Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

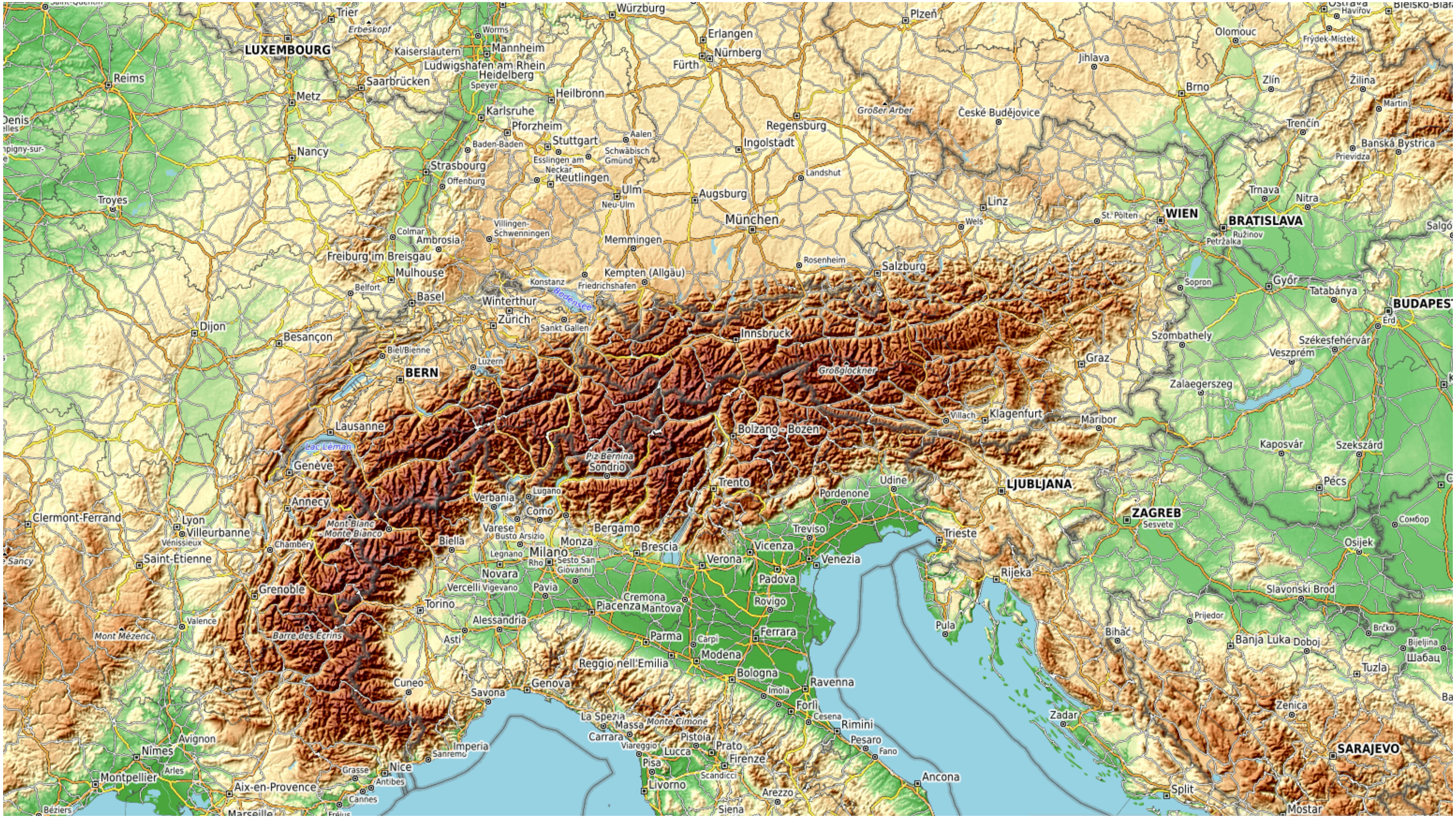
# Computational Geometry

## Delaunay Triangulations
or
## Height Interpolation

Lecture #8

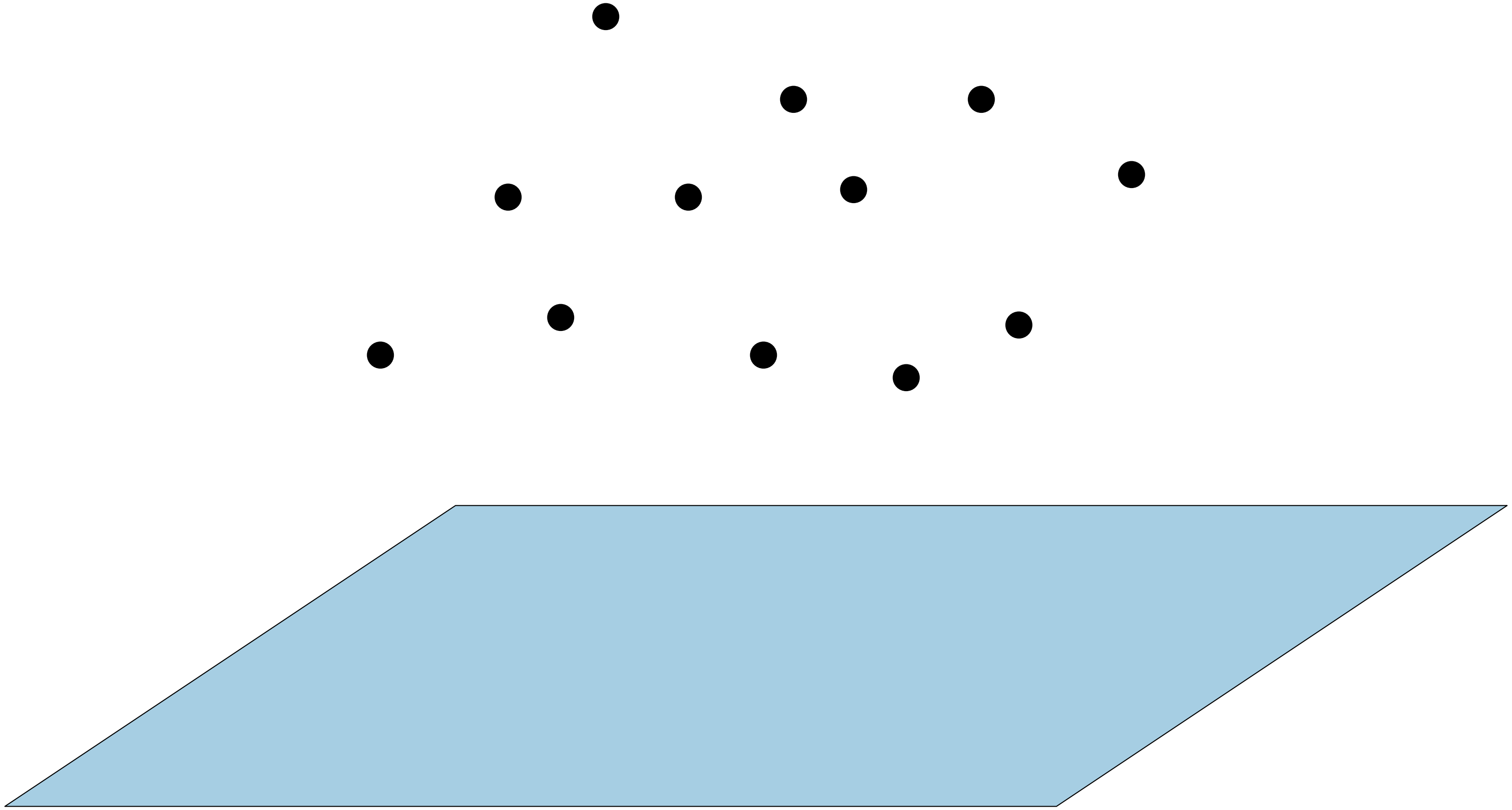
Thomas van Dijk

Winter Semester 2019/20




[opentopomap.org]

# Height Interpolation



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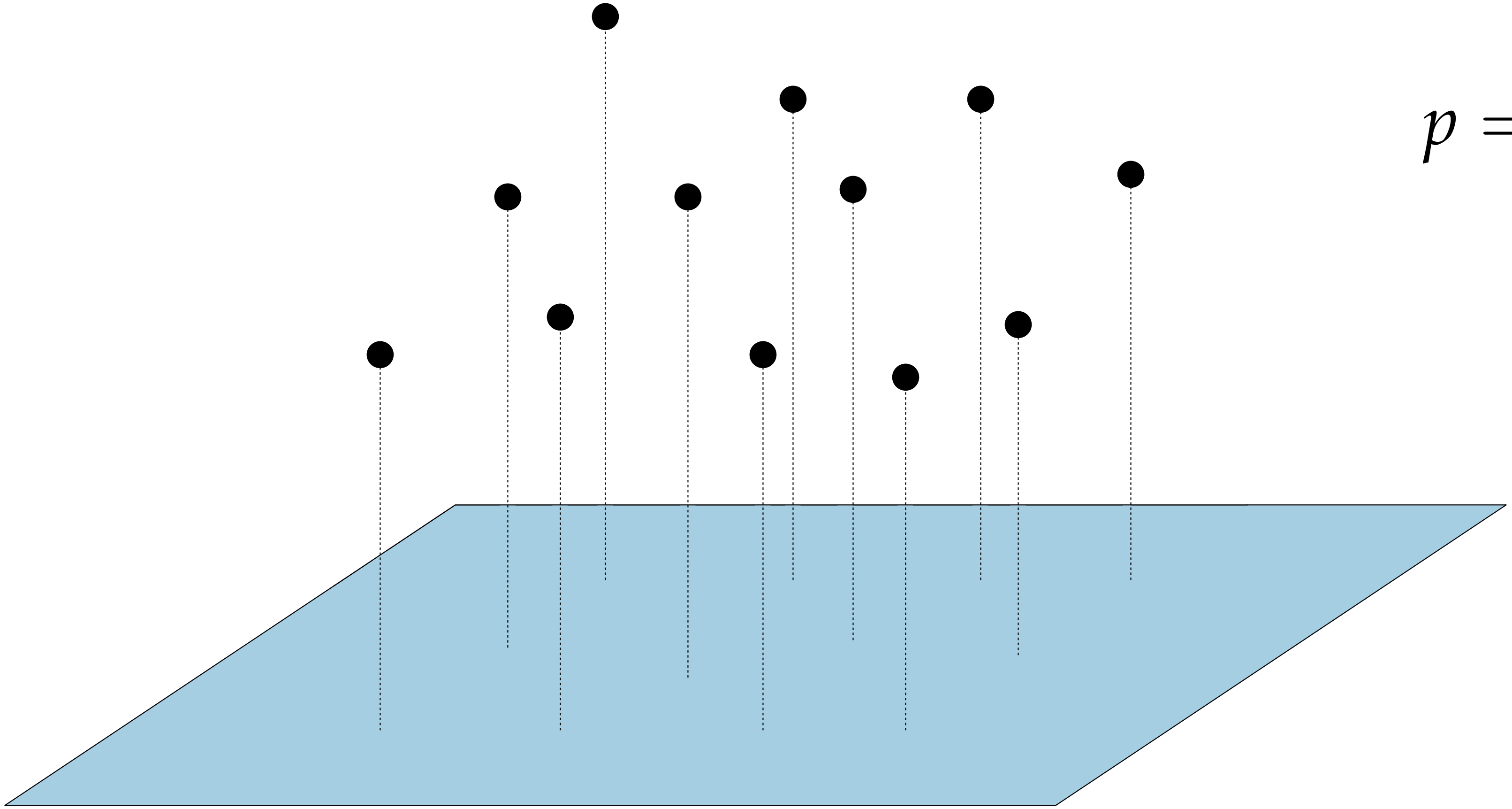

$$p = (x_p, y_p, z_p)$$

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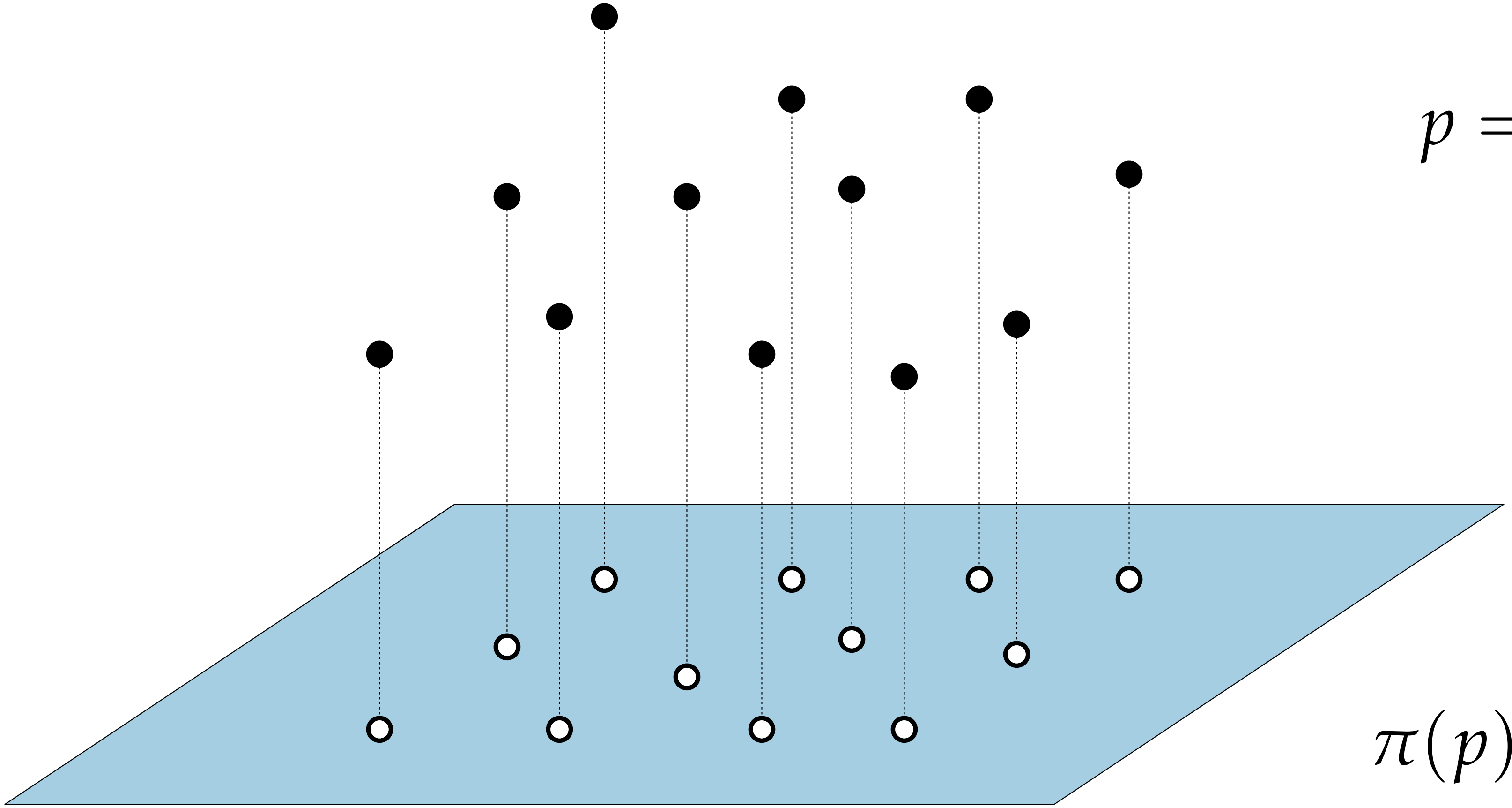

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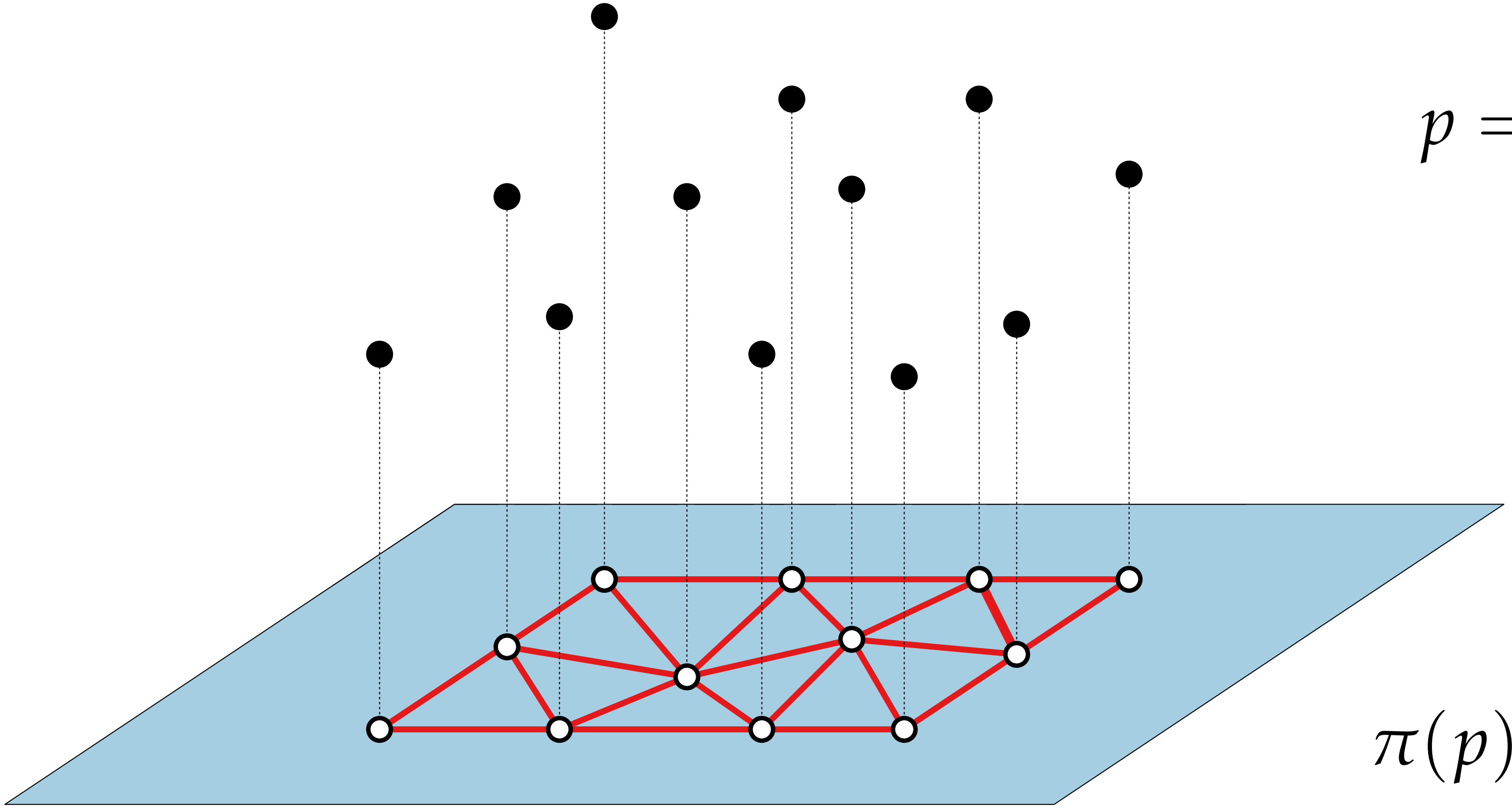

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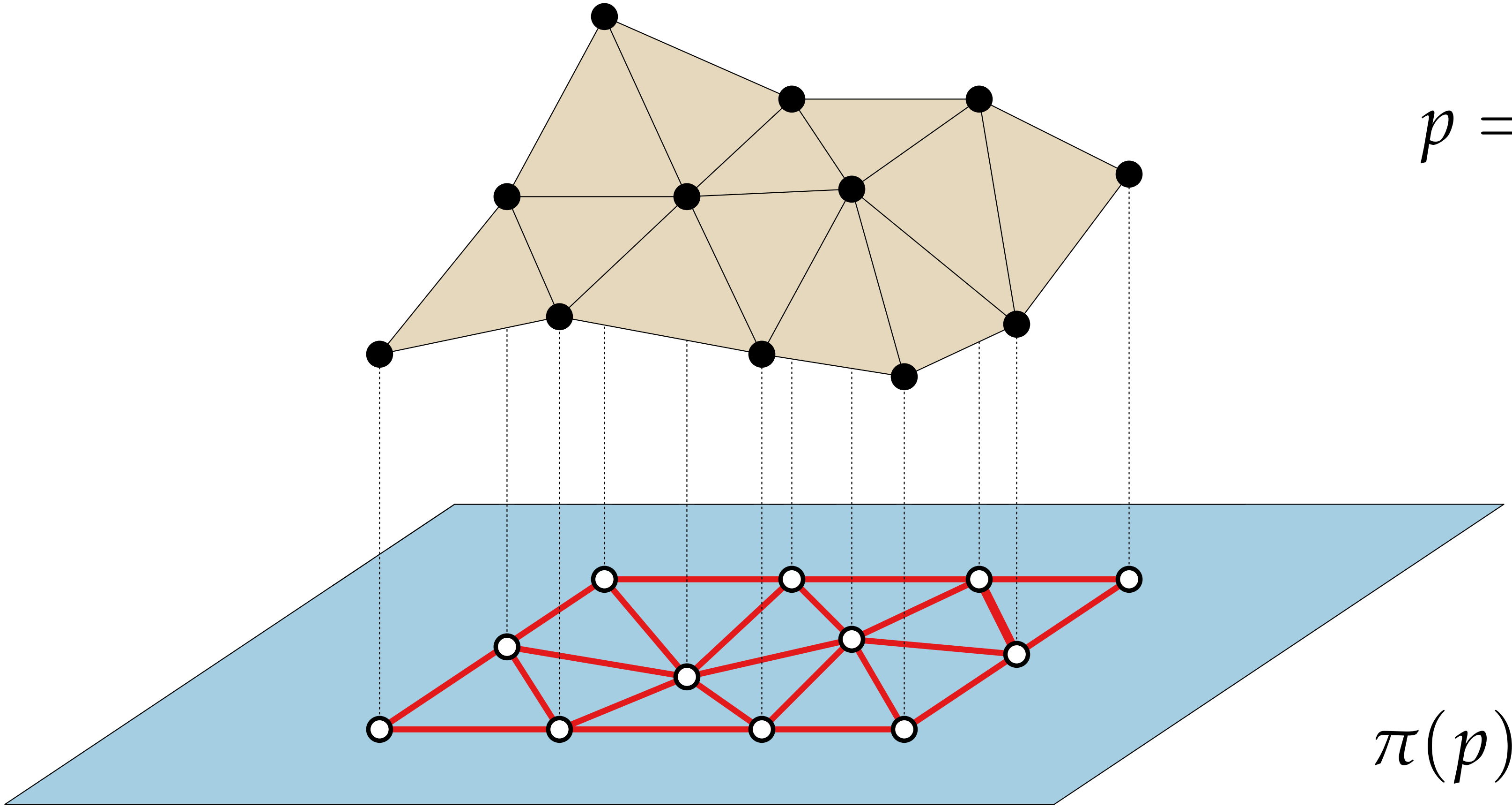

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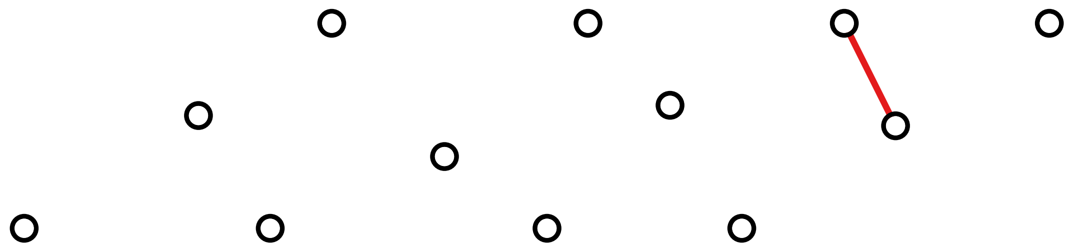
# Triangulation of Planar Point Sets

**Definition:** Given $P \subset \mathbb{R}^2$, a *triangulation* of $P$ is a maximal planar subdivision with vtx set $P$, that is, no edge can be added without crossing.



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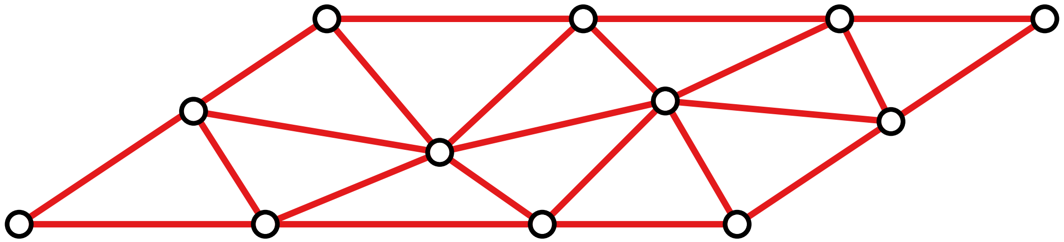

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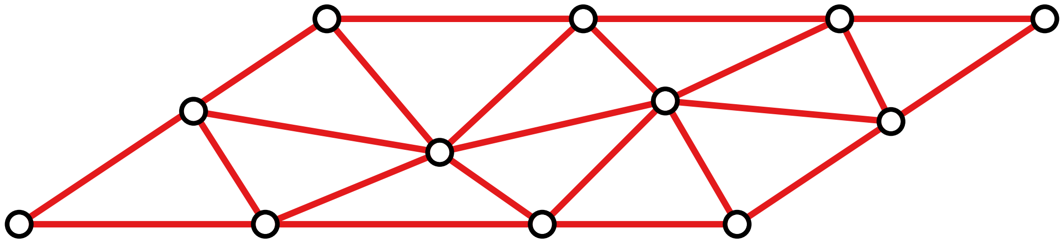

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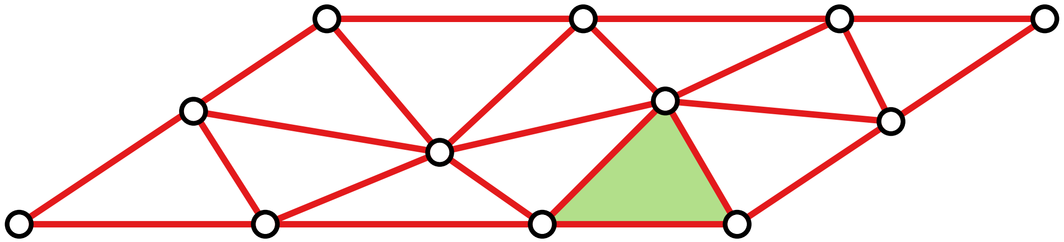

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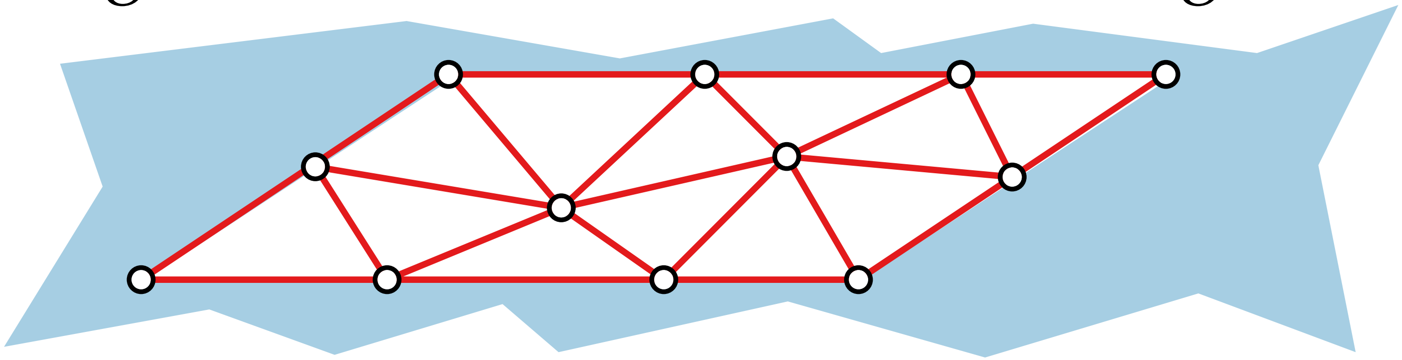

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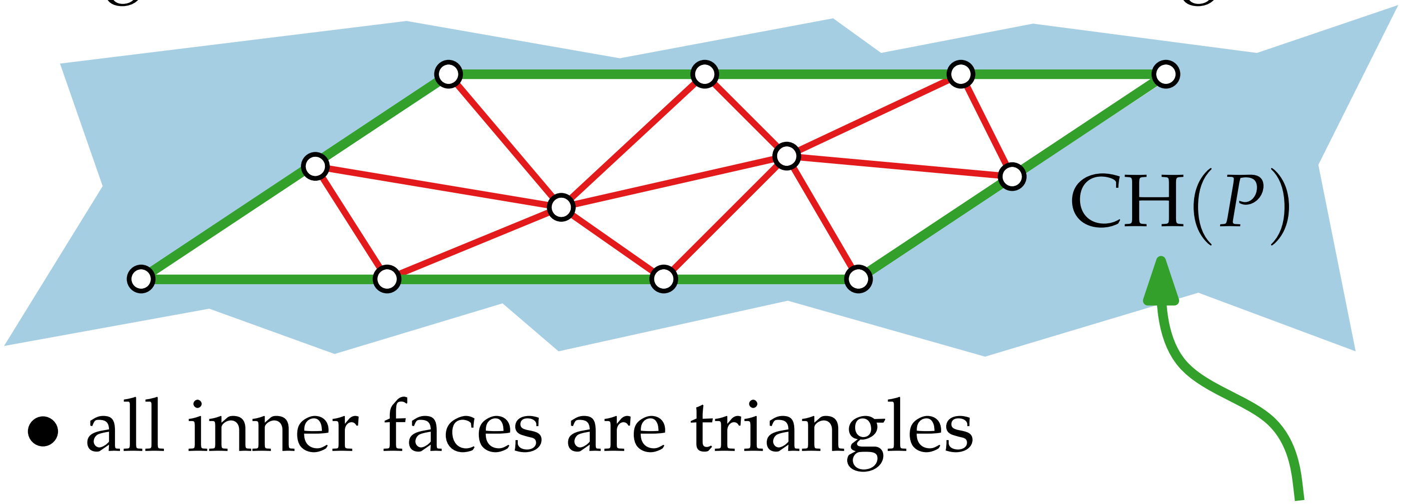


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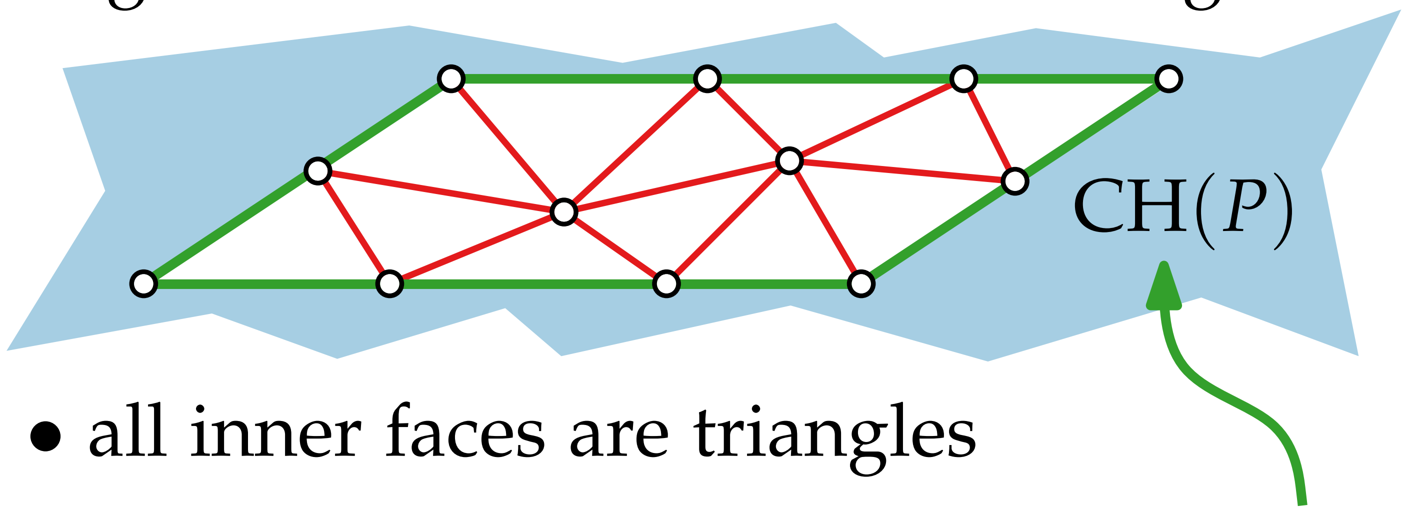


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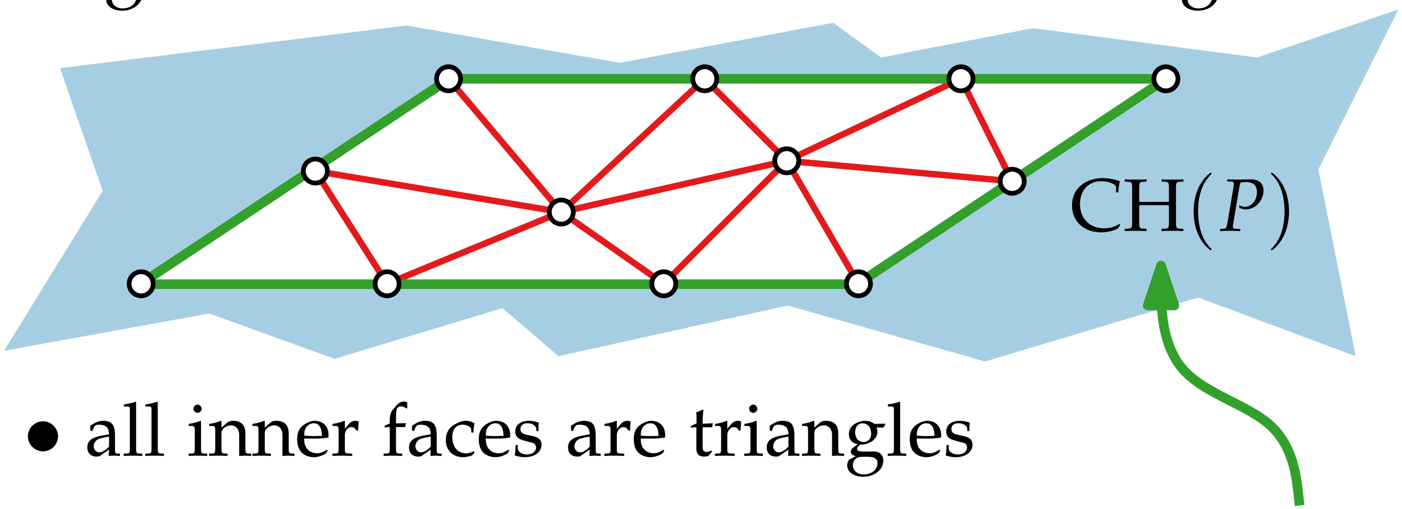


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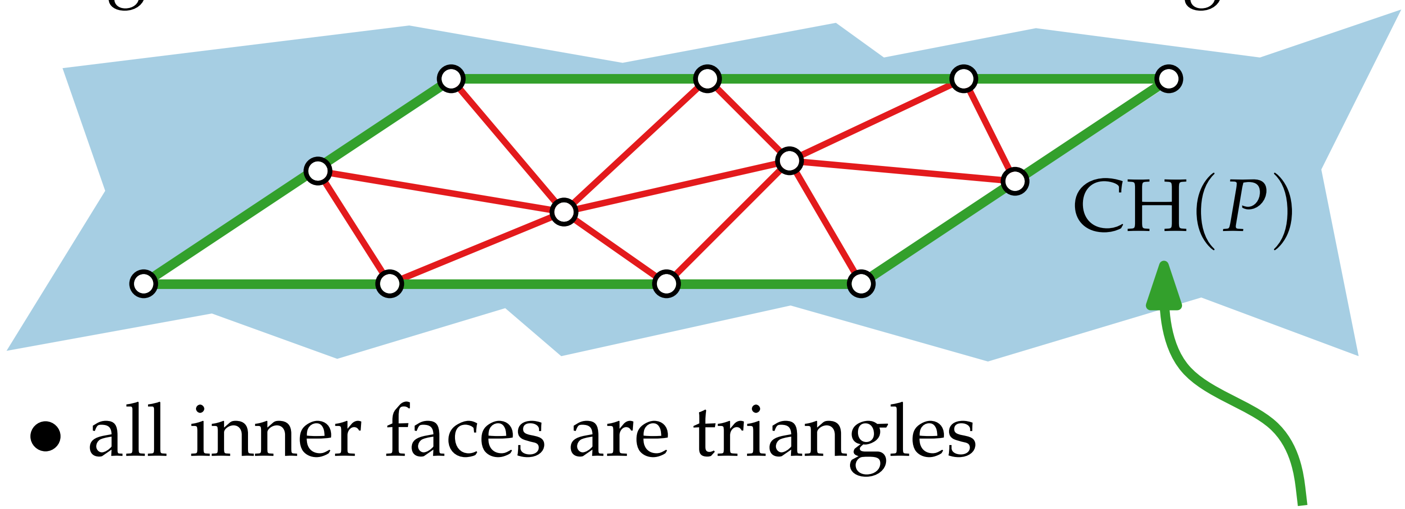


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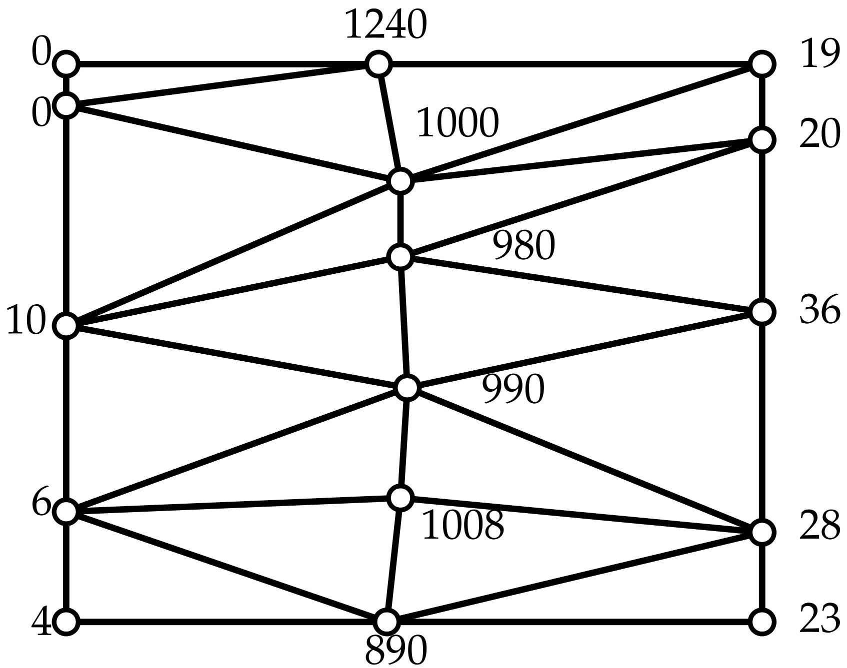
**Task:** Compute $t$ and $e$!

# Back to Height Interpolation



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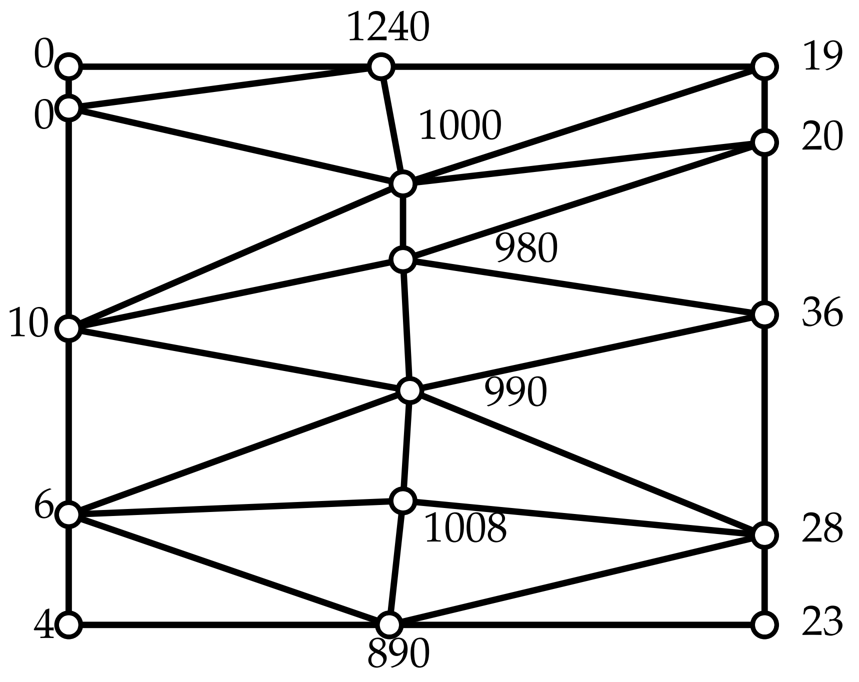


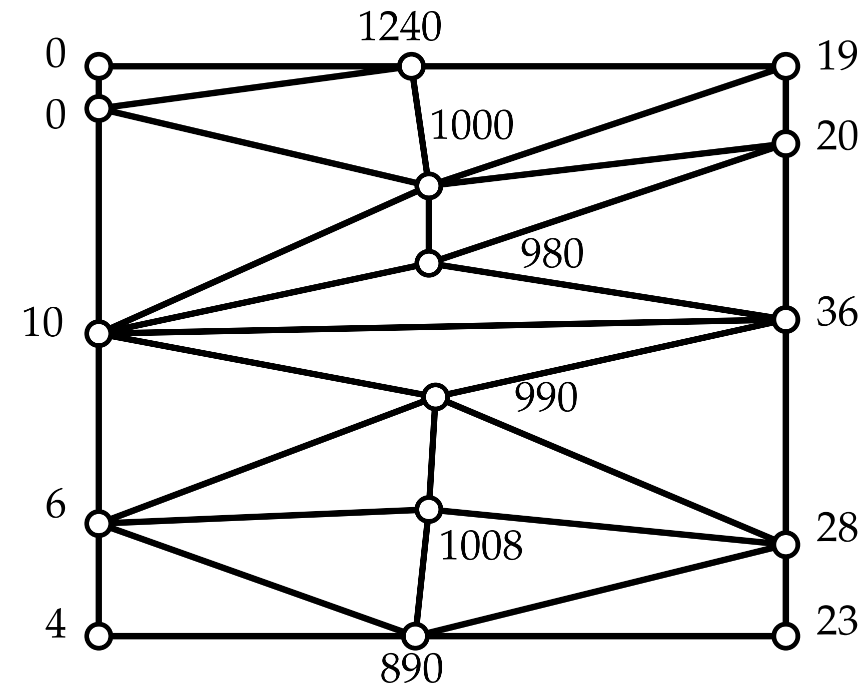

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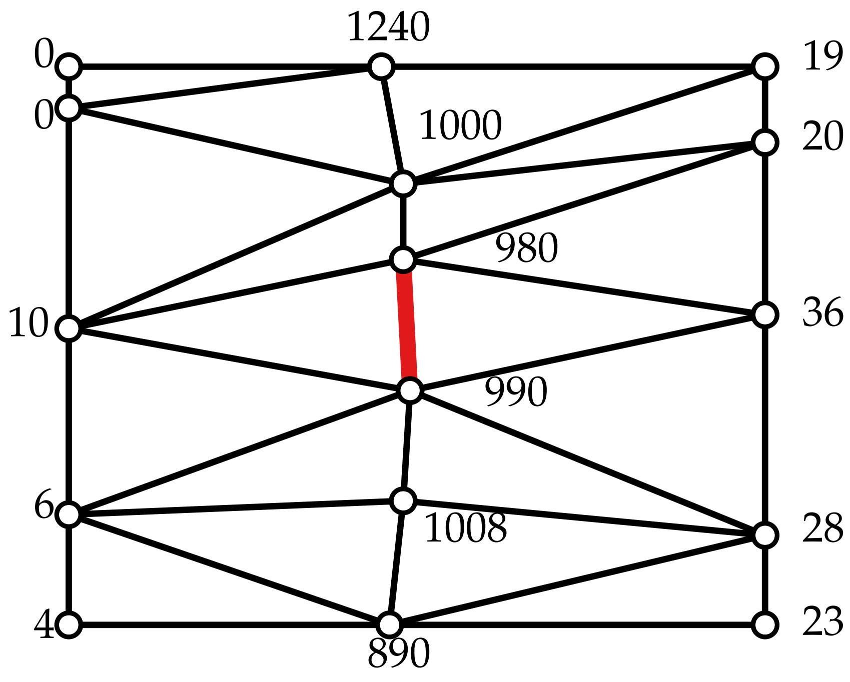


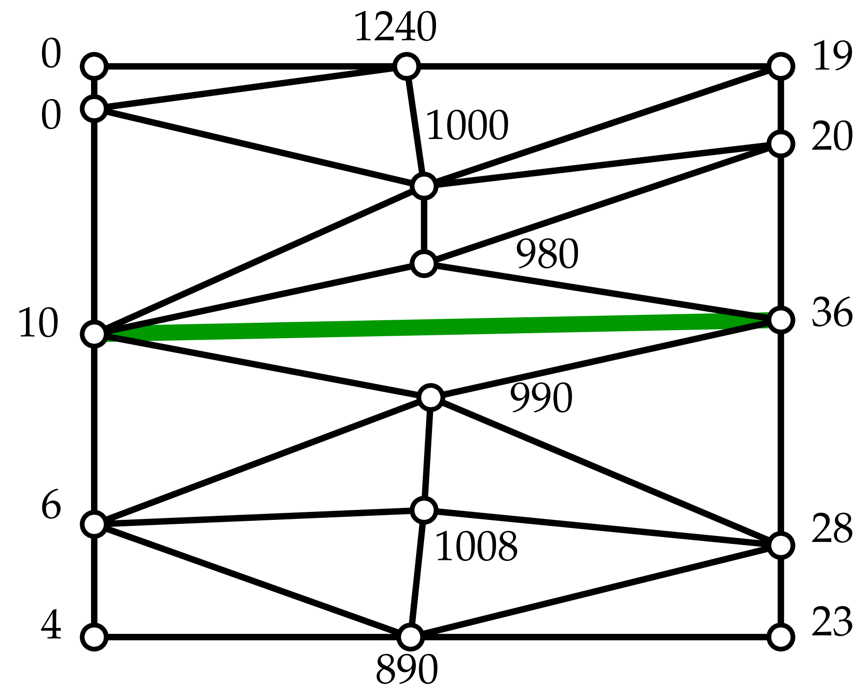

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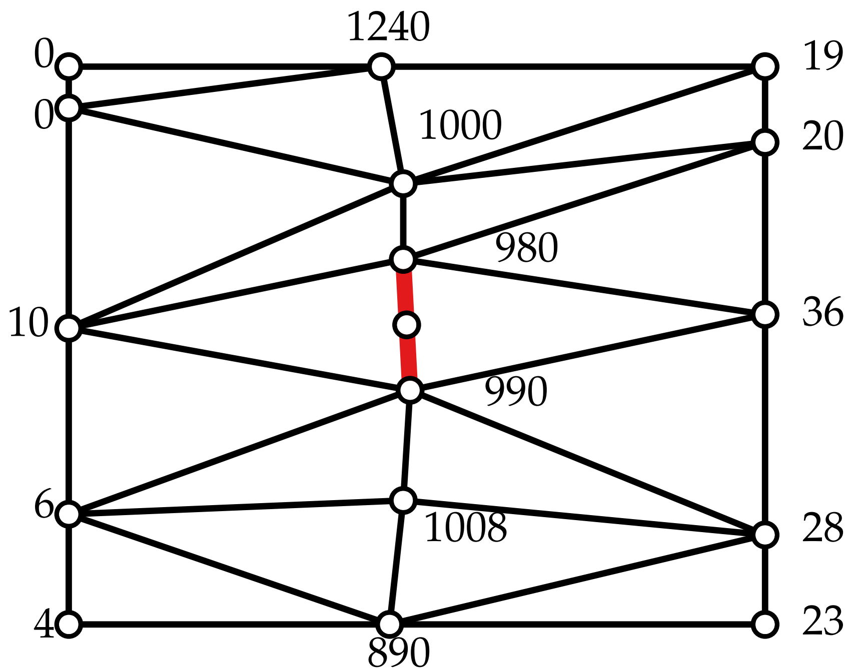


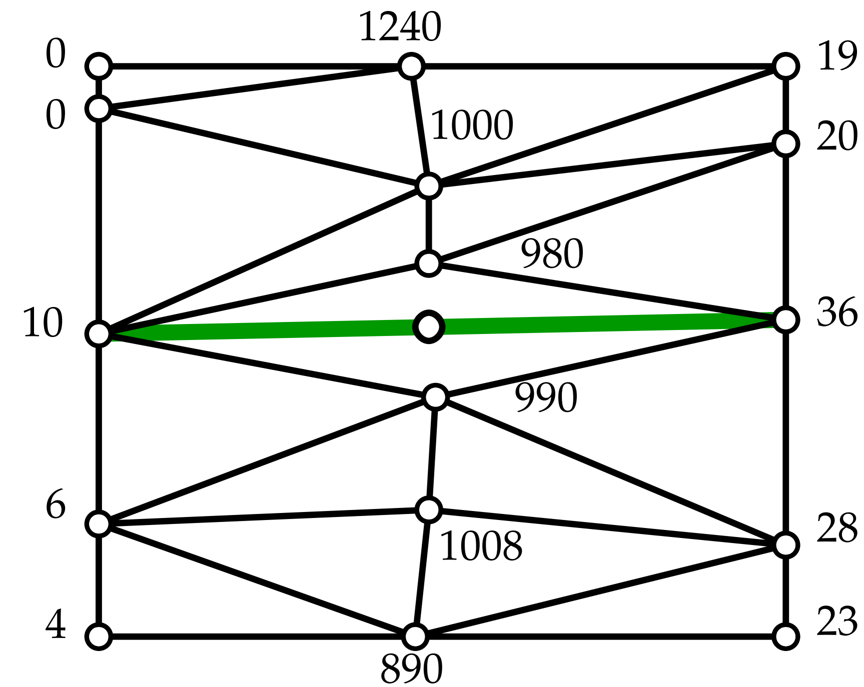

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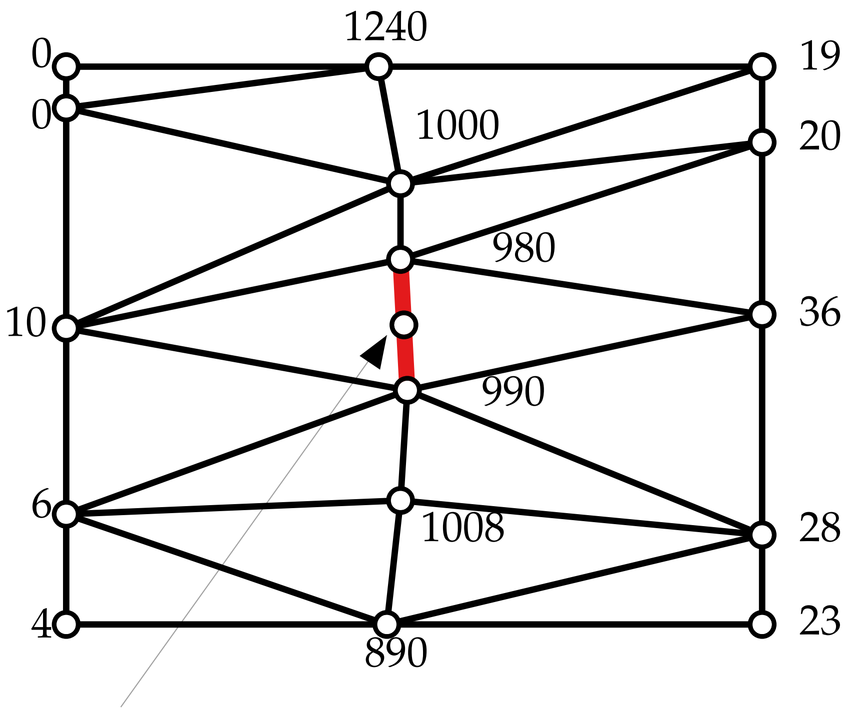
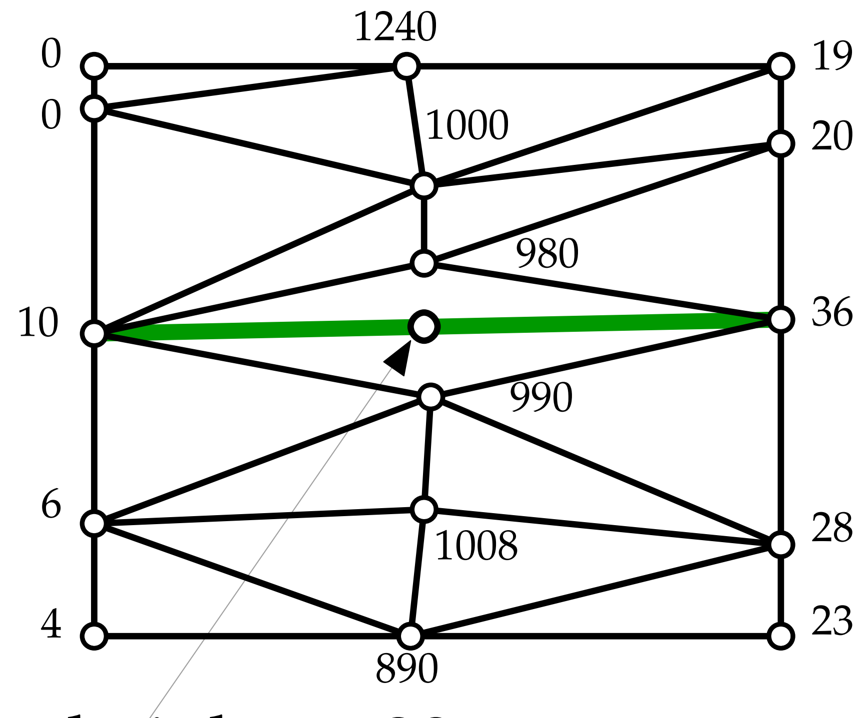



height = 985




height = 23

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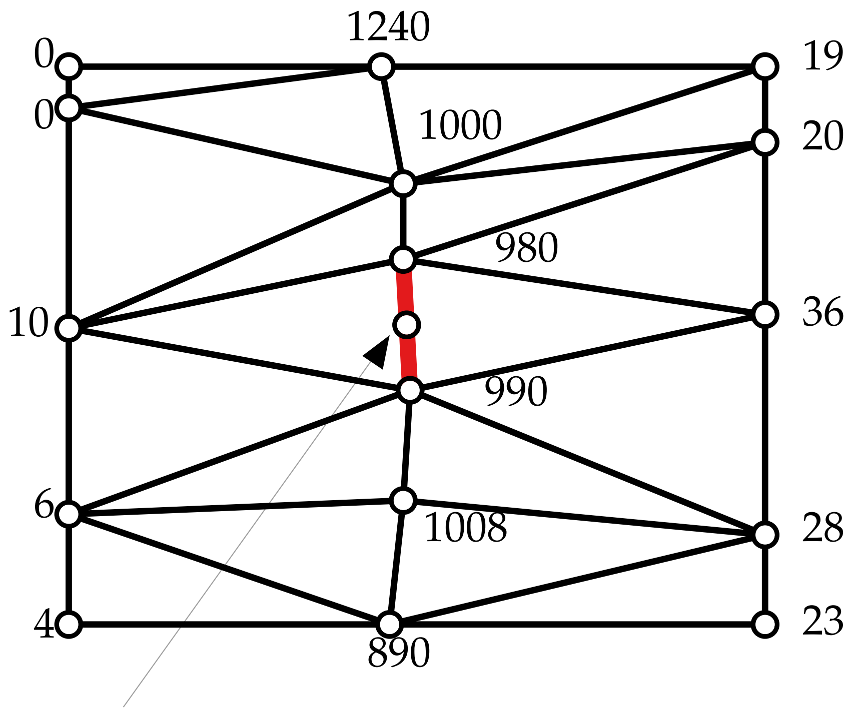


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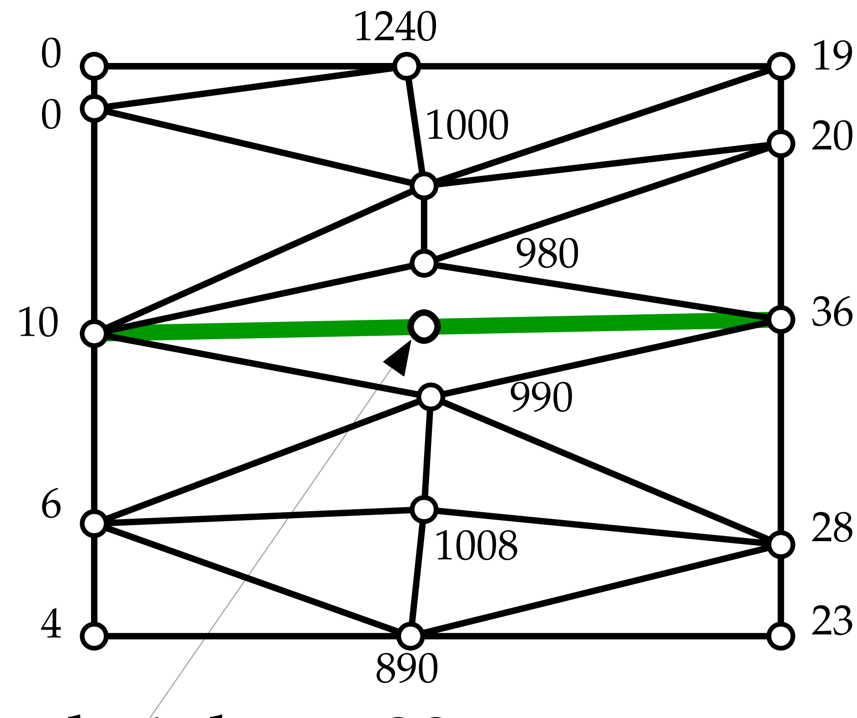


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**Intuition:** Avoid “skinny” triangles!

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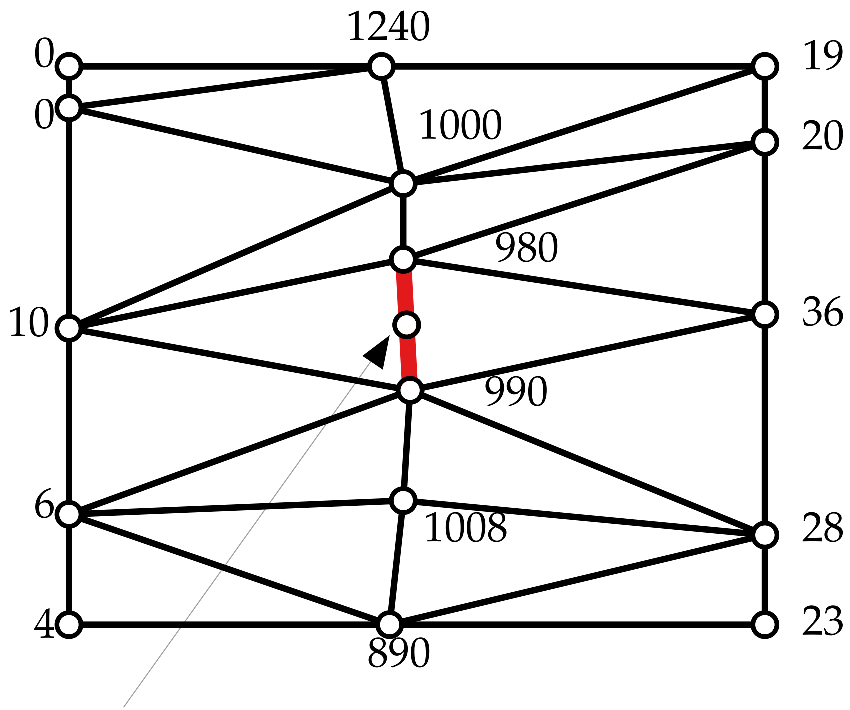


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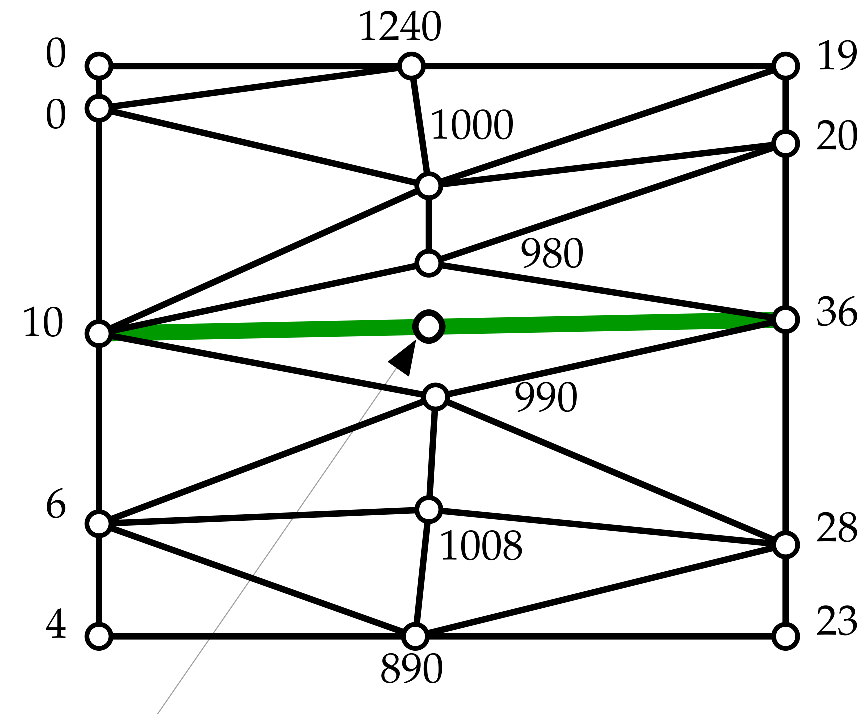


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**Intuition:** Avoid "skinny" triangles!

In other words: avoid small angles!

# Angle-Optimal Triangulations

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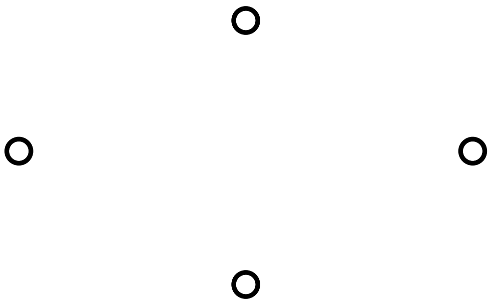

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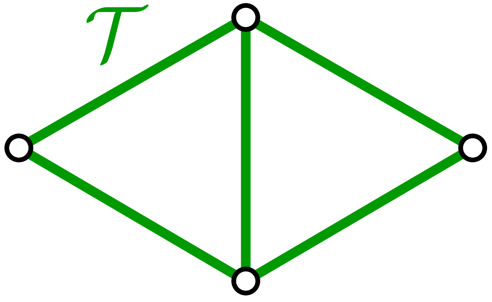

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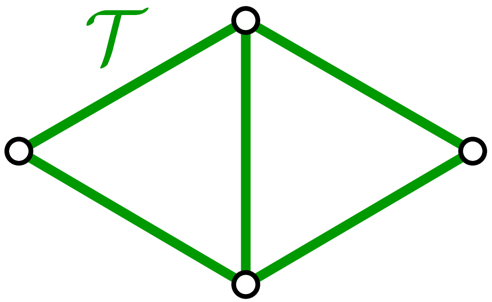

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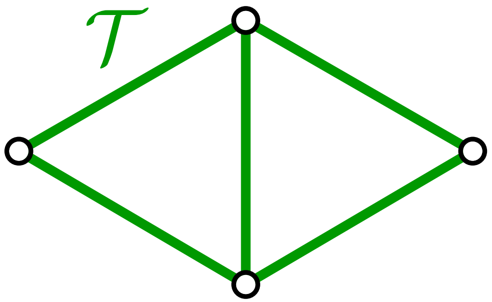

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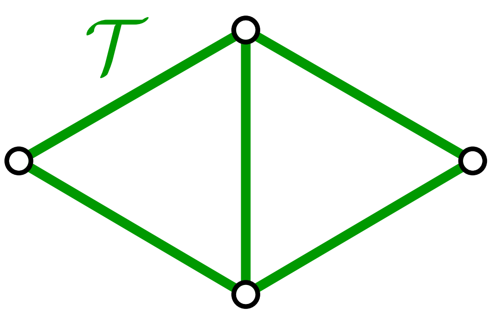

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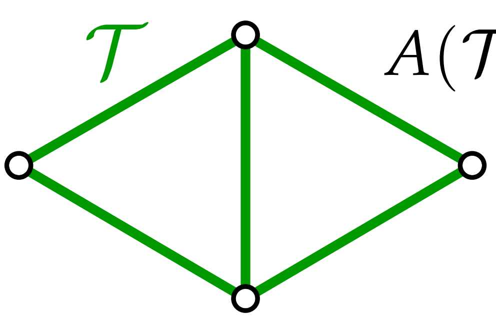


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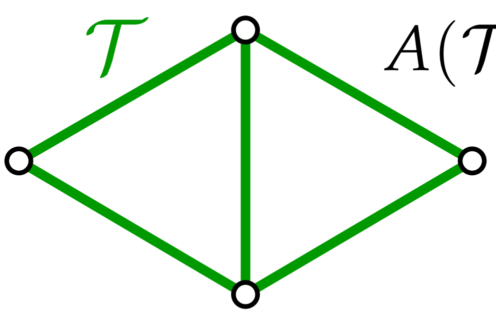


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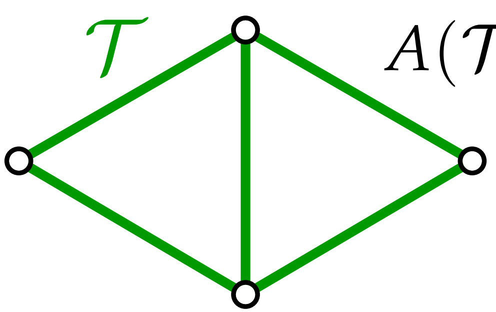


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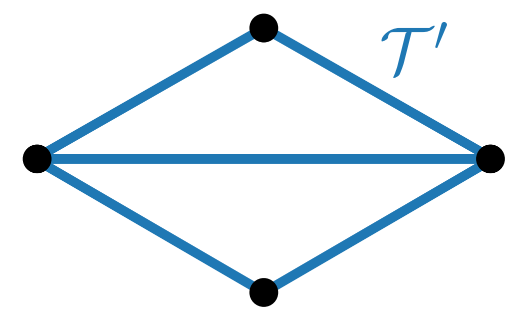

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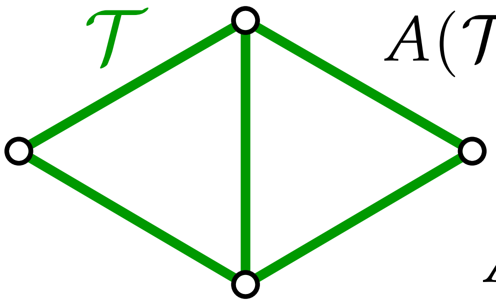


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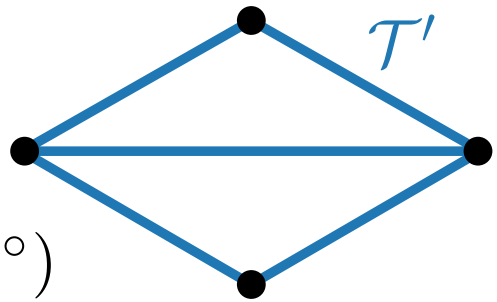

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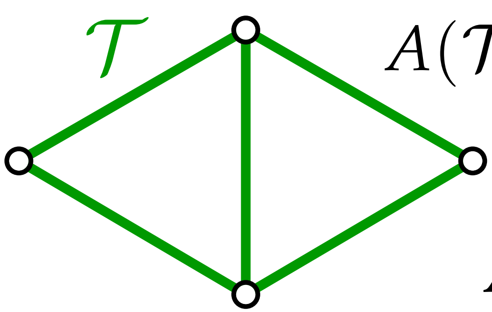


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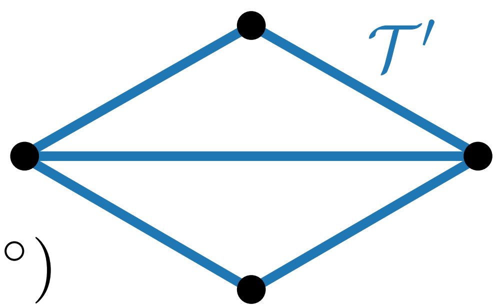


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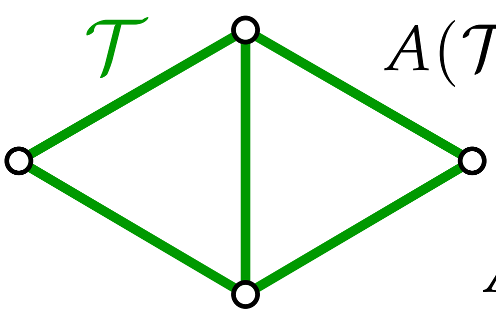


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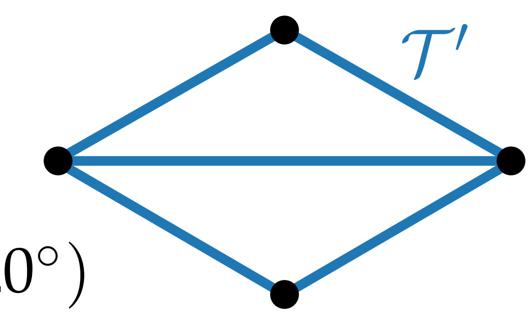


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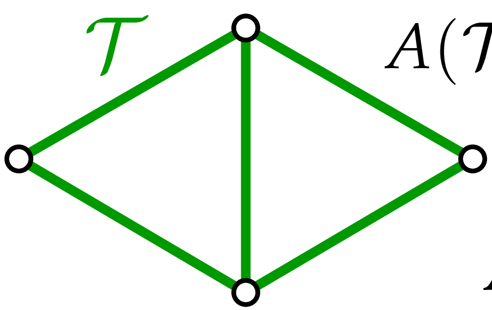
$\mathcal{T}$



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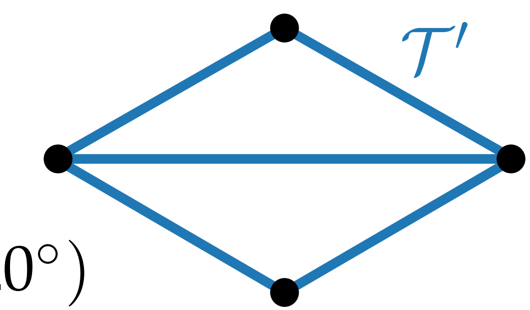
$\mathcal{T}'$



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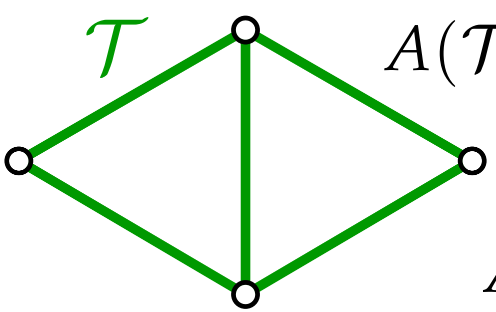


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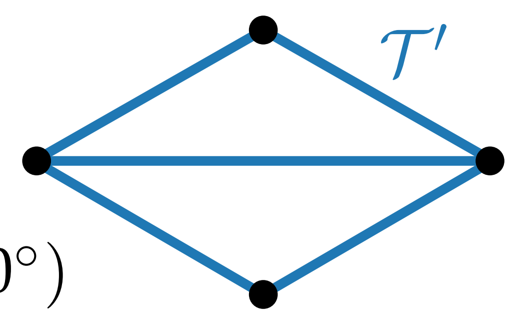
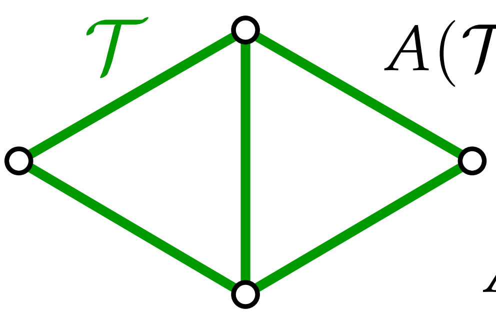


# Angle-Optimal Triangulations

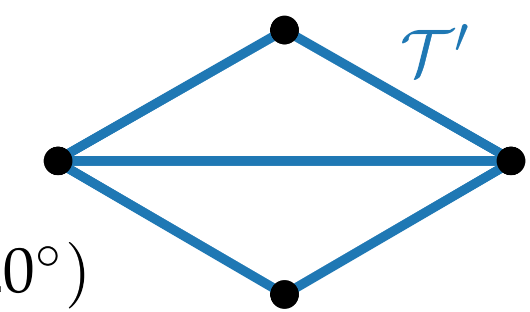
**Definition:** Given a set $P \subset \mathbb{R}^2$ and a triangulation $\mathcal{T}$ of $P$, let $m$ be the number of triangles in $\mathcal{T}$ and let $A(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3m})$ be the *angle vector* of $\mathcal{T}$, where $\alpha_1 \leq \cdots \leq \alpha_{3m}$ are the angles in the triangles of $\mathcal{T}$.

We say $A(\mathcal{T}) > A(\mathcal{T}')$
if $\exists i \in \{1, ..., 3m\} : \alpha_i > \alpha'_i$ and $\forall j < i : \alpha_j = \alpha'_j$.

$\mathcal{T}$ is *angle-optimal* if




$A(\mathcal{T}) = (60°, 60°, 60°, 60°, 60°, 60°)$

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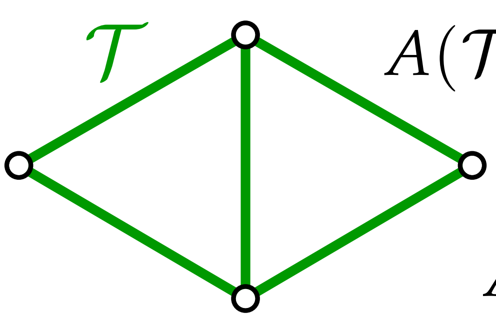


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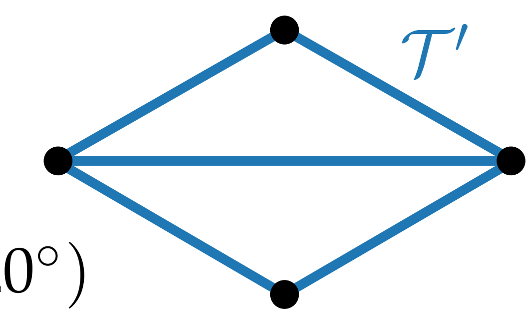


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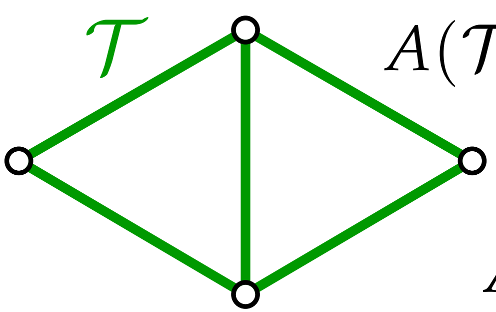


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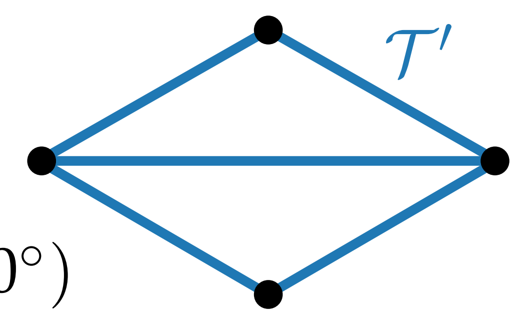


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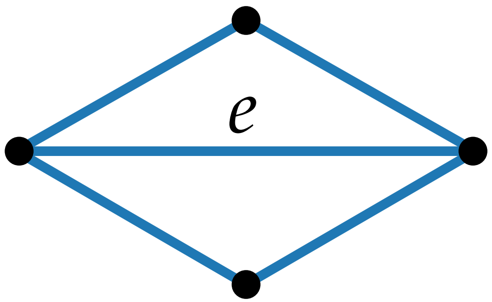
# Edge Flips

**Definition:** $\mathcal{T}$ a triangulation. An edge $e$ of $\mathcal{T}$ is *illegal* if the minimum angle in the two triangles adjacent to $e$ increases when flipping.



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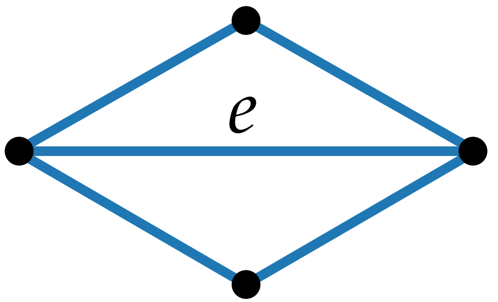
$\min_i \alpha_i = 30^\circ$



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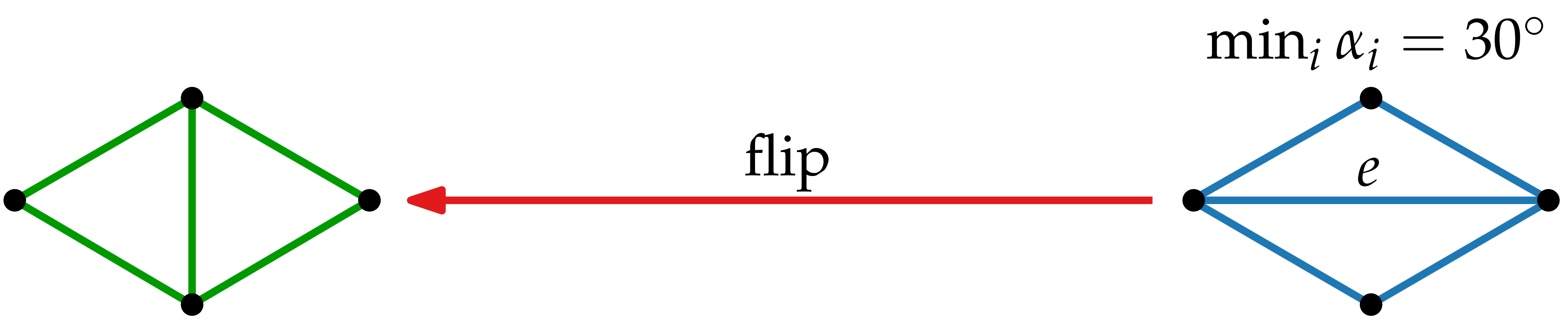

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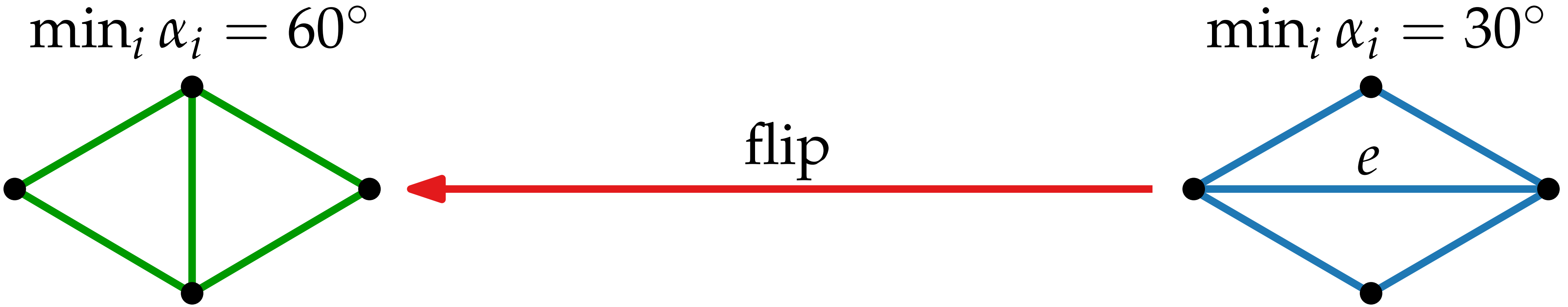

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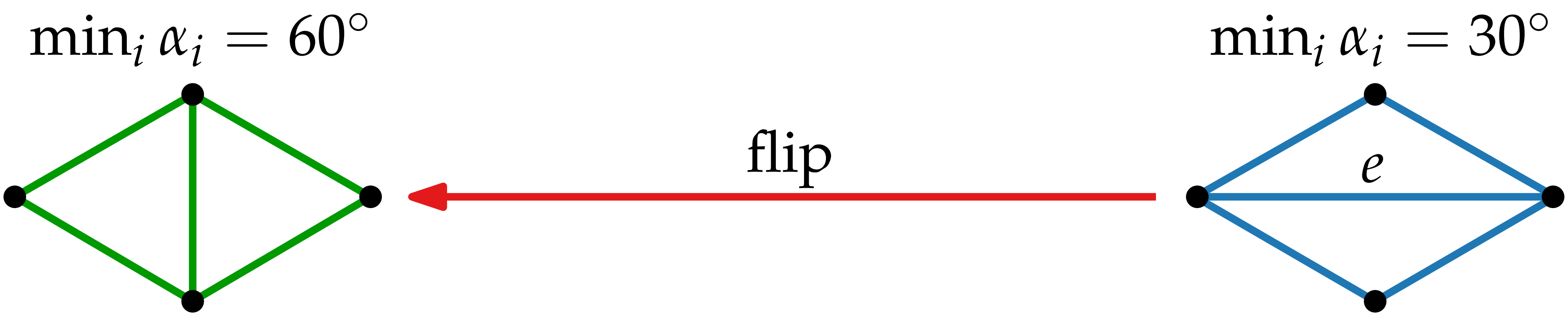

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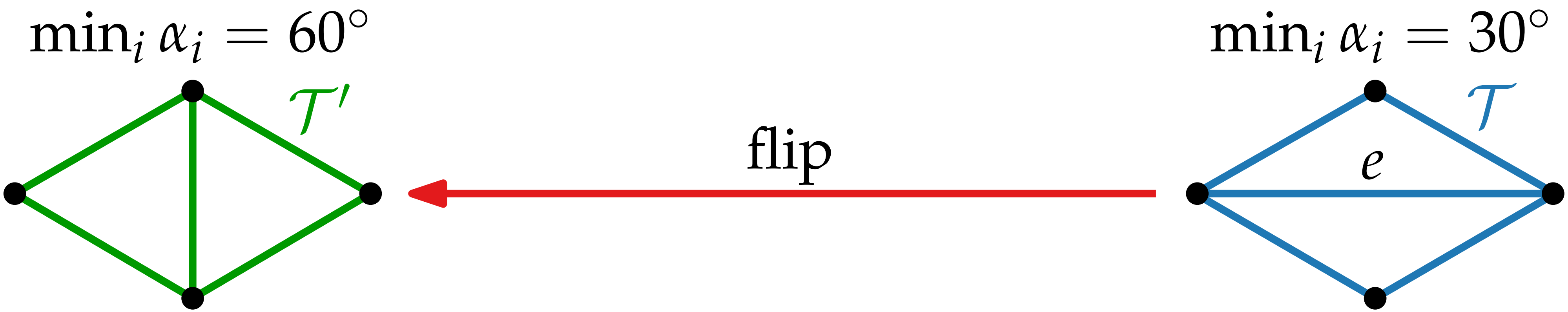

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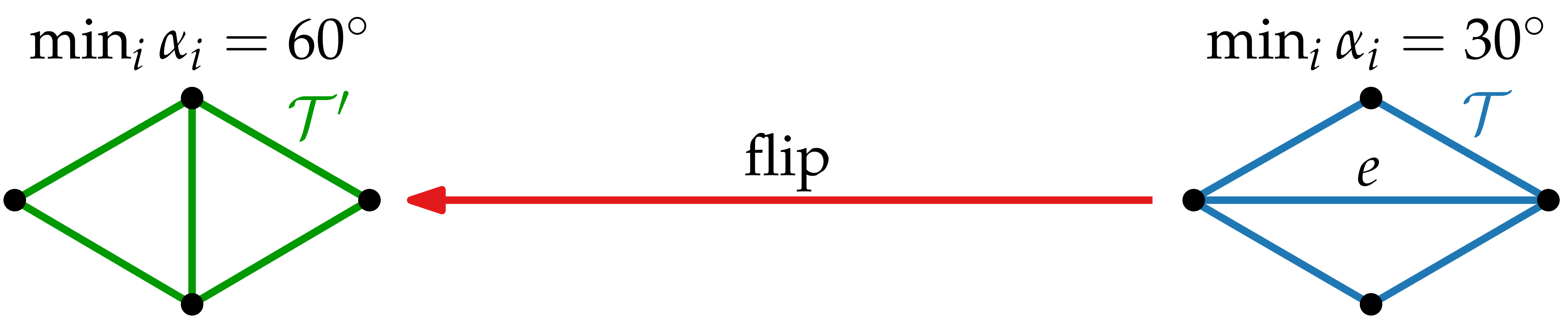

# This is all Greek to me...

**Theorem:**

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The diameter of a circle always subtends a right angle to any point on the circle.

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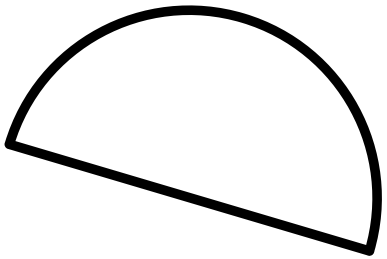

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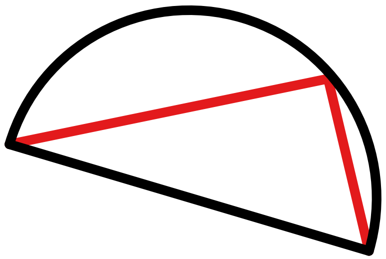

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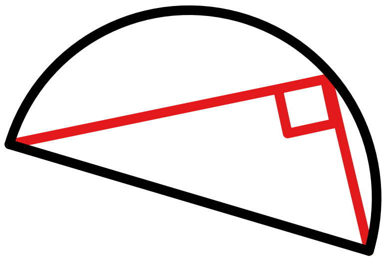

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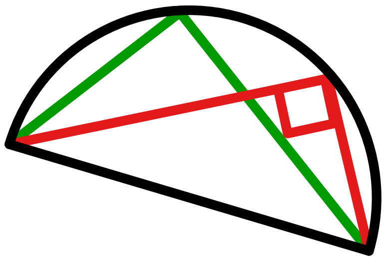

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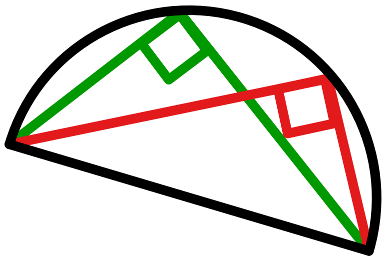

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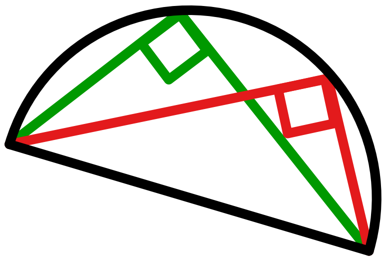

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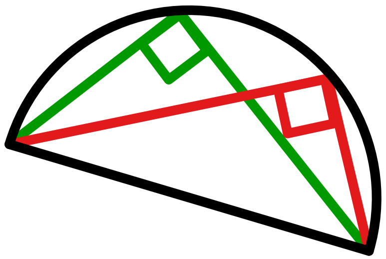

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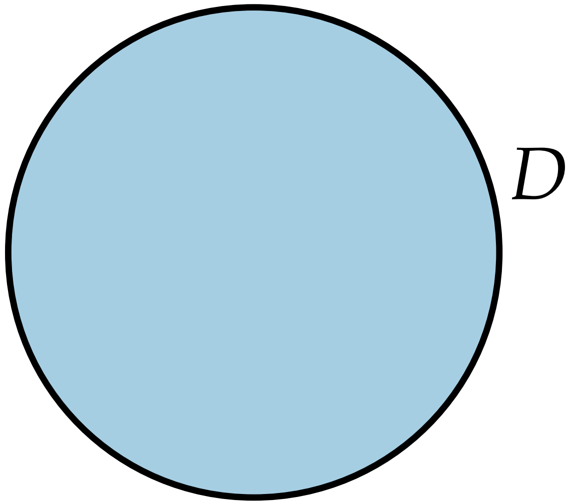

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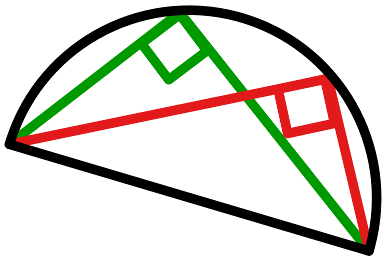

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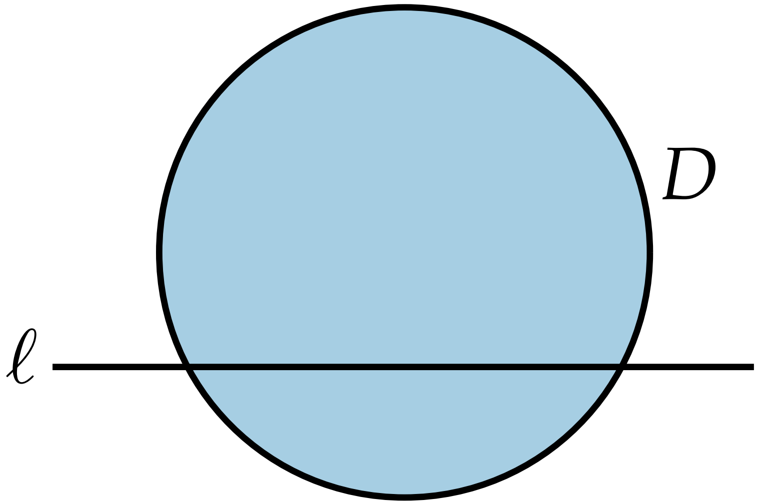

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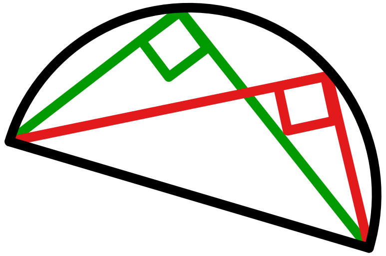

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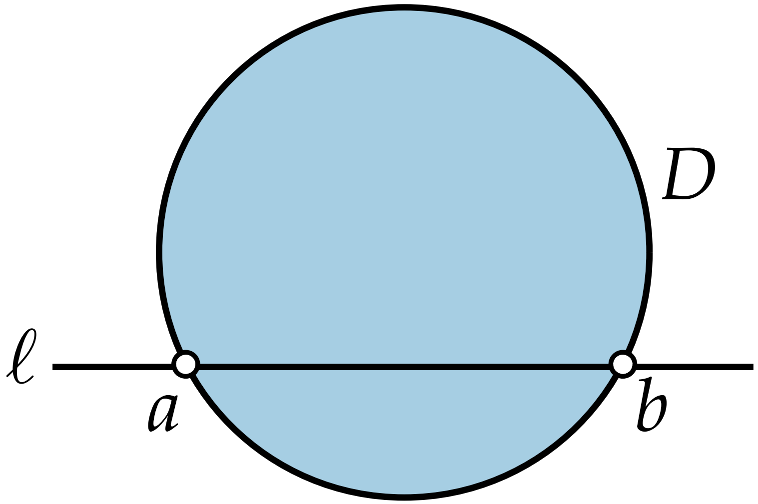


$$\{a, b\} := \ell \cap \partial D \ (a \neq b)$$

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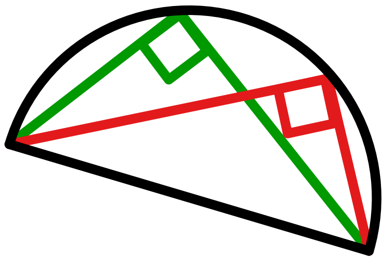

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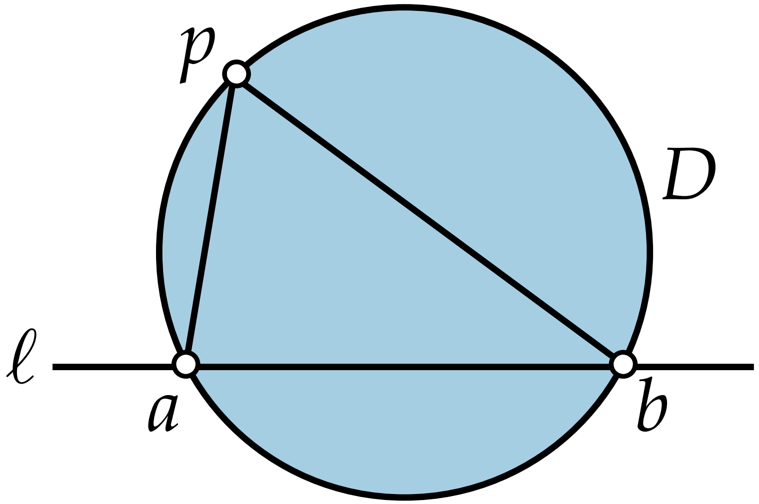


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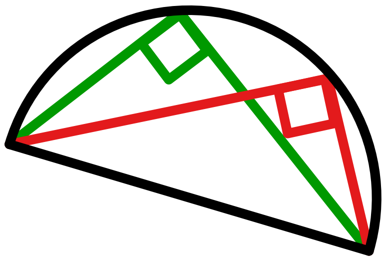

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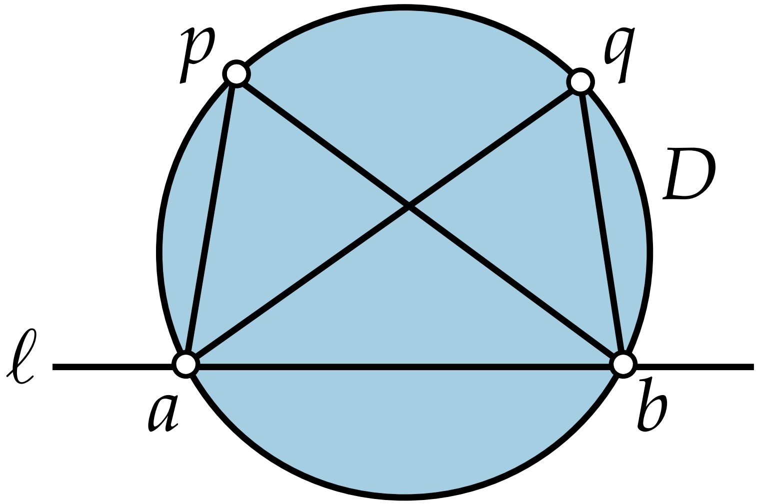


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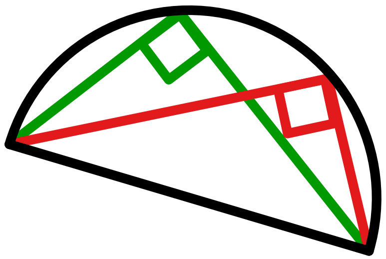

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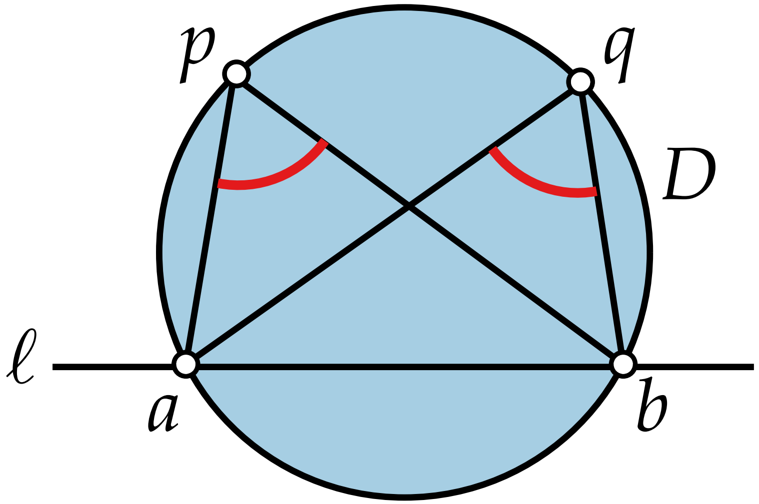


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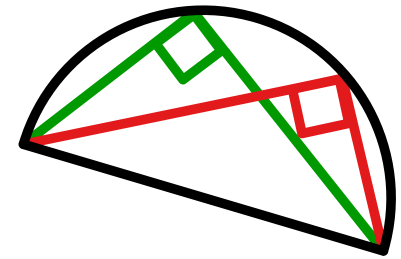

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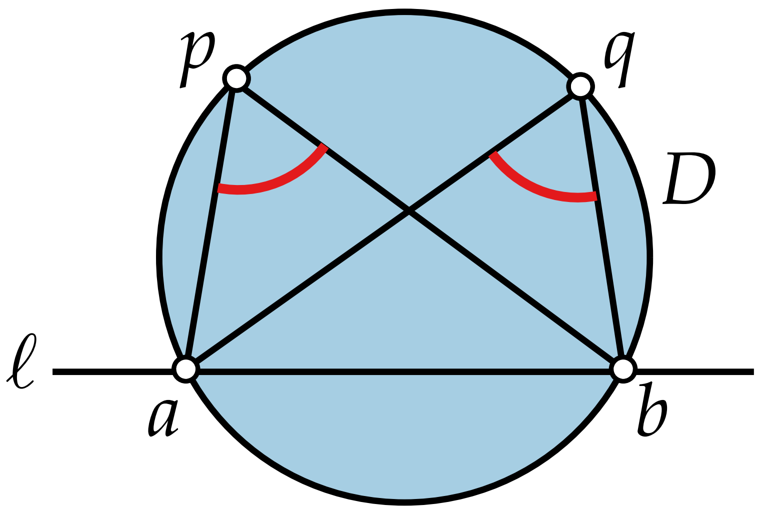


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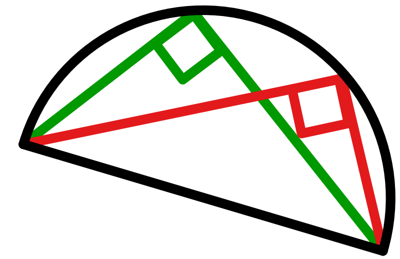

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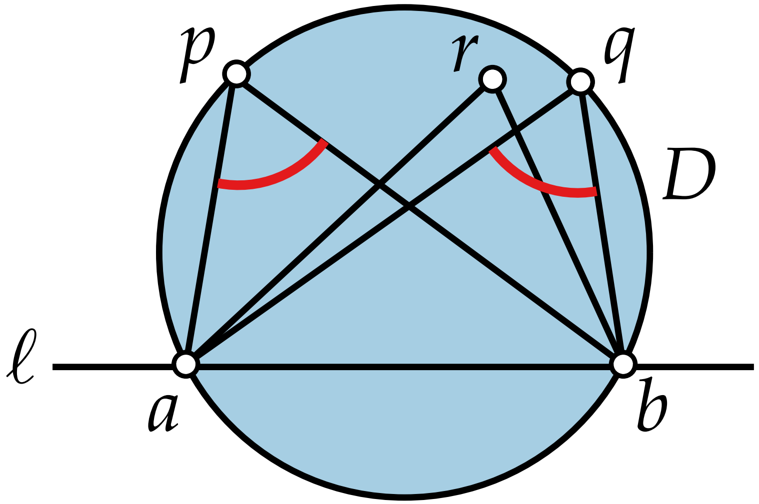


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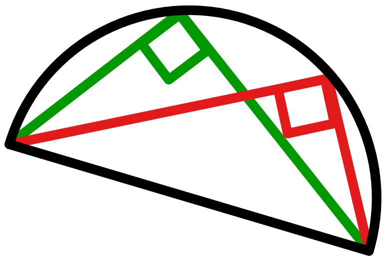

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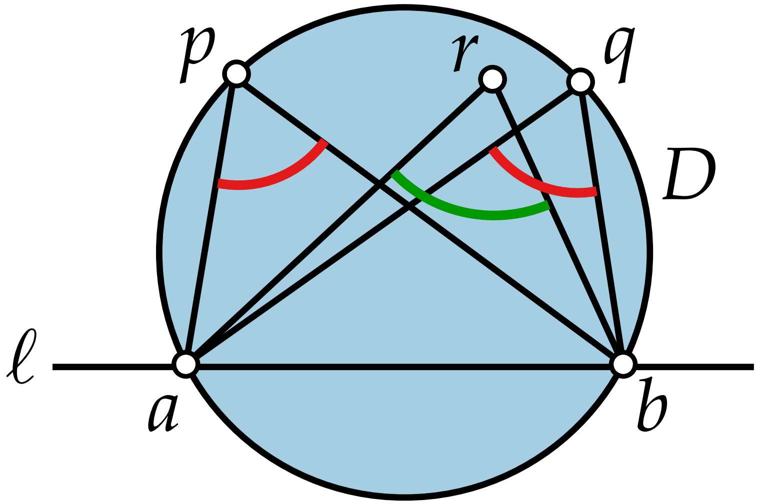


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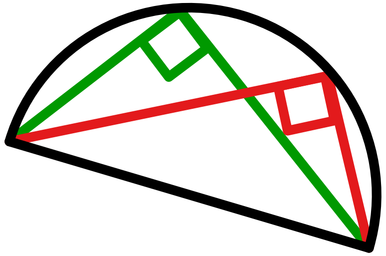

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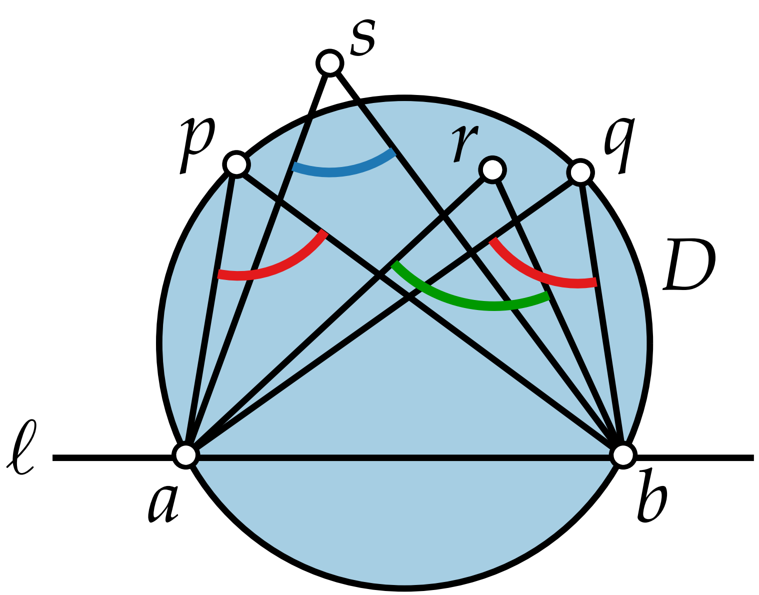


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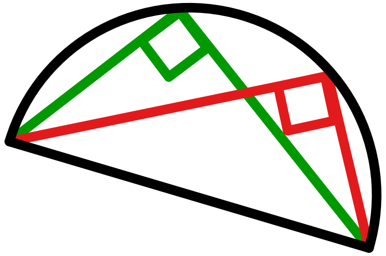

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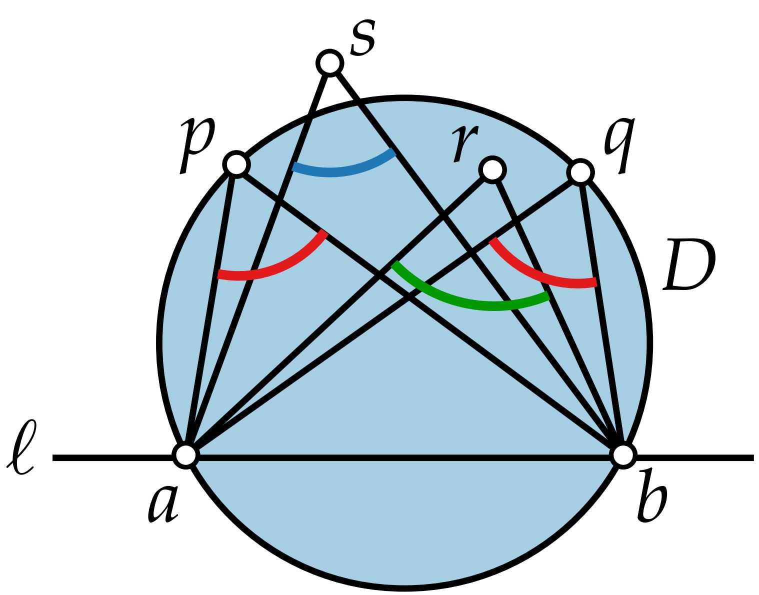


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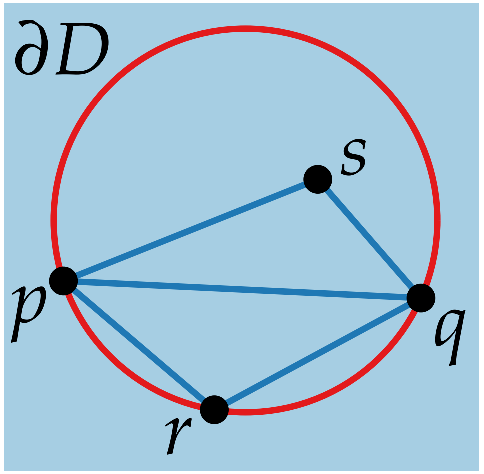
# Legal Triangulations

**Lemma:** Let $\Delta prq, \Delta pqs \in \mathcal{T}$ and $p, q, r \in \partial D$.
Then edge $pq$ is illegal iff $s \in \text{int}(D)$.



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If $p, q, r, s$ in convex position and $s \notin \partial D$,
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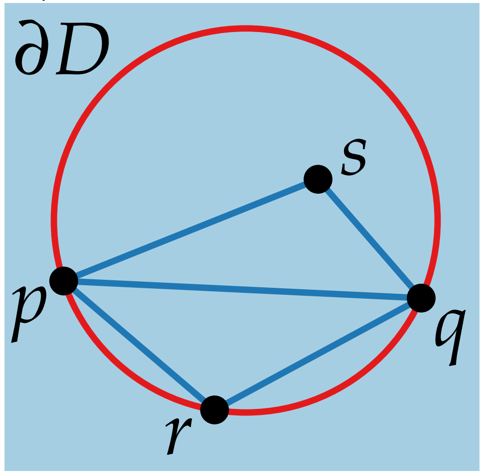

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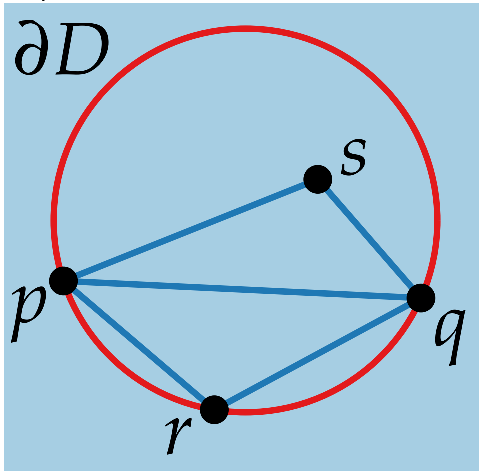


**Proof:**

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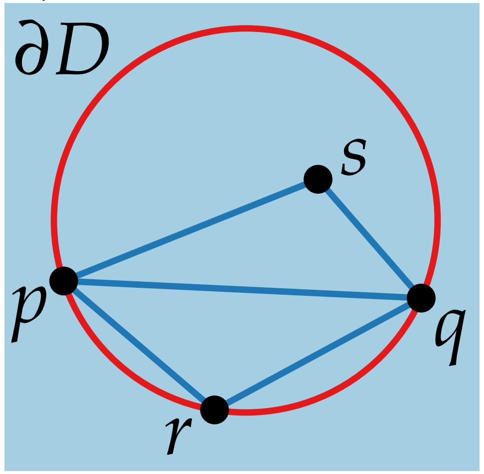


**Proof:** Show: $\forall \alpha'$ in $\mathcal{T}'$ $\exists \alpha$ in $\mathcal{T}$ s.t. $\alpha < \alpha'$.
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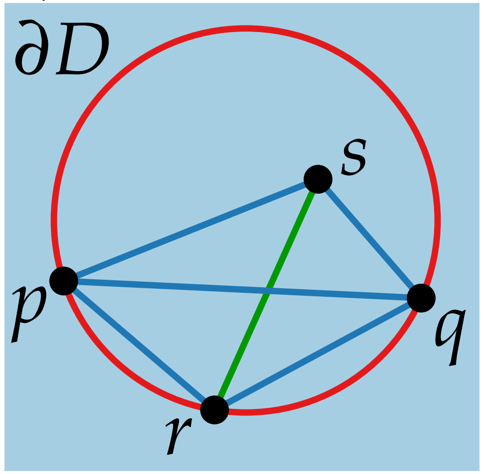


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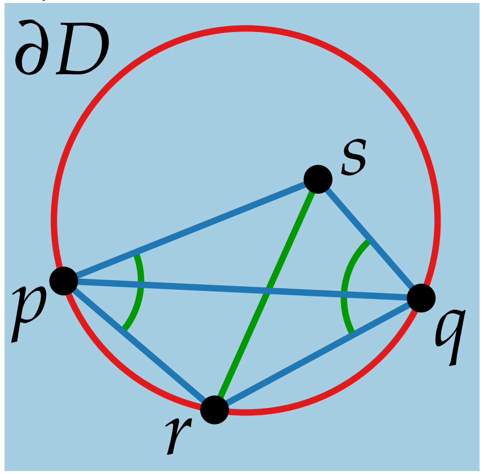


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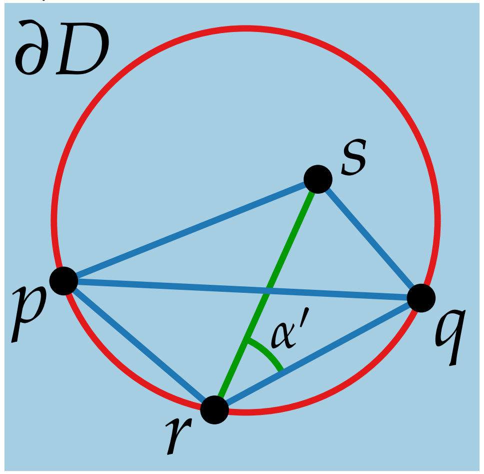


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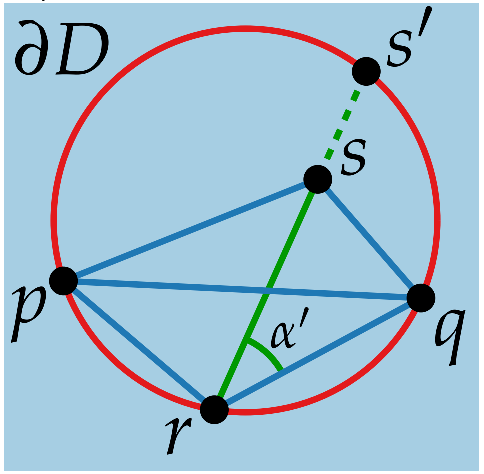


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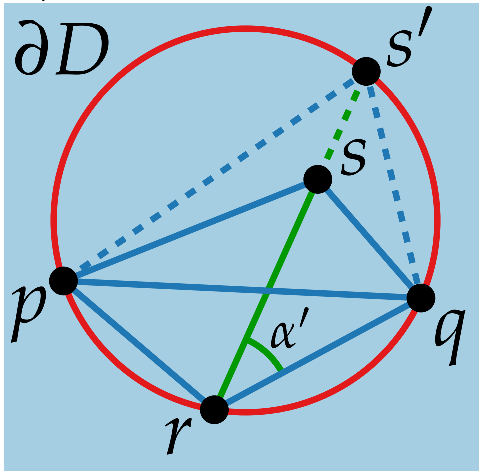


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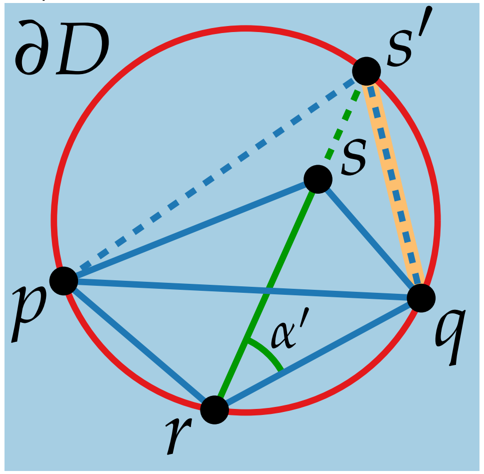


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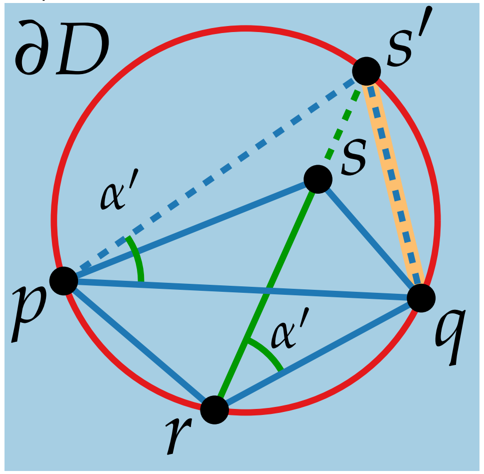


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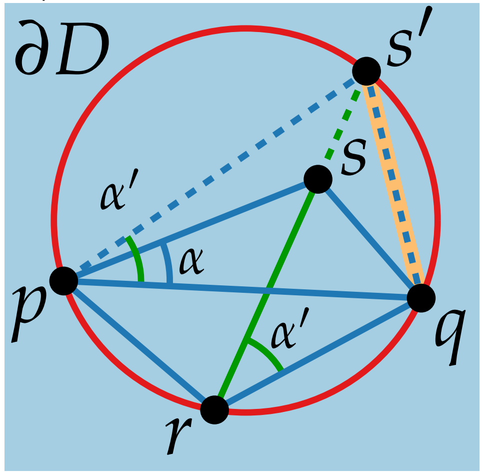


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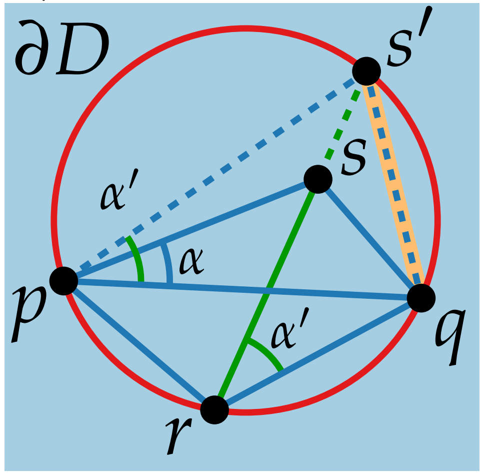

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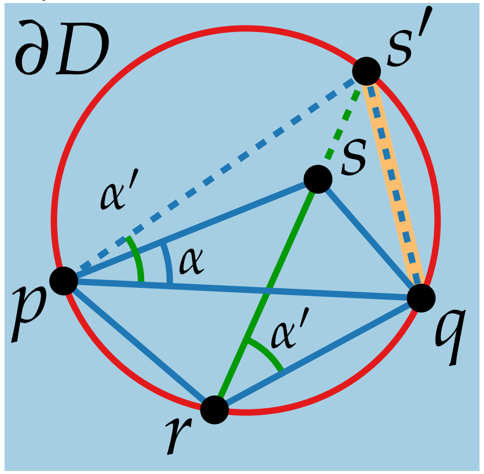


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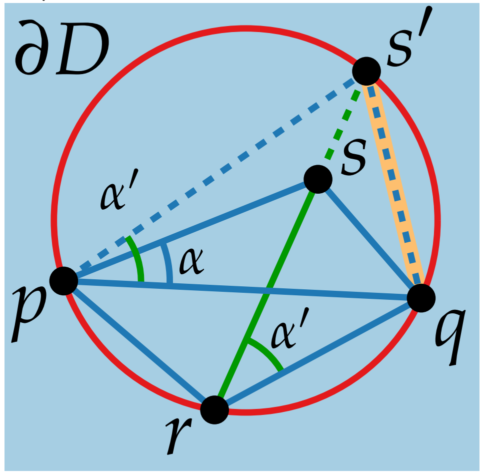


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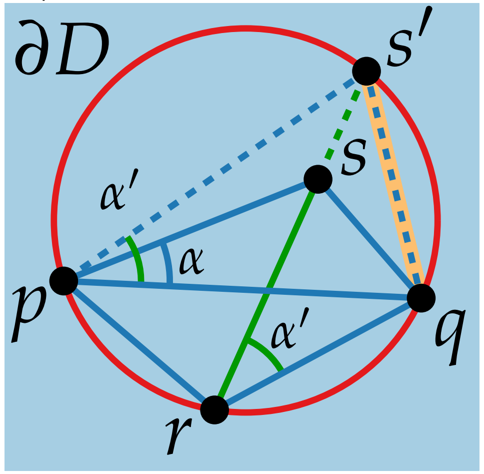


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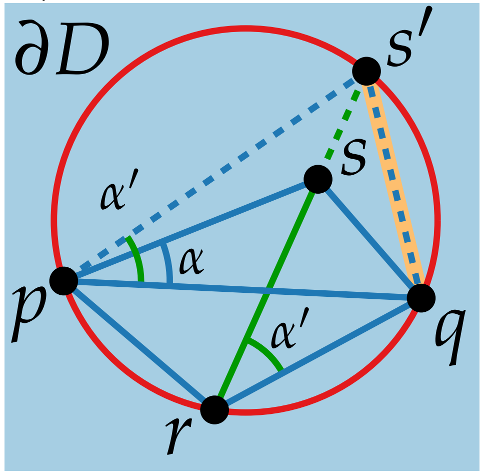


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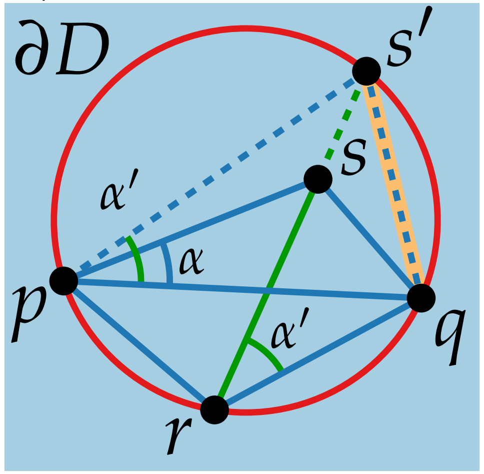


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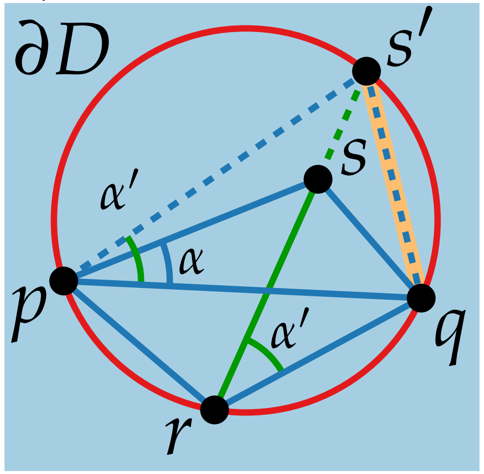


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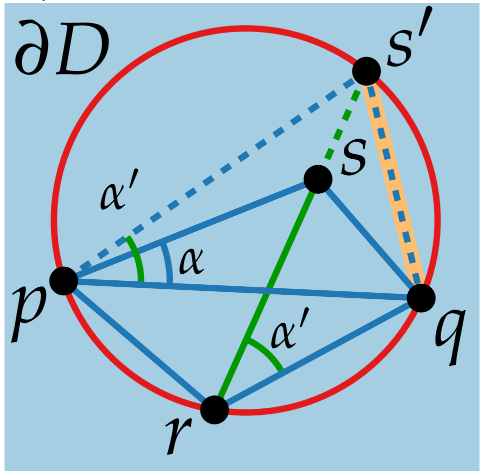


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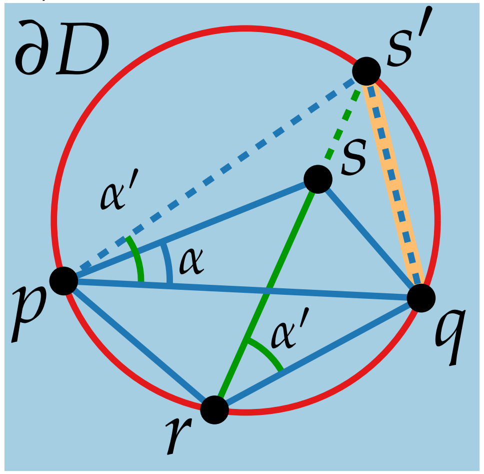


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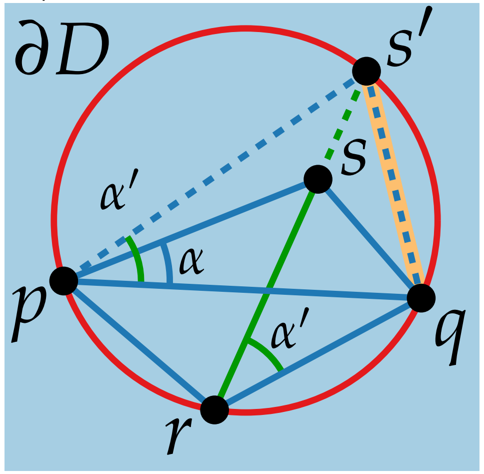


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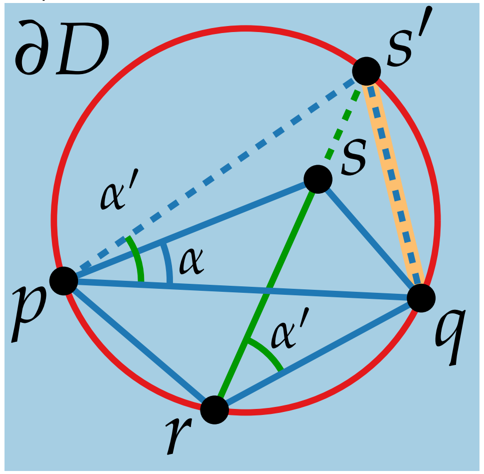


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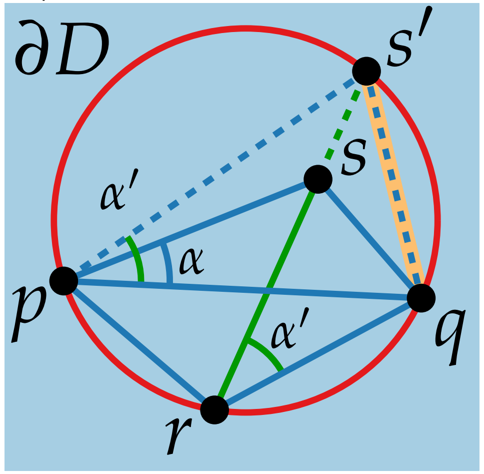


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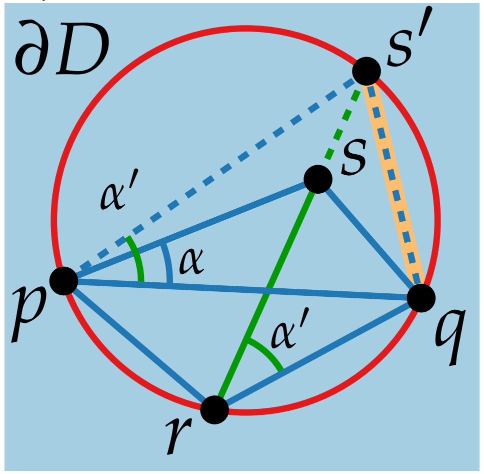


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To clarify things, we'll introduce yet another type of triangulation. . .

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**Definition:** The *Delaunay graph* $\mathcal{DG}(P)$ is the straight-line drawing of $\mathcal{G}$.

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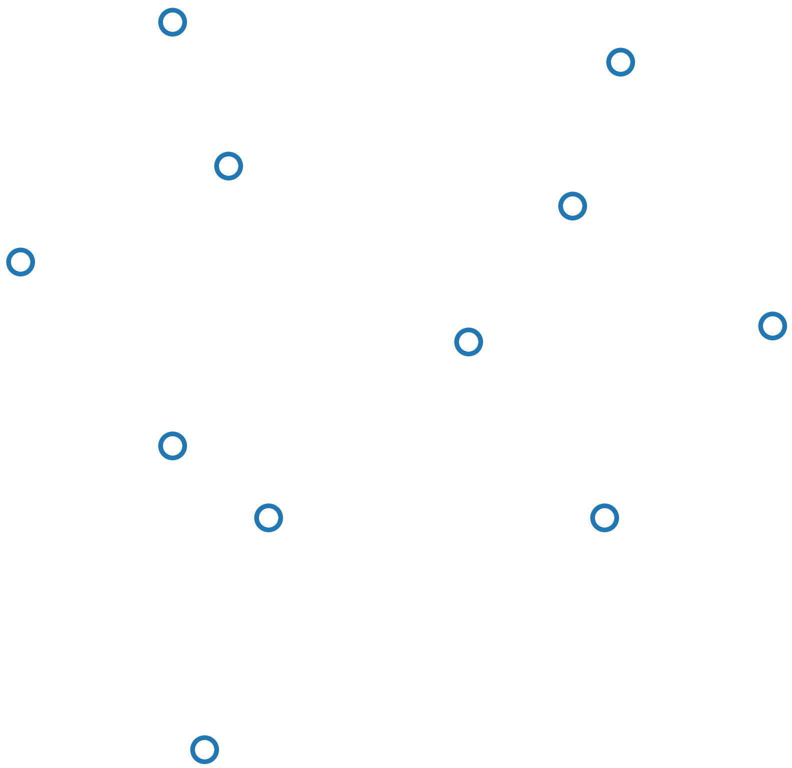
$P \subset \mathbb{R}^2$



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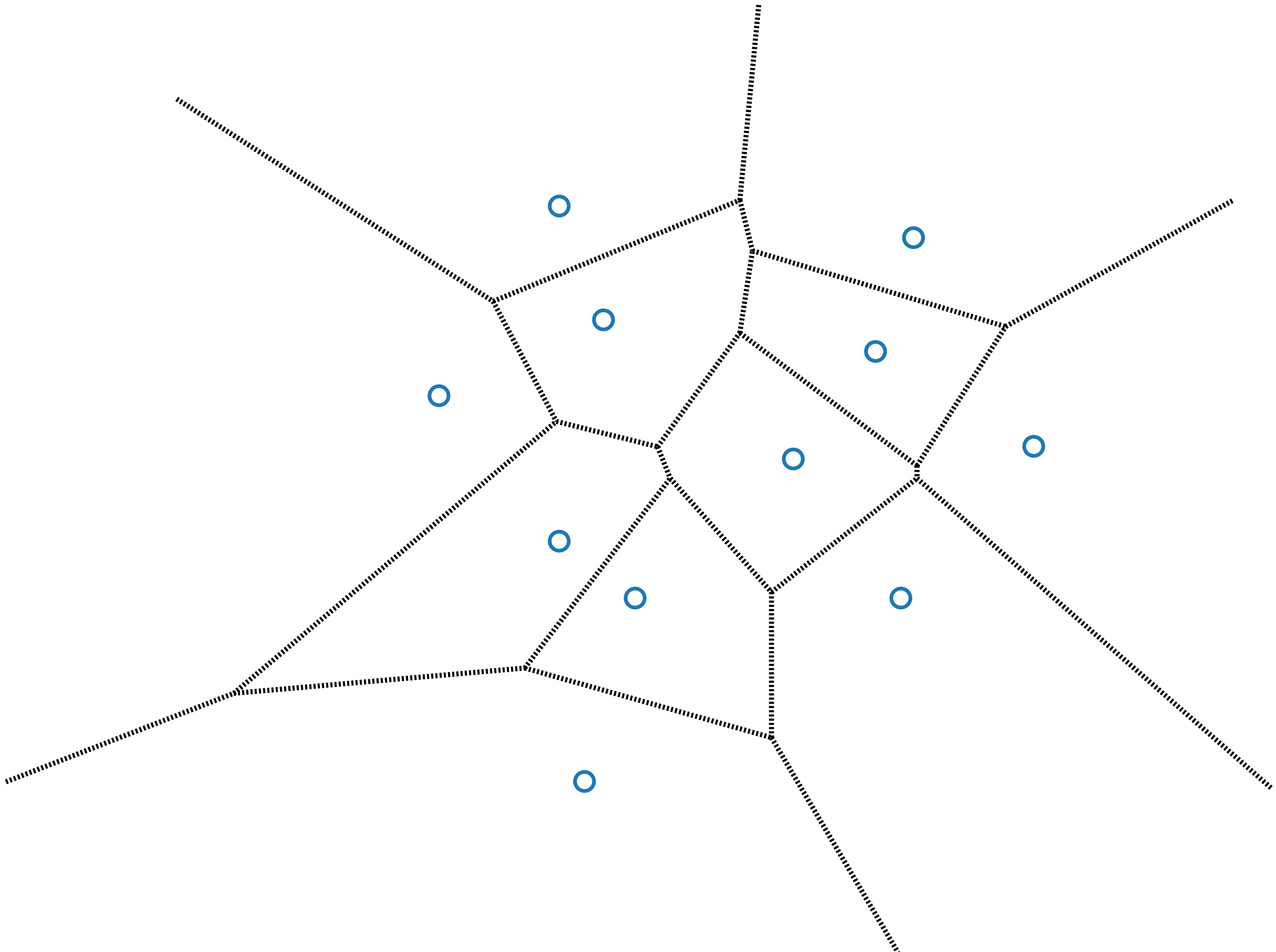

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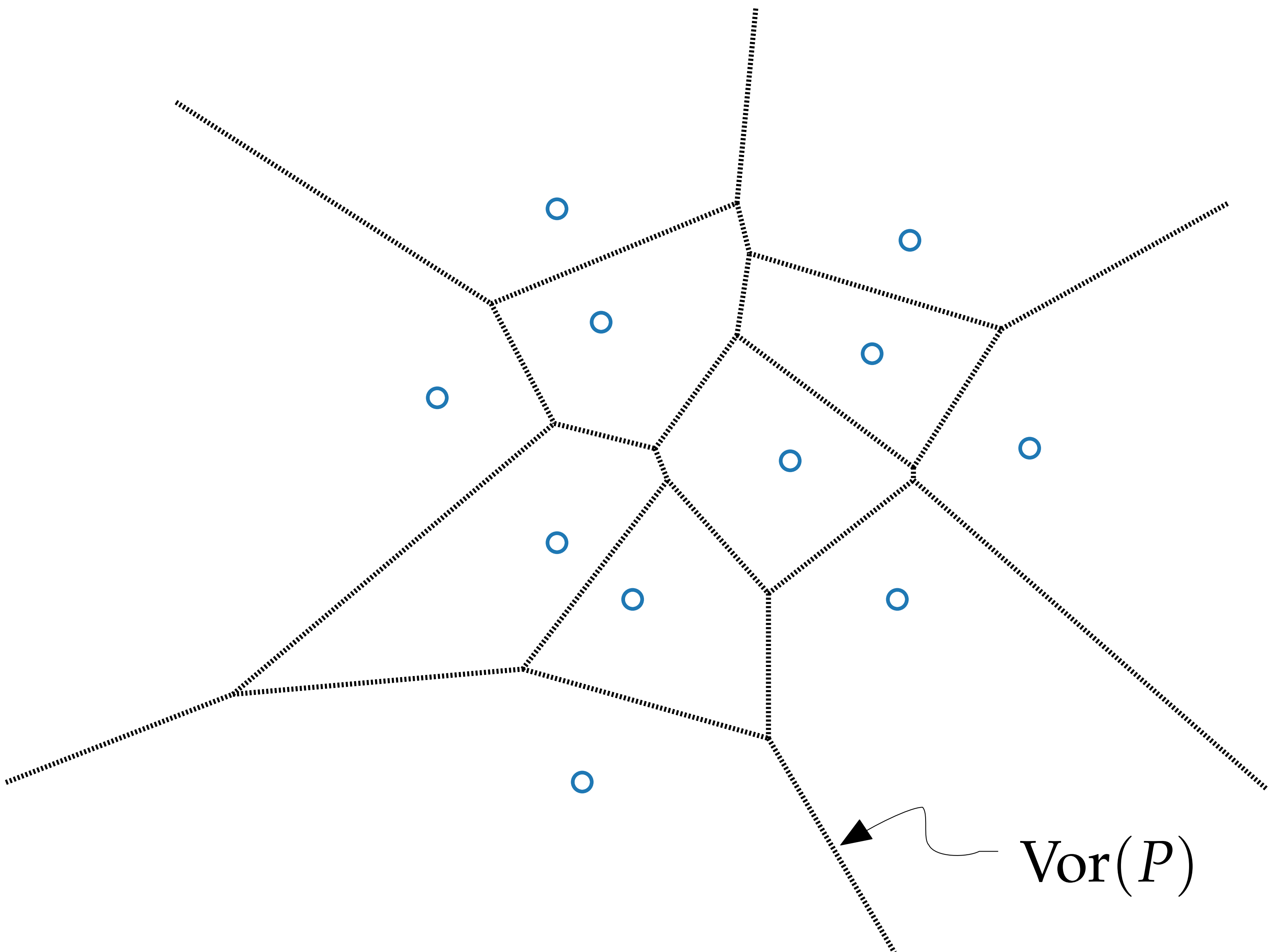

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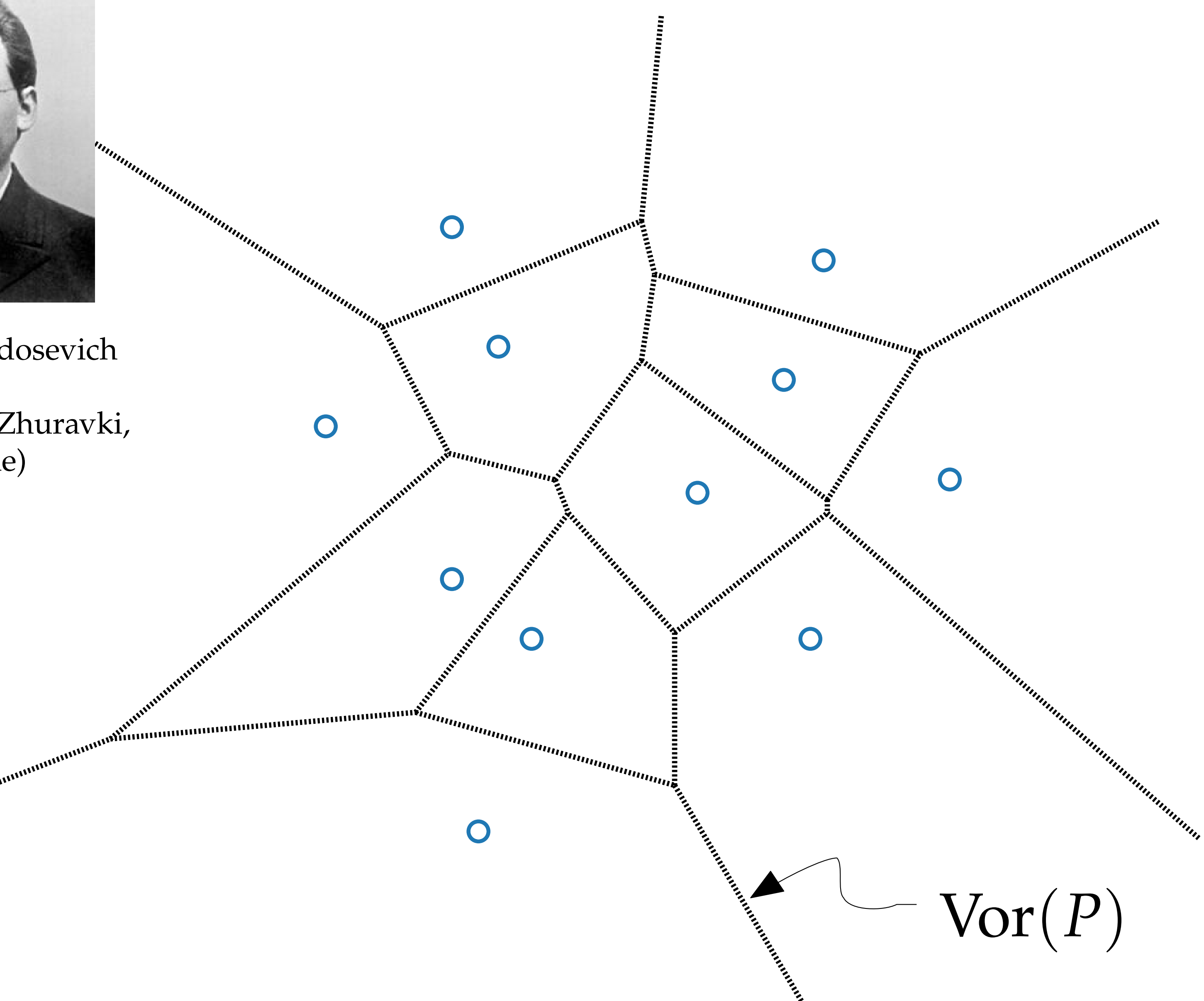

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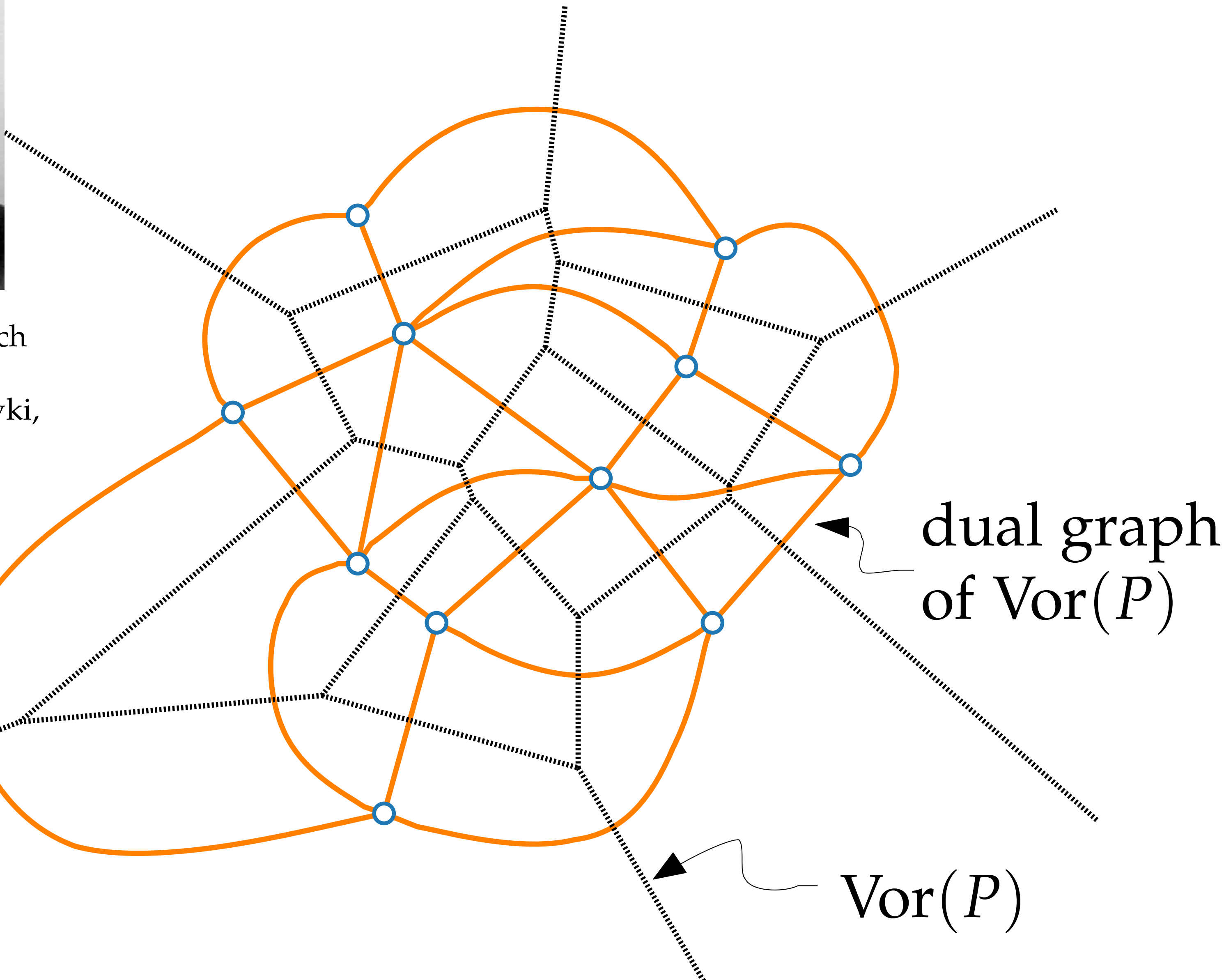

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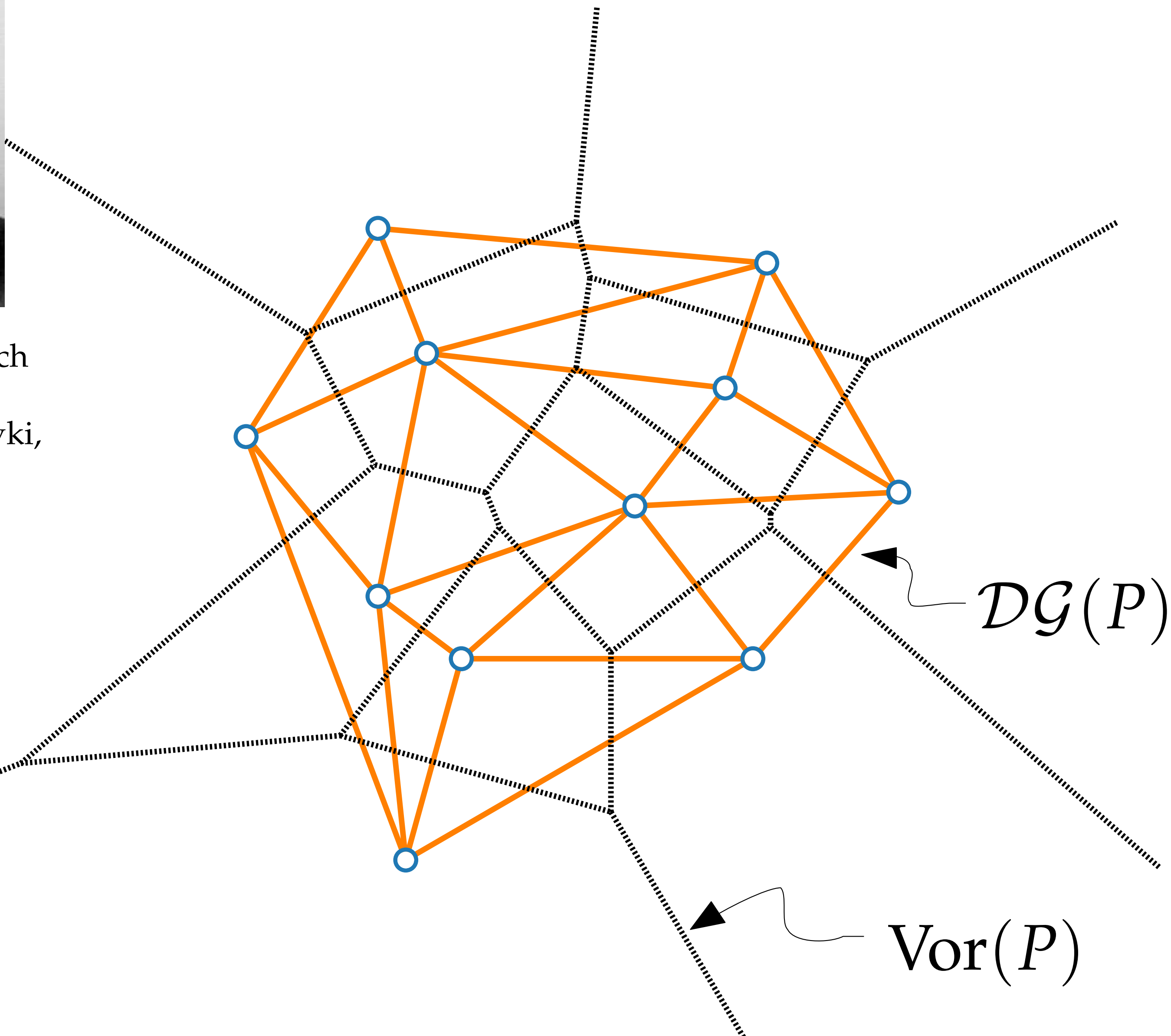

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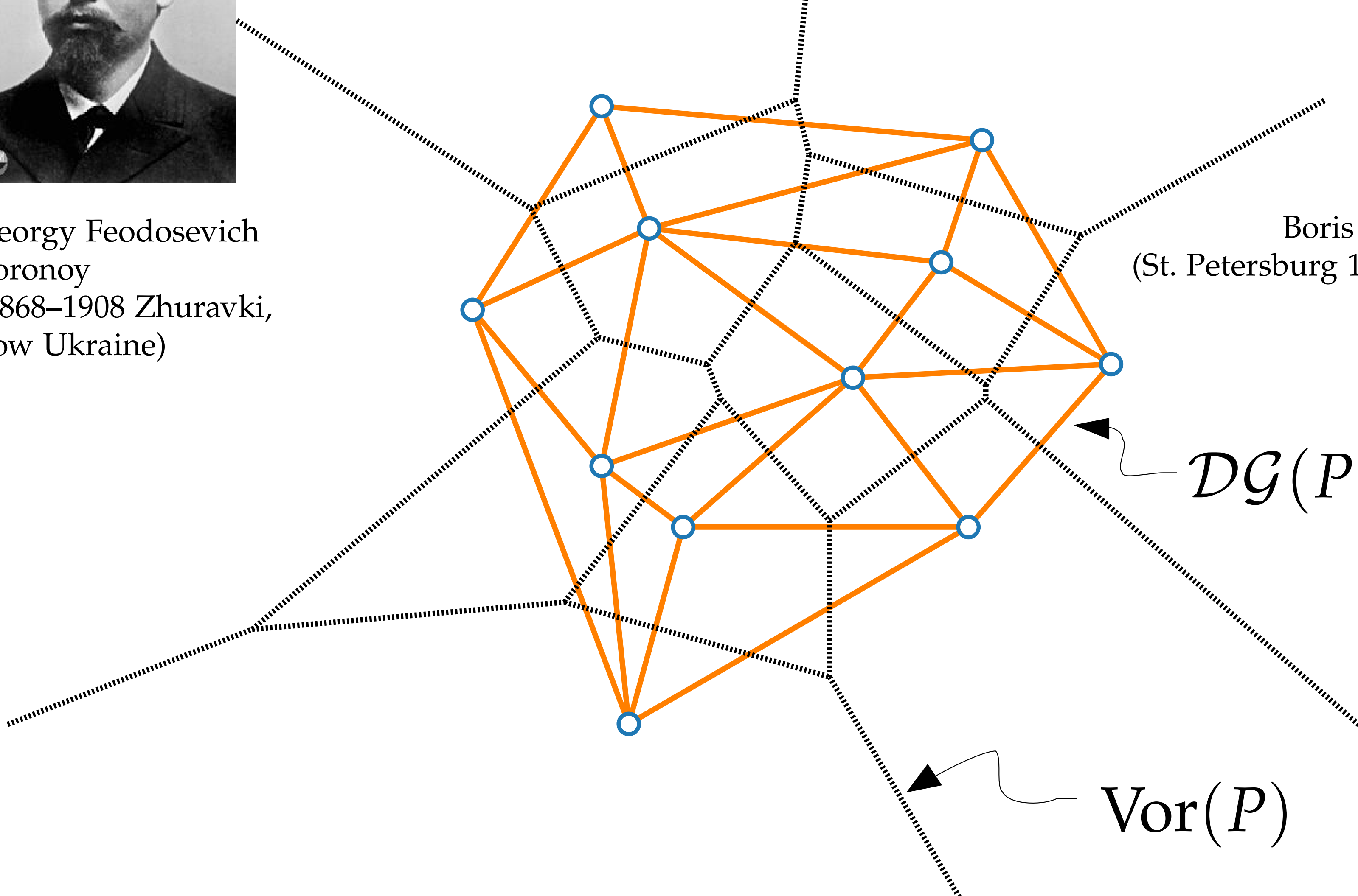



Boris Nikolaevich Delone (St. Petersburg 1890–1980 Moscow)

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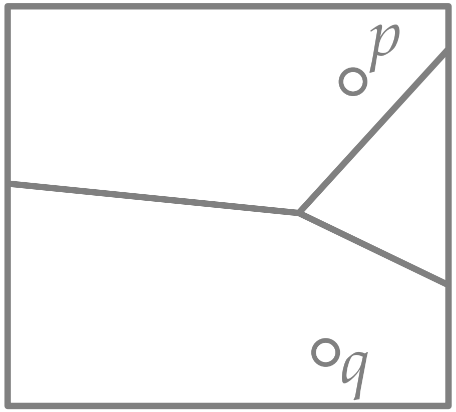

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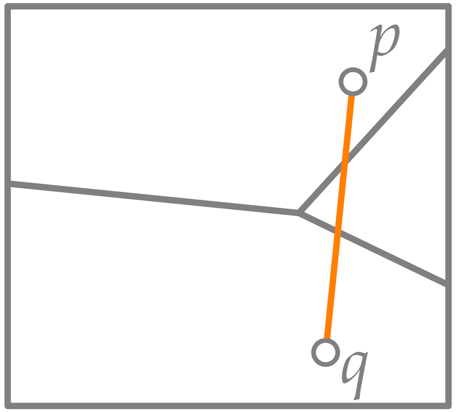

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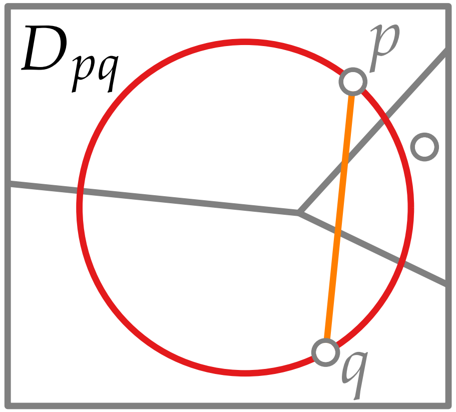

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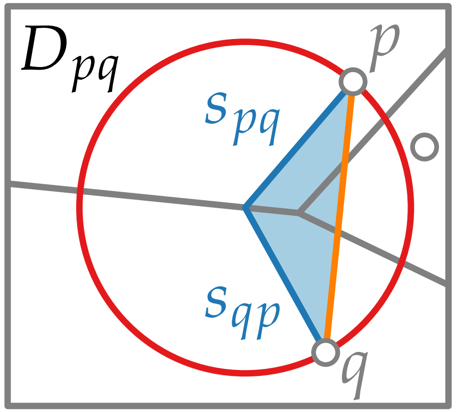

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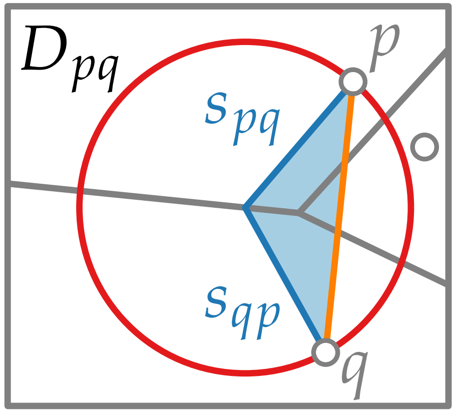

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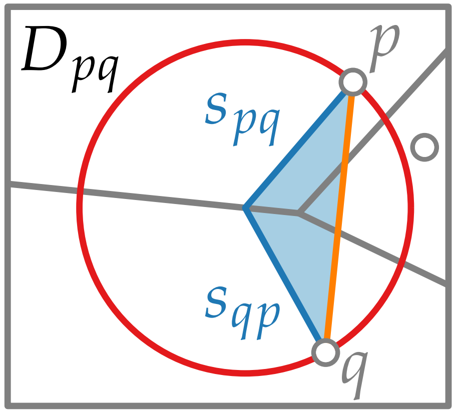

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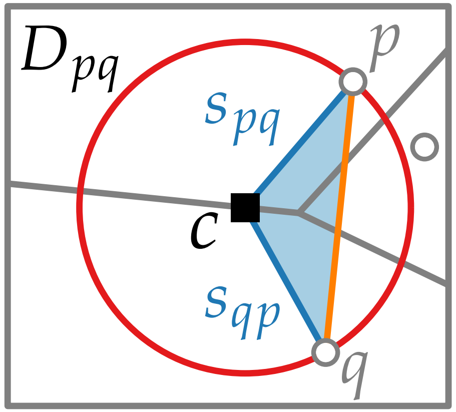


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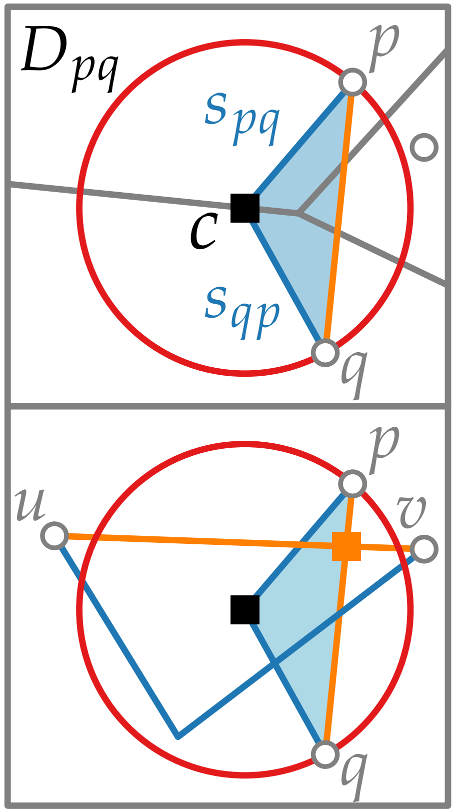


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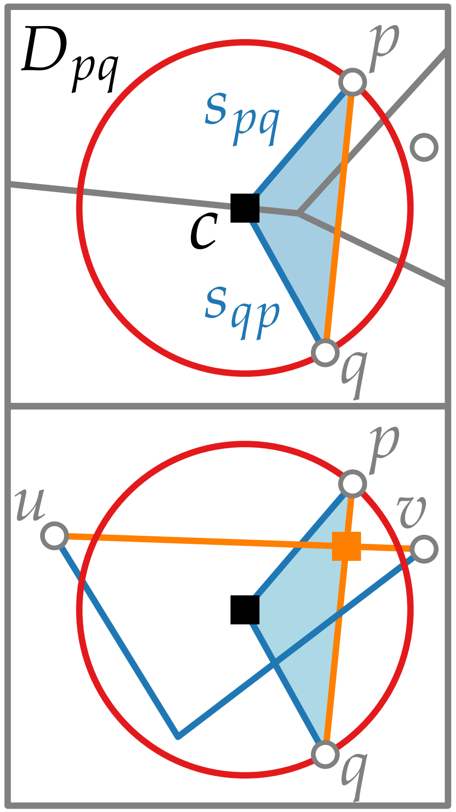


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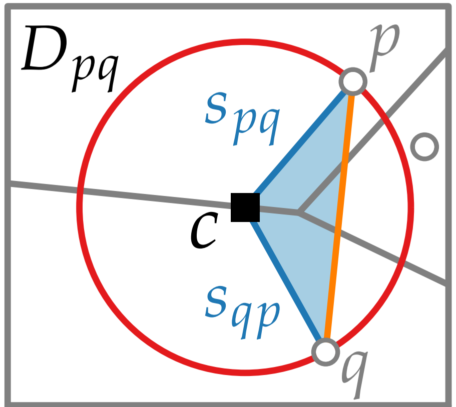


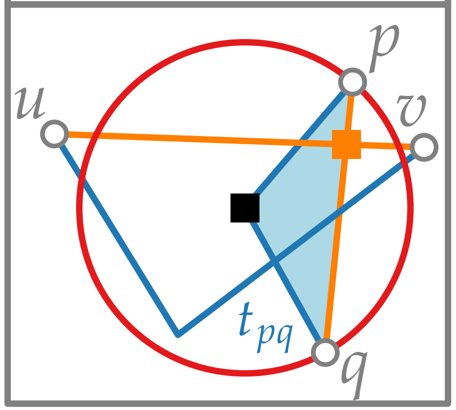


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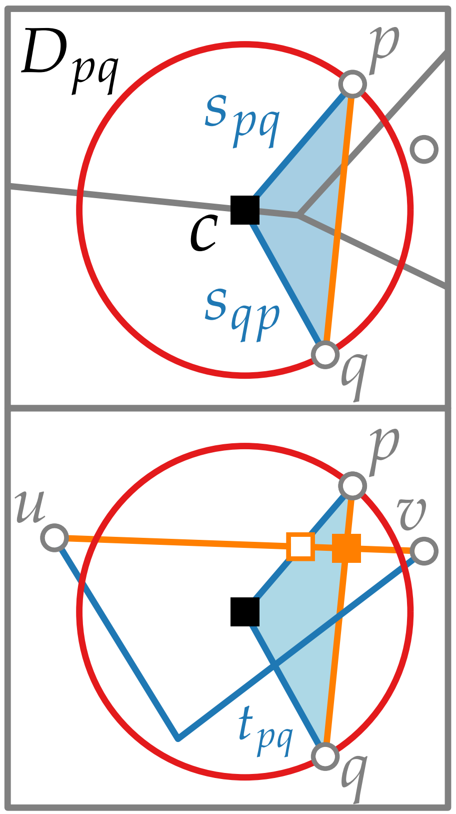


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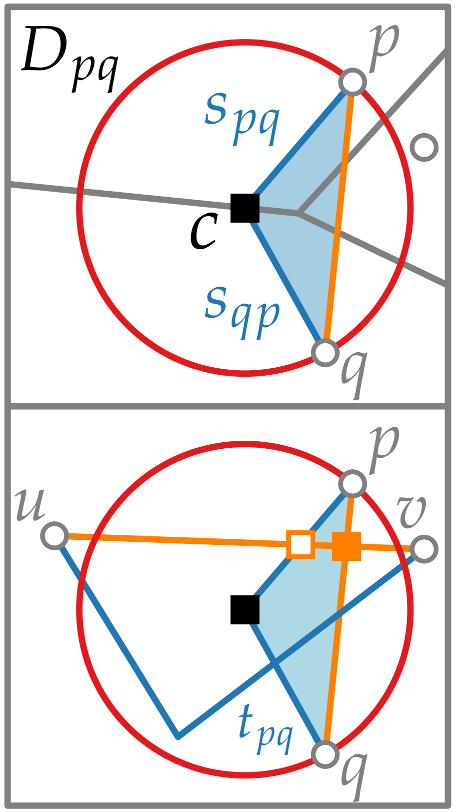


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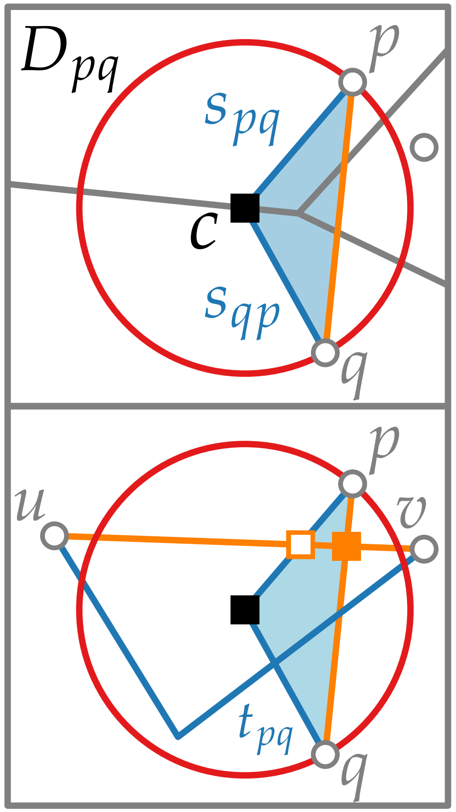


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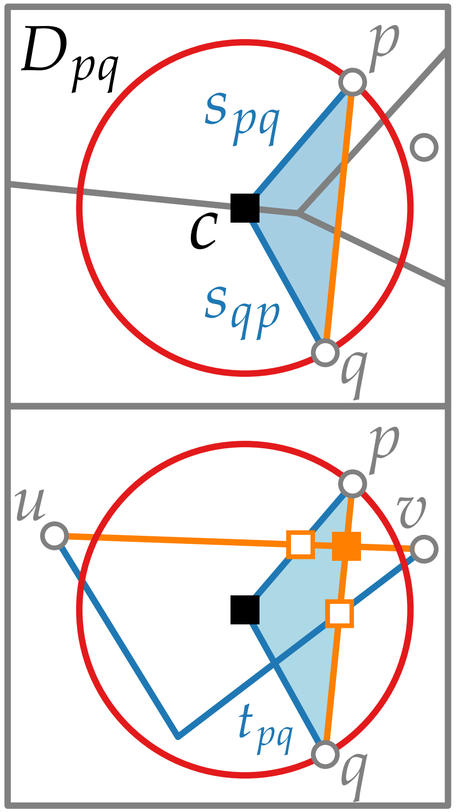


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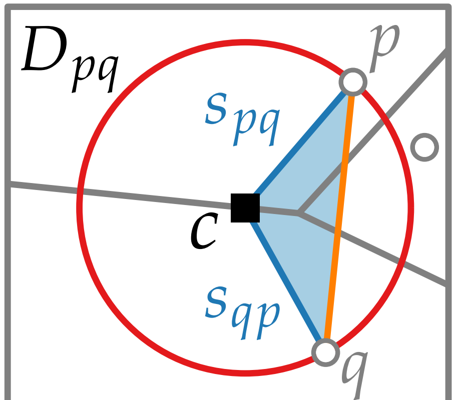


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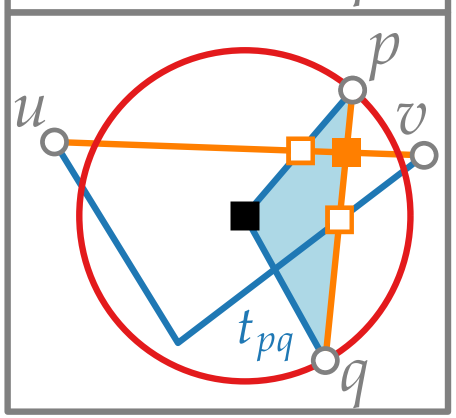


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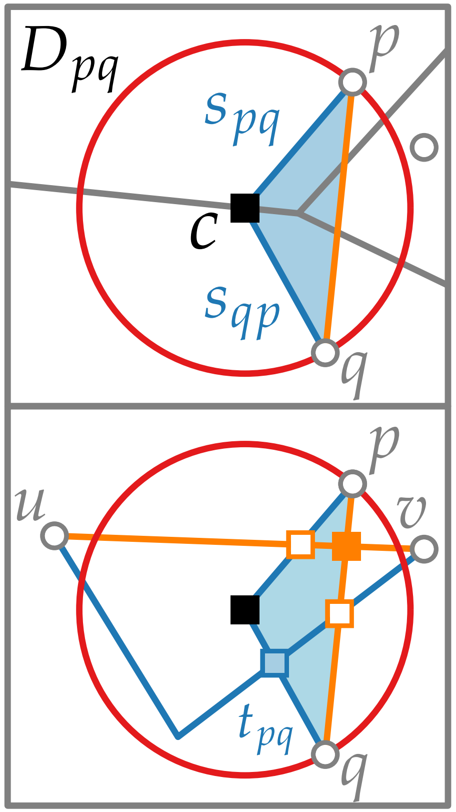


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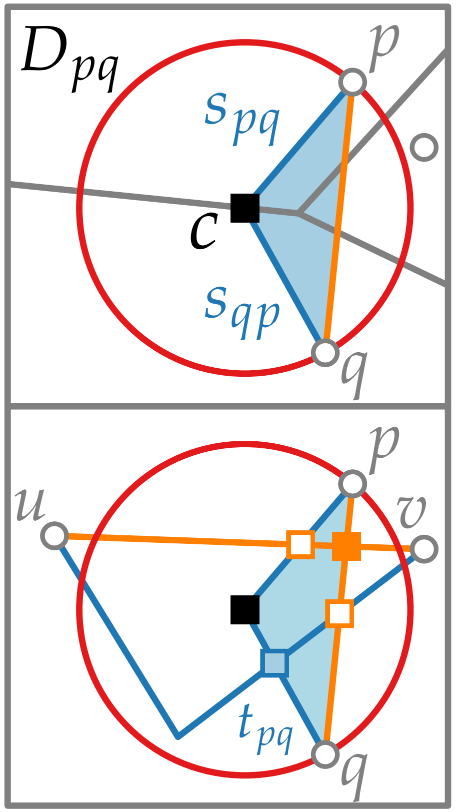


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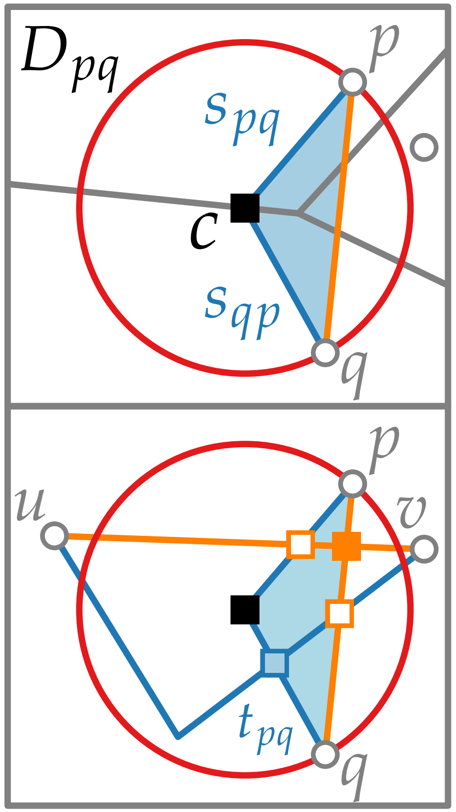


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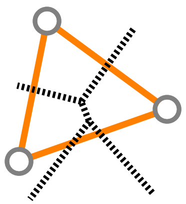

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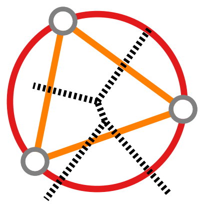

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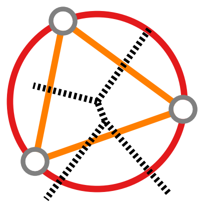

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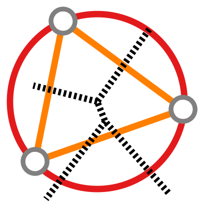

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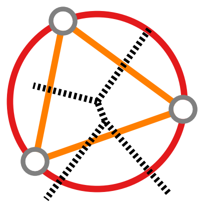

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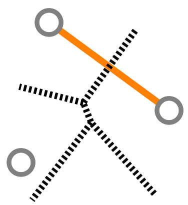

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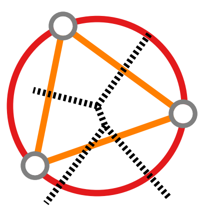

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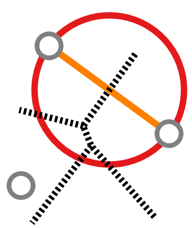

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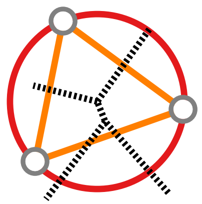

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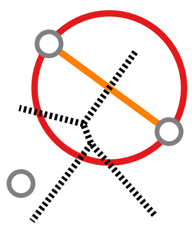

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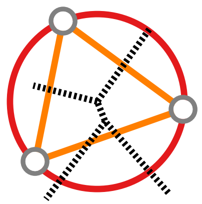

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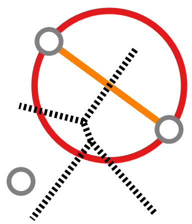

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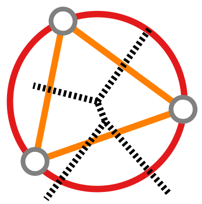

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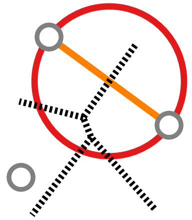

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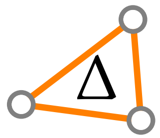
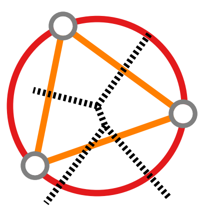


# Characterization

Characterization of Voronoi vertices and Voronoi edges $\Rightarrow$

**Theorem.** $P \subset \mathbb{R}^2$ finite. Then



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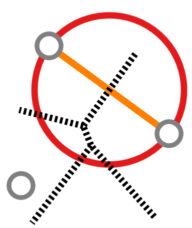

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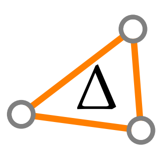

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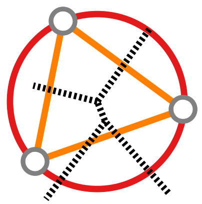

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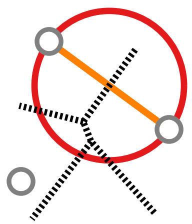

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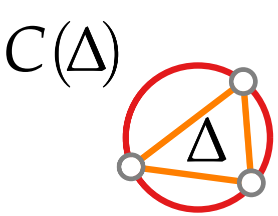

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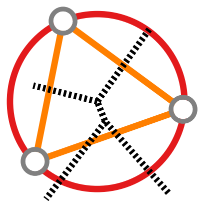

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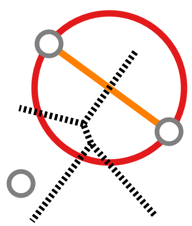

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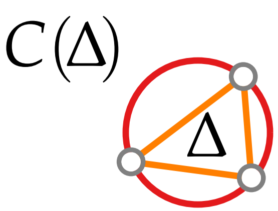


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(“empty-circumcircle property”)

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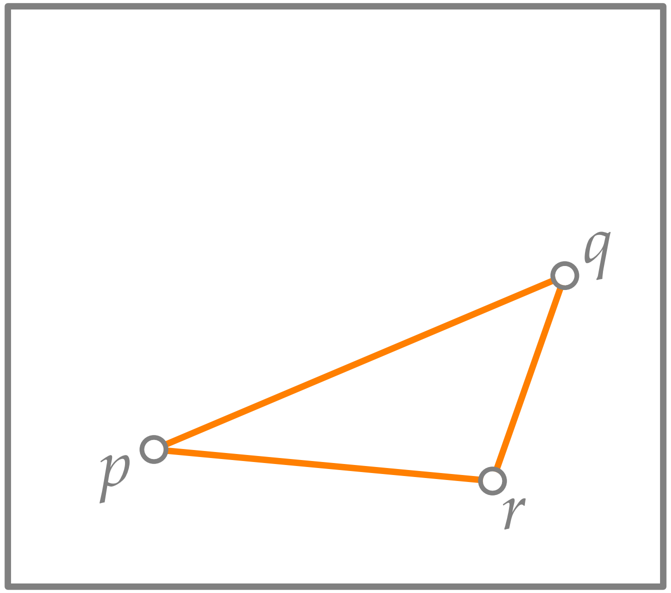

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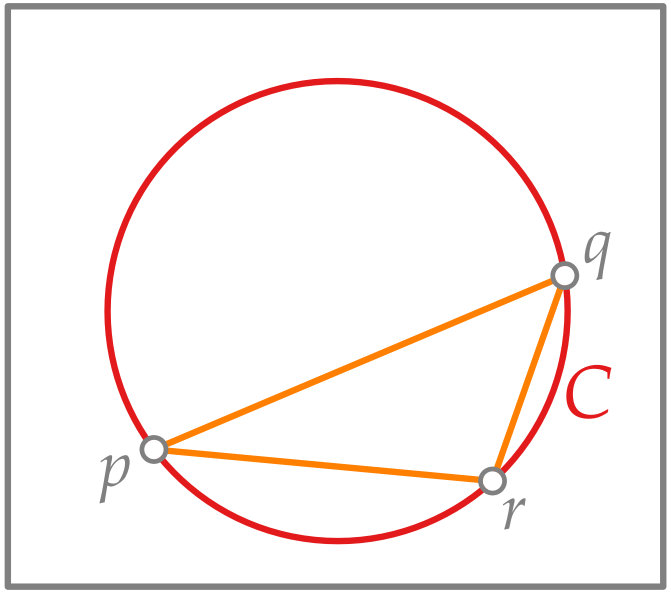

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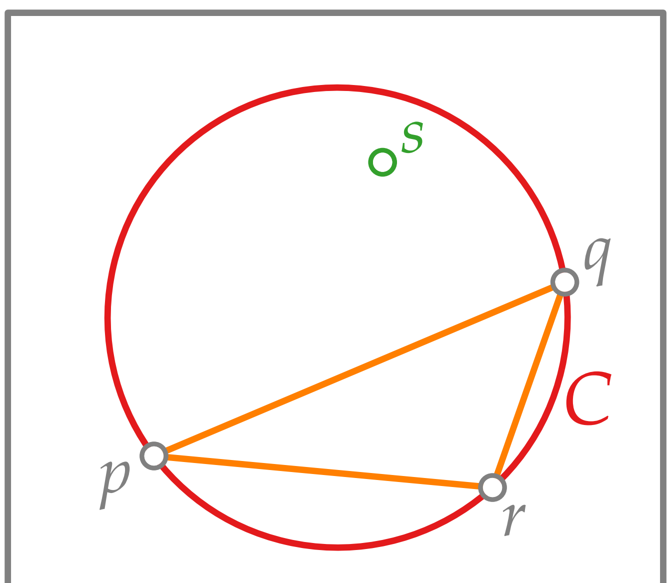

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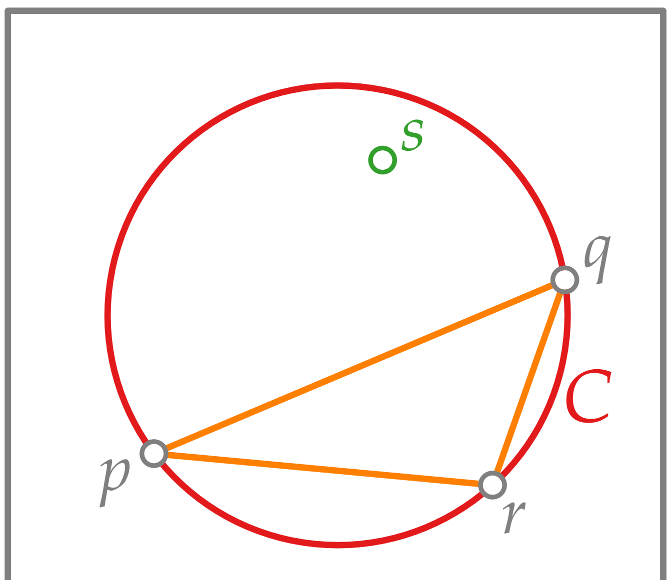


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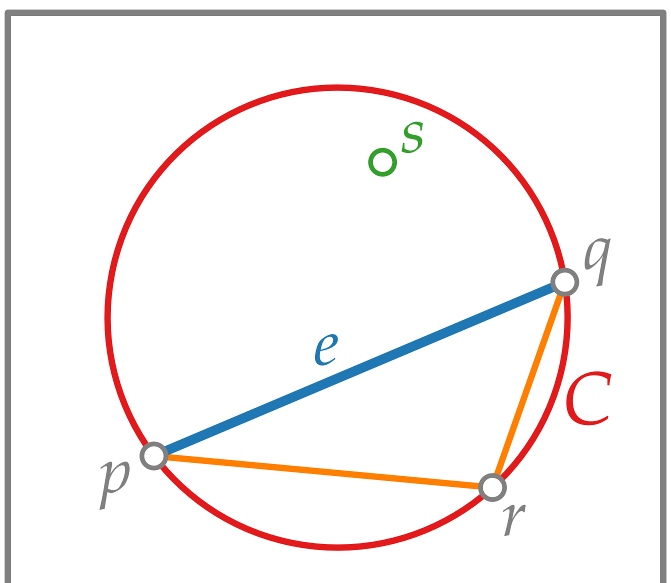


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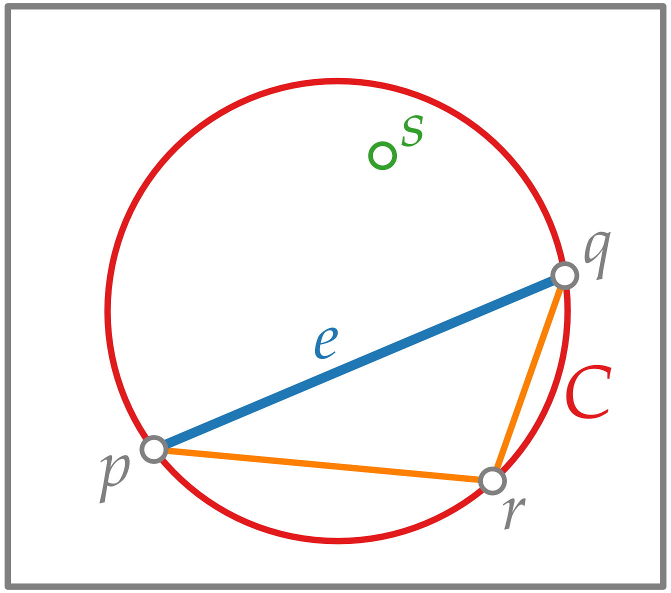


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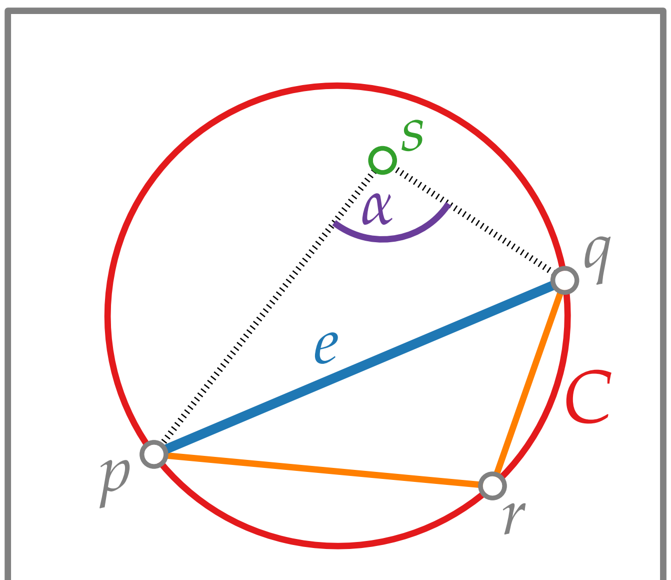


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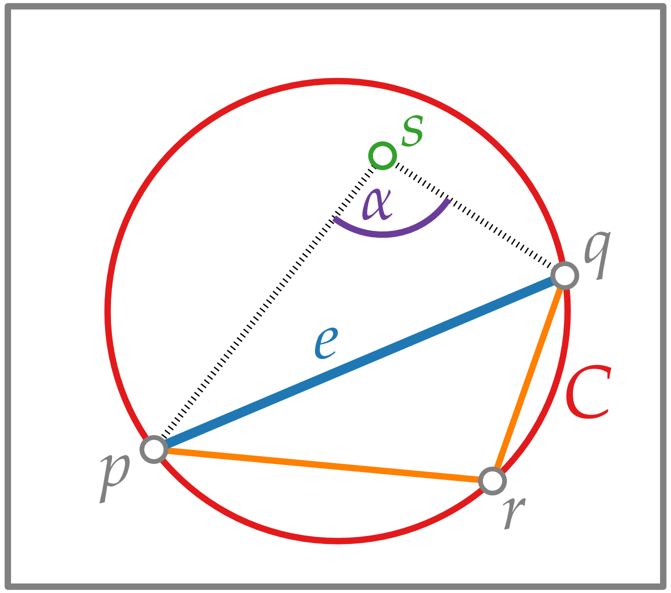

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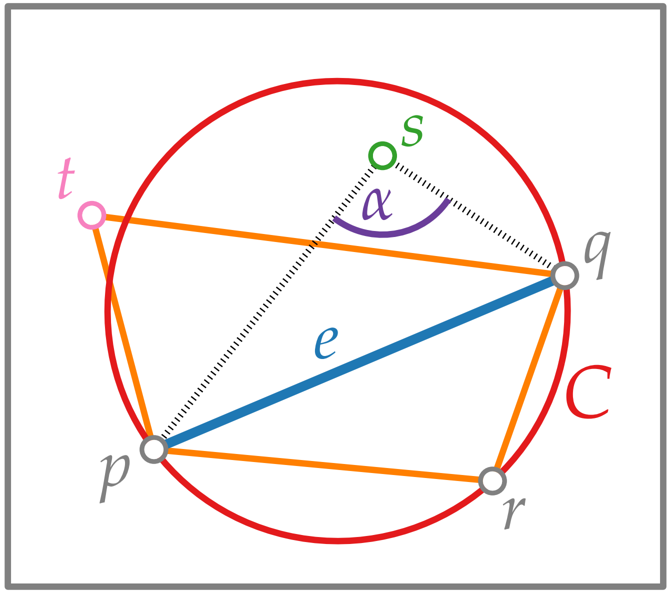

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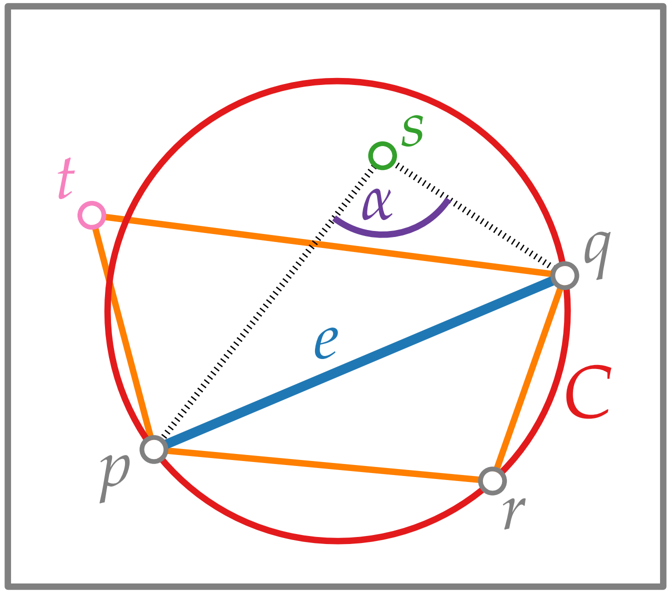

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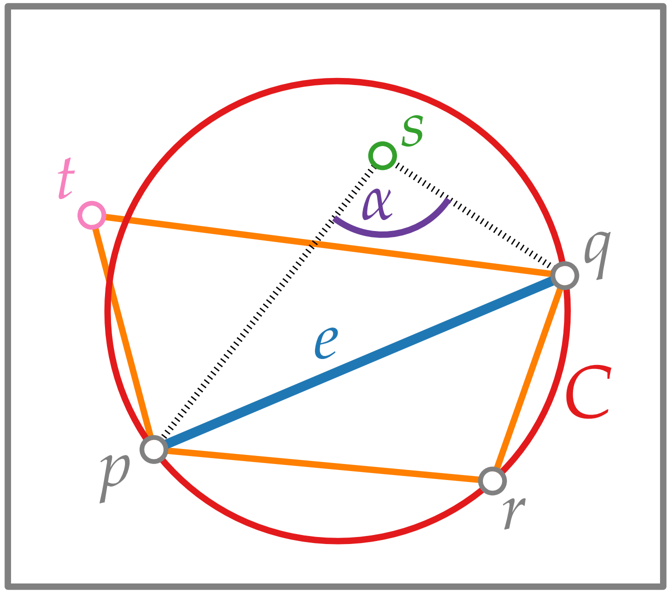

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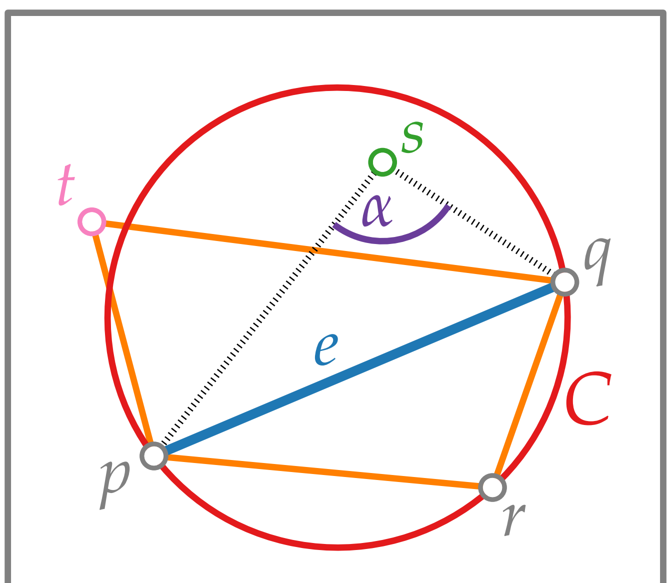

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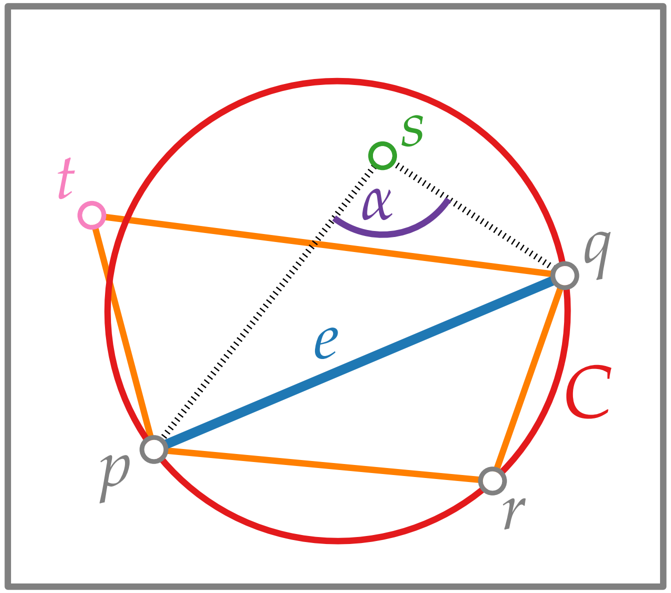

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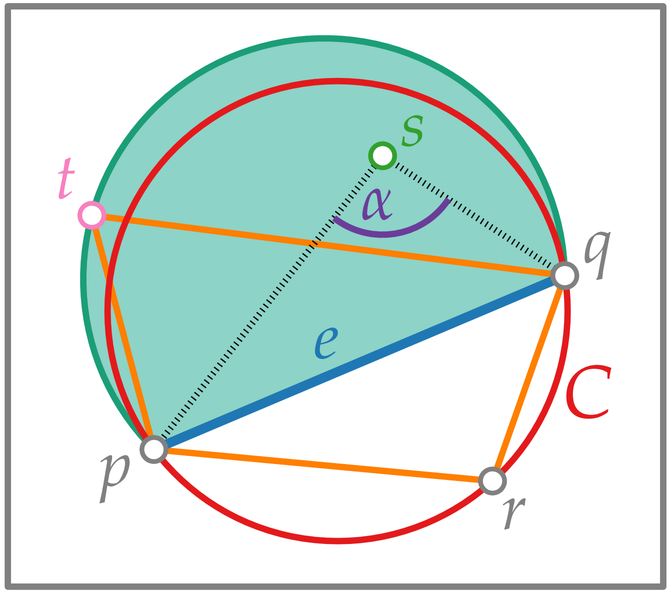

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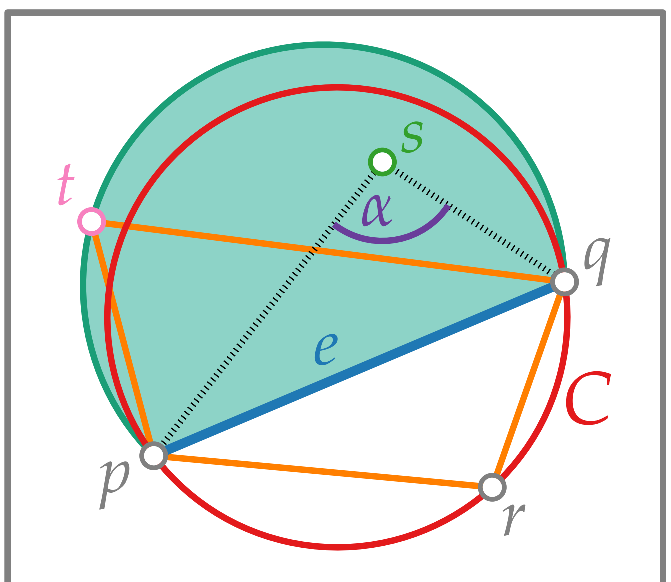

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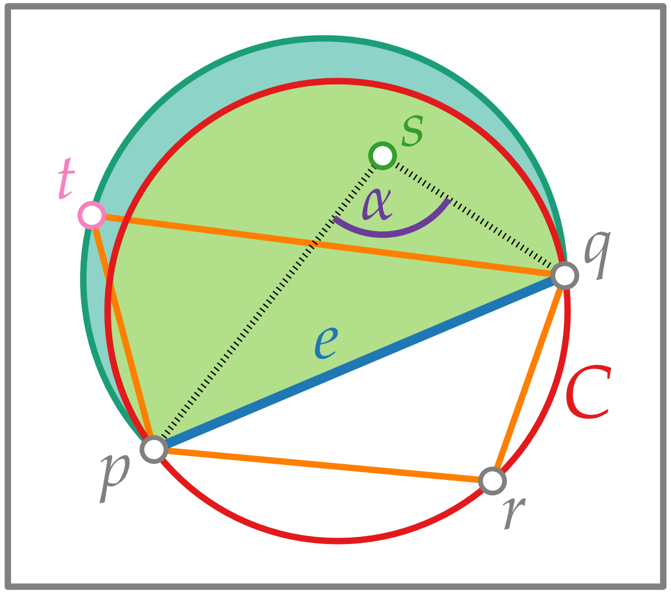

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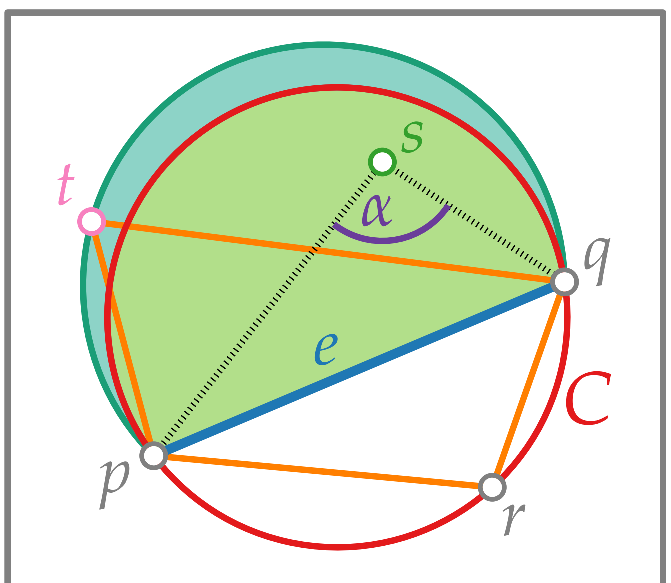

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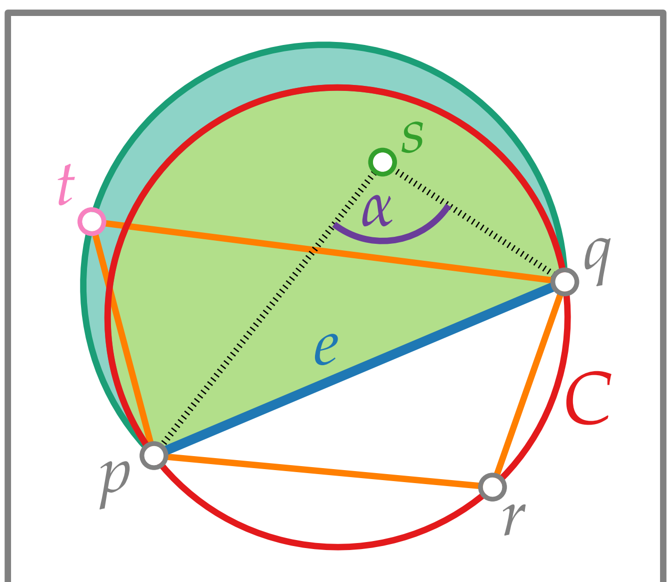

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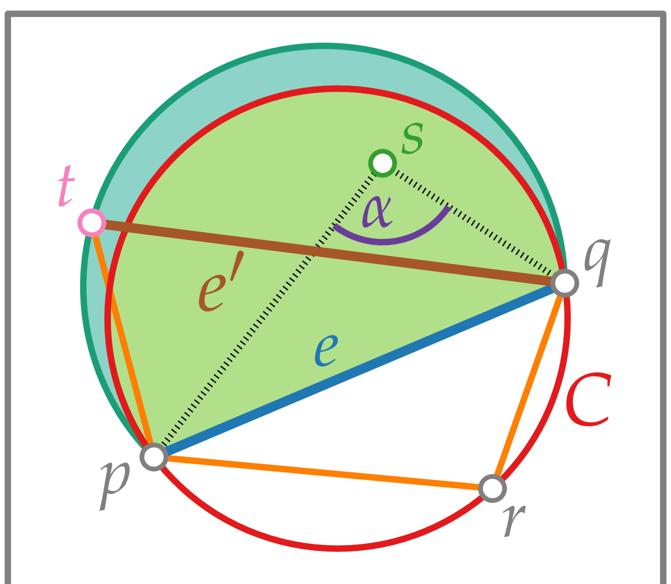

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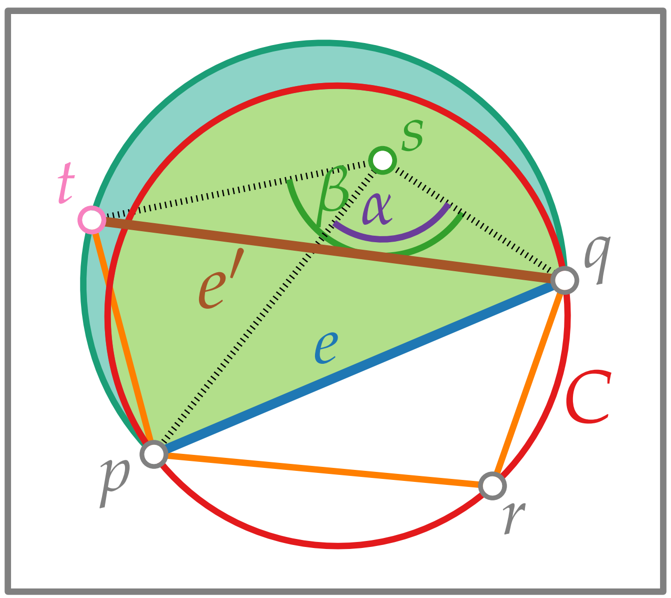

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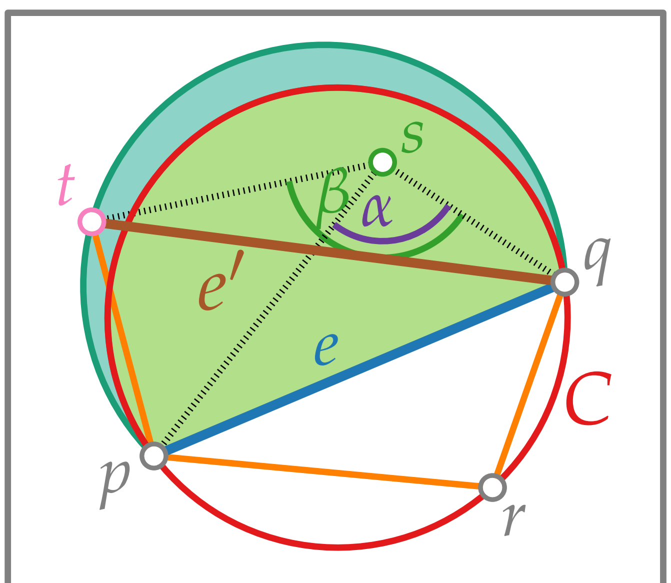


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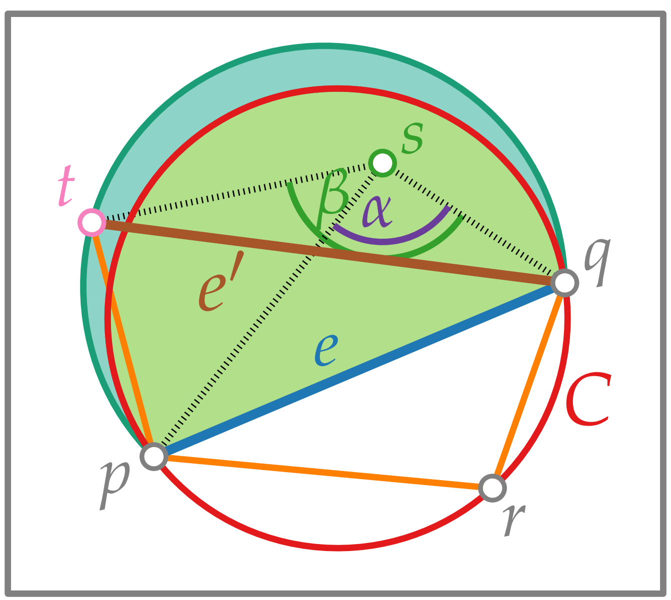


↯ Contradiction to choice of the pair $(\Delta pqr, s)$. □

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**Observation.** Suppose $P$ is in general position...

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All Delaunay triang. have same min. angle.

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**Fact.** A Delaunay triangulation of an arbitrary set of $n$ pts in the plane can be computed in $O(n \log n)$ time.

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