Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Lehrstuhl für
INFORMATIK I
Effiziente Algorithmen und
wissensbasierte Systeme

Institut für Informatik

Advanced Algorithms

Winter term 2019/20

Lecture 9. Exact Algorithms for Geometric Intersection Graphs

Slides by Paweł Rzążewski, Warsaw University of Technology

Steven Chaplick & Alexander Wolff *Chair for Computer Science I*

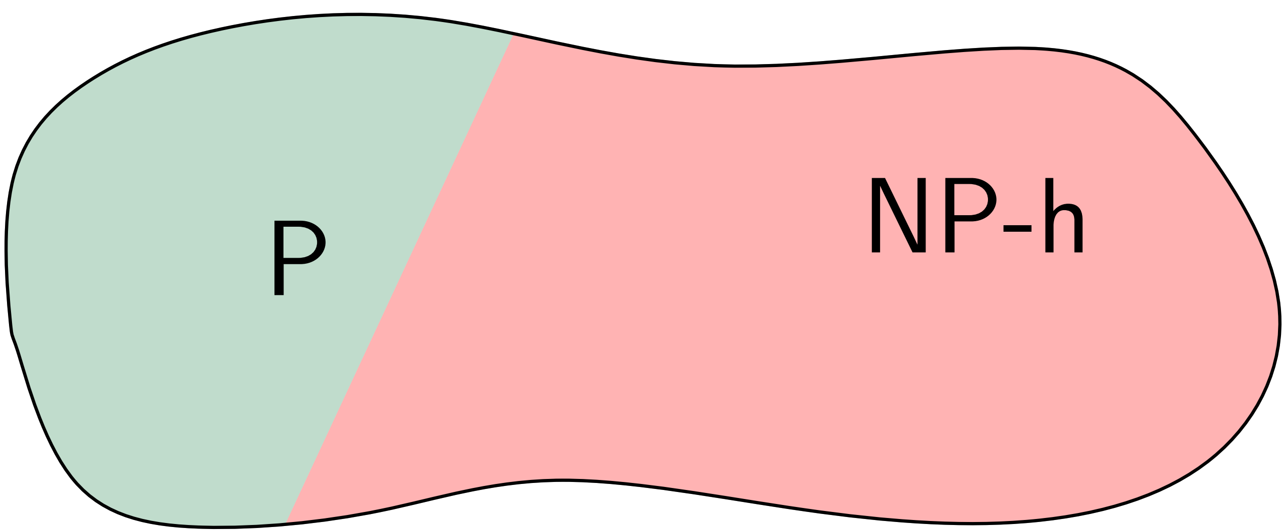
Fine-Grained Complexity and the Exponential-Time Hypothesis

Classical Approach to Complexity Theory

Assuming $P \neq NP$, we partition problems into two sets:

- P (solvable in polynomial time)
  proven by presenting an algorithm

- NP-hard (no polynomial algorithm)
  proven by polynomial reductions



Classical Approach to Complexity Theory

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- P (solvable in polynomial time)
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    worth attention,
    how fast can be solve them?

- NP-hard (no polynomial algorithm)
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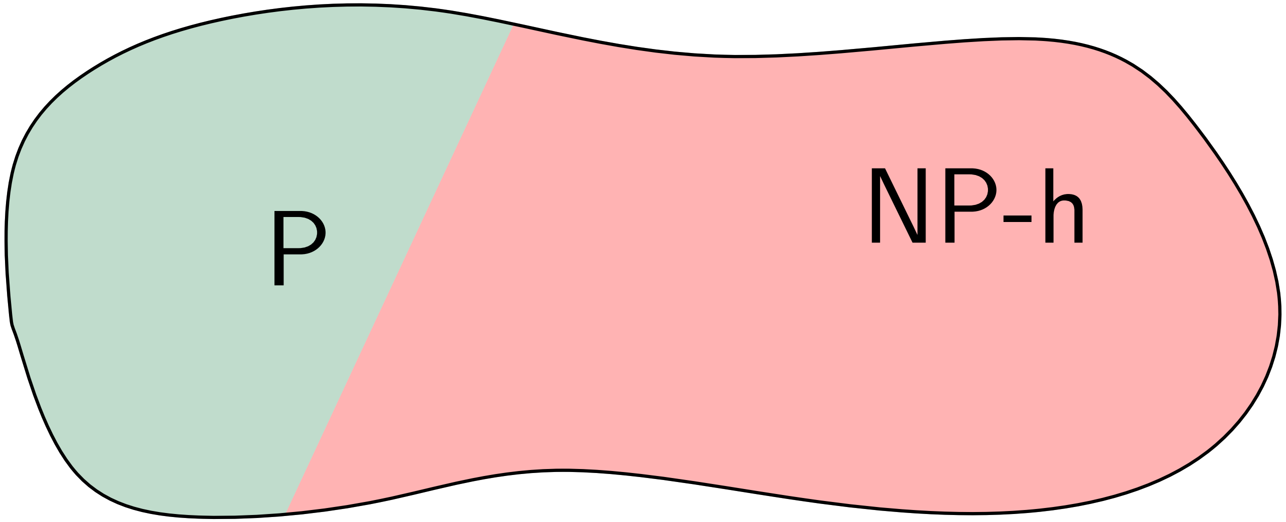
    hopeless, unsolvable



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Exponential Time Hypothesis (ETH) [Impagliazzo, Paturi, 1999]
There is no algorithm solving $3\text{-}\textsc{Sat}$ with n variables and $\mathcal{O}(n)$ clauses in time $2^{o(n)}$.

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subexponential time: $2^{o(n)}$,
e.g. $2^{\mathcal{O}(n^{0.99})}$ or $2^{\mathcal{O}(n/\log n)}$

A Closer Look

Being a stronger assumption than P $\neq$ NP, ETH allows for a finer analysis:

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  (no polynomial
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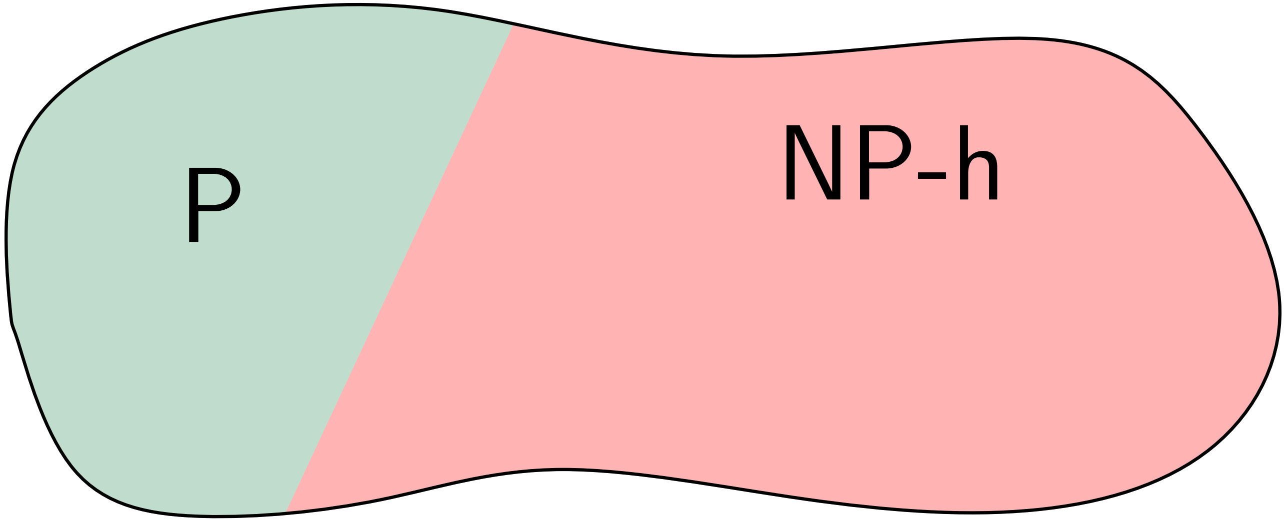

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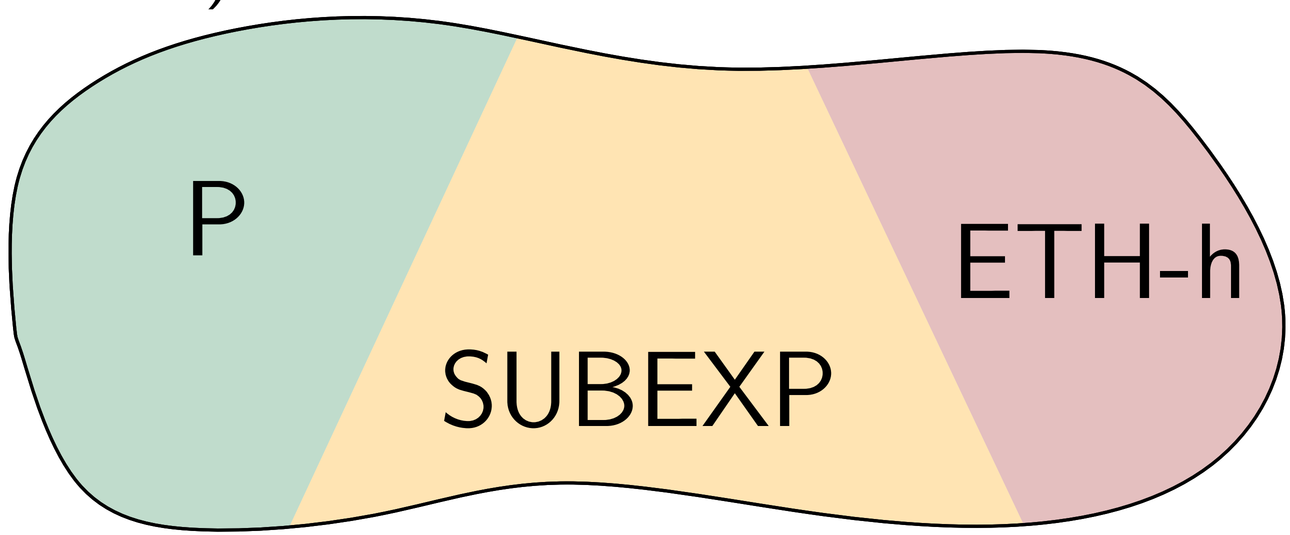
ETH-hard (no subexponential algorithm)



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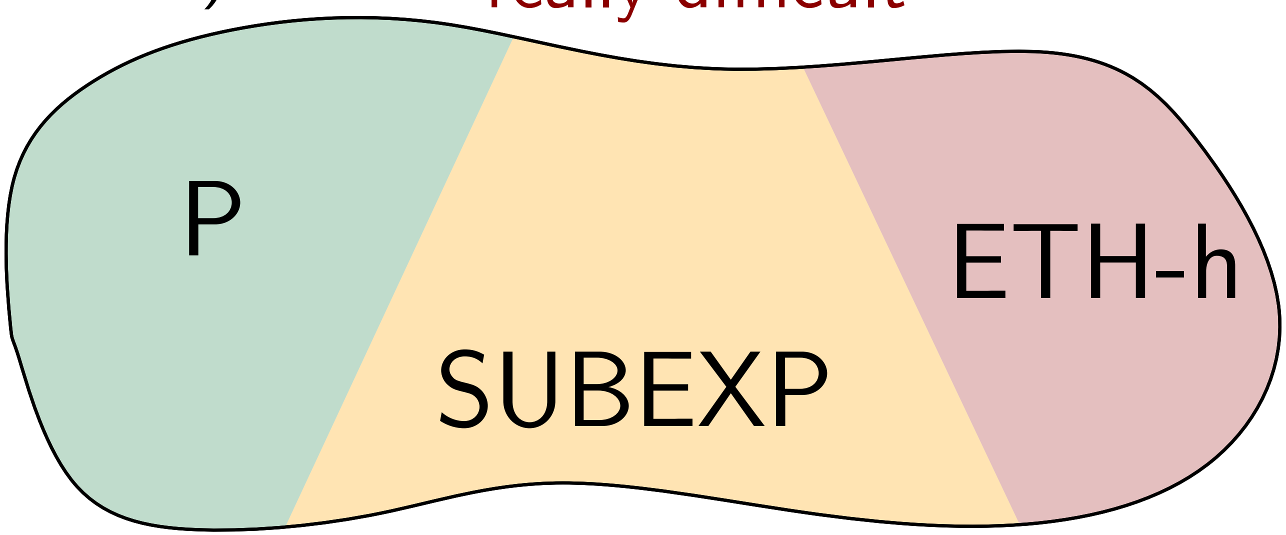
Being a stronger assumption than P $\neq$ NP, ETH allows for a finer analysis:

- P (solvable in polynomial time)
  easy
- NP-hard (no polynomial algorithm)
  - SUBEXP (solvable in subexponential time)
    not so super-hard
  - ETH-hard (no subexponential algorithm)
    really difficult



Lower Bounds

- Hardness is proven via reductions.
- Start from 3-SAT with n variables and $m = \mathcal{O}(n)$ clauses.
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$$\begin{array}{lllll} \alpha = 1 & \text{(linear reduction)} & \rightarrow & \text{no } 2^{o(n)} \text{ algorithm} \\ \alpha = 2 & \text{(quadratic reduction)} & \rightarrow & \text{no } 2^{o(\sqrt{n})} \text{ algorithm} \end{array}$$

What Can We Hope for?

- Bad news: Assuming the ETH, there are no subexponential algorithms for canonical graph problems.

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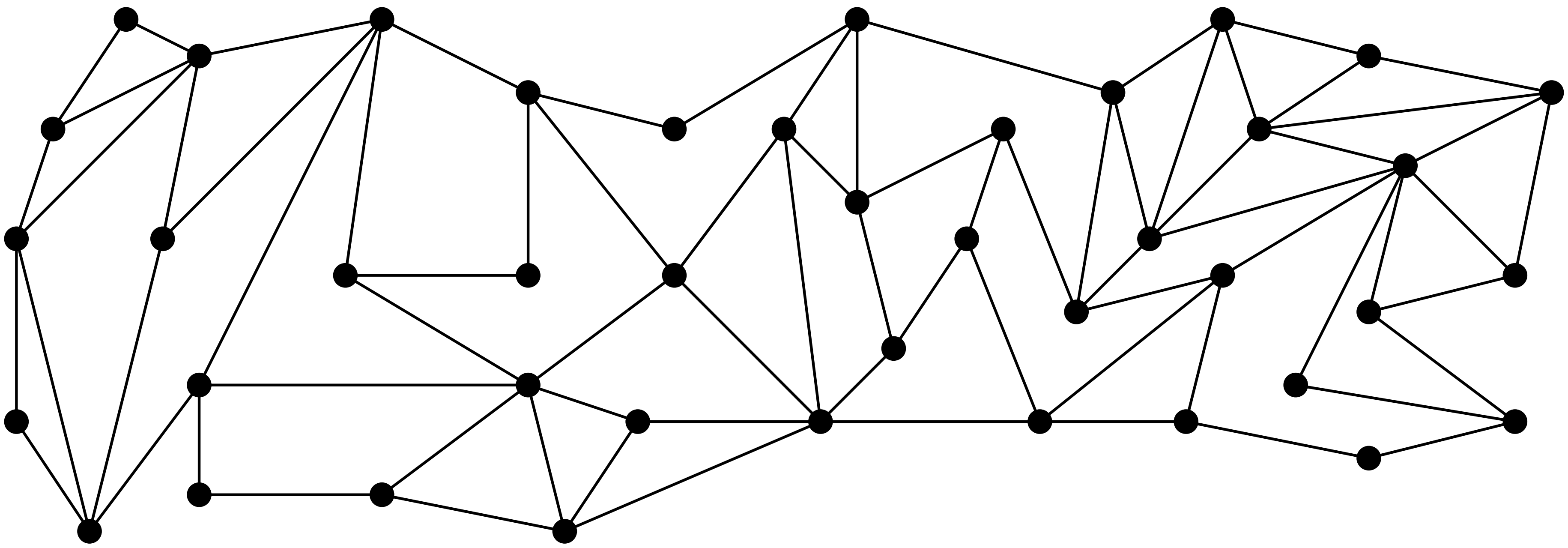
  Still boring!

Subexponential Algorithms for Planar Graphs

Planar separator theorem [Lipton & Tarjan 1979]

Every planar graph has a balanced separator of size $\mathcal{O}(\sqrt{n})$.

- also specialized versions, e.g. the separator is a cycle



Independent Set

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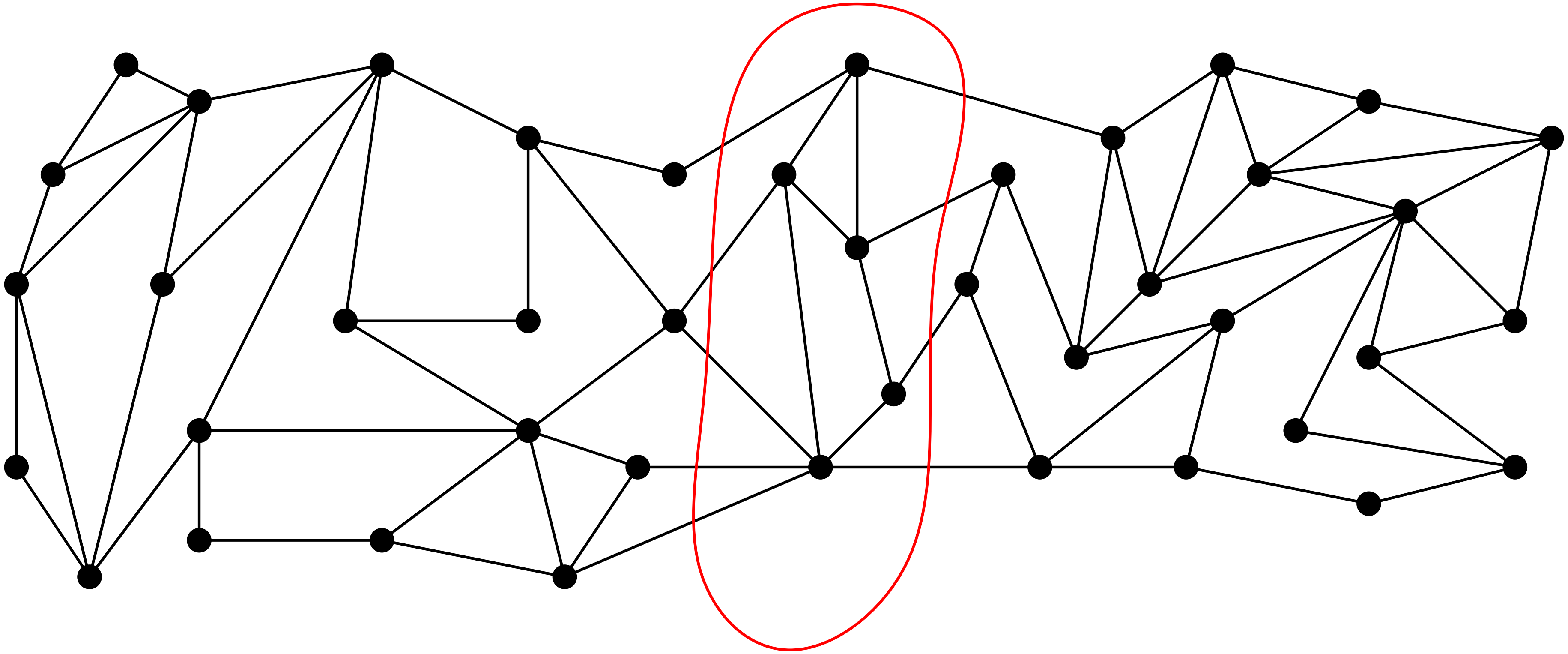

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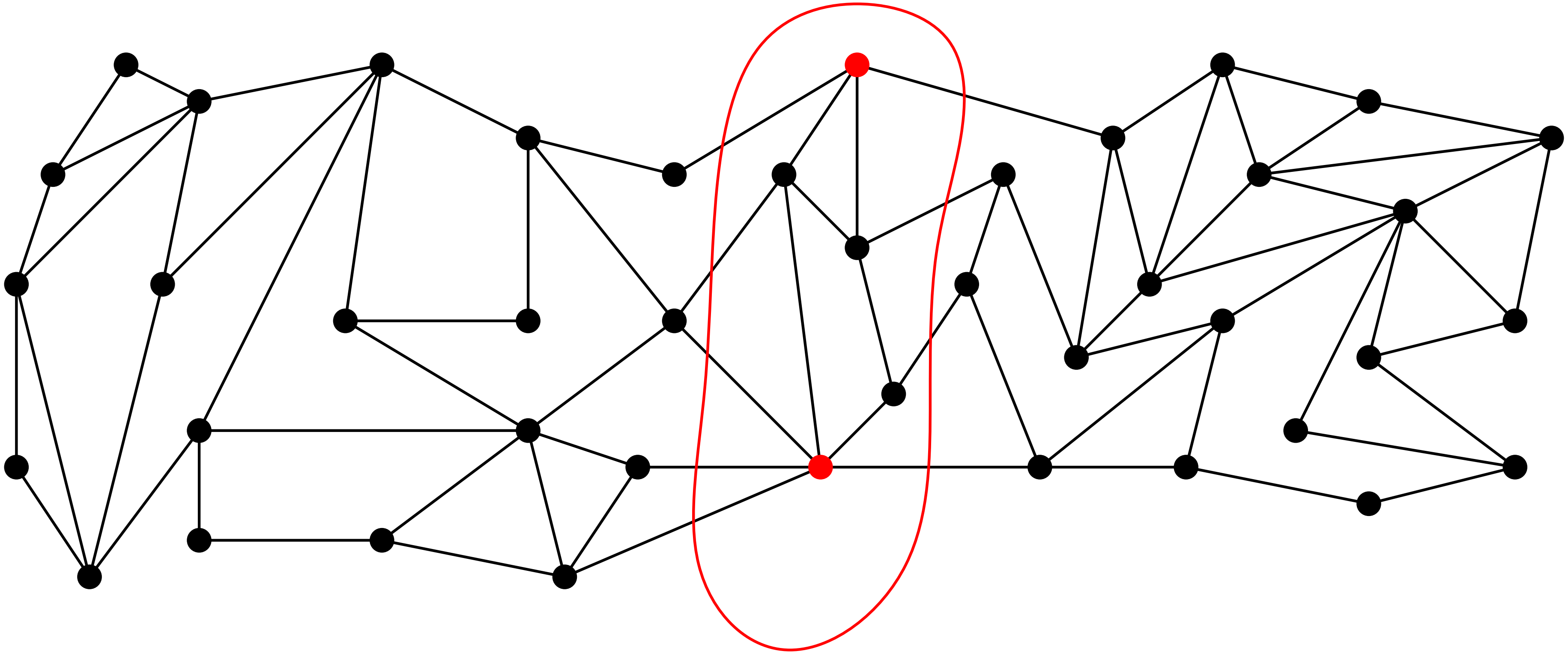

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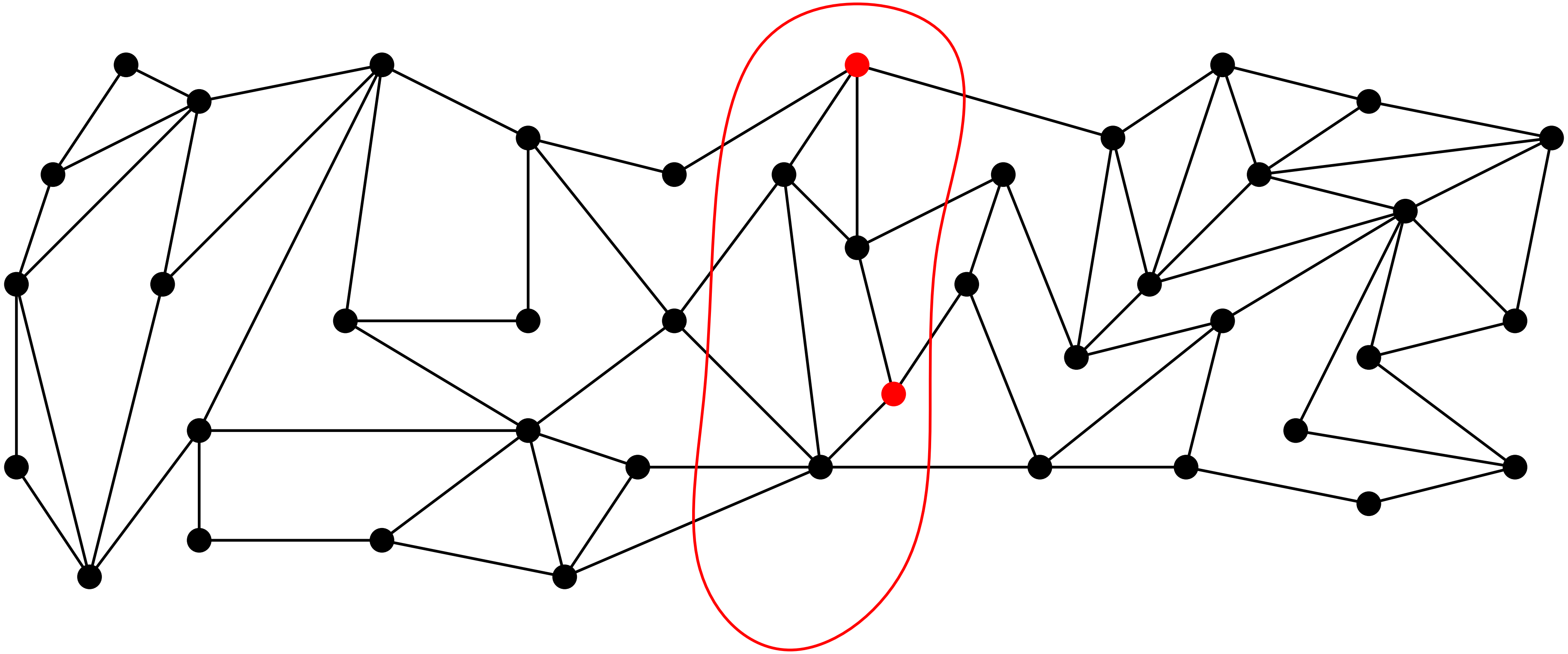

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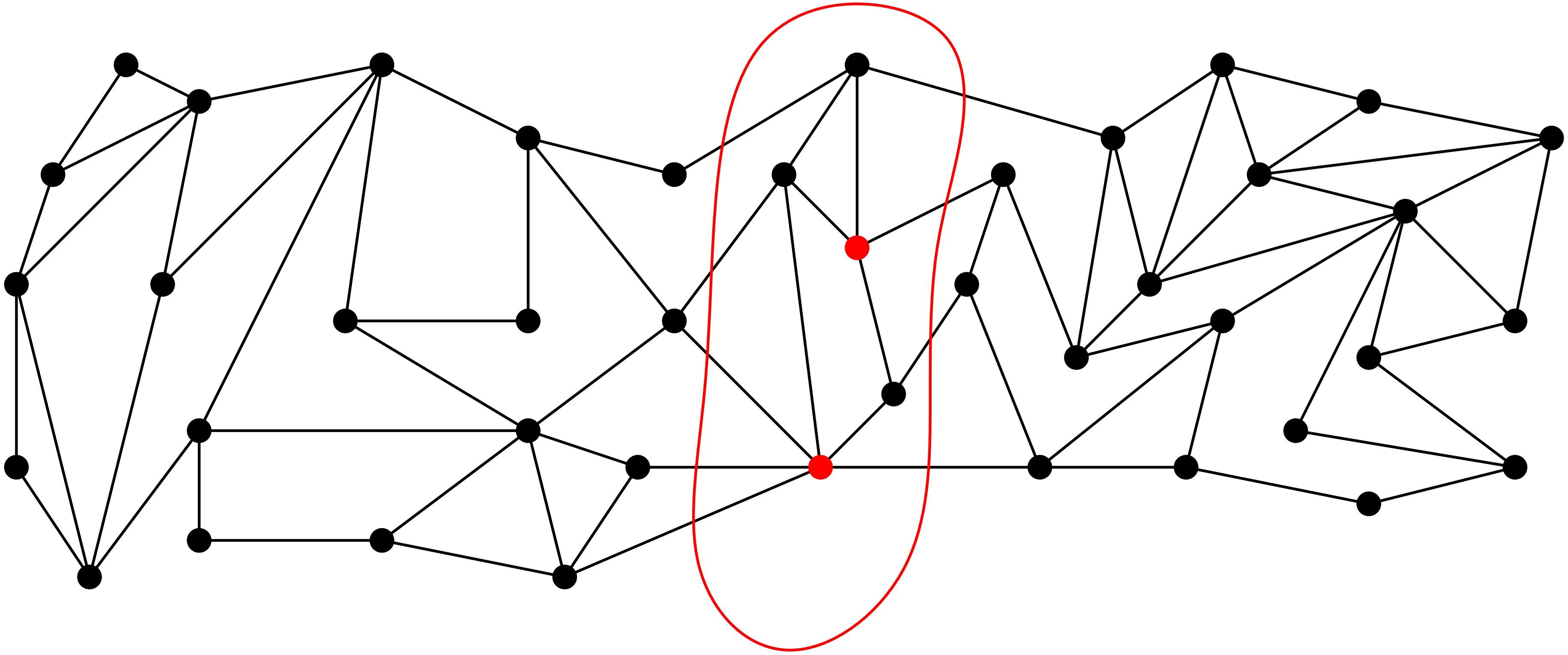

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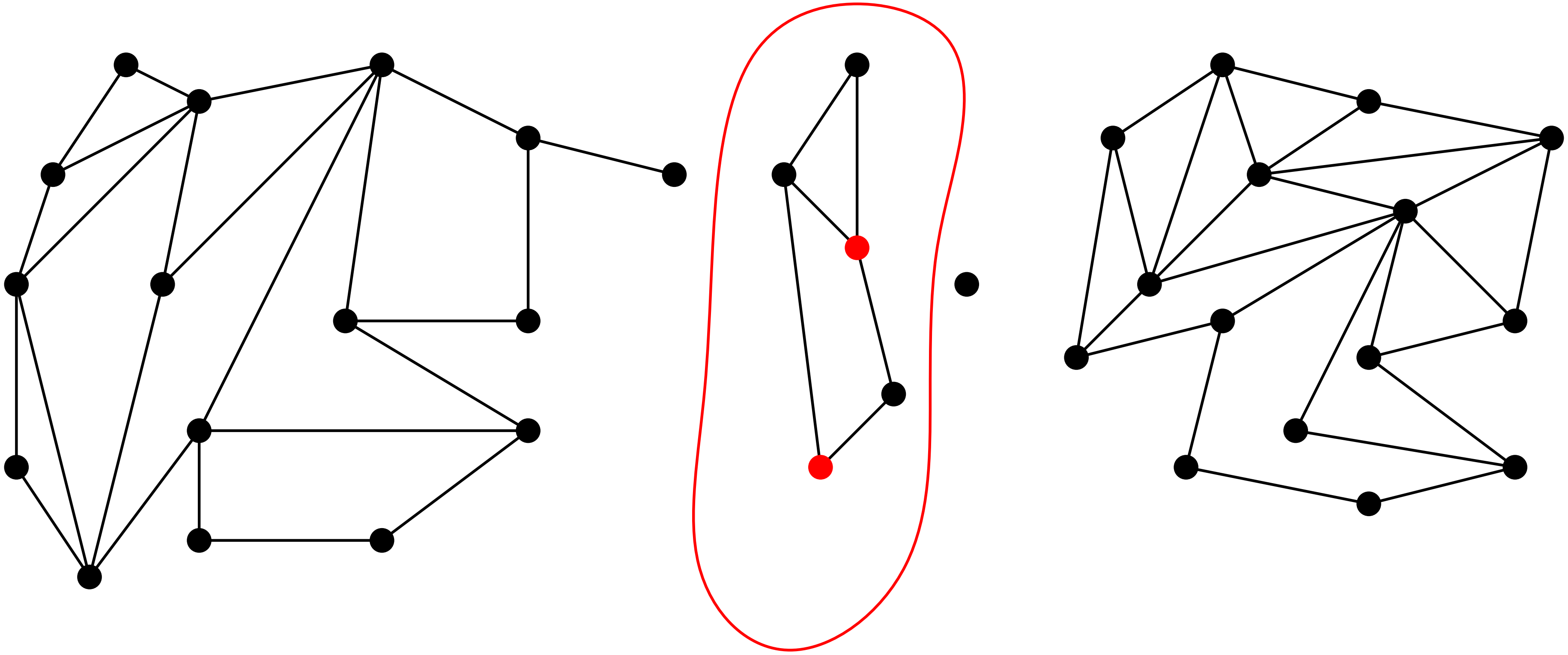

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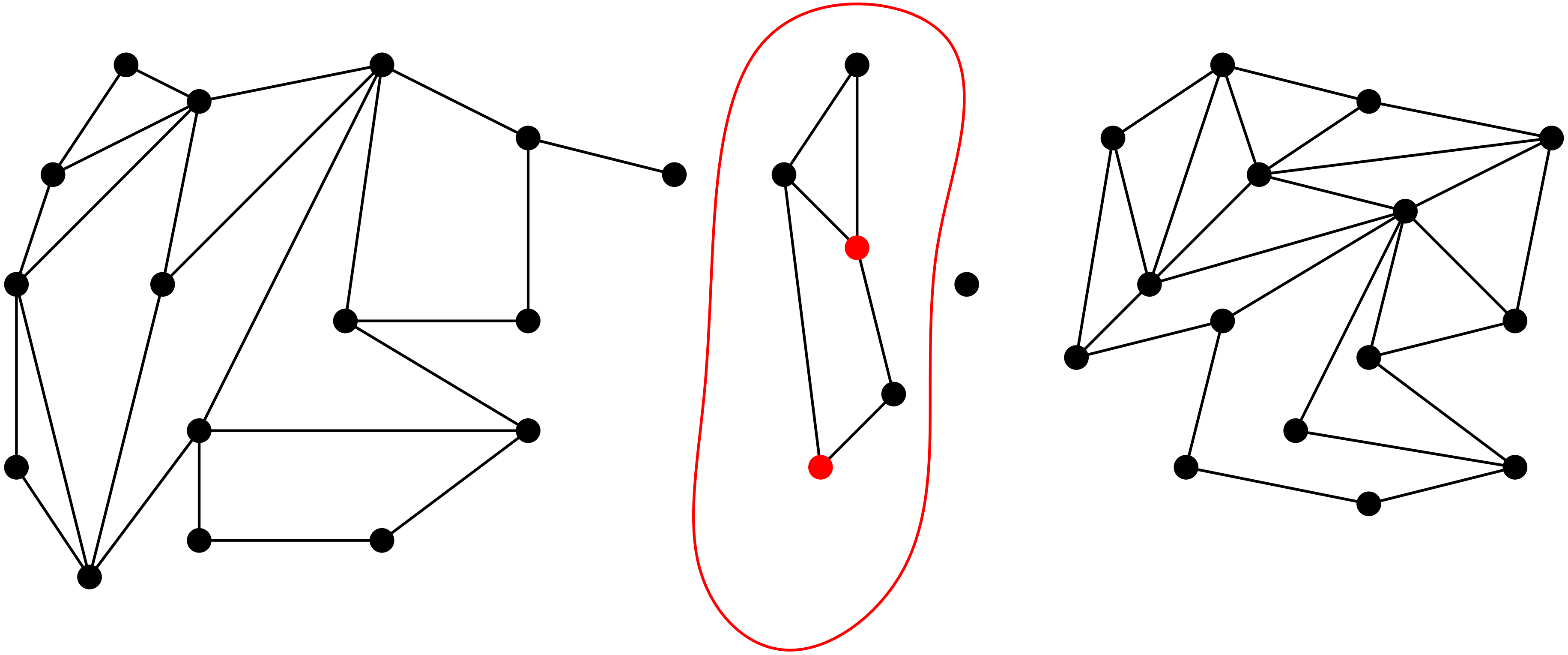
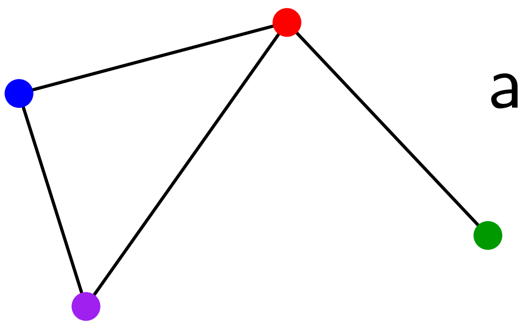


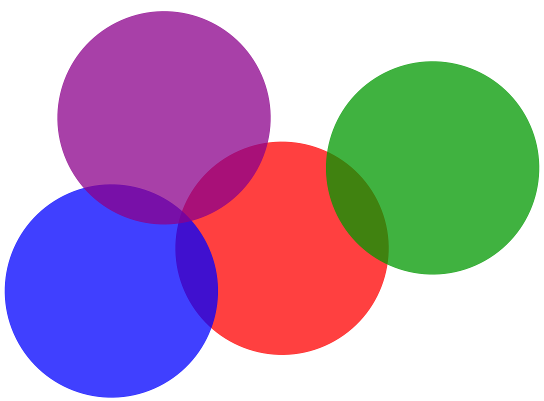
Independent Set

- Divide & conquer yields a $2^{\mathcal{O}(\sqrt{n})}$-time algorithm.

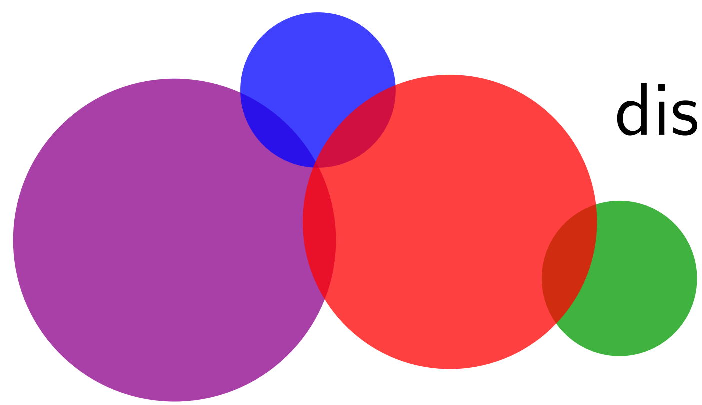
Geometric Intersection Graphs



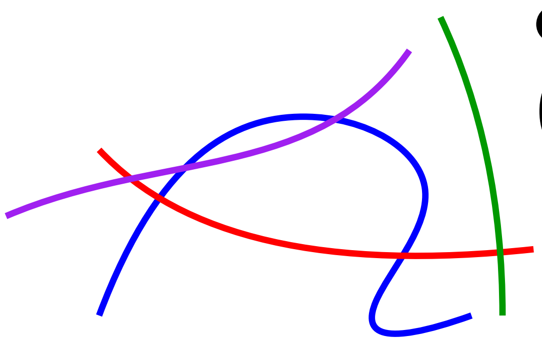
as an intersection graph of:



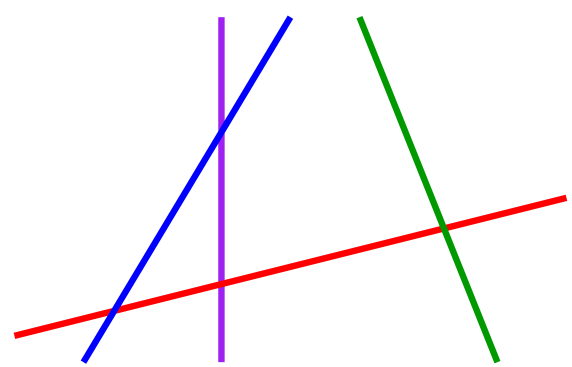
unit disks (UDG)



disks (DG)

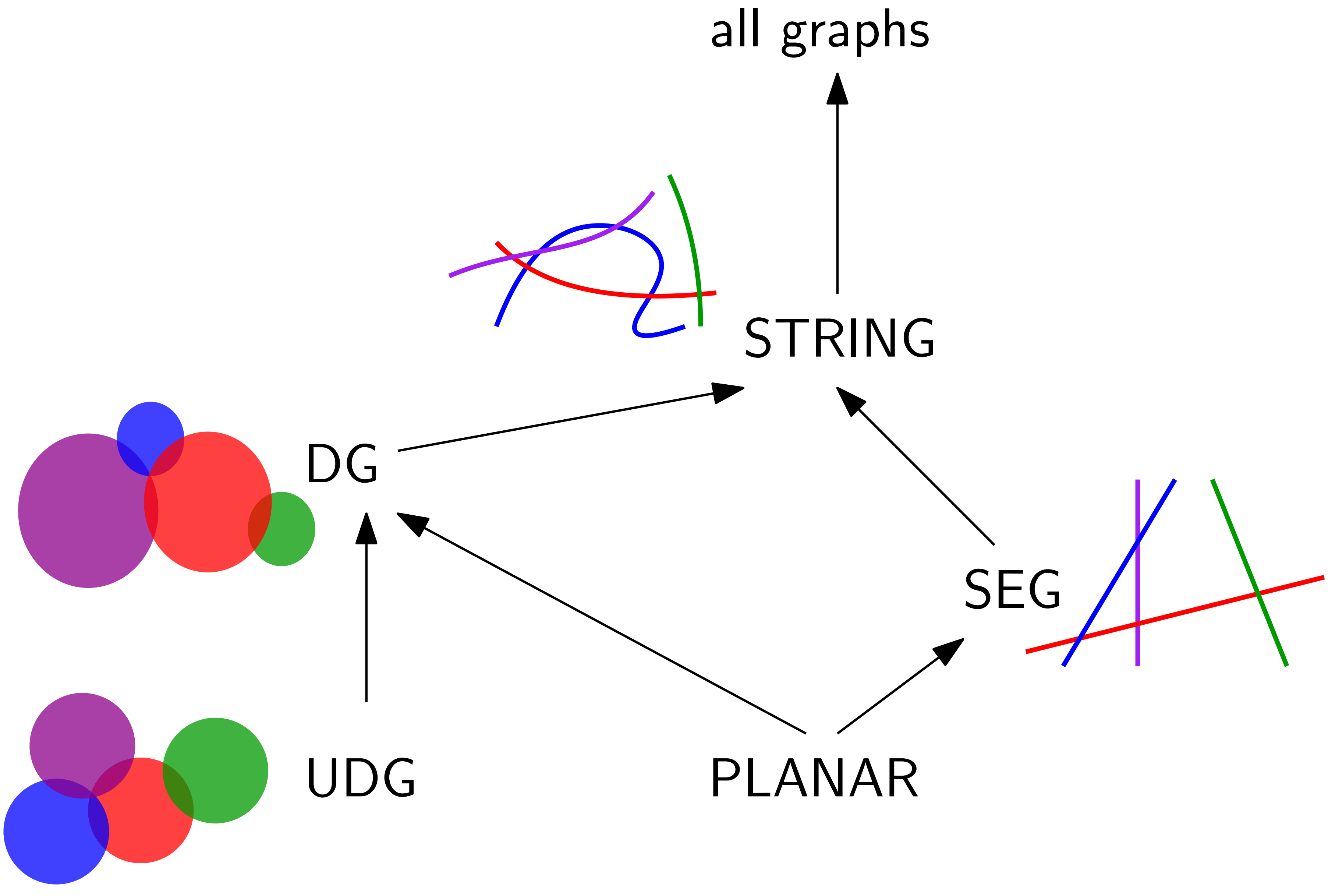


curves (STRING)



segments (SEG)

Relations Between Classes



Separator-Based Algorithms for Disk Intersection Graphs

k-Coloring Disk Graphs

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2. ply $\leq k \Rightarrow$ balanced separator S of size $\mathcal{O}(\sqrt{nk})$
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Theorem: For any fixed k, k-Coloring can be solved in time $2^{\mathcal{O}(\sqrt{n})}$ for disk graphs.

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Key observation:

Yes-instances of k-Coloring do not have large cliques.

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$$\begin{aligned} F(n) &\leq (\tau+1) \cdot F(n-\tau) \leq (\tau+1)^2 \cdot F(n-2\tau) \\ &\leq \ldots \leq (\tau+1)^{n/\tau} \cdot \mathcal{O}(1) = 2^{\mathcal{O}(n/\tau \cdot \log \tau)} = 2^{\widetilde{\mathcal{O}}(n/\tau)} \end{aligned}$$

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Algorithm.

1. ply $> \tau \Rightarrow$ clique of size $> \tau$, branch $(2^{\widetilde{\mathcal{O}}(n/\tau)})$
2. ply $\leq \tau \Rightarrow$ balanced separator S of size $\mathcal{O}(\sqrt{n\tau})$
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Independent Set in Disk Graphs (cont'd)

- We have two basic steps:
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Also, still quite boring!

Optimality for
Segment and String Graphs

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String separator theorem [Matoušek 2014, Lee 2016]

String graphs have balanced separators of size $\mathcal{O}(\sqrt{m})$.

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Theorem. [Fox & Pach 2011]

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Theorem. [Fox & Pach 2011]
Independ. Set in string graphs can be solved in time $2^{\widetilde{\mathcal{O}}(n^{2/3})}$.

1. There is a vertex v of degree at least $\tau = n^{1/3} \Rightarrow$ branch!
   - Either discard v or put it into the solution.

     $F(n) \leq F(n-1) + F(n - n^{1/3})$

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   - guessing a color for v does not mean we can discard $N(v)$!

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$\Rightarrow$ We cannot use the separator theorem!

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Theorem [Bonnet & Rz. 2018] k-Coloring for string graphs:

1. for $k = 3$, can be solved in time $2^{\widetilde{\mathcal{O}}(n^{2/3})}$,
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Even though G is dense, almost all its edges are meaningless!

Hardness of List 4-Coloring

- Reduce from 3-Sat with n variables and $m = \mathcal{O}(n)$ clauses.
- Variables: $v_1, v_2 \ldots, v_n$, clauses: $C_1, C_2, \ldots, C_m$
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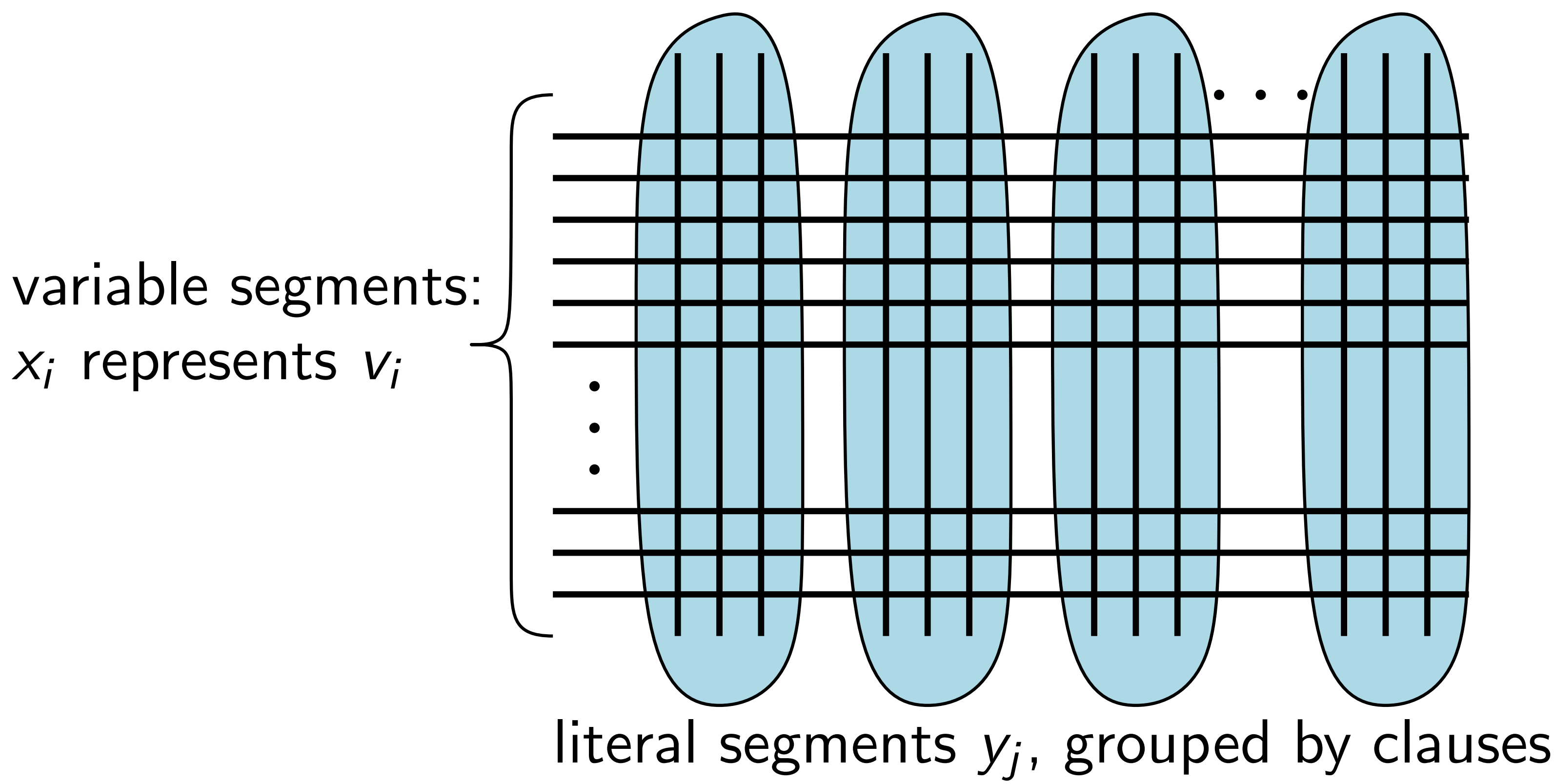

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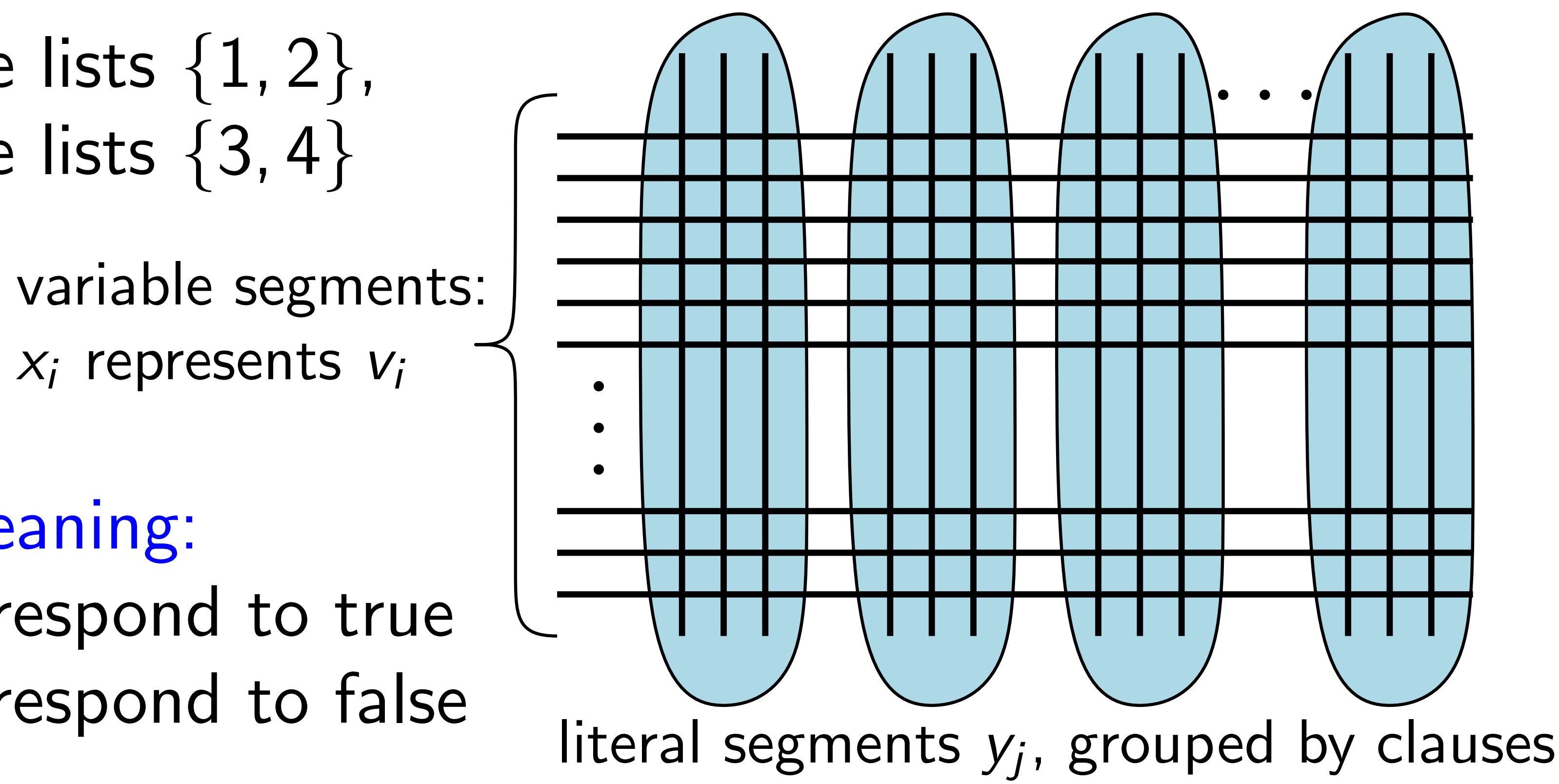
- x_i's have lists $\{1, 2\}$, y_i's have lists $\{3, 4\}$



Intended meaning:
1 and 3 correspond to true
2 and 4 correspond to false

Hardness of List 4-Coloring (cont'd)

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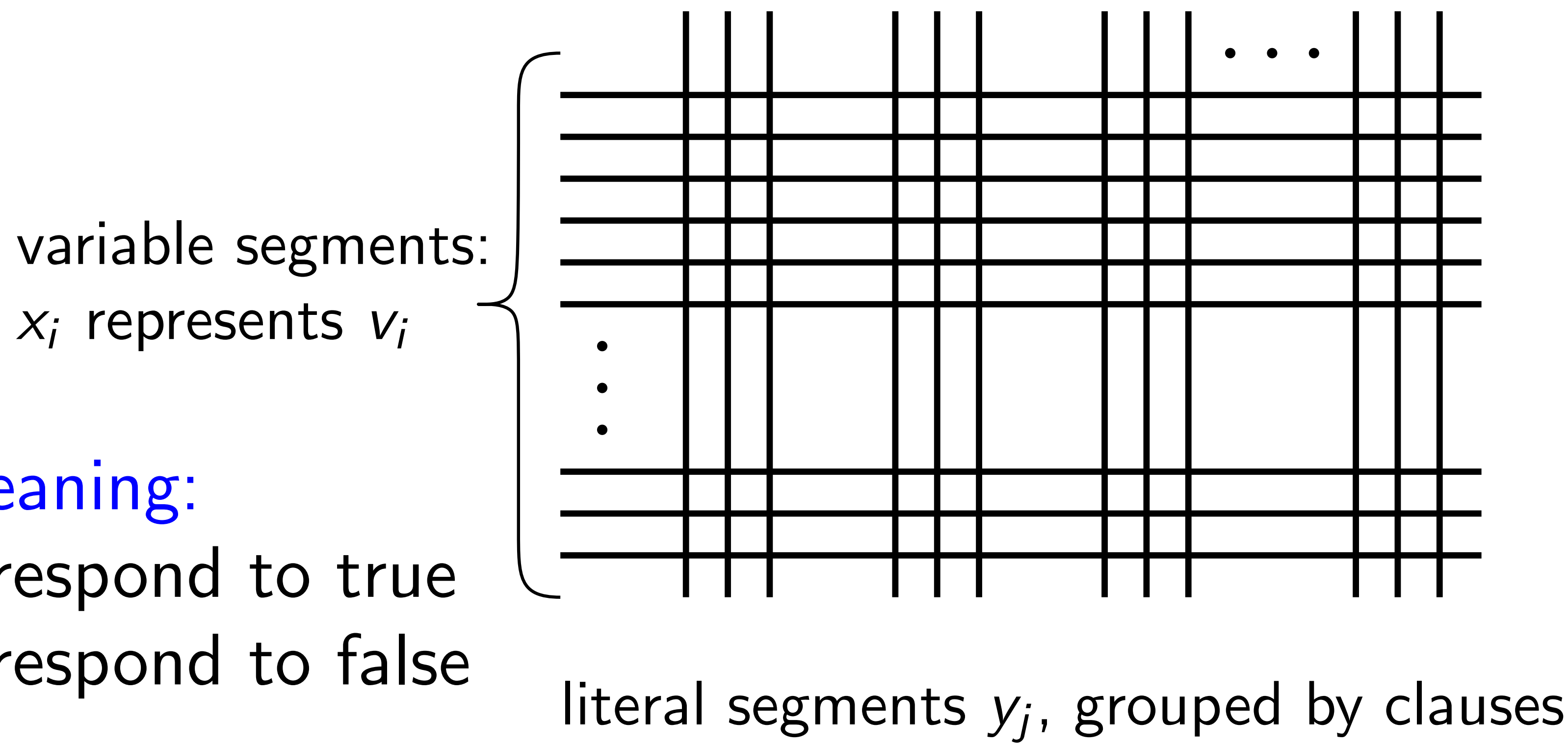

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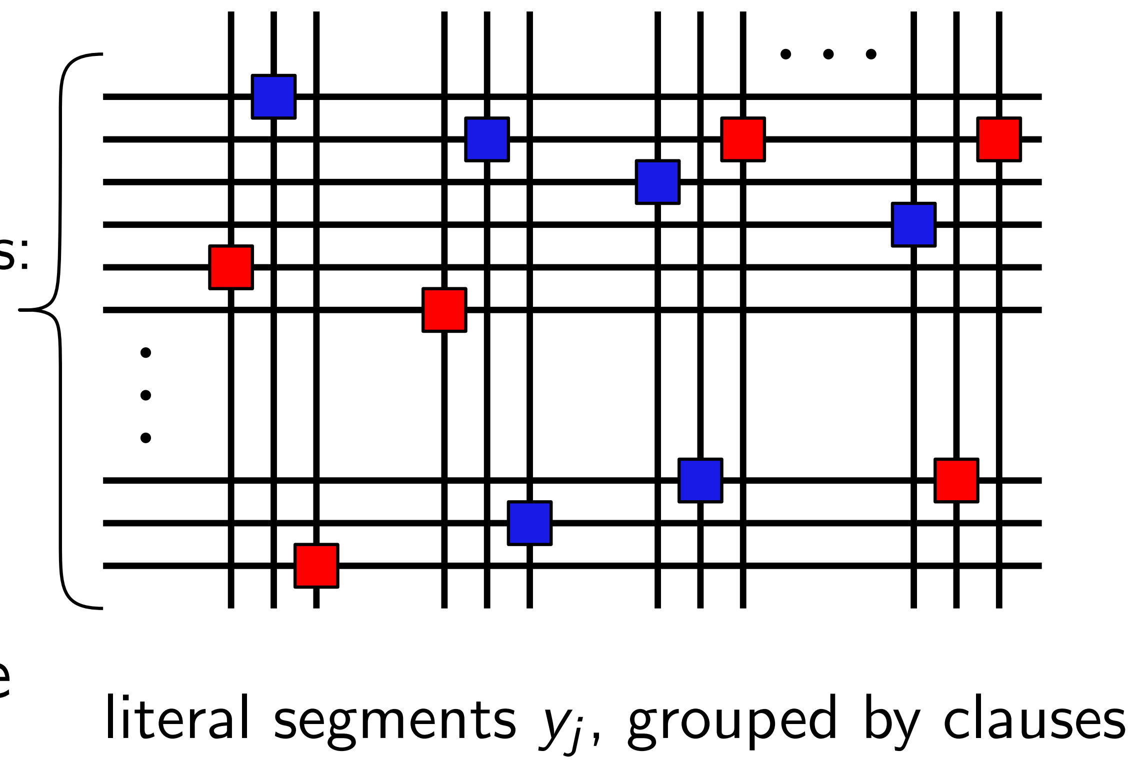

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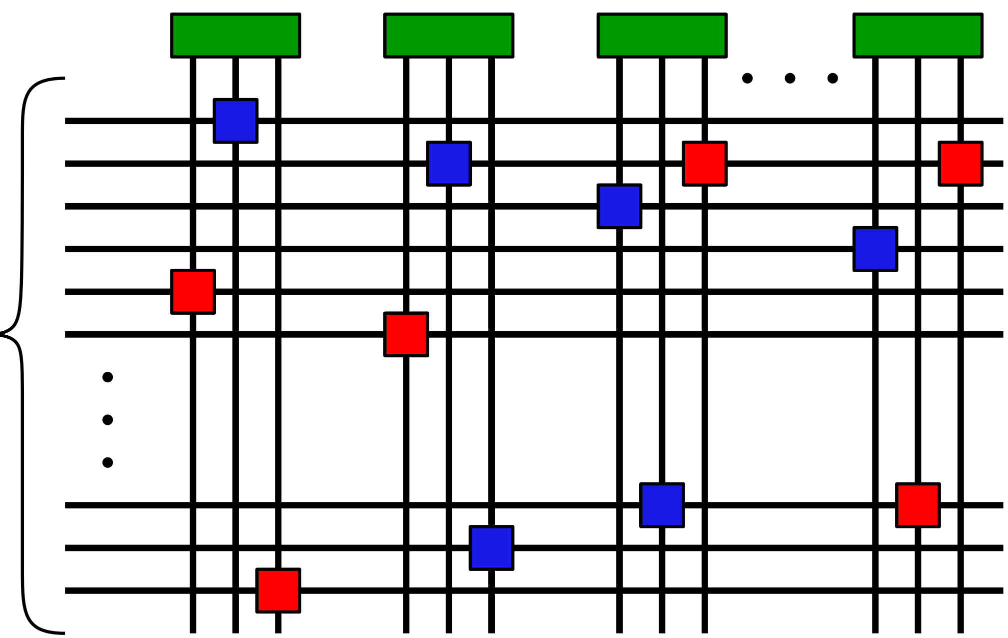
- Satisfiability

  at least one of y's must be colored 3

variable segments:
x_i represents v_i



literal segments y_j, grouped by clauses

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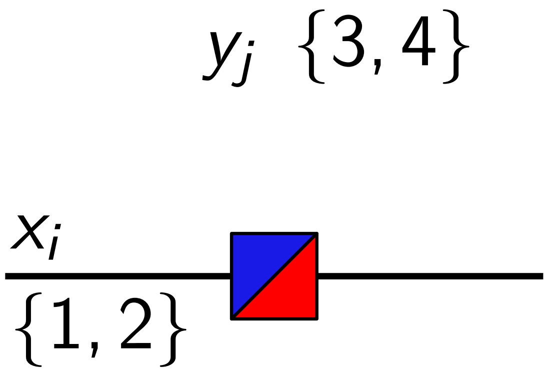
Consistency Gadgets



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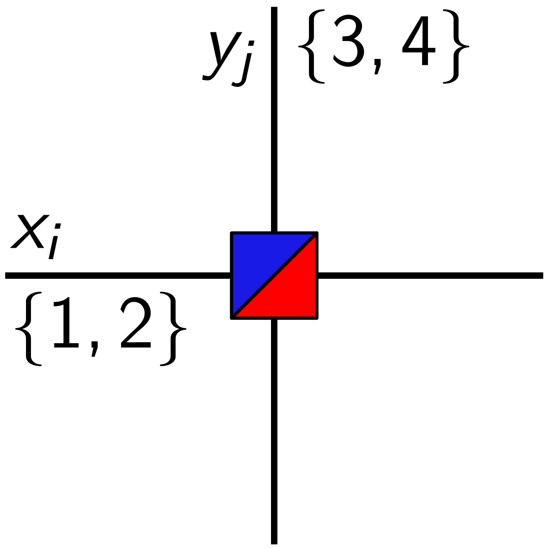
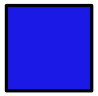
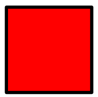
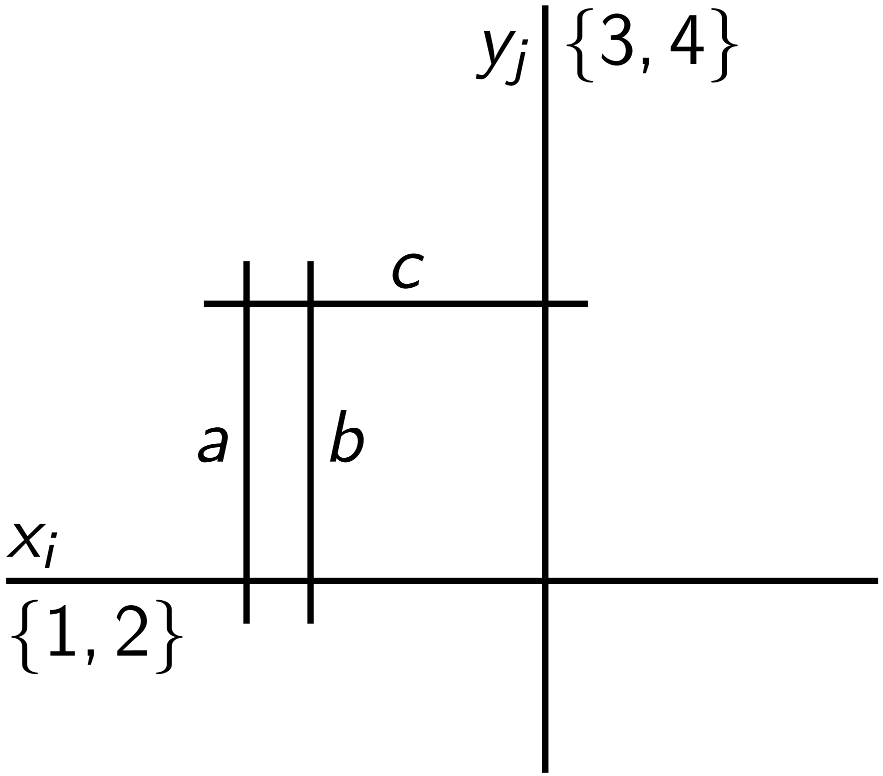
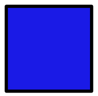
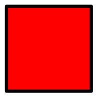


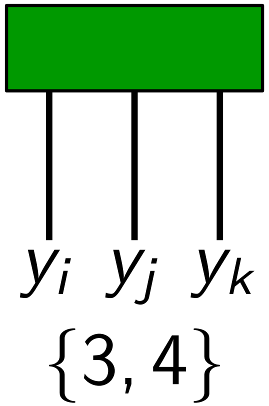
 x_i gets color 1 iff y_j gets color 3

 x_i gets color 1 iff y_j gets color 4



		
a	$1, 3$	$1, 4$
b	$2, 4$	$2, 3$
c	$3, 4$	$3, 4$

Satisfiability Gadget



At least one of u_i, y_j, y_k must get color 3.

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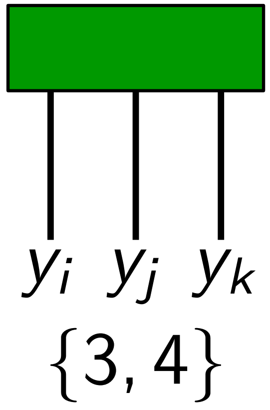

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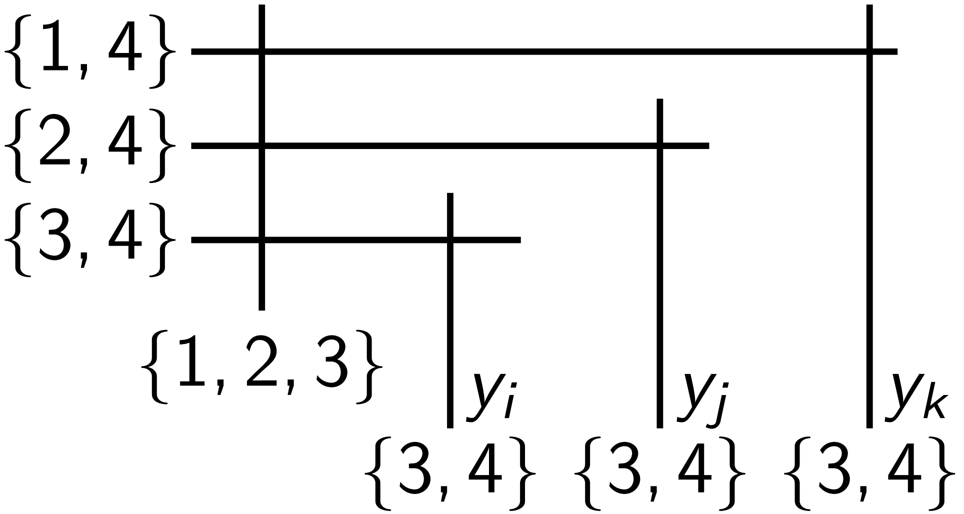

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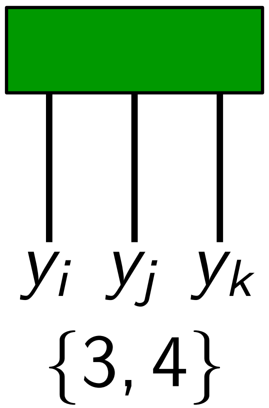

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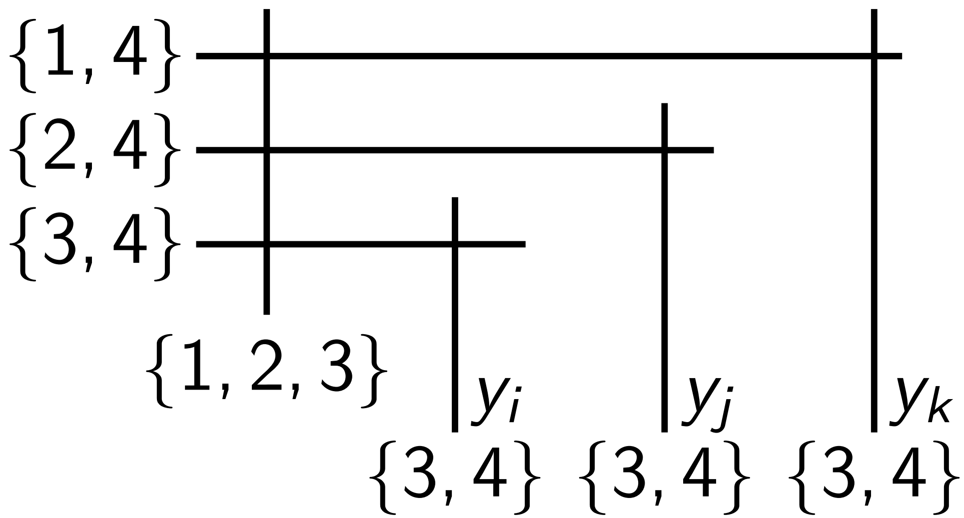
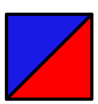
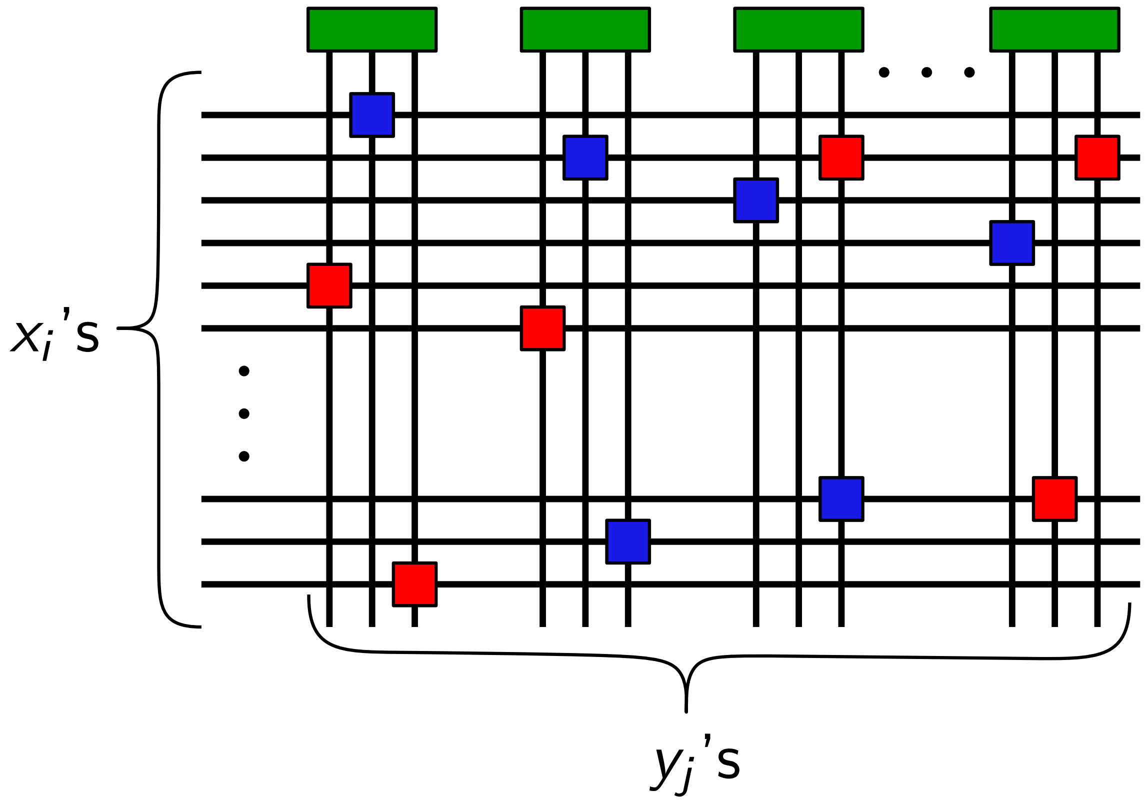


- Note that there are segments with 3-element lists – if all lists have ≤ 2 elements, then the problem is in P!

Wrap-Up

- We've reduced from 3-SAT with n variables and $m = \mathcal{O}(n)$ clauses.
- How many segments do we have?

x_i's	
y_j's	
	
	
total	



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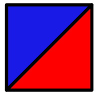
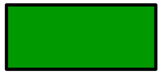
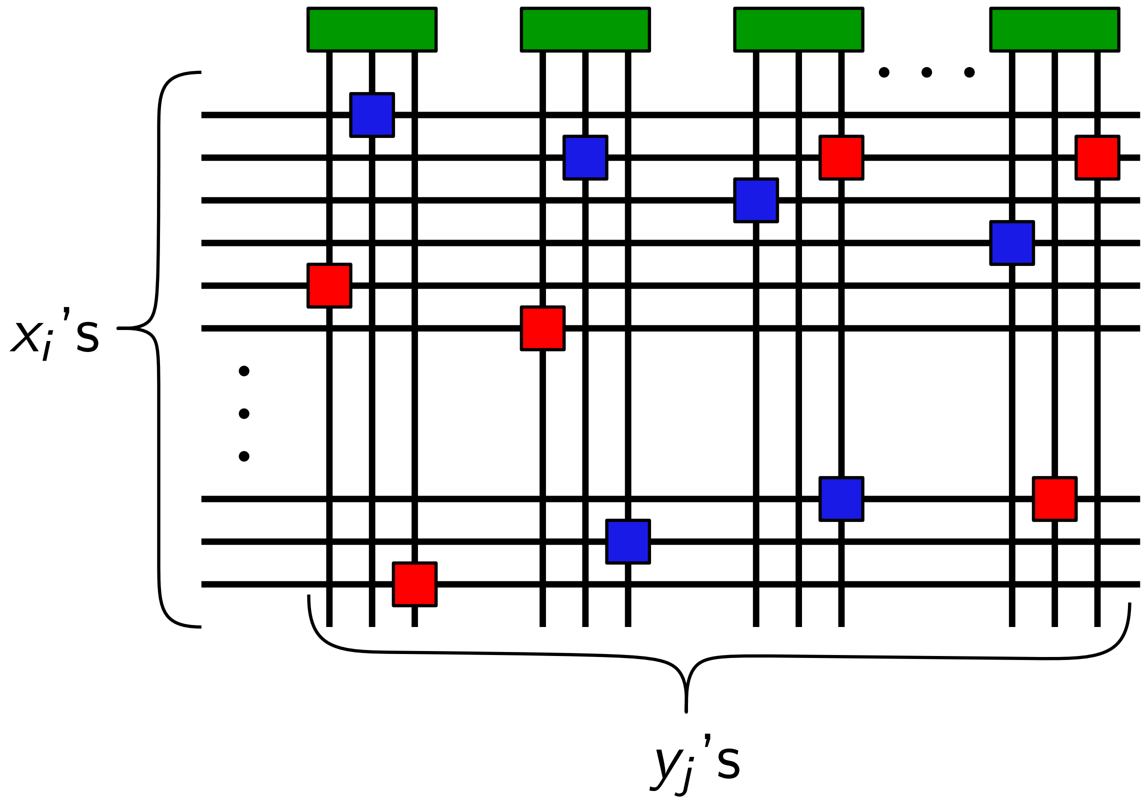
y_j's	$3m$
	$3m \times 3$
	$m \times 4$
total	$n + 16m$ $\in \mathcal{O}(n)$



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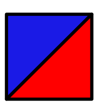	$3m \times 3$
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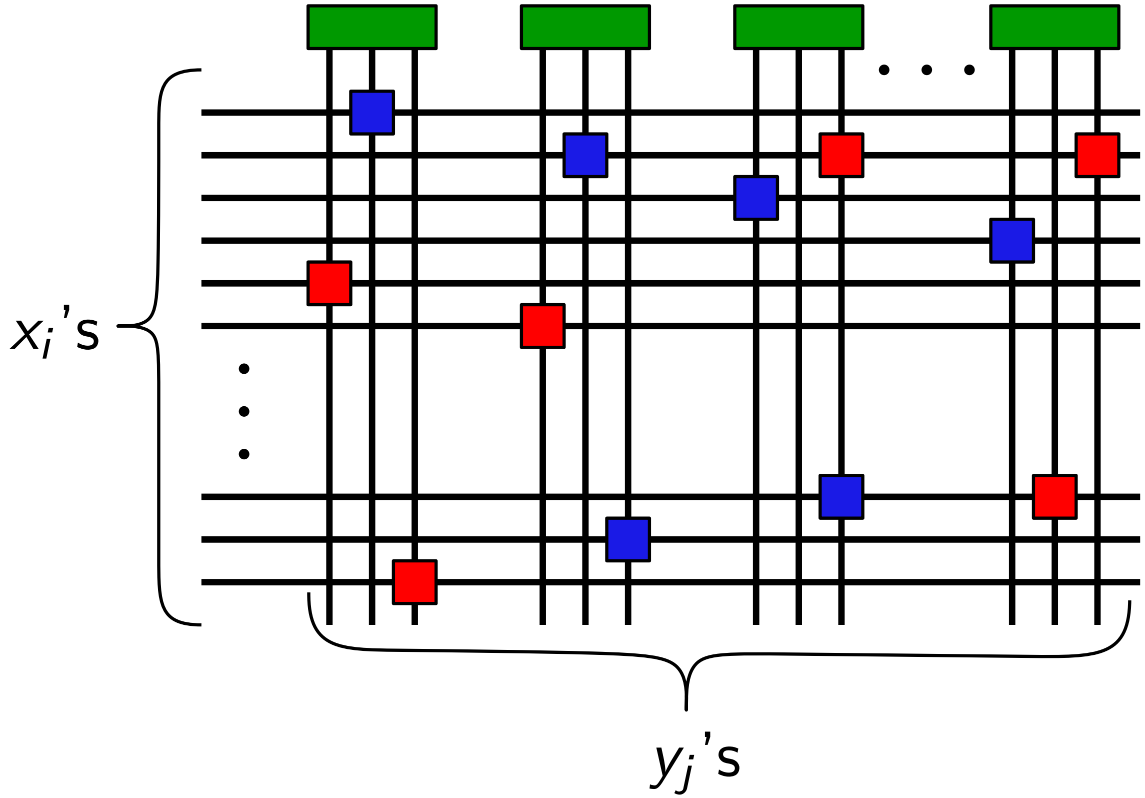

- Assume we could LIST 4-COLOR segment graphs with N vertices in time $2^{o(N)}$.

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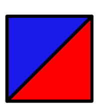	$3m \times 3$
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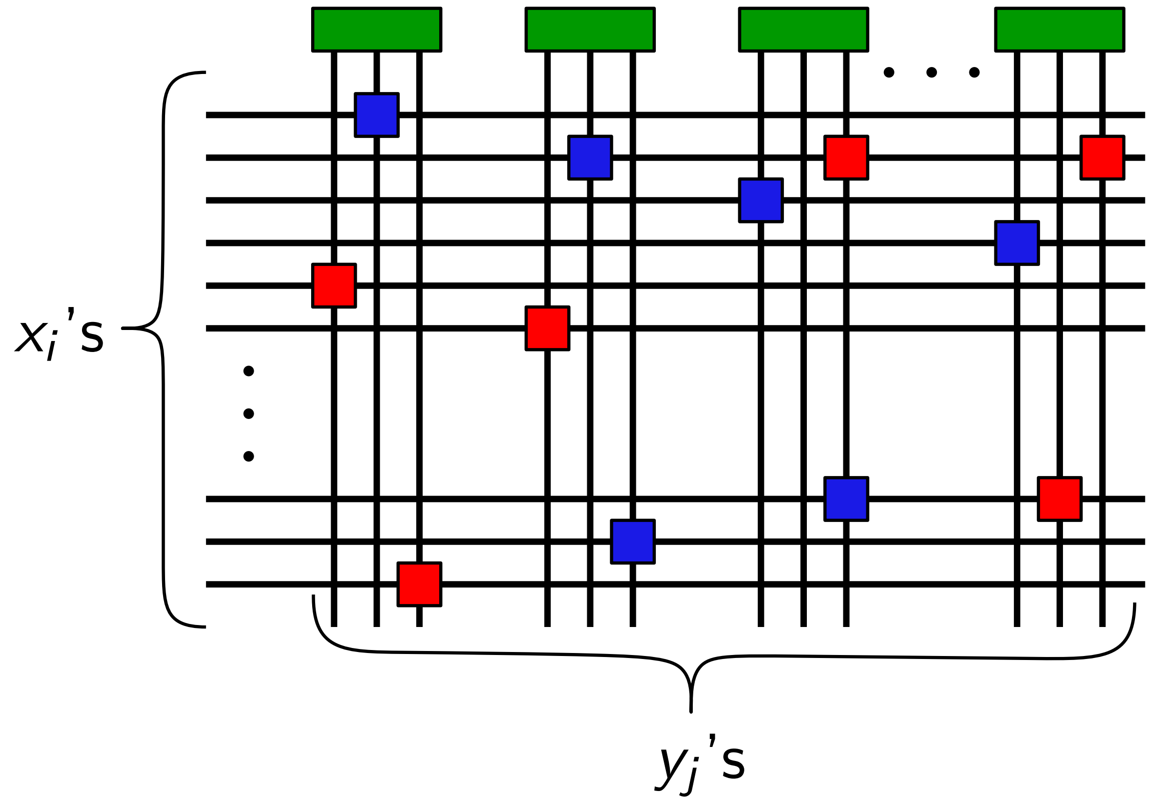

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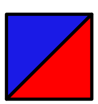	$3m \times 3$
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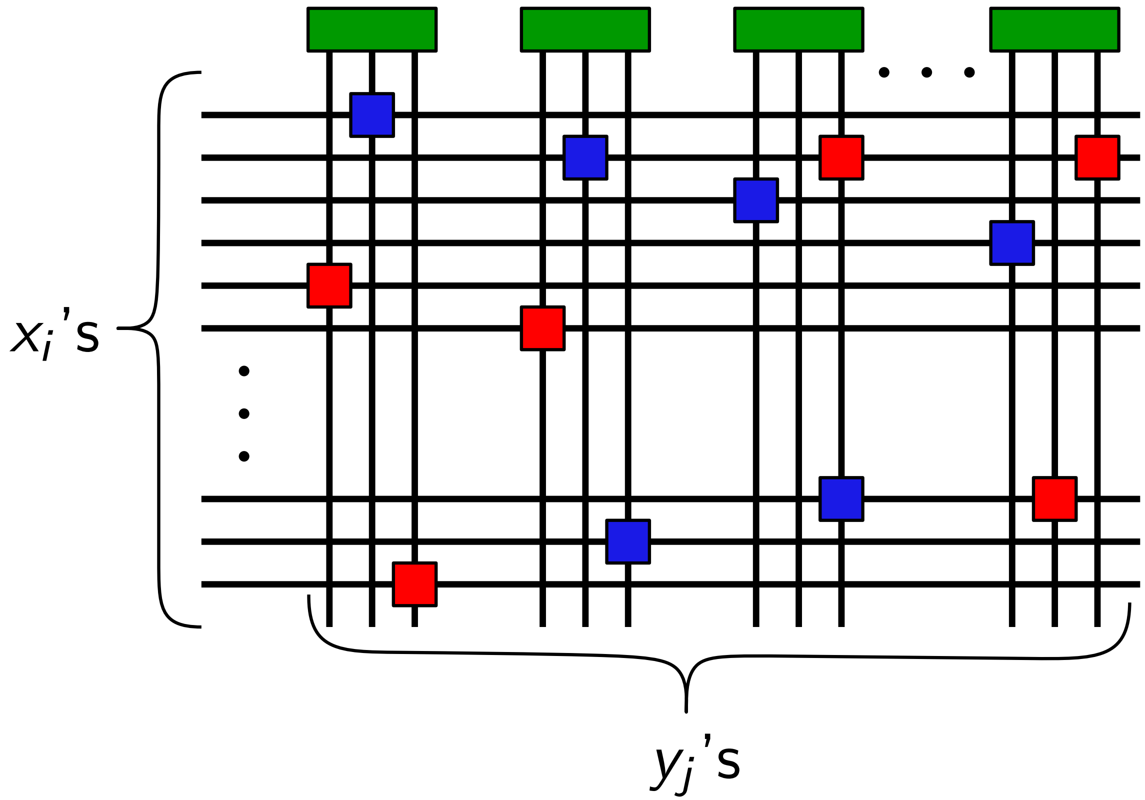

- Assume we could LIST 4-COLOR segment graphs with N vertices in time $2^{o(N)}$.
  $\Rightarrow$ Could solve 3-SAT in time $2^{o(n)}$.
  $\Rightarrow$ ETH would fail.

– End of Lecture –

Feedback Vertex Set in string graphs

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- if we have a small separator, the divide & conquer works
- what if we have a vertex of large degree?

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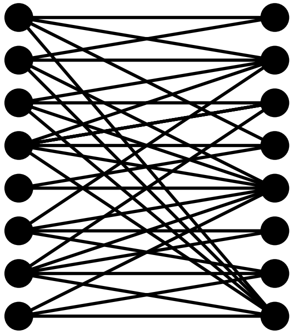

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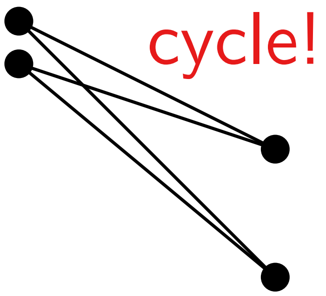

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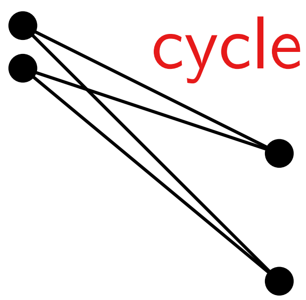

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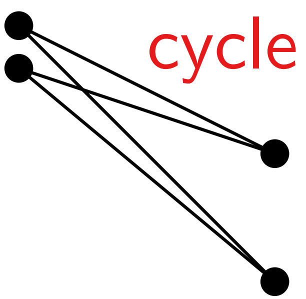

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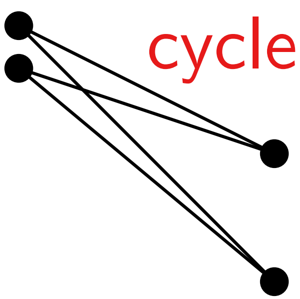

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- But no $2^{o(n)}$ algorithm for Odd Cycle Transversal

A detour: the need of representation and robust algorithms

Finding geometric representations

- How fast can we find representations?

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Recognizing string graphs is in NP.

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- a strong indication that the problem is not in NP!
- similar for unit disk graphs [Kang, Müller, '12]

What about our algorithms?

Independent Set in disk graphs

1. ply $> n^{1/3} \rightarrow$ a clique of size $> n^{1/3}$, branch
2. ply $\leq n^{1/3} \rightarrow$ a balanced separator S of size $\mathcal{O}(n^{2/3})$
3. guess the solution on S
4. recurse using divide & conquer

Total running time: $2^{\widetilde{\mathcal{O}}(n^{2/3})}$.

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- where do we need a representation?
- enumerating all possibilities takes time $n^{n^{2/3}} = 2^{\widetilde{\mathcal{O}}(n^{2/3})}$
- we do not really need a representation!

Robust algorithms

- An algorithm is robust, if it either
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- on the other hand, our hardness results hold even if a geometric representation is given

When large cliques do not help

CLIQUE in disk graphs

- CLIQUE is polynomially solvable in UDG [Clark et al., 1990]
- the complexity for DG is open

- the existence of a large clique does not make the problem any easier!

Clique in disk graphs

- Clique is polynomially solvable in UDG [Clark et al., 1990]
- the complexity for DG is open
- the existence of a large clique does not make the problem any easier!
- we need to make our hands dirty and look at the properties of geometric representations
- by some epsilon-perturbation we can assume that no three centers are aligned

Notation: vertex v_i is represented by a disk with the center c_i

C_4's in disk graphs

Simple observation.

In any disk representation of of C_4 with vertices v_1, v_2, v_3, v_4:

the line $\ell(c_2 c_4)$ crosses the segment $c_1 c_3$, or

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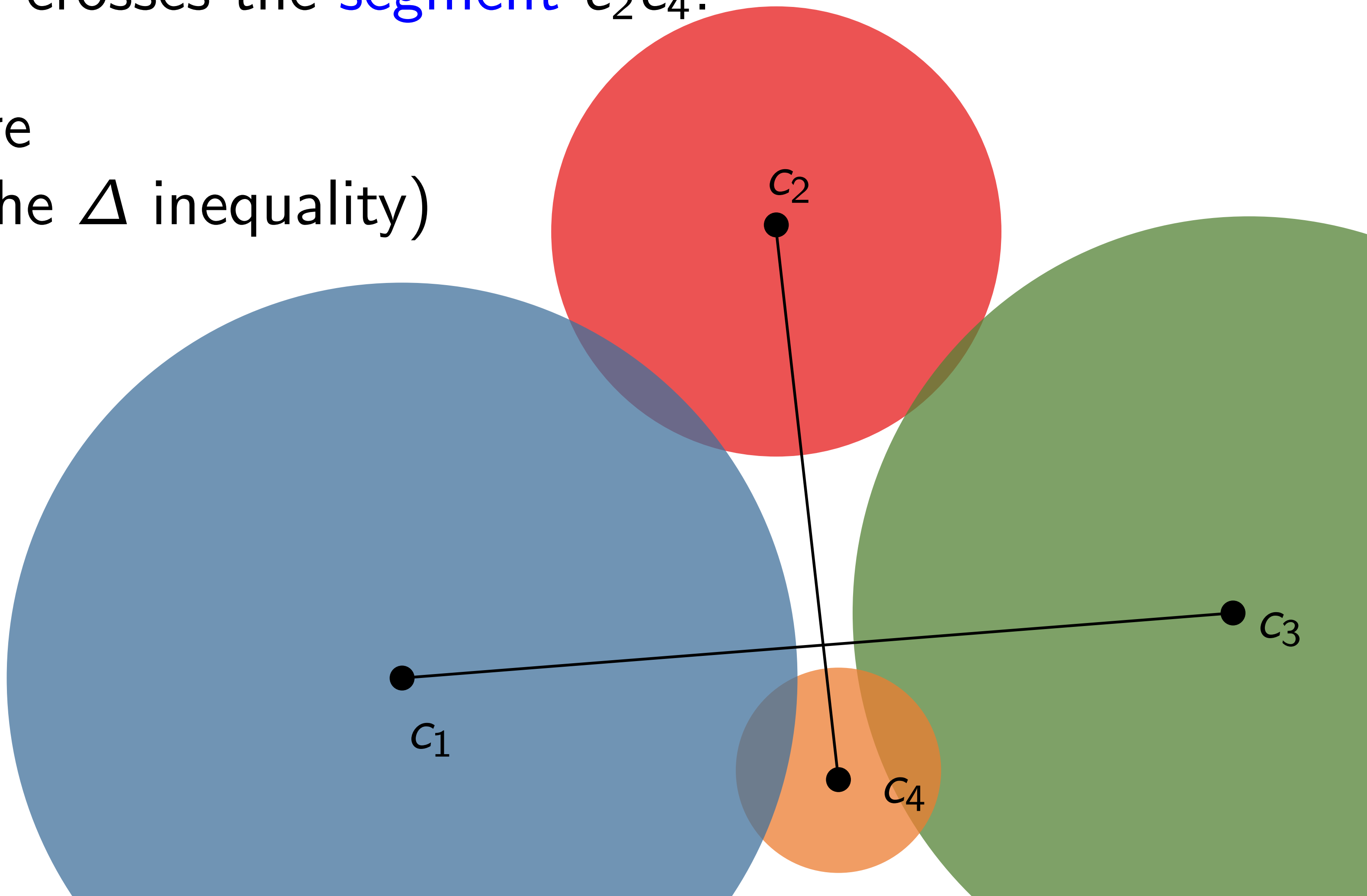

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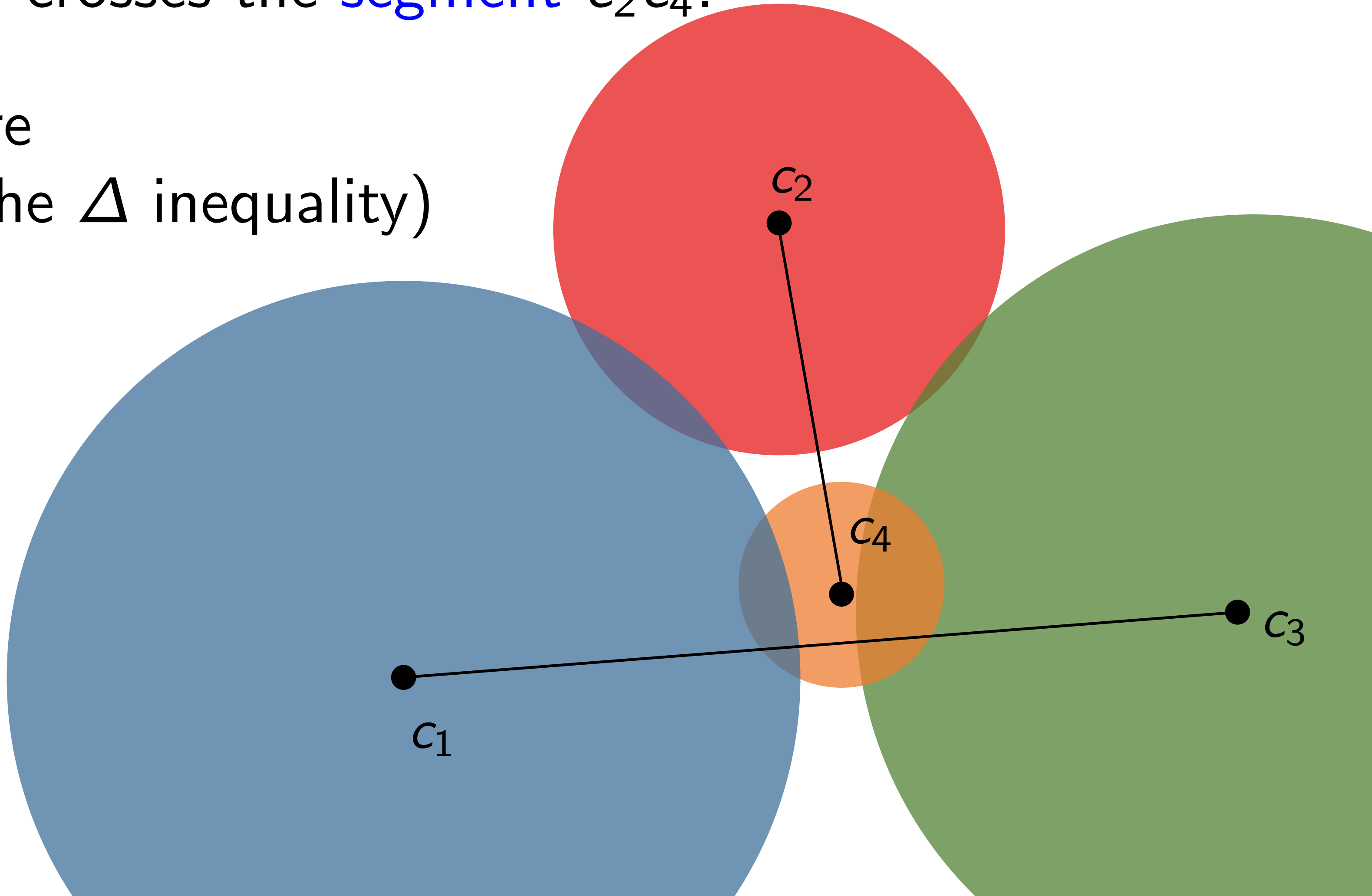
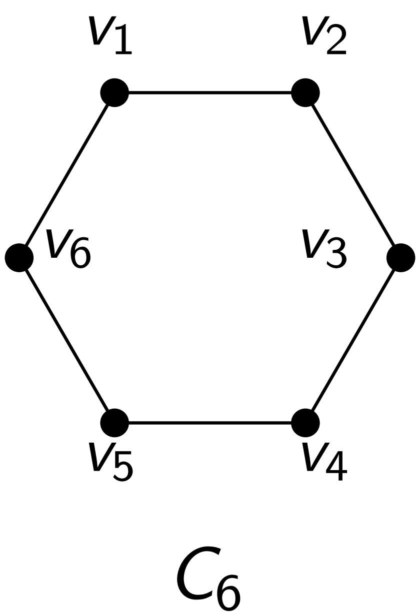


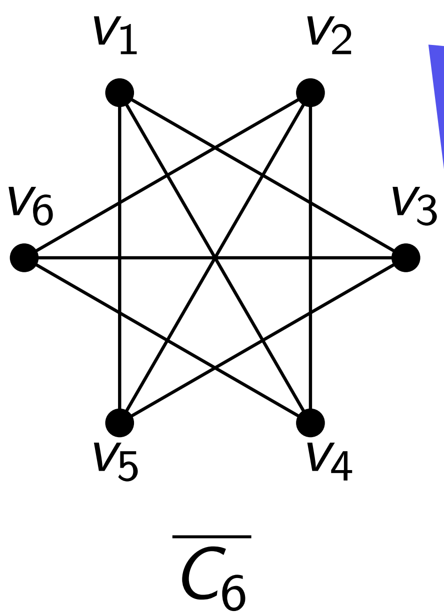
Non-disk graphs

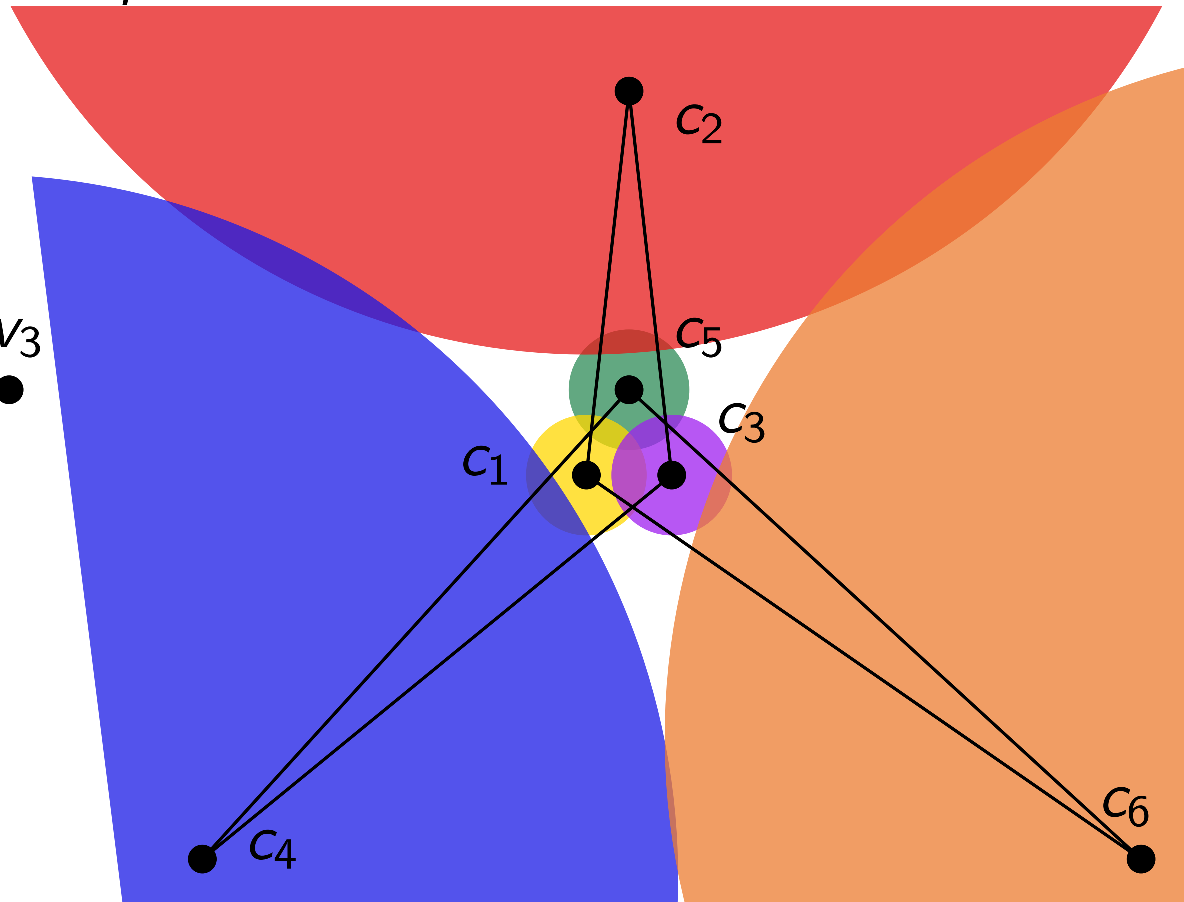
Observation [Bonnet, Giannopoulos, Kim, Rz. Sikora, 2018].
For odd p, q, the graph $G = \overline{C_p + C_q}$ is not a disk graph.

Proof by contradiction.

- suppose there is a representation
- let $S_1, \ldots, S_p$ and $S'_1, \ldots, S'_q$ be segments of the co-cycles







Non-disk graphs

Observation [Bonnet, Giannopoulos, Kim, Rz. Sikora, 2018].
For odd p, q, the graph $G = \overline{C_p + C_q}$ is not a disk graph.

Proof by contradiction.

- suppose there is a representation
- let $S_1, \ldots, S_p$ and $S'_1, \ldots, S'_q$ be segments of the co-cycles
- every S_i and every S'_j correspond to $2K_2$ in $\overline{G}$
  $\rightarrow$ their endpoints induce a C_4 in G
  $\rightarrow \ell(S_i)$ crosses S_j or $\ell(S_j)$ crosses S_j

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- $\sum_{i=1}^{p}(a_i + b_i - c_i) = pq$ is even $\rightarrow$ contradiction $\square$

$\textsc{Clique}$ for disk graphs

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Independent Set in a co-disk graph:

1. vertex of degree at least $n^{1/3} \to$ branching
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$2^{\widetilde{\mathcal{O}}(n^{2/3})}$ (for item 1)

guess the solution on X or $N[C]$ and finish in poly time (for items 2 and 3)

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Theorem [BGKRzS '18].
Clique in disk graphs can be solved in time $2^{\widetilde{\mathcal{O}}(n^{2/3})}$.

Open problem: MAX CUT in disk graphs

- partition vertices into two sets, to maximize the number of crossing edges
- NP-hard on unit disk graphs, reduction is quadratic $\rightarrow$ no $2^{o(\sqrt{n})}$algorithm
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- complexity even unclear for (unit) interval graphs

Episode 2: parameterized algorithms

Geometric separators

k-Independent Set in unit disk graphs

- is there an independent set of size at least k?
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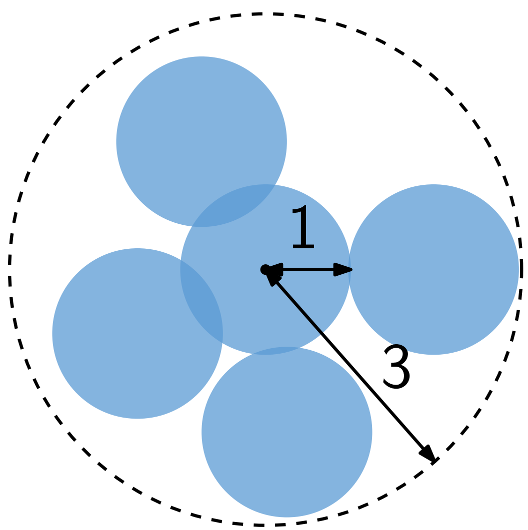

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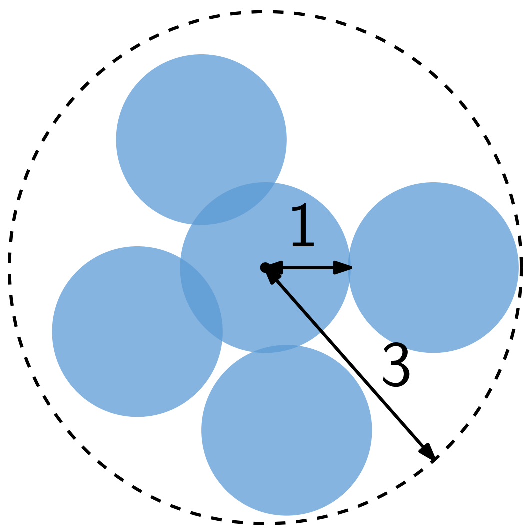

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- assume that $\pi \cdot k \leq$ total area $\leq 9\pi \cdot k$

Geometric separator theorem for unit disks

Geometric separator theorem [Alber, Fiala, '04].

Given a collection of unit disks with total area A, there exists a set S of disks, such that:

- total area of disks in S is $\mathcal{O}(\sqrt{A})$,
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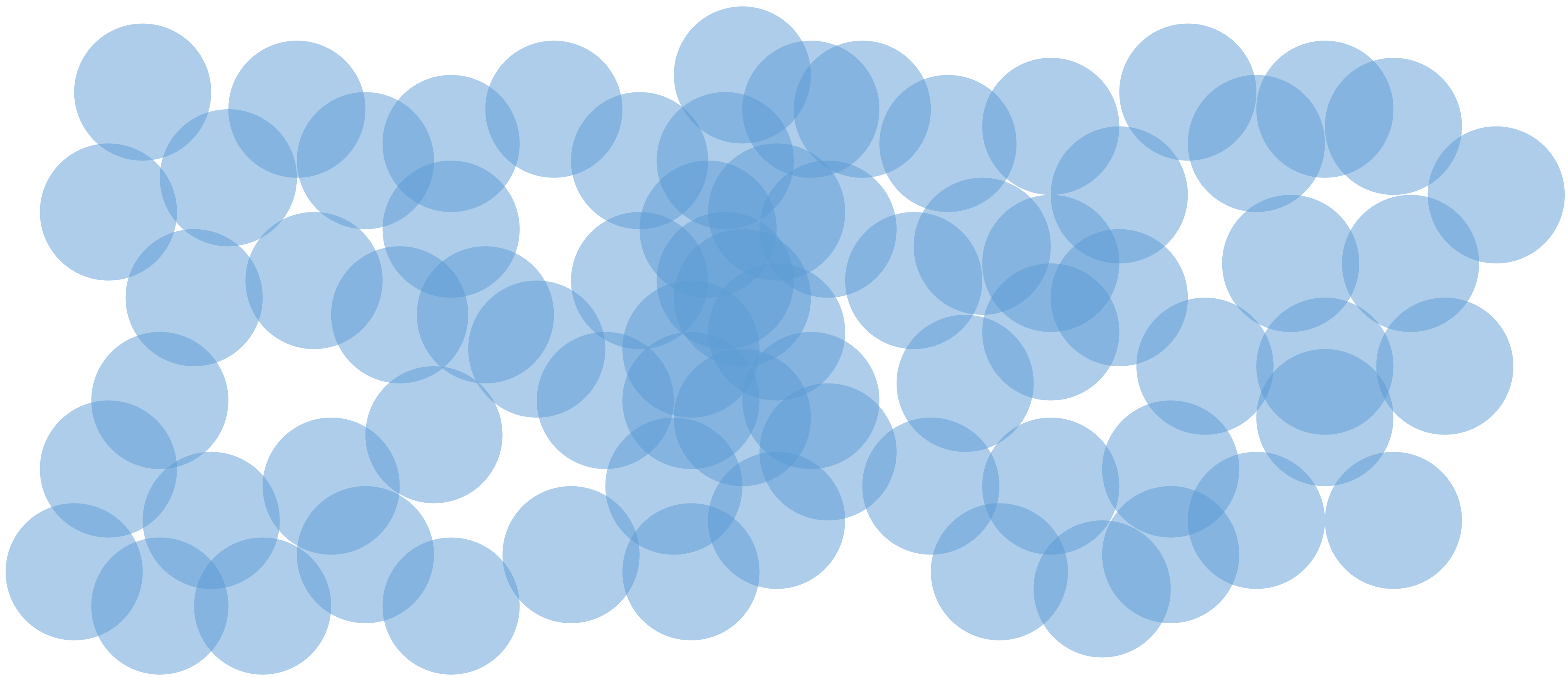

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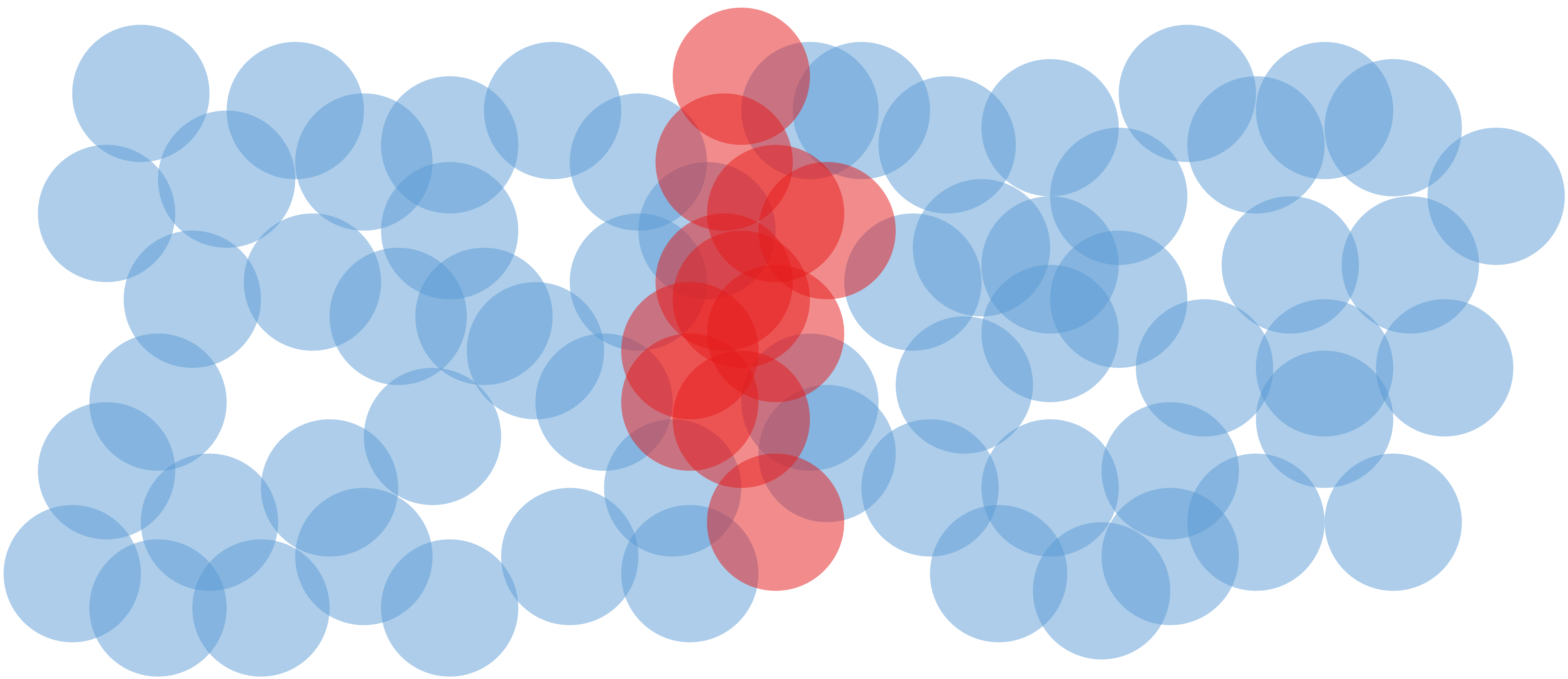

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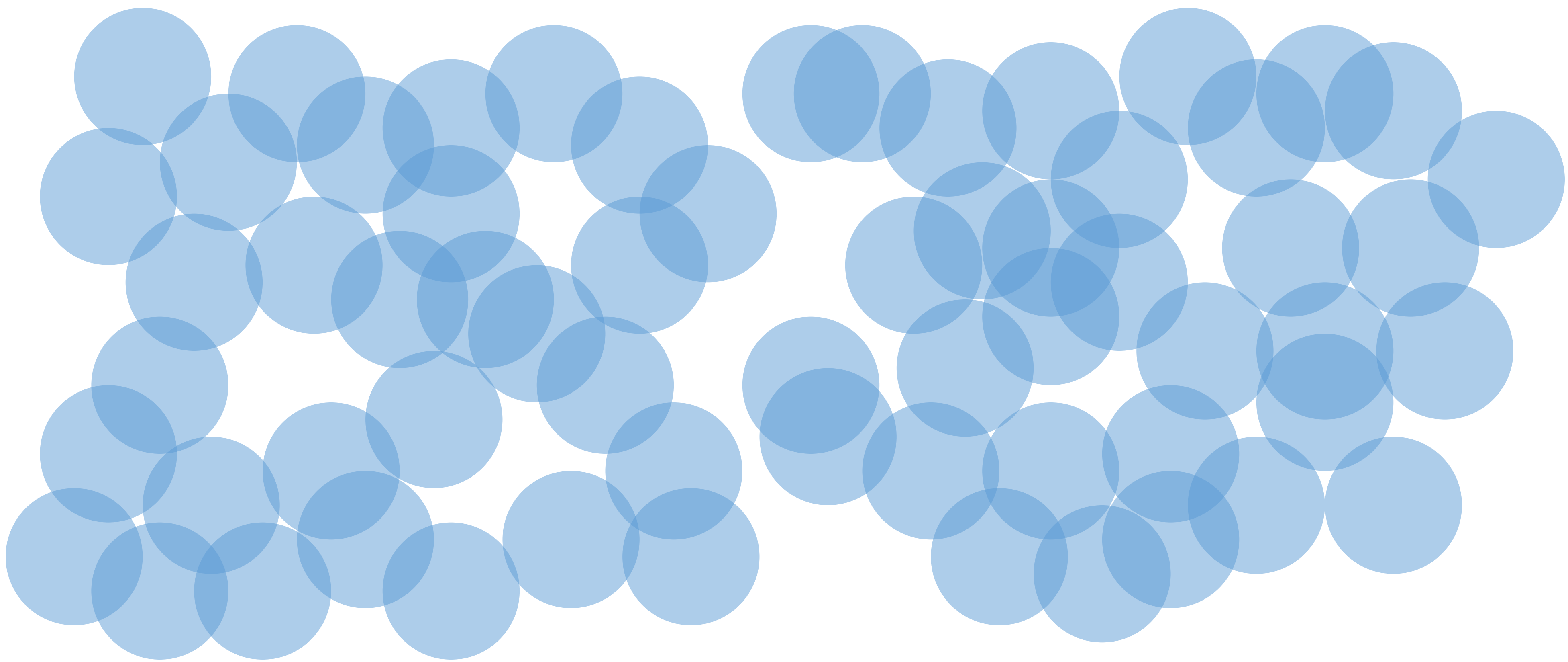

Divide & conquer using geometric separators

Algorithm [Alber, Fiala, '04].

1. $A =$ total area
2. if $A < \pi \cdot k$, return NO
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- overall complexity is $n^{\mathcal{O}(\sqrt{k})}$

Evaluation

Strengths

- simple
- parameterized
- faster than what we had in the classical setting: $\sum_{k=1}^{n} n^{\mathcal{O}(\sqrt{k})} = 2^{\widetilde{\mathcal{O}}(\sqrt{n})}$, compared to $2^{\widetilde{\mathcal{O}}(n^{2/3})}$
- optimal (under ETH)
- works also for disks and other shapes with bounded area

Weaknesses

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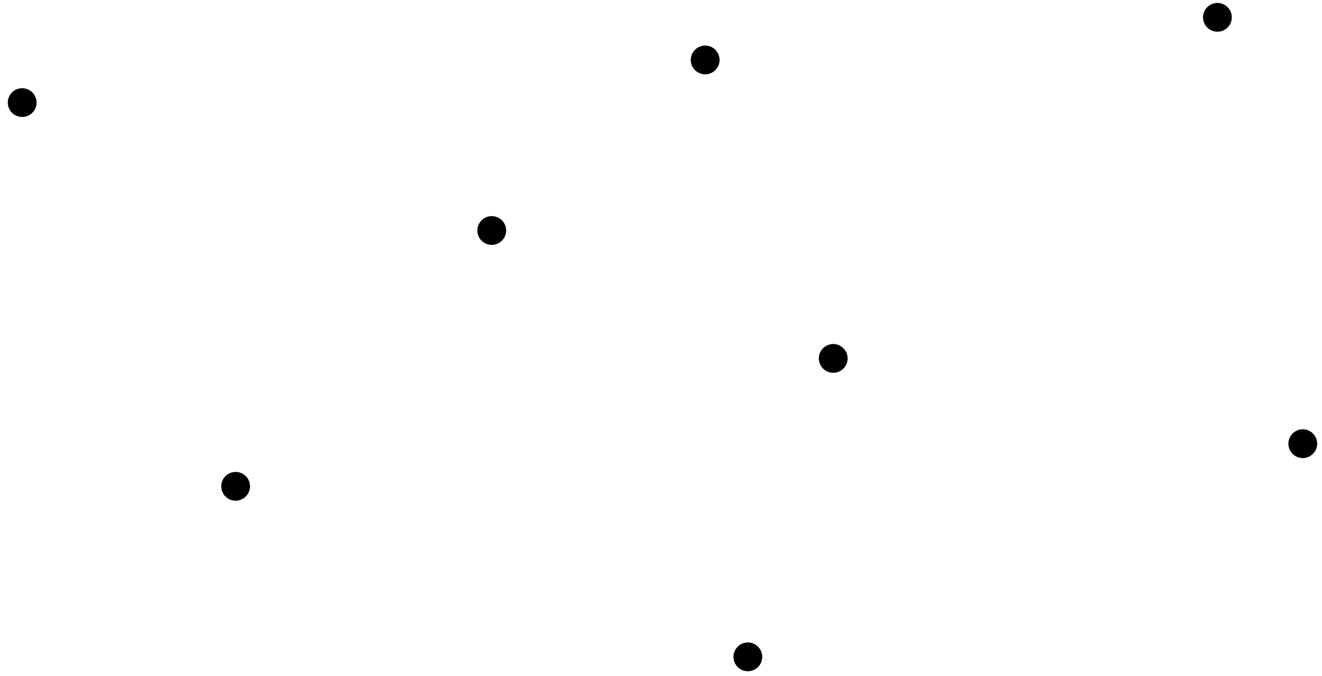
- in the remainder of this part we will learn how to address the first weakness, using a different approach

Voronoi-diagram approach

Voronoi diagrams

- we are given n points in the plane (objects)
- each point of the plane is assigned to the closest object



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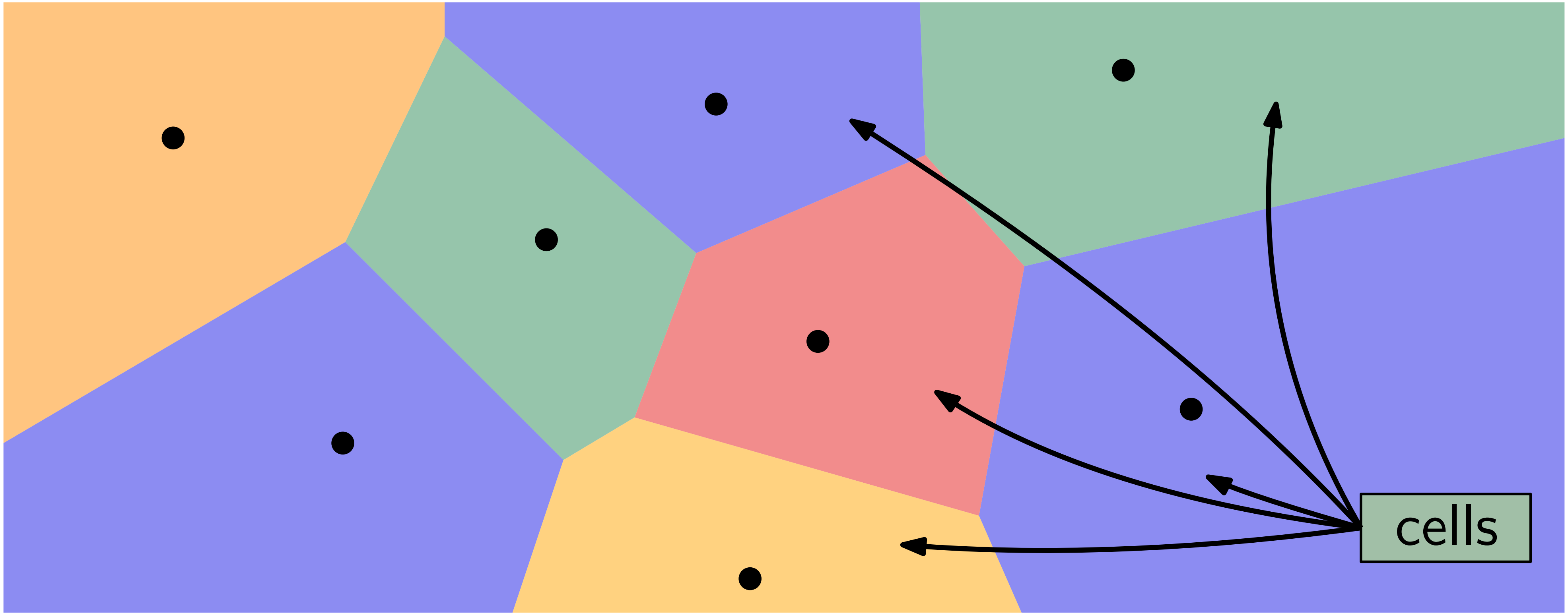

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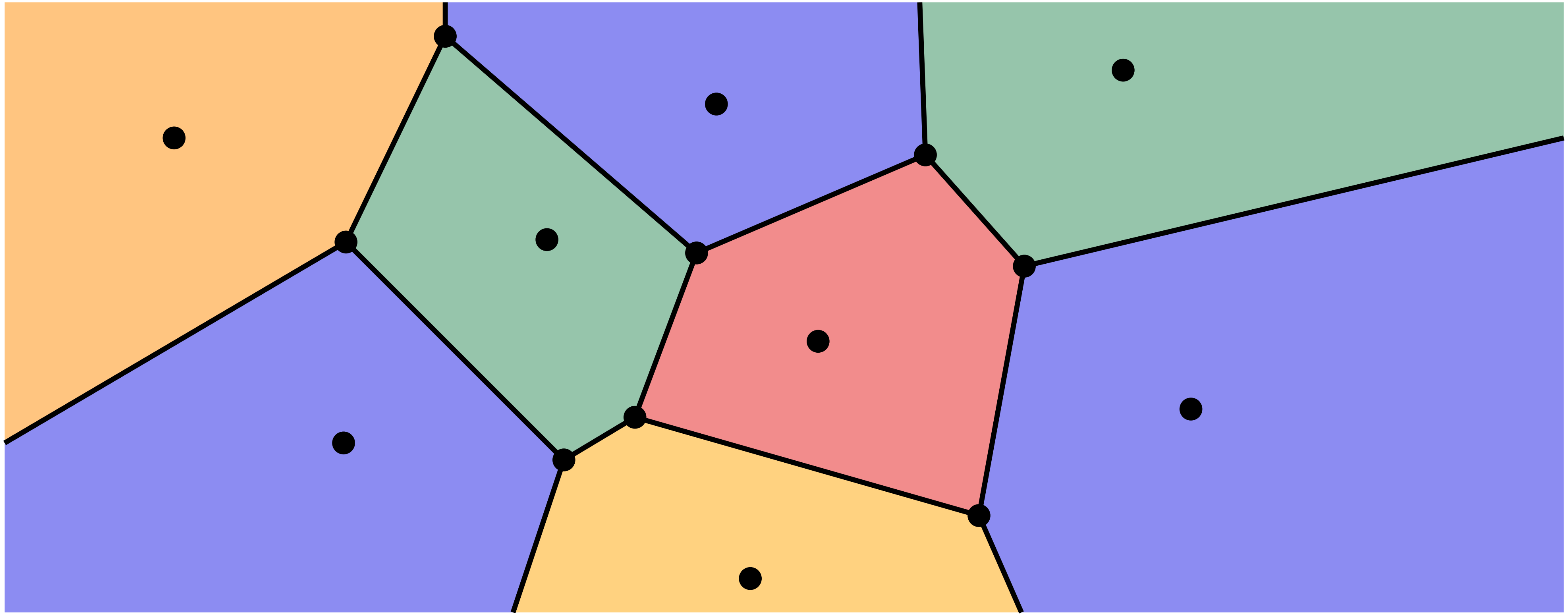

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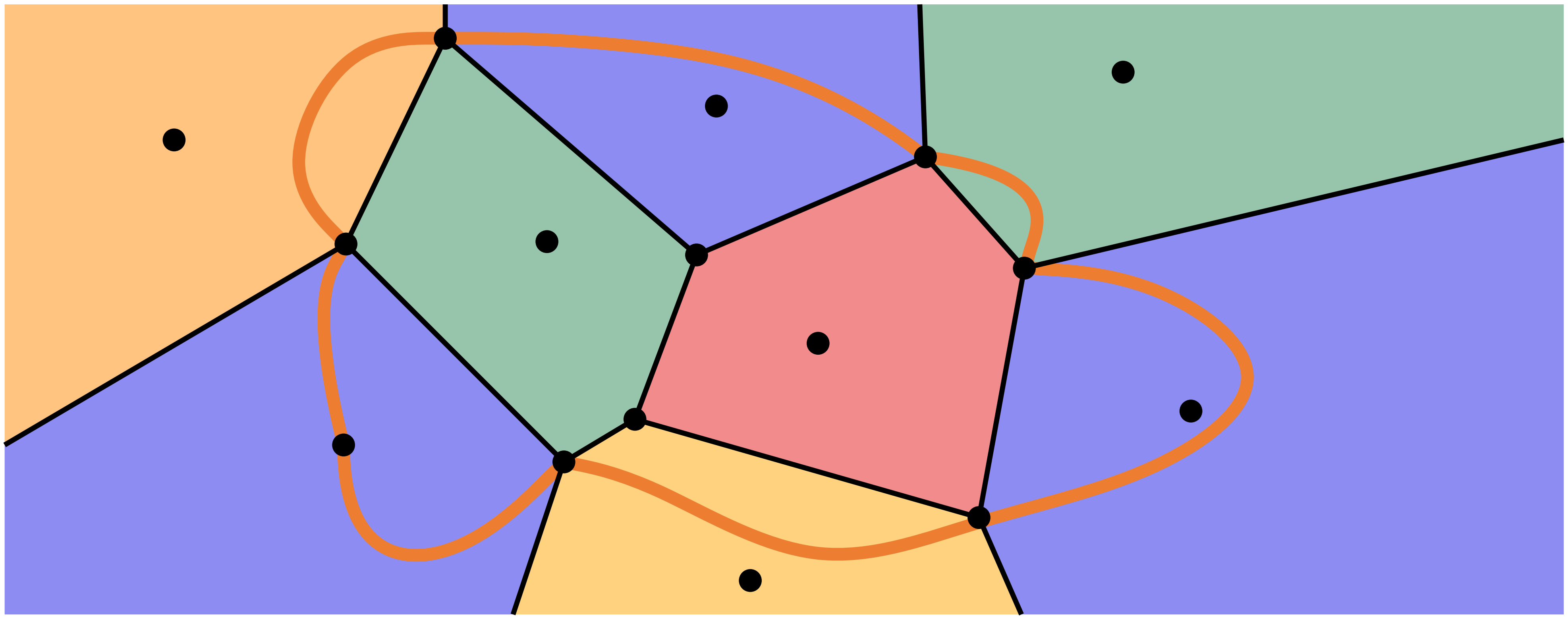

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Theorem [Marx, Pilipczuk '15]. Each graph like this has a balanced noose separator of size $\mathcal{O}(\sqrt{n})$.

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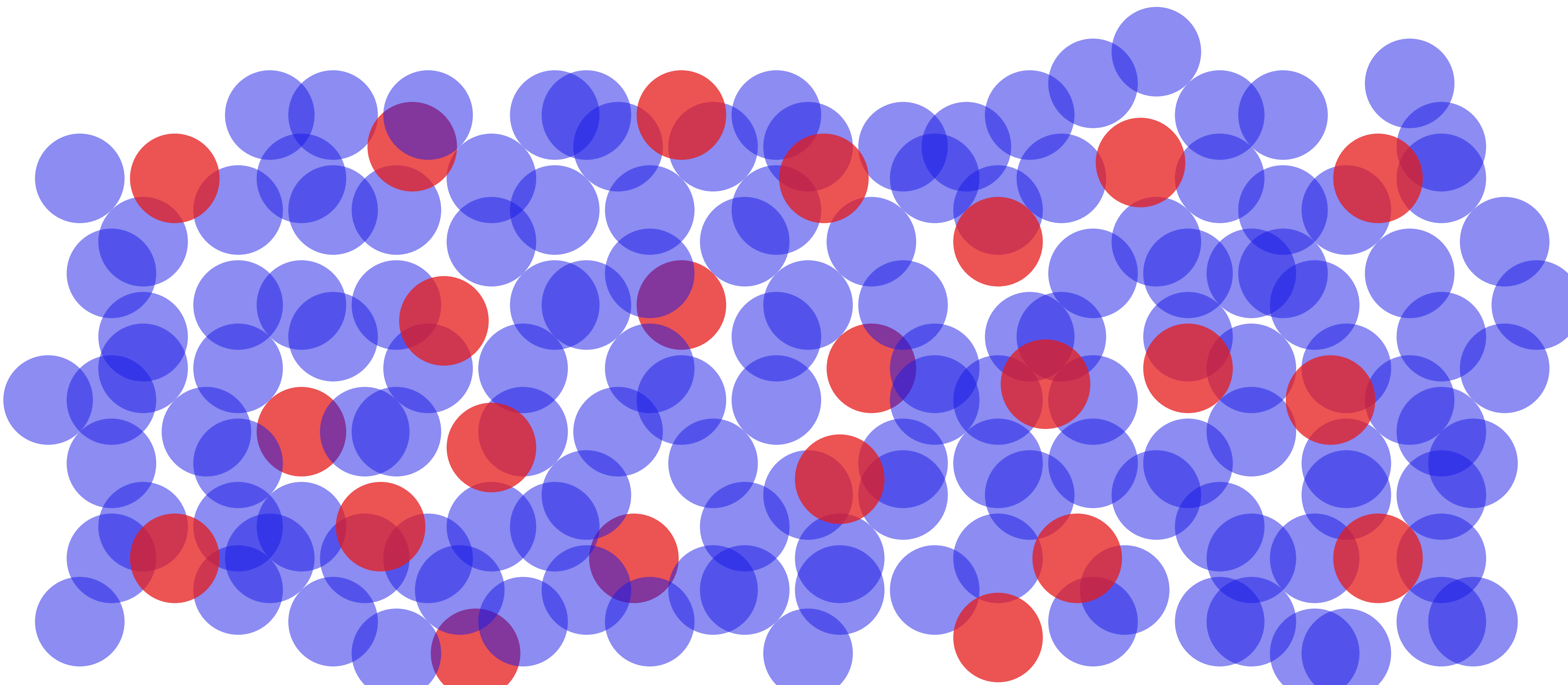
- consider a solution to the problem – k disjoint disks



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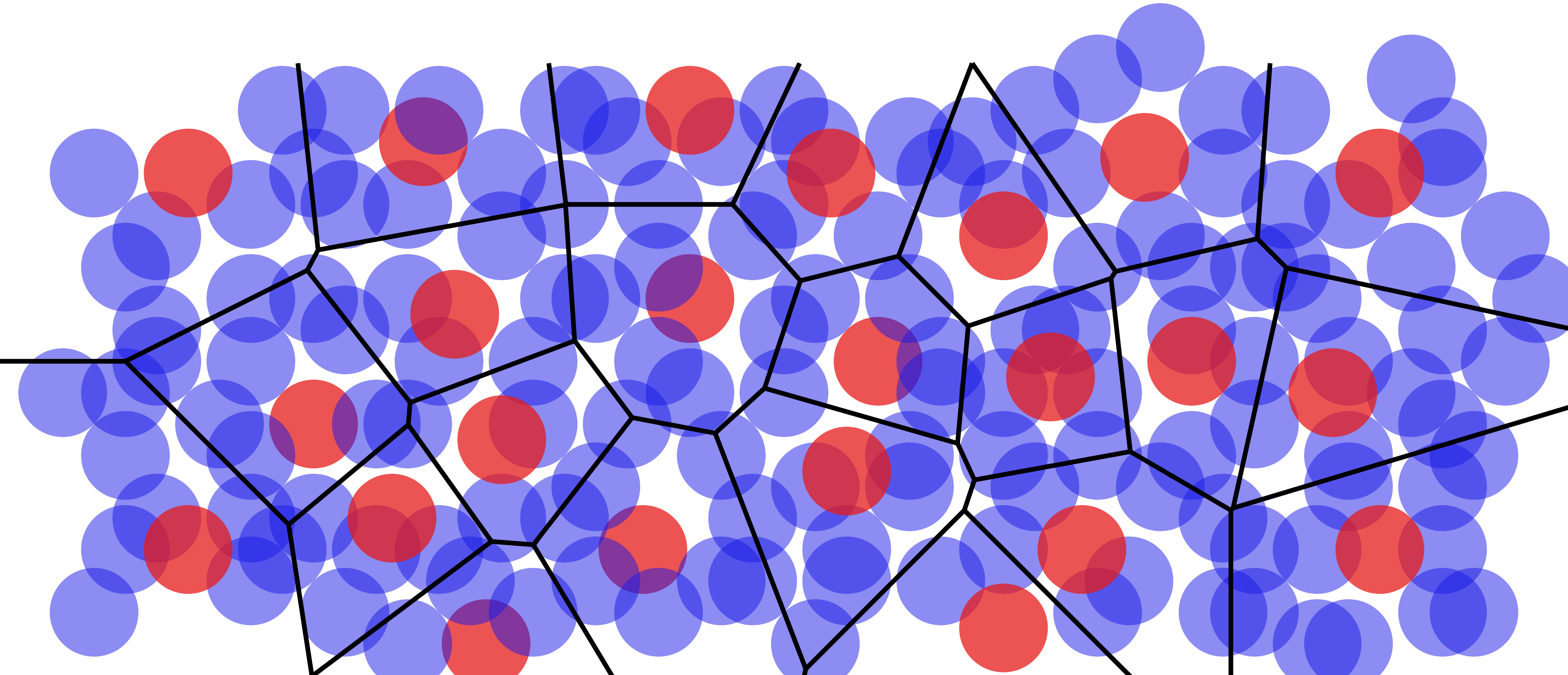

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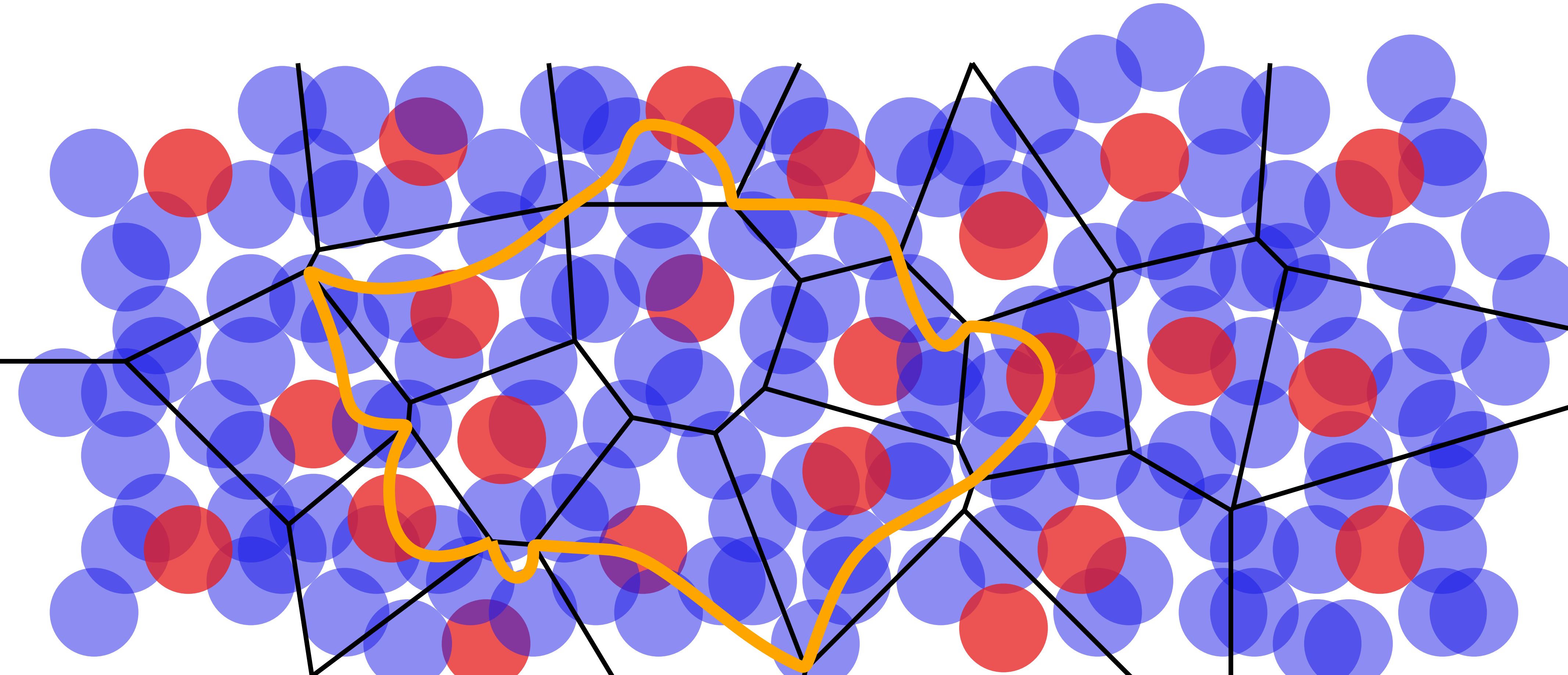

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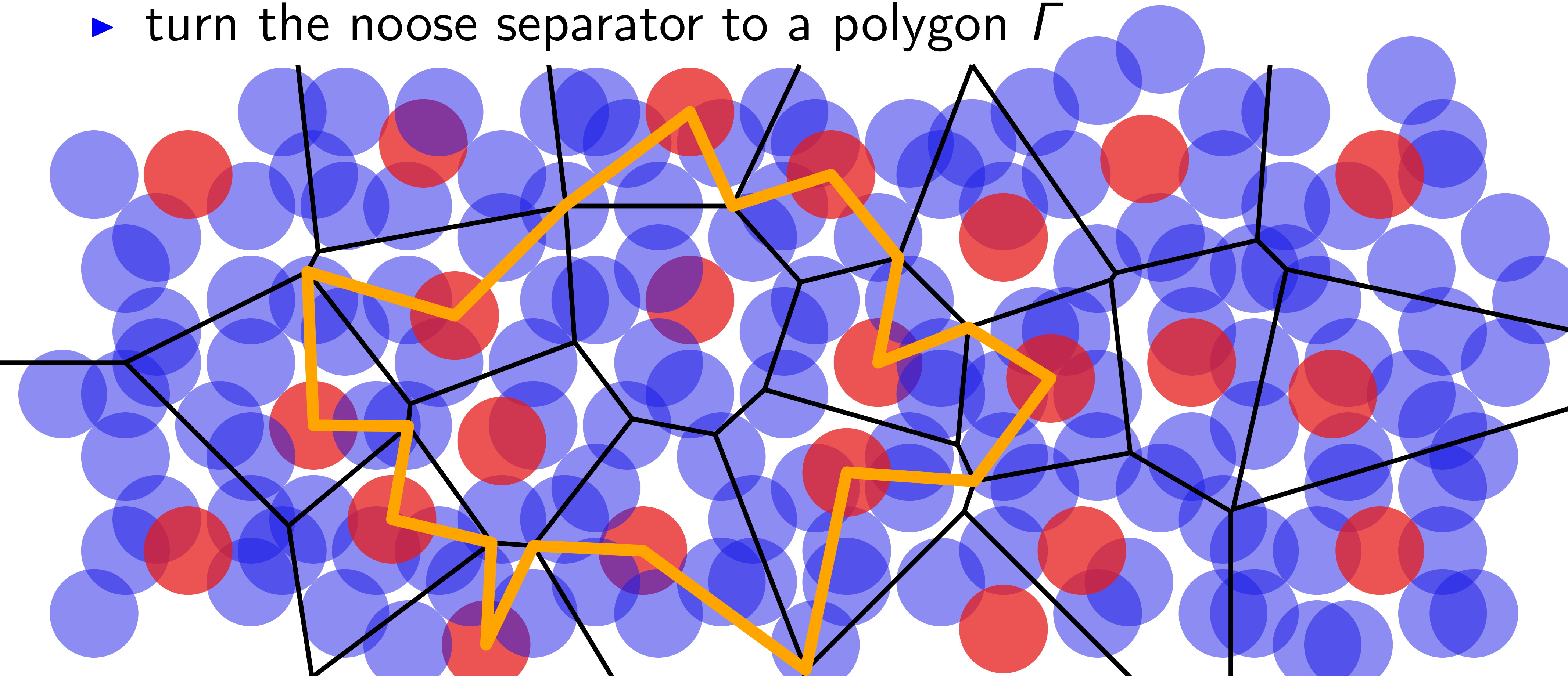

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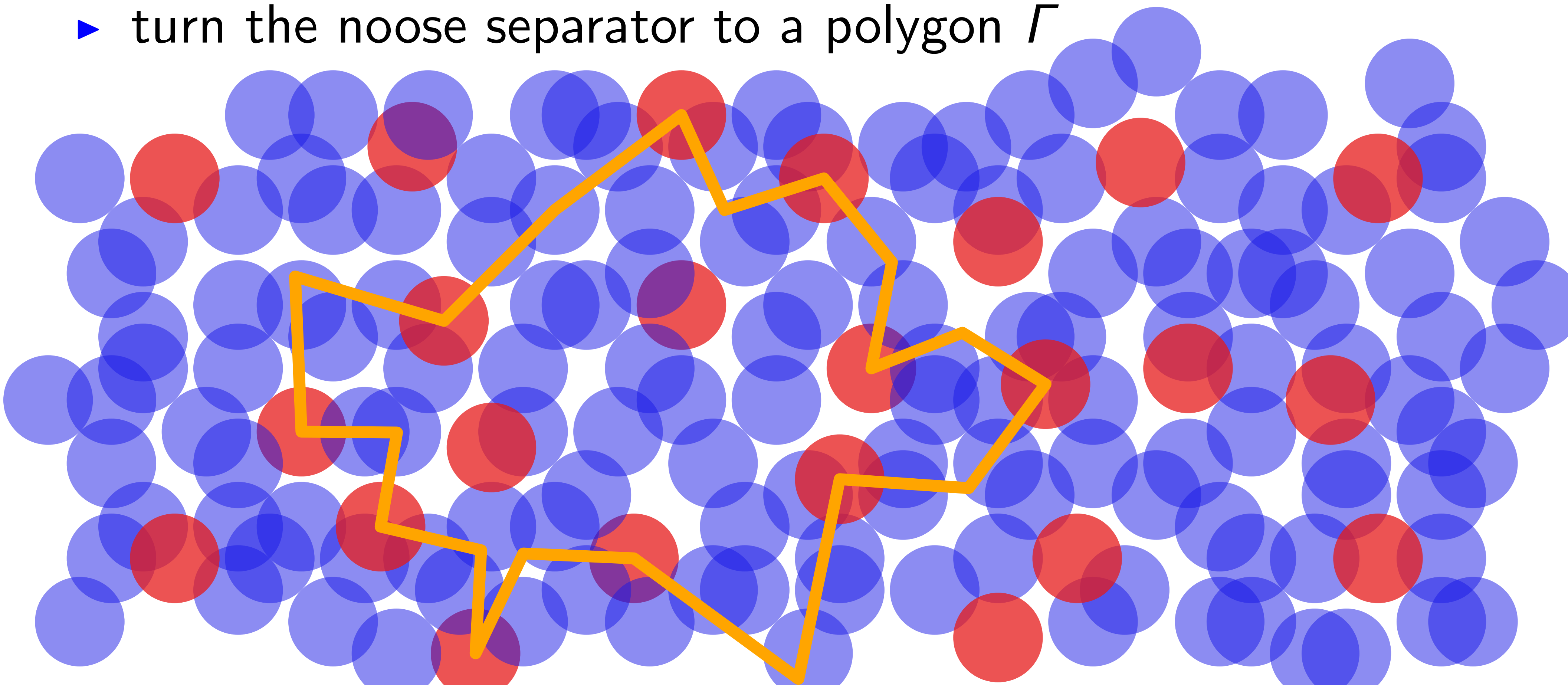
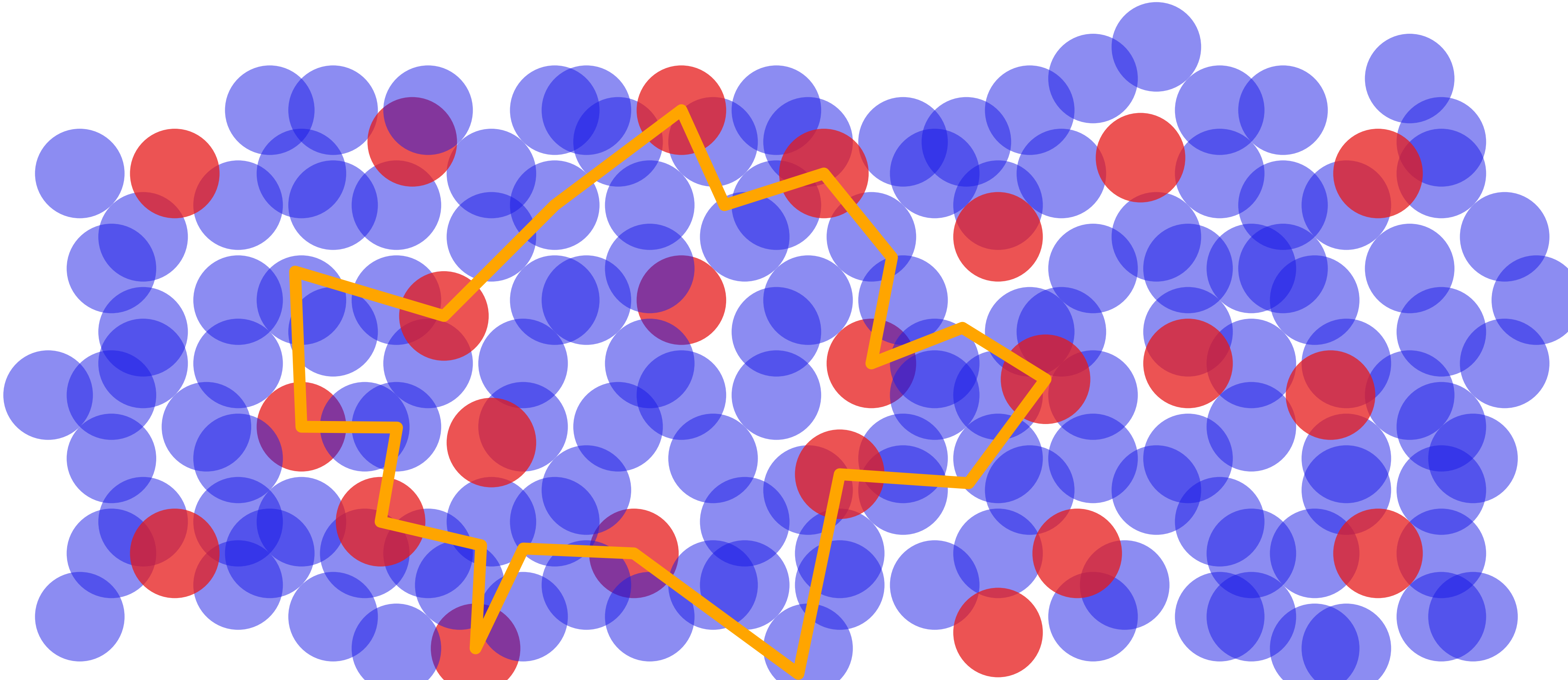


Separators in a solution Voronoi diagram

- every disk touching the outline of the polygon or any of the disks on its vertices can be discarded



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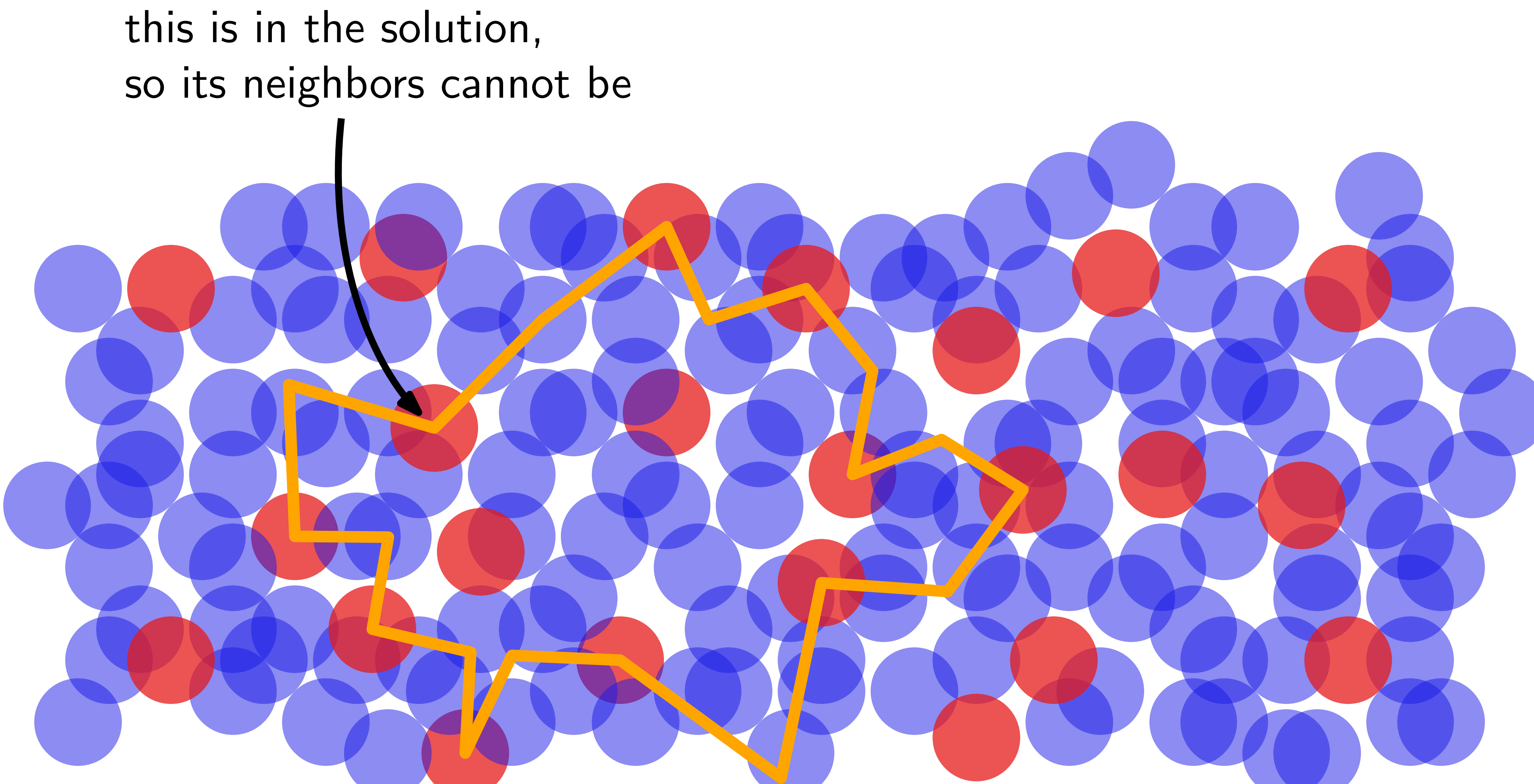

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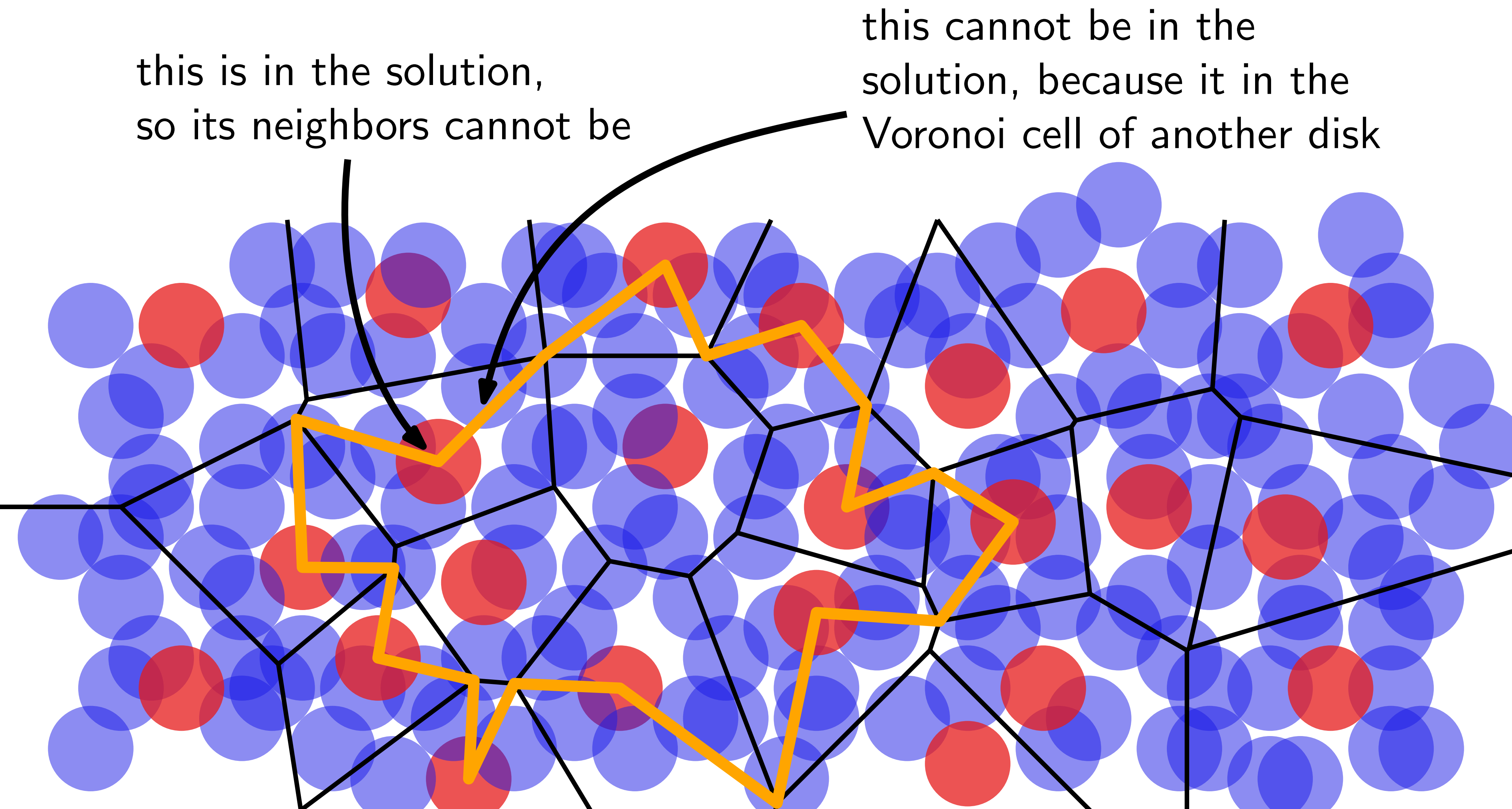

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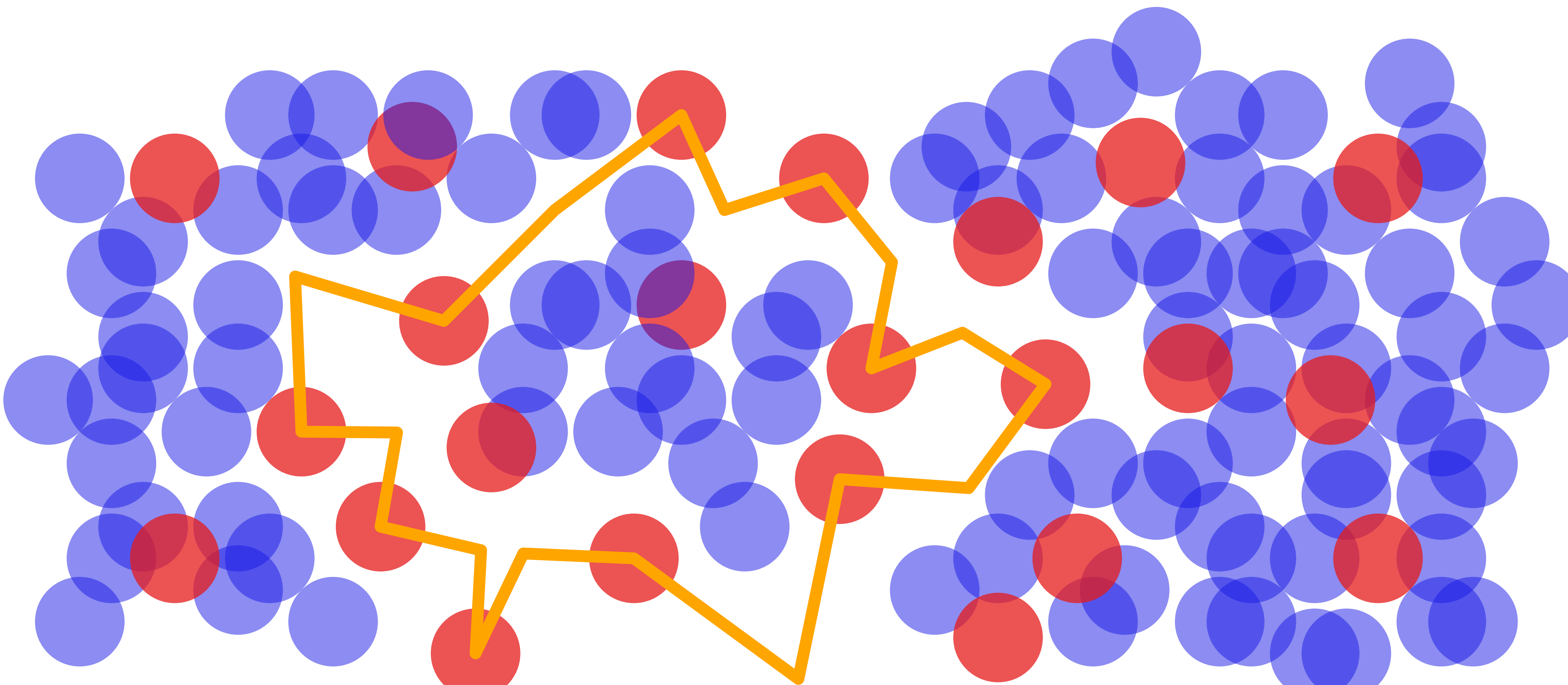

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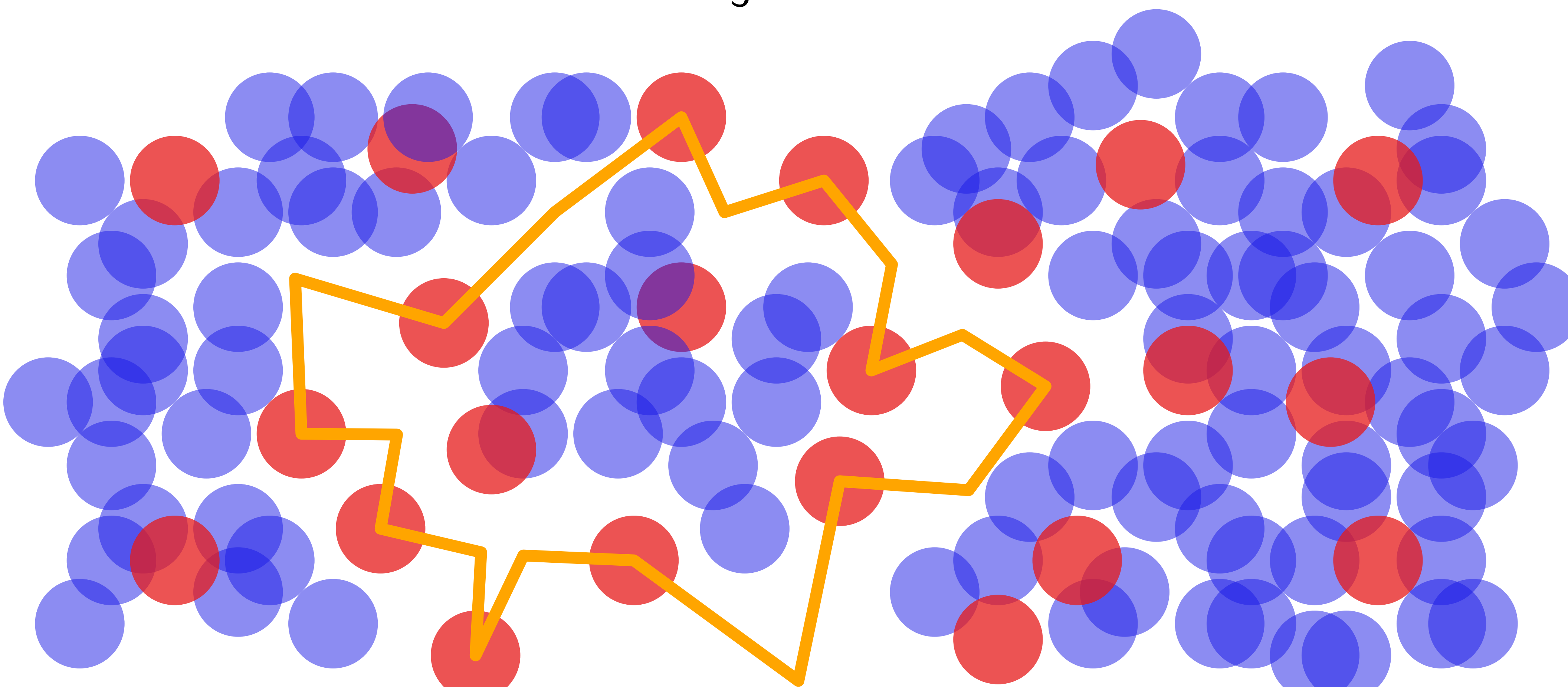
- apply recursion to disks inside and outside the polygon, we look for a solutions of size k_1, k_2, where $k_1 + k_2 = k$ and $k_1, k_2 \leq \frac{2}{3}k$



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- we can't, but we can still guess the polygon separator Γ

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  - $\mathcal{O}(\sqrt{k})$ centers of disks
  - $\mathcal{O}(\sqrt{k})$ vertices the Voronoi diagram
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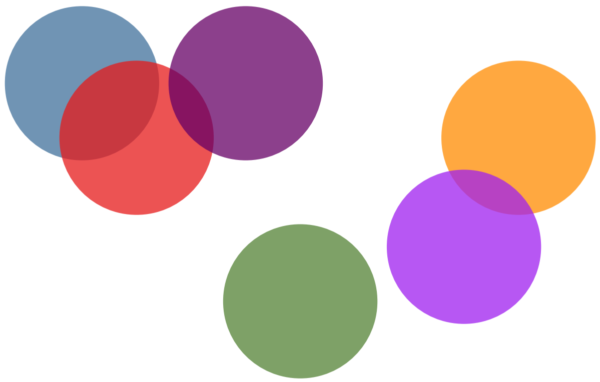
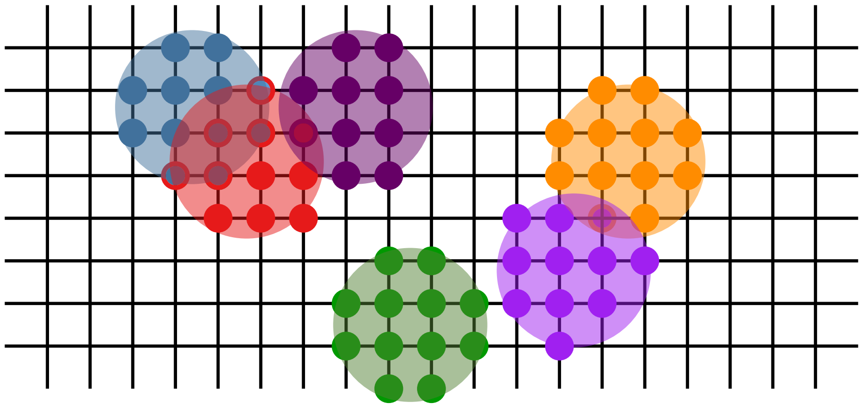
$$T(n,k) \leq n^{\mathcal{O}(\sqrt{k})} \cdot k^2 \cdot 2T(n, \tfrac{2}{3}k) = n^{\mathcal{O}(\sqrt{k})}$$

From disks to other geometric objects

- disks can be seen as connected subgraphs of a fine grid





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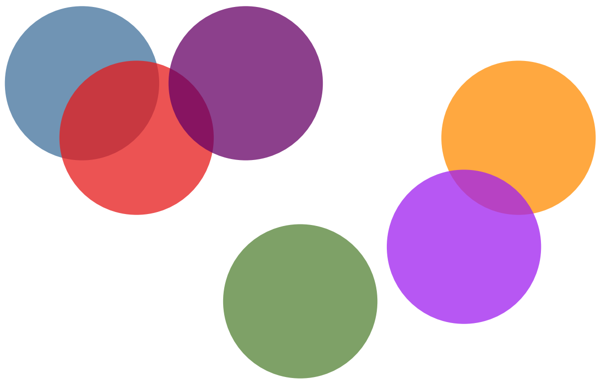
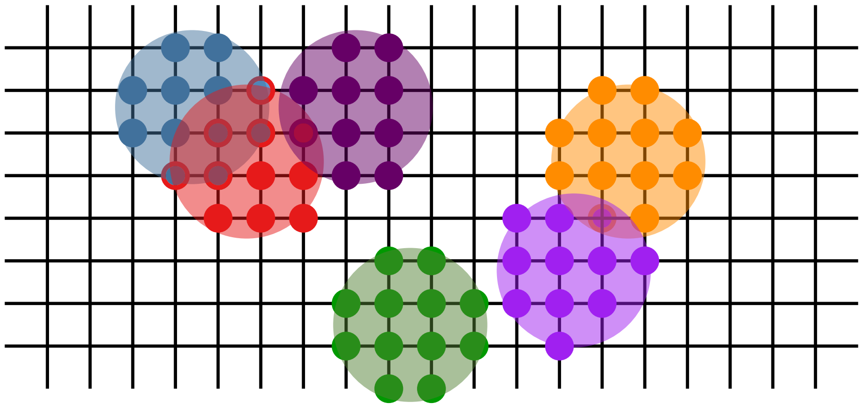
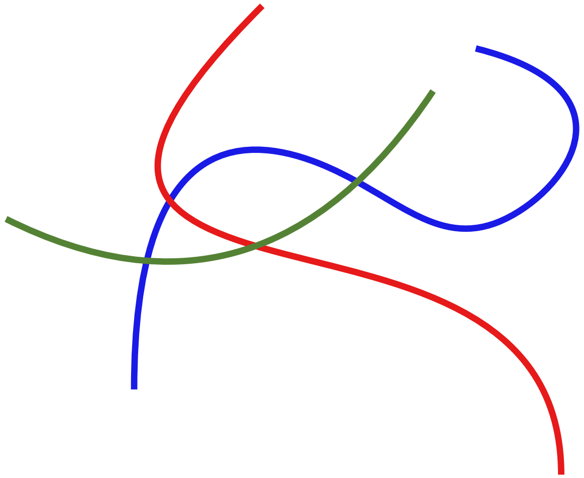
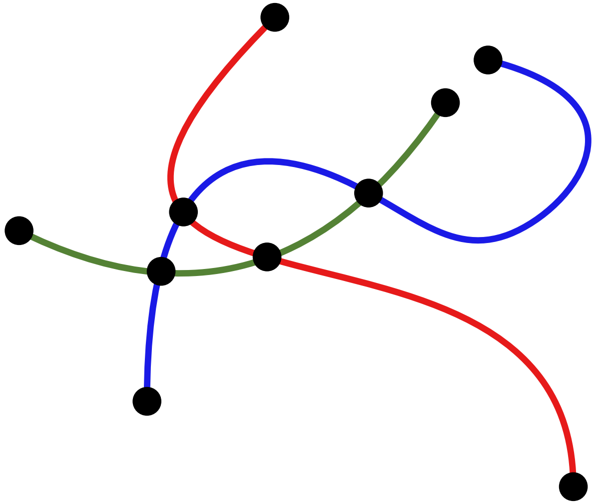




- string graphs = intersection graphs of connected subgraphs of planar graphs





General statement

- the whole approach can be re-interpreted in terms of packing disjoint subgraphs of planar graphs

Theorem [Marx, Pilipczuk '15].
Given a planar graph G with r vertices and n connected subgraphs of G, in time $n^{\mathcal{O}(\sqrt{k})} \cdot \mathrm{poly}(r)$ we can decide if there is a collection of k disjoint subgraphs.

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- works for weighted variants
- to some extent works also for covering variant (domination)

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- for disks and segments $r = \text{poly}(n)$
- Open question: For disk graphs, is there a robust algorithm for Independent Set with complexity $2^{o(k)}$ or $2^{\widetilde{\mathcal{O}}(\sqrt{n})}$?

Lower bounds
for parameterized algorithms

Parameterized lower bounds

- we know that k-Independent Set can be solved in time $n^{\mathcal{O}(\sqrt{k})}$ in disk graphs
- we aim to show that this is asymptotically optimal

Parameterized lower bounds

- we know that k-Independent Set can be solved in time $n^{\mathcal{O}(\sqrt{k})}$ in disk graphs
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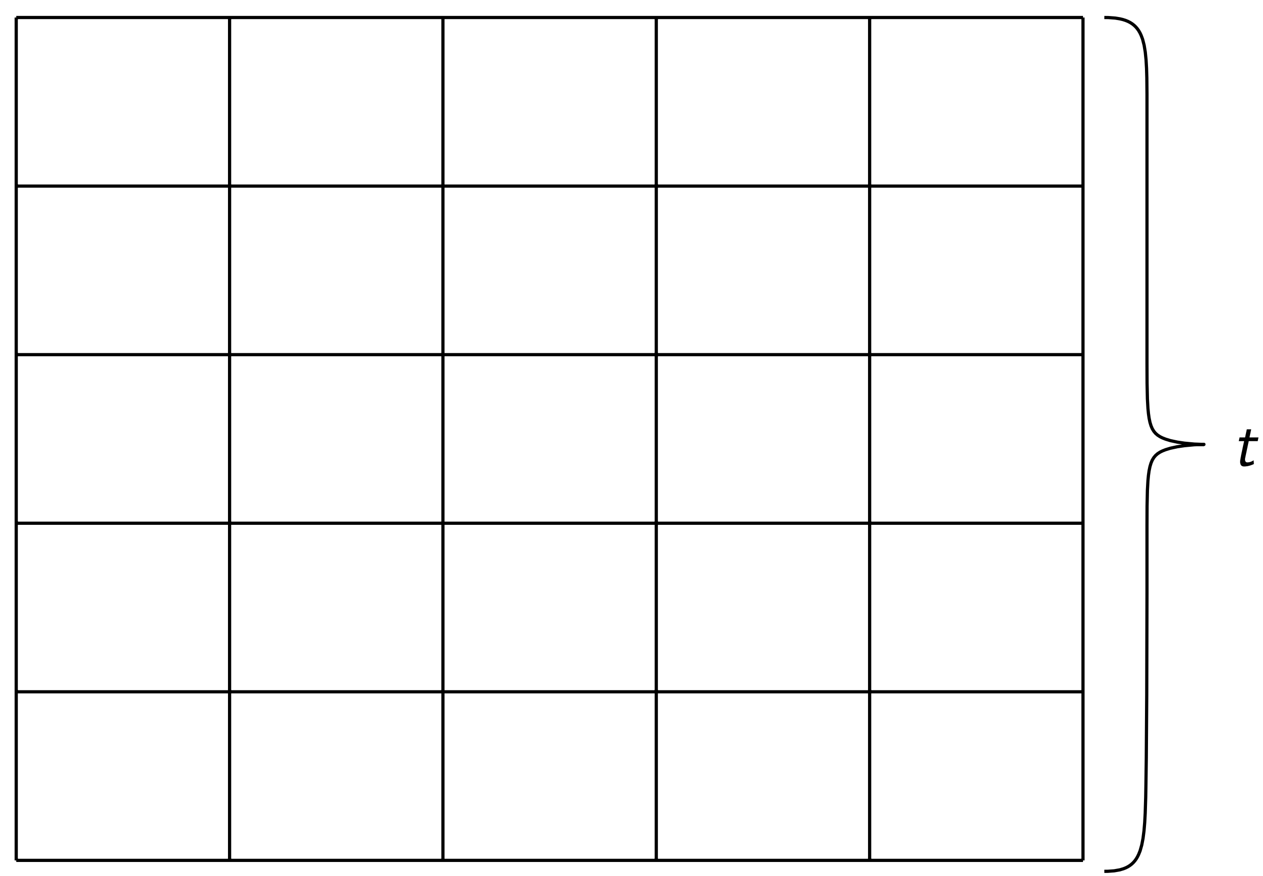
- we will need the following

Theorem.
Assuming the ETH, k-Clique cannot be solved in time $n^{o(k)}$.

- proof by a textbook reduction from 3-Sat

Grid Tiling

- we are given a square $t \times t$ grid



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(1,1)(1,2) (2,2)(2,3)	(1,1)(1,3) (1,4)(2,4) (3,1)	(1,4)(2,3) (2,4)(4,1)	(1,1)(1,4) (2,2)(2,3)	(1,1)(1,2) (2,2)(2,3)
(1,2)(1,3) (3,2)(4,1)	(2,1)(2,2) (3,3)(3,5)	(2,1)(2,3) (3,4)(3,5)	(2,5)(3,4) (4,1)(4,2)	(1,1)(1,2) (3,2)
(1,1)(1,2) (1,3)(1,4)	(1,1)(1,3) (2,4)(3,4)	(1,4)(2,1) (2,2)(2,3)	(1,2)(1,4) (3,1)(3,3)	(1,1)(1,2) (1,3)(2,2)
(1,2)(1,3) (2,2)(2,3)	(1,3)(2,1) (2,3)(2,4)	(2,1)(2,4) (3,1)(3,2)	(1,3)(2,3) (2,4)(4,1)	(1,4)(2,1) (2,2)(3,1)
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- how fast can we solve it?

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(1,1)**(1,2)** (1,3)(1,4)	(1,1)**(1,3)** (2,4)(3,4)	**(1,4)**(2,1) (2,2)(2,3)	(1,2)**(1,4)** (3,1)(3,3)	(1,1)**(1,2)** (1,3)(2,2)
(1,2)(1,3) **(2,2)**(2,3)	(1,3)(2,1) **(2,3)**(2,4)	(2,1)**(2,4)** (3,1)(3,2)	(1,3)(2,3) **(2,4)**(4,1)	(1,4)(2,1) **(2,2)**(3,1)
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- we will show that this is optimal

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Hardness of GRID TILING

- $t \times t$ grid, each cell with some pairs from $[n] \times [n]$

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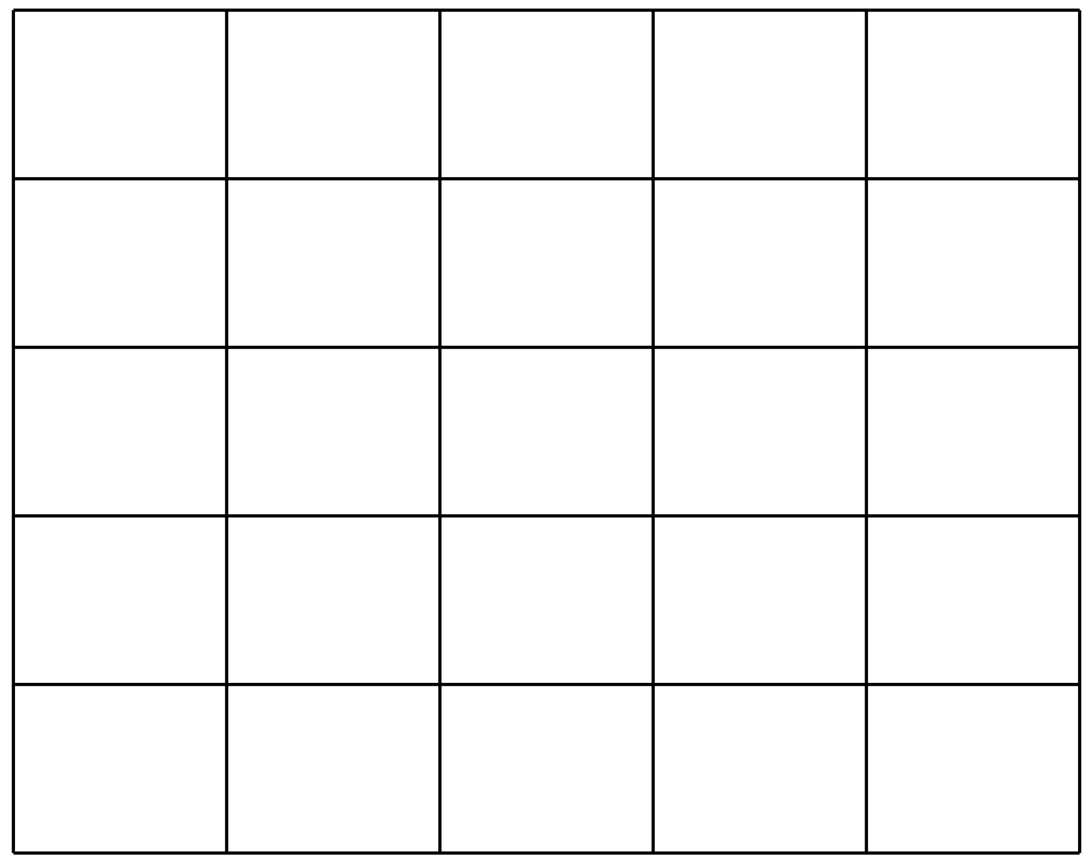

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(i,i) $i \in [n]$	(i,j) $ij \in [n]$	(i,j) $ij \in [n]$	(i,j) $ij \in [n]$	(i,j) $ij \in [n]$
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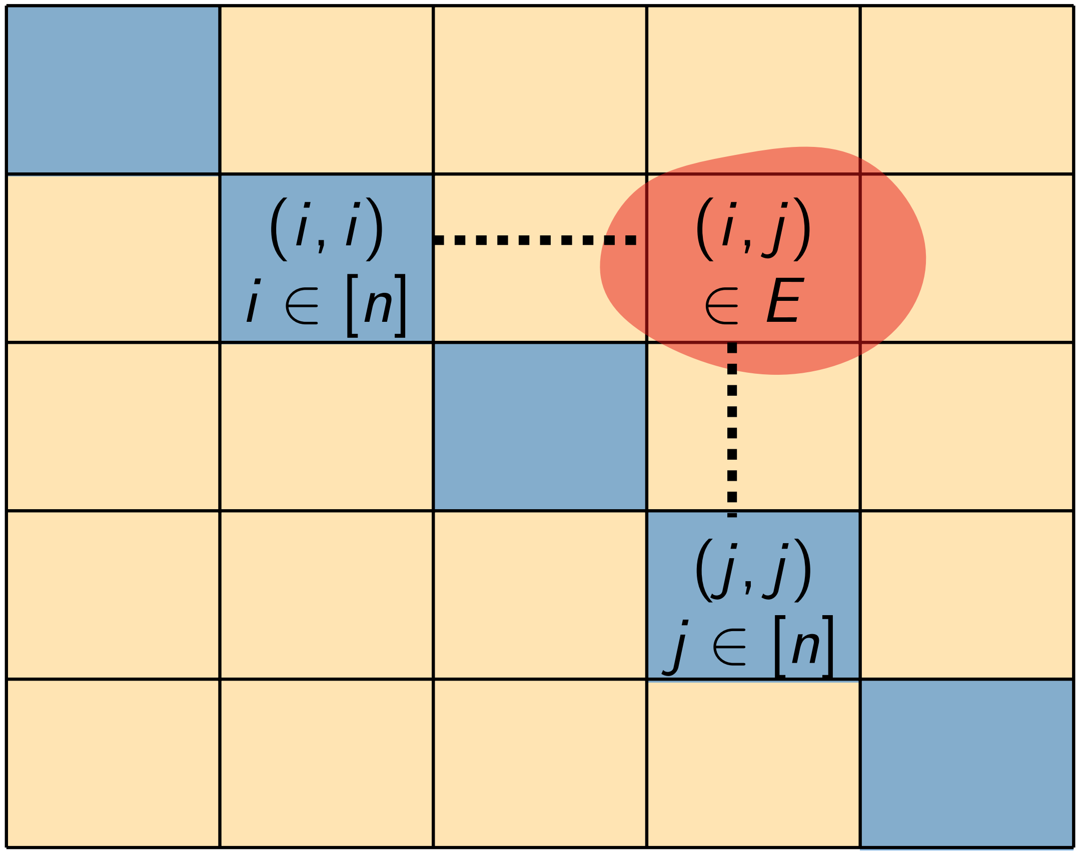

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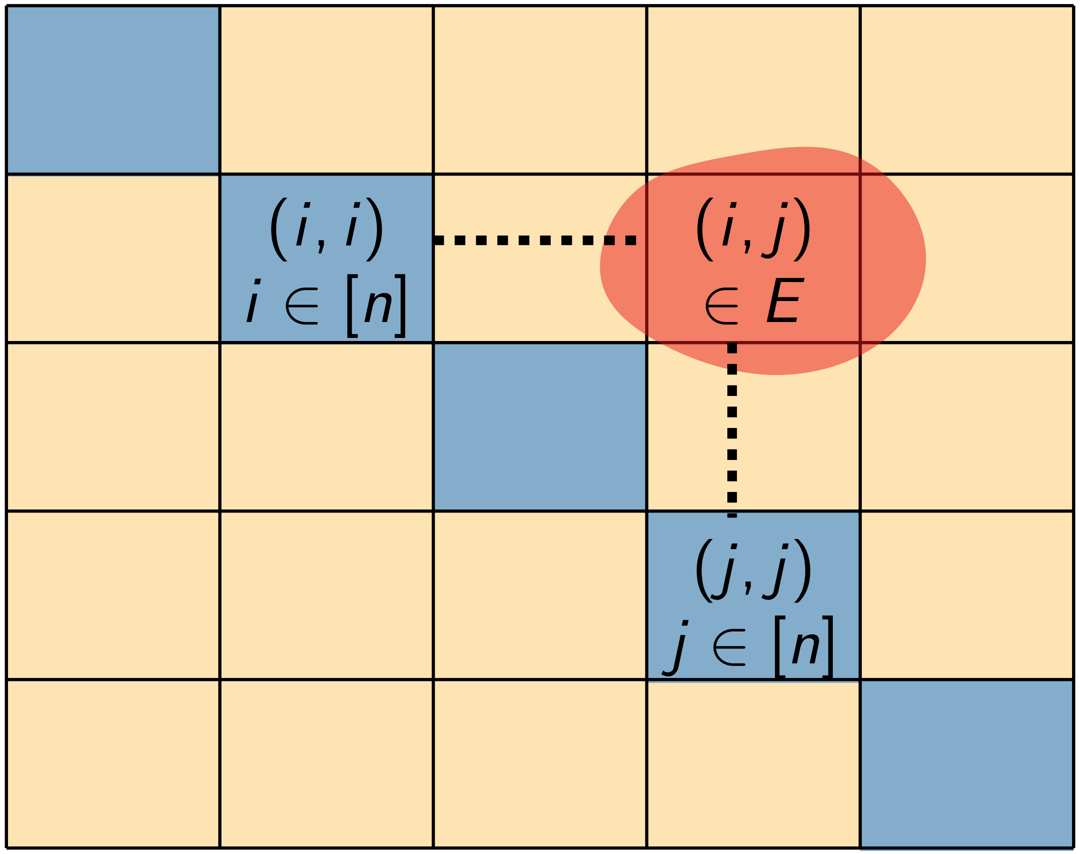
- Selected pairs on the diagonal correspond to a clique
- solving GRID TILING in time $n^{o(t)} \rightarrow$ solving k-CLIQUE in time $n^{o(k)}$



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Theorem. Assuming the ETH, there is no algorithm solving Grid Tiling in time $n^{o(t)}$.

Grid Tiling with $\leq$

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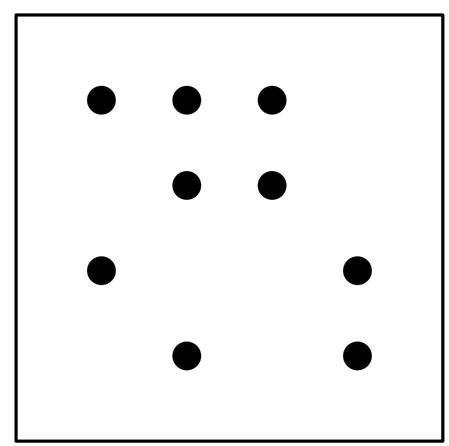
Theorem. Assuming the ETH, there is no algorithm solving Grid Tiling with $\leq$ in time $n^{o(t)}$.

- each set $S_{i,j}$ can be seen as points of $n \times n$ grid

(1,1)(1,2)(1,3)
(2,2)(2,3)
(3,1)(3,4)
(4,2)(4,4)

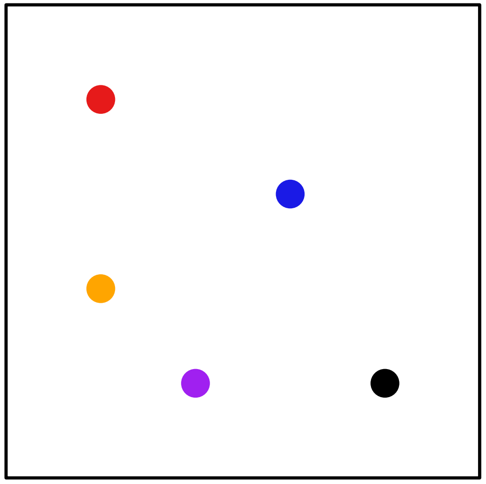


Hardness of Independent Set in UDGs

Theorem. Grid Tiling with $\leq$ cannot be solved in time $n^{o(t)}$, unless the ETH fails.

- $t \times t$ outer grid, $n \times n$ inner grids

a single cell:



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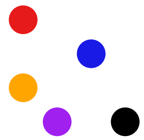

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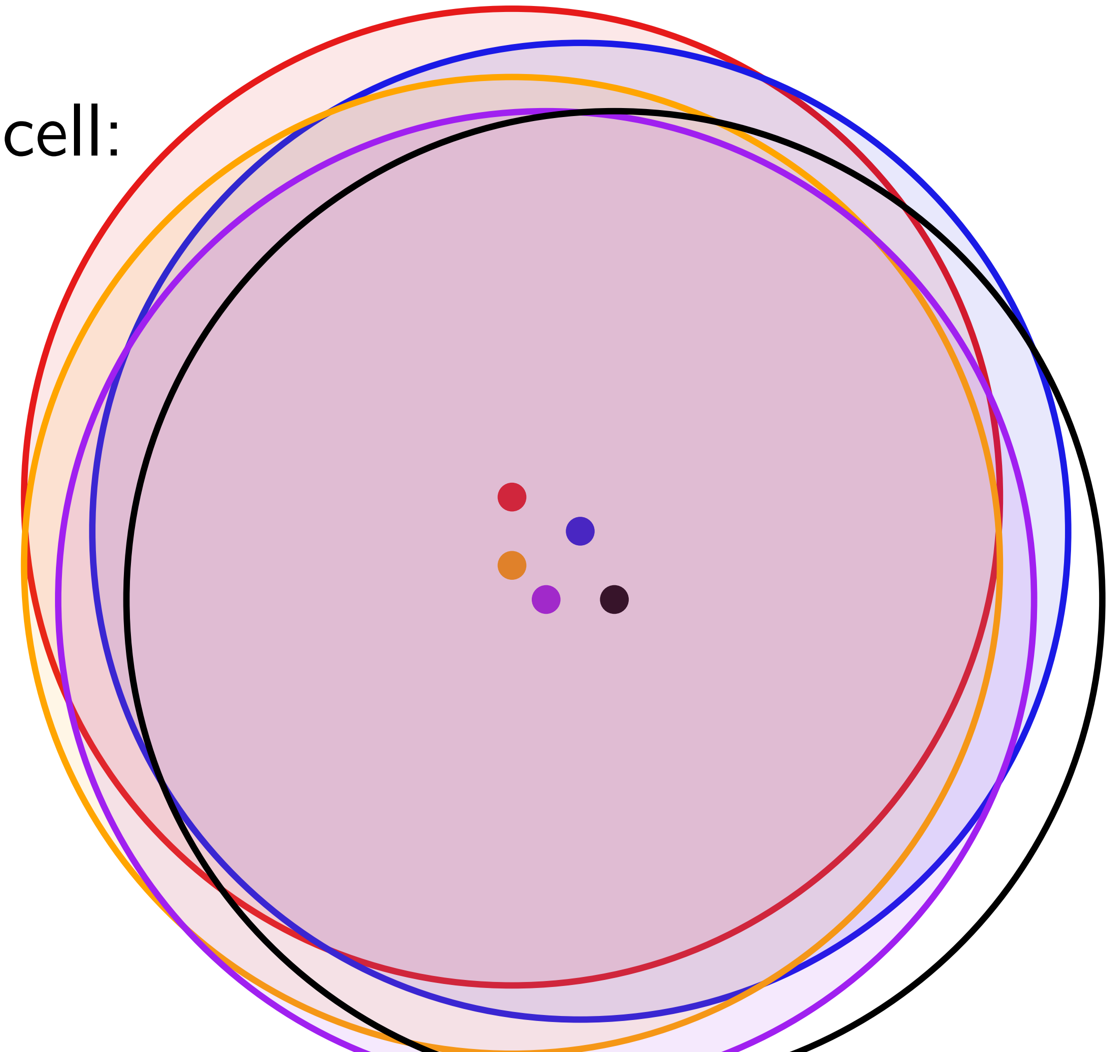
introduce unit disks centered at these points



Hardness of INDEPENDENT SET in UDGs

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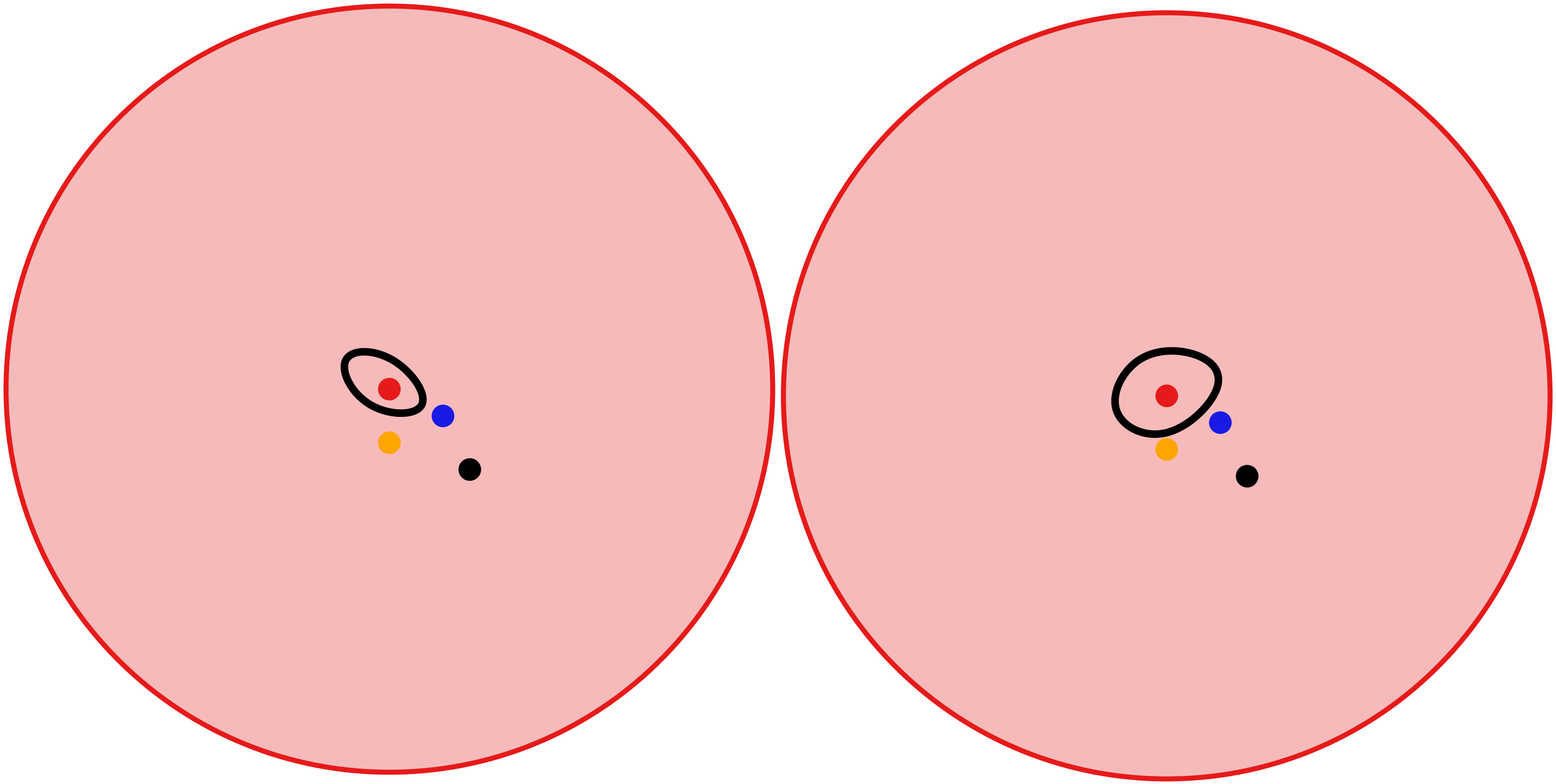

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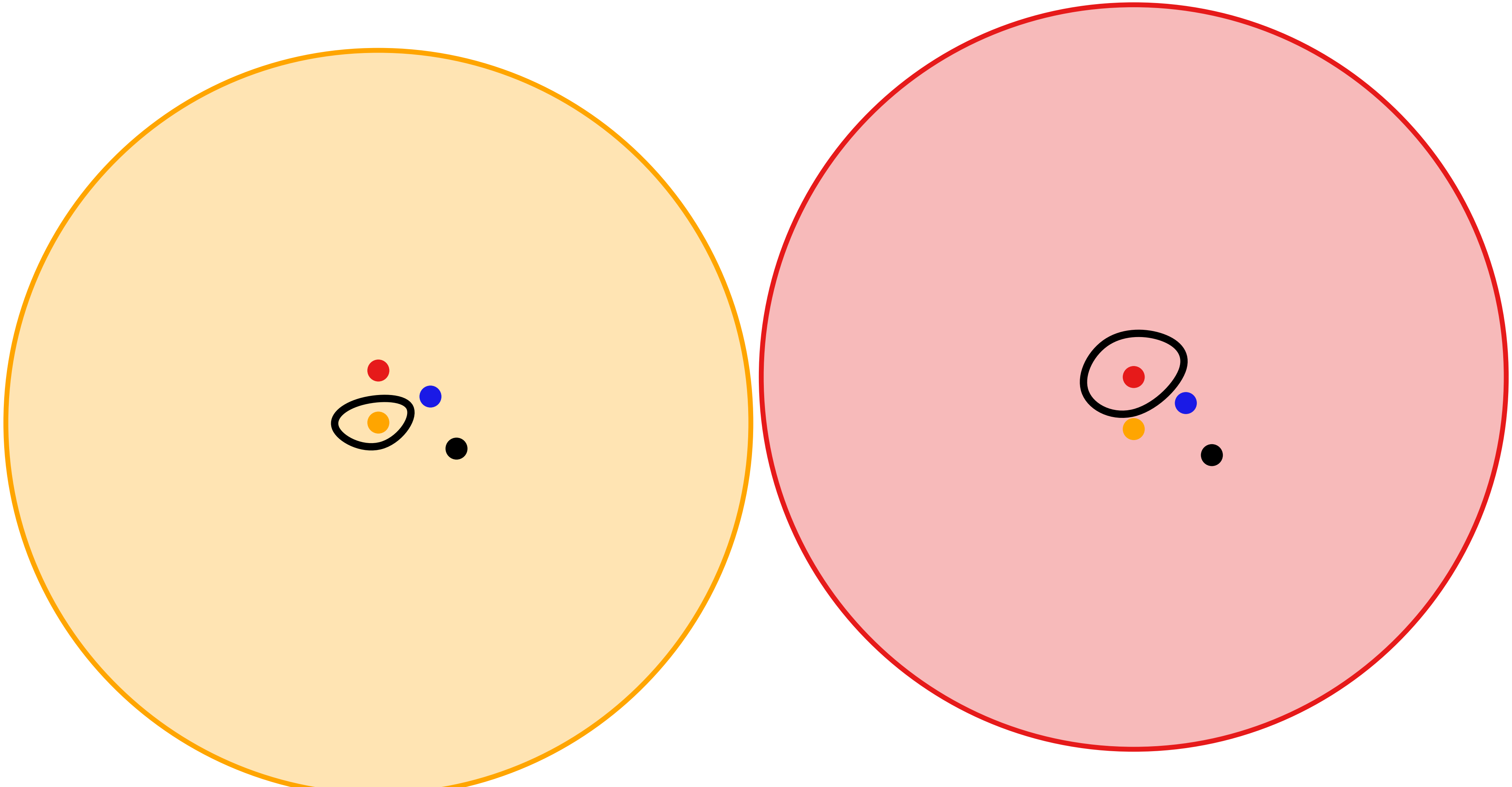

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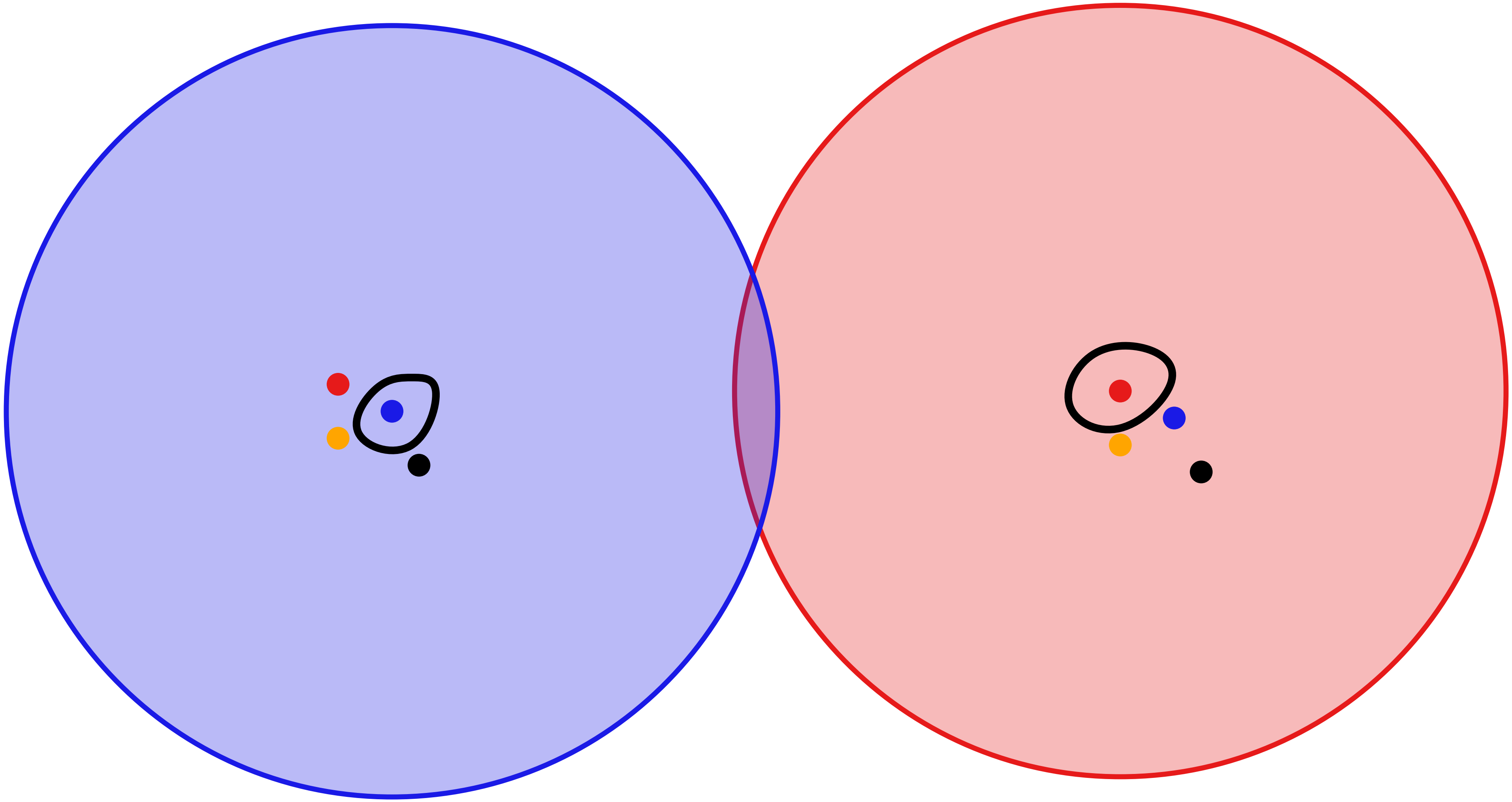

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- number of disks $N \leq t^2 \cdot n^2$
- solving Independent Set in time $N^{o(\sqrt{k})}$
  $\to$ solving Grid Tiling in time $n^{o(t)}$ $\to$ the ETH fails $\square$

Other faces of GRID TILING

- similar approach can be used to show lower bounds for (CONNECTED) DOMINATING SET [Marx + Kisfaludi-Bak]
- reductions are not specific to disks: in general they can be adjusted for any convex fat shapes

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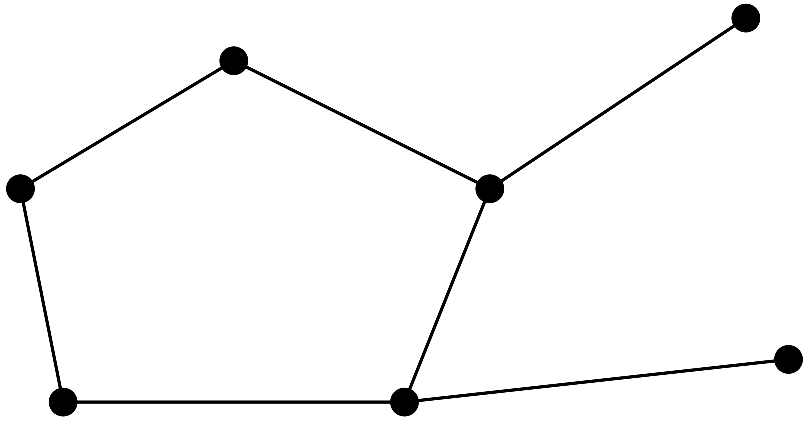
... but it's a different story

Bidimensionality in geometric graphs

Minors

- minor = a graph obtained by deleting vertices/edges and contracting edges



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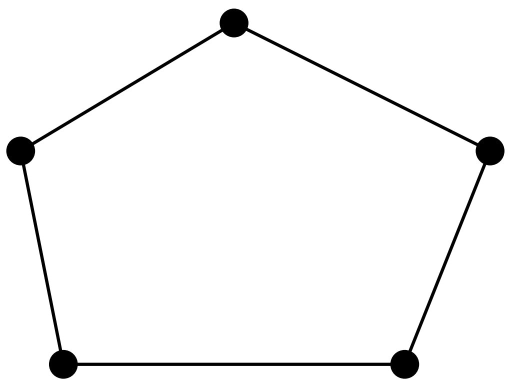

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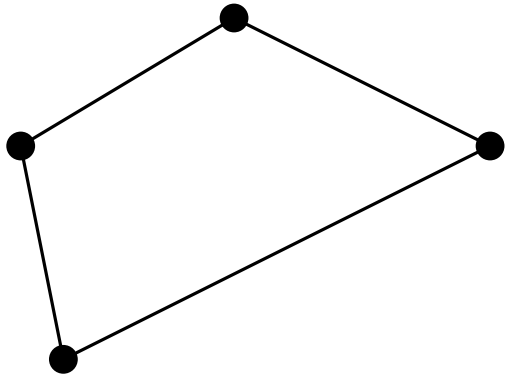

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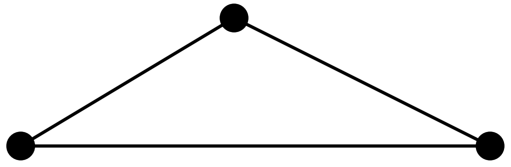

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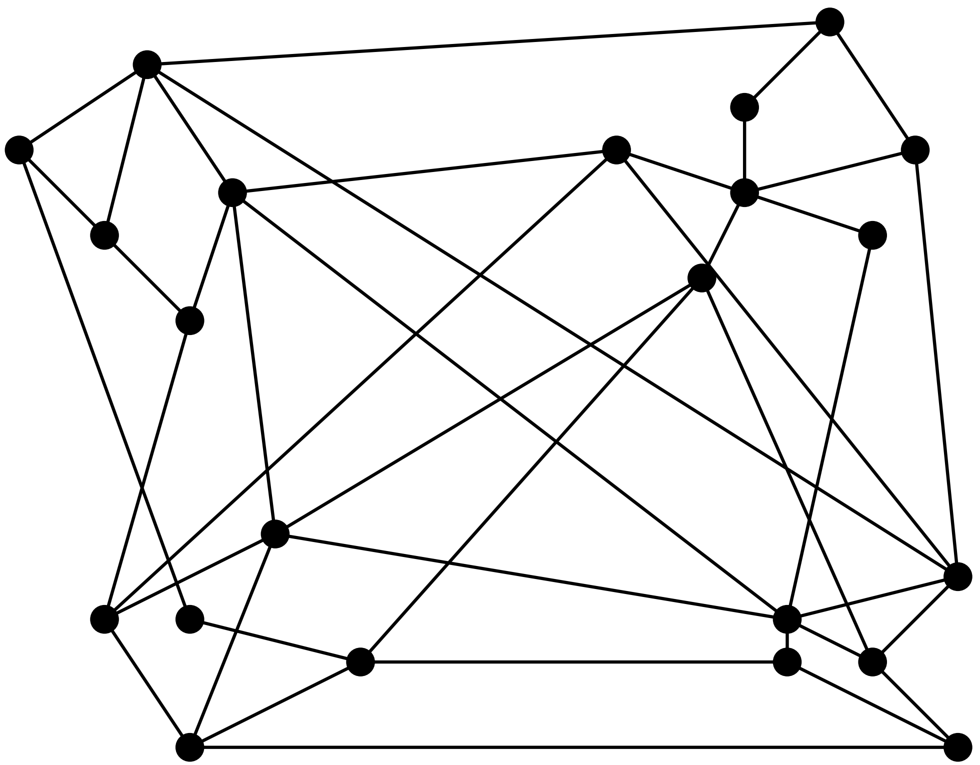

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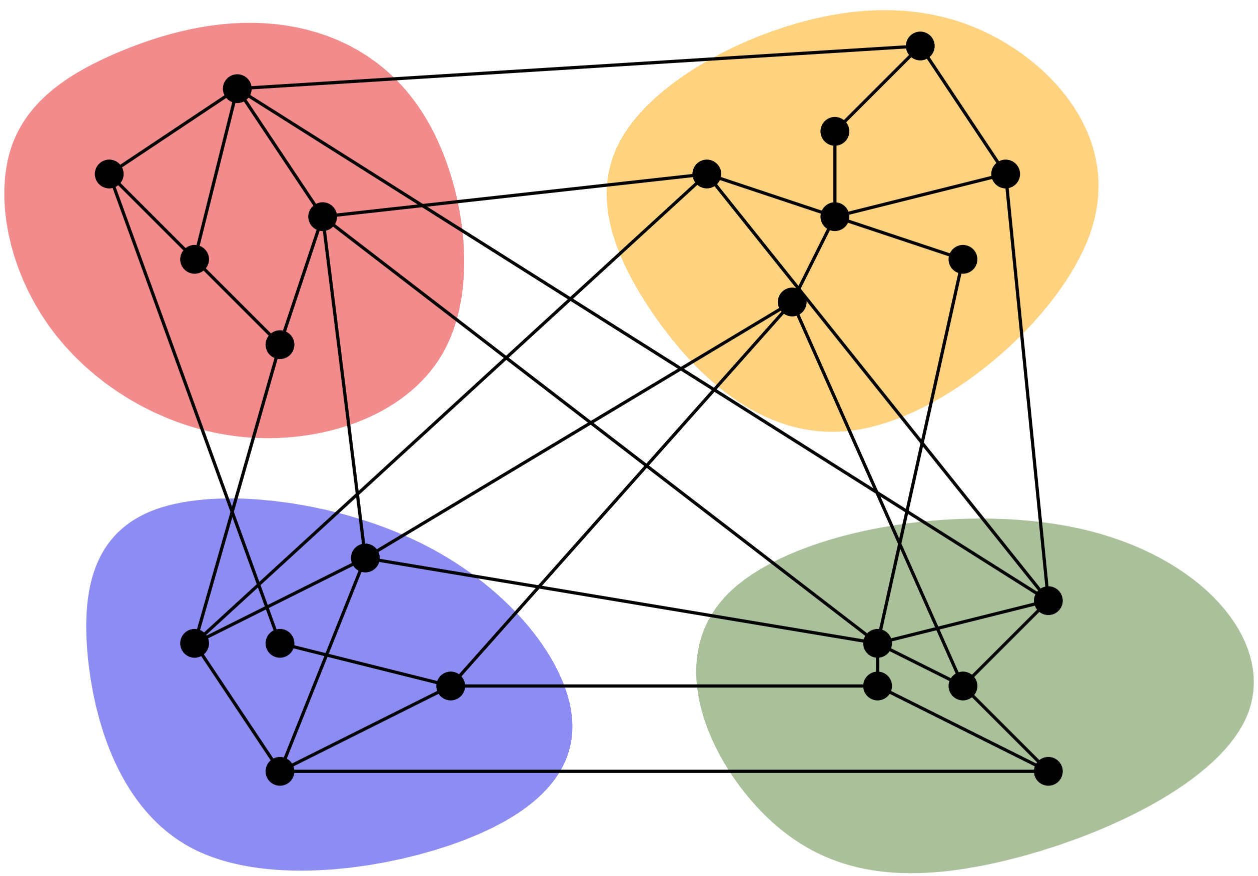

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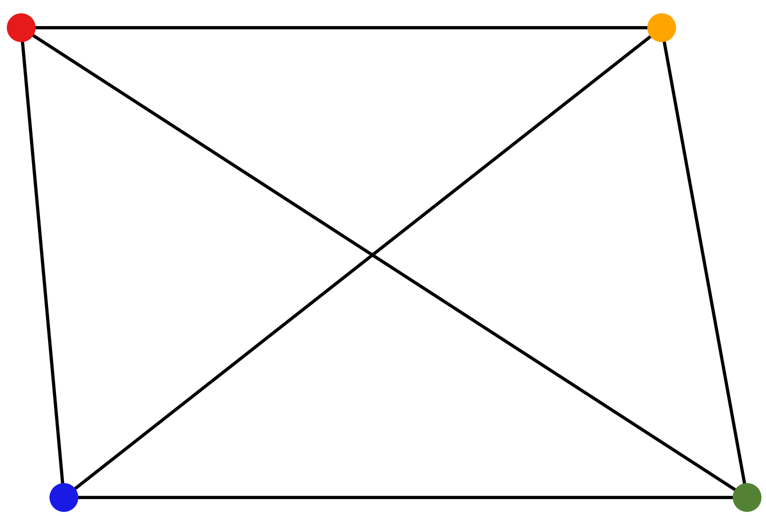
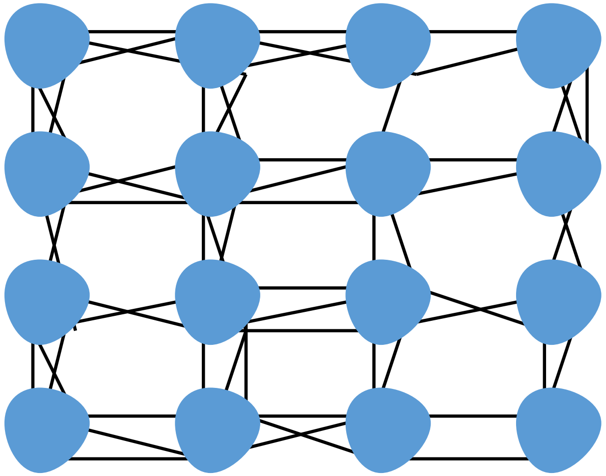


Grid minor theorem

- the presence of $t \times t$ grid minor forces treewidth $\geq t$



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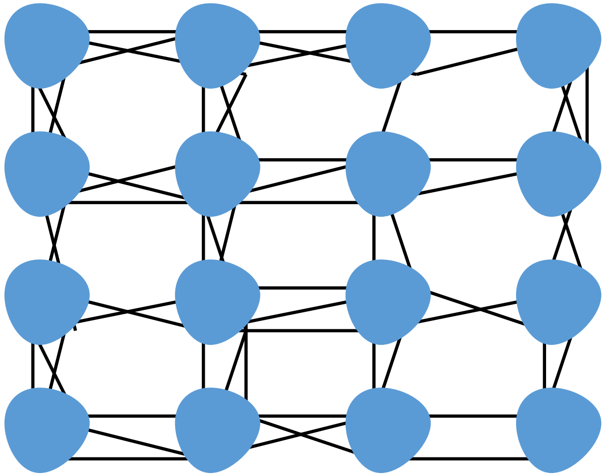

Grid minor theorem [Robertson, Seymour '86].
Every graph with treewidth $\geq f(t)$ contains a $t \times t$ grid minor.

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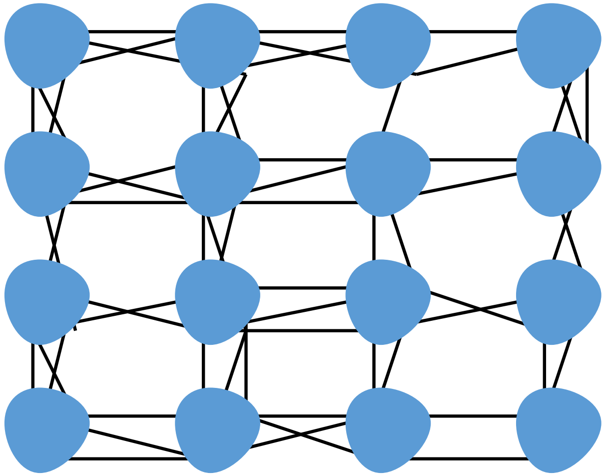

Grid minor theorem [Chuzhoy, Tan '19].
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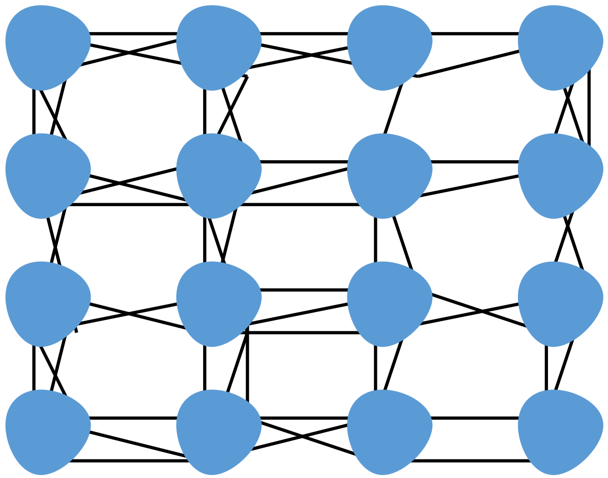

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Planar grid minor theorem [Robertson, Seymour, Thomas '94, Gu, Tamaki '12].
Every planar graph with treewidth $\geq 9/2 \cdot t$ contains a $t \times t$ grid minor. There is a poly-time algorithm for finding a grid or a tree decomposition.

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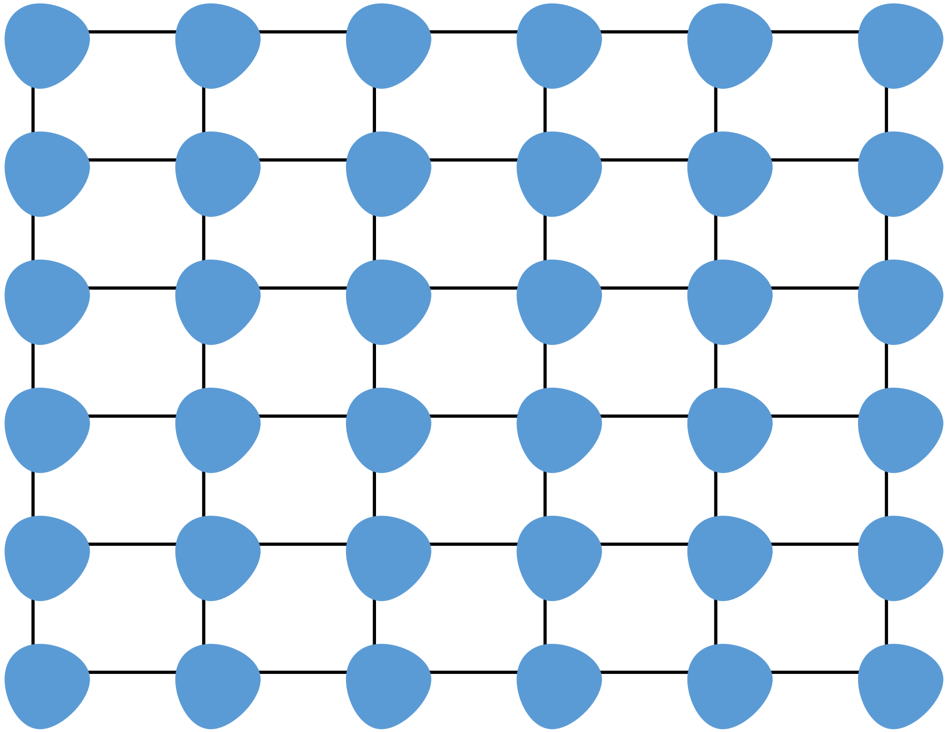

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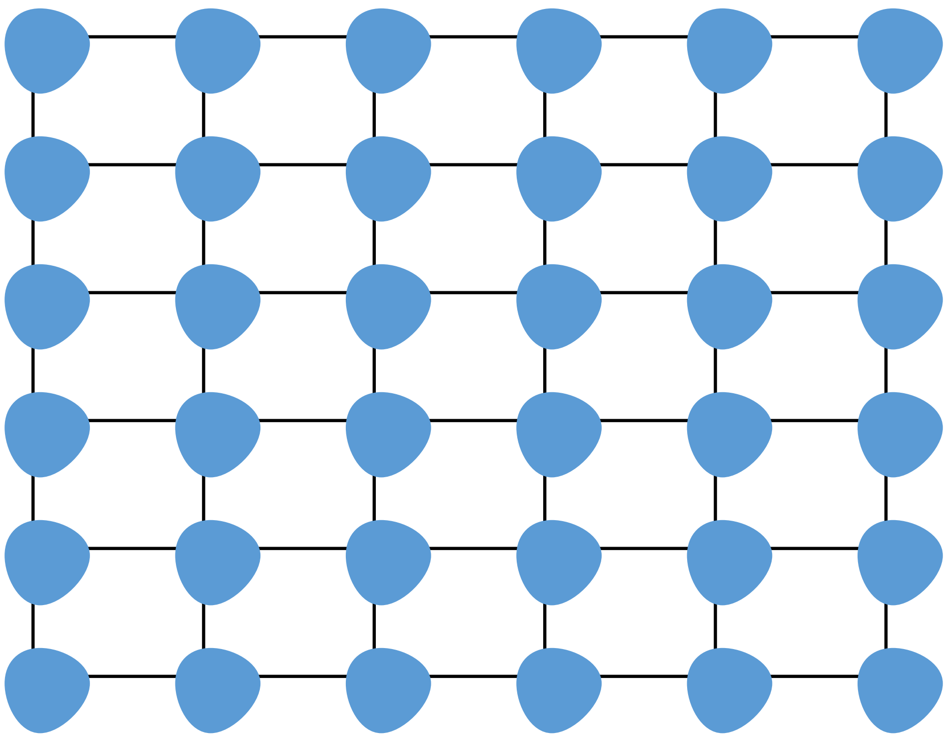

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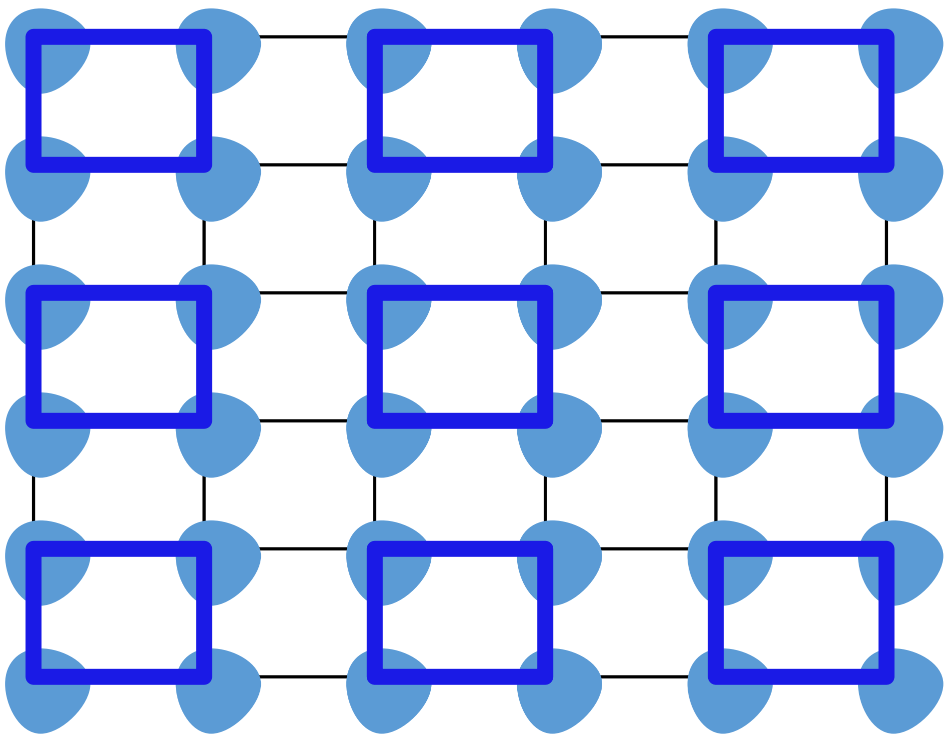

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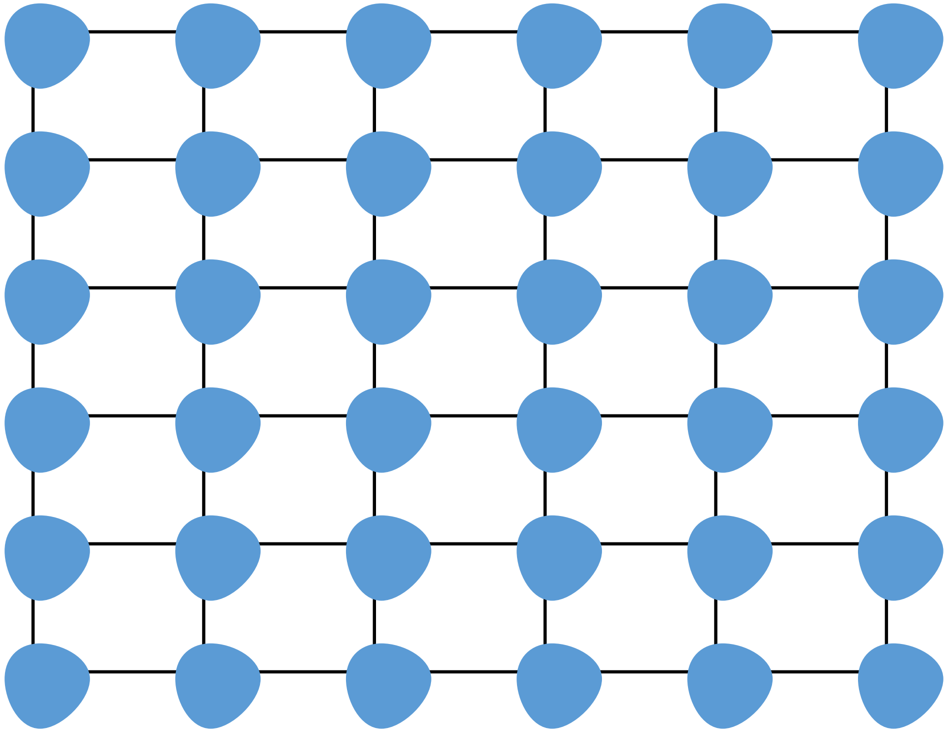

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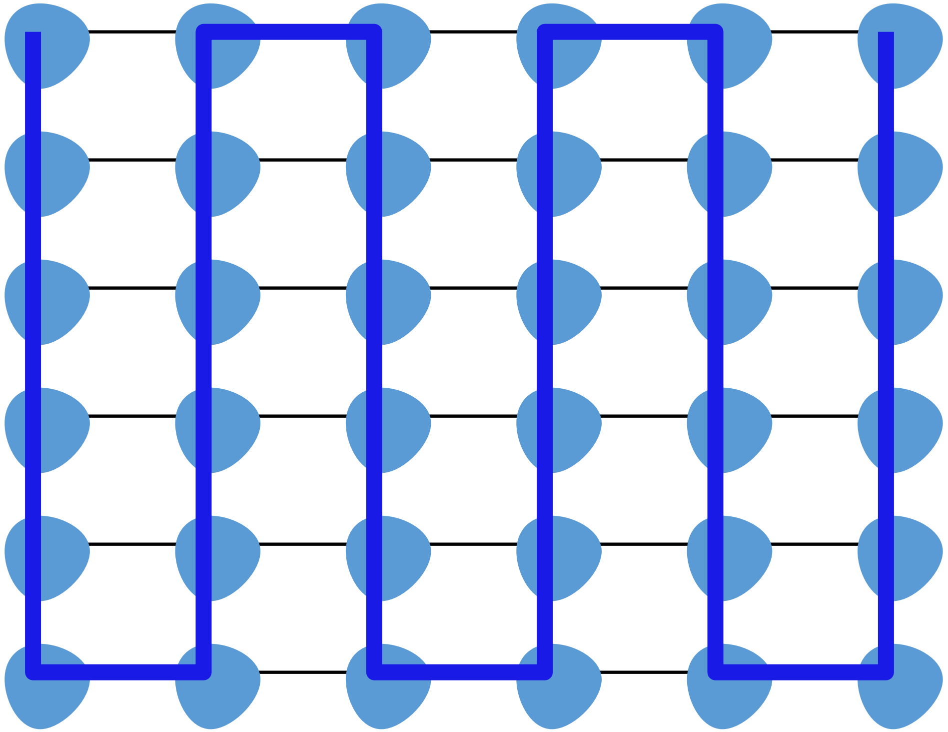

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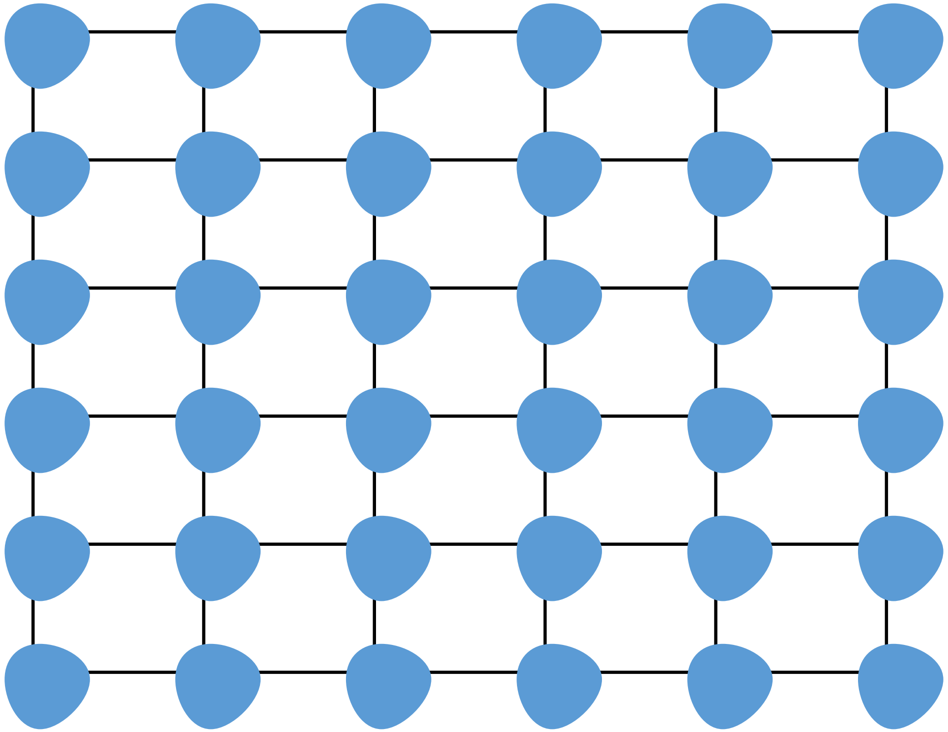

- $2^{\widetilde{\mathcal{O}}(\sqrt{k})} \cdot \text{poly}(n)$-algorithms for many parameterized problems

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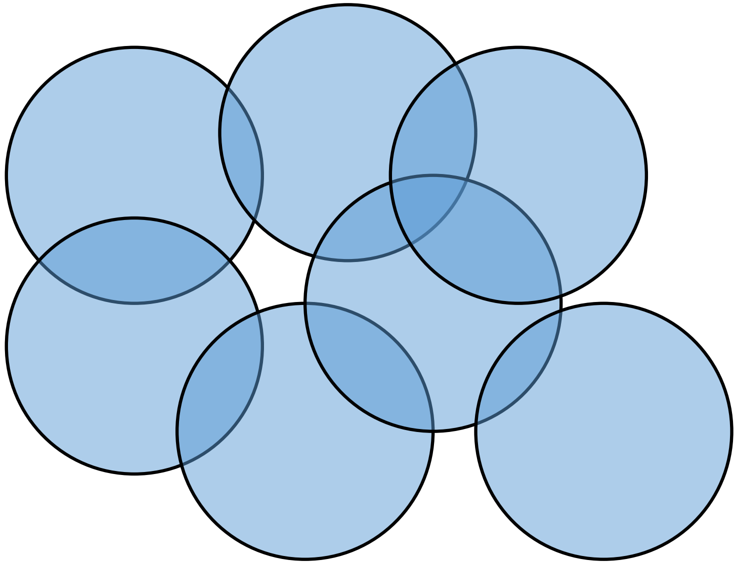

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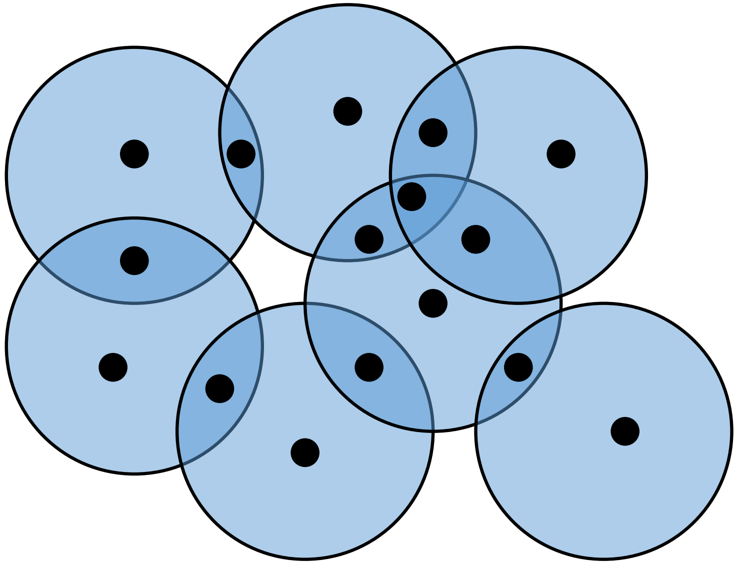

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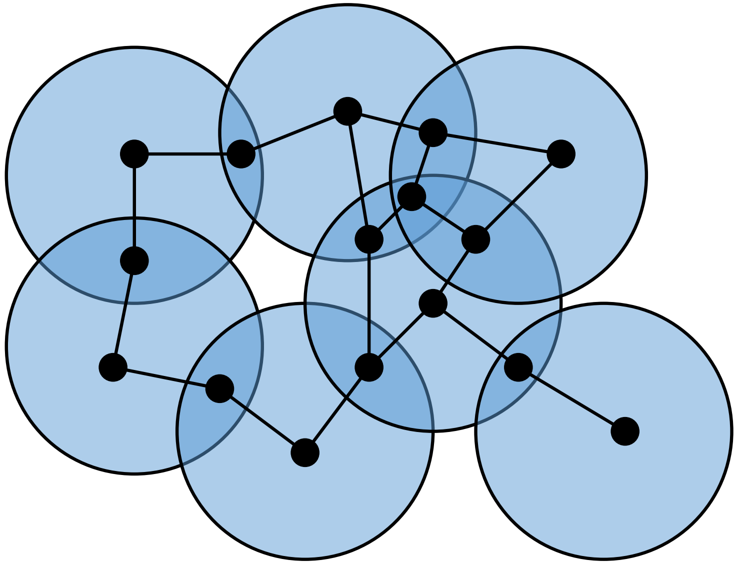

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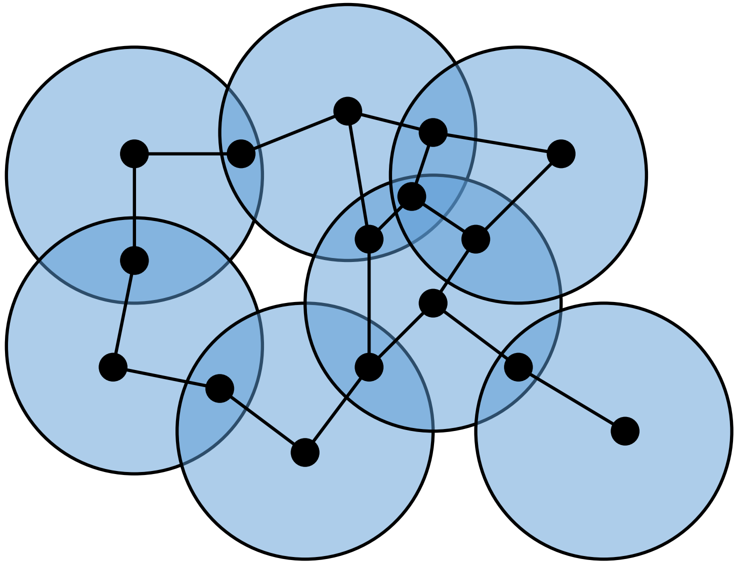

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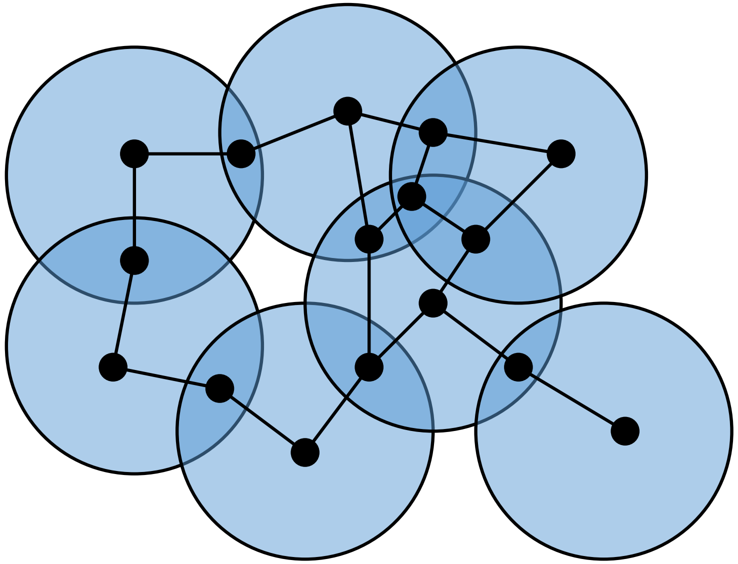

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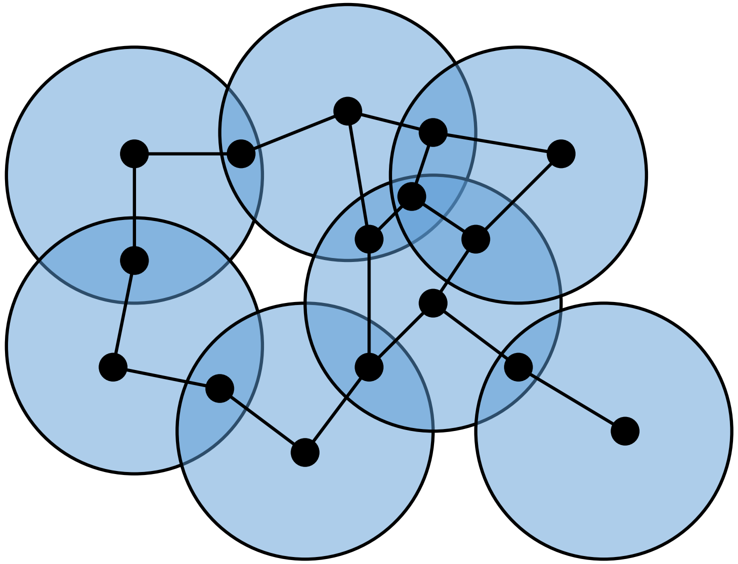

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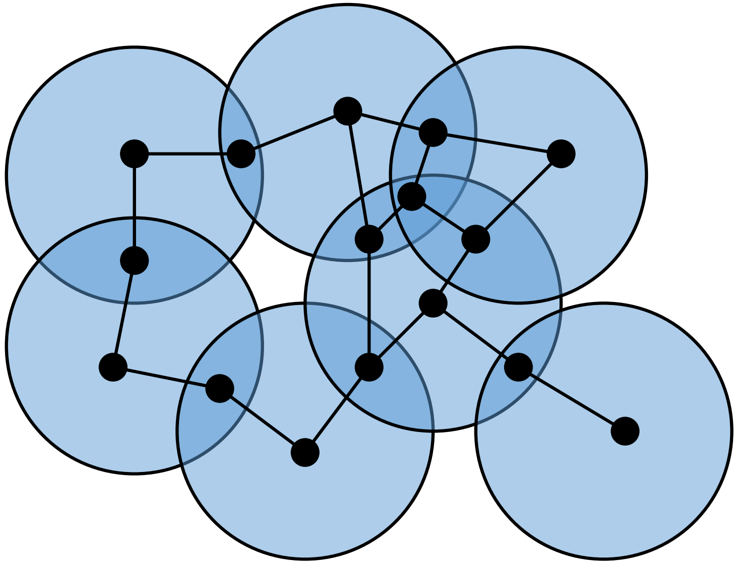

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- using this, we construct a $t' \times t'$ grid minor in G, where $t' = \mathcal{O}(t) = \mathcal{O}(\mathrm{tw}(G))$

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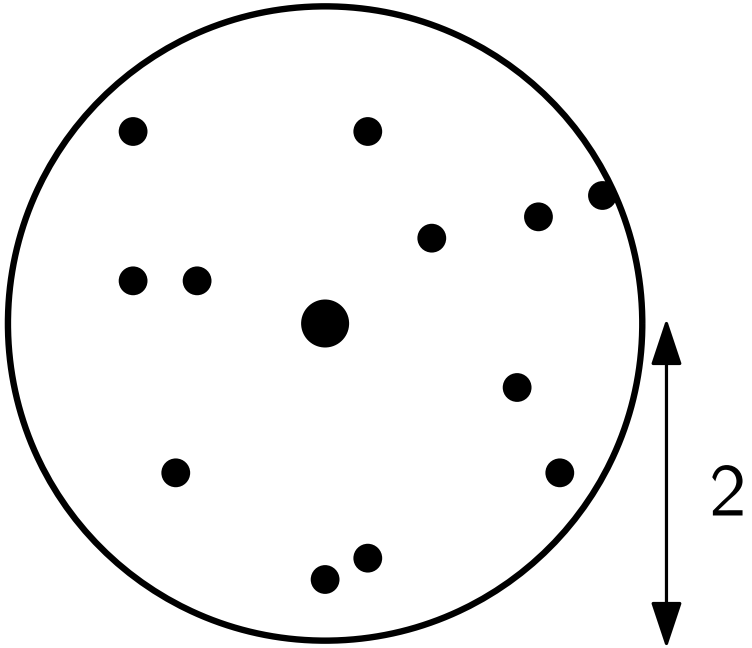

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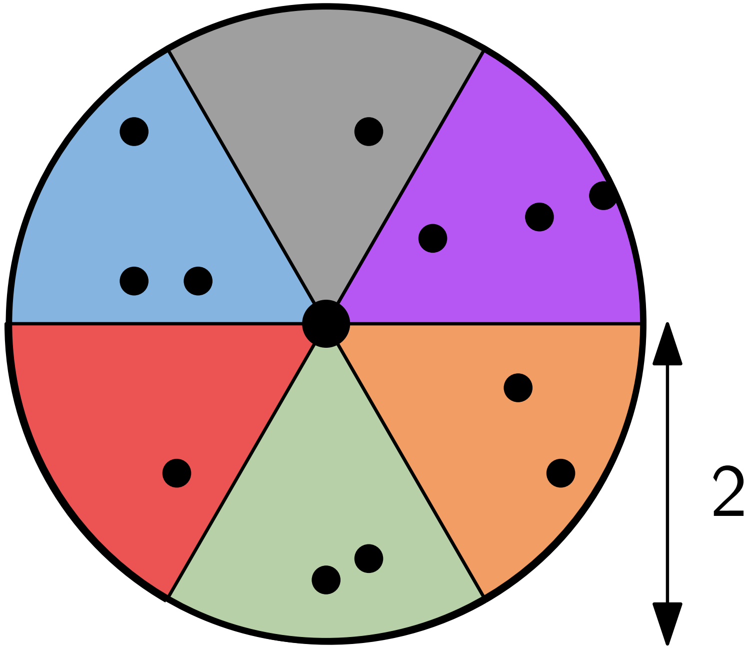

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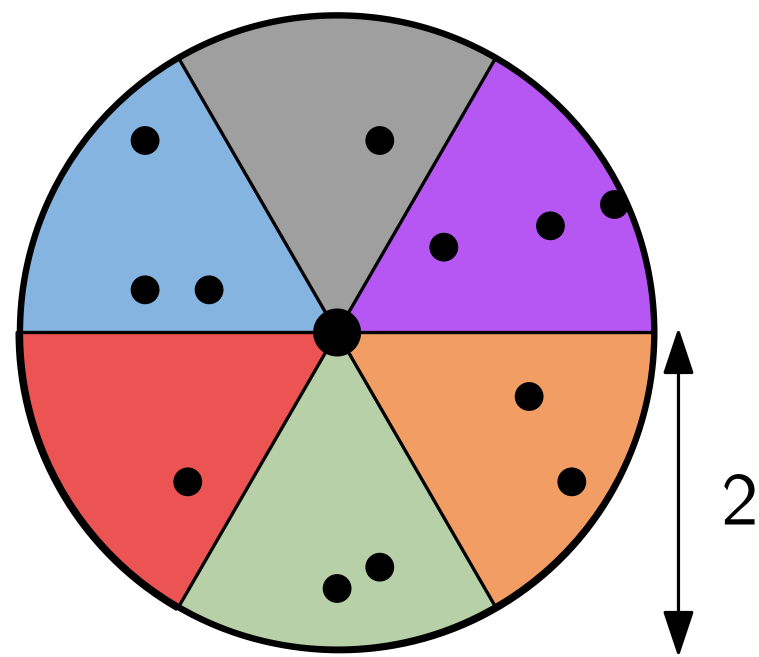
- take a vertex of degree Δ
- centers or all neighbors are in the radius-2 disk
- centers in each region correspond to a clique
- add some technical magic



Theorem [FLS '11].
Every unit disk graph with no p-clique and treewidth $\Omega(p \cdot t)$ has a $t \times t$ grid minor.

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$$T(n,k) \leq k^{2\varepsilon} \cdot T(n, k-k^{\varepsilon}) \leq \exp\{k^{1-\epsilon} \log k\} \cdot \text{poly}(n)$$

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3. If $|C| < k^{\varepsilon}$, then one of the following occurs:
   (a) treewidth $= \mathcal{O}(k^{\varepsilon} \cdot t) = k^{\mathcal{O}(1/2+\varepsilon)}$, divide & conquer
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Overall running time is $2^{\mathcal{O}(k^{0.75} \cdot \log k)} \cdot \mathrm{poly}(n)$.

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