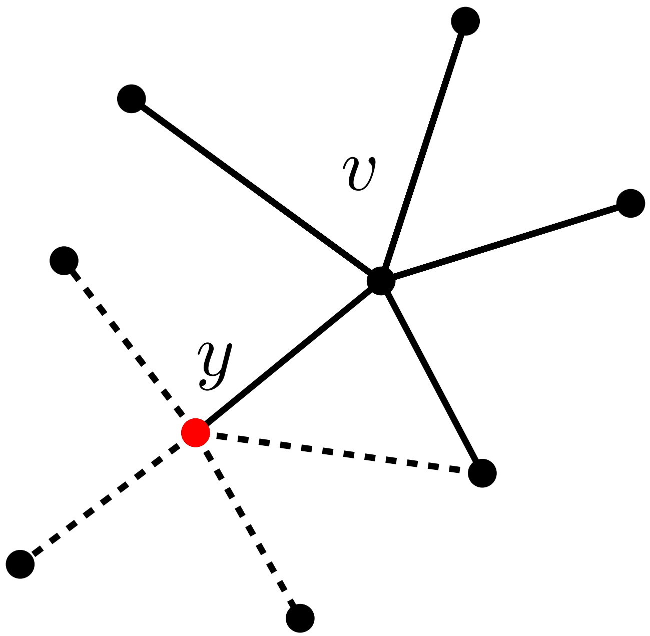
1 + 0
3
2
1
0

Observation

Lemma. Let U be a *maximum* independent set in G.



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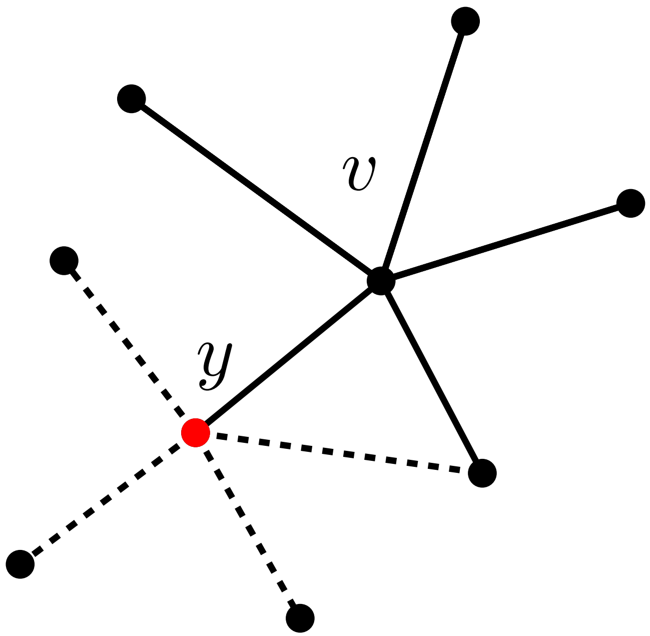

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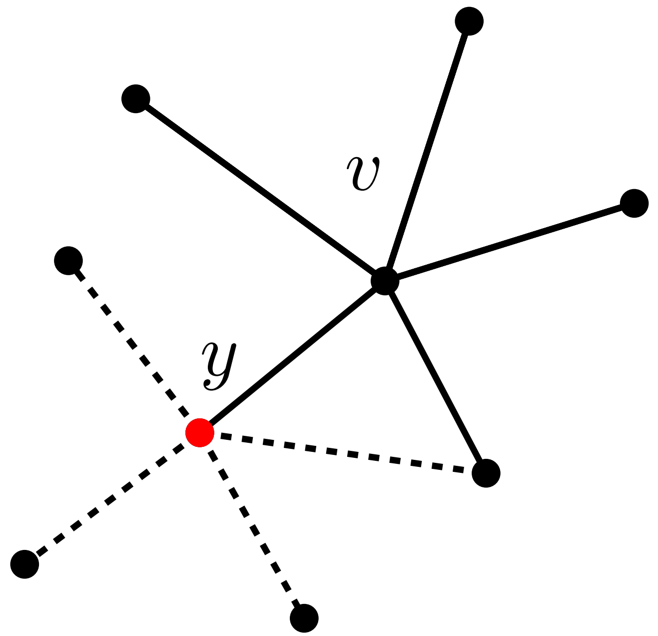

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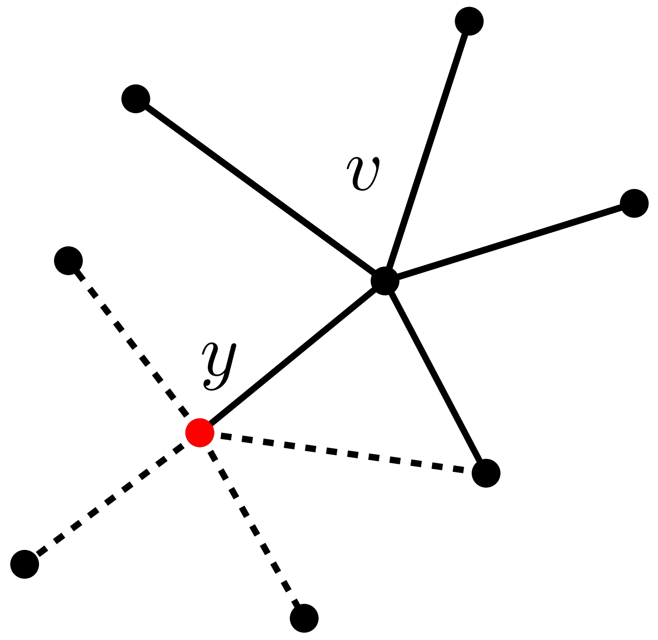
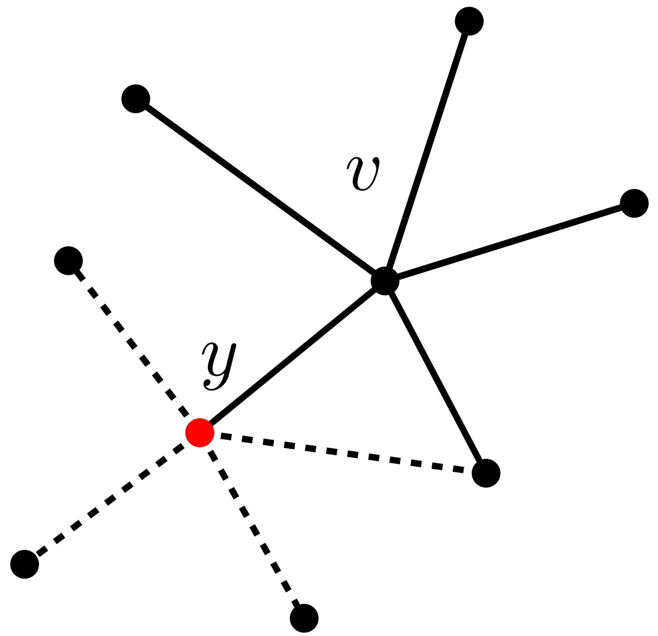
(i) $v \in U \Rightarrow\ N(v) \cap U\ = \emptyset$
(ii) $v \notin U \Rightarrow$



Observation

Lemma. Let U be a *maximum* independent set in G. Then, for each vertex $v \in V$:

(i) $v \in U \Rightarrow \; N(v) \cap U \; = \emptyset$
(ii) $v \notin U \Rightarrow |N(v) \cap U| \geq 1$



Observation

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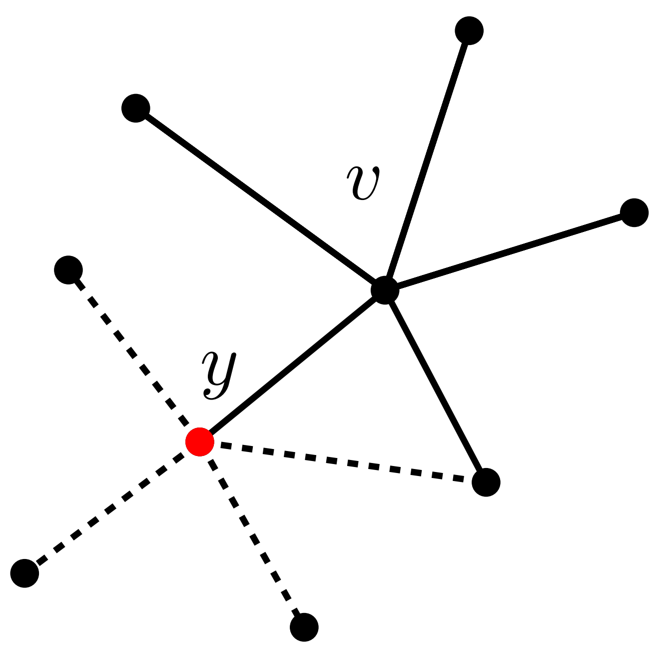
Then, for each vertex $v \in V$:

(i) $v \in U \Rightarrow \; N(v) \cap U \; = \emptyset$
(ii) $v \notin U \Rightarrow |N(v) \cap U| \geq 1$

Thus, $N[v] := N(v) \cup \{v\}$ contains some $y \in U$,
and no other vertex of $N[y]$ is in U.



Smarter Branching Algorithm

Algorithm MIS(G)
  if $V = \emptyset$ **then**
    return 0
  $v \leftarrow$ vertex of minimum degree in $V(G)$
  return $1 + \max\{\text{MIS}(G - N[y]) \mid y \in N[v]\}$

Smarter Branching Algorithm

```
Algorithm MIS(G)
  if V = ∅ then
    return 0
  v ← vertex of minimum degree in V(G)
  return 1 + max{MIS(G − N[y]) | y ∈ N[v]}
```

Correctness: follows from the previous lemma.

Smarter Branching Algorithm

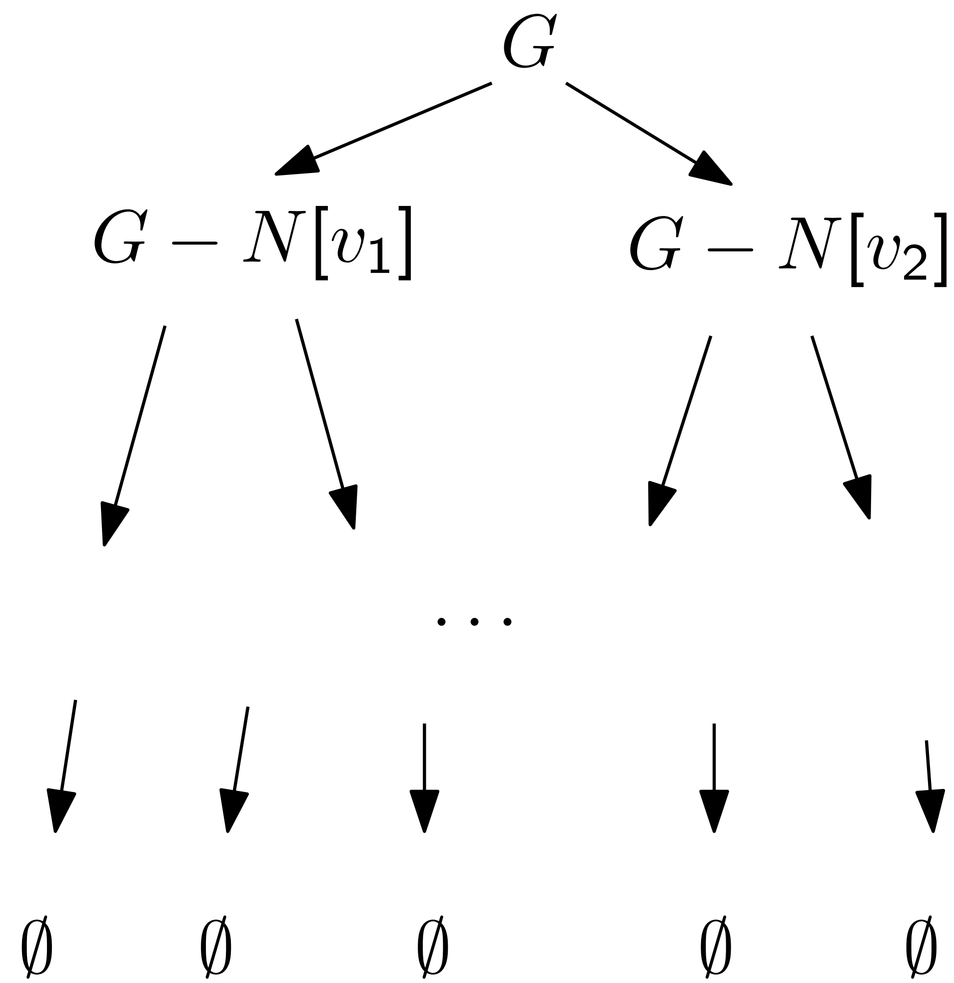
Algorithm MIS(G)
if $V = \emptyset$ **then**
  return 0
$v \leftarrow$ vertex of minimum degree in $V(G)$
return $1 + \max\{\text{MIS}(G - N[y]) \mid y \in N[v]\}$

Correctness: follows from the previous lemma.

We will now prove a runtime of $O^*(3^{n/3}) = O^*(1.4423^n)$

Runtime

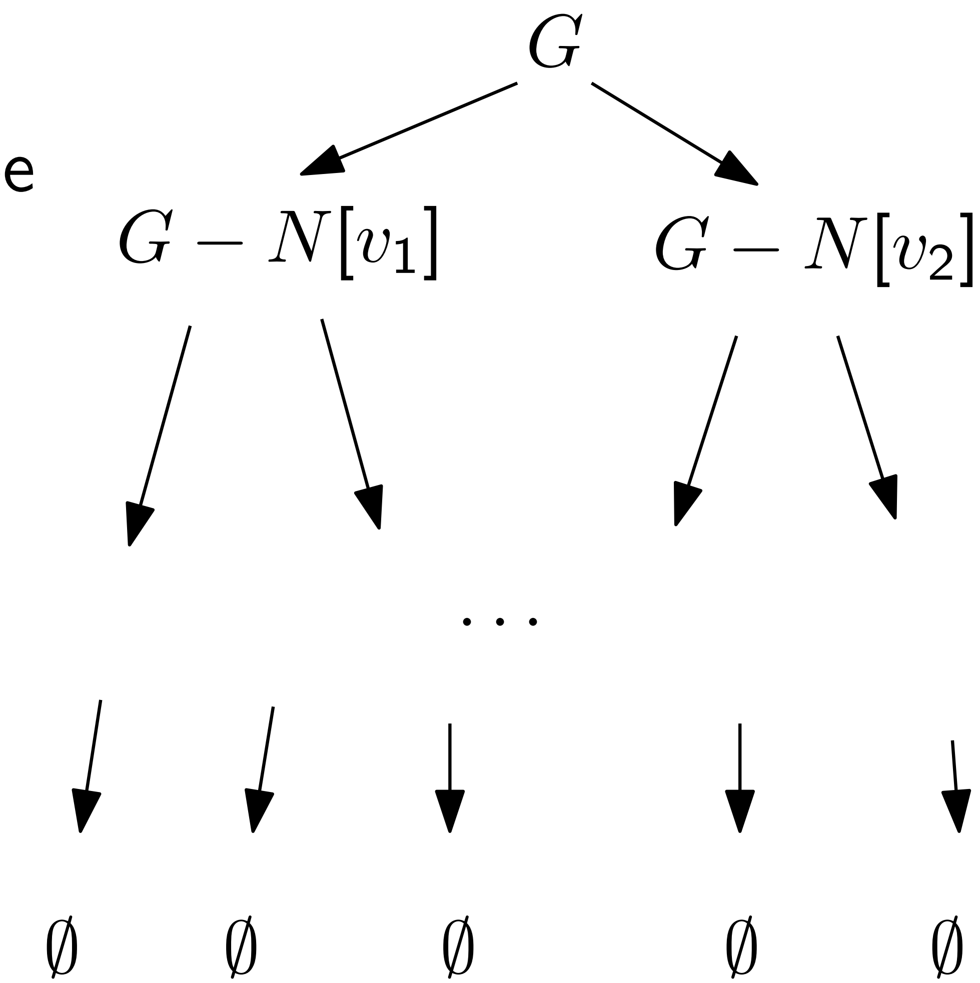
Execution corresponds to a *search tree* whose nodes are labeled with the input of the respective recursive call.



Runtime

Execution corresponds to a *search tree* whose nodes are labeled with the input of the respective recursive call.

Let $B(n)$ be the maximum number of leaves of a search tree for a graph with n vertices.

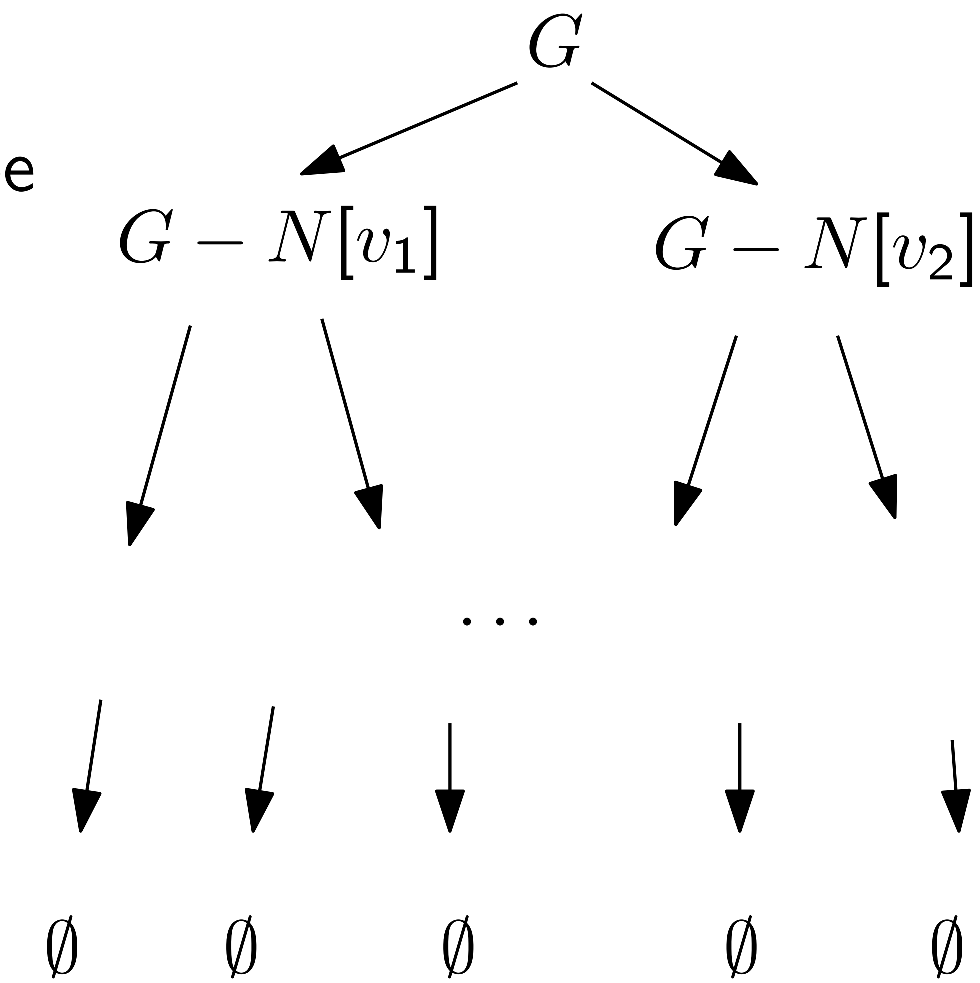


Runtime

Execution corresponds to a *search tree* whose nodes are labeled with the input of the respective recursive call.

Let $B(n)$ be the maximum number of leaves of a search tree for a graph with n vertices.

The search tree has height $\leq$



Runtime

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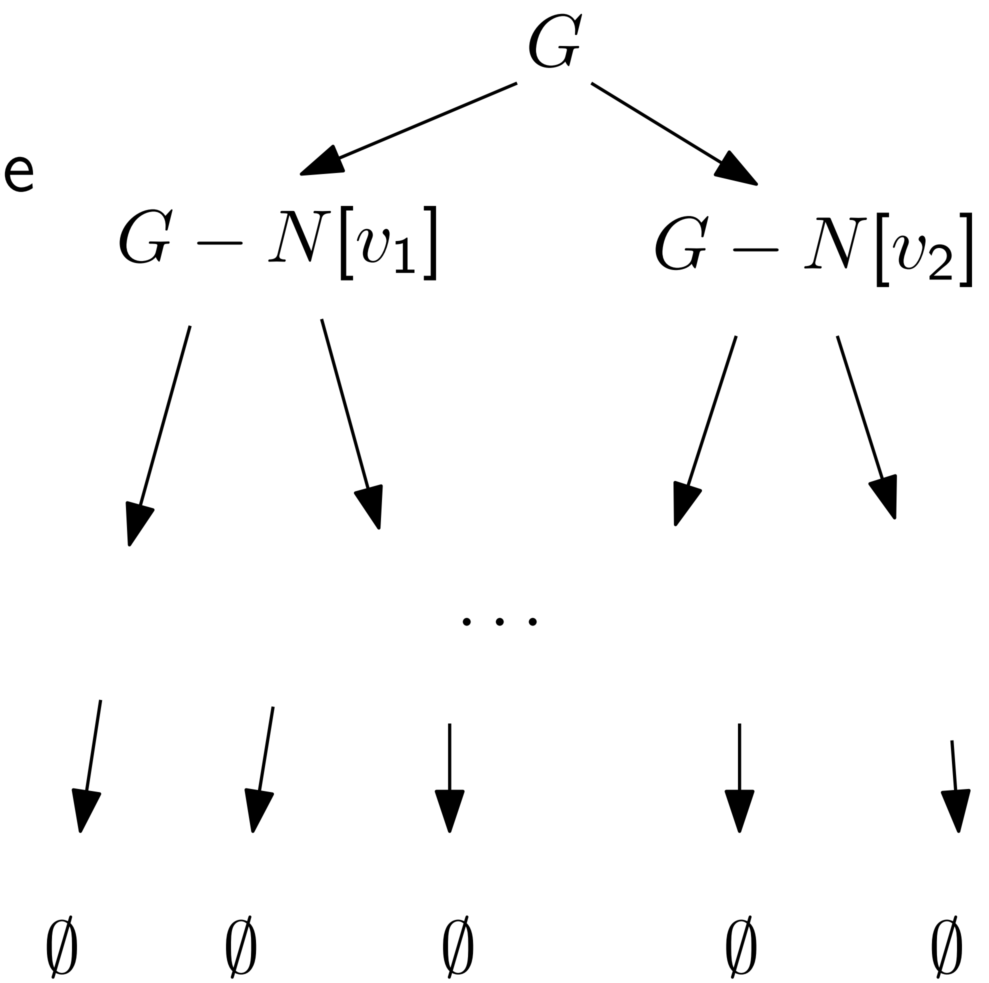
Let $B(n)$ be the maximum number of leaves of a search tree for a graph with n vertices.

The search tree has height $\leq n$.



Runtime

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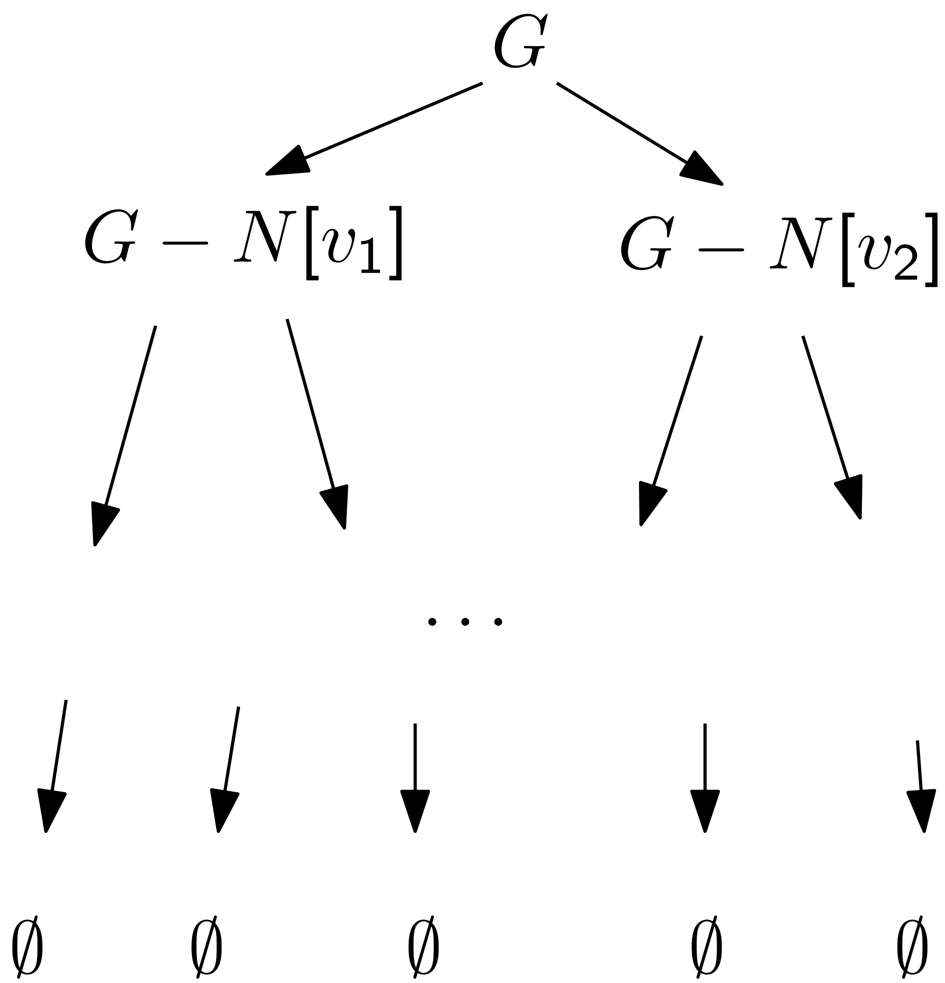


The search tree has height $\leq n$.

$\Rightarrow$ Algorithm runs in time $T(n) \in$

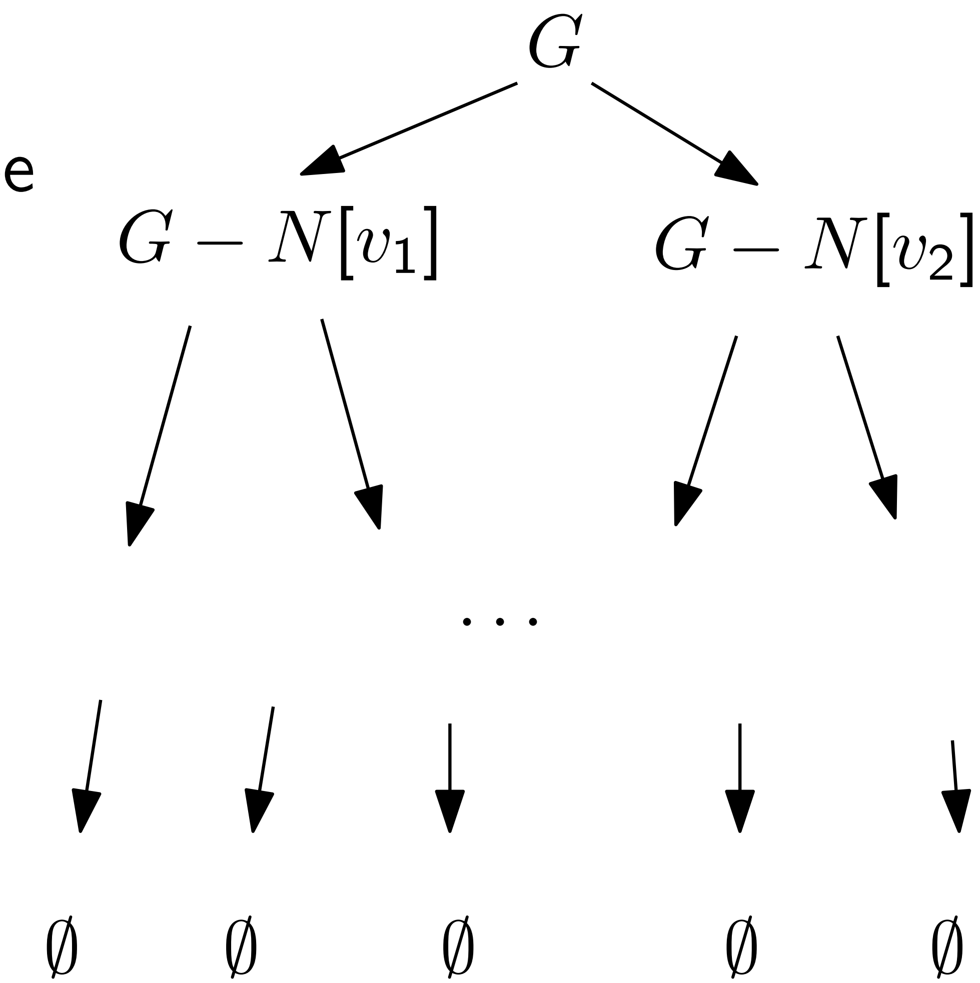
Runtime

Execution corresponds to a *search tree* whose nodes are labeled with the input of the respective recursive call.

Let $B(n)$ be the maximum number of leaves of a search tree for a graph with n vertices.

The search tree has height $\leq n$.

$\Rightarrow$ Algorithm runs in time $T(n) \in O^*(nB(n)) =$



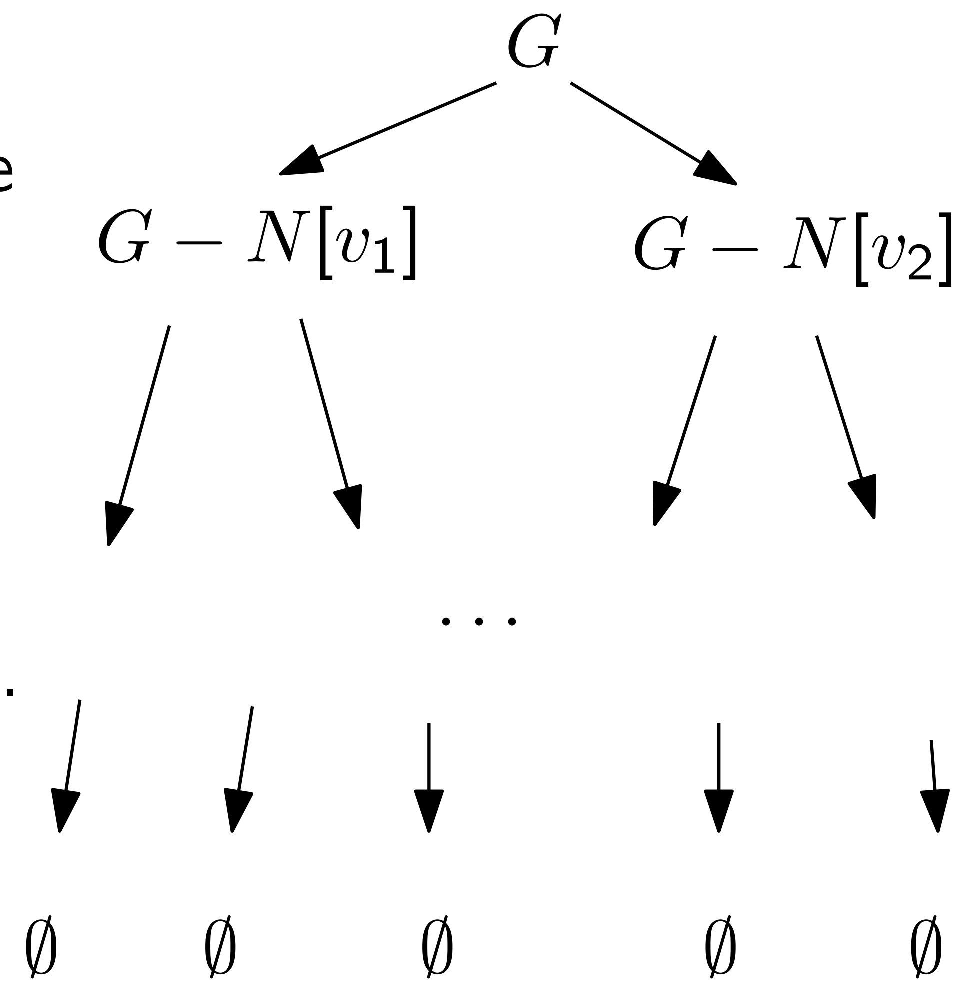
Runtime

Execution corresponds to a *search tree* whose nodes are labeled with the input of the respective recursive call.

Let $B(n)$ be the maximum number of leaves of a search tree for a graph with n vertices.

The search tree has height $\leq n$.

$\Rightarrow$ Algorithm runs in time $T(n) \in O^*(nB(n)) = O^*(B(n))$.



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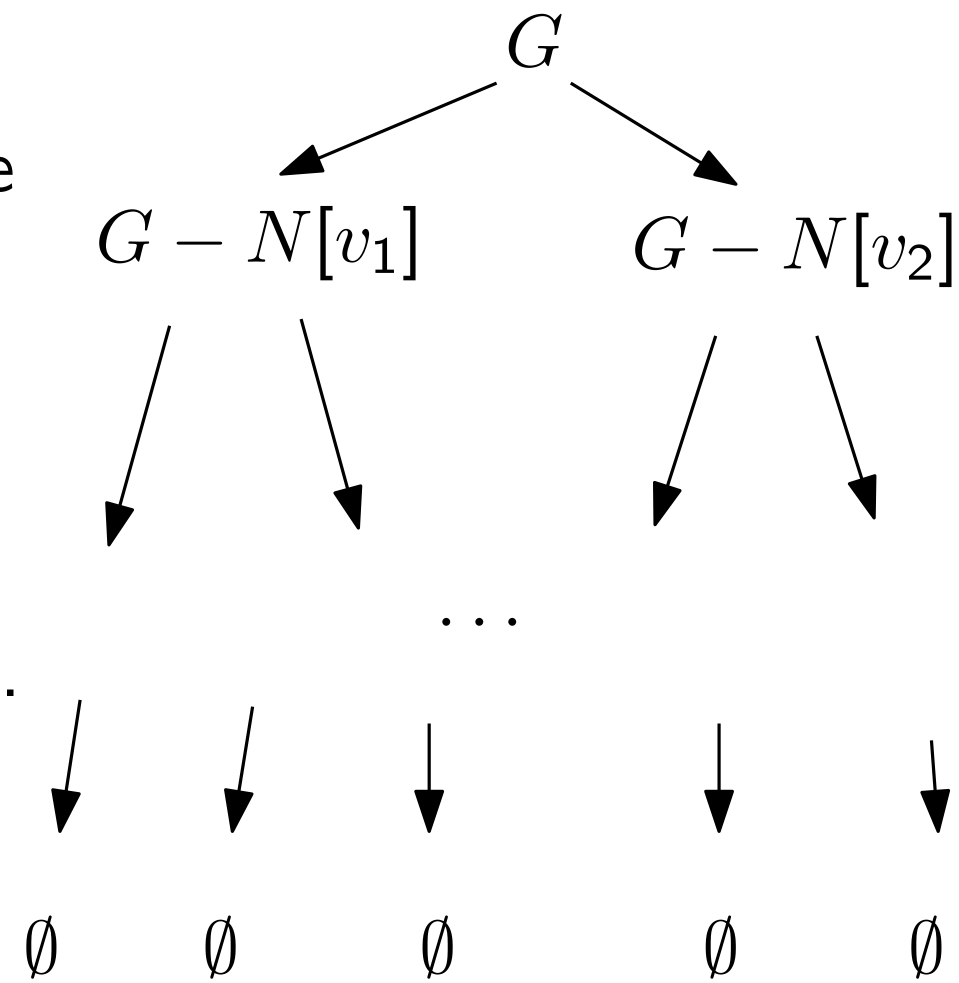
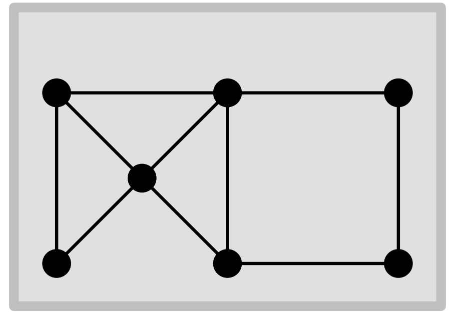
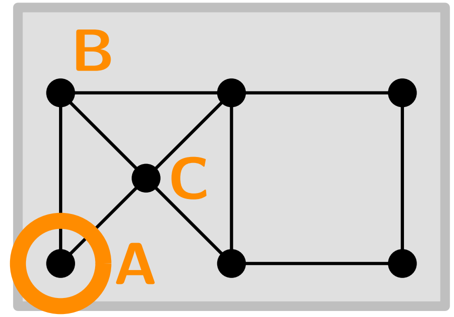


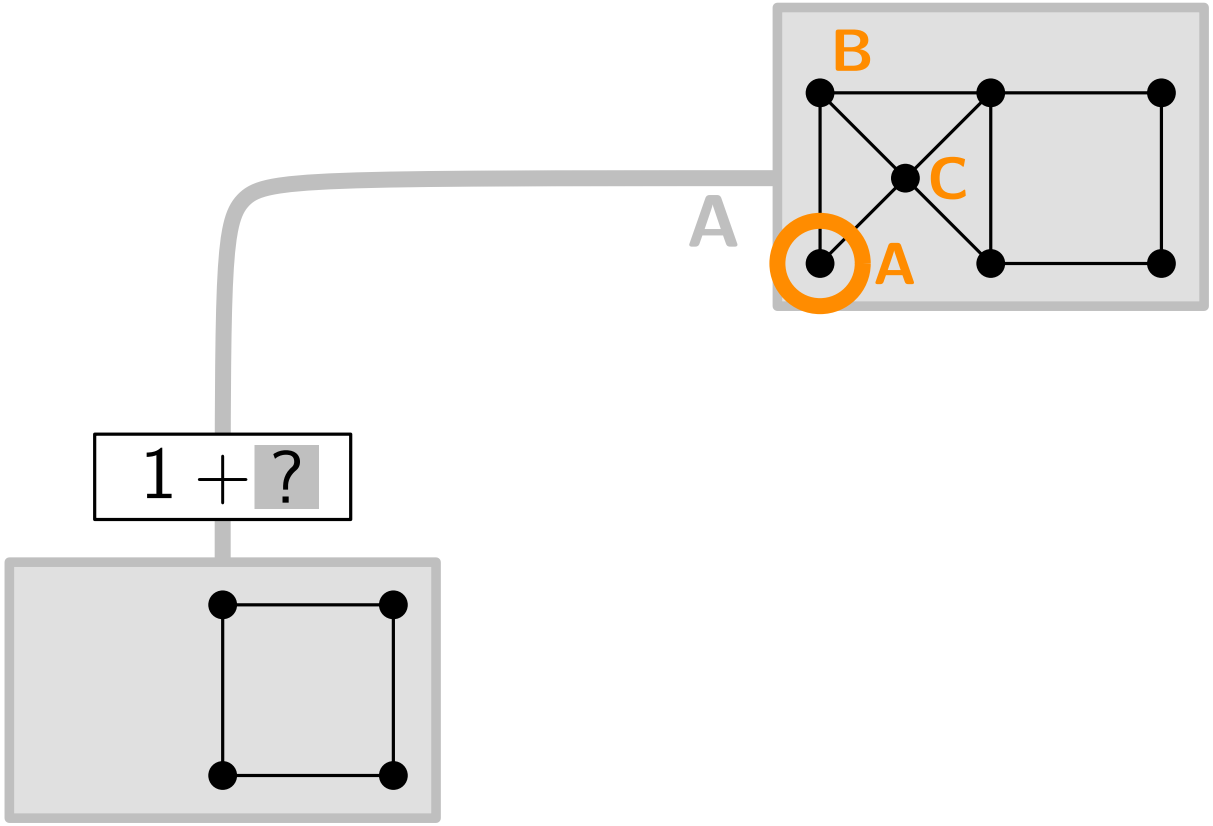
The search tree has height $\leq n$.

$\Rightarrow$ Algorithm runs in time $T(n) \in O^*(nB(n)) = O^*(B(n))$.

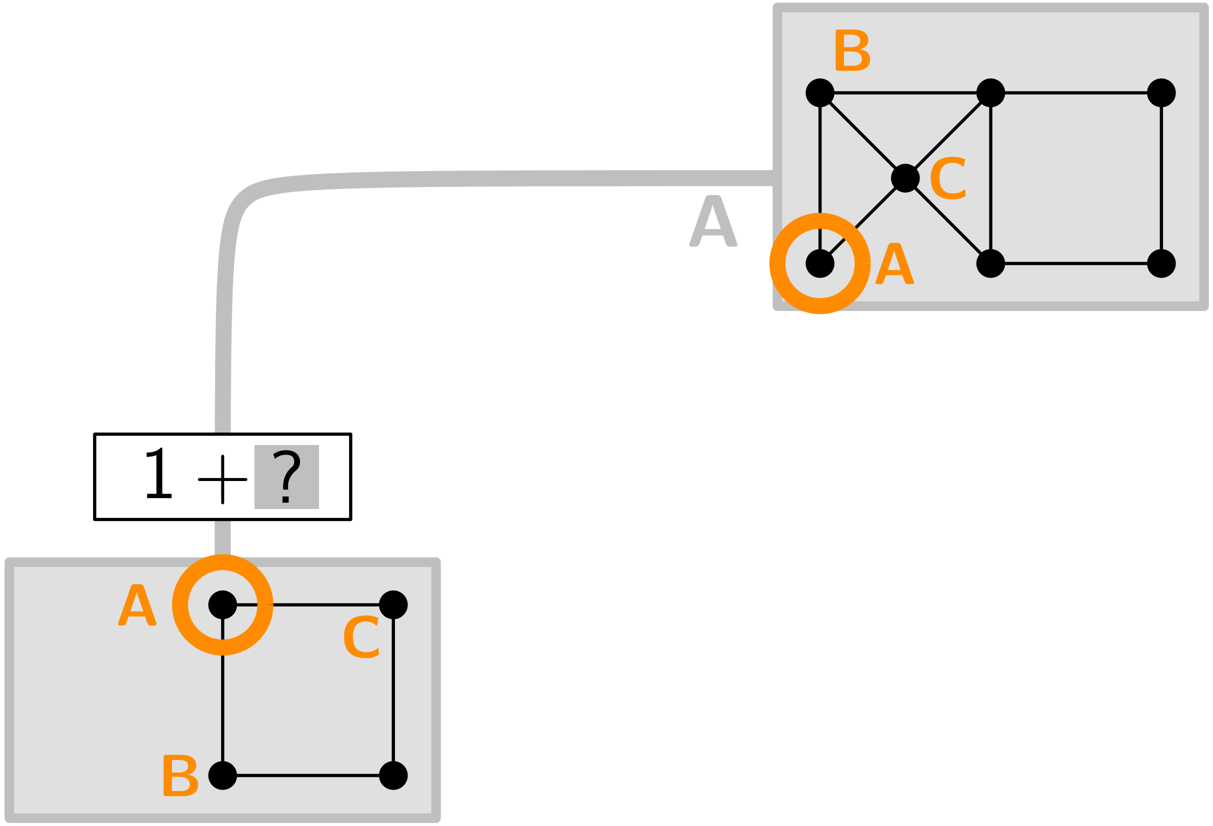
Let's consider an example run.



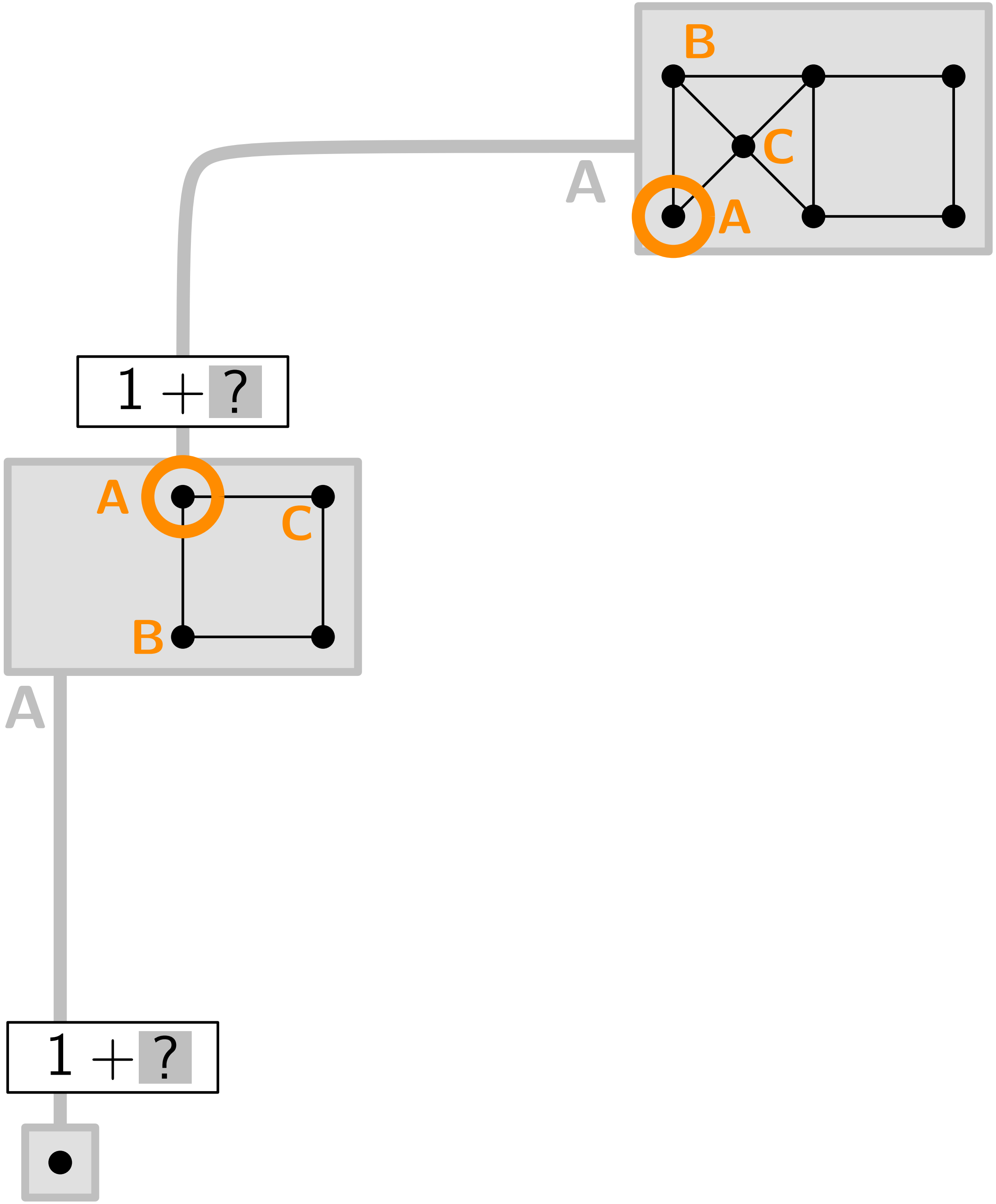

B
C
A

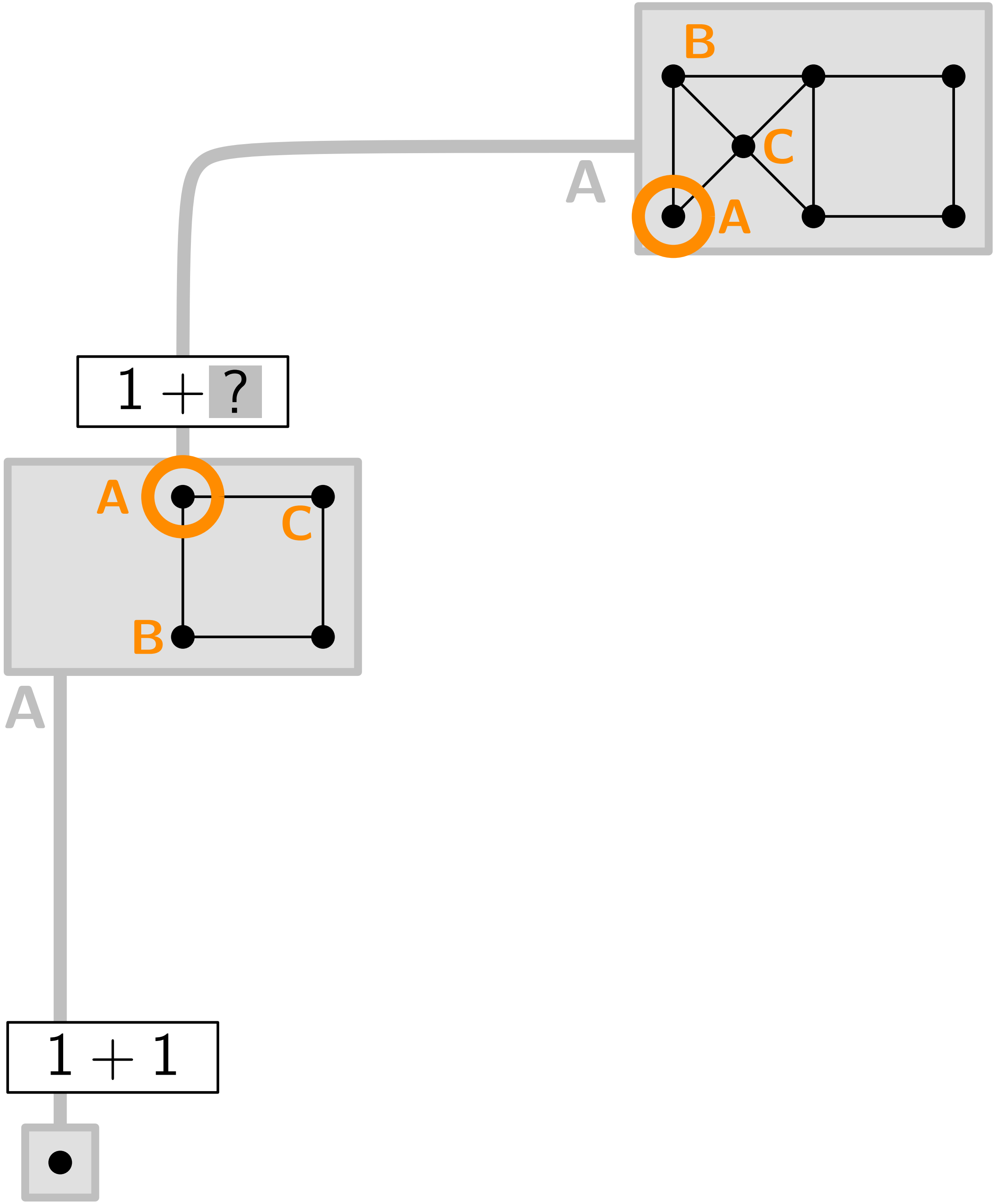


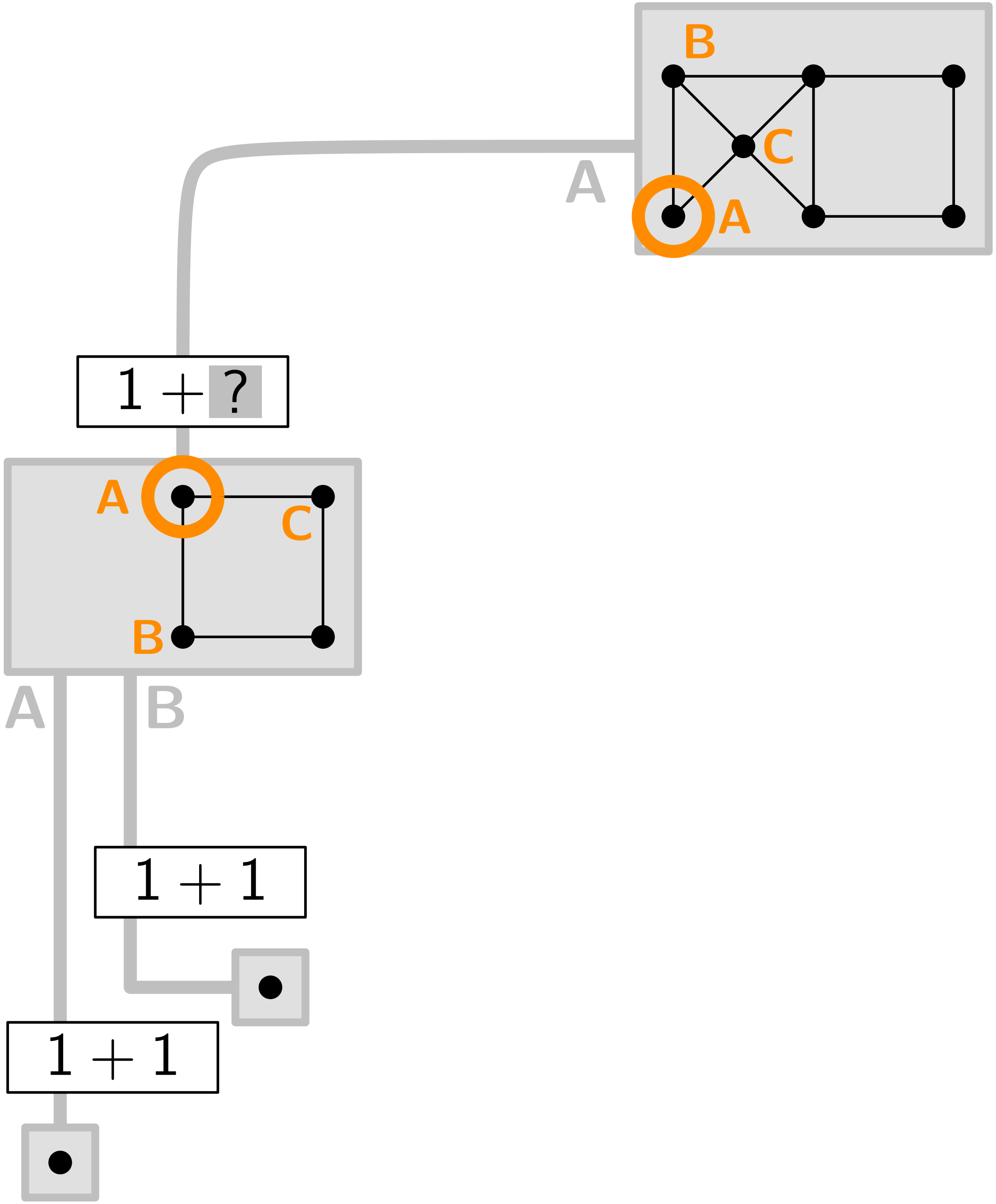
B
C
A
A
1 + ?

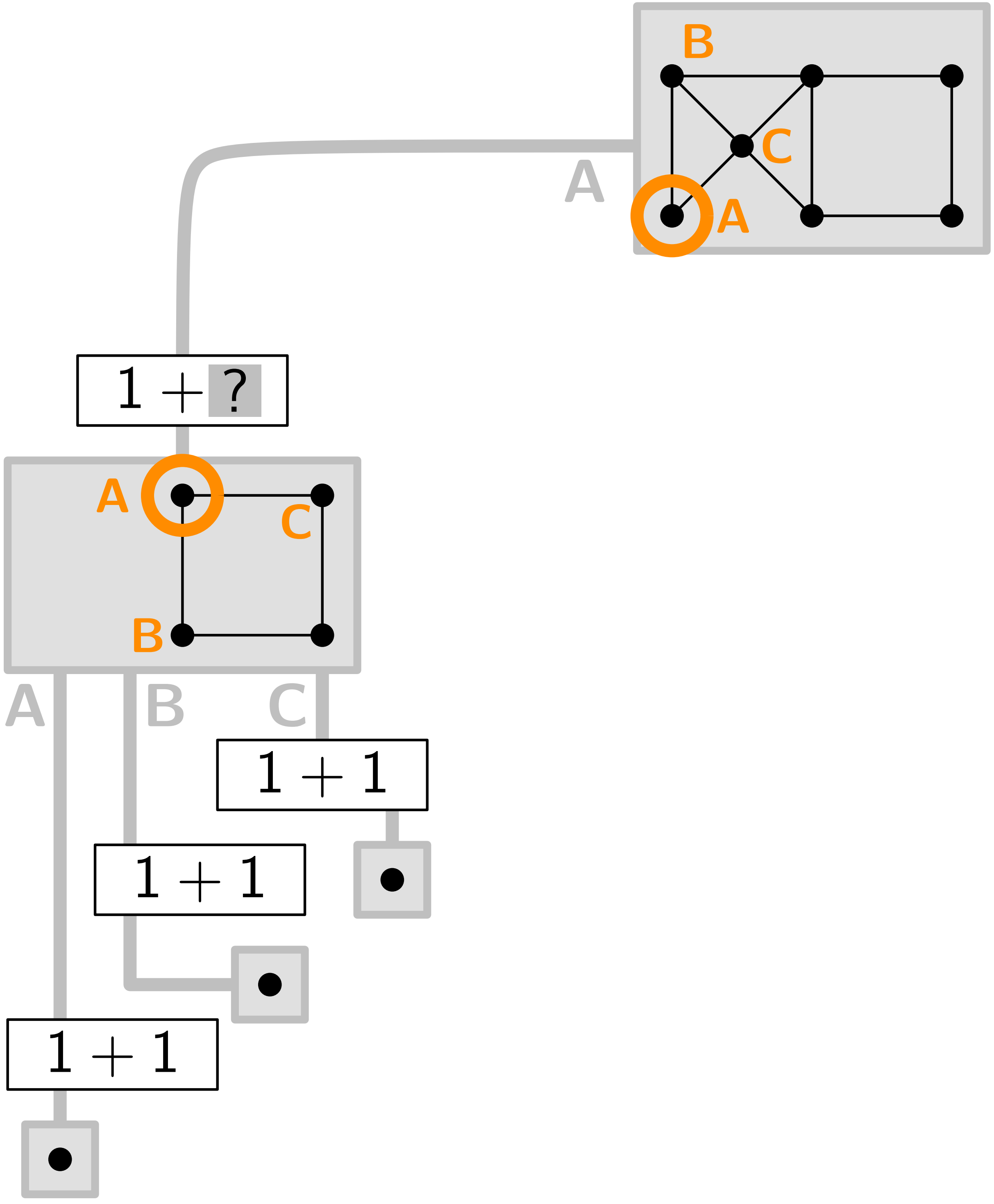


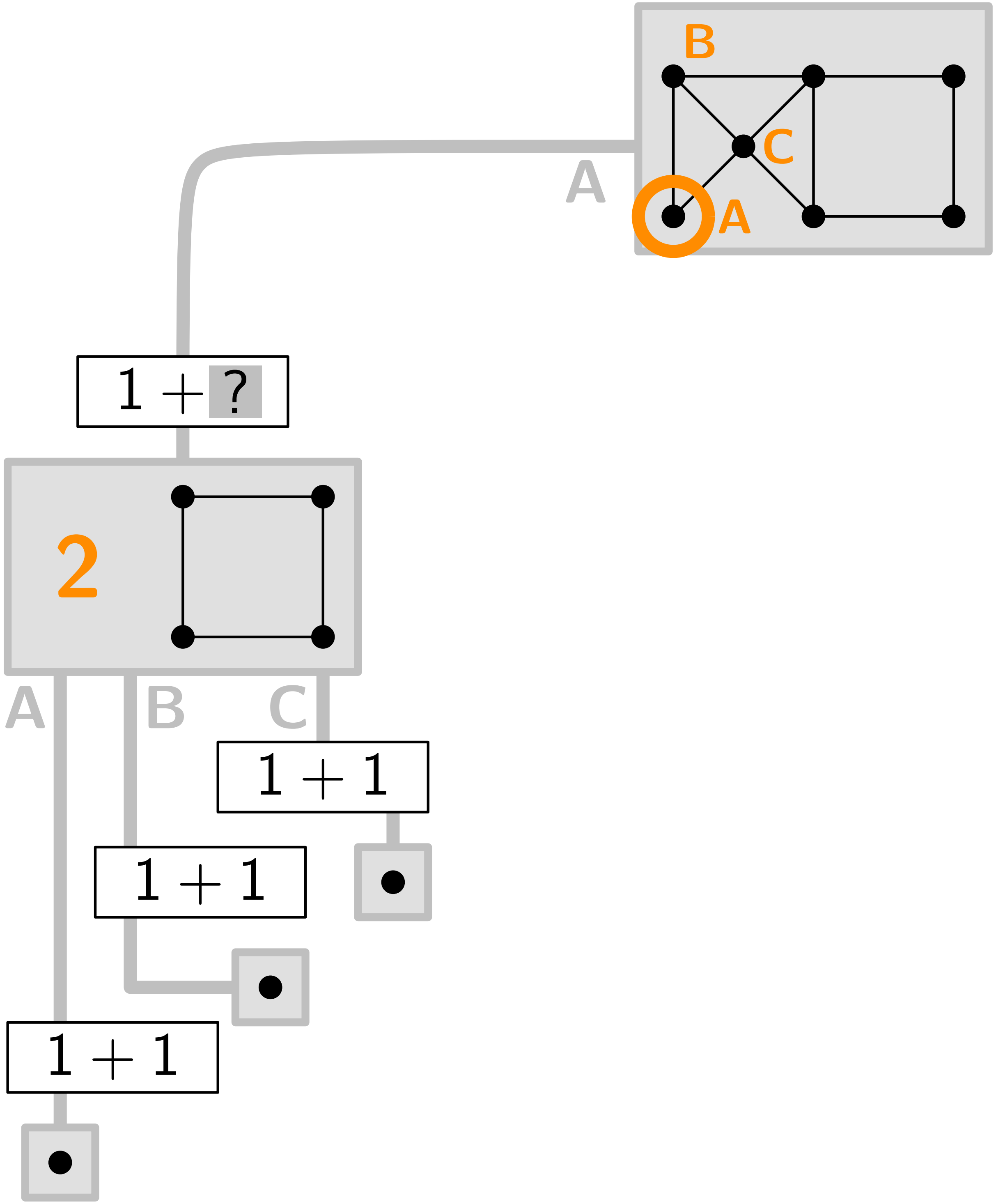
B
C
A
A
1 + ?
A
C
B

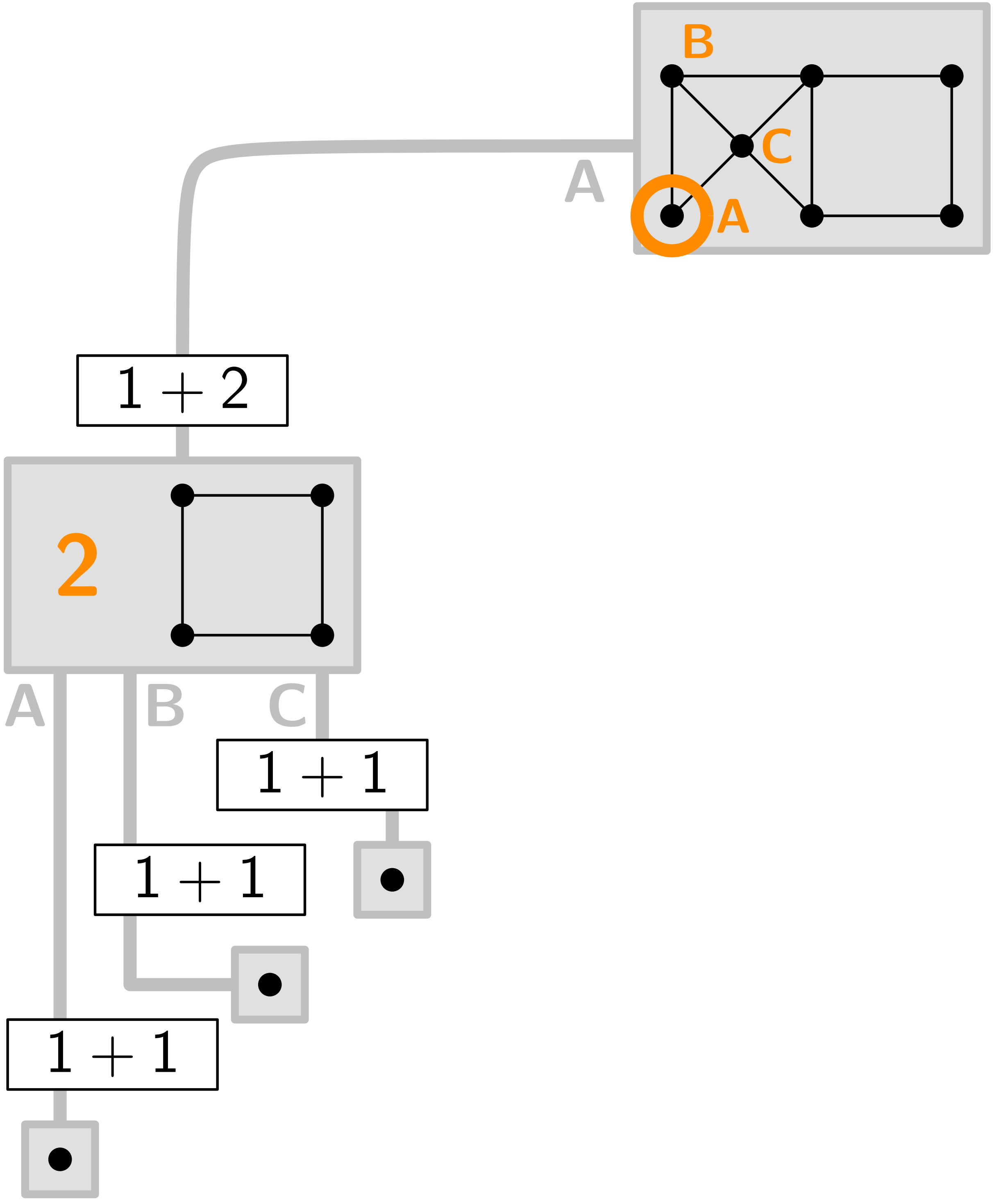

B
C
A
A
1 + ?
A
C
B
A
1 + ?

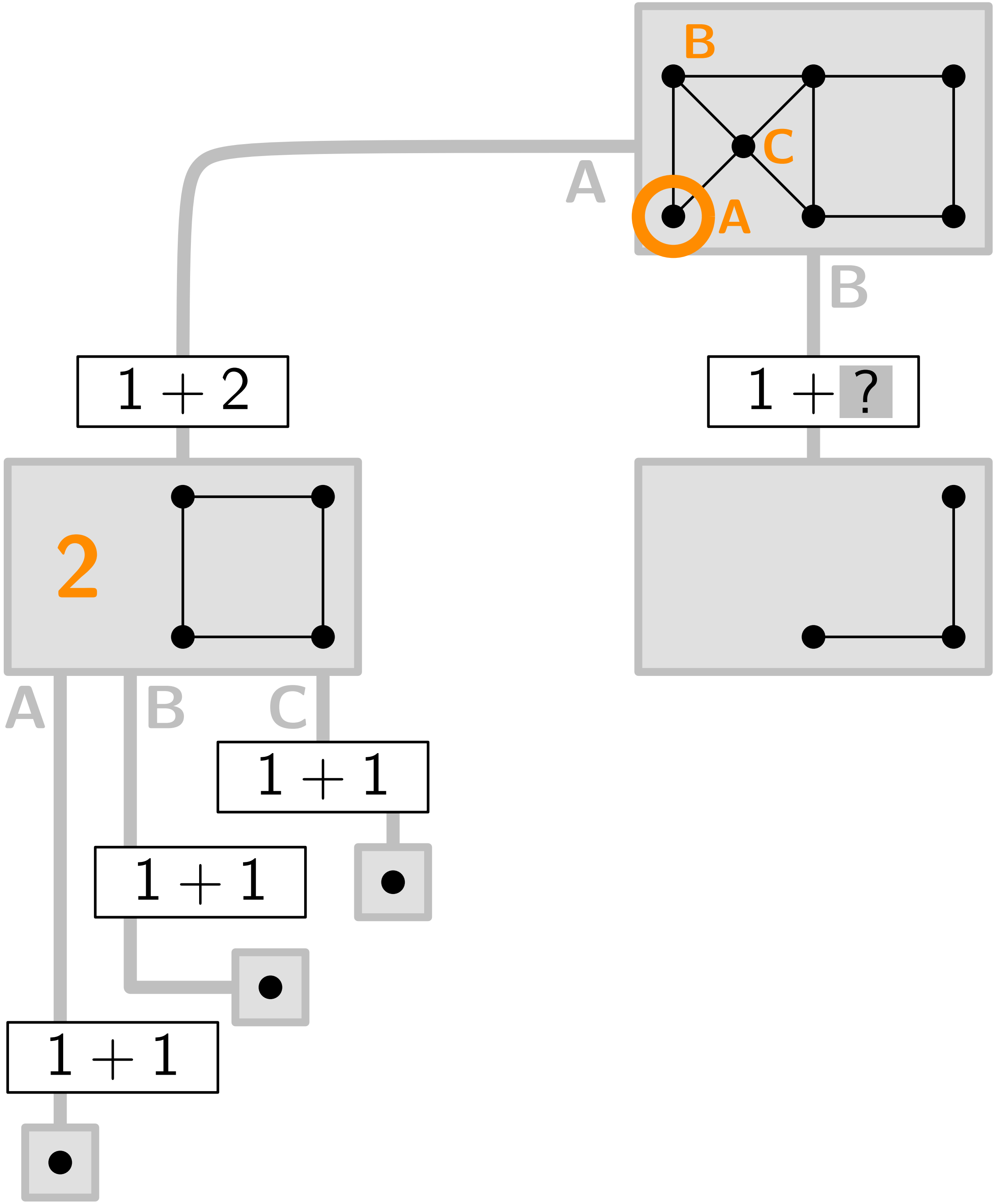

B
C
A
A
1 + ?
A
C
B
A
1 + 1

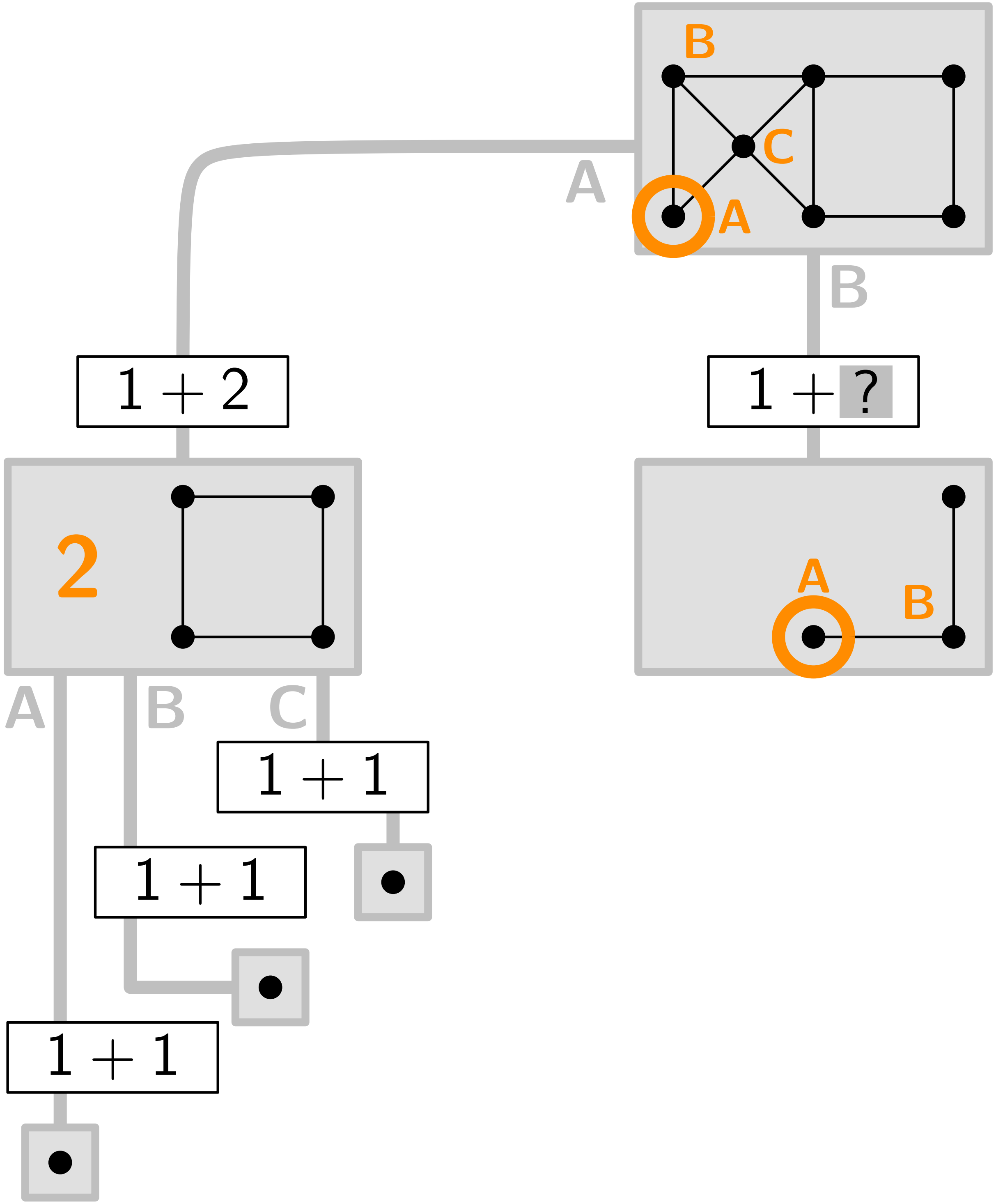

B
C
A
A
1 + ?
A
C
B
A
B
1 + 1
1 + 1

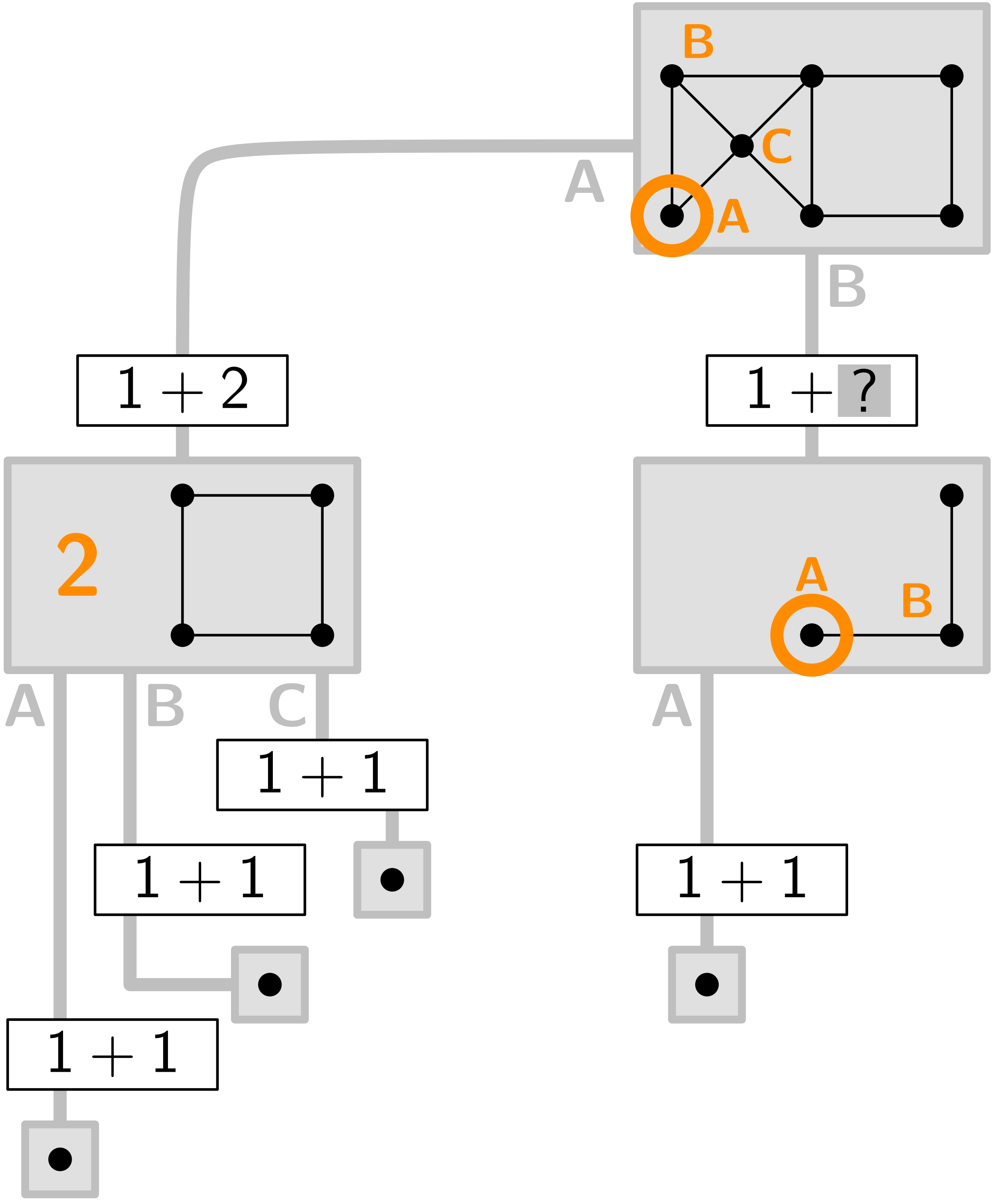

B
C
A
A
$1 + ?$
A
C
B
A
B
C
$1 + 1$
$1 + 1$
$1 + 1$

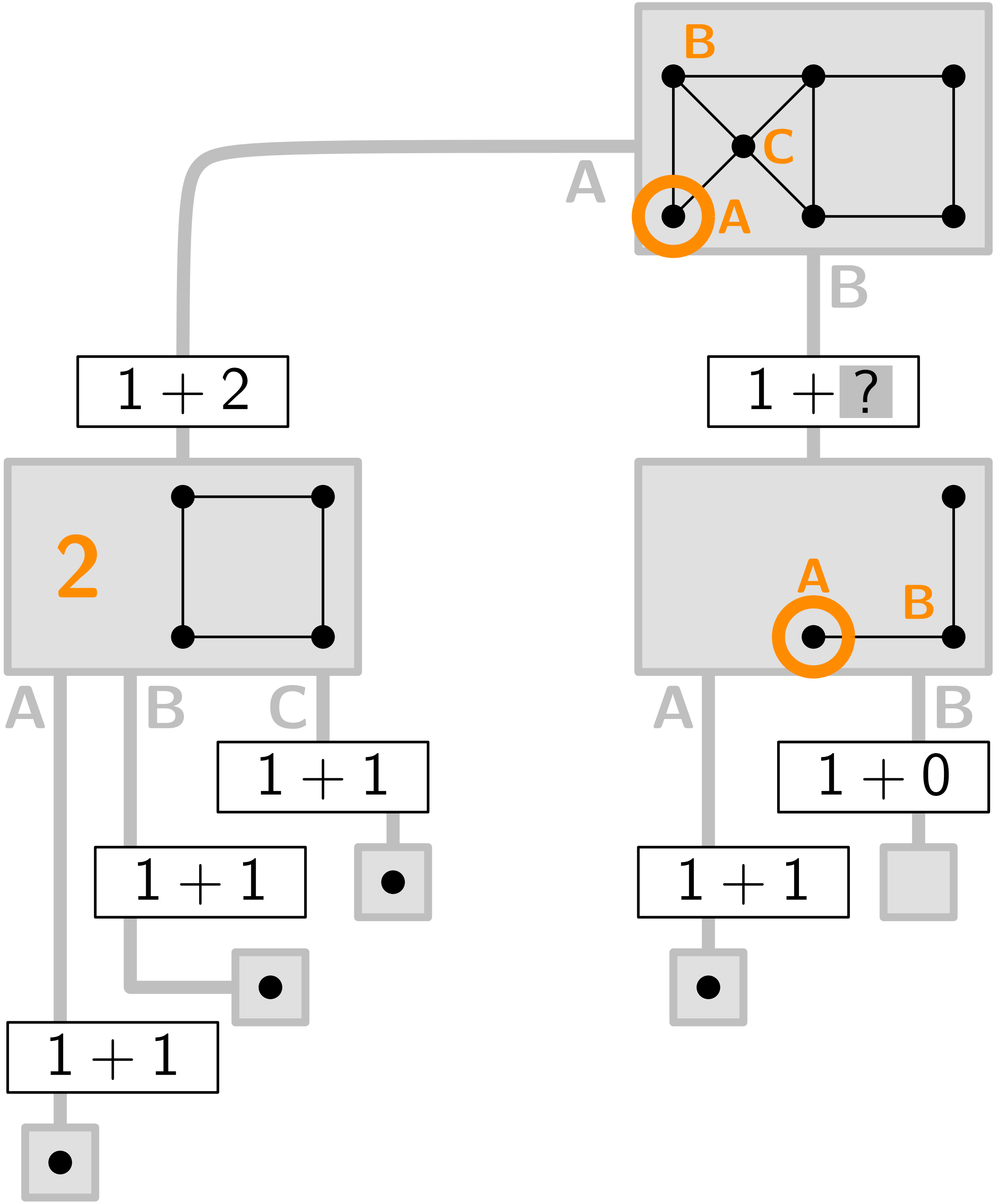

B
C
A
A
1 + ?
2
A
B
C
1 + 1
1 + 1
1 + 1


B
C
A
A
1 + 2
2
A
B
C
1 + 1
1 + 1
1 + 1

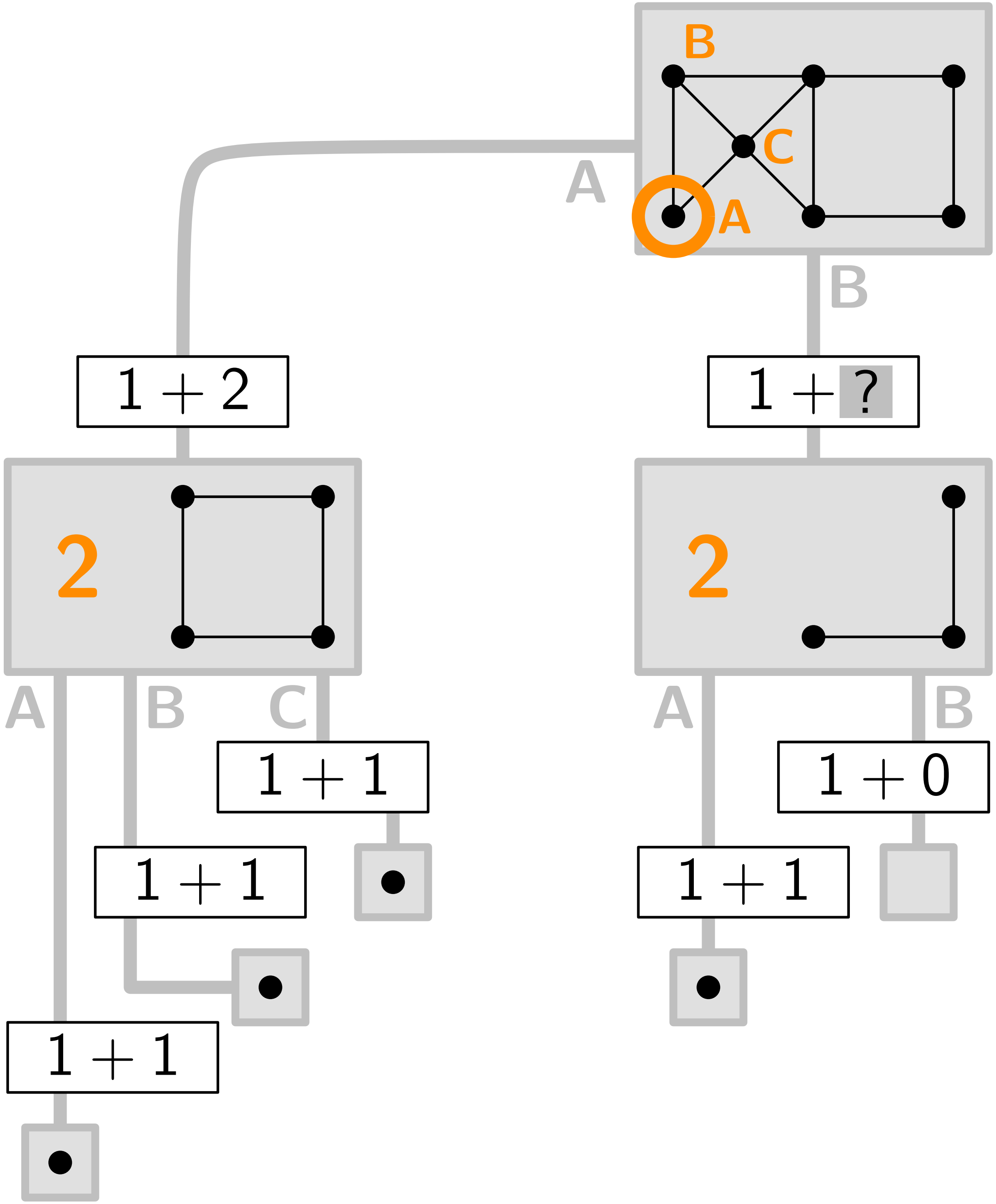

B
C
A
A
B
1 + 2
1 + ?
2
A
B
C
1 + 1
1 + 1
1 + 1

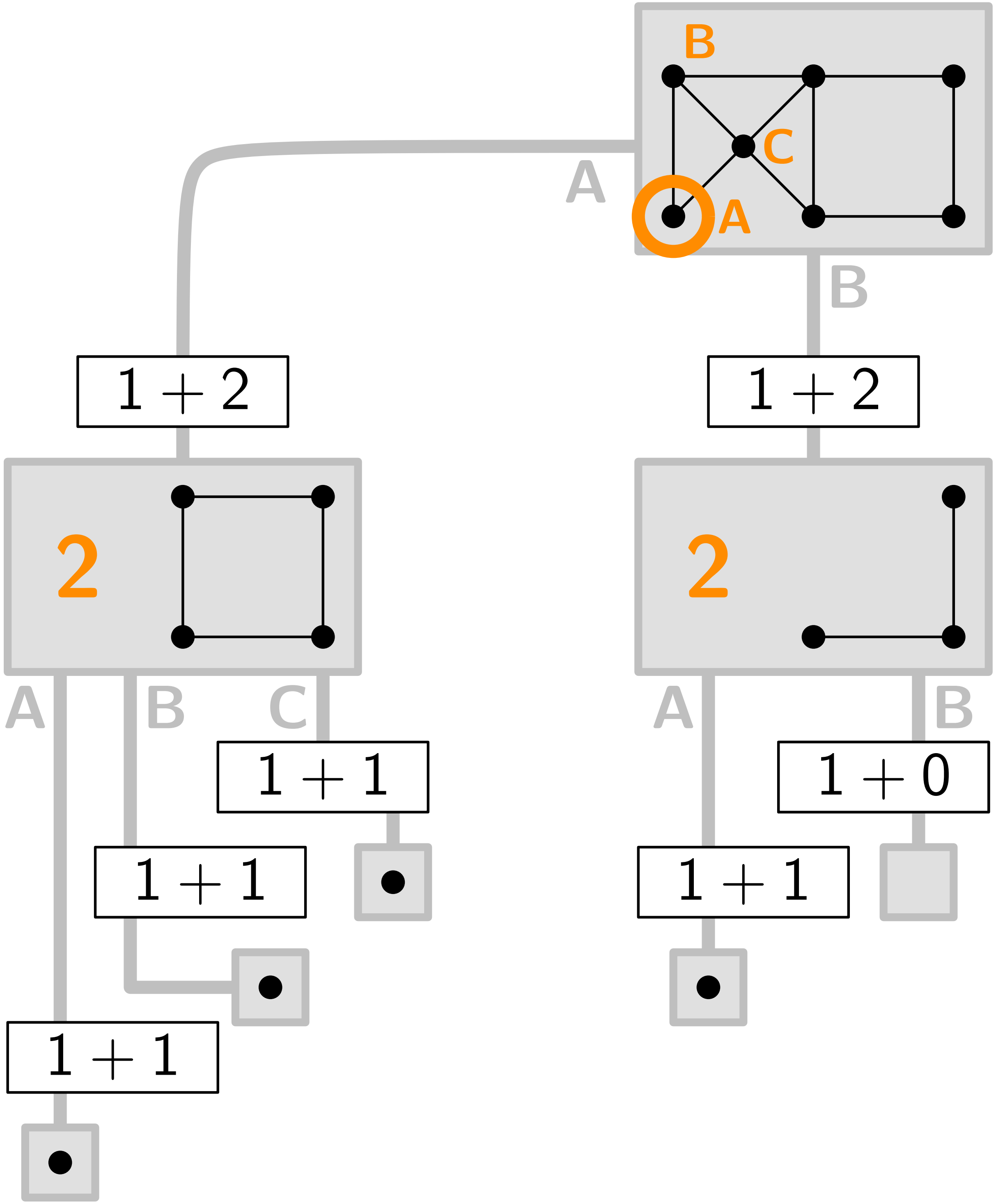

B
C
A
A
B
1 + 2
1 + ?
2
A
B
A
B
C
1 + 1
1 + 1
1 + 1

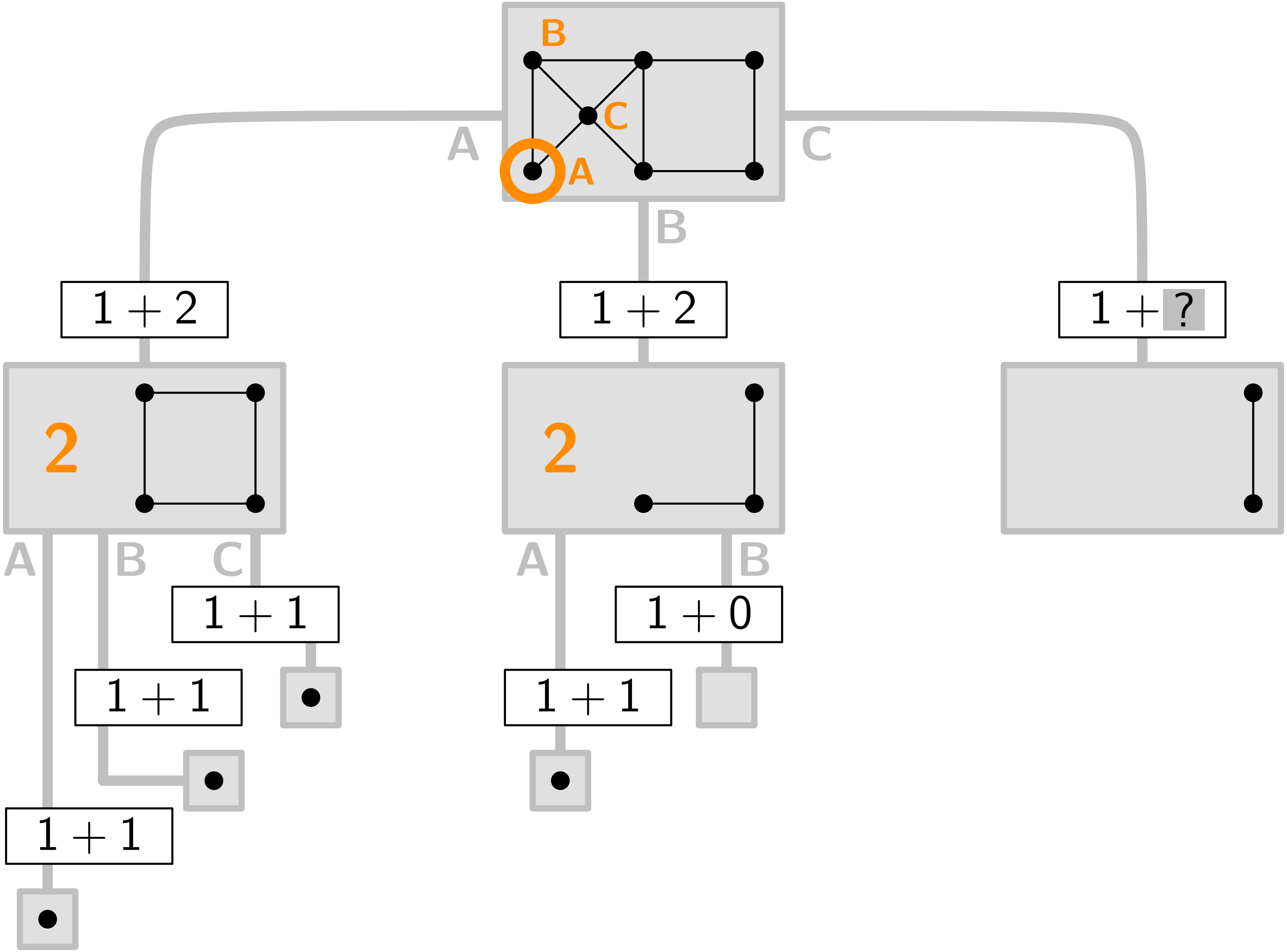

B
C
A
A
B
1 + 2
1 + ?
2
A
B
A
B
C
A
1 + 1
1 + 1
1 + 1
1 + 1



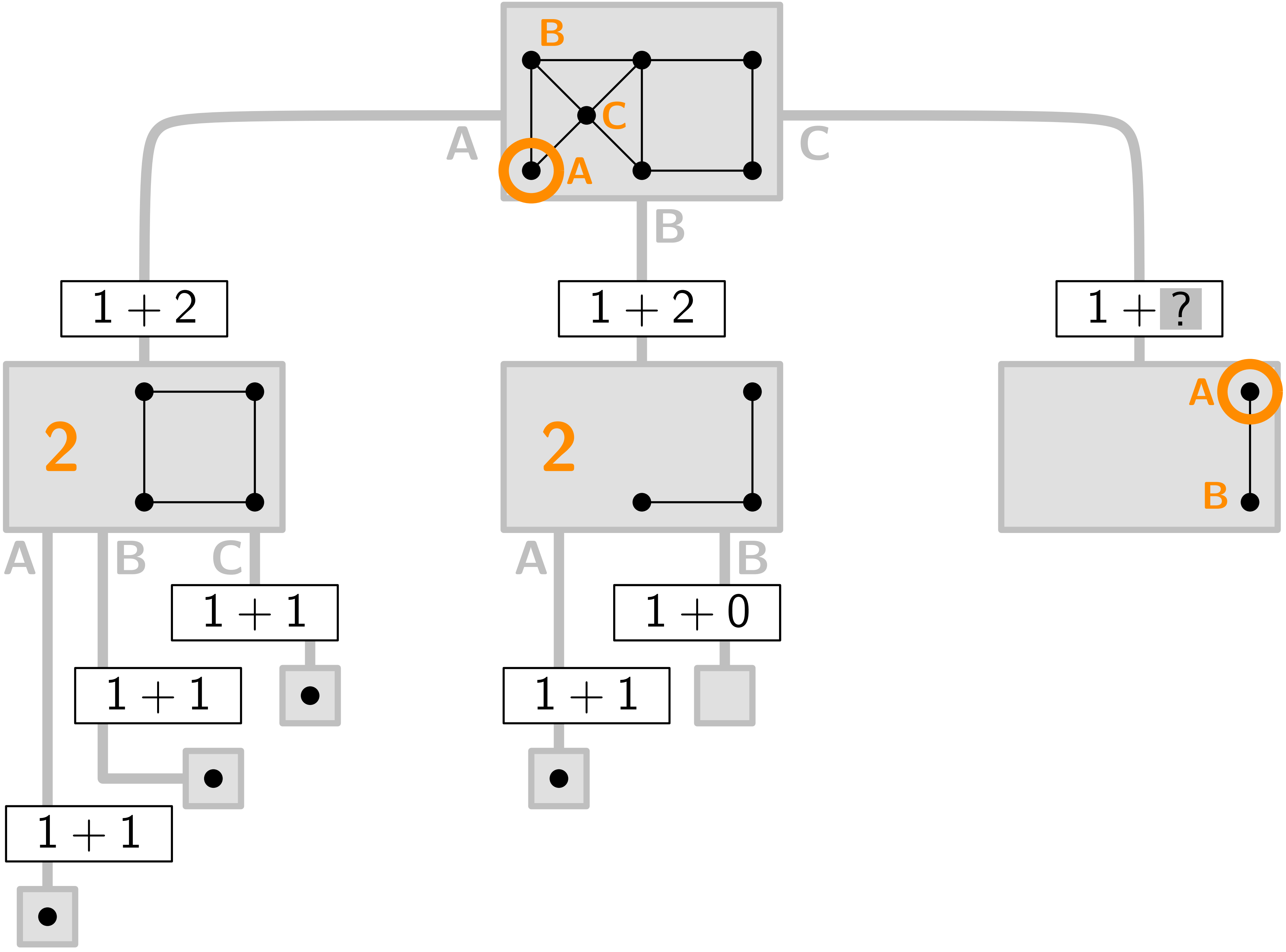
B
C
A
A
B
1 + 2
1 + ?
2
A
B
A
B
C
A
B
1 + 1
1 + 0
1 + 1
1 + 1
1 + 1

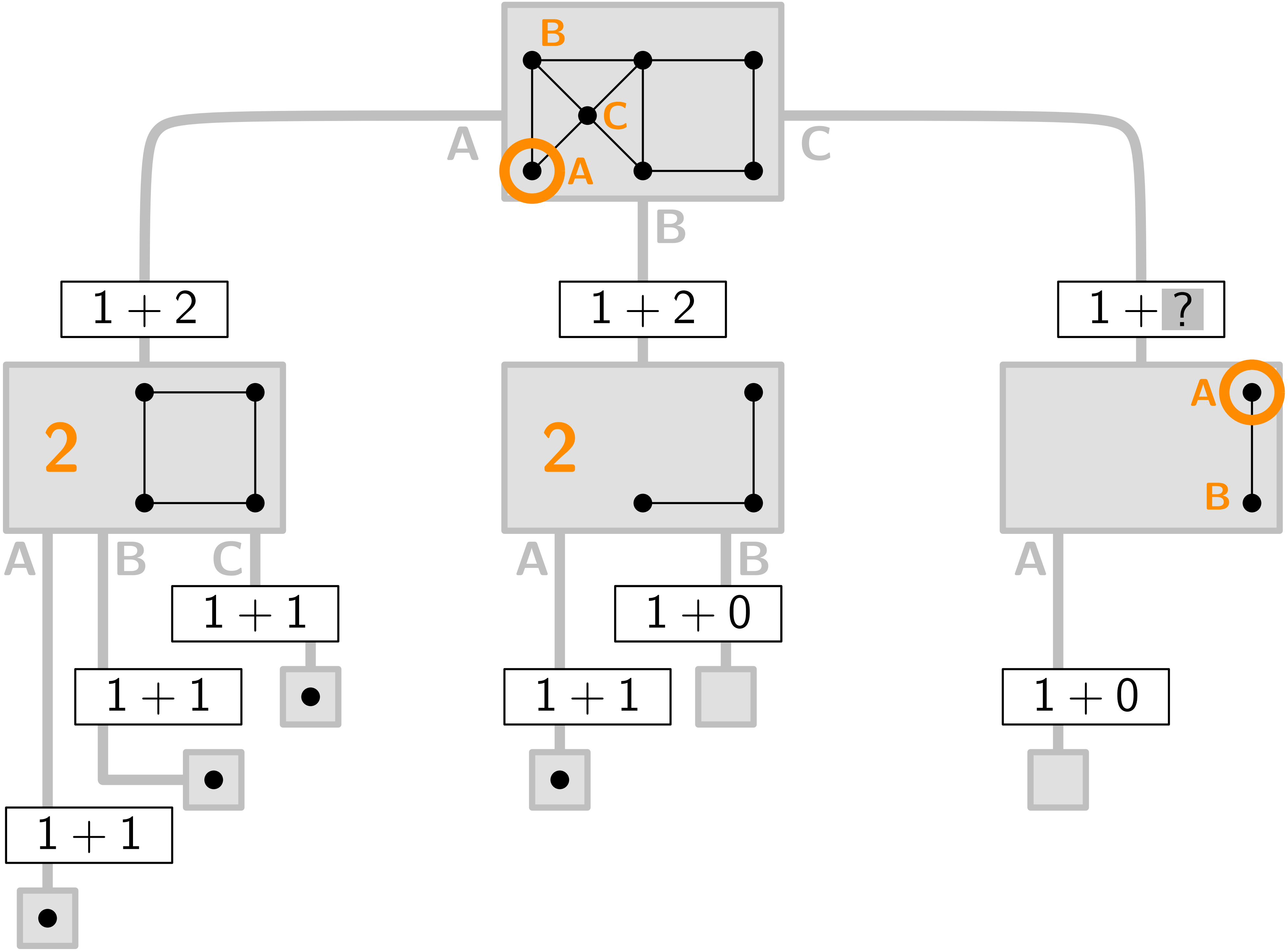

B
C
A
A
B
1 + 2
1 + ?
2
2
A
B
C
A
B
1 + 1
1 + 0
1 + 1
1 + 1
1 + 1

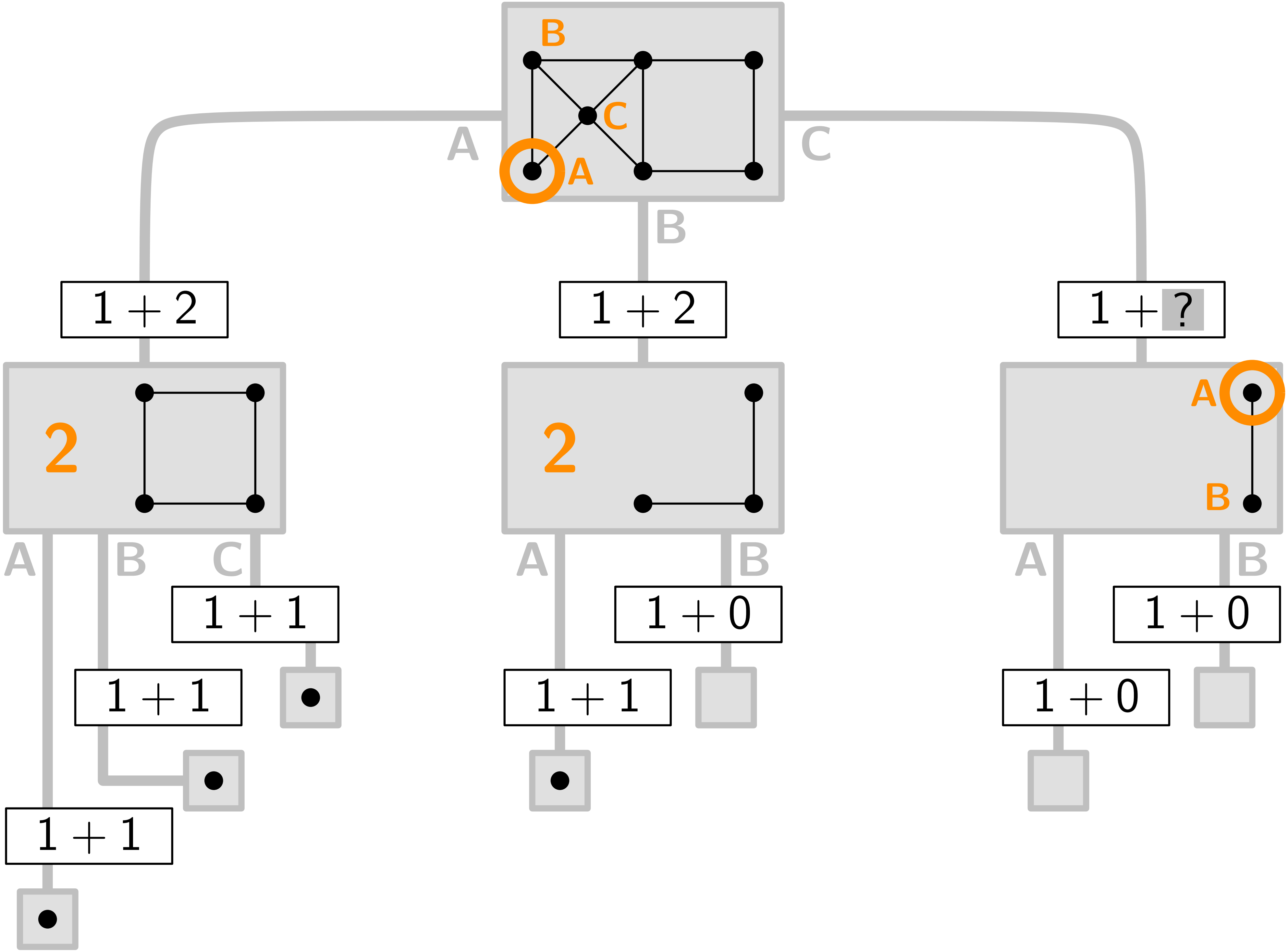

B
C
A
A
B
1 + 2
1 + 2
2
2
A
B
C
A
B
1 + 1
1 + 0
1 + 1
1 + 1
1 + 1



B
C
A
A
B
C
1 + 2
1 + 2
1 + ?
2
2
A
B
C
A
B
1 + 1
1 + 0
1 + 1
1 + 1
1 + 1


B
C
A
A
B
C
1 + 2
1 + 2
1 + ?
2
2
A
B
A
B
C
A
B
1 + 1
1 + 0
1 + 1
1 + 1
1 + 1


B
C
A
A
B
C
1 + 2
1 + 2
1 + ?
2
2
A
B
A
B
C
A
B
A
1 + 1
1 + 0
1 + 1
1 + 1
1 + 0
1 + 1


B
C
A
A
B
C
1 + 2
1 + 2
1 + ?
2
2
A
B
A
B
C
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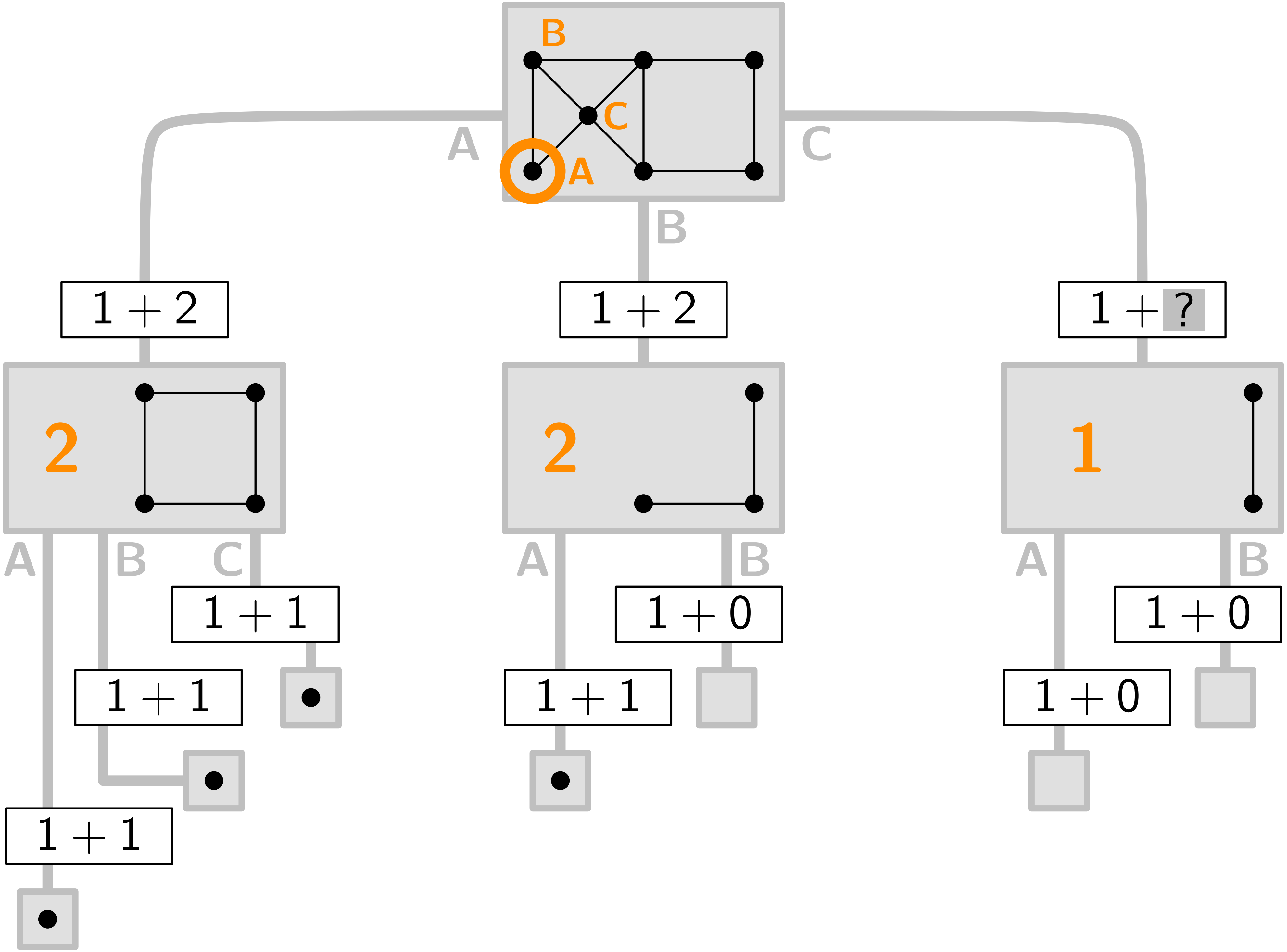
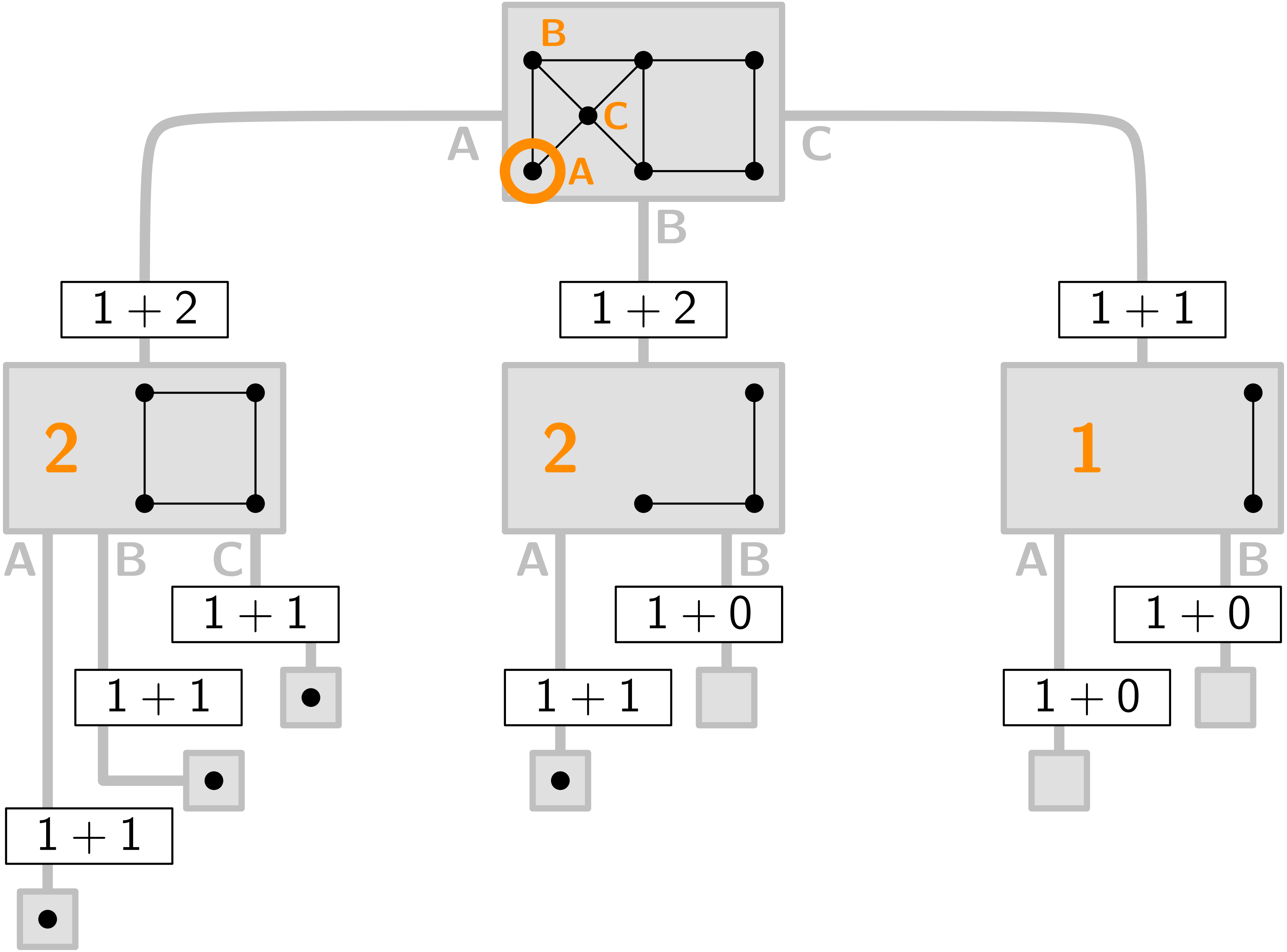
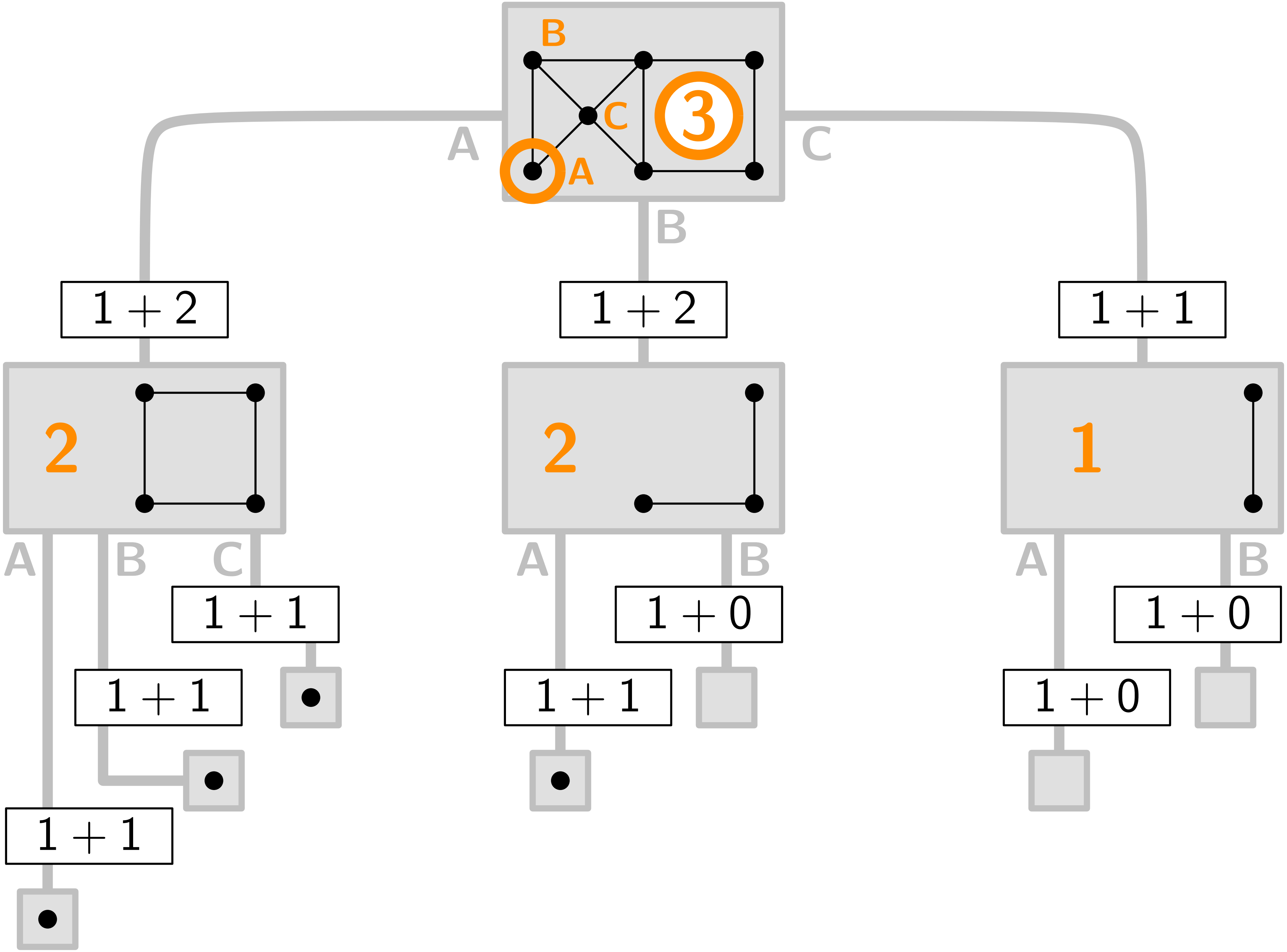
B
A
B
1 + 1
1 + 0
1 + 0
1 + 1
1 + 1
1 + 0
1 + 1


B
C
A
A
B
C
1 + 2
1 + 2
1 + ?
2
2
1
A
B
C
A
B
A
B
1 + 1
1 + 0
1 + 0
1 + 1
1 + 1
1 + 0
1 + 1


B
C
A
A
B
C
1 + 2
1 + 2
1 + 1
2
2
1
A
B
C
A
B
A
B
1 + 1
1 + 0
1 + 0
1 + 1
1 + 1
1 + 0
1 + 1


B
C
A
3
A
B
C
1 + 2
1 + 2
1 + 1
2
2
1
A
B
C
A
B
A
B
1 + 1
1 + 0
1 + 0
1 + 1
1 + 1
1 + 0
1 + 1

Runtime Analysis

For a worst-case n-vertex graph G ($n \geq 1$):

$$B(n) \leq \sum_{y \in N[v]} B(\qquad)$$

where v is a minimum-degree vertex of G.

Runtime Analysis

For a worst-case n-vertex graph G ($n \geq 1$):

$$B(n) \quad \leq \quad \sum_{y \in N[v]} B(n - (\mathsf{deg}(y) + 1))$$

where v is a minimum-degree vertex of G.

Runtime Analysis

For a worst-case n-vertex graph G ($n \geq 1$):

$$B(n) \leq \sum_{y \in N[v]} B(n - (\deg(y) + 1))$$

$$\leq$$

where v is a minimum-degree vertex of G.

Runtime Analysis

For a worst-case n-vertex graph G ($n \geq 1$):

$$\begin{aligned} B(n) &\leq \sum_{y \in N[v]} B(n - (\text{deg}(y) + 1)) \\ &\leq (\text{deg}(v) + 1) \cdot \end{aligned}$$

where v is a minimum-degree vertex of G.

Runtime Analysis

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$$B(n) \leq \sum_{y \in N[v]} B(n - (\text{deg}(y) + 1))$$

$$\leq (\text{deg}(v) + 1) \cdot B(\qquad\qquad\qquad),$$

where v is a minimum-degree vertex of G.

Runtime Analysis

For a worst-case n-vertex graph G ($n \geq 1$):

$$B(n) \leq \sum_{y \in N[v]} B(n - (\text{deg}(y) + 1)) \leq (\text{deg}(v) + 1) \cdot B(\, n - (\text{deg}(v) + 1) \,),$$

where v is a minimum-degree vertex of G.

Runtime Analysis

For a worst-case n-vertex graph G $(n \geq 1)$:

$$B(n) \leq \sum_{y \in N[v]} B(n - (\mathsf{deg}(y) + 1))$$
$$\leq (\mathsf{deg}(v) + 1) \cdot B(\ n - (\mathsf{deg}(v) + 1)\ )\,,$$

where v is a minimum-degree vertex of G.

For the second inequality, we still need to argue that B is *monotone*, that is, $B(n') \leq B(n)$ for any $n' \leq n$.

Runtime Analysis

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$$B(n) \leq \sum_{y \in N[v]} B(n - (\mathrm{deg}(y) + 1))$$
$$\leq (\mathrm{deg}(v) + 1) \cdot B(\ n - (\mathrm{deg}(v) + 1)\ ),$$

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This is not difficult: Let G' be a graph with n' vertices and a search tree with the maximum number of leaves, $B(n')$.

Runtime Analysis

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$$B(n) \leq \sum_{y \in N[v]} B(n - (\text{deg}(y) + 1)) \leq (\text{deg}(v) + 1) \cdot B(\ n - (\text{deg}(v) + 1)\ )\,,$$

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This is not difficult: Let G' be a graph with n' vertices and a search tree with the maximum number of leaves, $B(n')$.

Add to G' $n - n'$ independent vertices.

Runtime Analysis

For a worst-case n-vertex graph G ($n \geq 1$):

$$B(n) \leq \sum_{y \in N[v]} B(n - (\deg(y) + 1))$$

$$\leq (\deg(v) + 1) \cdot B(\ n - (\deg(v) + 1)\ )\,,$$

where v is a minimum-degree vertex of G.

For the second inequality, we still need to argue that B is *monotone*, that is, $B(n') \leq B(n)$ for any $n' \leq n$.

This is not difficult: Let G' be a graph with n' vertices and a search tree with the maximum number of leaves, $B(n')$.

Add to G' $n - n'$ independent vertices.

This yields an n-vertex graph witnessing that $B(n) \geq B(n')$.

Runtime Analysis (cont'd)

Recall: $B(n) \leq (\deg(v) + 1) \cdot B(\ n - (\deg(v) + 1)\ )$

Runtime Analysis (cont'd)

Recall: $B(n) \leq (\mathsf{deg}(v) + 1) \cdot B(\ n - (\mathsf{deg}(v) + 1)\ )$

We proceed by induction to show that $B(n) \leq 3^{n/3}$.

Runtime Analysis (cont'd)

Recall: $B(n) \leq (\deg(v) + 1) \cdot B(\ n - (\deg(v) + 1)\ )$

We proceed by induction to show that $B(n) \leq 3^{n/3}$.

Base case: $B(0) = 1 \leq 3^{0/3}$

Runtime Analysis (cont'd)

Recall: $B(n) \leq (\text{deg}(v)+1) \cdot B(\ n - (\text{deg}(v)+1)\ )$

We proceed by induction to show that $B(n) \leq 3^{n/3}$.

Base case: $B(0) = 1 \leq 3^{0/3}$

Hypothesis: for $n \geq 1$, set $s = \text{deg}(v) + 1$ in

Runtime Analysis (cont'd)

Recall: $B(n) \leq (\text{deg}(v)+1) \cdot B(\ n - (\text{deg}(v)+1)\ )$

We proceed by induction to show that $B(n) \leq 3^{n/3}$.

Base case: $B(0) = 1 \leq 3^{0/3}$

Hypothesis: for $n \geq 1$, set $s = \text{deg}(v)+1$ in

Thus,

$B(n) \leq$

Runtime Analysis (cont'd)

Recall: $B(n) \leq (\deg(v)+1) \cdot B(\ n - (\deg(v)+1)\ )$

We proceed by induction to show that $B(n) \leq 3^{n/3}$.

Base case: $B(0) = 1 \leq 3^{0/3}$

Hypothesis: for $n \geq 1$, set $s = \deg(v)+1$ in

Thus,

$B(n) \leq s \cdot B(n-s) \leq$

Runtime Analysis (cont'd)

Recall: $B(n) \leq (\deg(v)+1) \cdot B(\ n - (\deg(v)+1)\ )$

We proceed by induction to show that $B(n) \leq 3^{n/3}$.

Base case: $B(0) = 1 \leq 3^{0/3}$

Hypothesis: for $n \geq 1$, set $s = \deg(v)+1$ in

Thus,

$B(n) \leq s \cdot B(n-s) \leq s \cdot 3^{(n-s)/3} =$

Runtime Analysis (cont'd)

Recall: $B(n) \leq (\deg(v)+1) \cdot B(\ n - (\deg(v)+1)\ )$

We proceed by induction to show that $B(n) \leq 3^{n/3}$.

Base case: $B(0) = 1 \leq 3^{0/3}$

Hypothesis: for $n \geq 1$, set $s = \deg(v)+1$ in

Thus,

$B(n) \leq s \cdot B(n-s) \leq s \cdot 3^{(n-s)/3} = \frac{s}{3^{s/3}} \cdot 3^{n/3} \leq$

Runtime Analysis (cont'd)

Recall: $B(n) \leq (\deg(v)+1) \cdot B(\ n - (\deg(v)+1)\ )$

We proceed by induction to show that $B(n) \leq 3^{n/3}$.

Base case: $B(0) = 1 \leq 3^{0/3}$

Hypothesis: for $n \geq 1$, set $s = \deg(v)+1$ in

Thus,

$$B(n) \leq s \cdot B(n-s) \leq s \cdot 3^{(n-s)/3} = \frac{s}{3^{s/3}} \cdot 3^{n/3} \overset{?}{\leq} 3^{n/3}$$

Runtime Analysis (cont'd)

Recall: $B(n) \leq (\deg(v)+1) \cdot B(\ n - (\deg(v)+1)\ )$

We proceed by induction to show that $B(n) \leq 3^{n/3}$.

Base case: $B(0) = 1 \leq 3^{0/3}$

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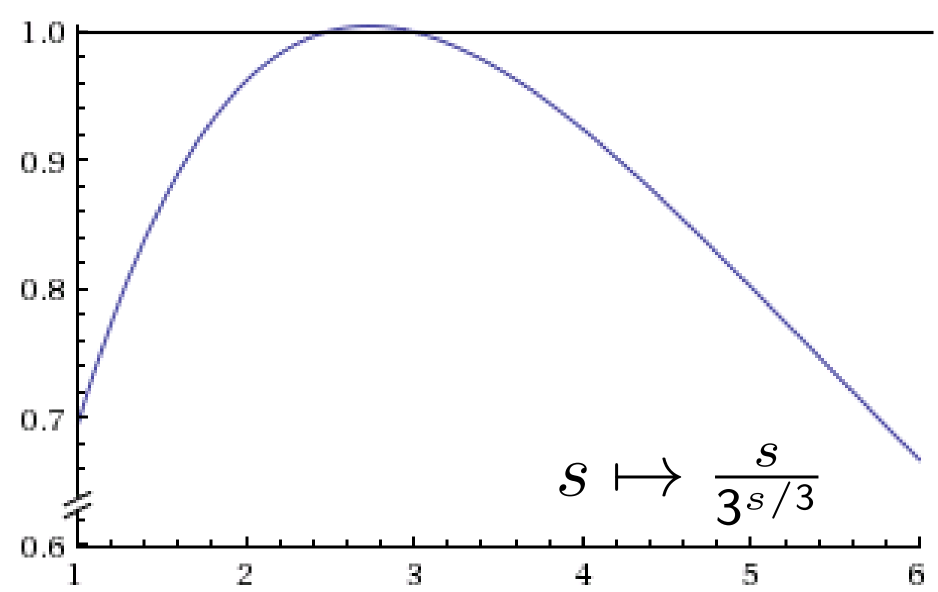

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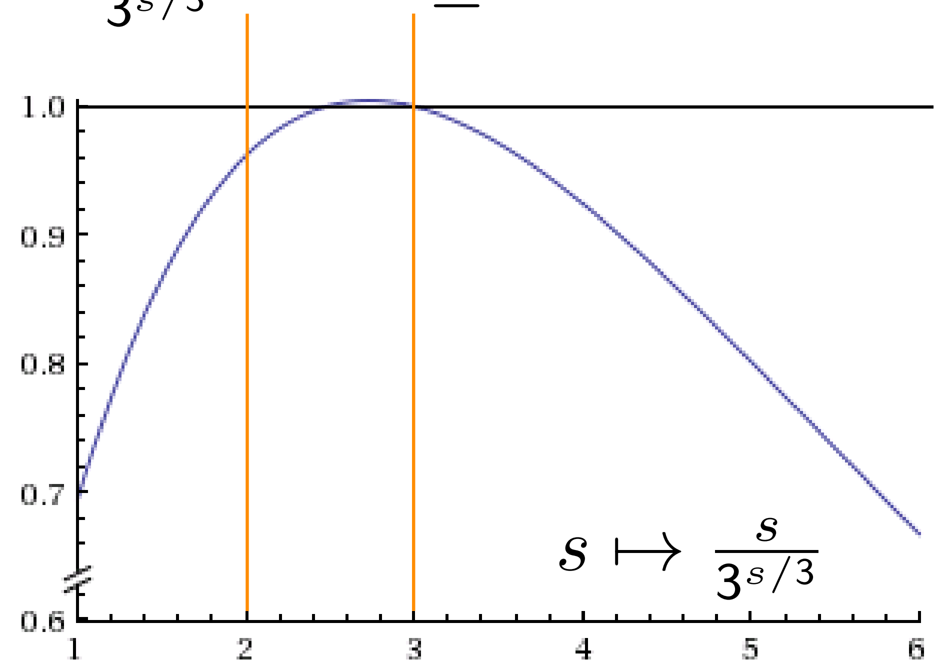

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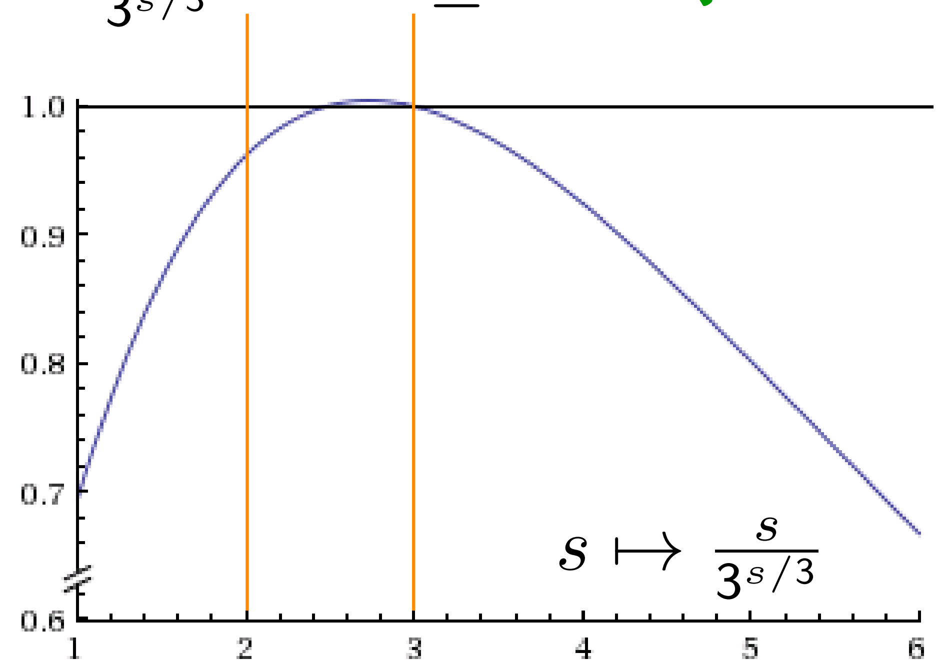

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$$B(n) \in O^*(\sqrt[3]{3}^n) \subset O^*(1.44225^n)$$