Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Lehrstuhl für
INFORMATIK I
Effiziente Algorithmen und
wissensbasierte Systeme

Institut für Informatik

Advanced Algorithms

Winter term 2019/20

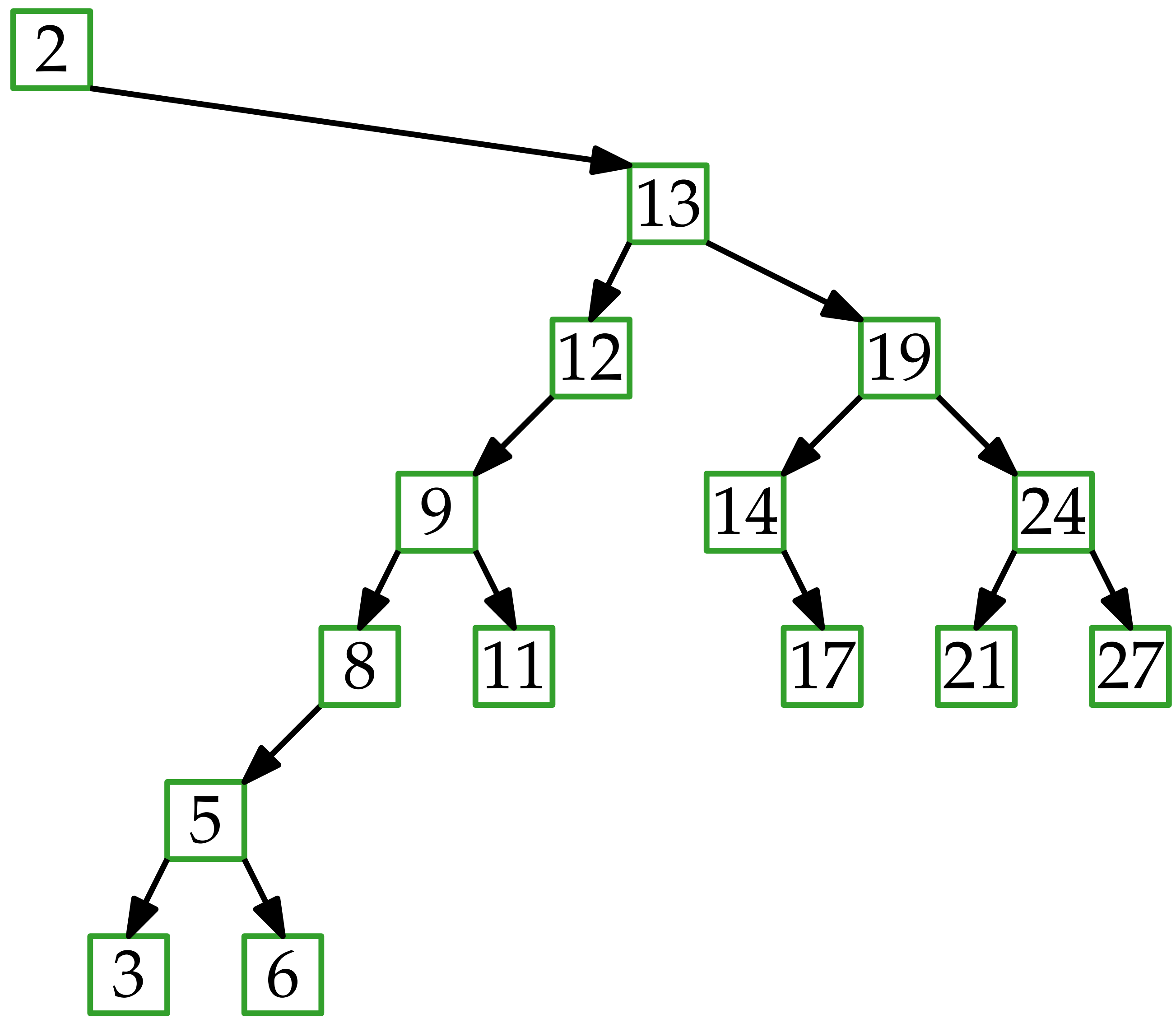
Lecture 8. Optimal binary search trees

Philipp Kindermann

Chair for Computer Science I

How good is a binary search tree?

Binary search tree:



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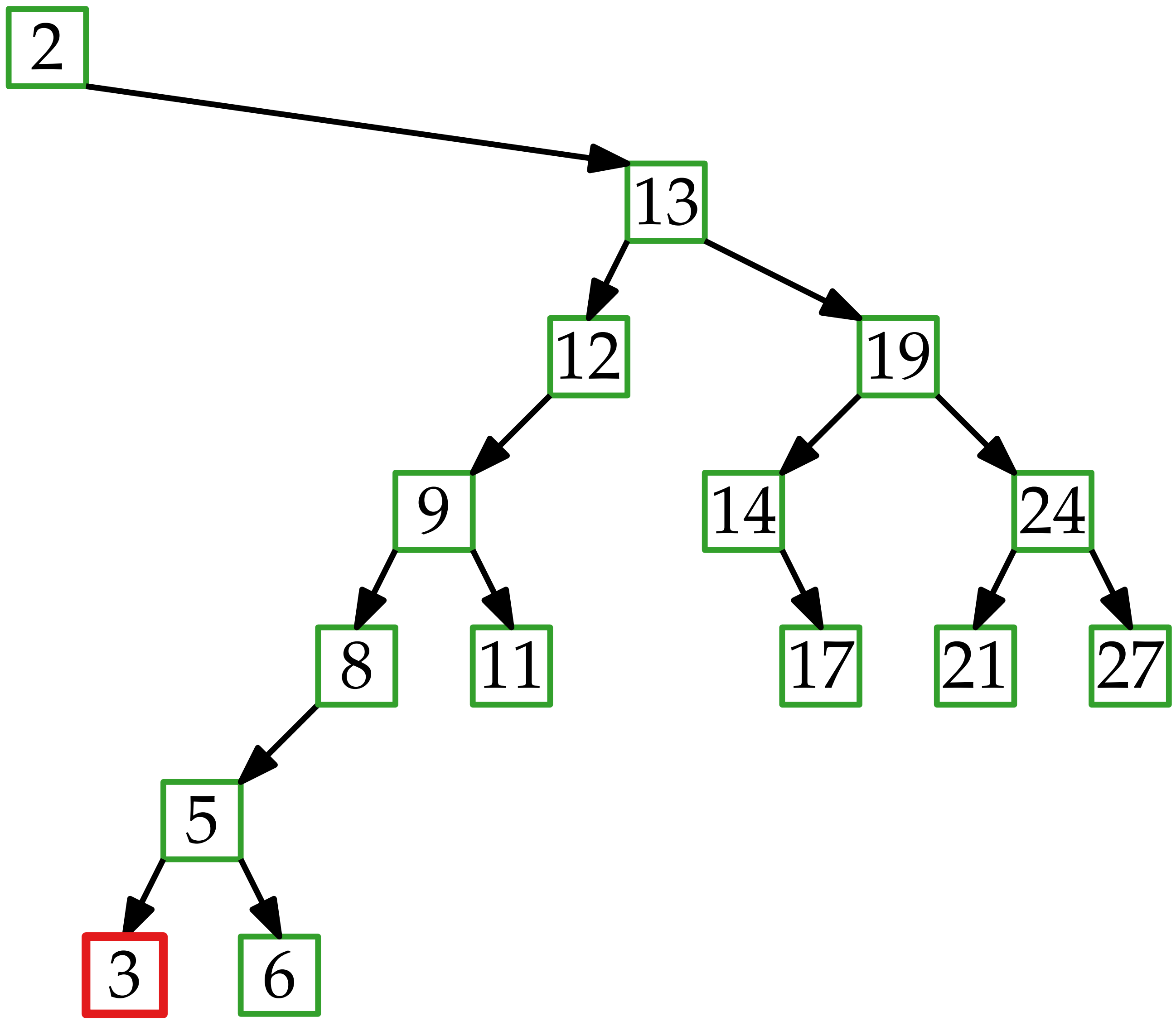

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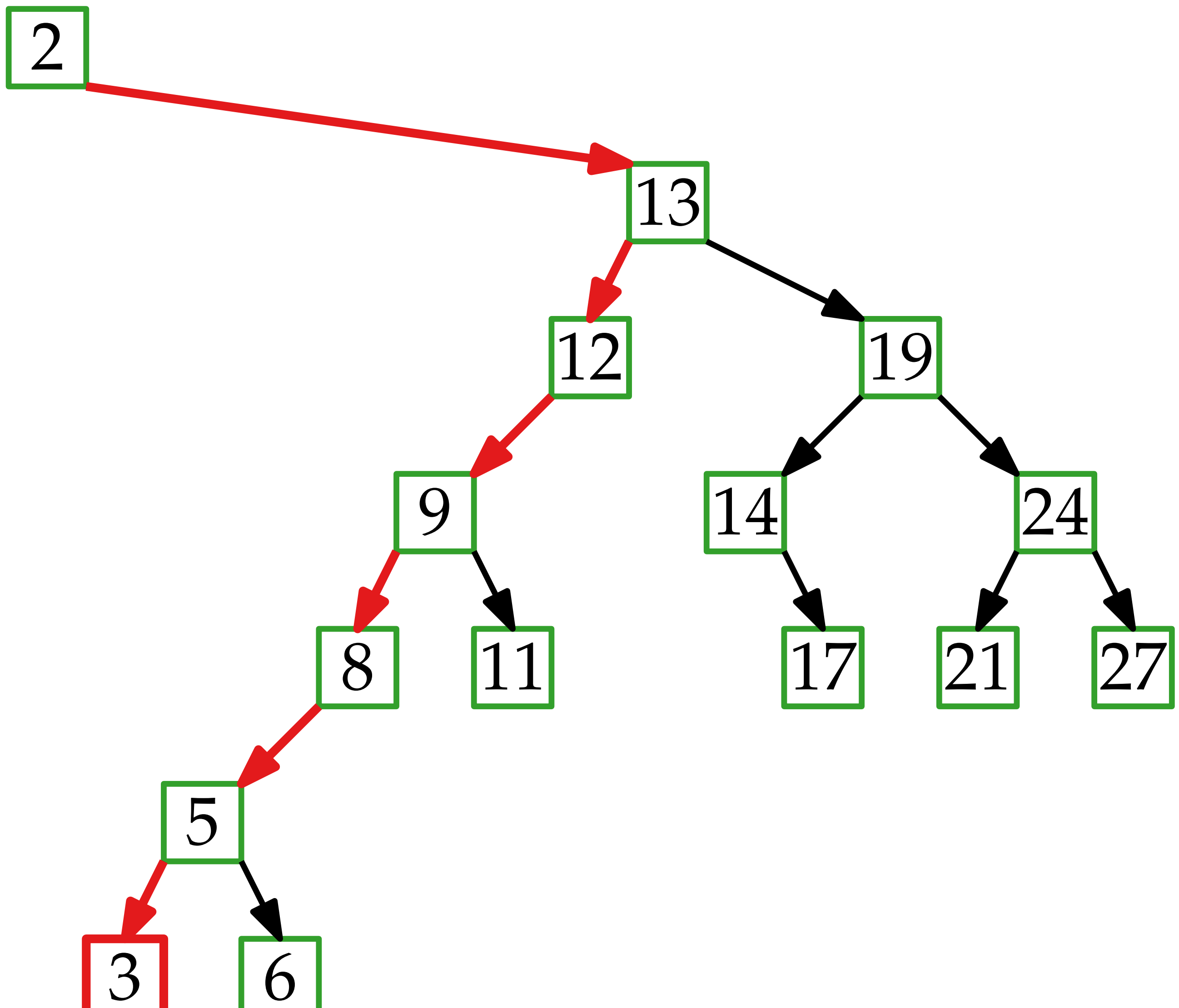

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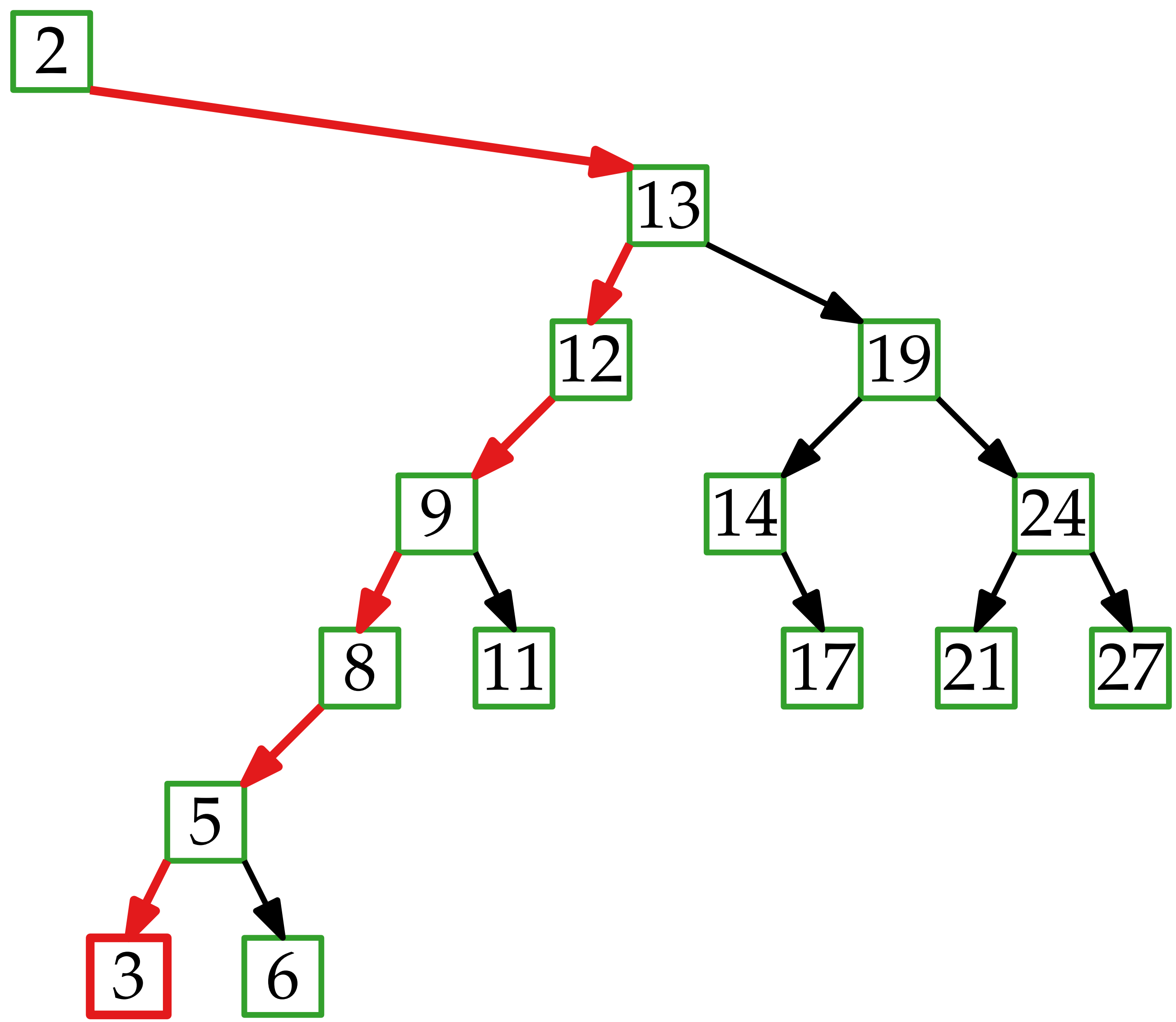

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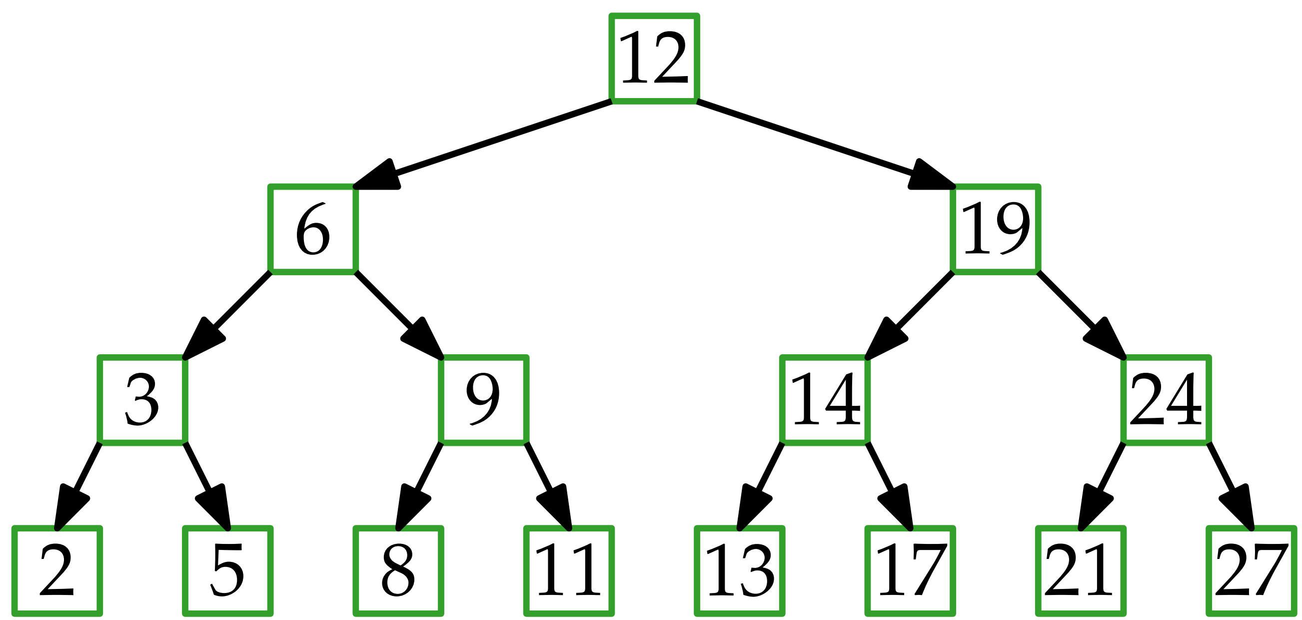

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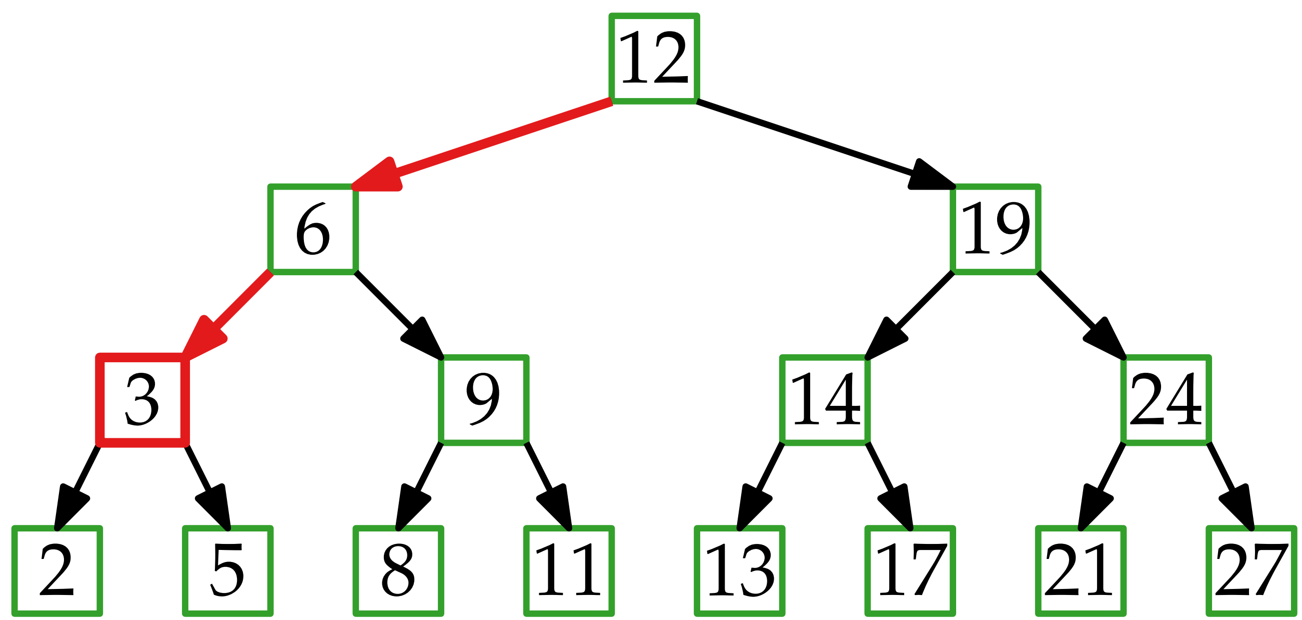

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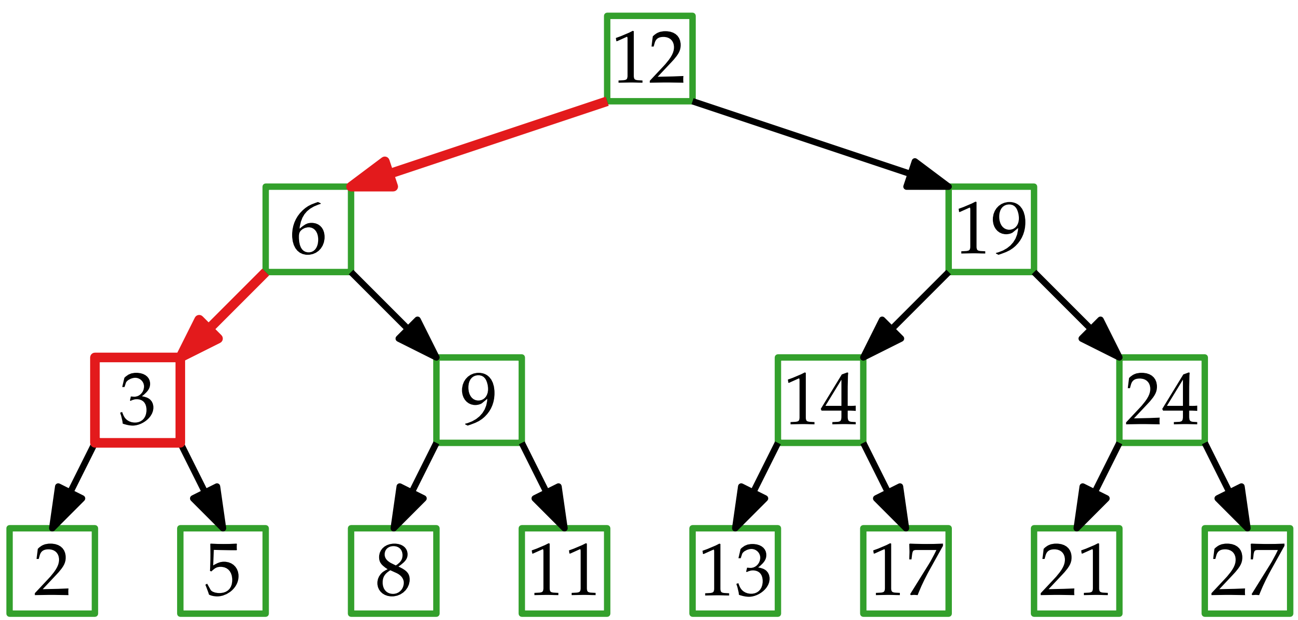

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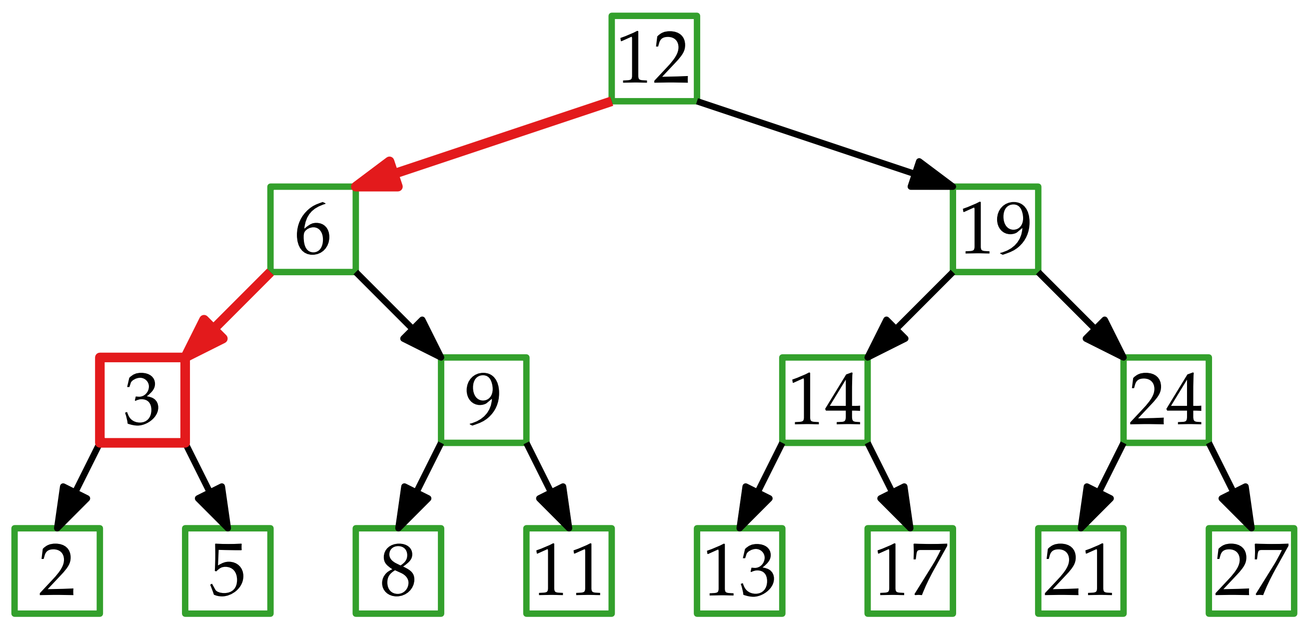

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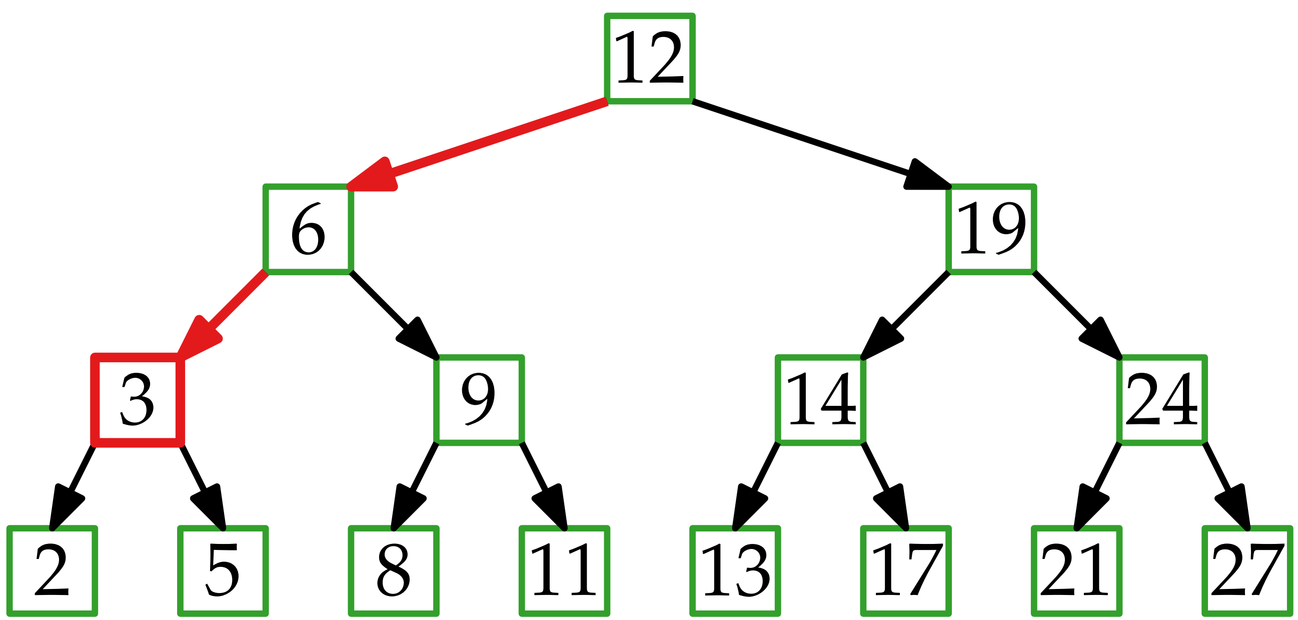

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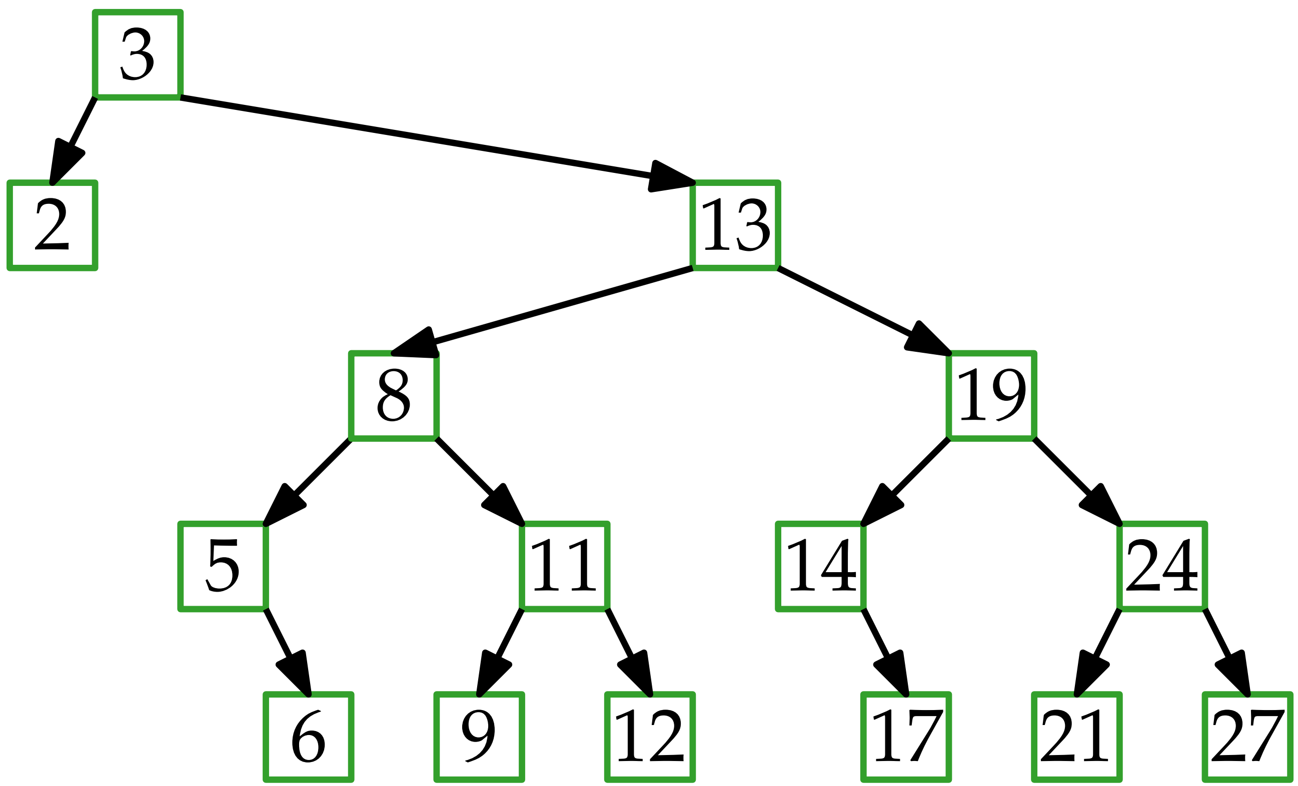

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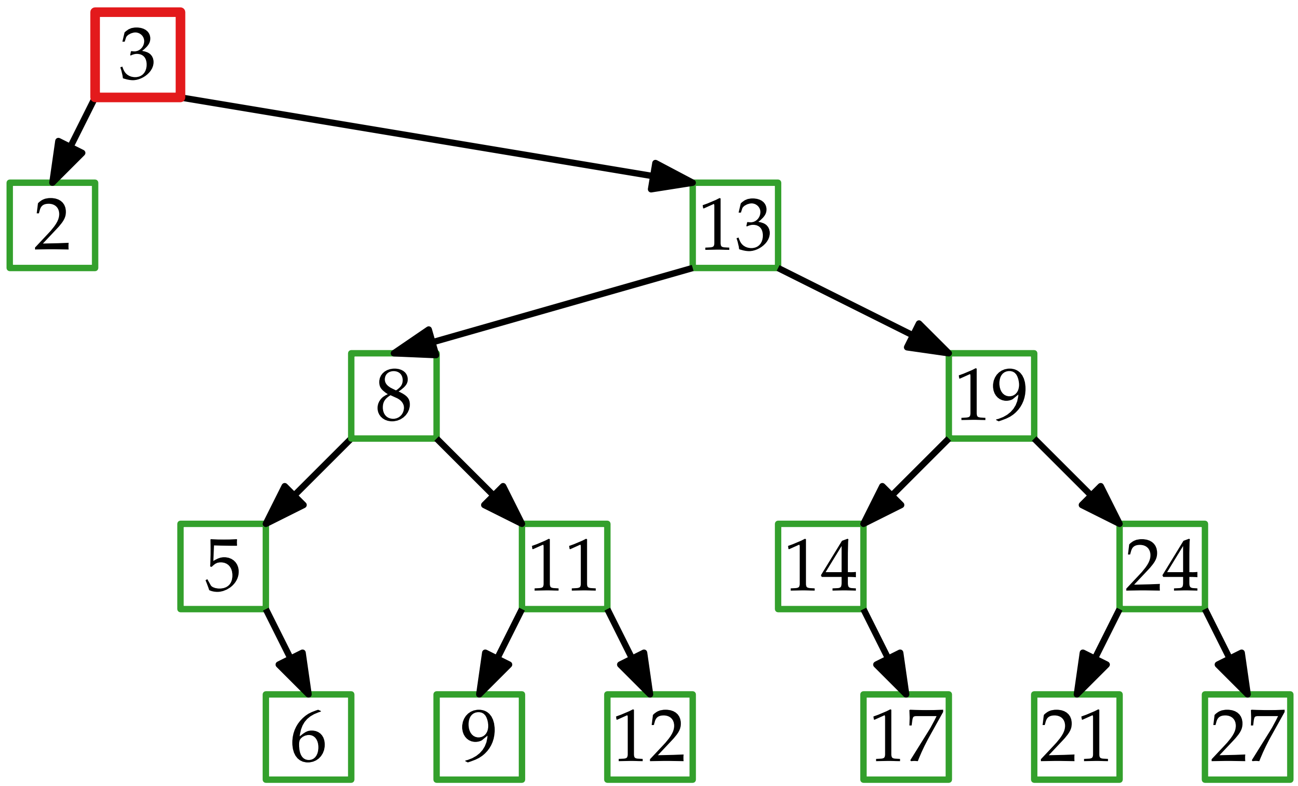

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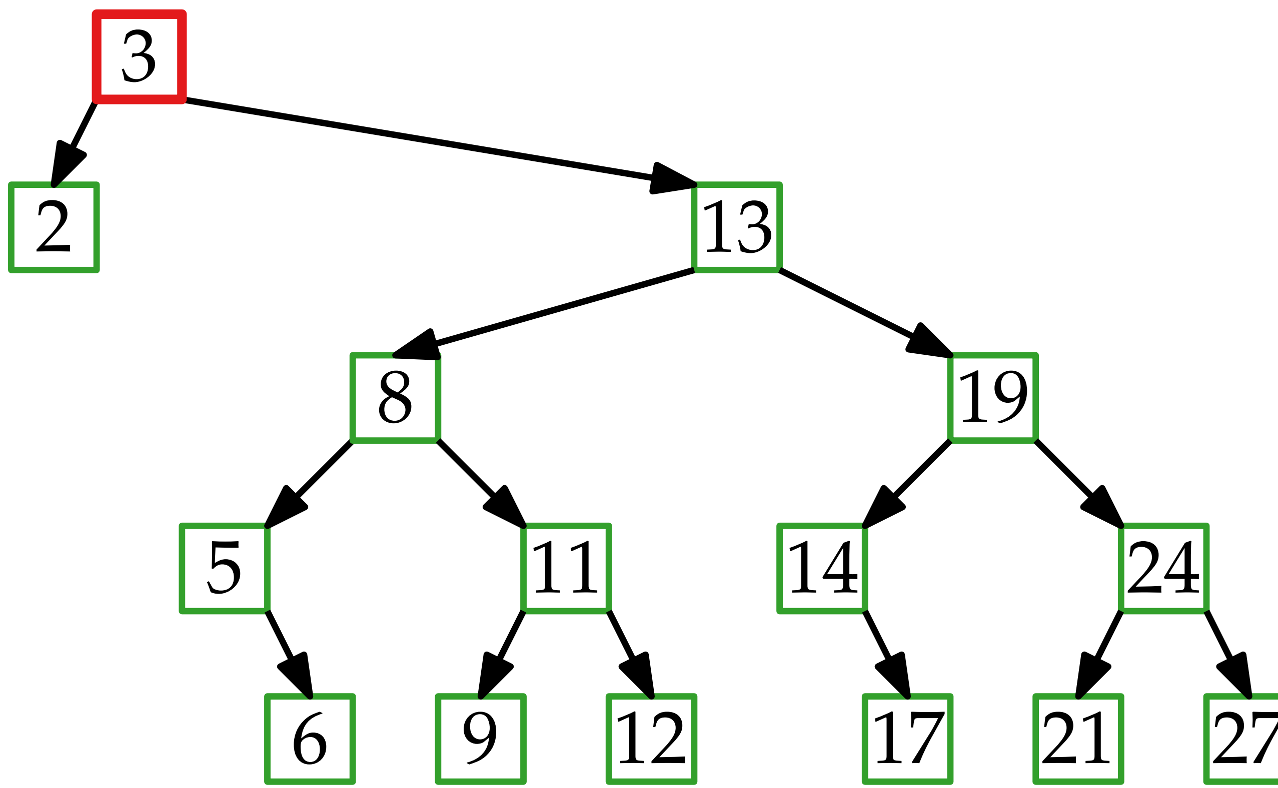

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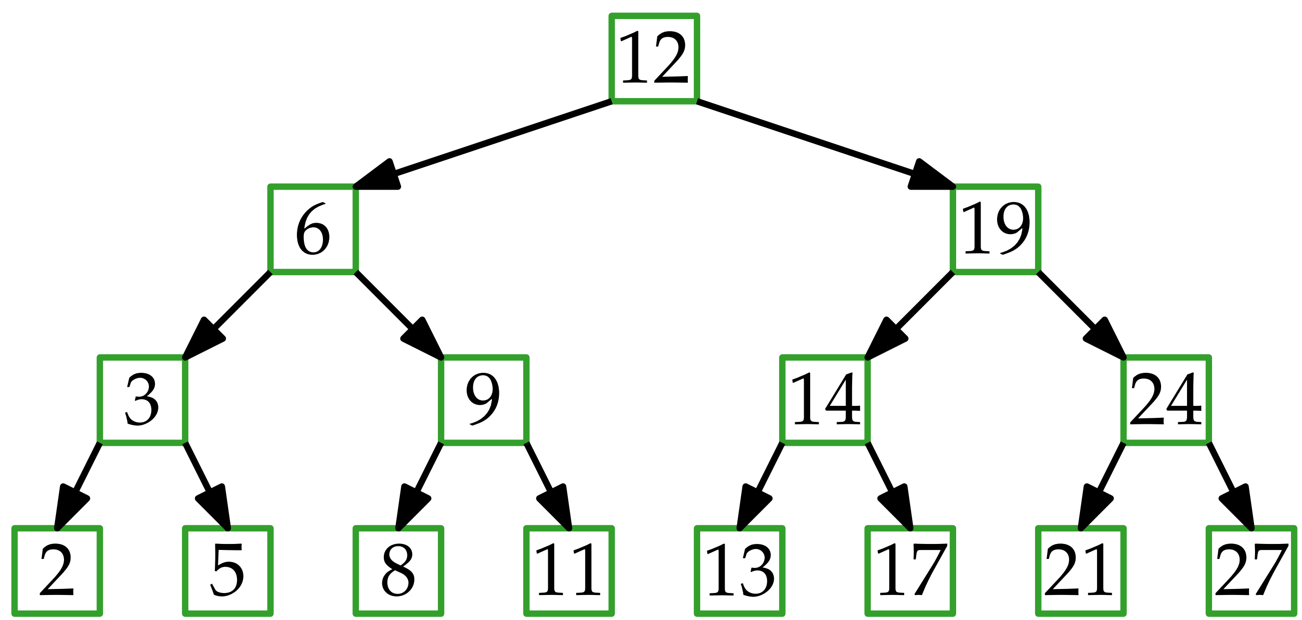

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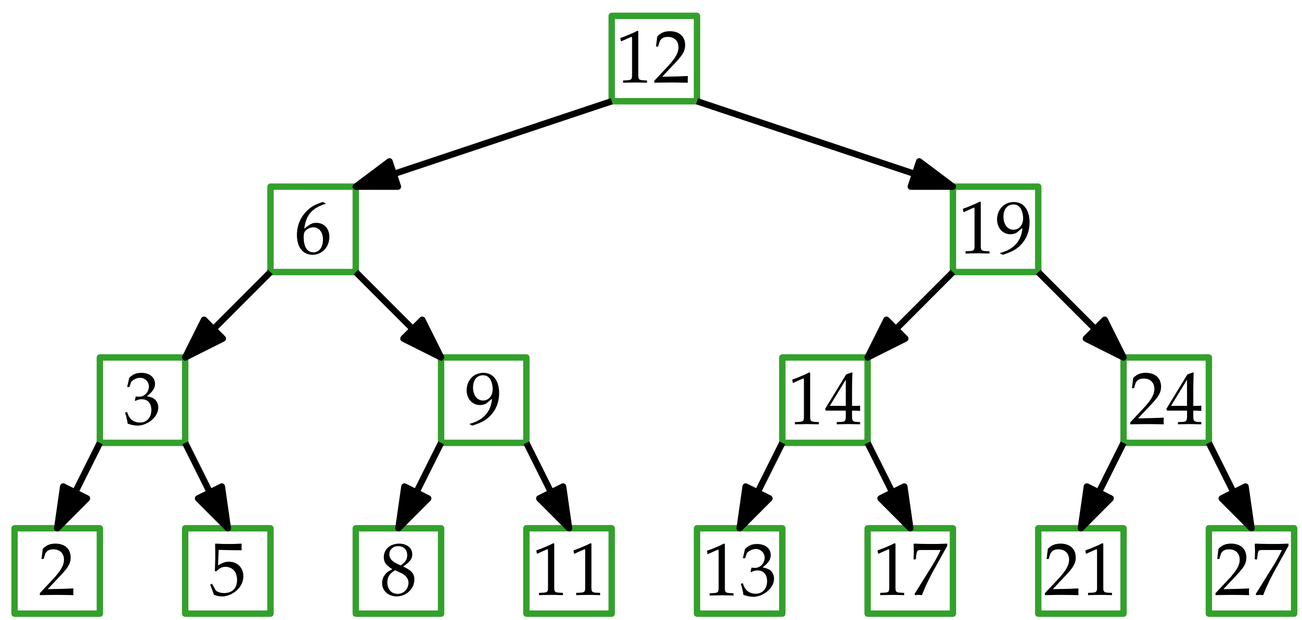

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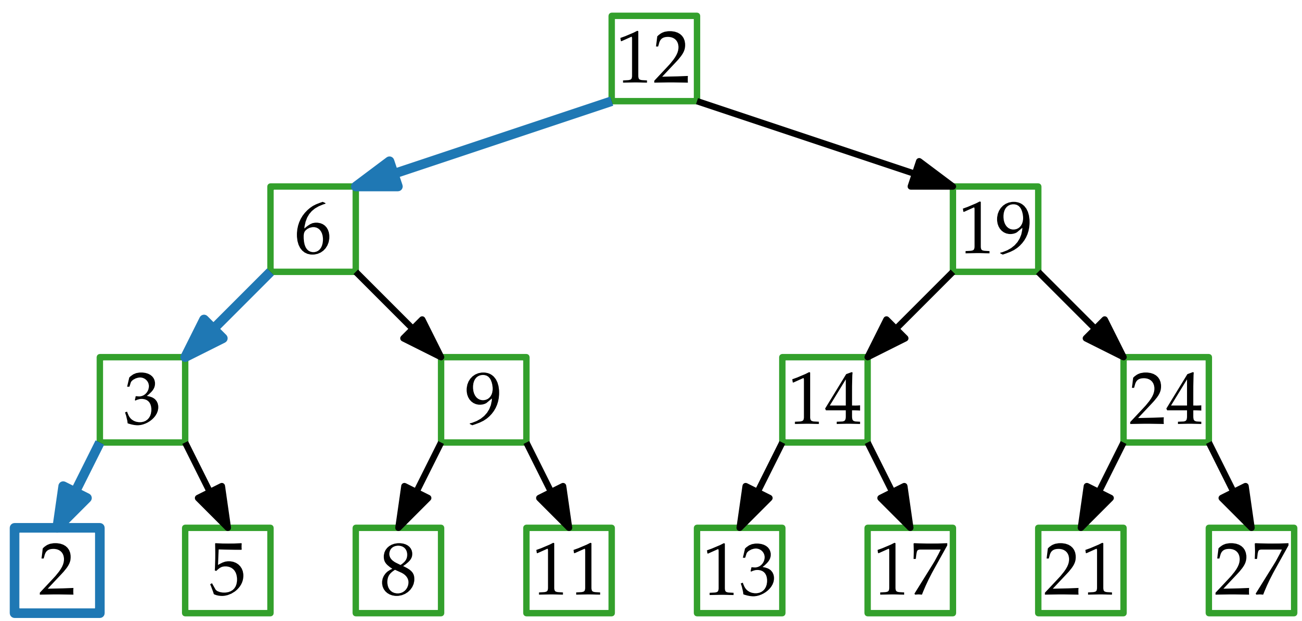

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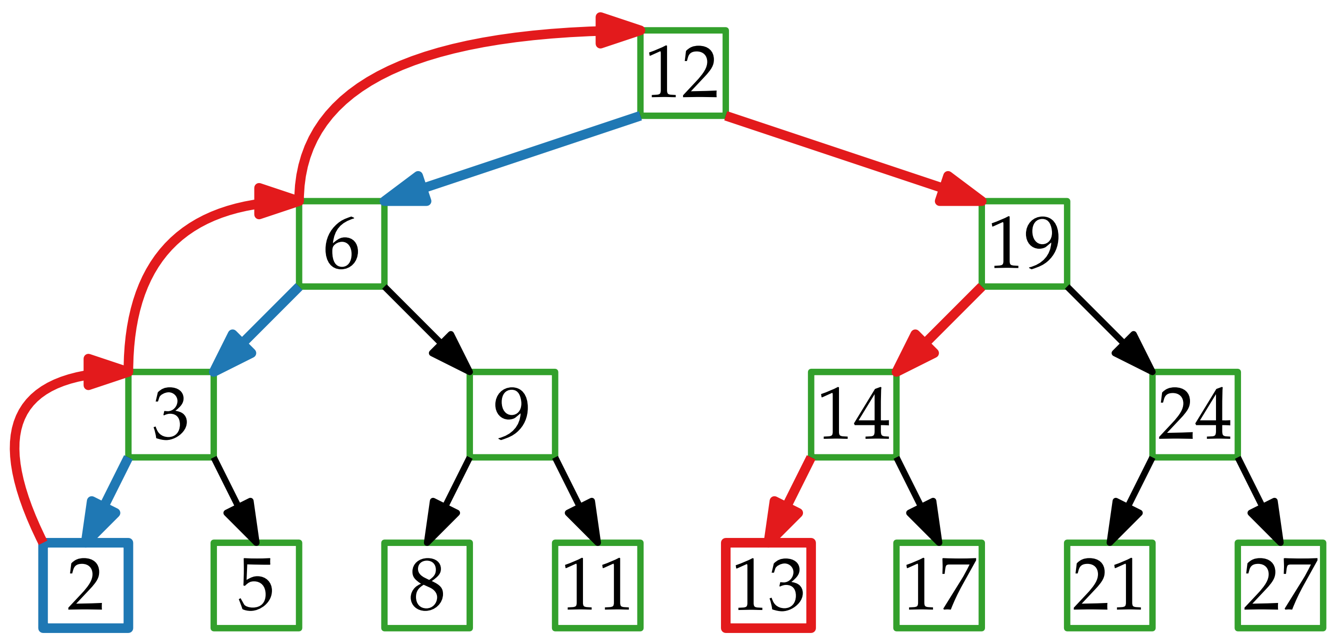

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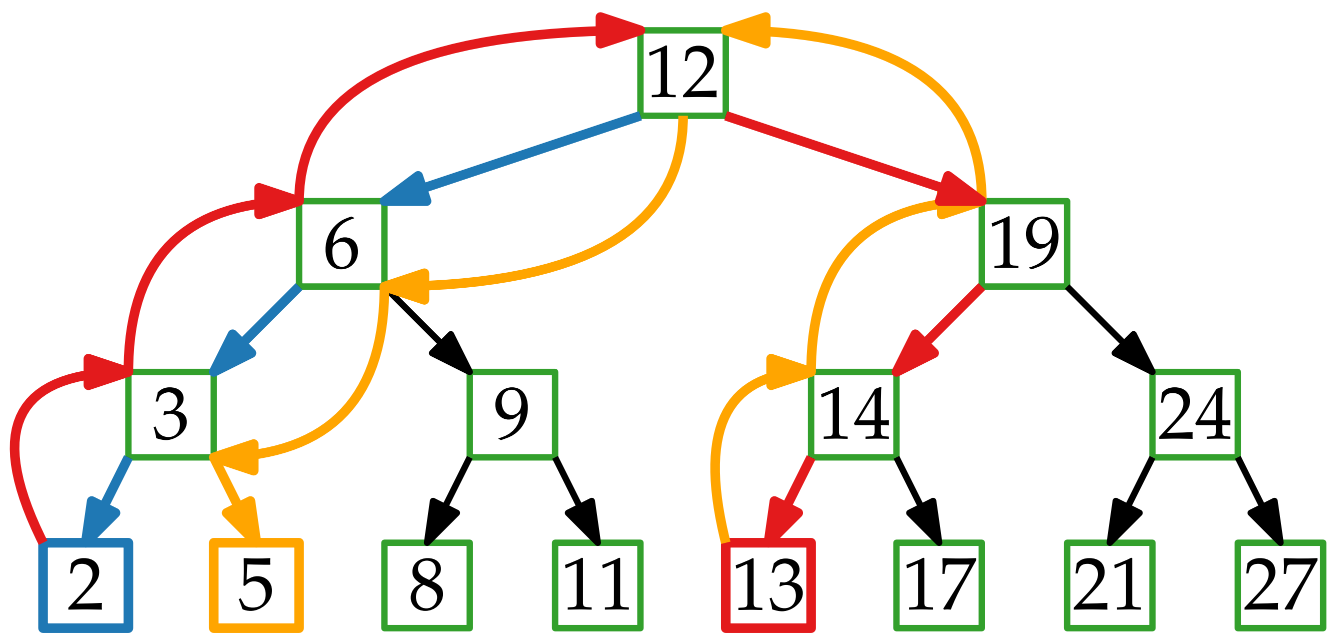

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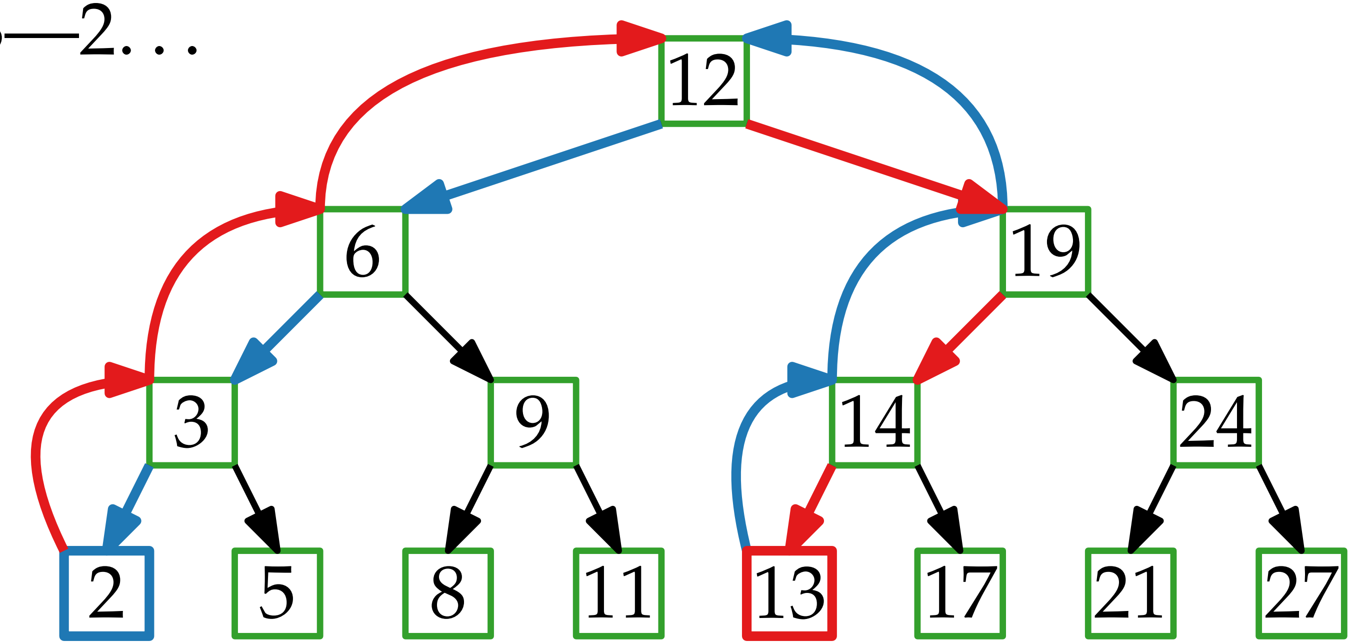

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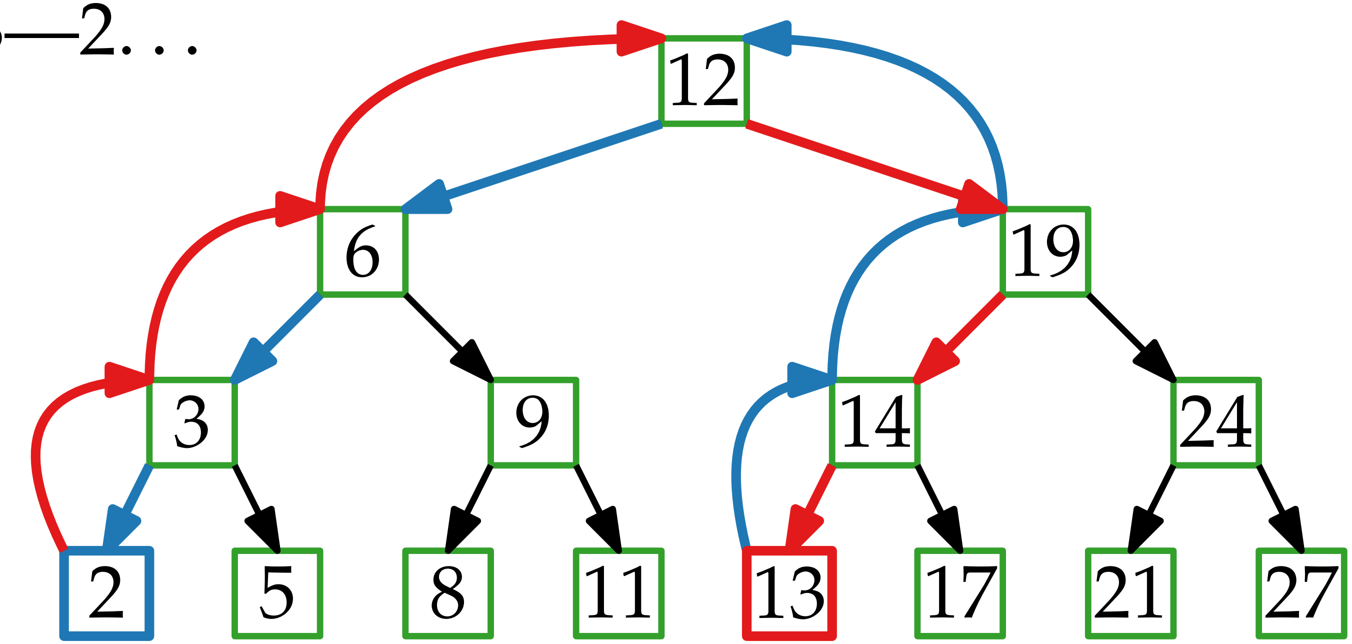

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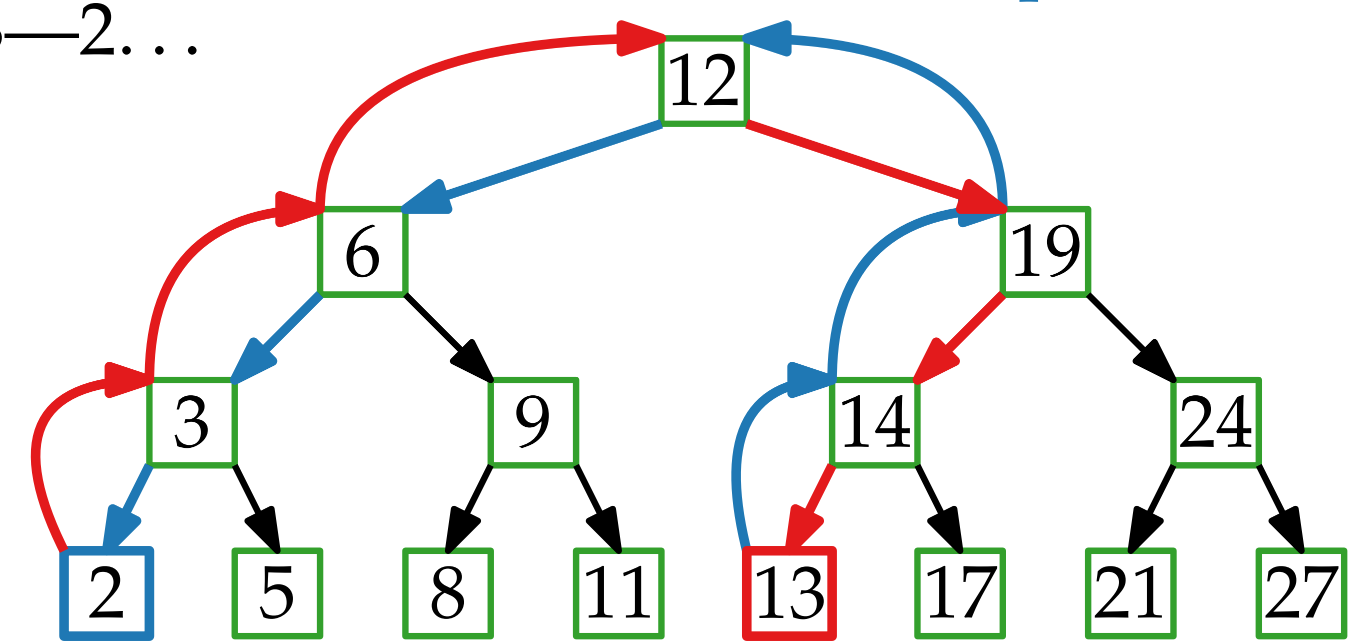

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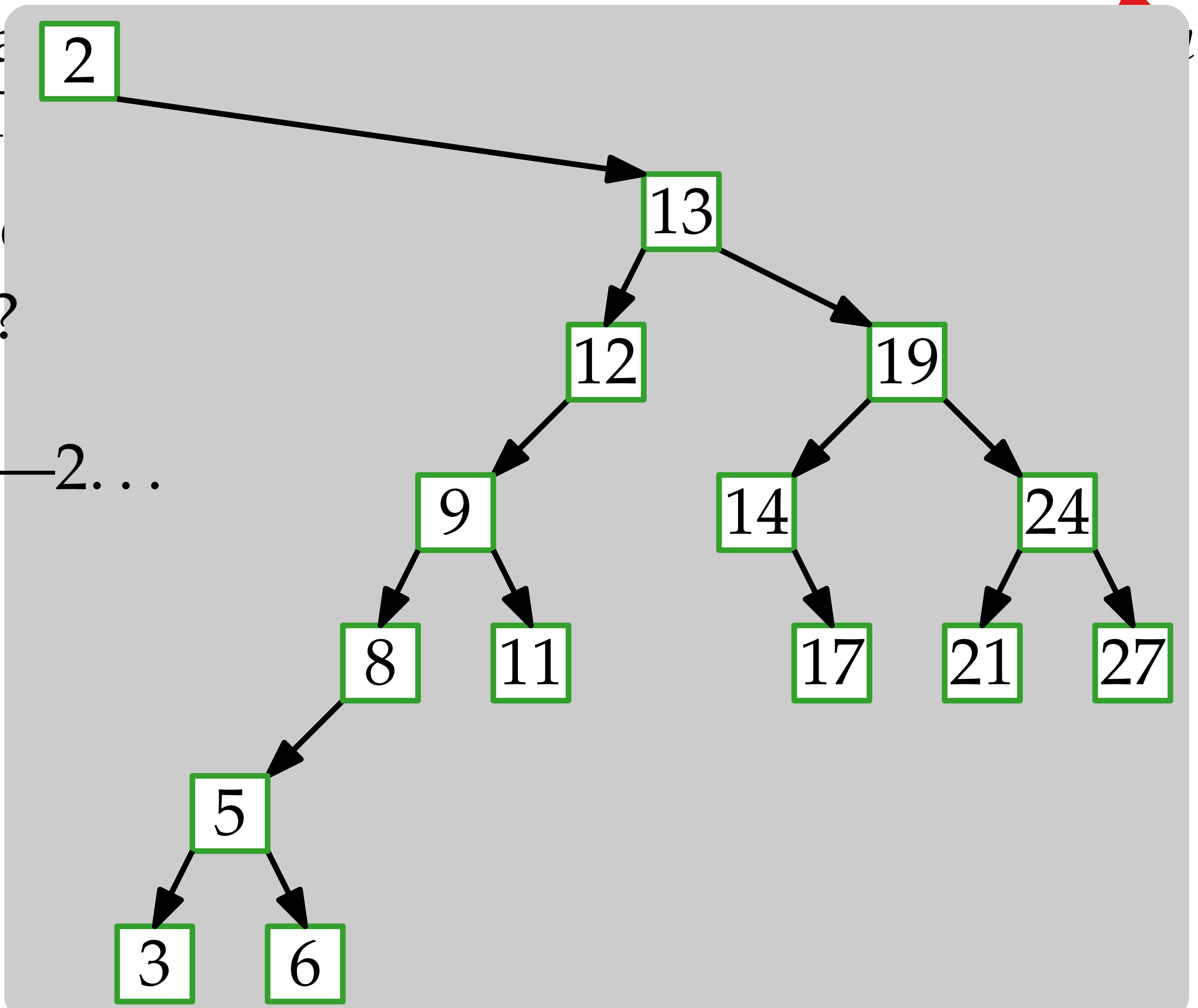

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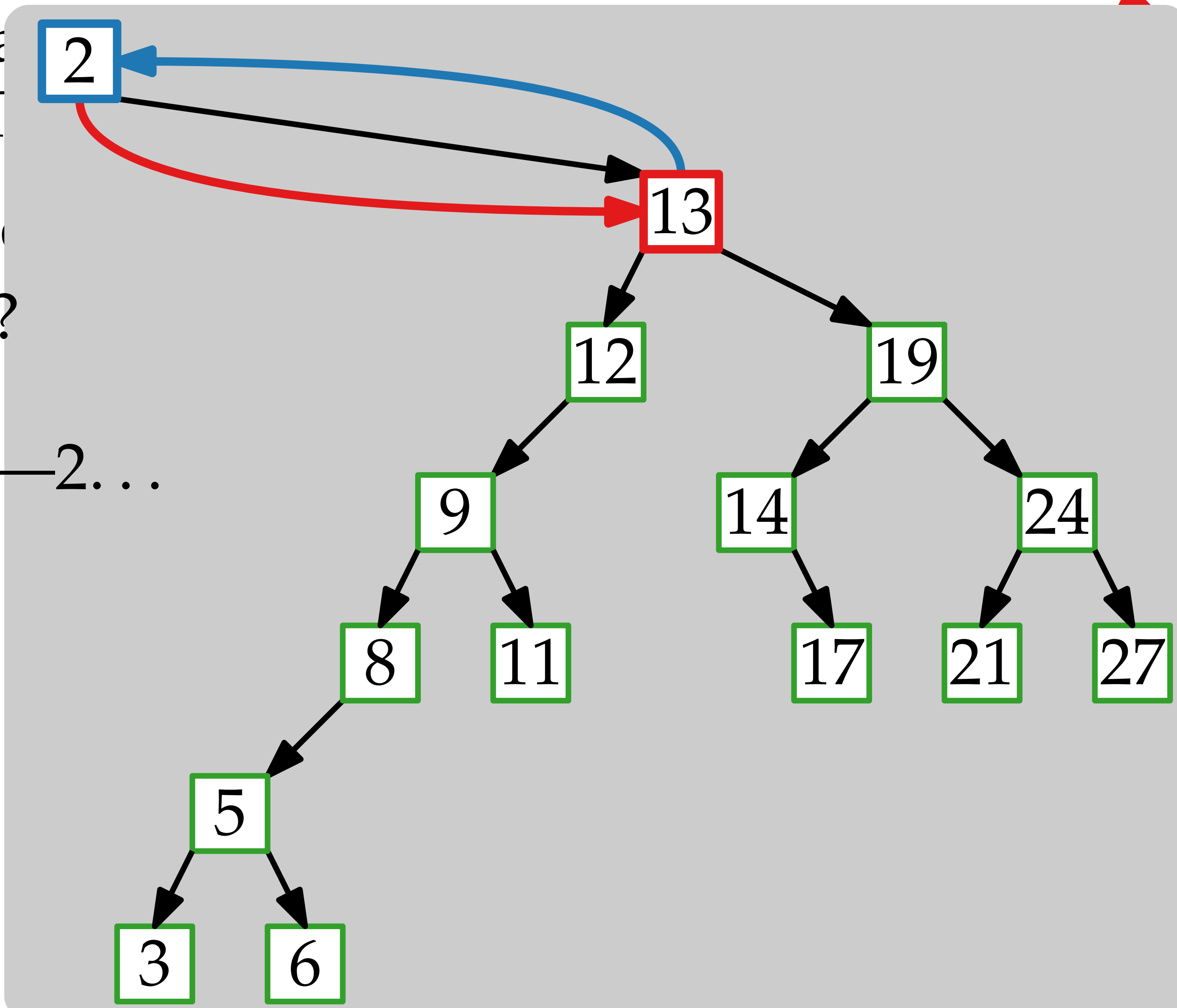

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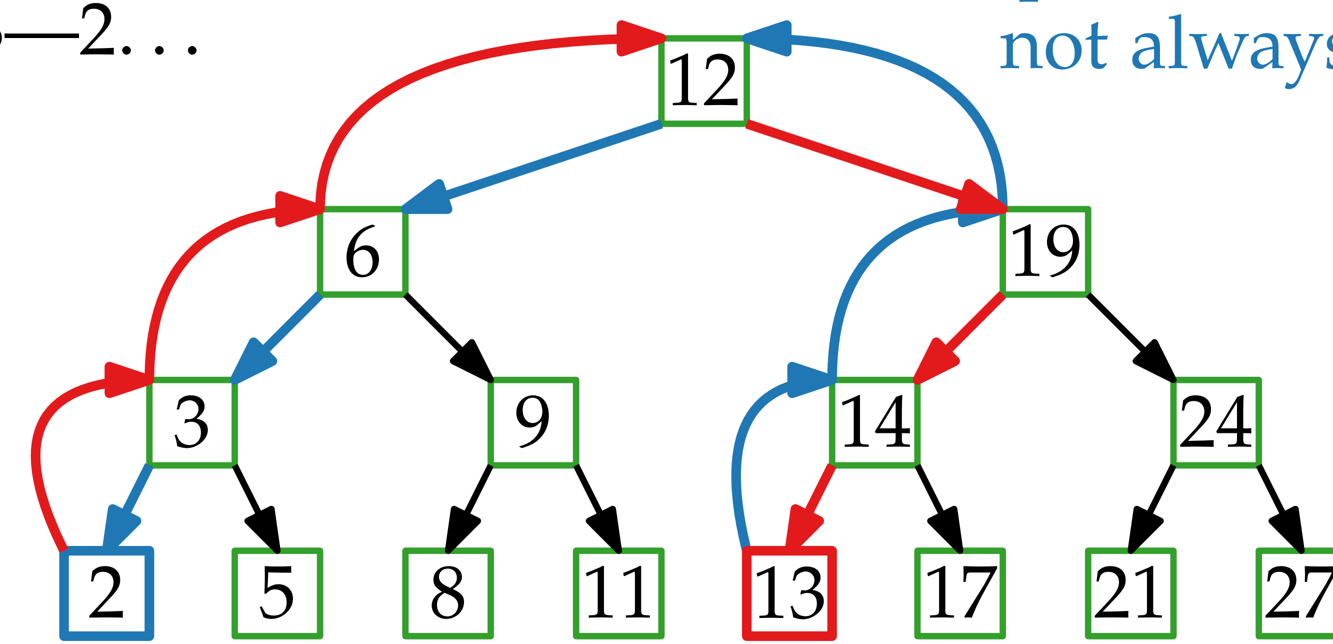
not always!



The performance of a BST depends on the model!

Model 1: Malicious Queries

Given a BST, what is the worst sequence of queries?

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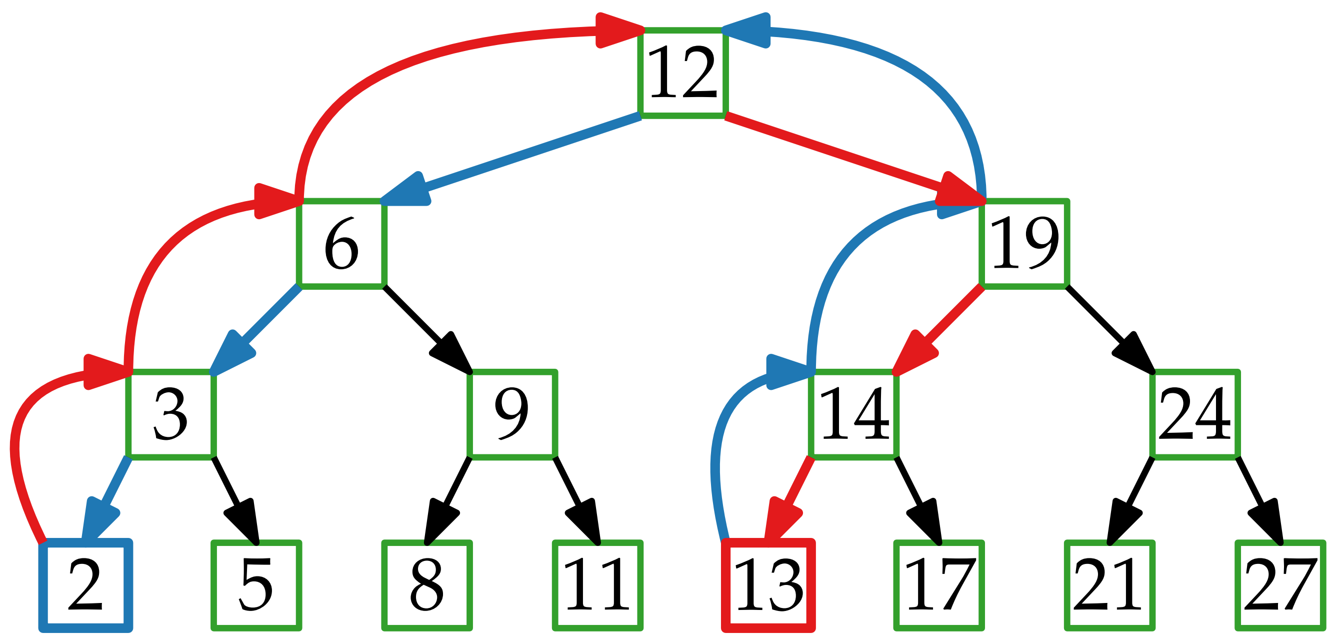

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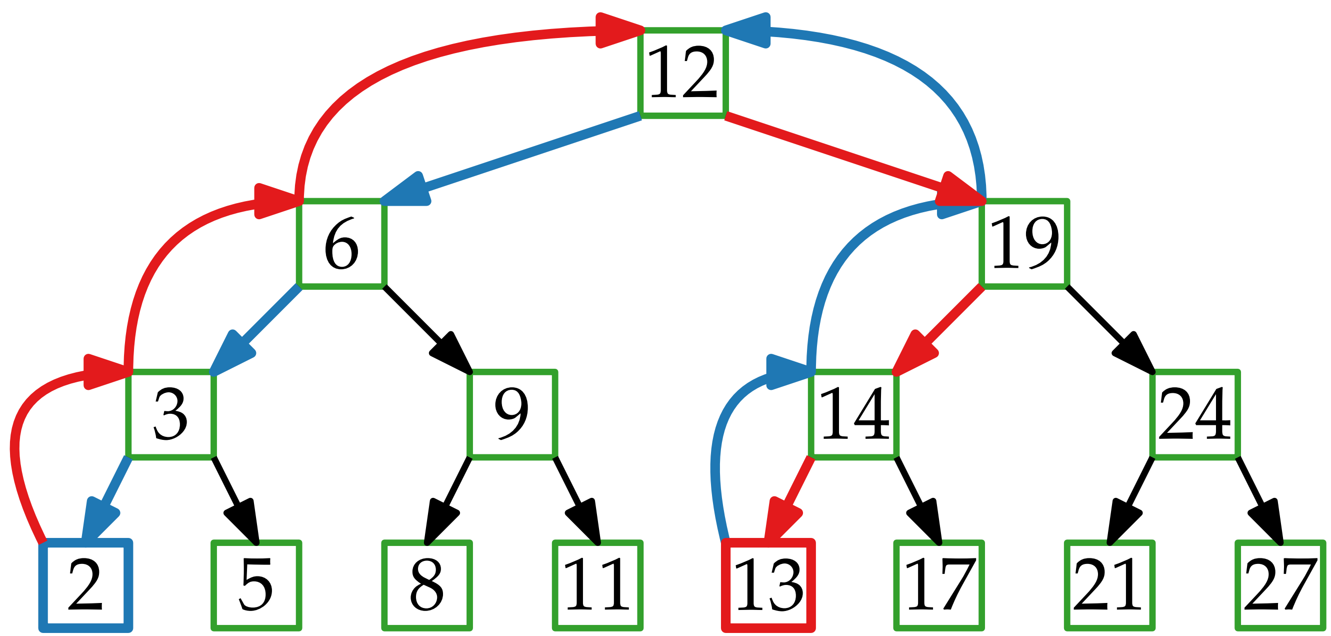
Lemma. The worst-case malicious query cost in any BST with n nodes is at least $\Omega(\log n)$ per query.



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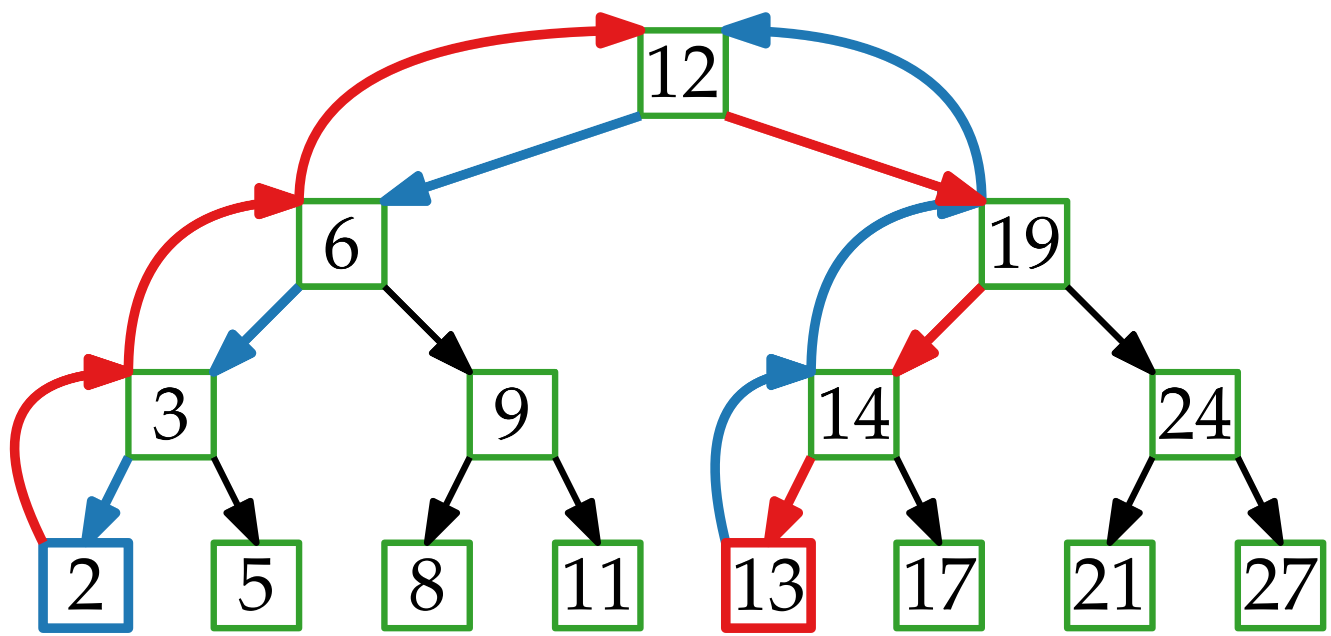
Definition. A BST is **balanced** if the (amortized) cost of *any* query is $O(\log n)$.



Model 2: Known Probability Distribution

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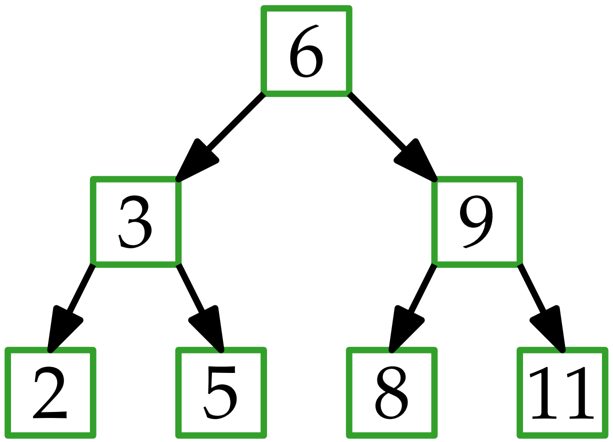

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Access Probabilities:

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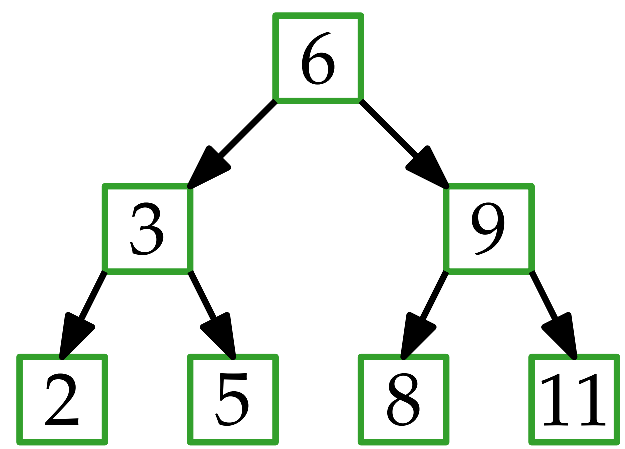

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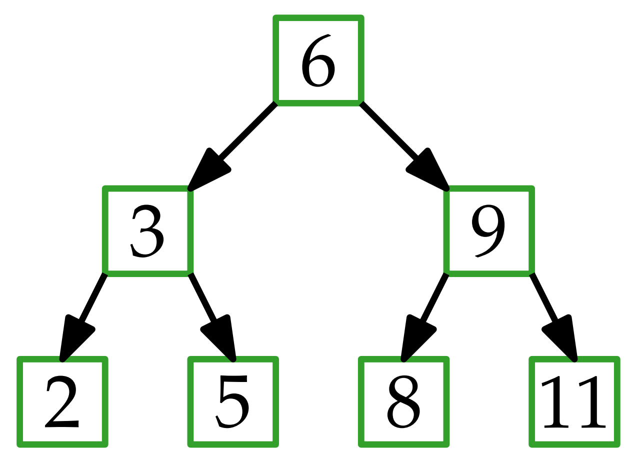

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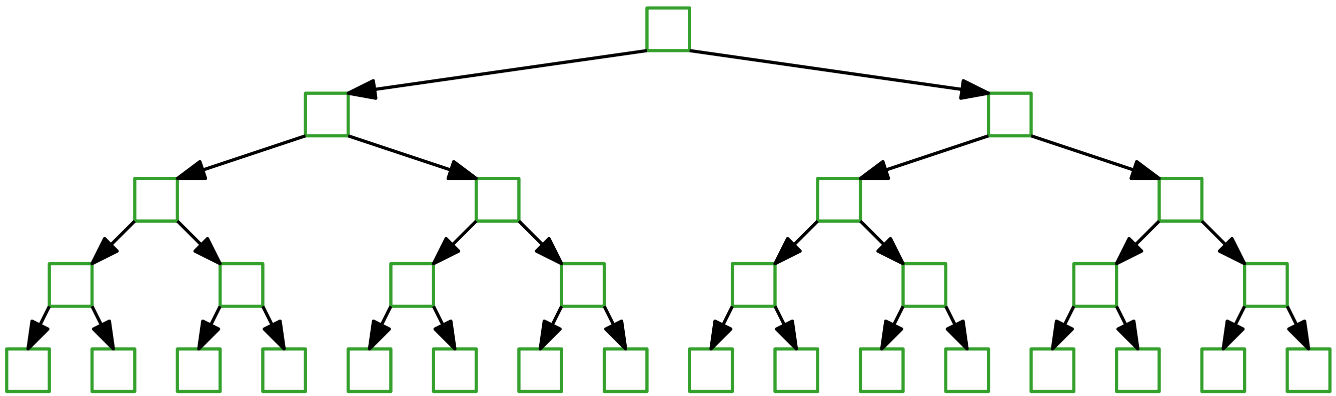

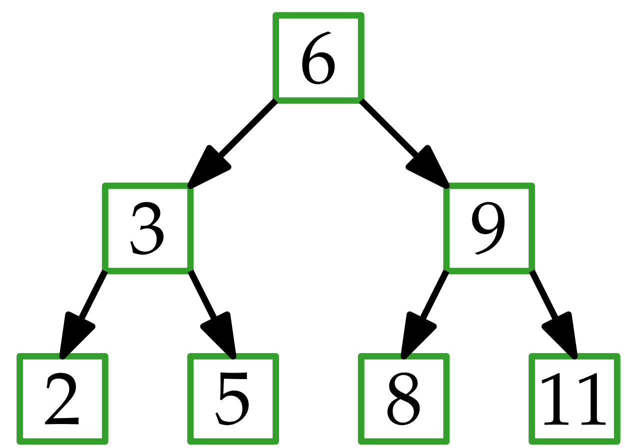

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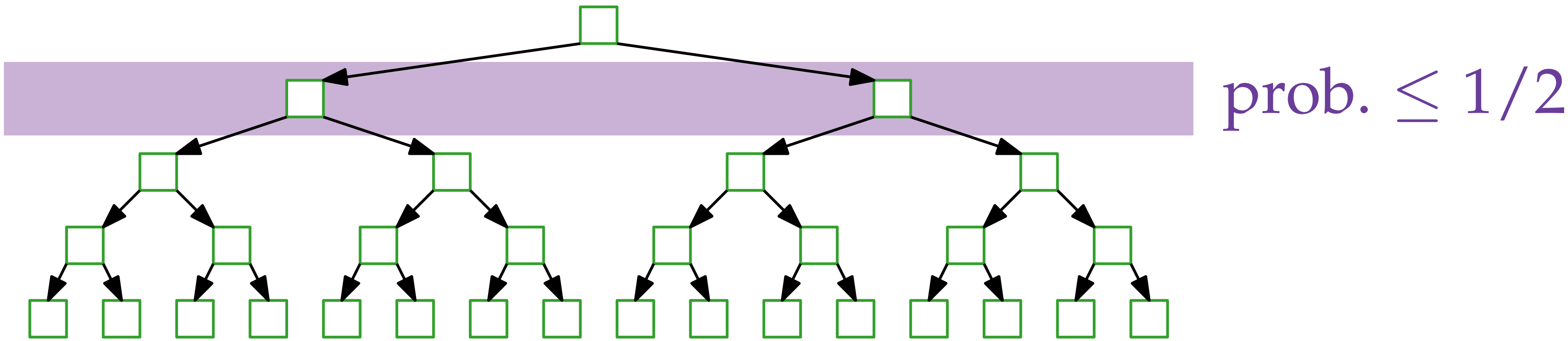

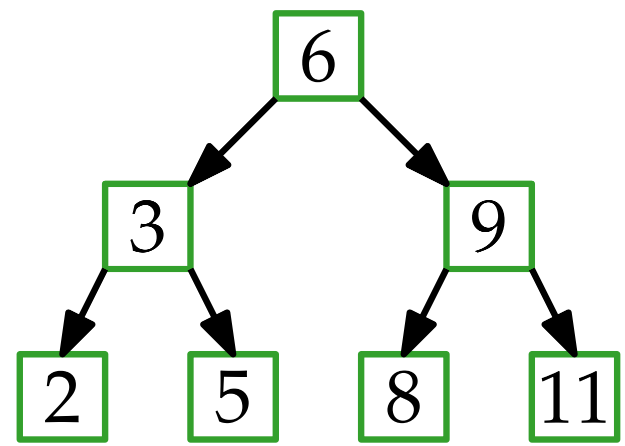

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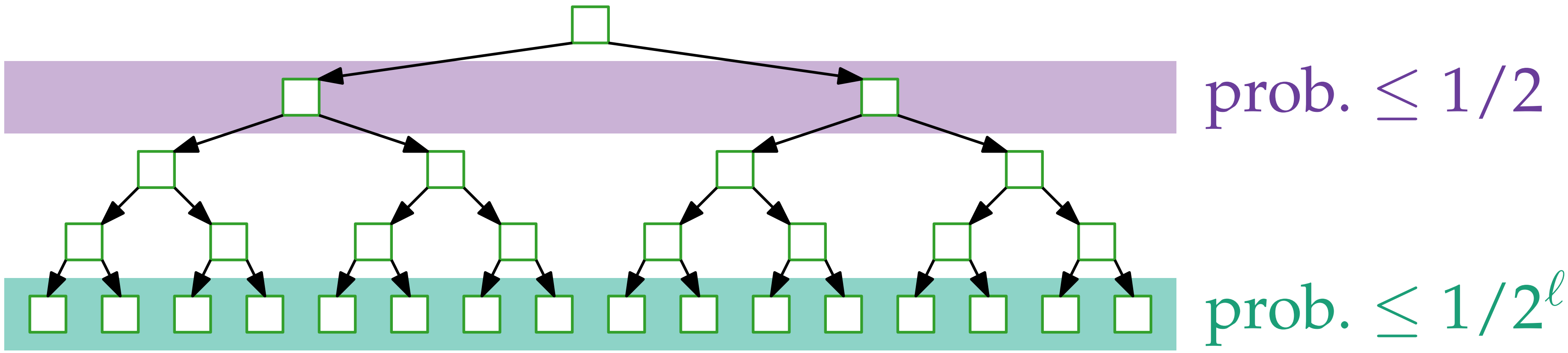

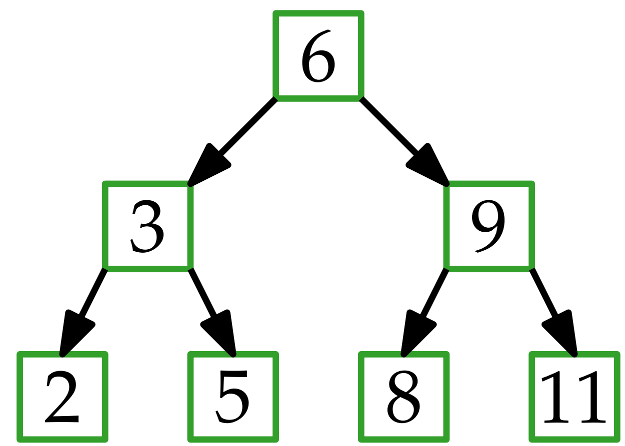

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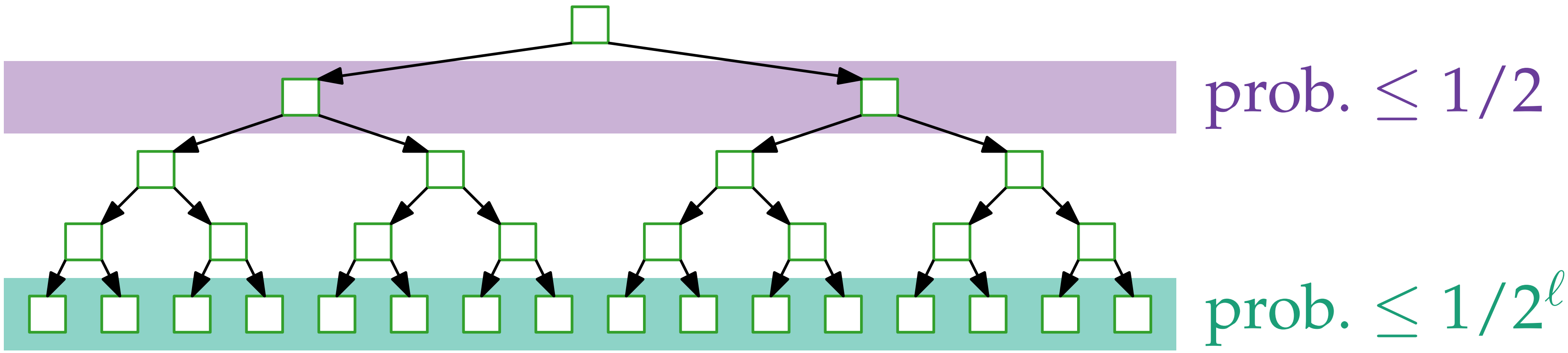

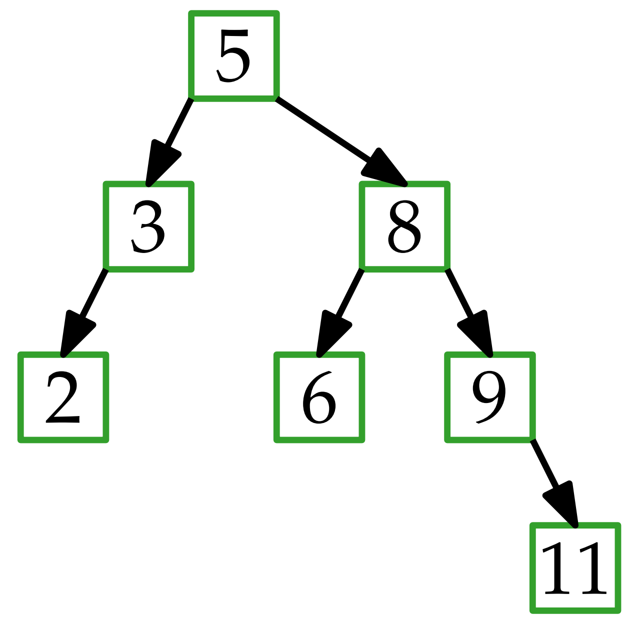

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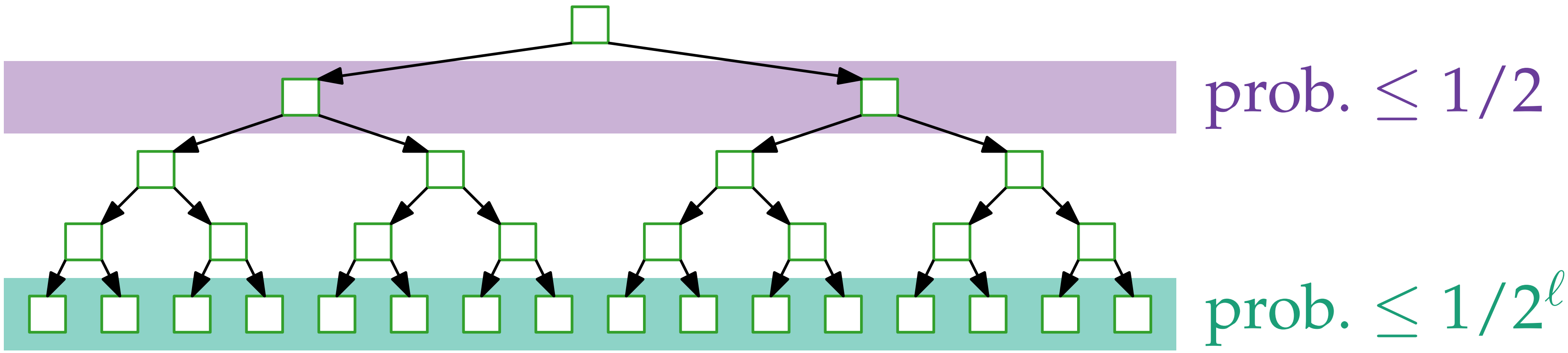

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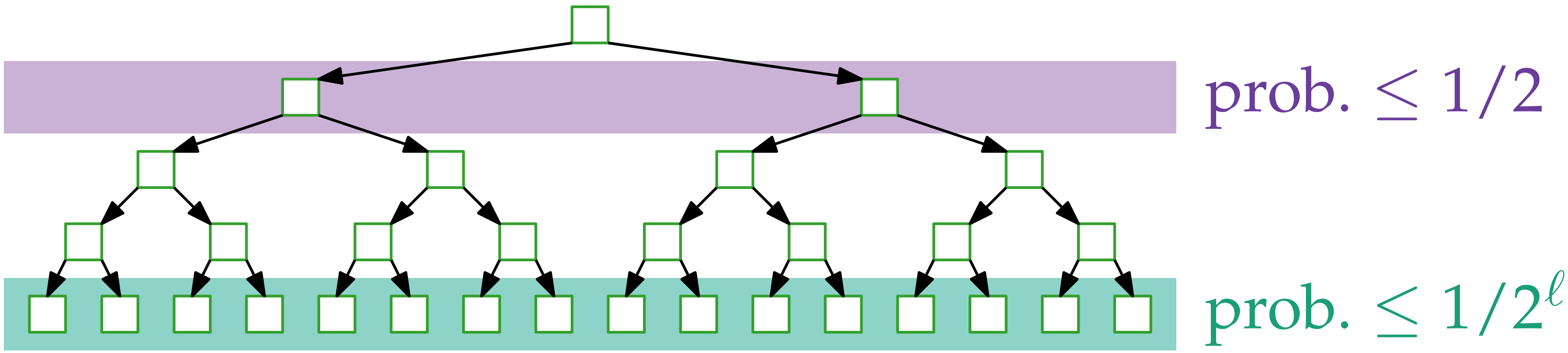

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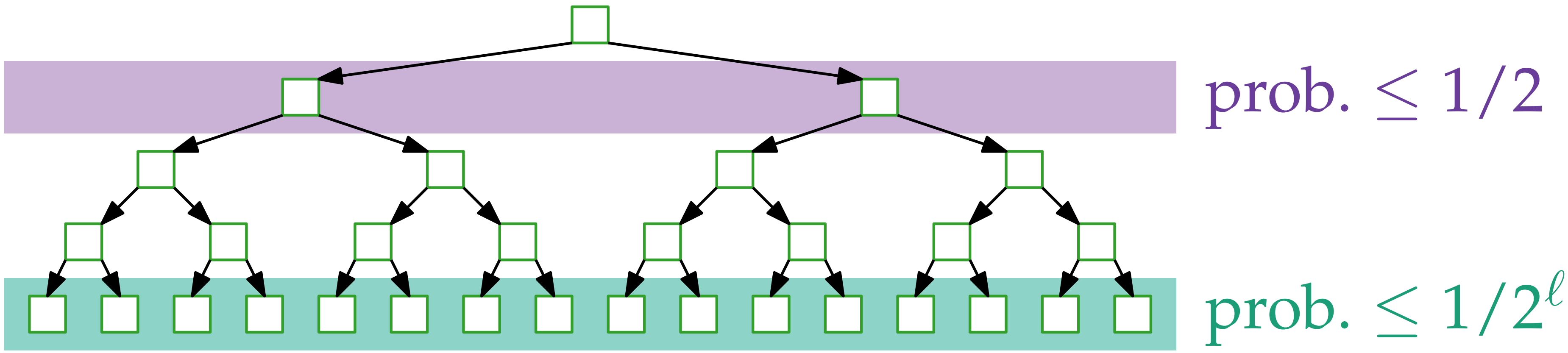

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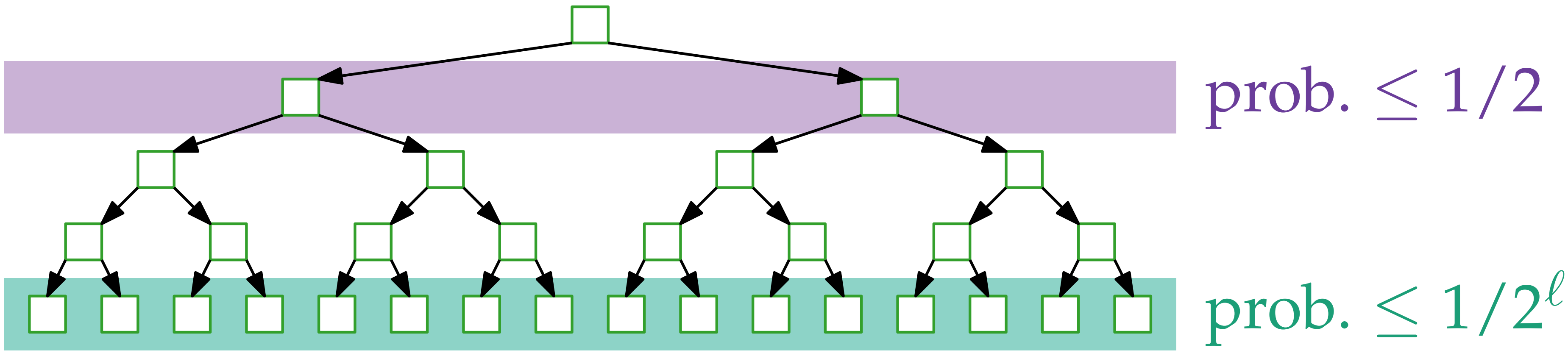

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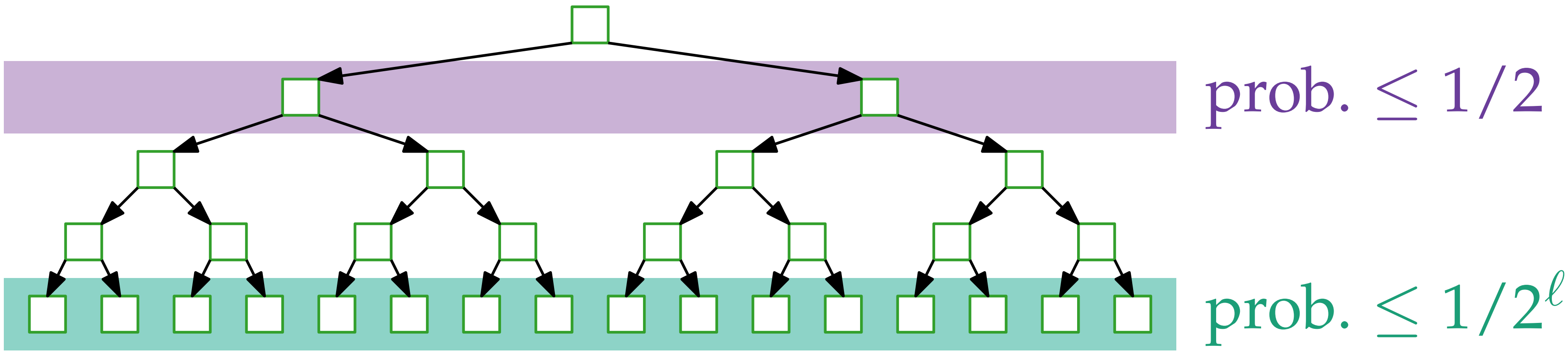

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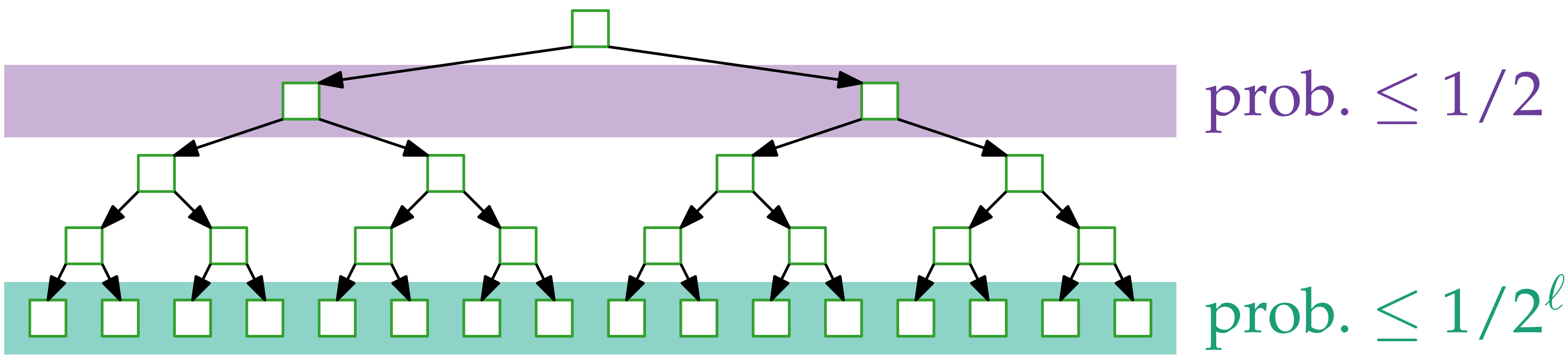

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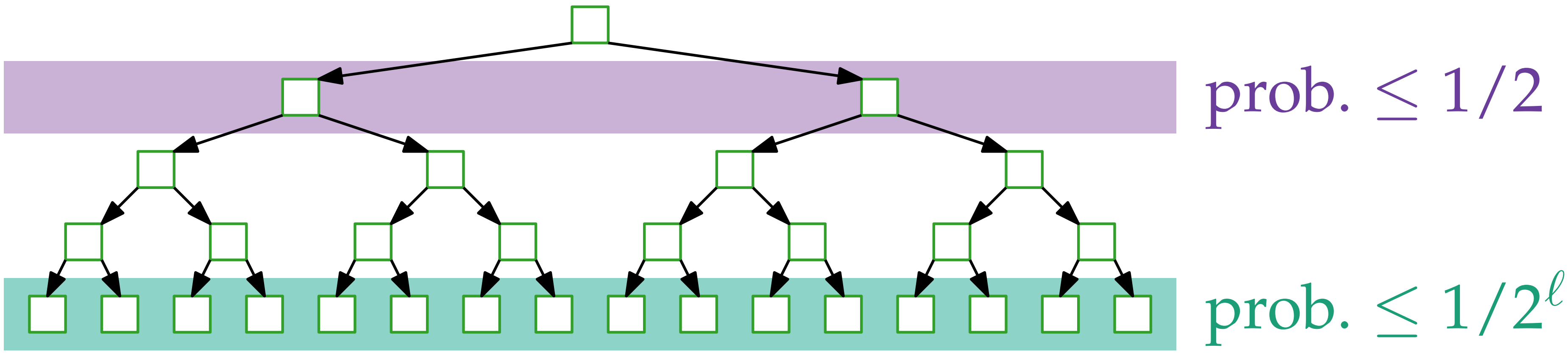

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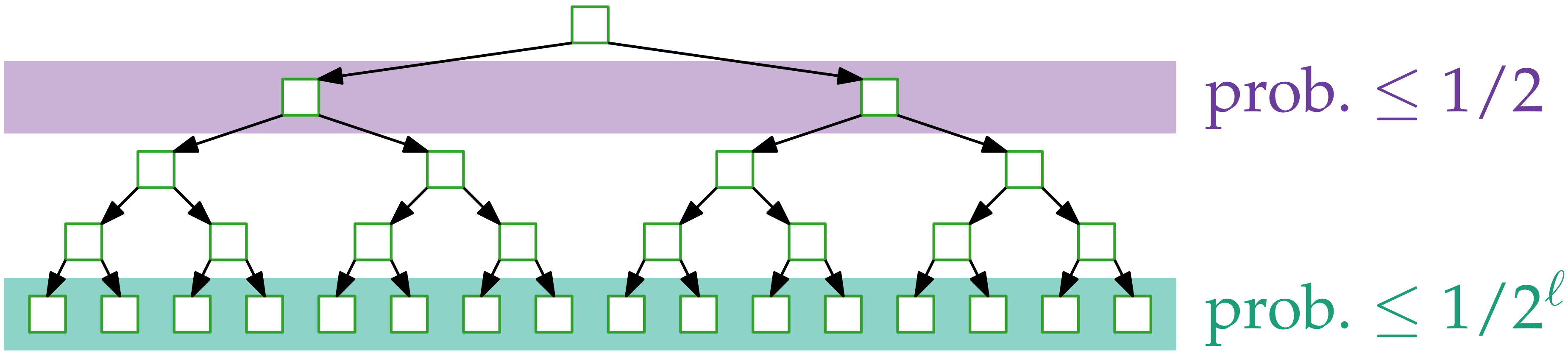
$$p_1 = 1, p_i = 0$$

Model 2: Known Probability Distribution

Access Probabilities:

2	3	5	6	8	9	11
2%	20%	30%	8%	20%	15%	5%

Idea: Place nodes with higher propability higher in the tree.



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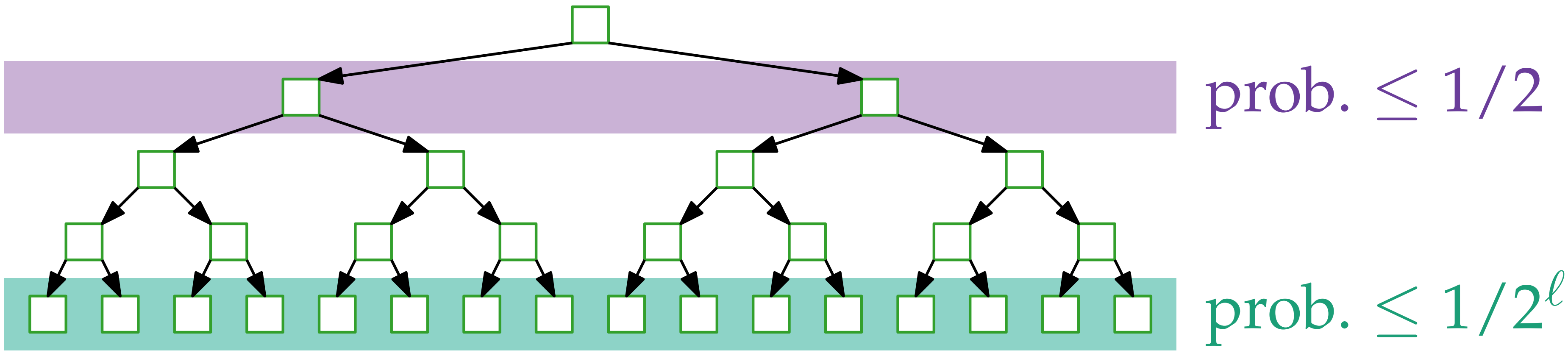

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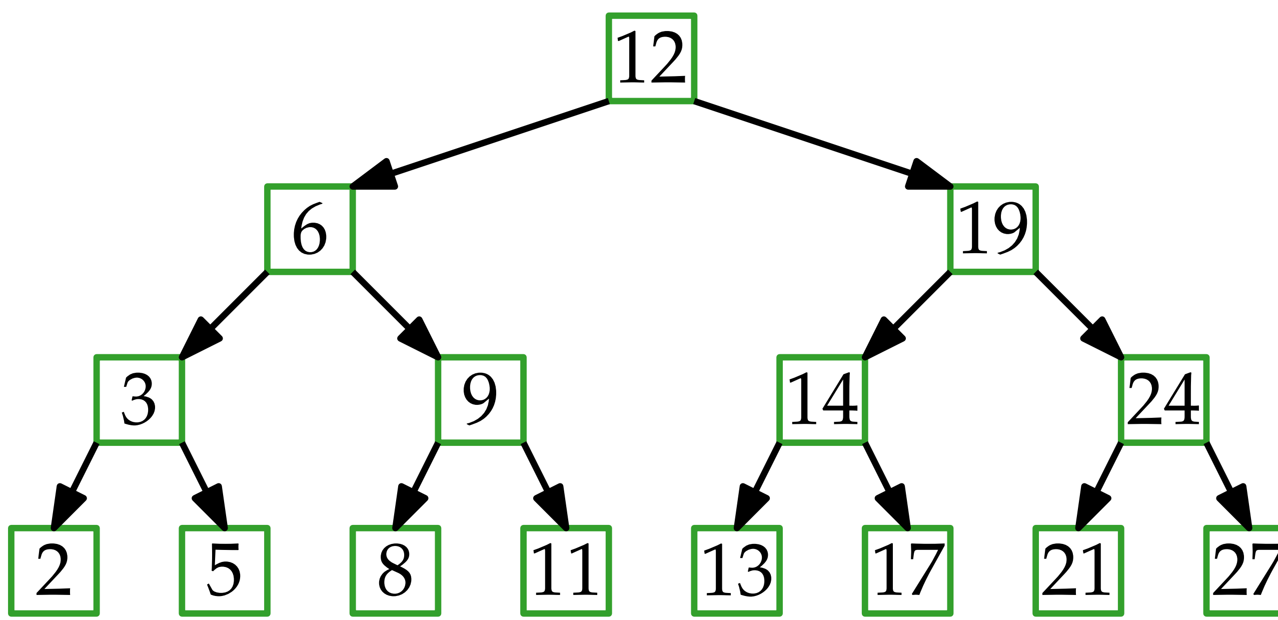

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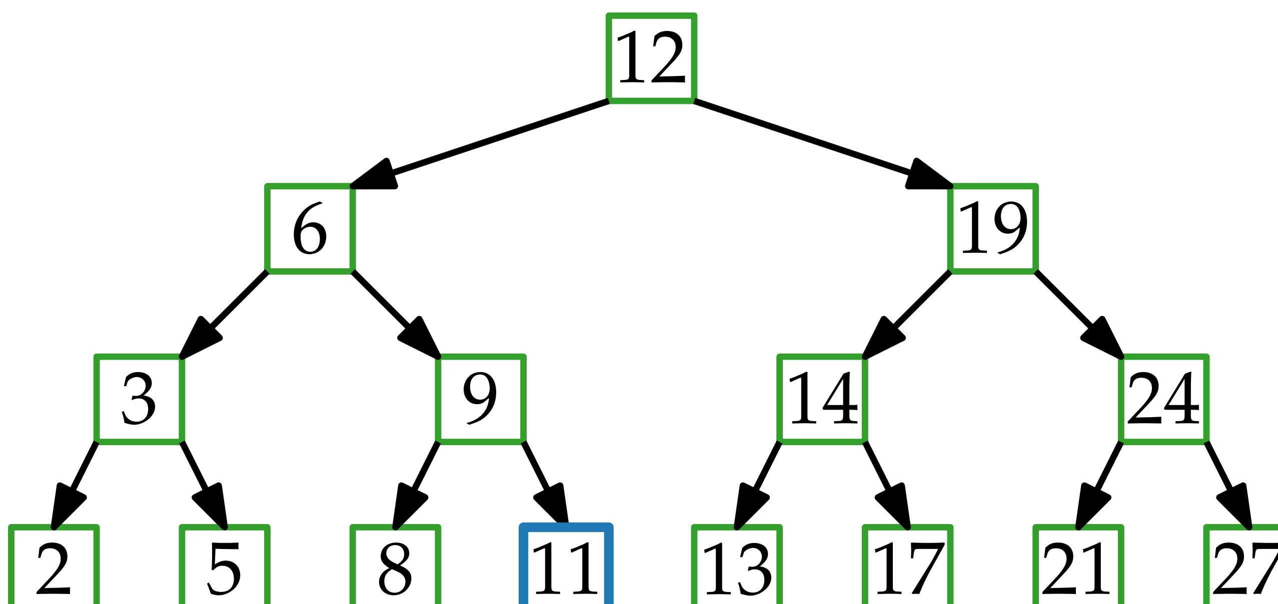

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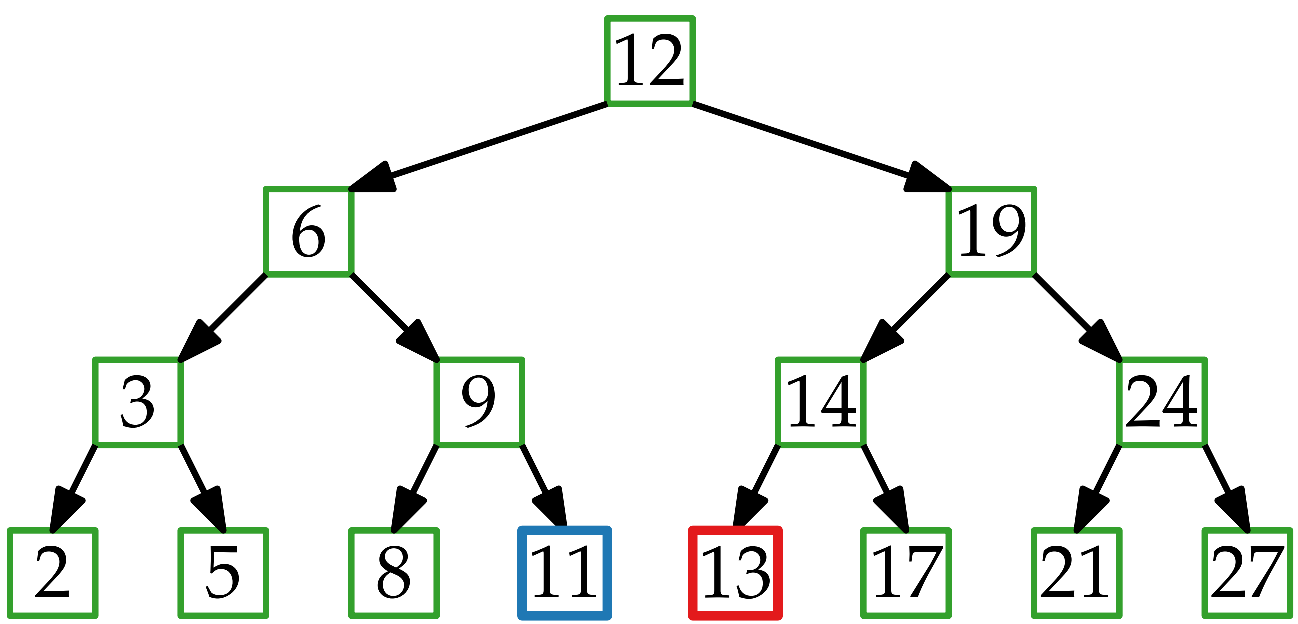

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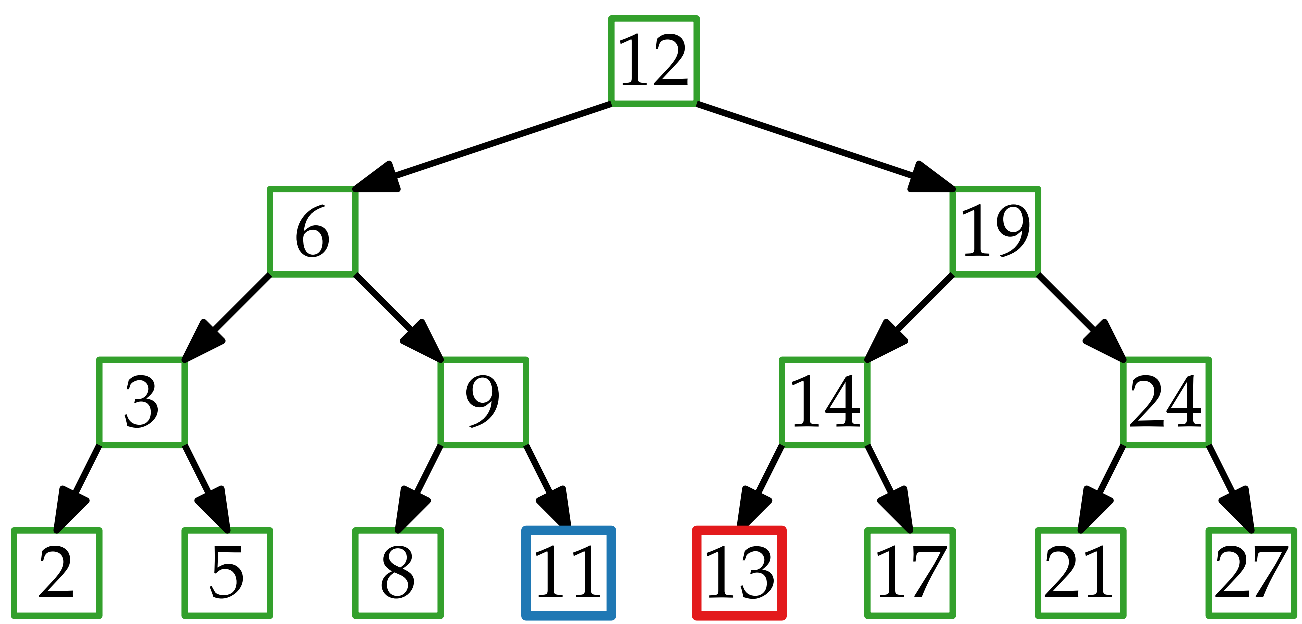

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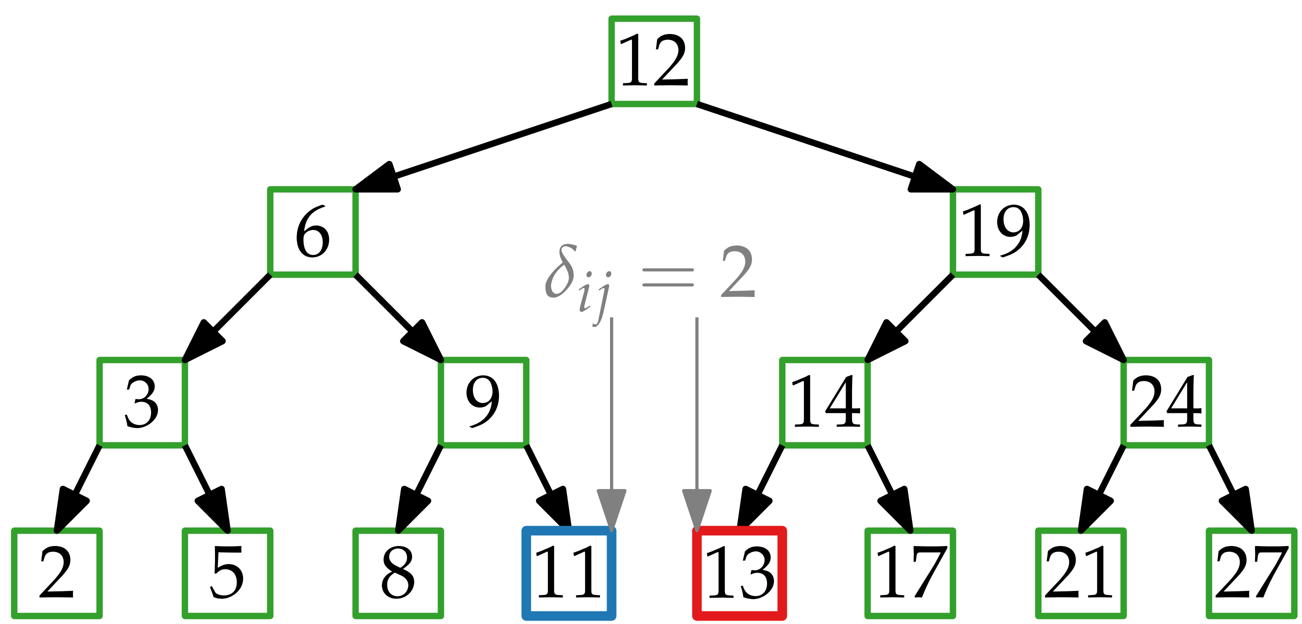

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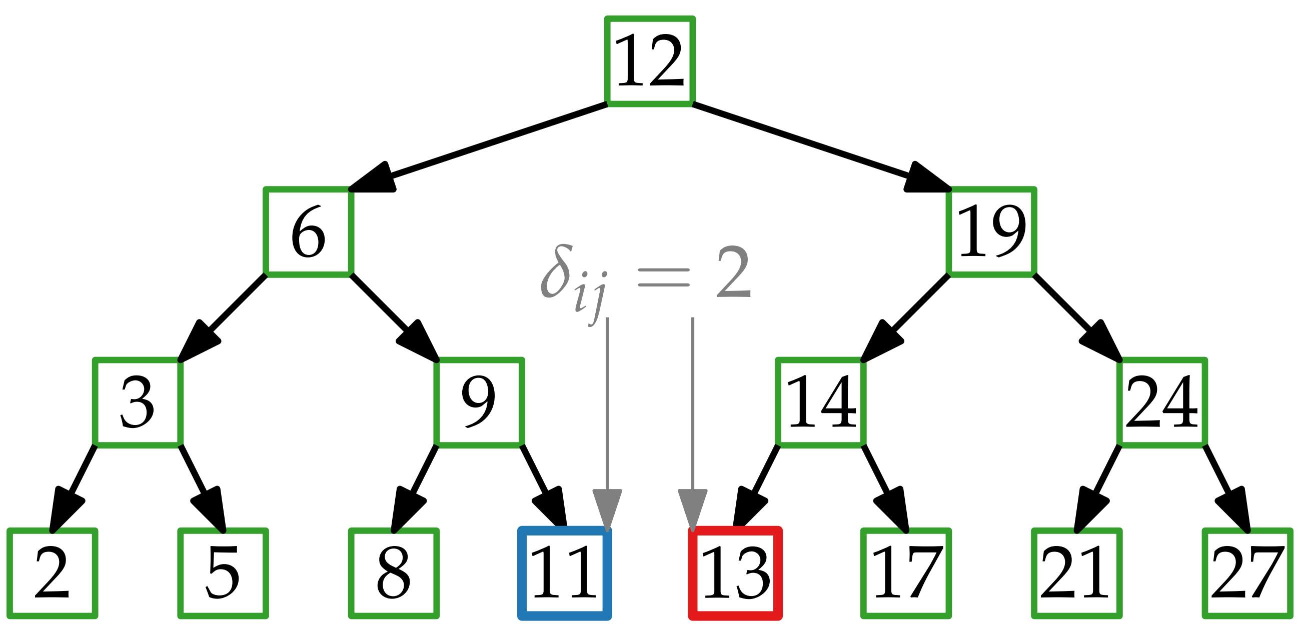

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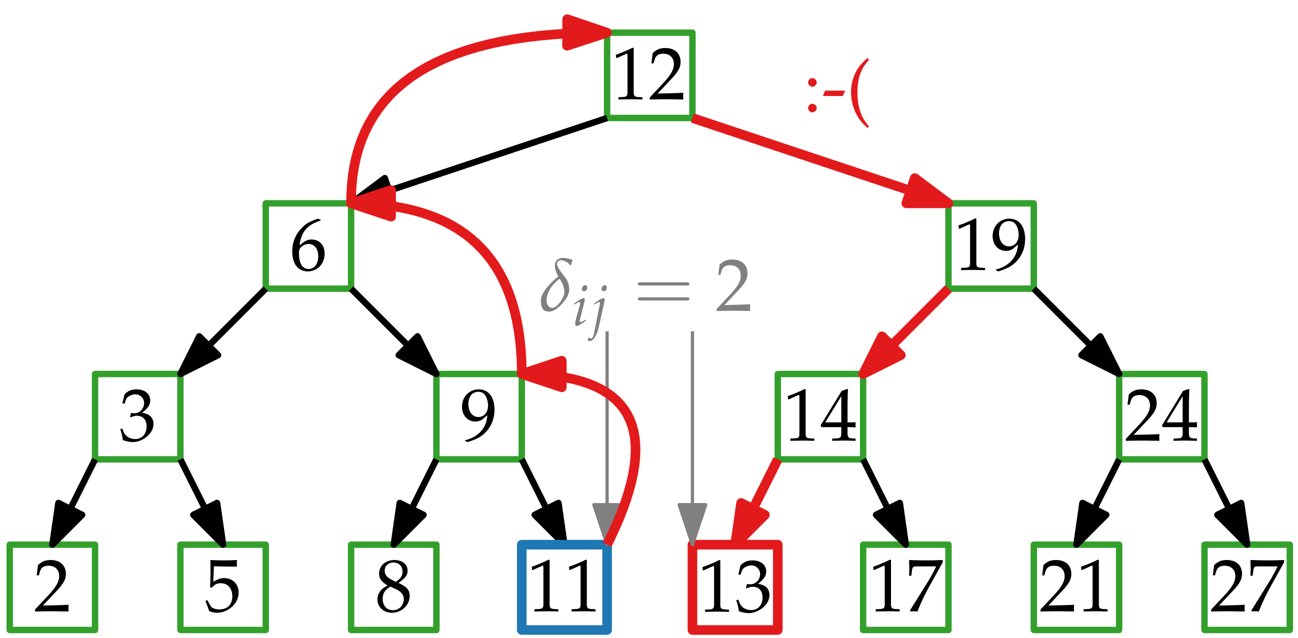

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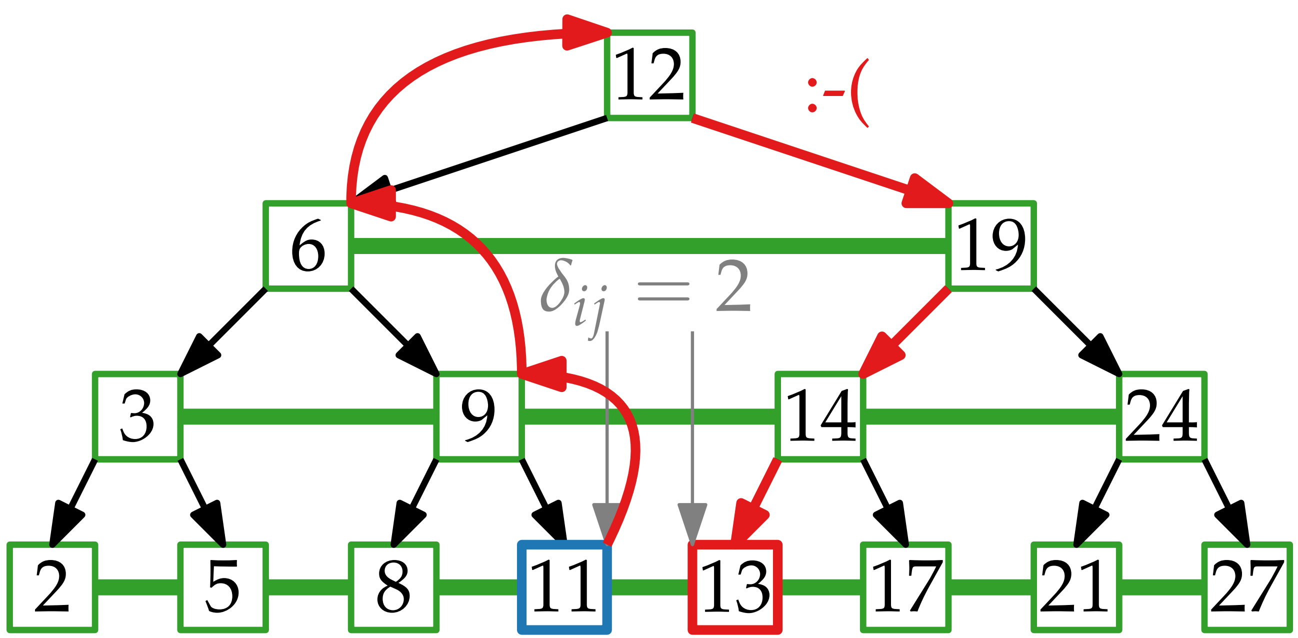

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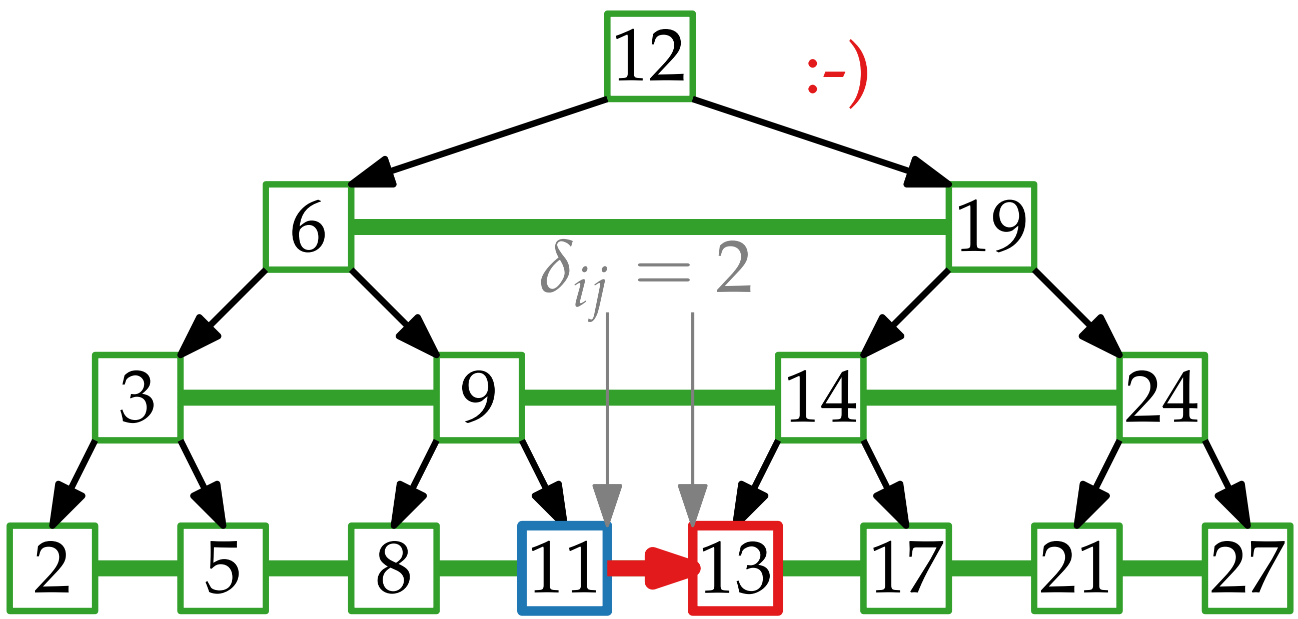

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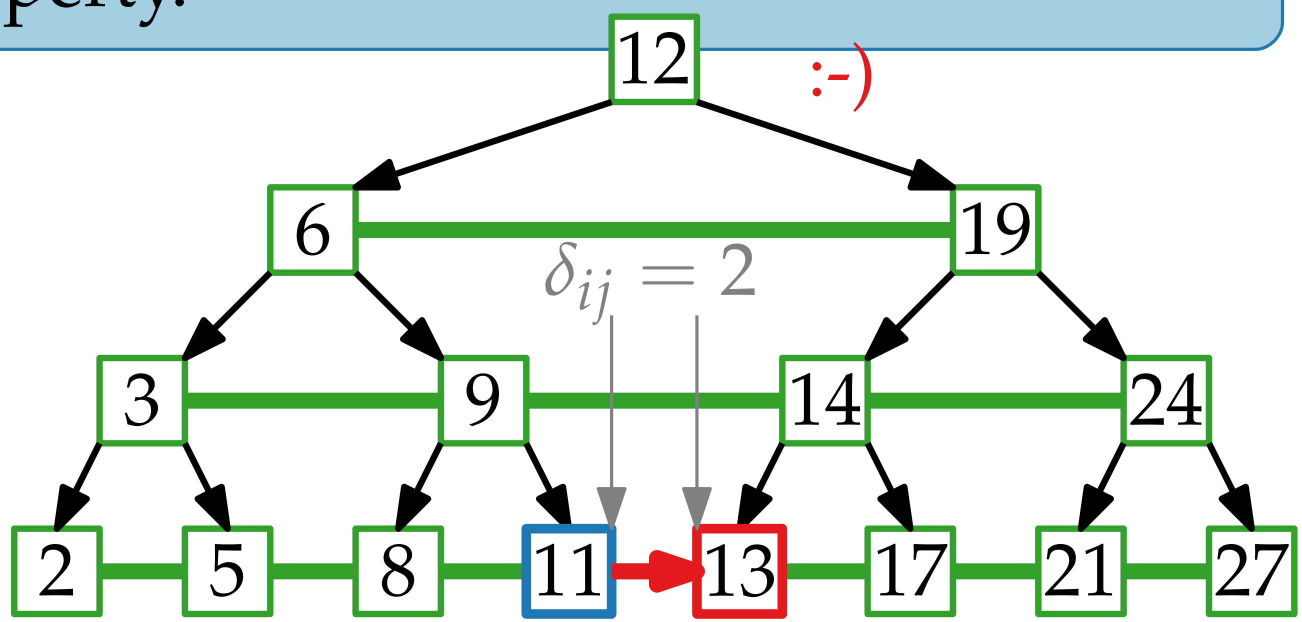
Lemma. A level-linked Red-Black-Tree has the dynamic finger property.



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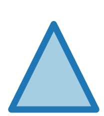

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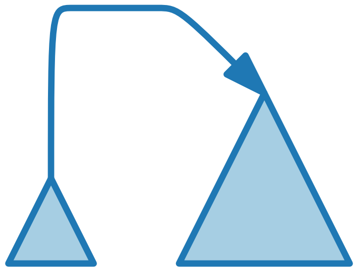

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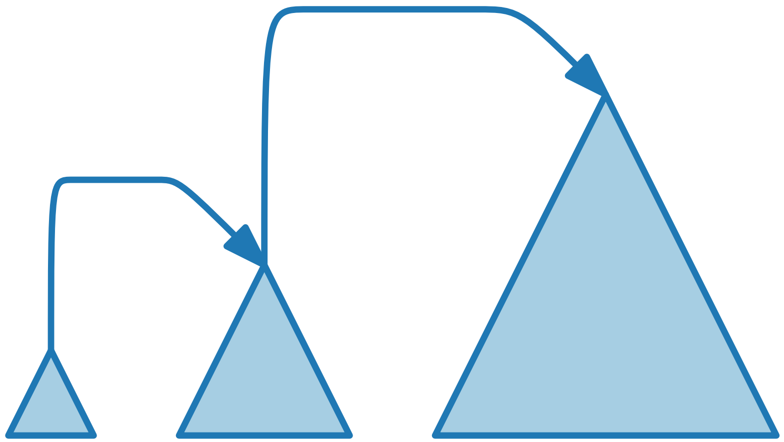

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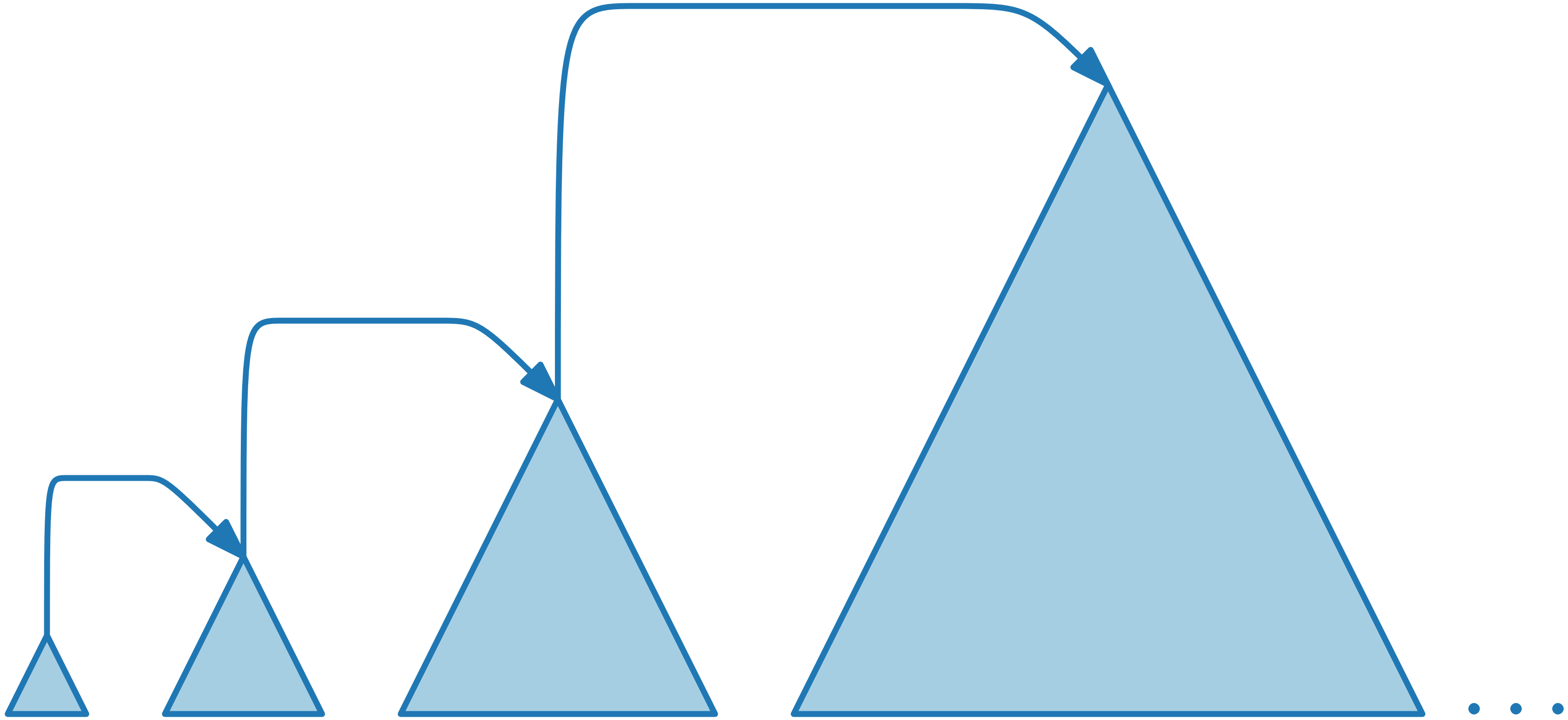

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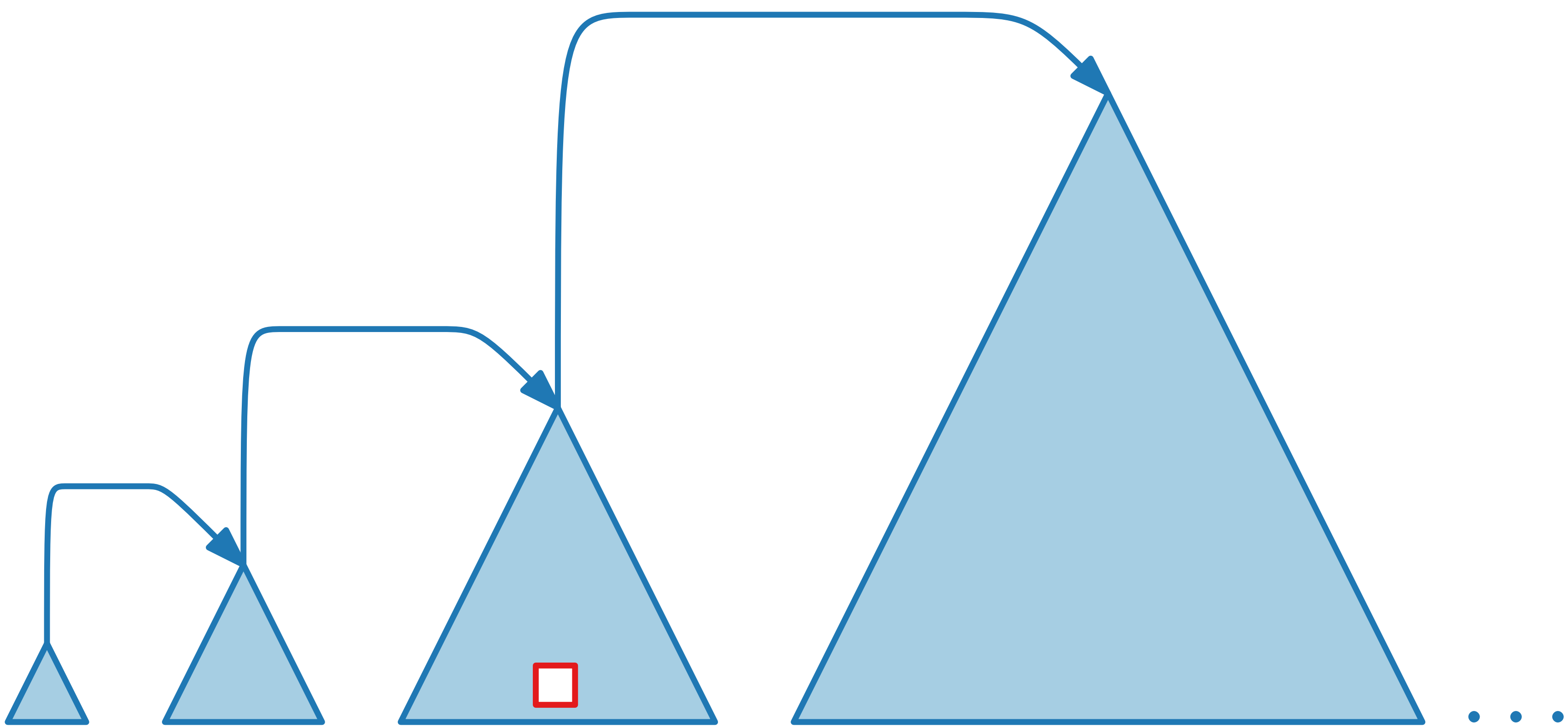

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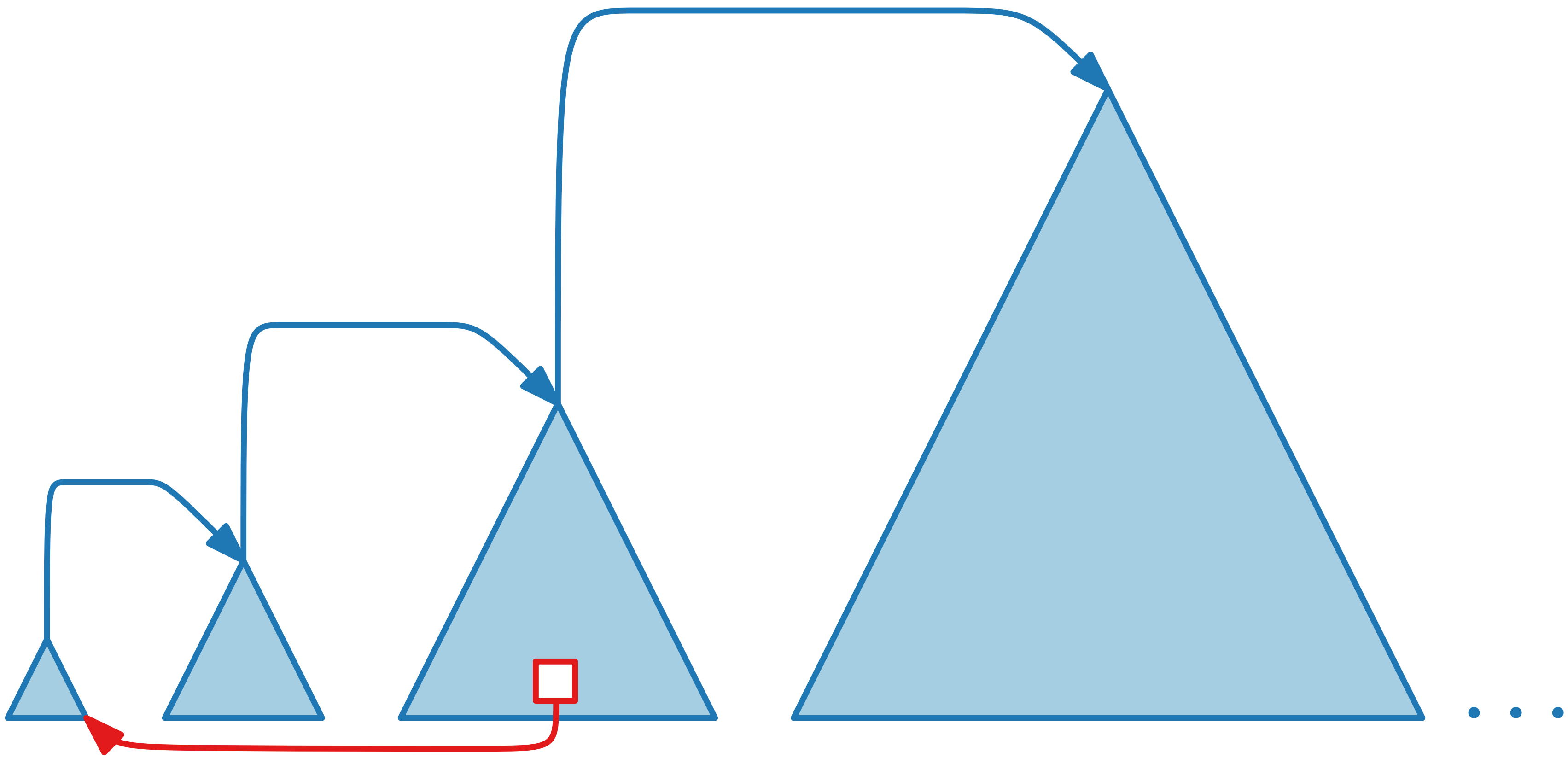

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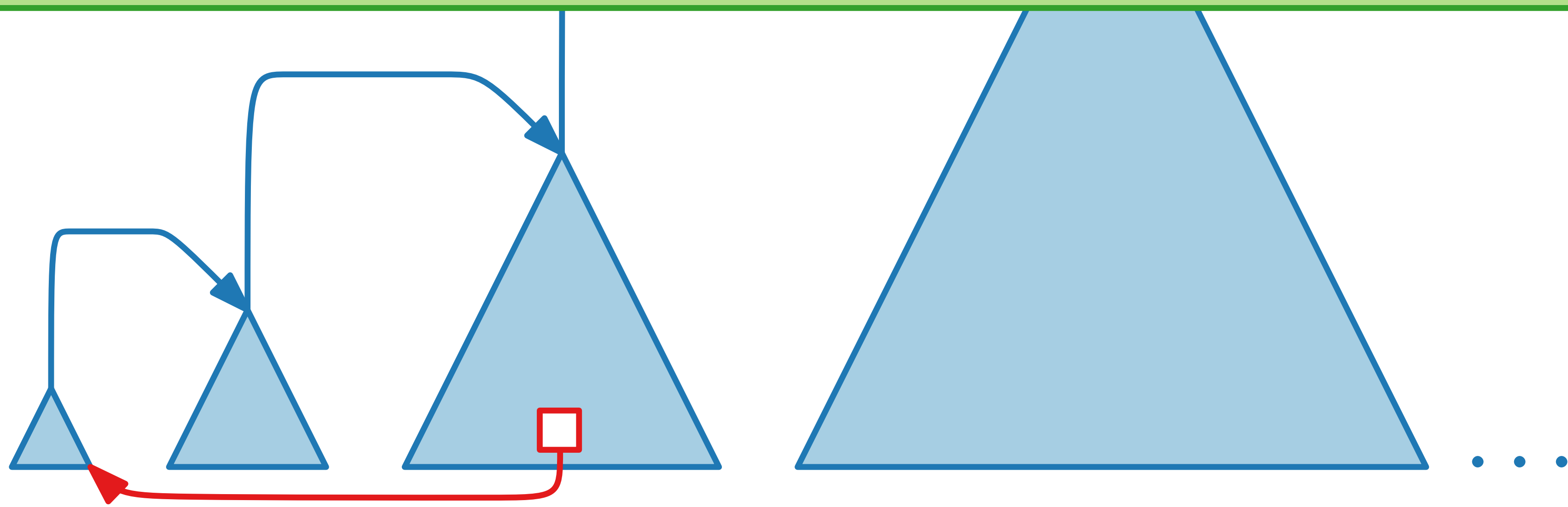
Definition. A BST has the **working set property** if the (amortized) cost of a query for key x is $O(\log t)$, where t is the number of keys queried more recently than x.



All these properties...

Balanced:	Queries take (amort.) $O(\log n)$ time
Entropy:	Queries take expected $O(1 + H)$ time
Dynamic Finger:	Queries take $O(\log \delta_i)$ time (δ_i: rank diff.)
Working Set:	Queries take $O(\log t)$ time (t: recency)

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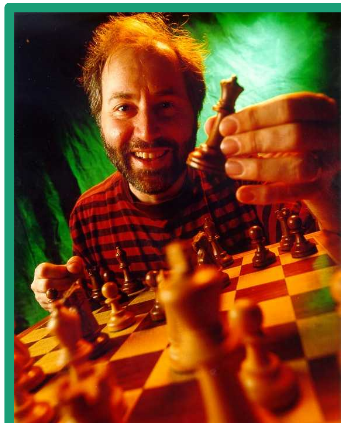
Yes!

Splay Trees



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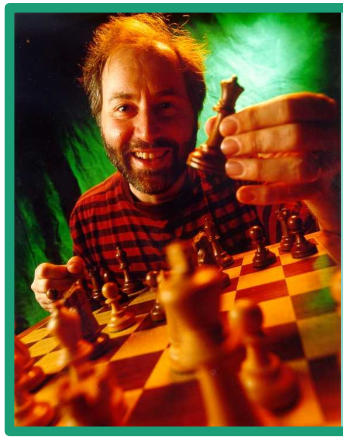

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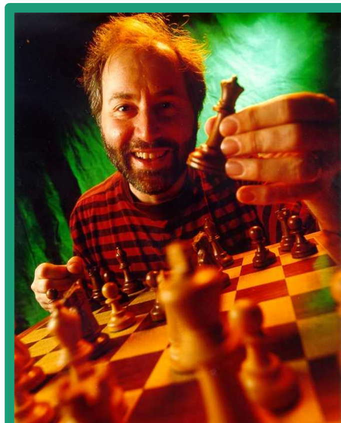

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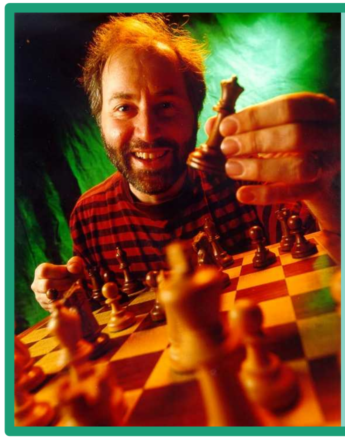

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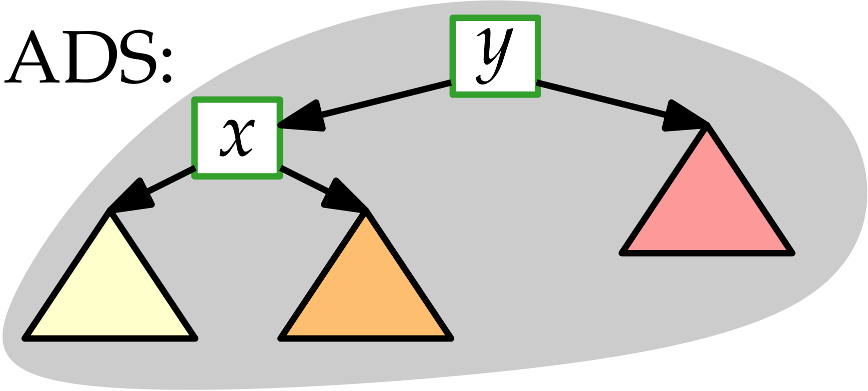

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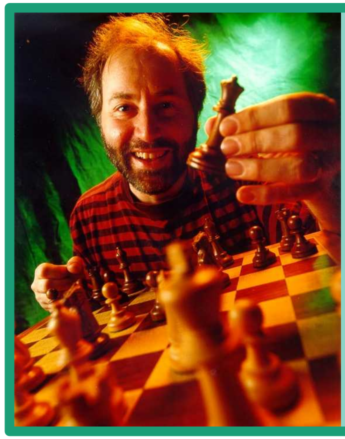

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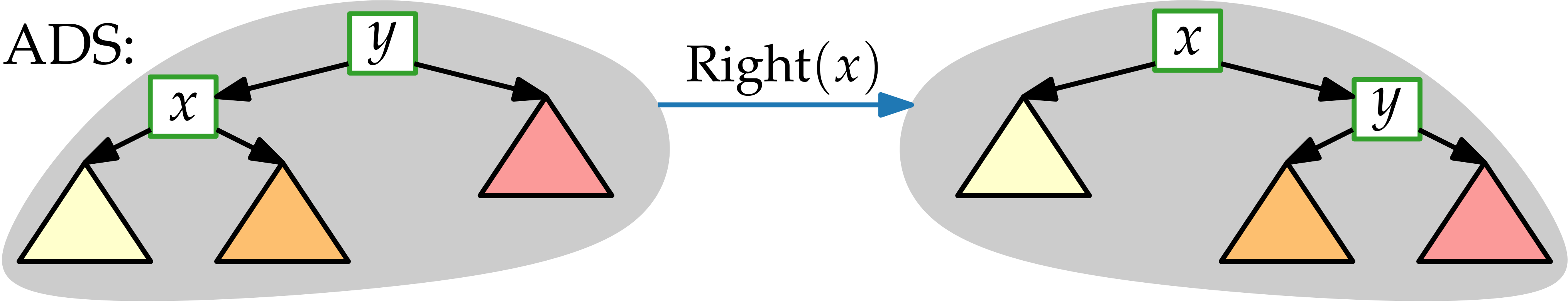

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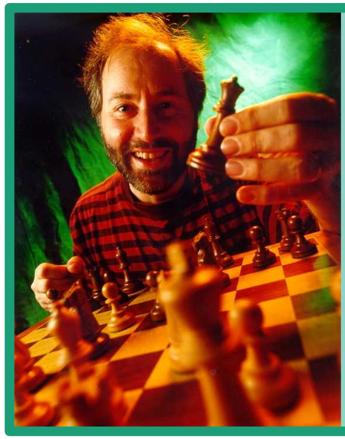

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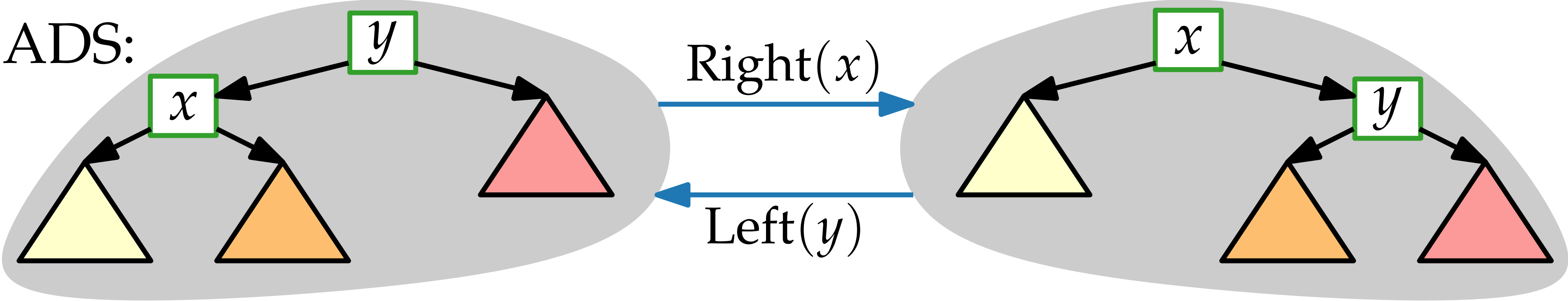

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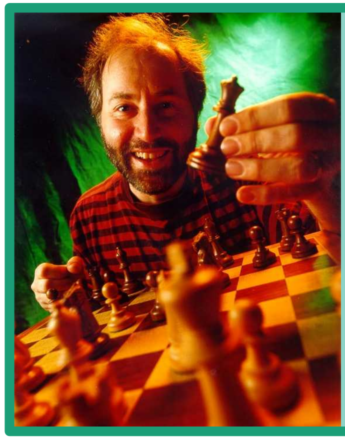

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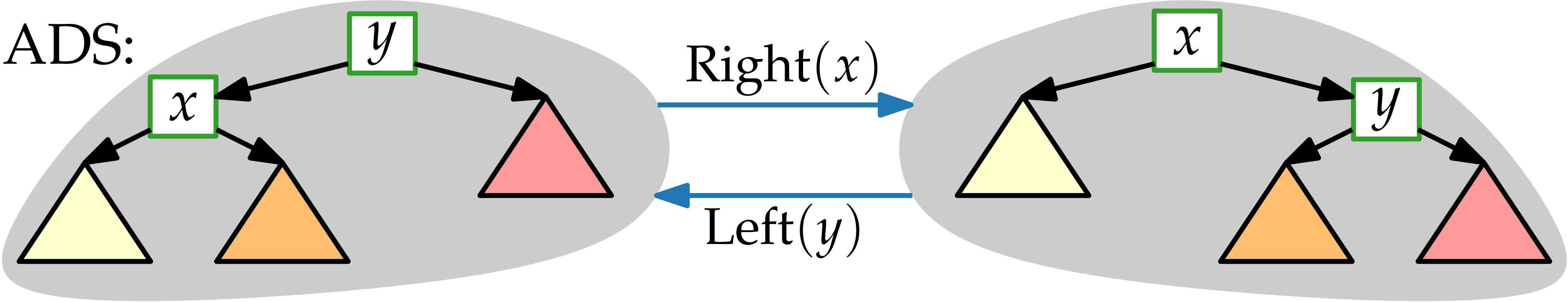

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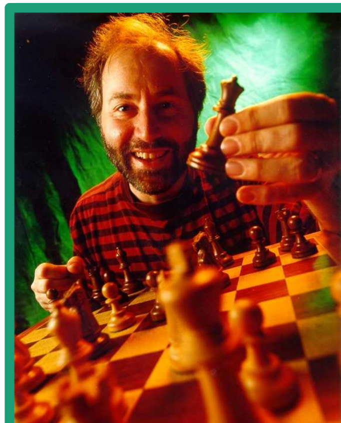

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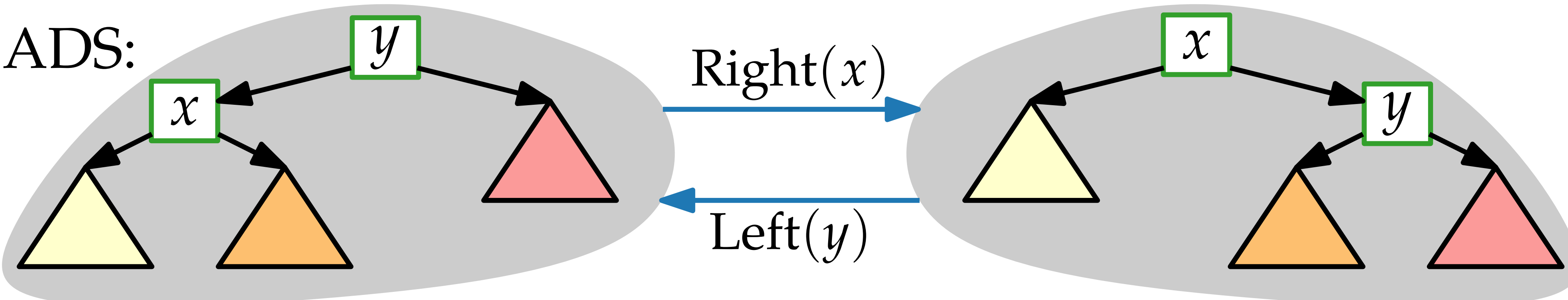

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Query(x): Splay(x), then return root

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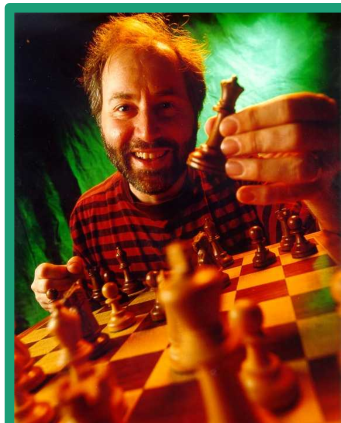

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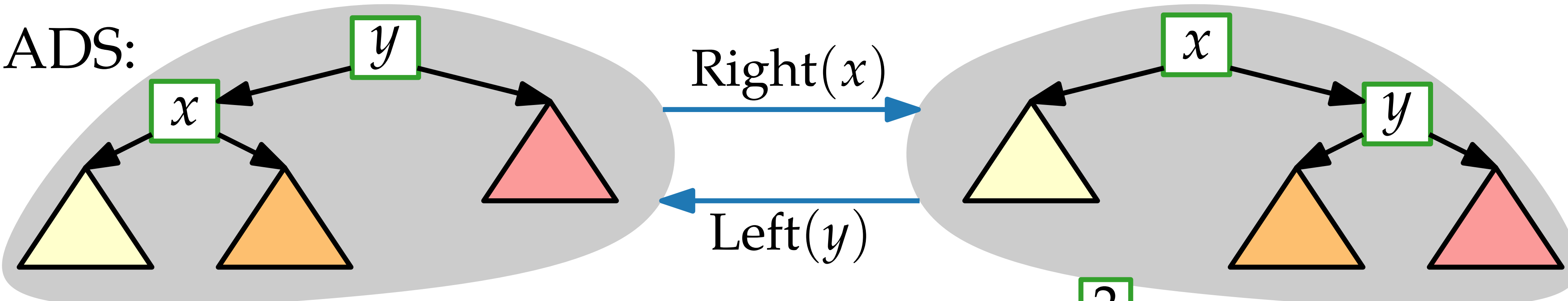

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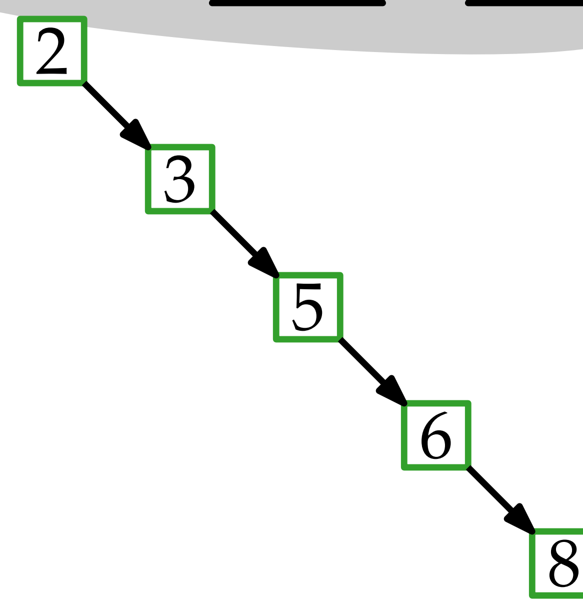

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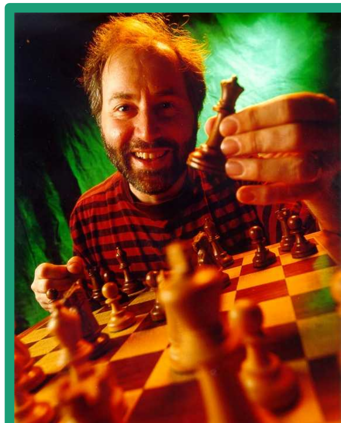

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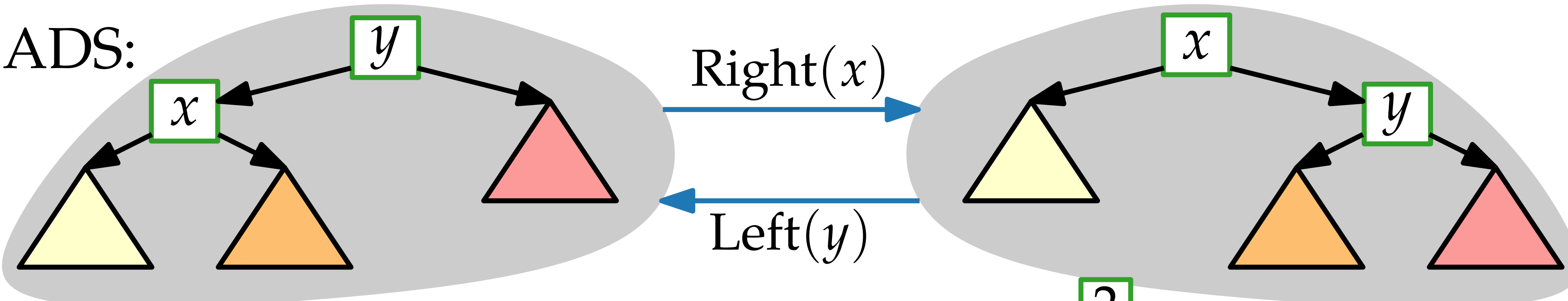

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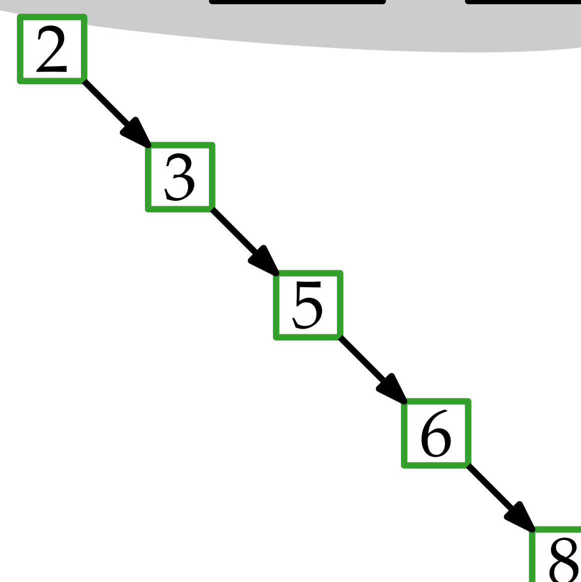
Query(8)



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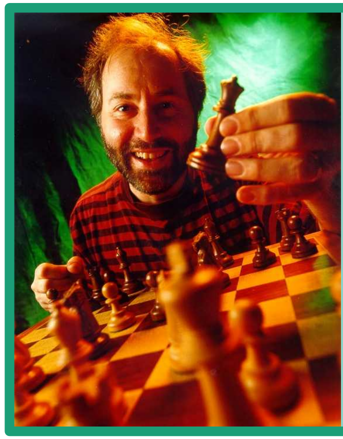

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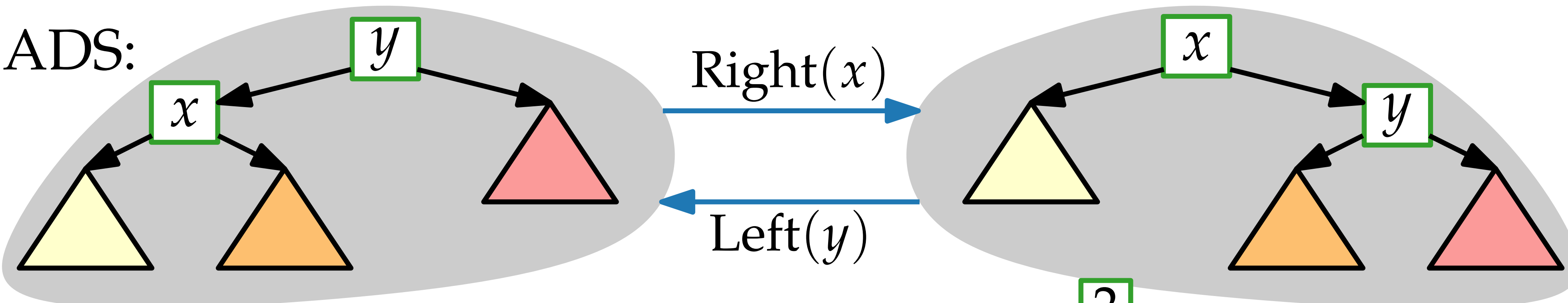

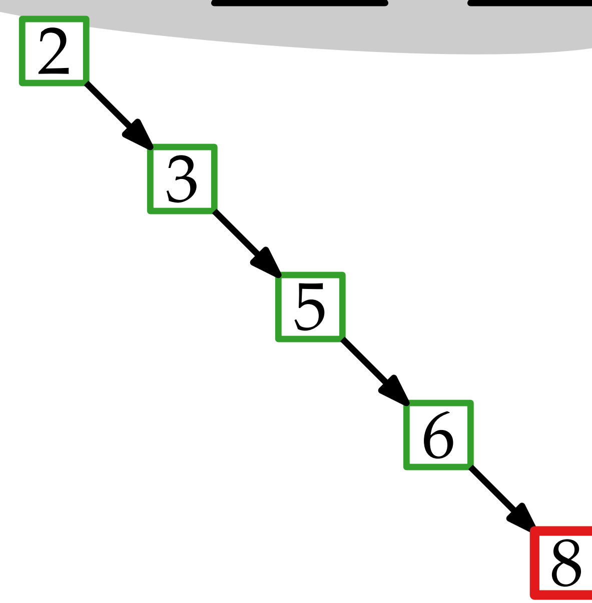

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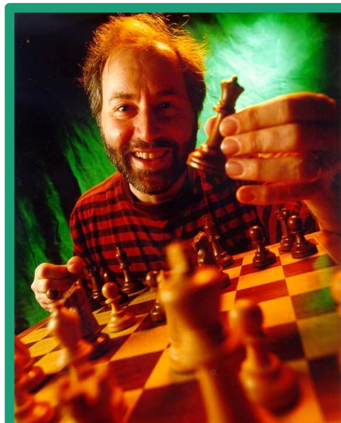

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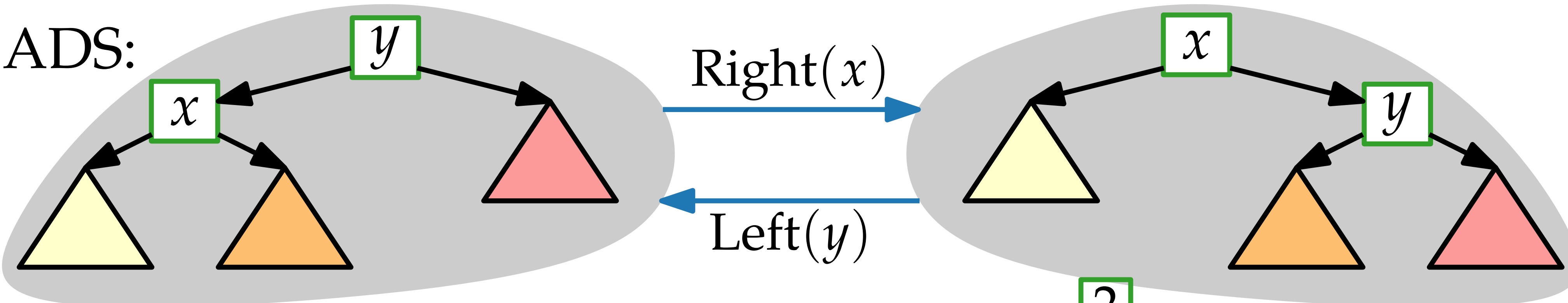

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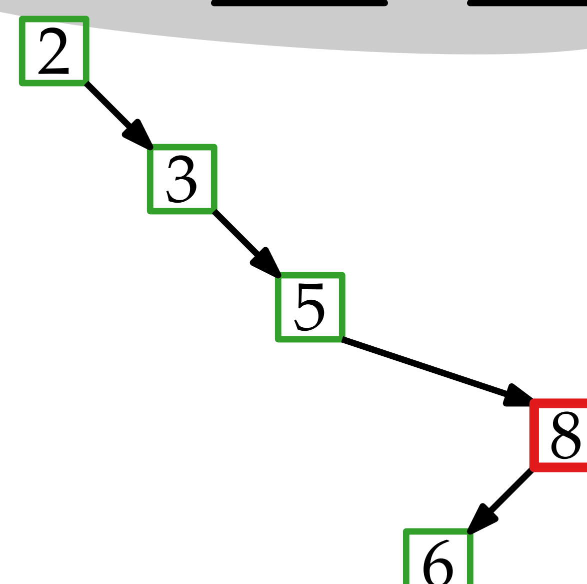

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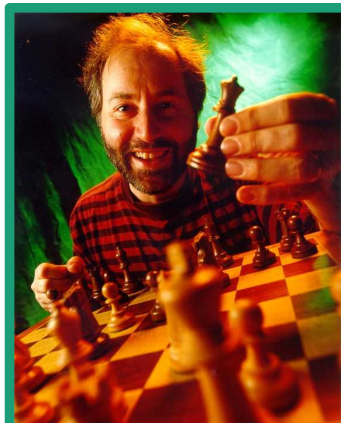

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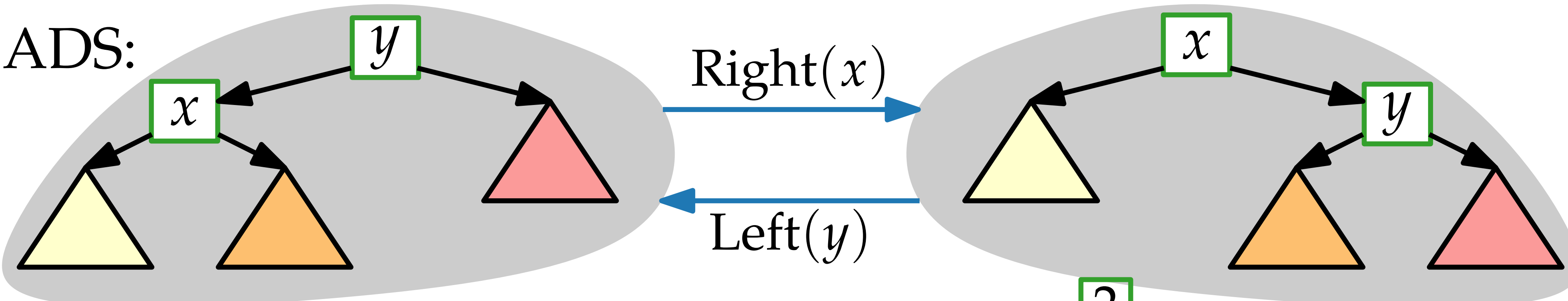

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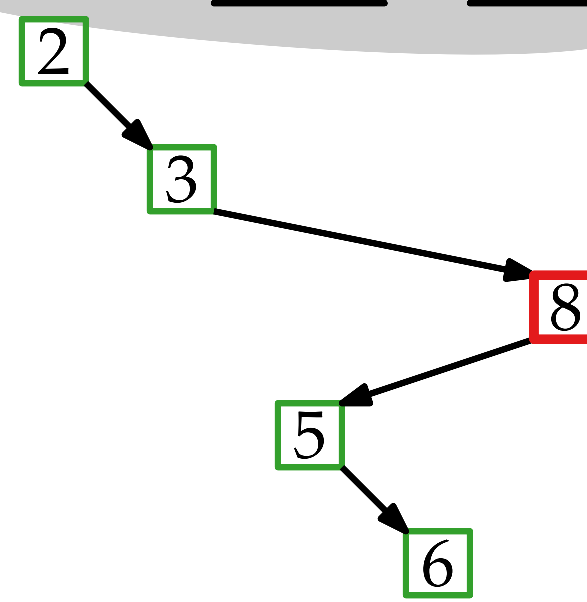
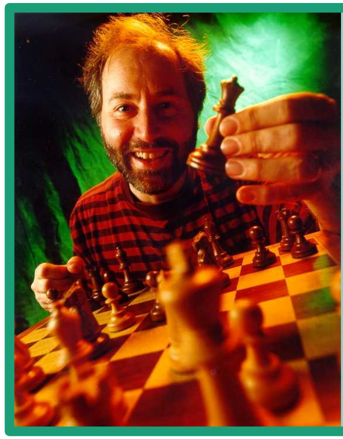


Splay Trees



Daniel D. Sleator

Robert E. Tarjan

J. ACM 1985

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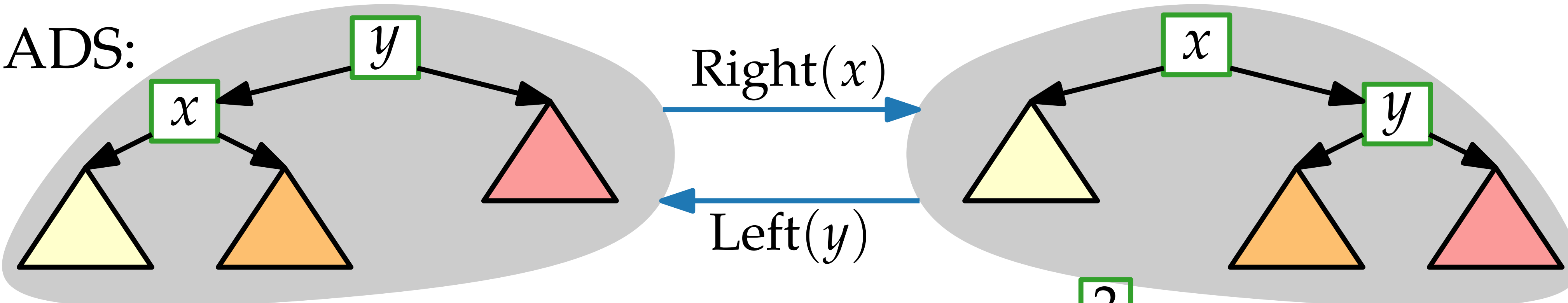
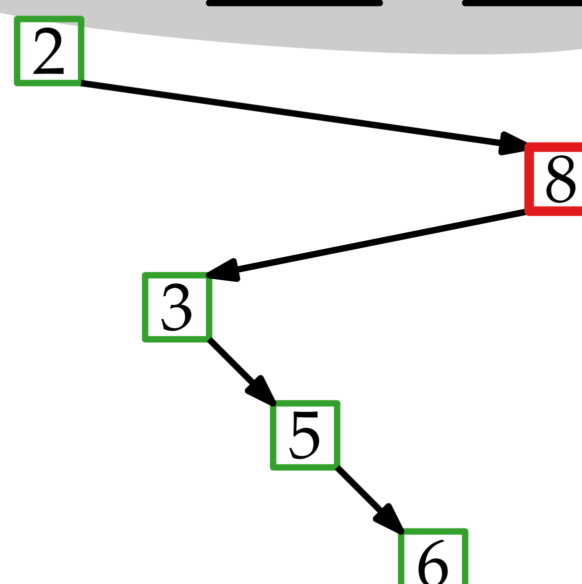


Splay(x): Rotate x to the root

Query(x): Splay(x), then return root

Query(8)



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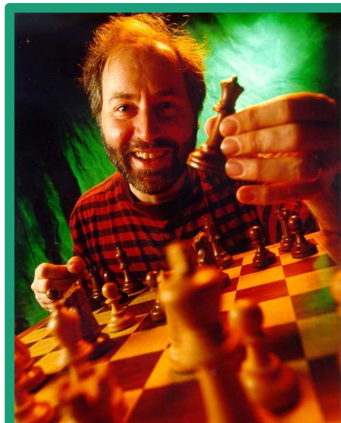

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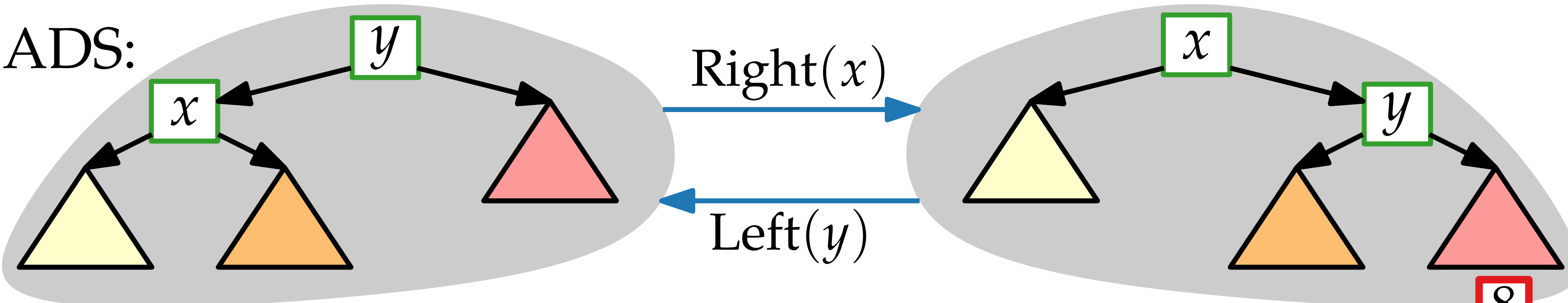

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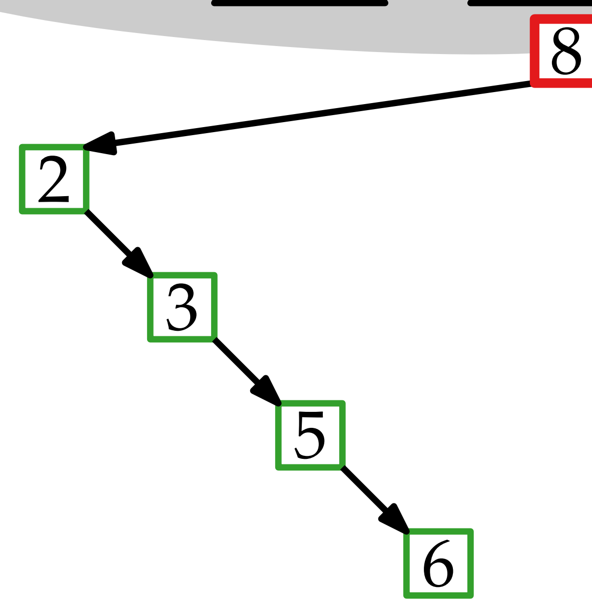
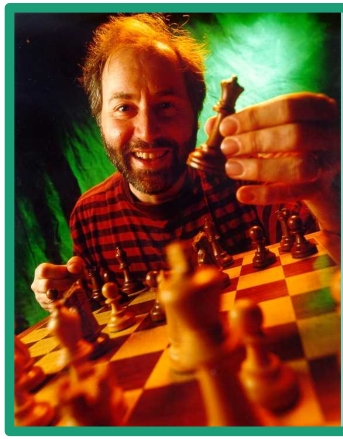


Splay Trees



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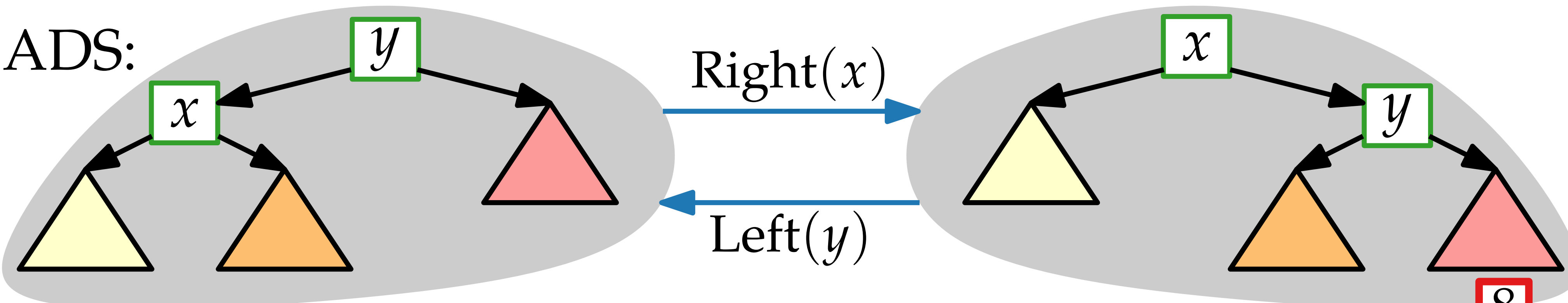

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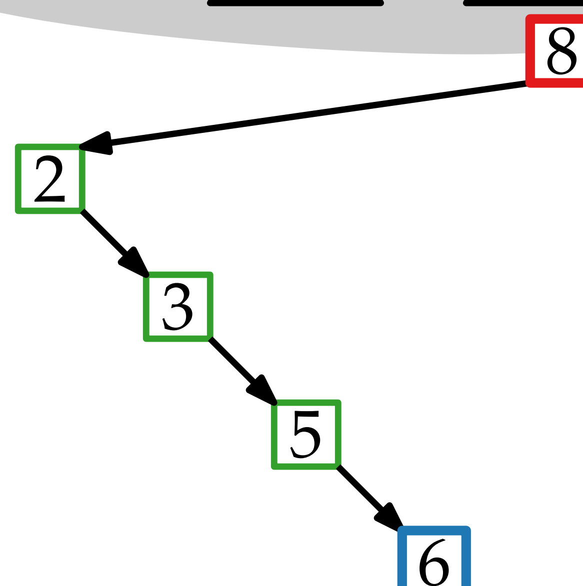
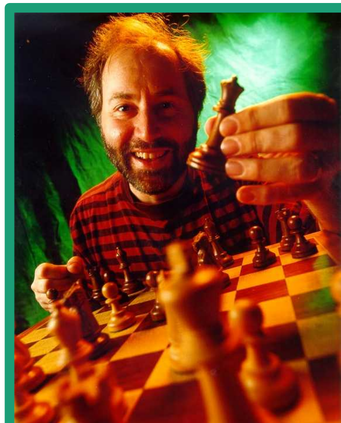
Query(8) Query(6)



Splay Trees



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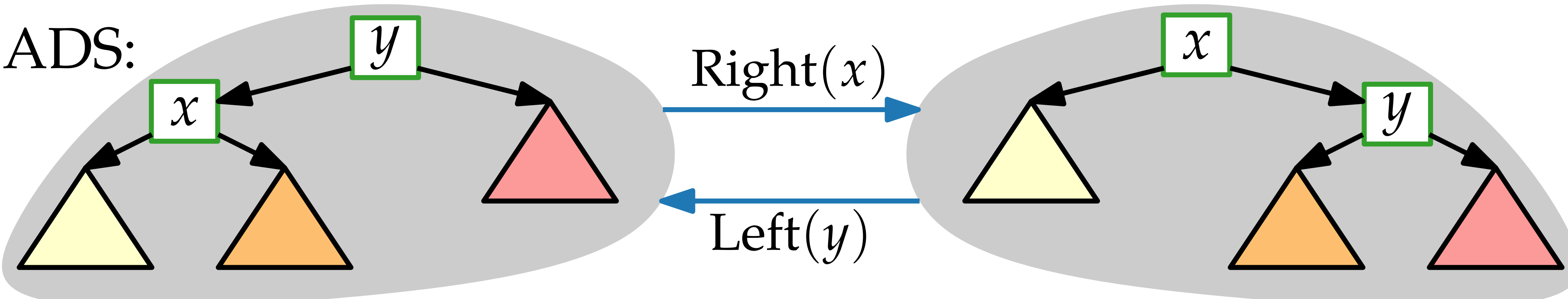

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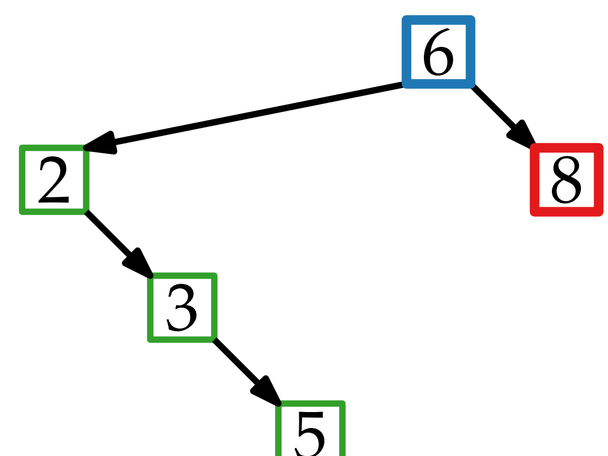
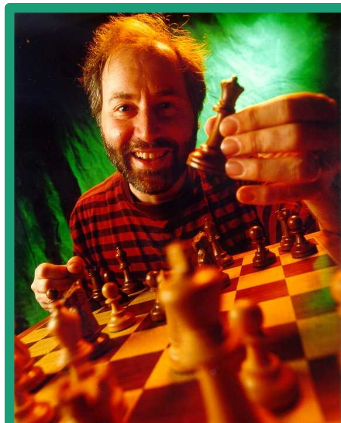
Query(x): Splay(x), then return root

Query(8) Query(6)



Splay Trees



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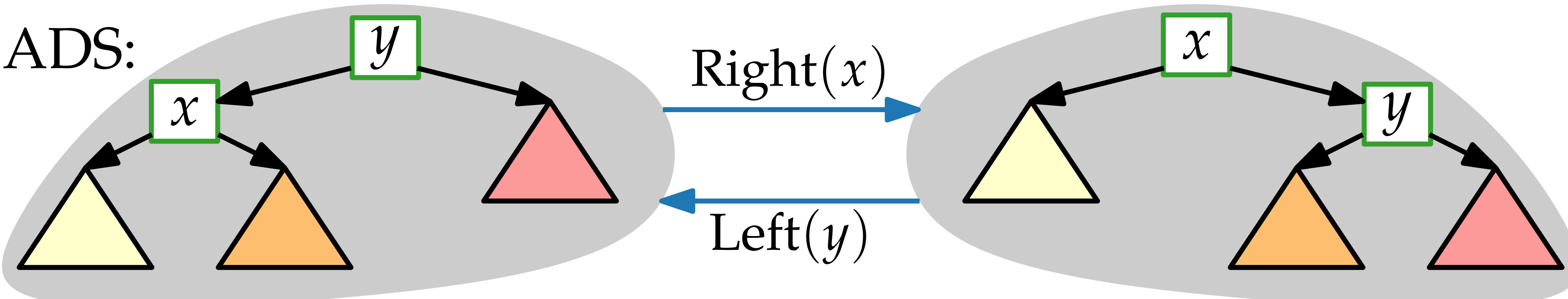
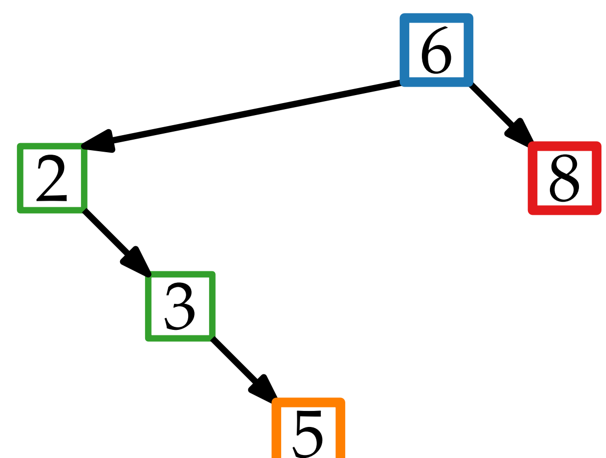


Splay(x): Rotate x to the root

Query(x): Splay(x), then return root

Query(8) Query(6) Query(5)



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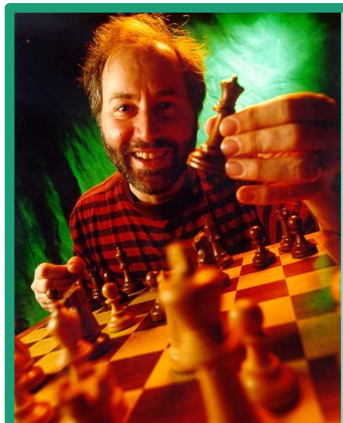

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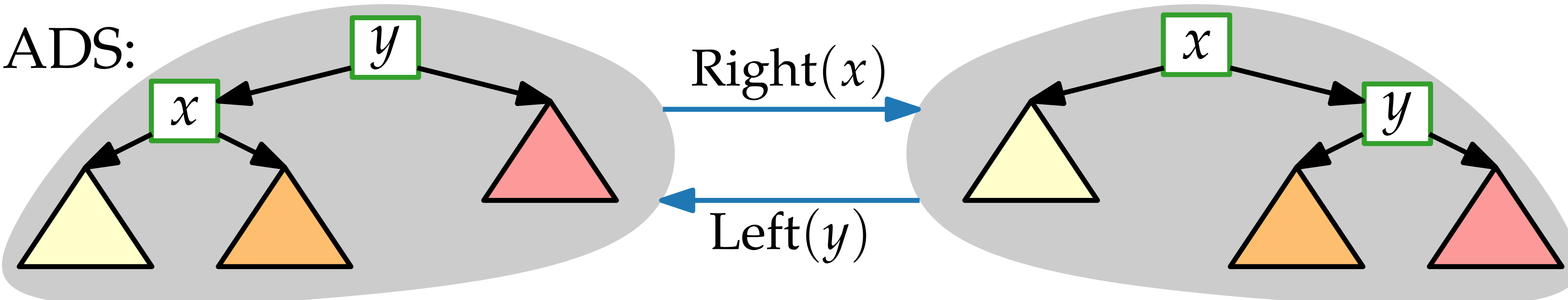

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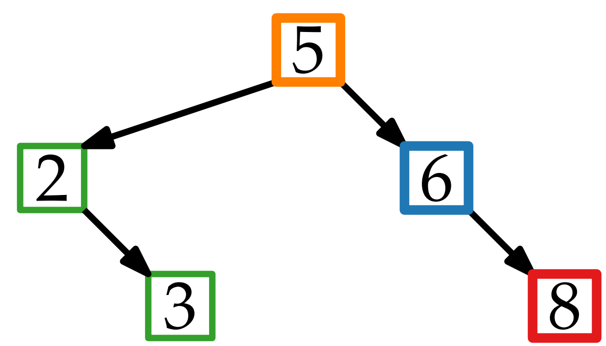

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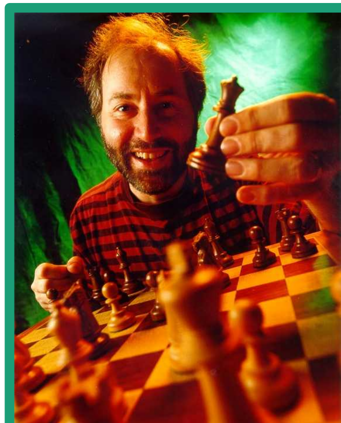

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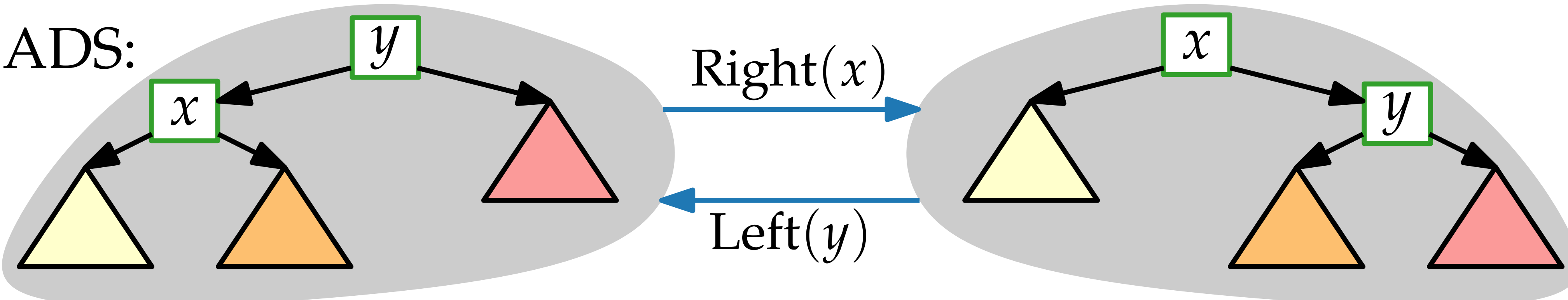

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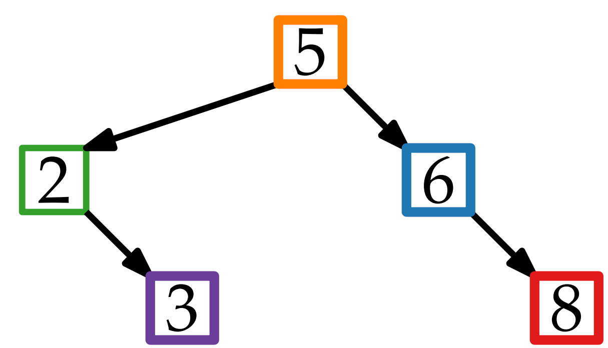
Query(8) Query(6) Query(5)
Query(3)



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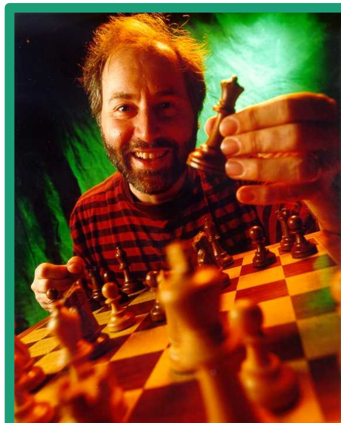

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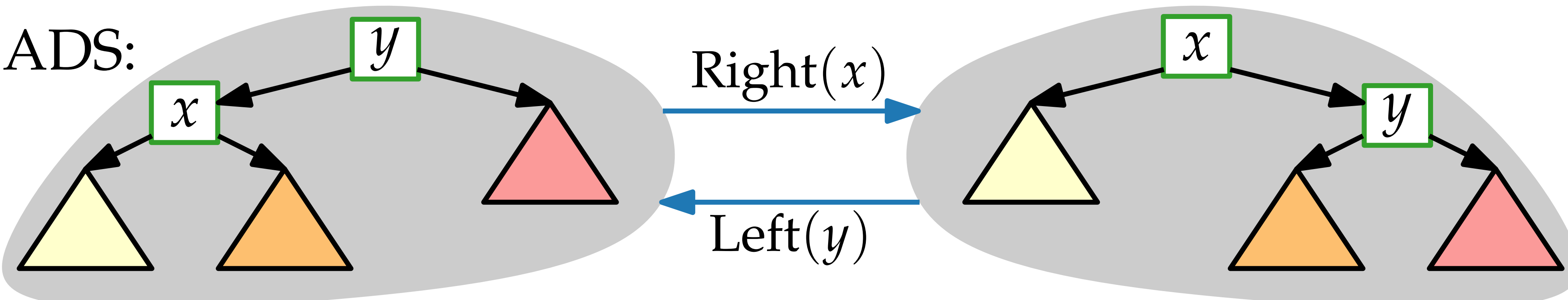

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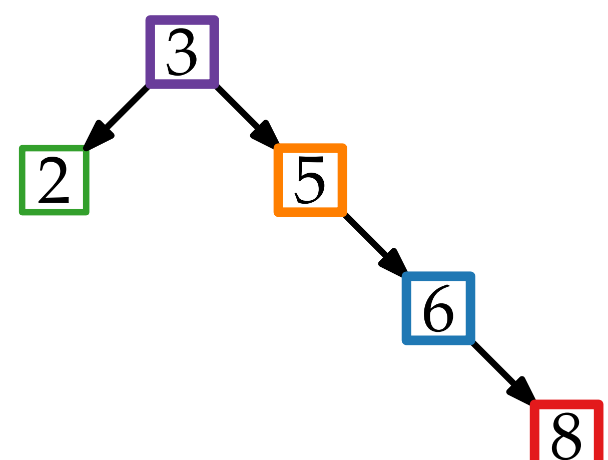

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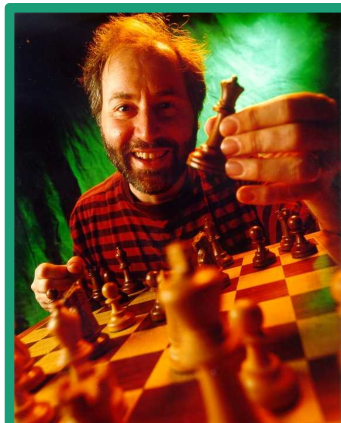

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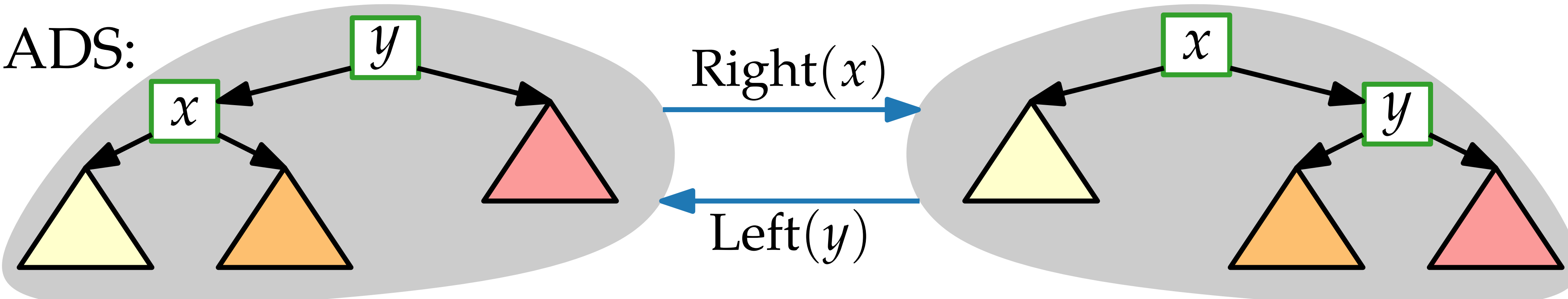

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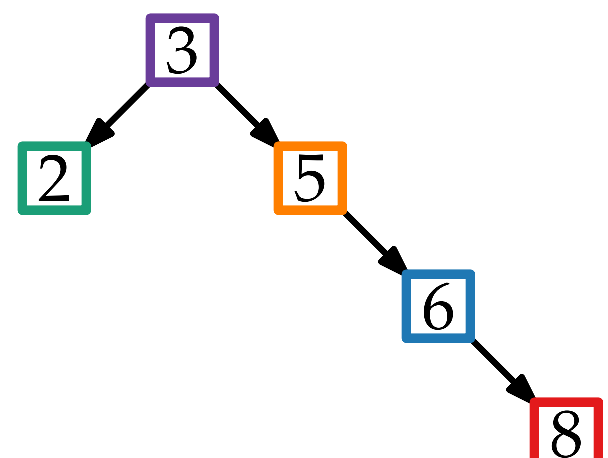

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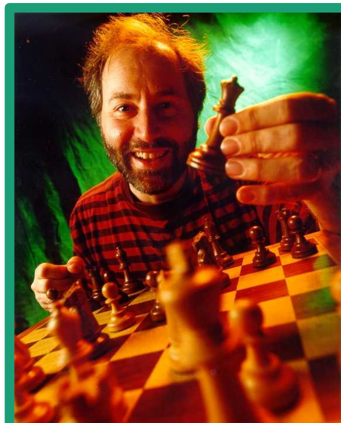

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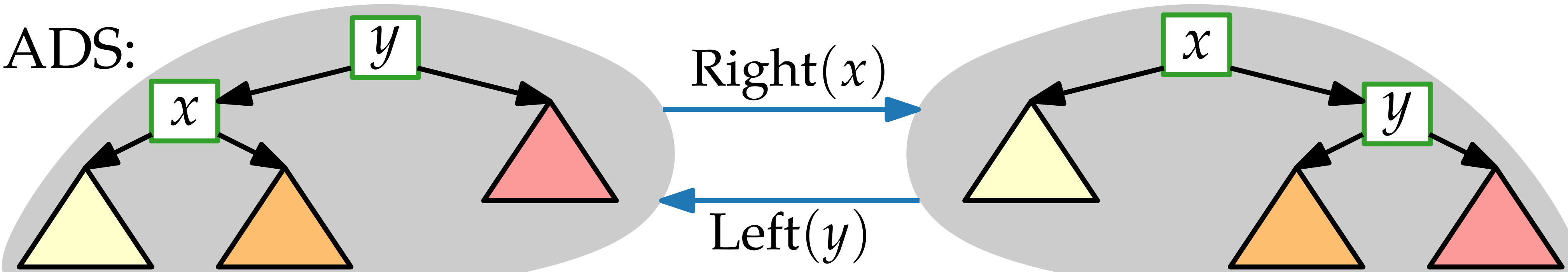

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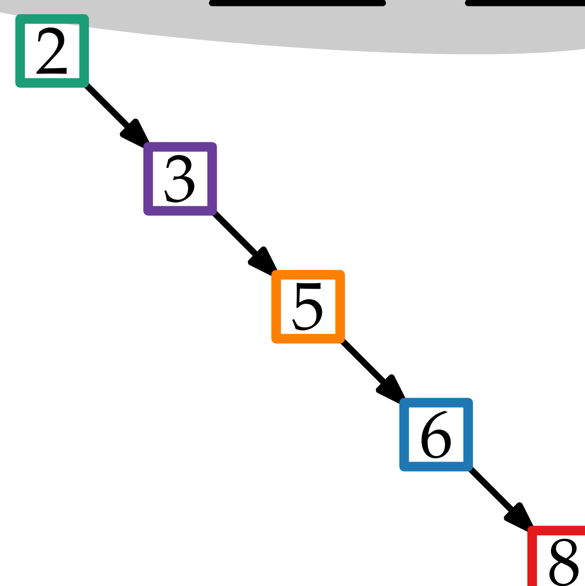

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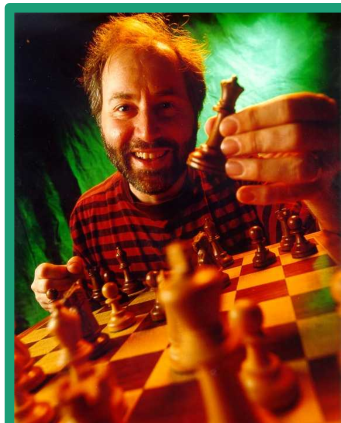

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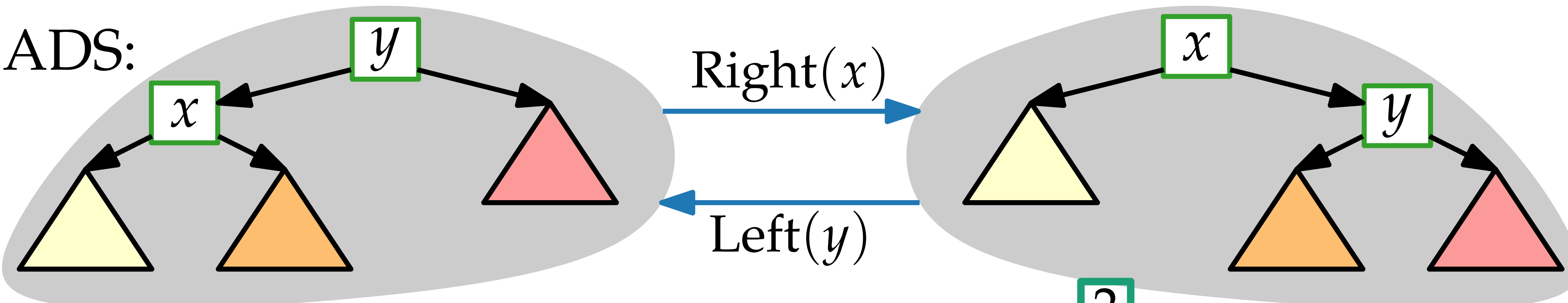

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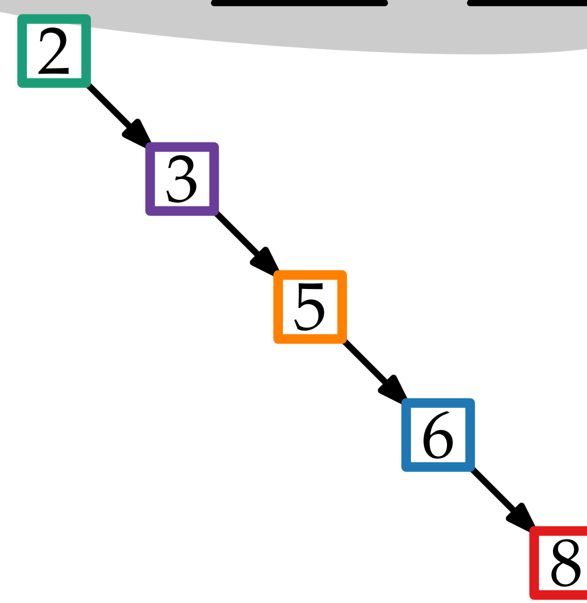
Query(8) Query(6) Query(5)

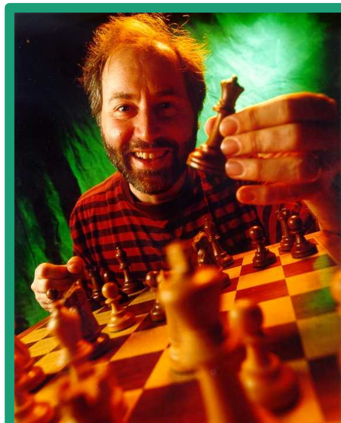
Query(3)

Query(2)

We're back at the start...



Splay Trees



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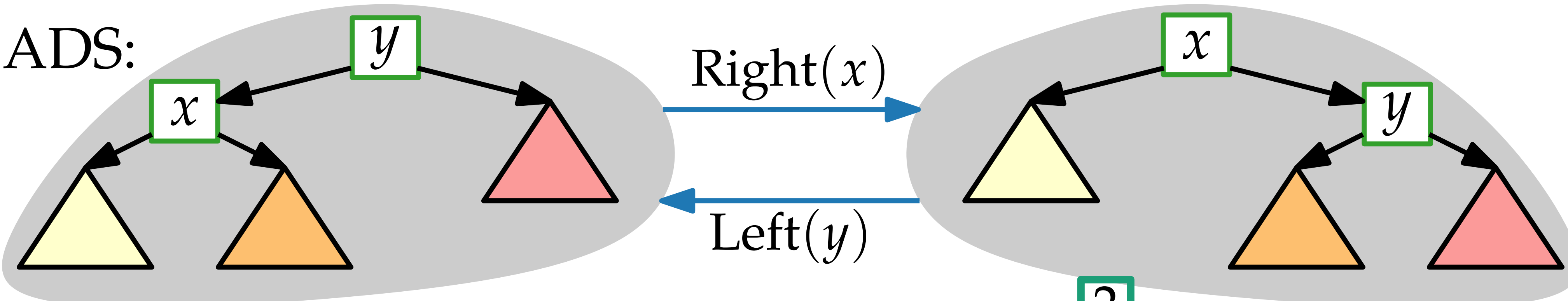

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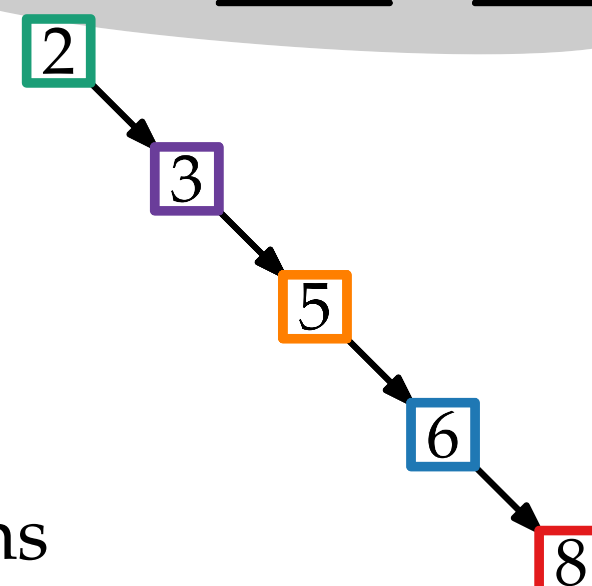
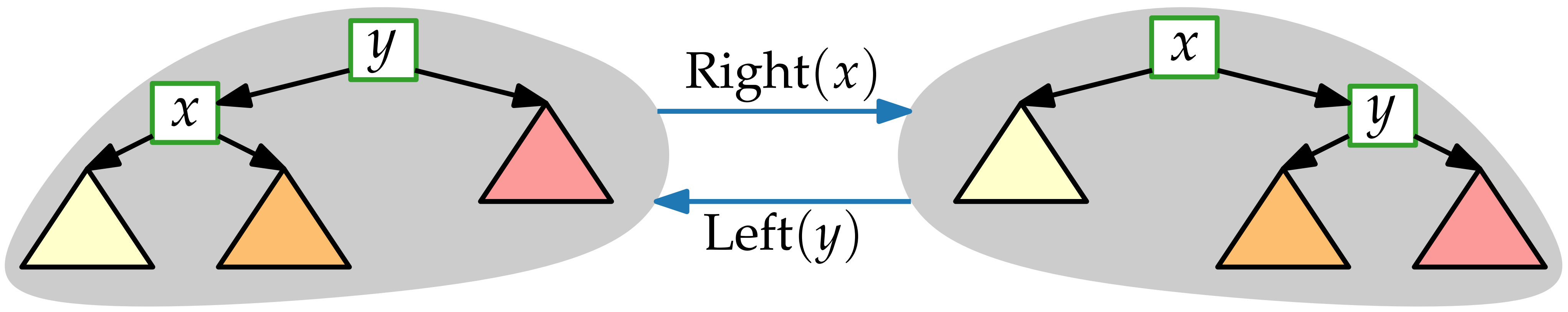
Query(x): Splay(x), then return root

Query(8) Query(6) Query(5)
Query(3)
Query(2)

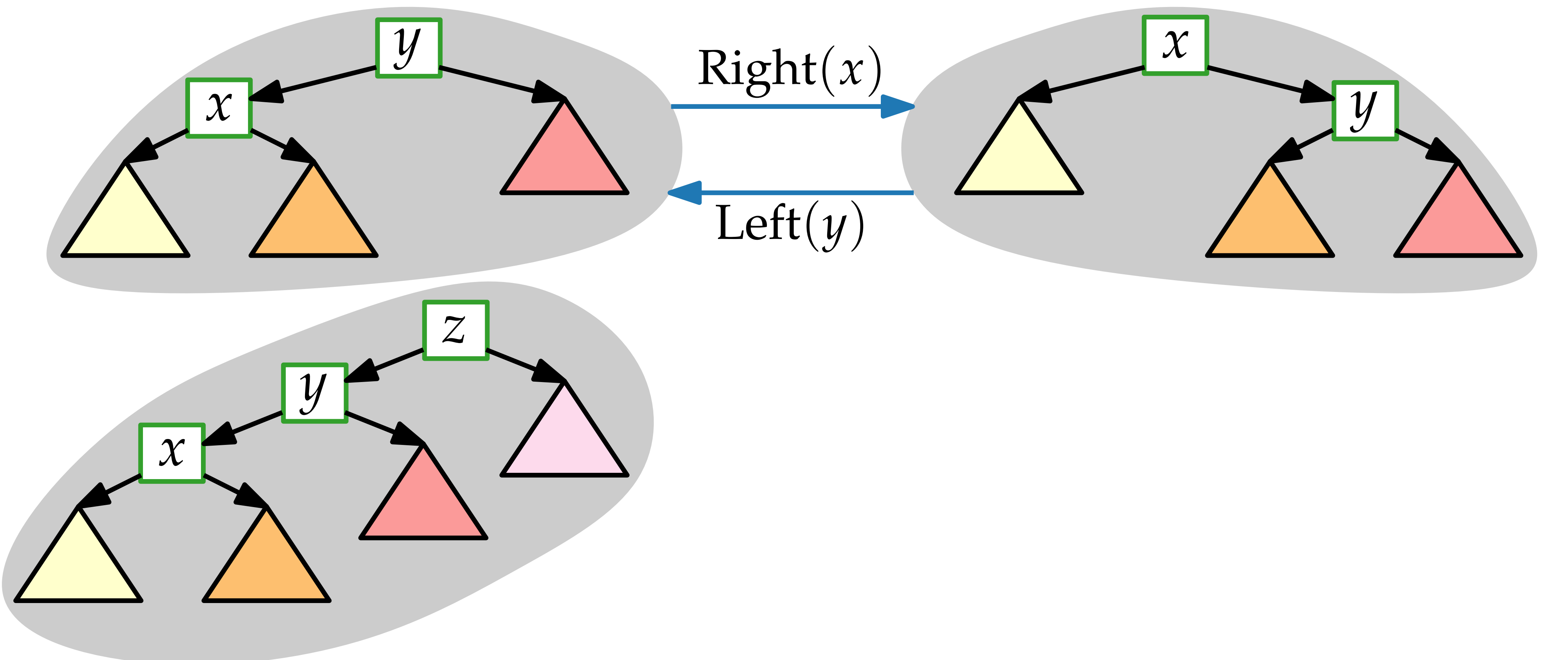
We're back at the start...
and we did $\Theta(n^2)$ rotations



Rotations II



Rotations II



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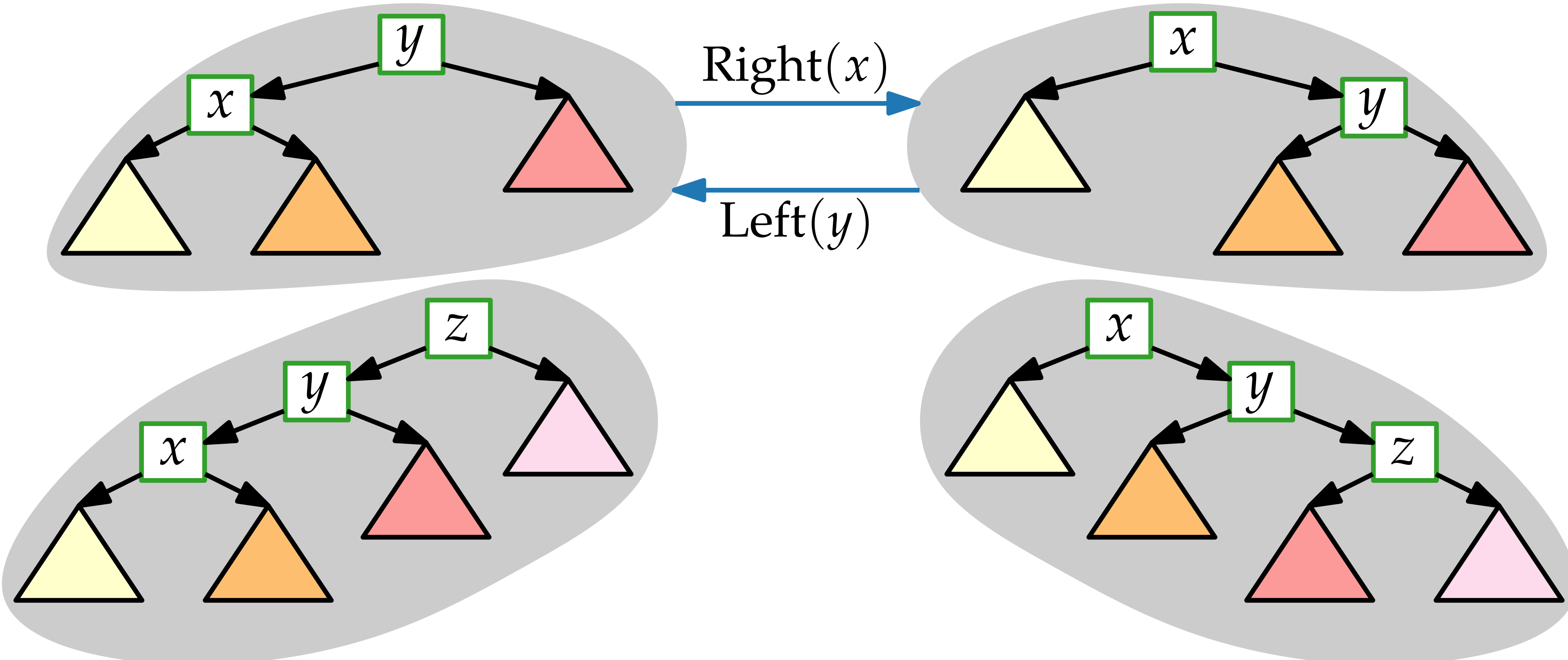
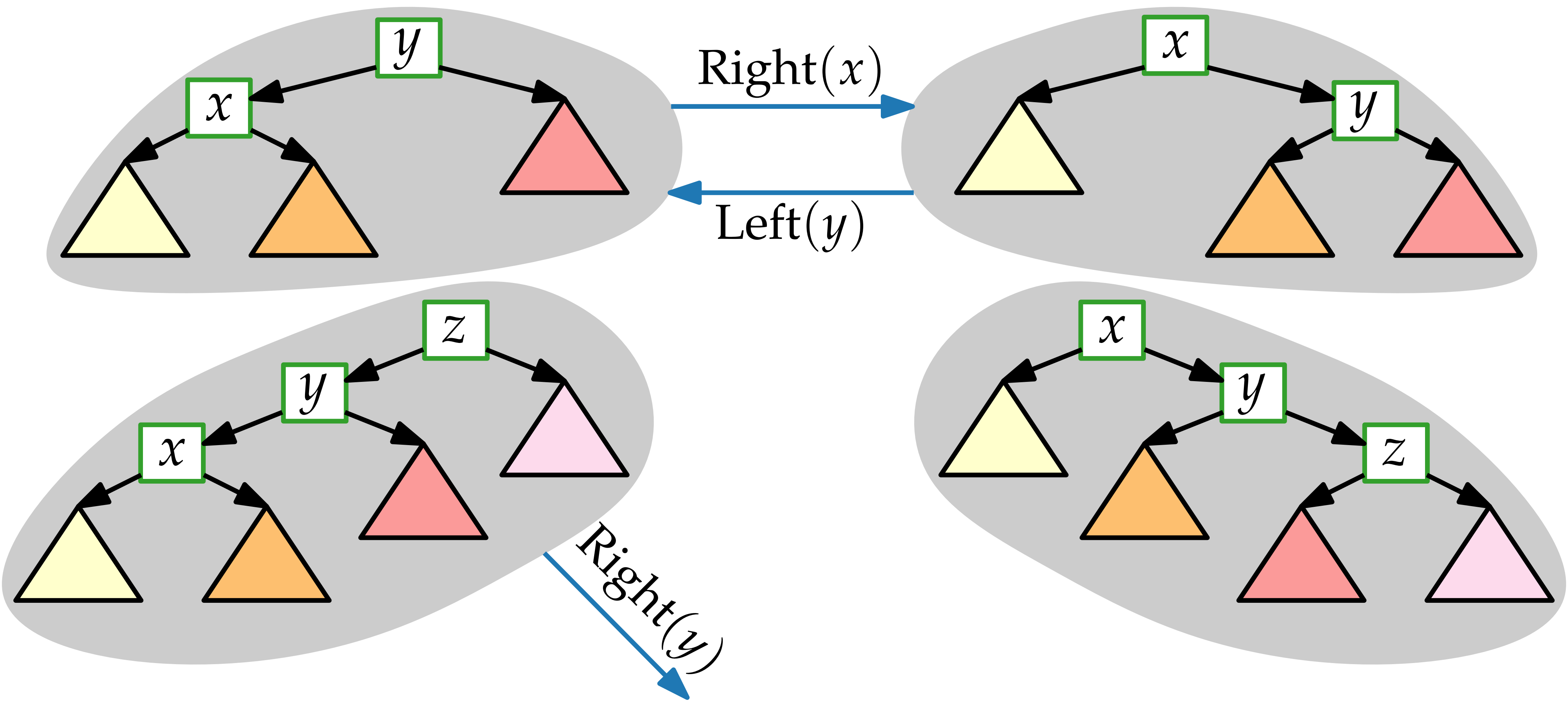


Rotations II



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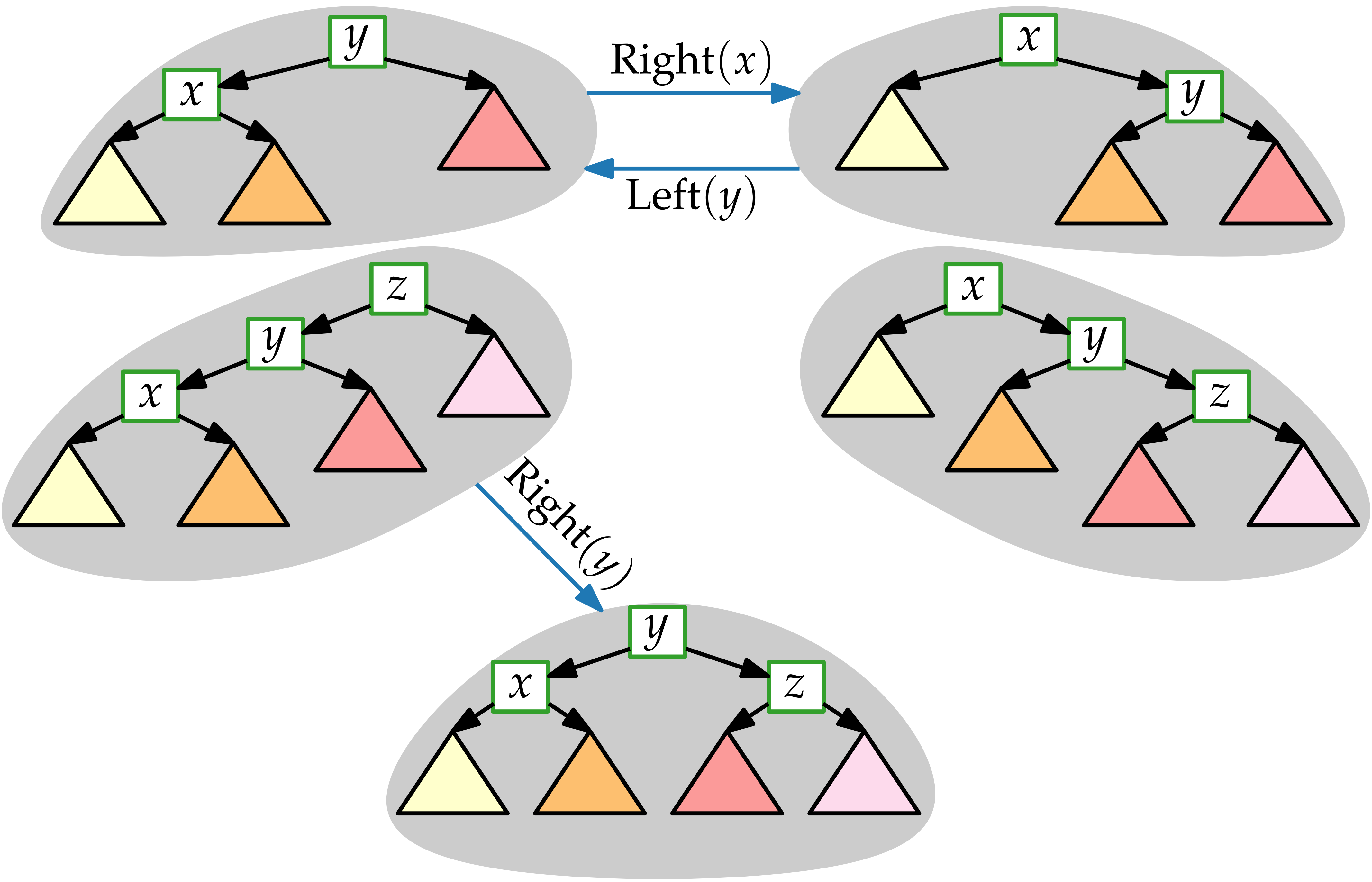

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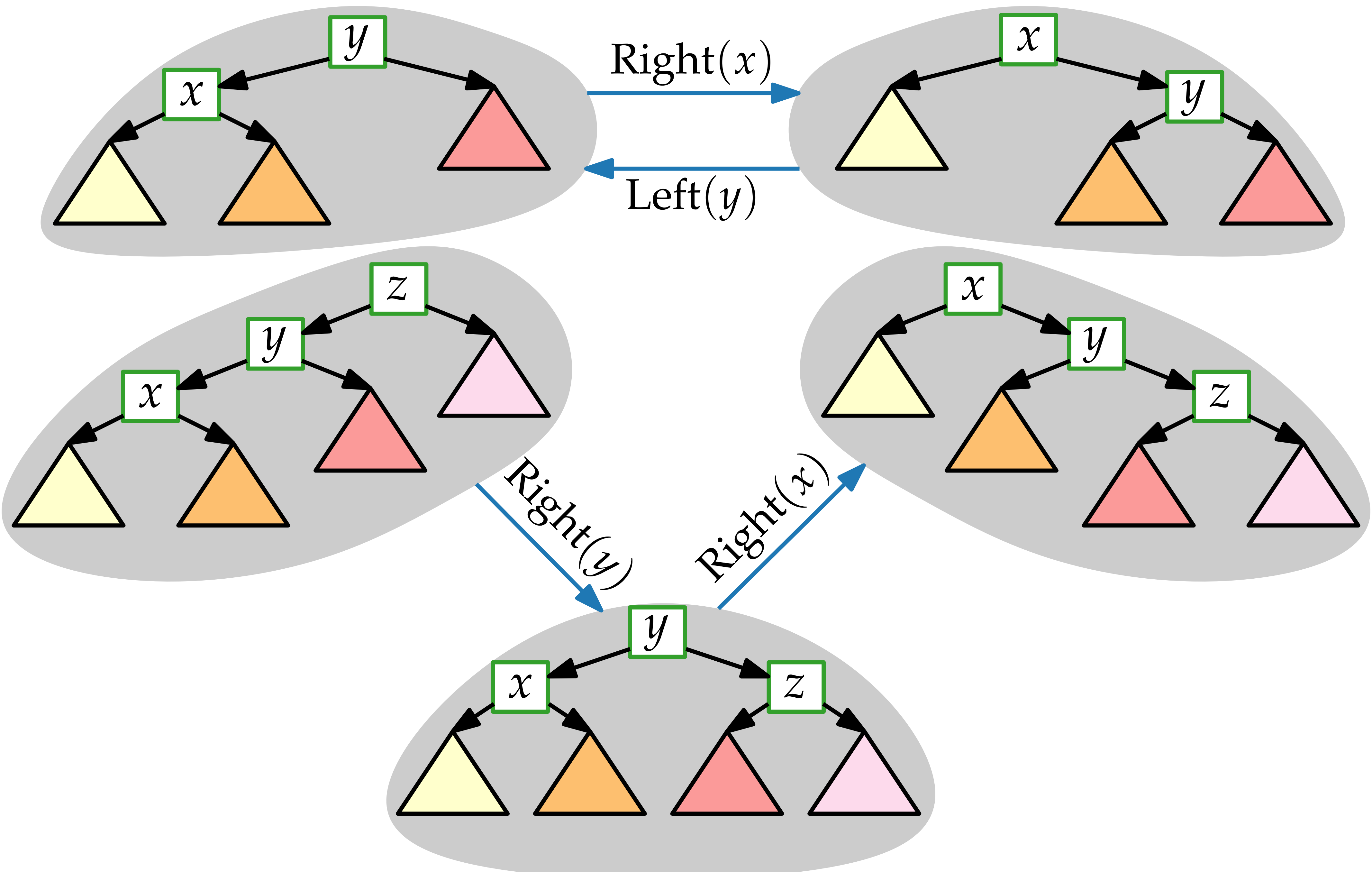

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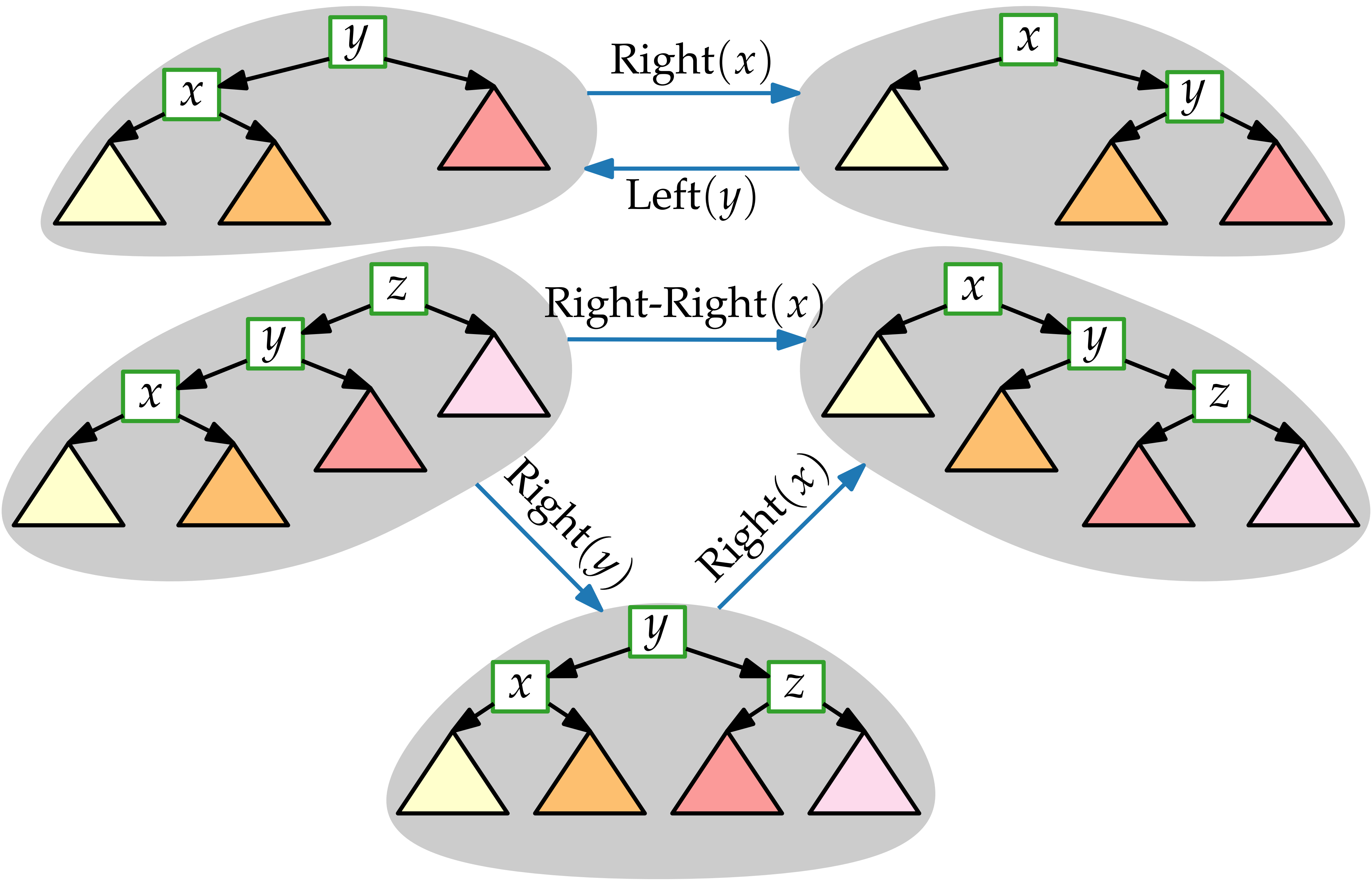

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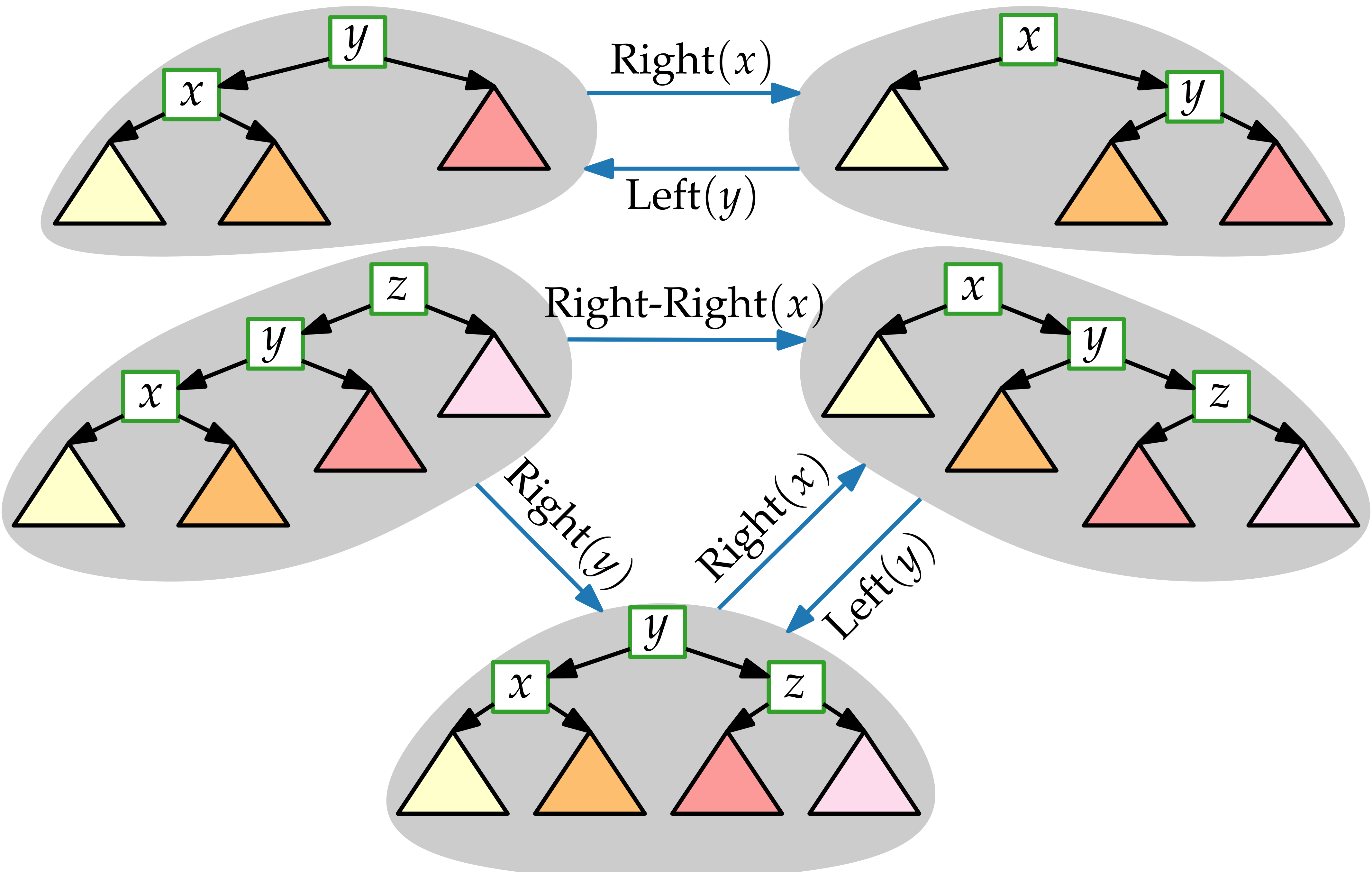

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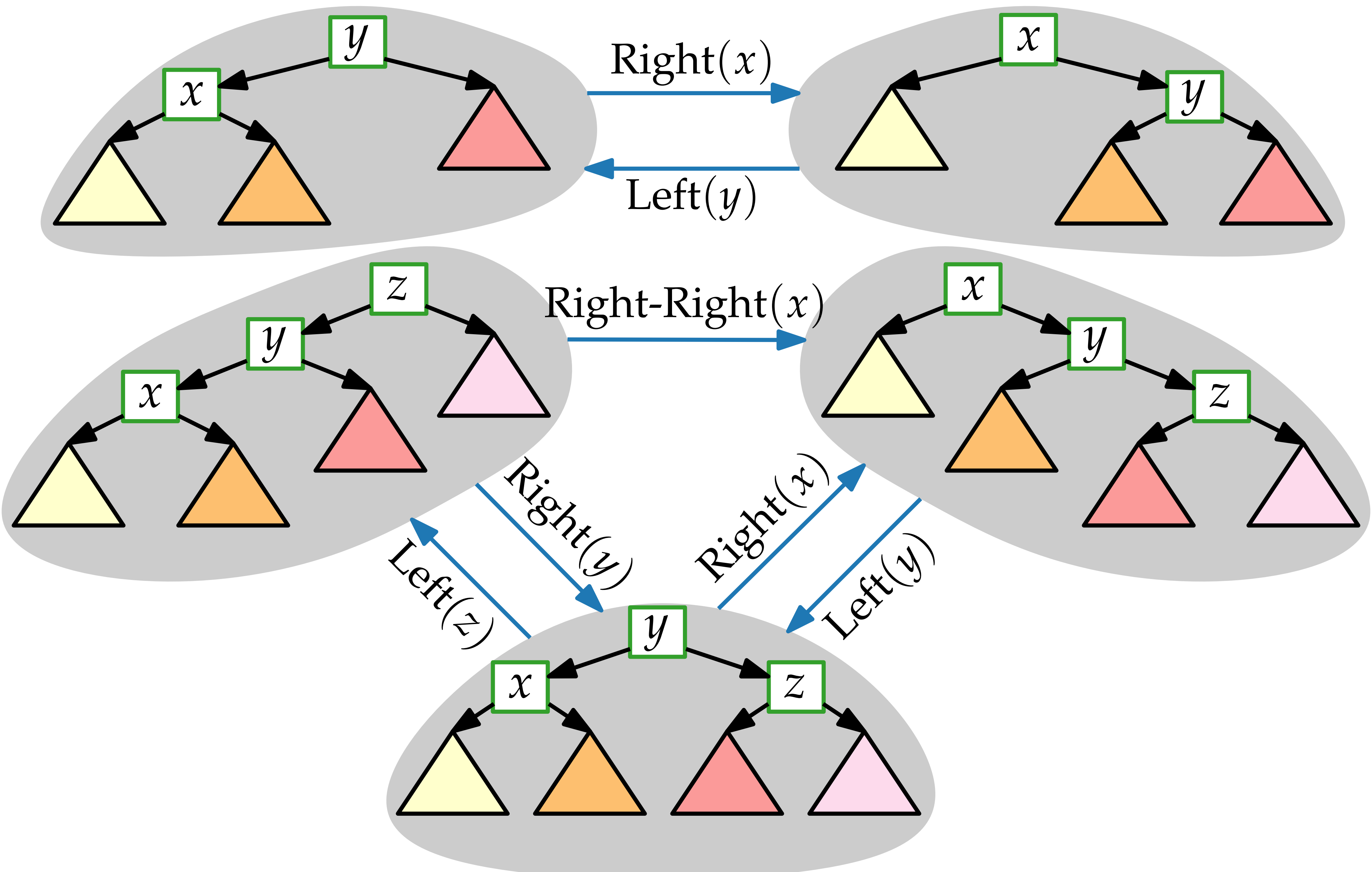

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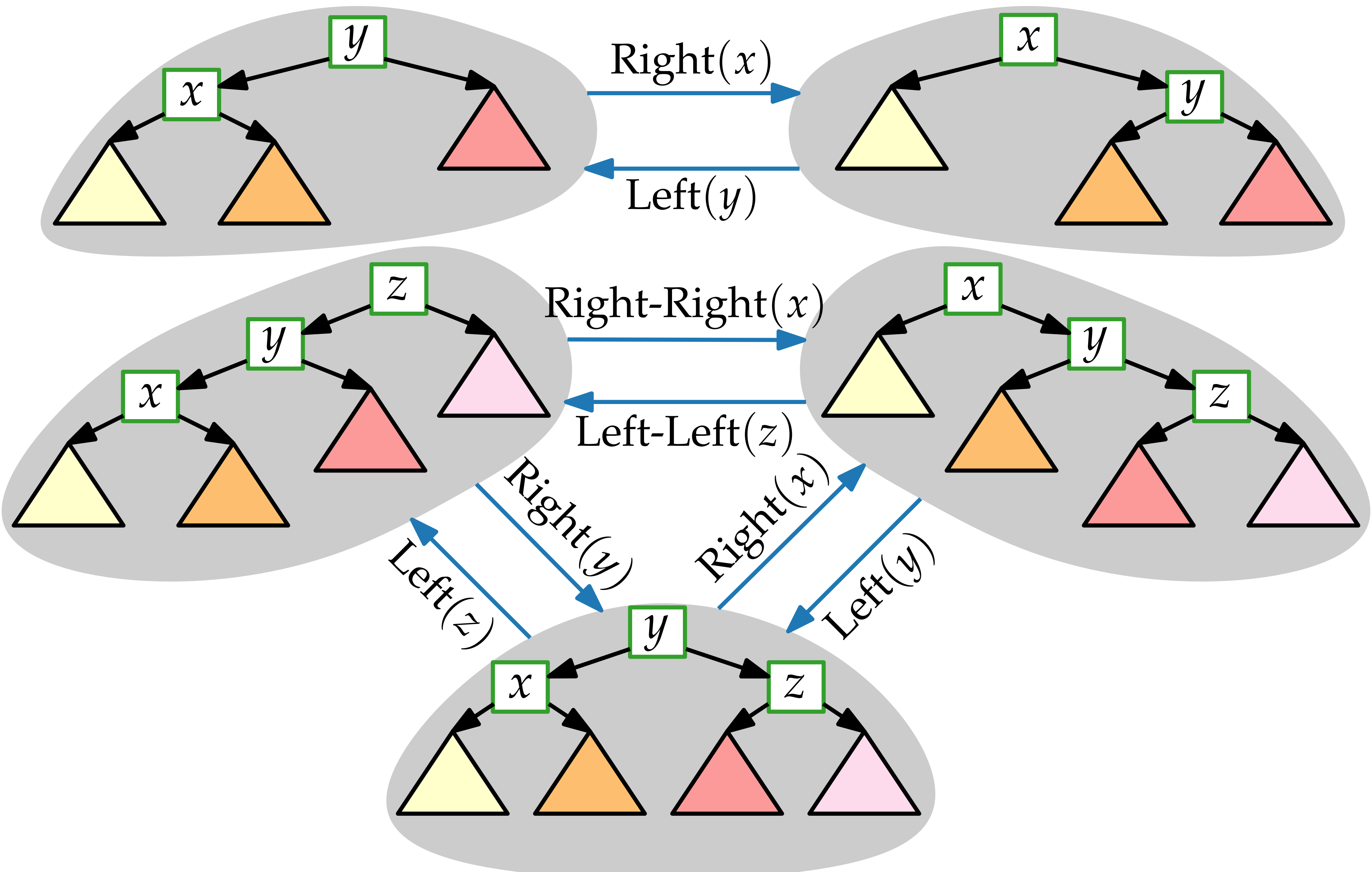

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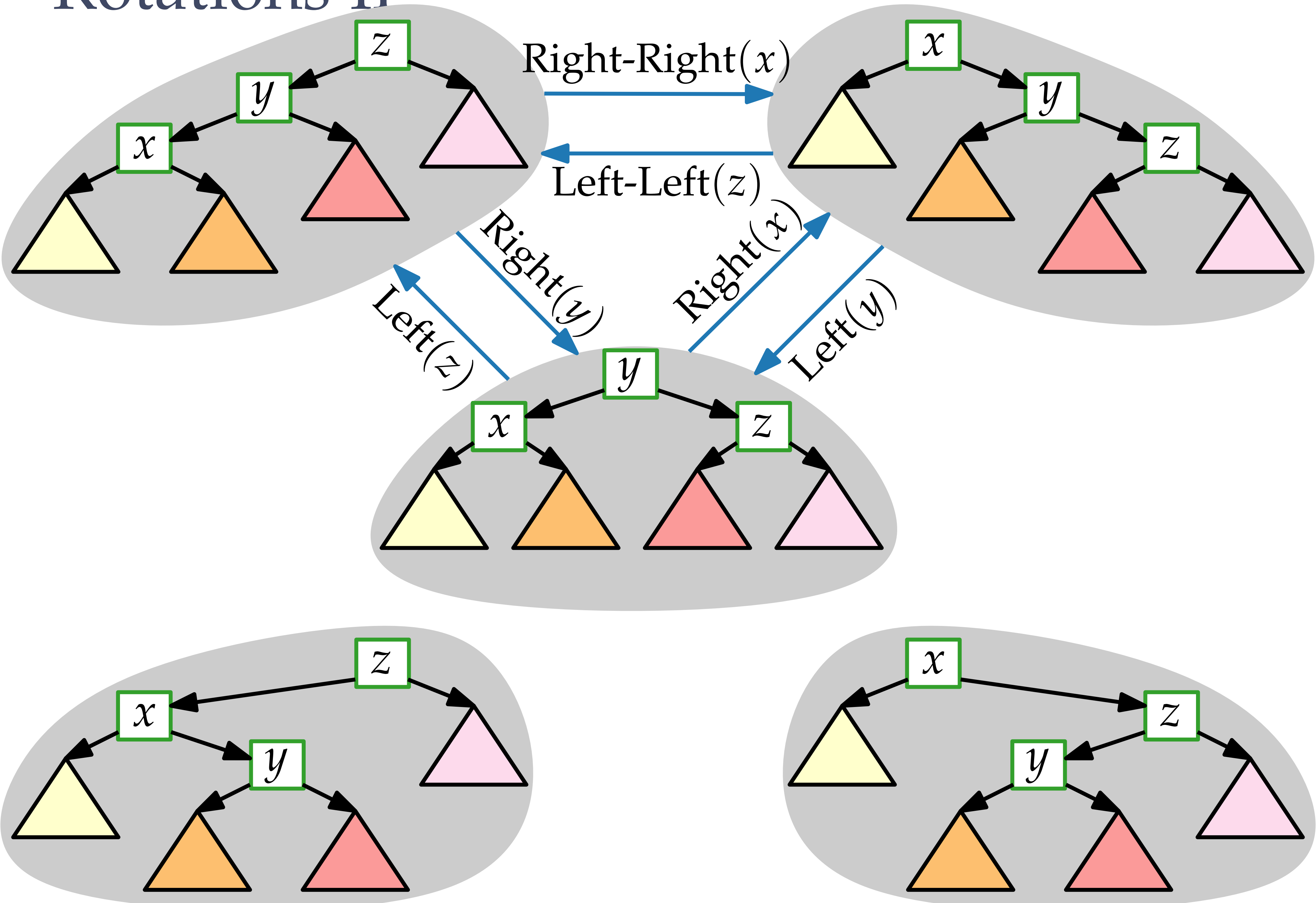

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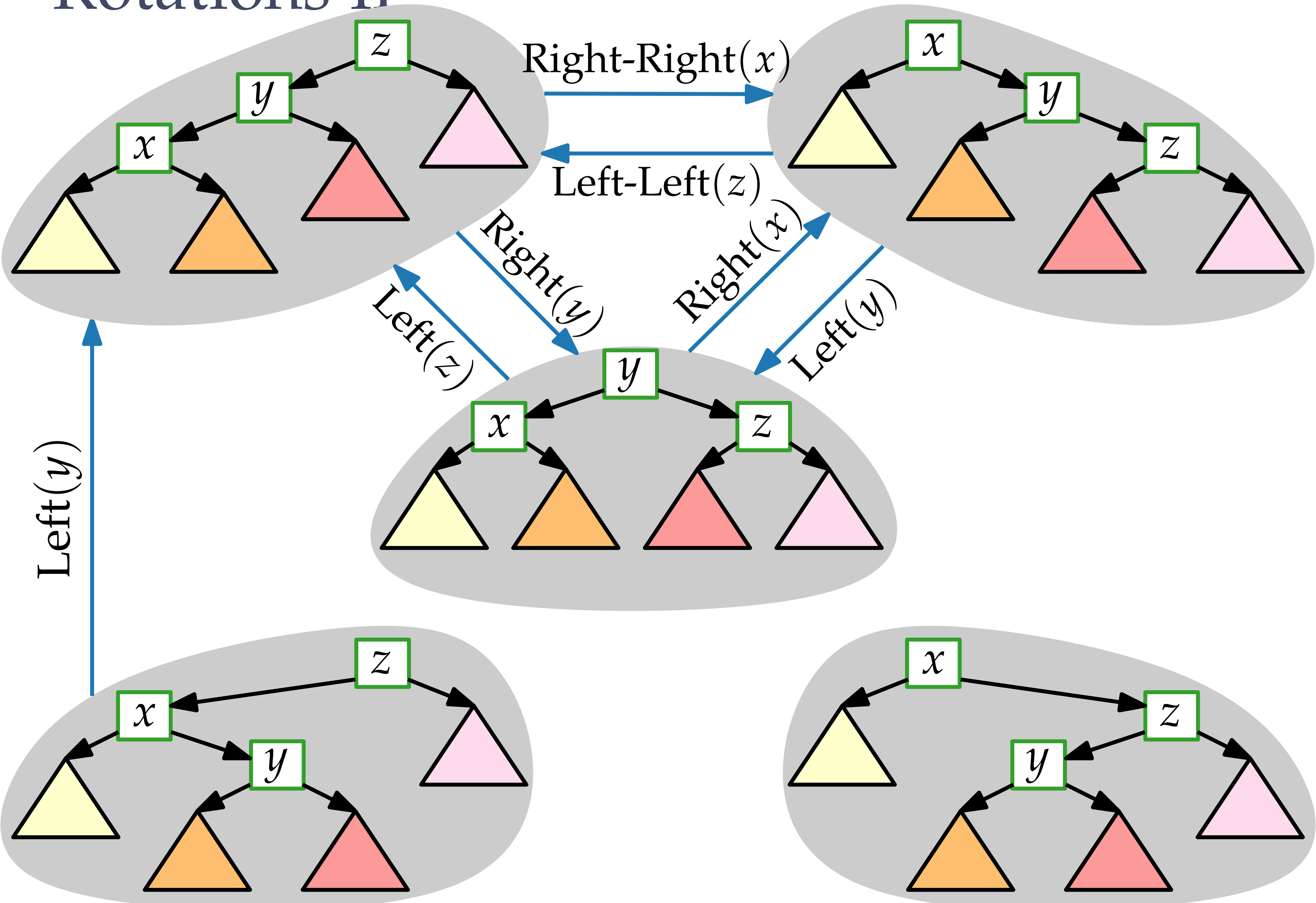

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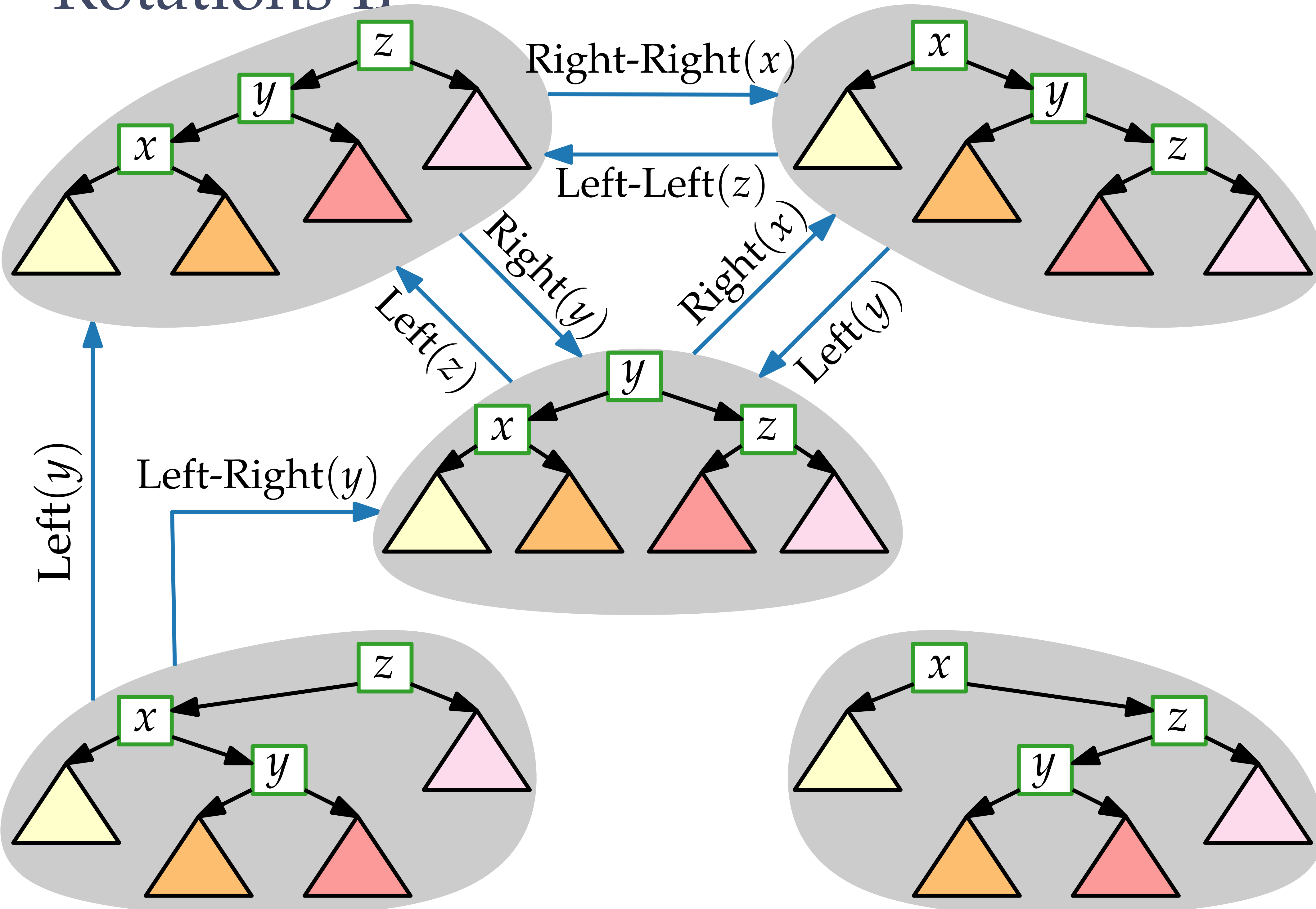
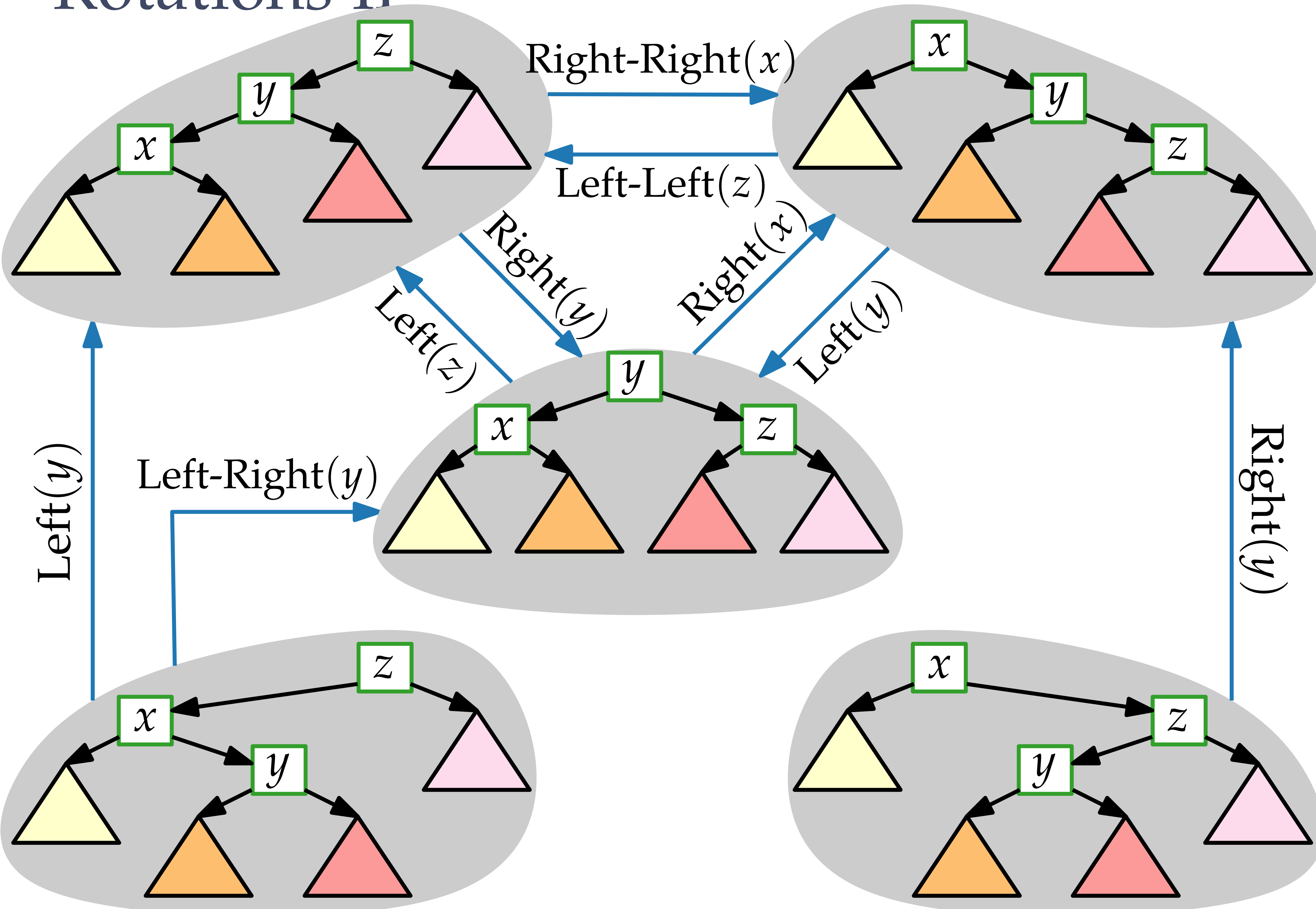


Rotations II



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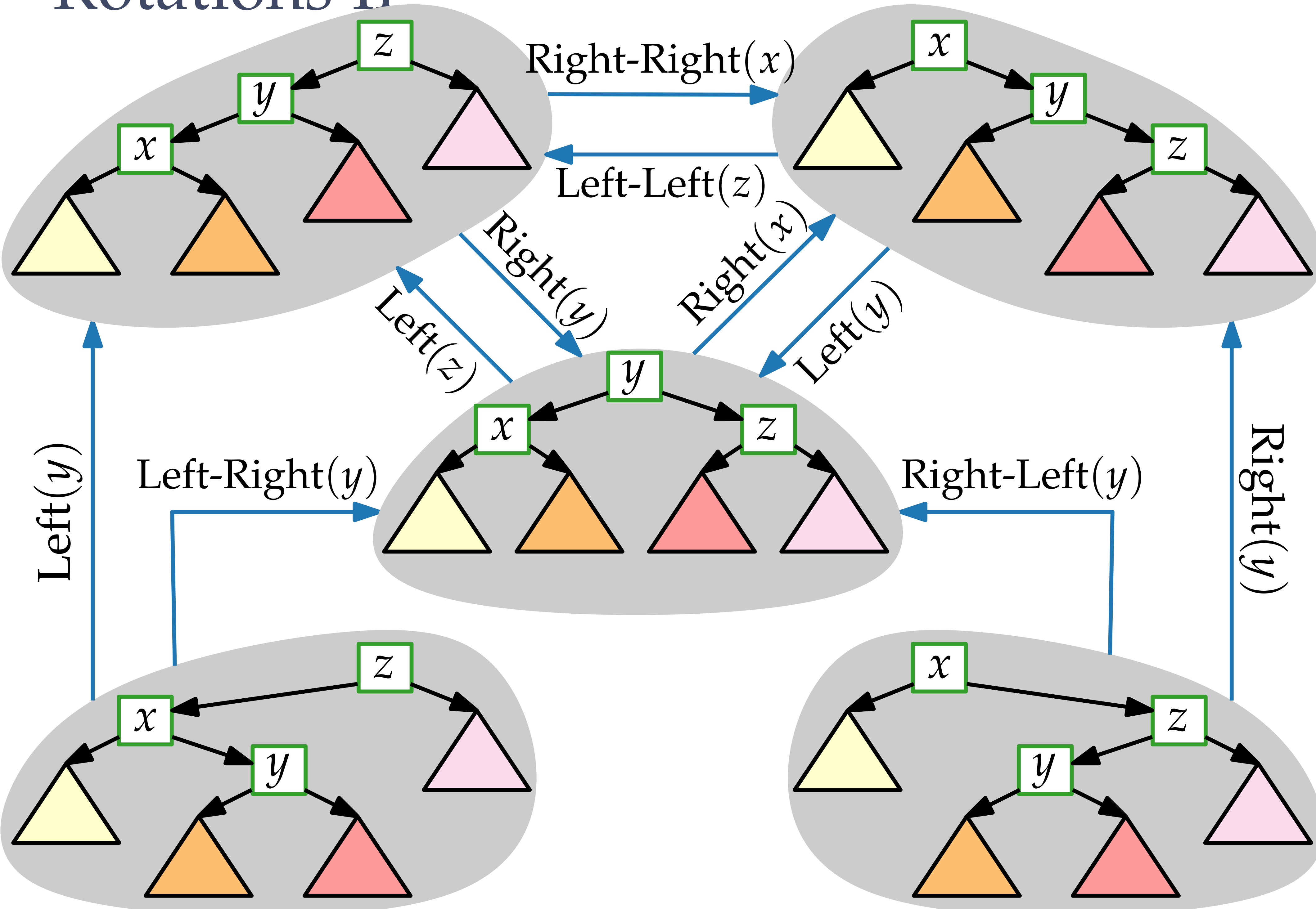

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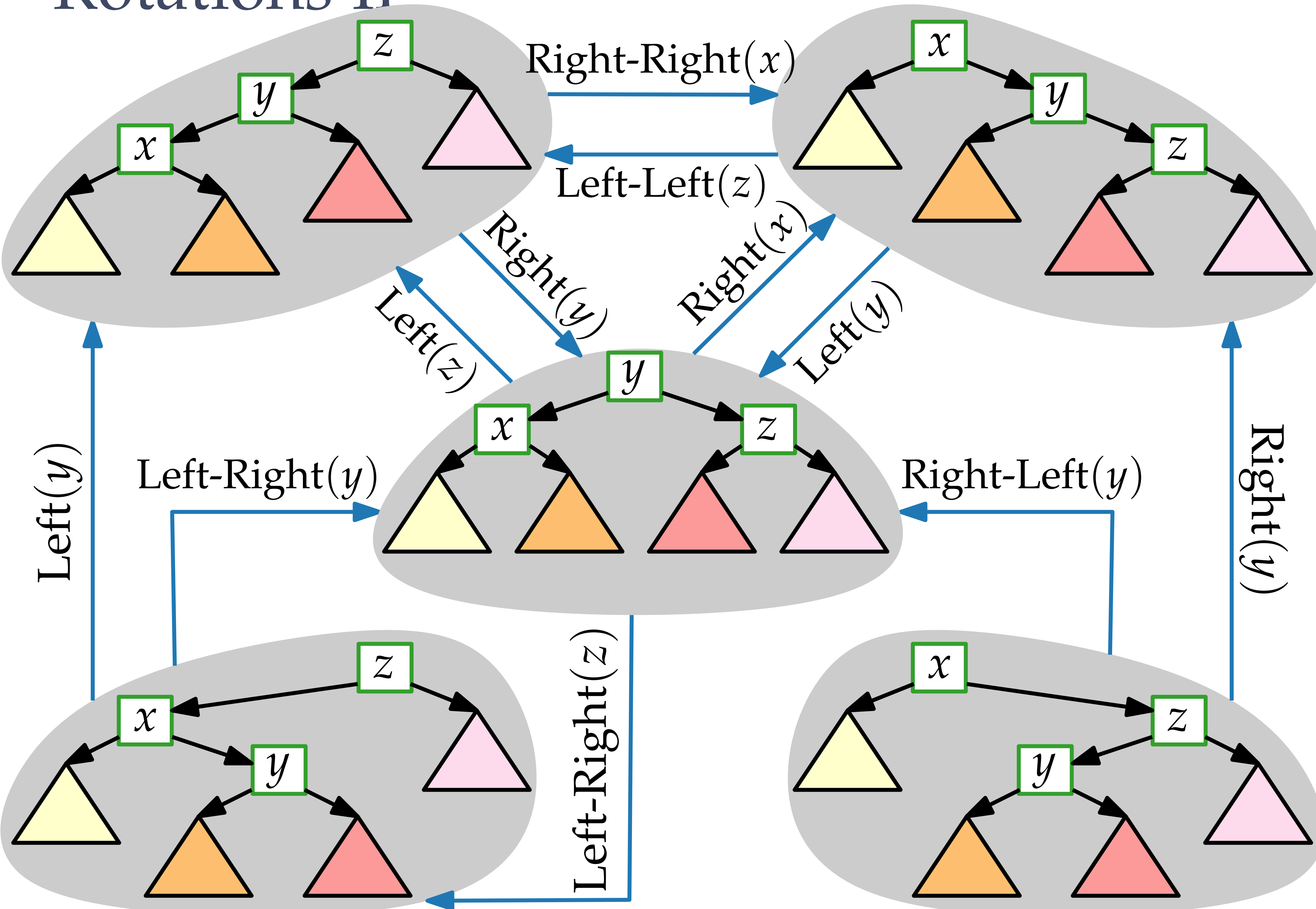

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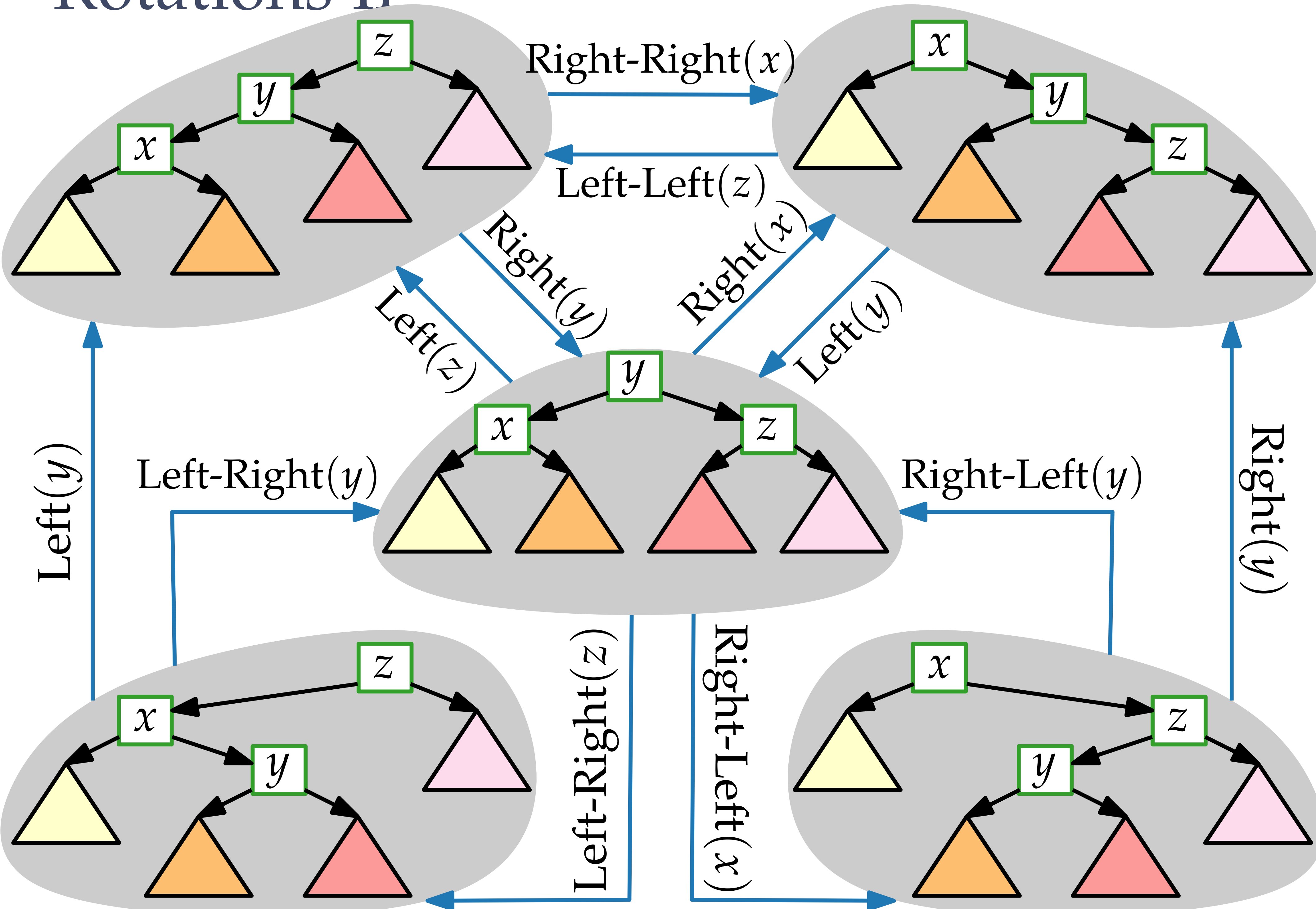

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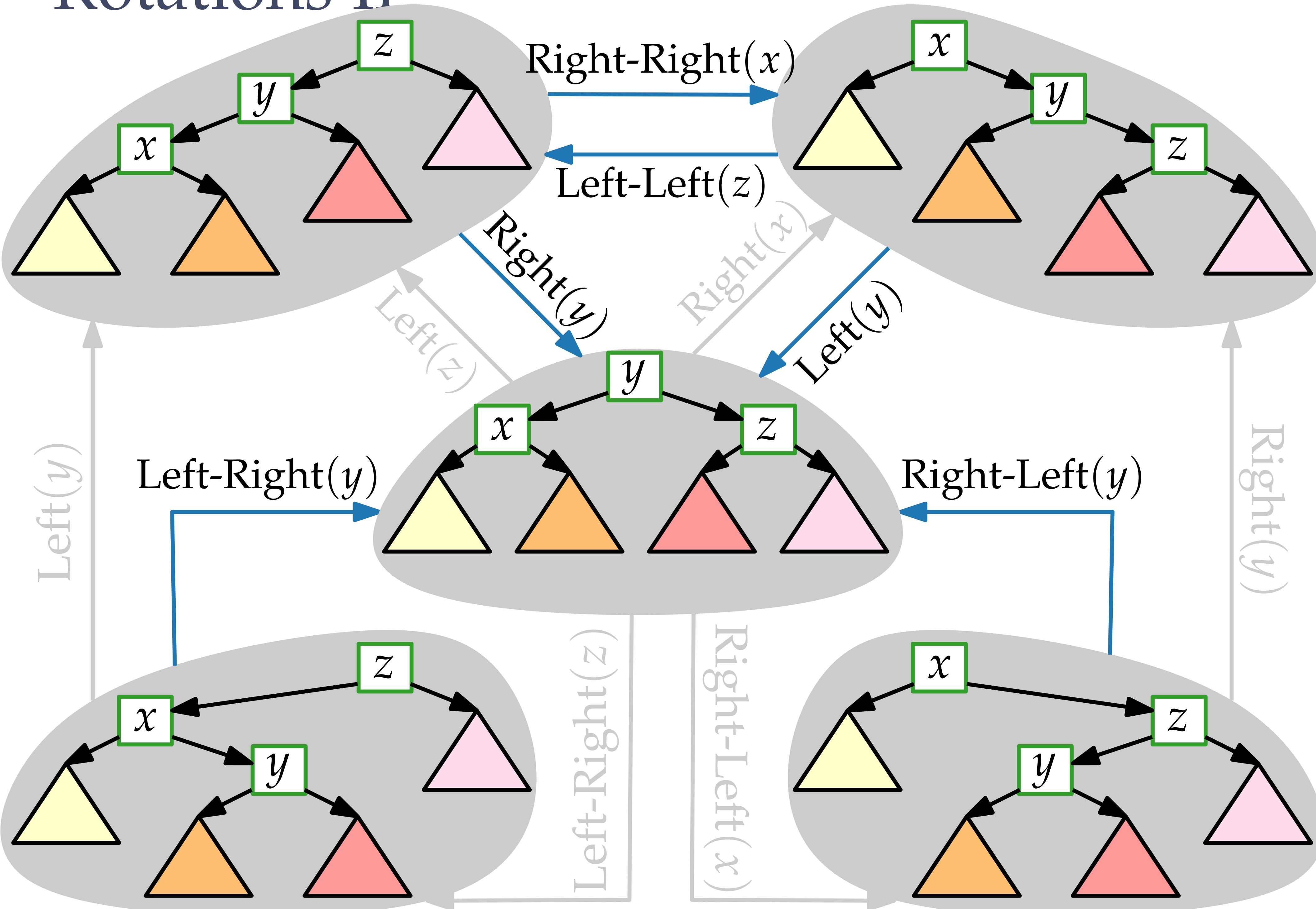

Splay

Algorithm: Splay(x)

Splay

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if $x \neq root$ **then**

Splay

Algorithm: Splay(x)

if $x \neq \mathit{root}$ **then**

  $y =$ parent of x

Splay

Algorithm: Splay(x)

if $x \neq root$ **then**

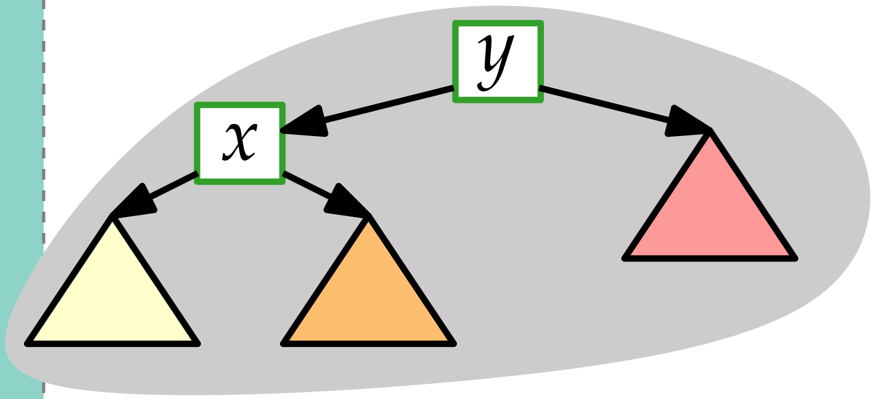
  $y =$ parent of x

  if $y = root$ **then**

Splay

Algorithm: Splay(x)

if $x \neq \mathit{root}$ **then**
  $y =$ parent of x
  if $y = \mathit{root}$ **then**
    if $x < y$ **then**



Splay

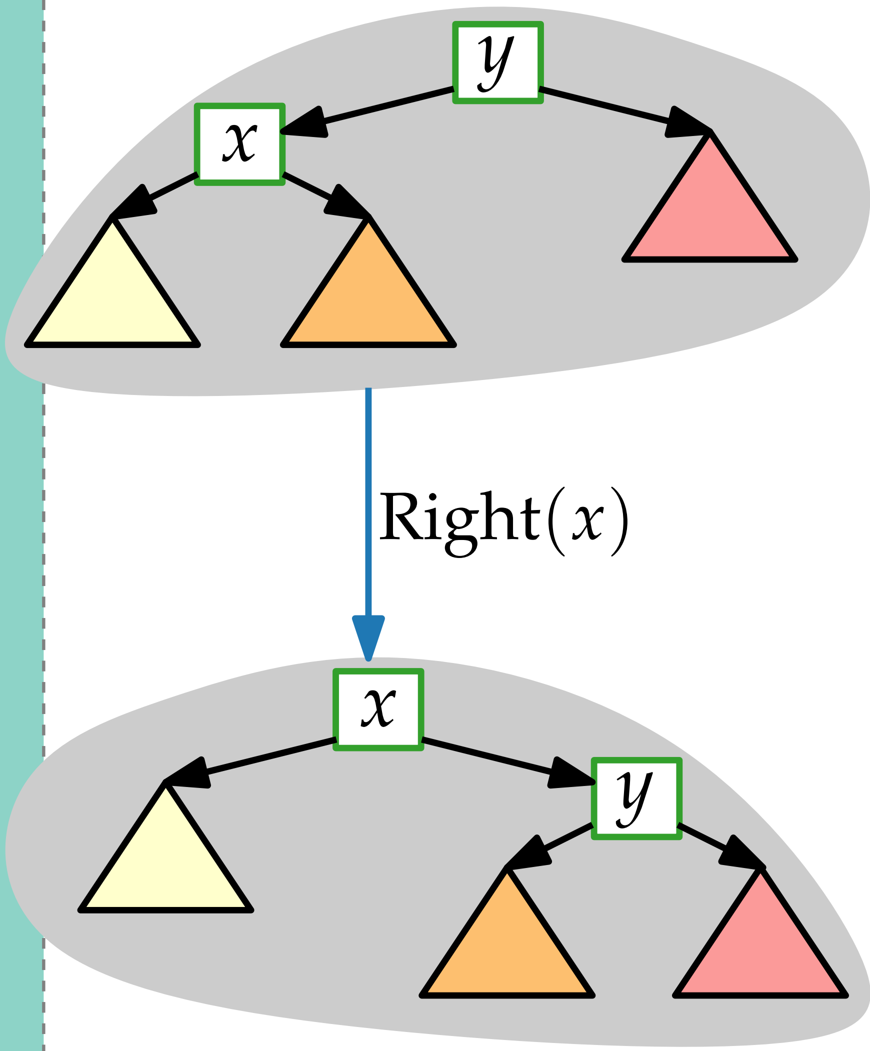
Algorithm: Splay(x)

if $x \neq \textit{root}$ **then**

  y = parent of x

  if $y = \textit{root}$ **then**

    if $x < y$ **then** Right(x)



Splay

Algorithm: Splay(x)

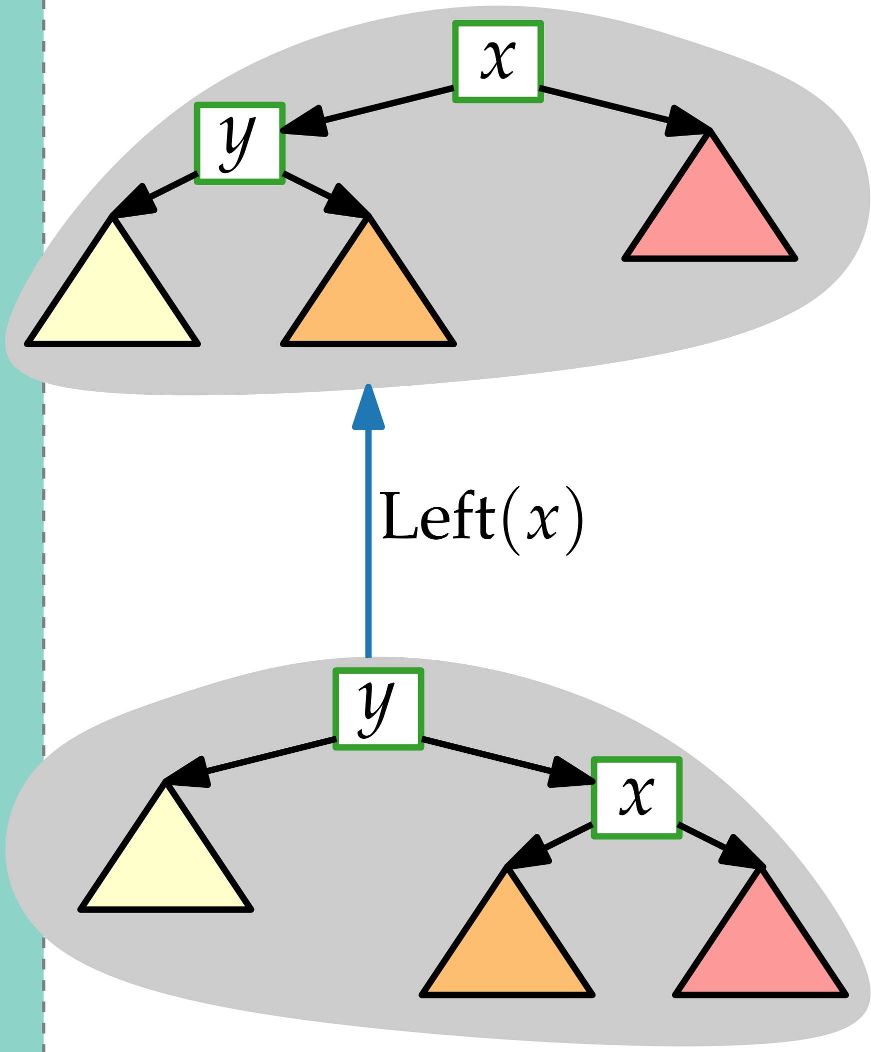
if $x \neq \mathit{root}$ **then**

  $y =$ parent of x

  if $y = \mathit{root}$ **then**

    if $x < y$ **then** Right(x)

    if $y < x$ **then** Left(x)



Splay

Algorithm: Splay(x)

if $x \neq root$ **then**

  $y =$ parent of x

  if $y = root$ **then**

    if $x < y$ **then** Right(x)

    if $y < x$ **then** Left(x)

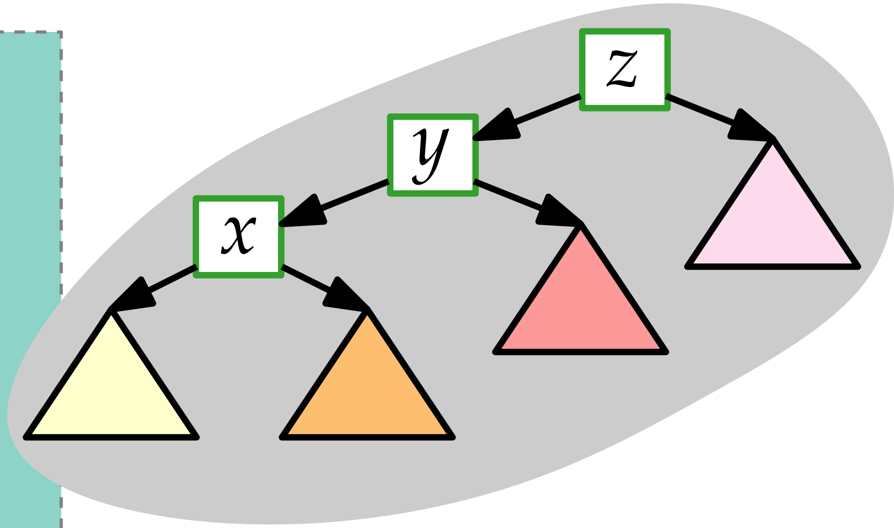
  else

    $z =$ parent of y

Splay

Algorithm: Splay(x)

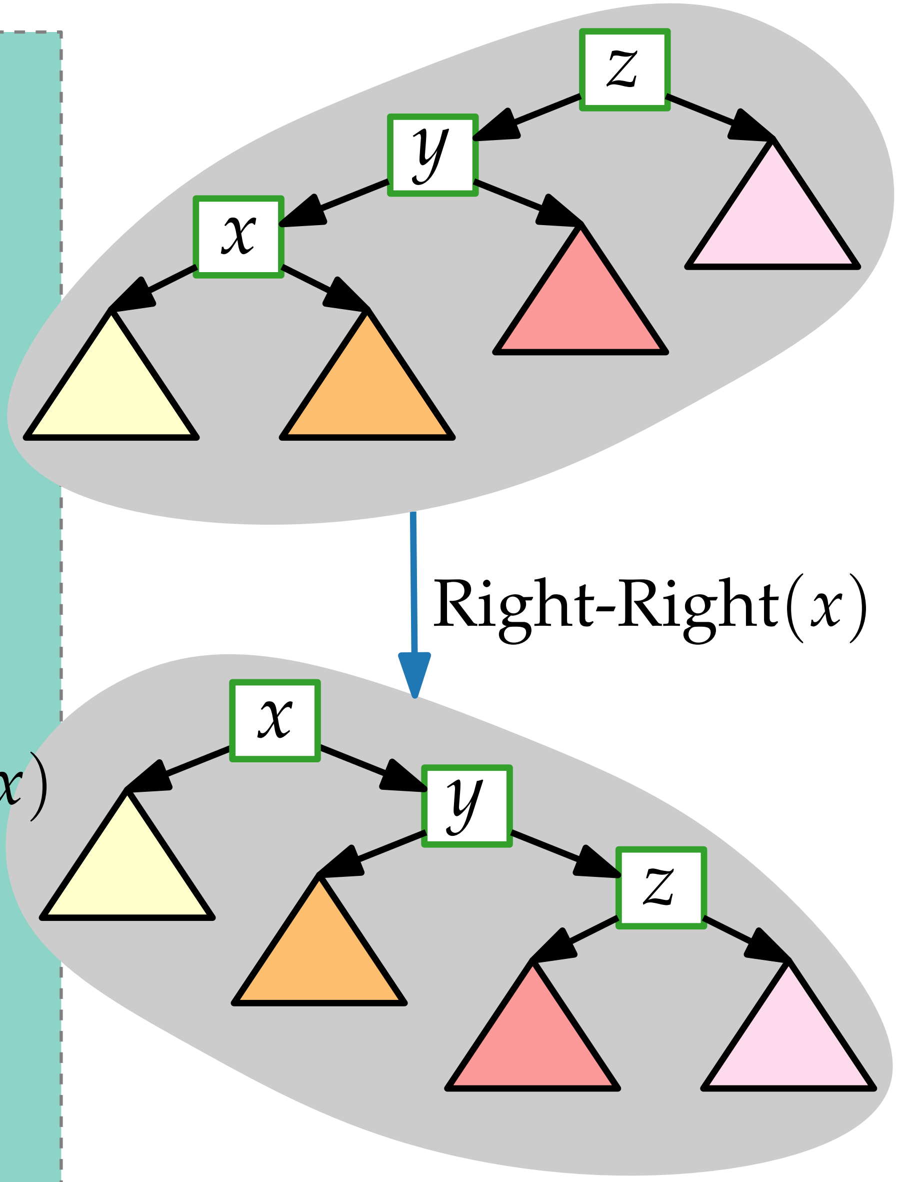
if $x \neq root$ **then**
- y = parent of x
- **if** $y = root$ **then**
  - **if** $x < y$ **then** Right(x)
  - **if** $y < x$ **then** Left(x)
- **else**
  - z = parent of y
  - **if** $x < y < z$ **then**



Splay

Algorithm: Splay(x)

if $x \neq \mathit{root}$ **then**
- $y =$ parent of x
- **if** $y = \mathit{root}$ **then**
  - **if** $x < y$ **then** Right(x)
  - **if** $y < x$ **then** Left(x)
- **else**
  - $z =$ parent of y
  - **if** $x < y < z$ **then** Right-Right(x)



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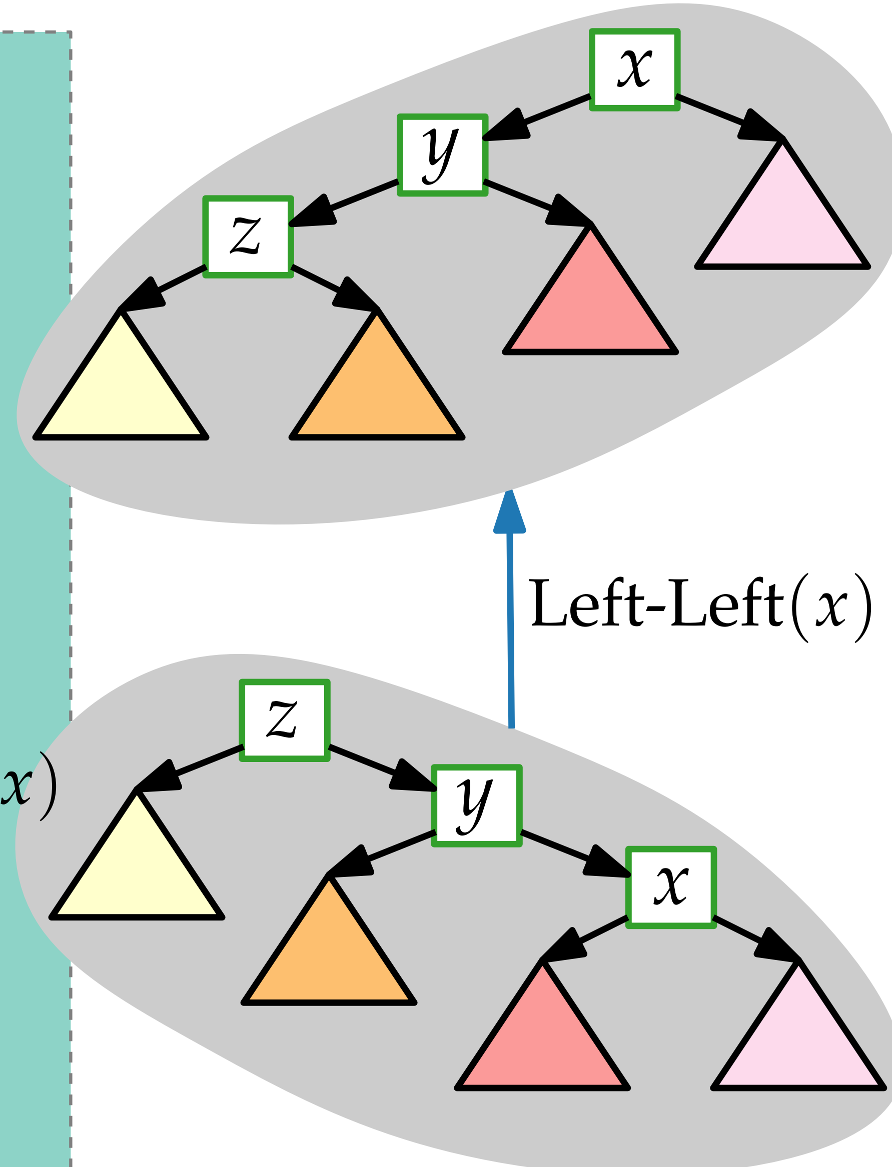
- y = parent of x
- **if** $y = \mathit{root}$ **then**
  - **if** $x < y$ **then** Right(x)
  - **if** $y < x$ **then** Left(x)
- **else**
  - z = parent of y
  - **if** $x < y < z$ **then** Right-Right(x)
  - **if** $z < y < x$ **then** Left-Left(x)



Splay

Algorithm: Splay(x)

if $x \neq \mathit{root}$ **then**
- y = parent of x
- **if** $y = \mathit{root}$ **then**
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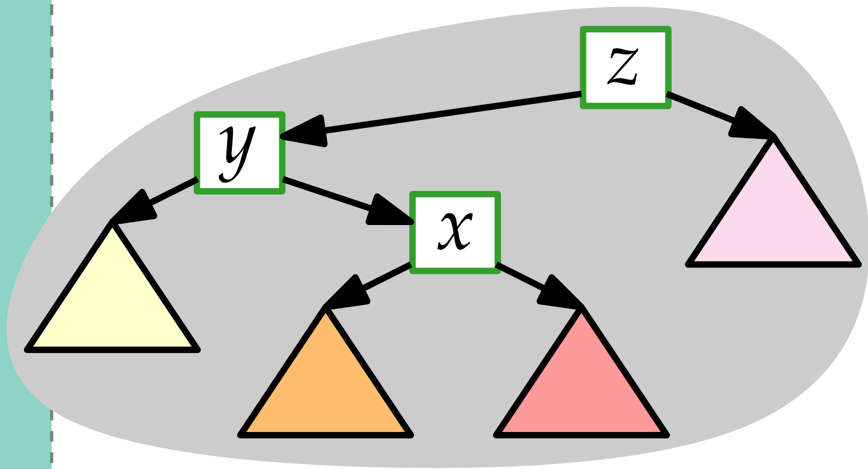
  - **if** $y < x < z$ **then**



Splay

Algorithm: Splay(x)

if $x \neq root$ **then**

  y = parent of x

  if $y = root$ **then**

    if $x < y$ **then** Right(x)

    if $y < x$ **then** Left(x)

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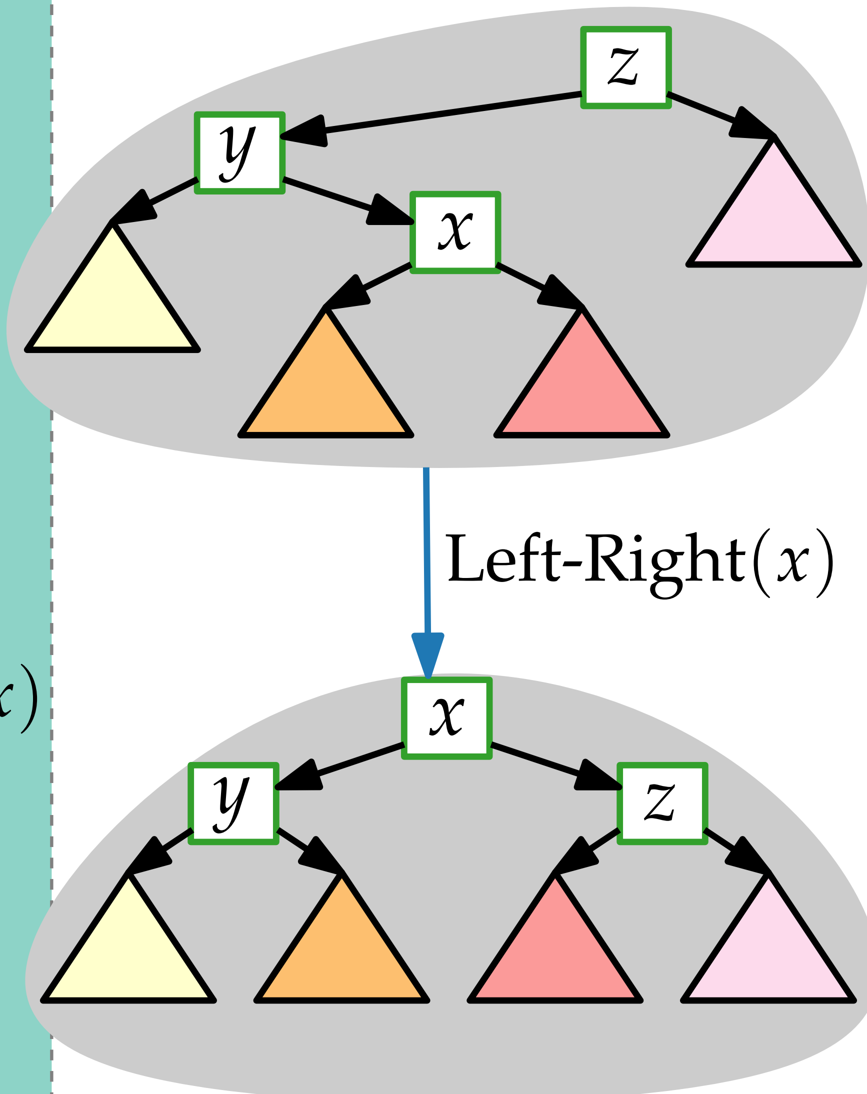

Splay

Algorithm: Splay(x)

if $x \neq \mathit{root}$ **then**
- $y =$ parent of x
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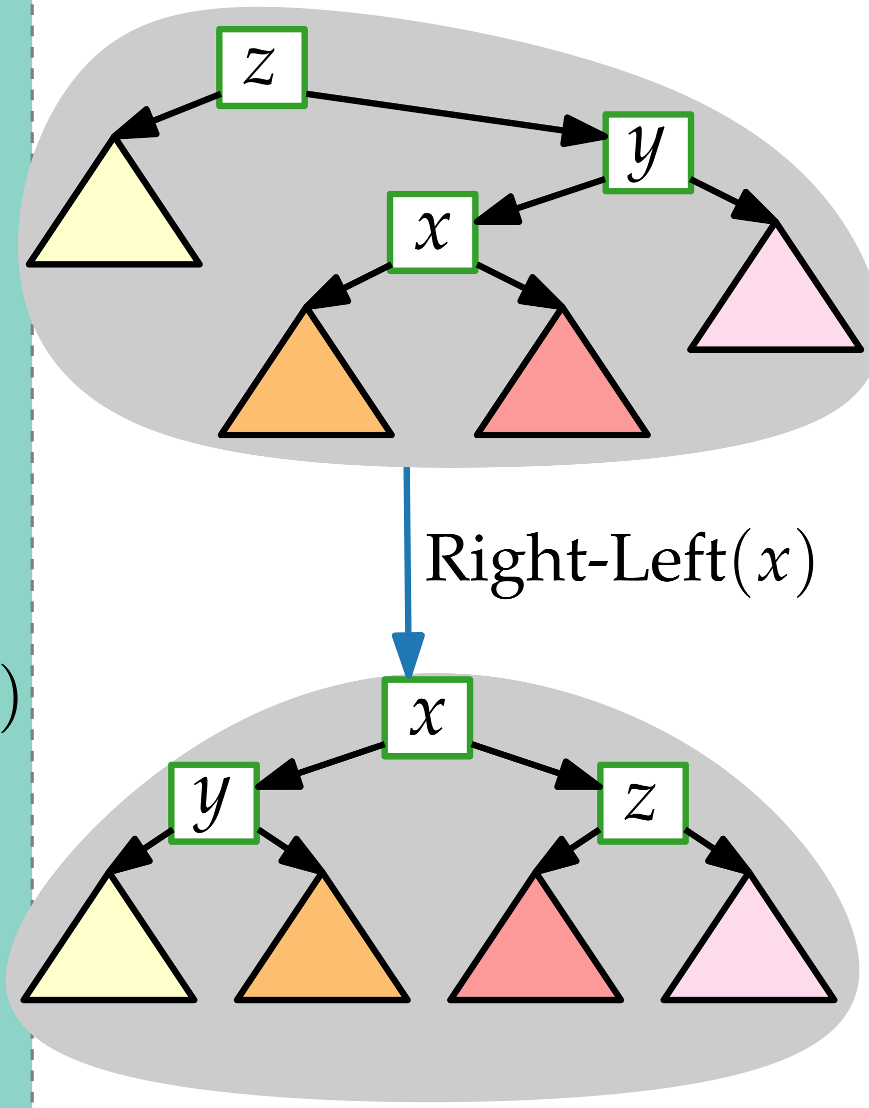

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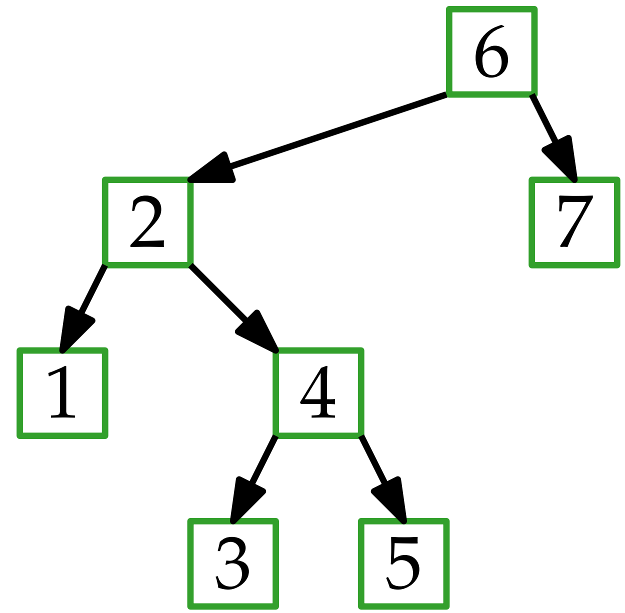
- Splay(x)

Splay(3):



Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
    y = parent of x
    if $y = root$ **then**
        if $x < y$ **then** Right(x)
        if $y < x$ **then** Left(x)
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        z = parent of y
        if $x < y < z$ **then** Right-Right(x)
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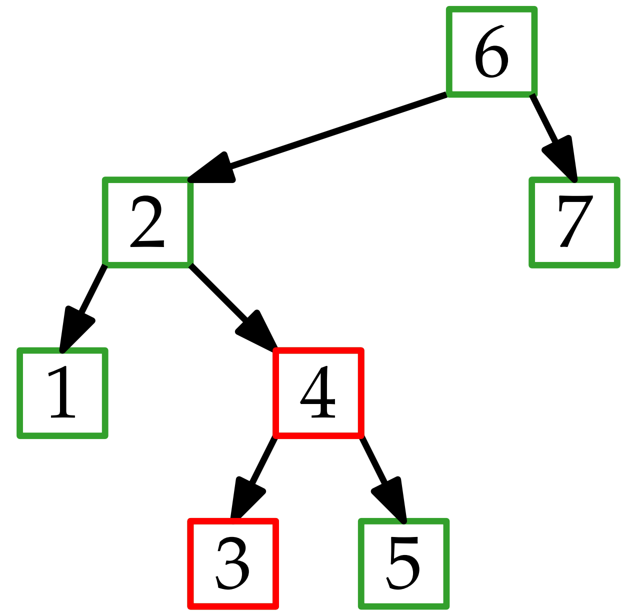

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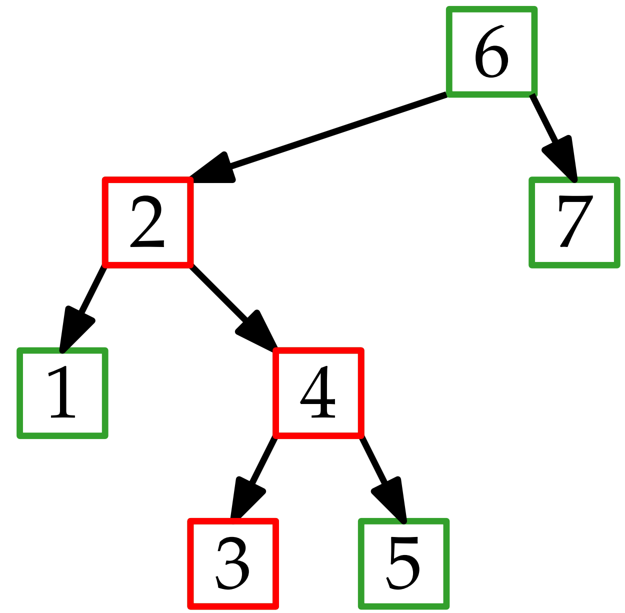

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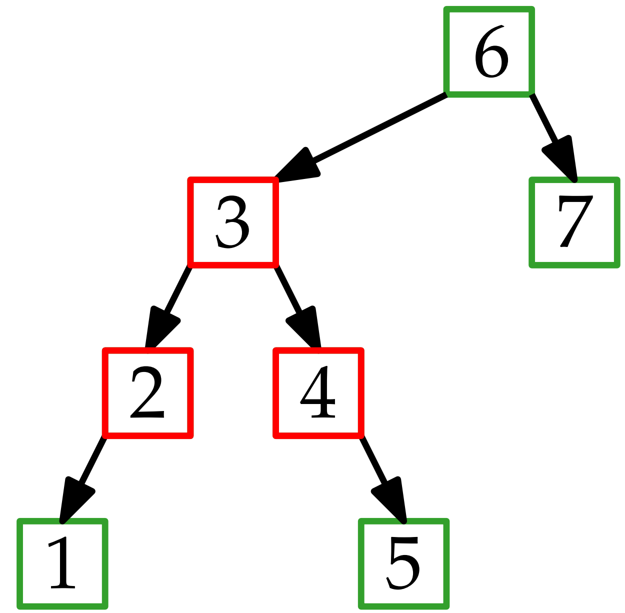

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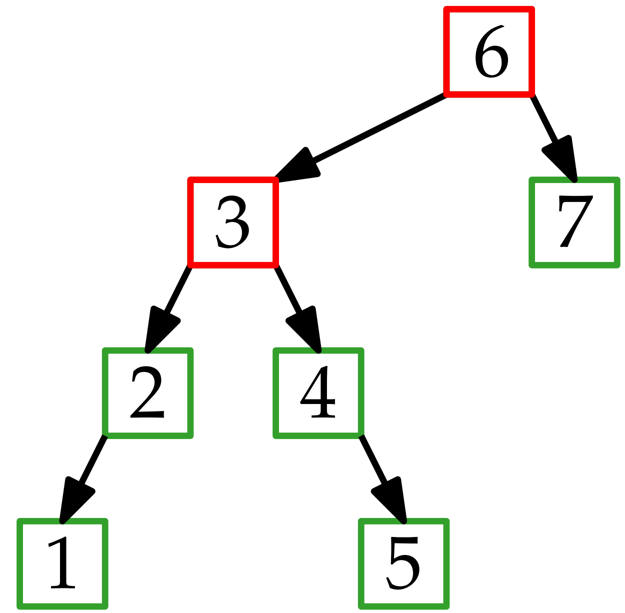

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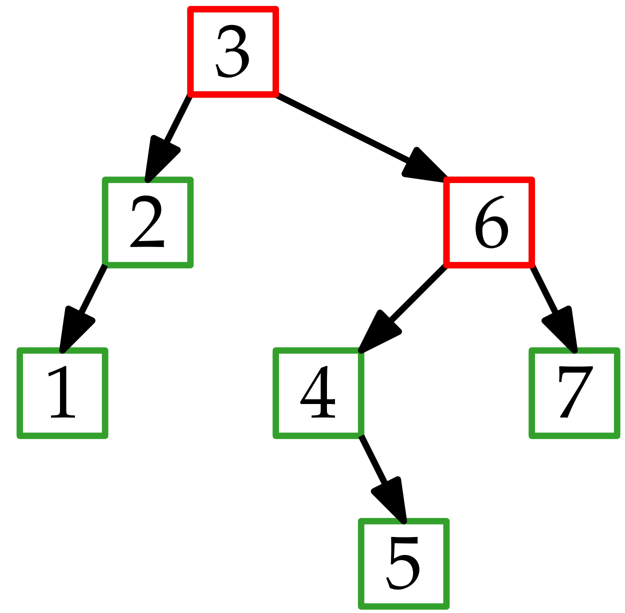

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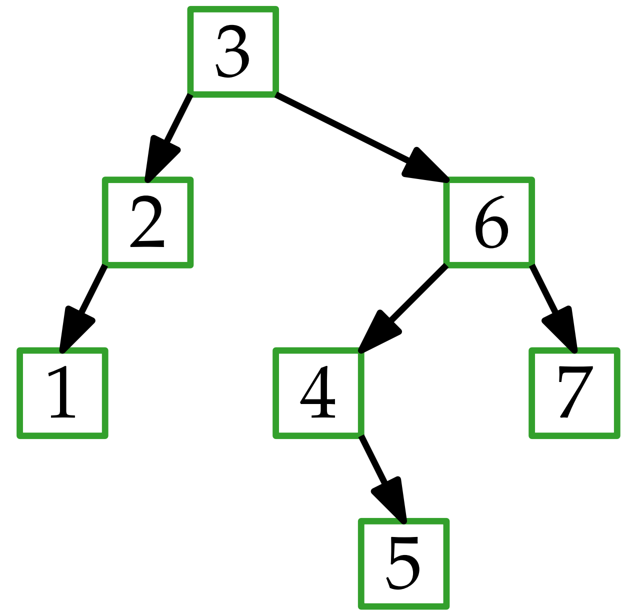

Call Splay(x):

Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
  $y =$ parent of x
  if $y = root$ **then**
    if $x < y$ **then** Right(x)
    if $y < x$ **then** Left(x)
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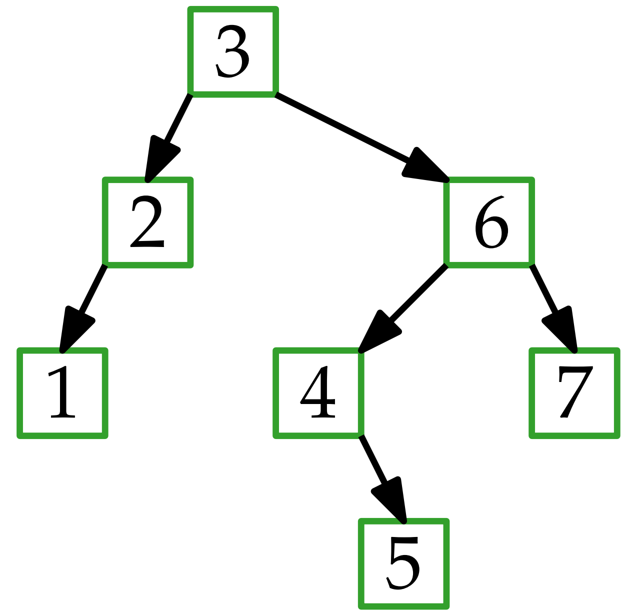

Call Splay(x):

- after Search(x)

Splay

Algorithm: Splay(x)

if $x \neq \mathit{root}$ **then**
- y = parent of x
- **if** $y = \mathit{root}$ **then**
  - **if** $x < y$ **then** Right(x)
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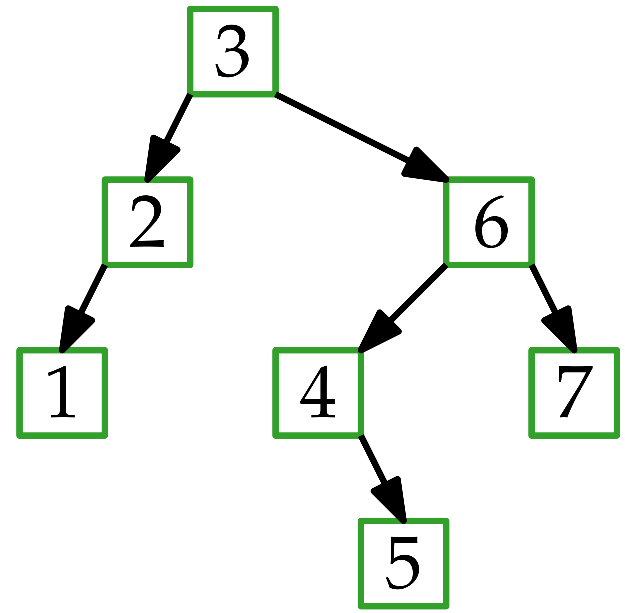
  - **if** $z < x < y$ **then** Right-Left(x)
- Splay(x)

Splay(3):



Call Splay(x):

- after Search(x)
- after Insert(x)

Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
- $y =$ parent of x
- **if** $y = root$ **then**
  - **if** $x < y$ **then** Right(x)
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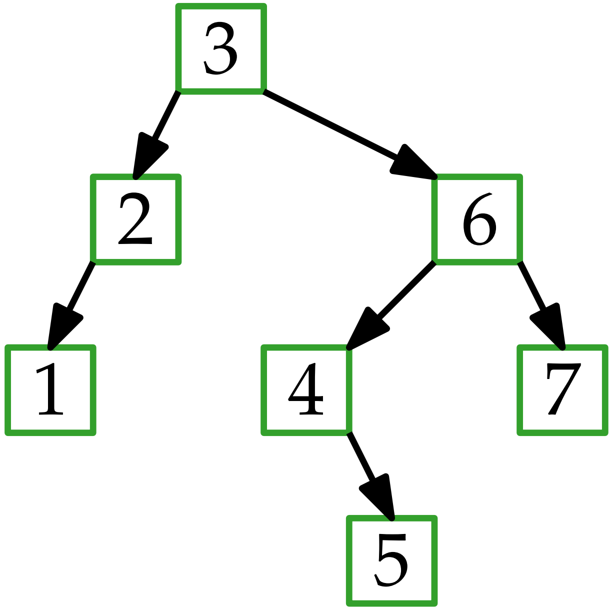
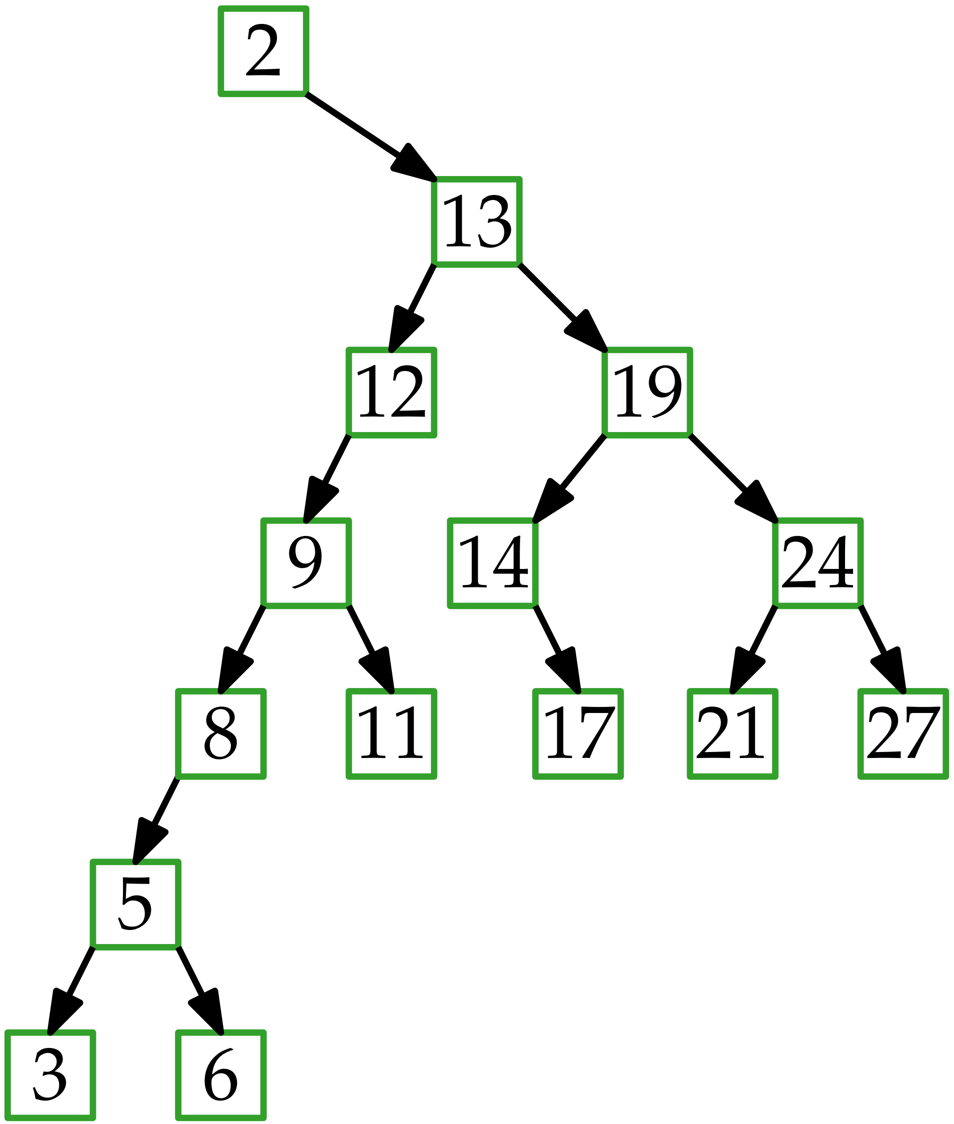
  - **if** $z < x < y$ **then** Right-Left(x)
- Splay(x)

Splay(3):



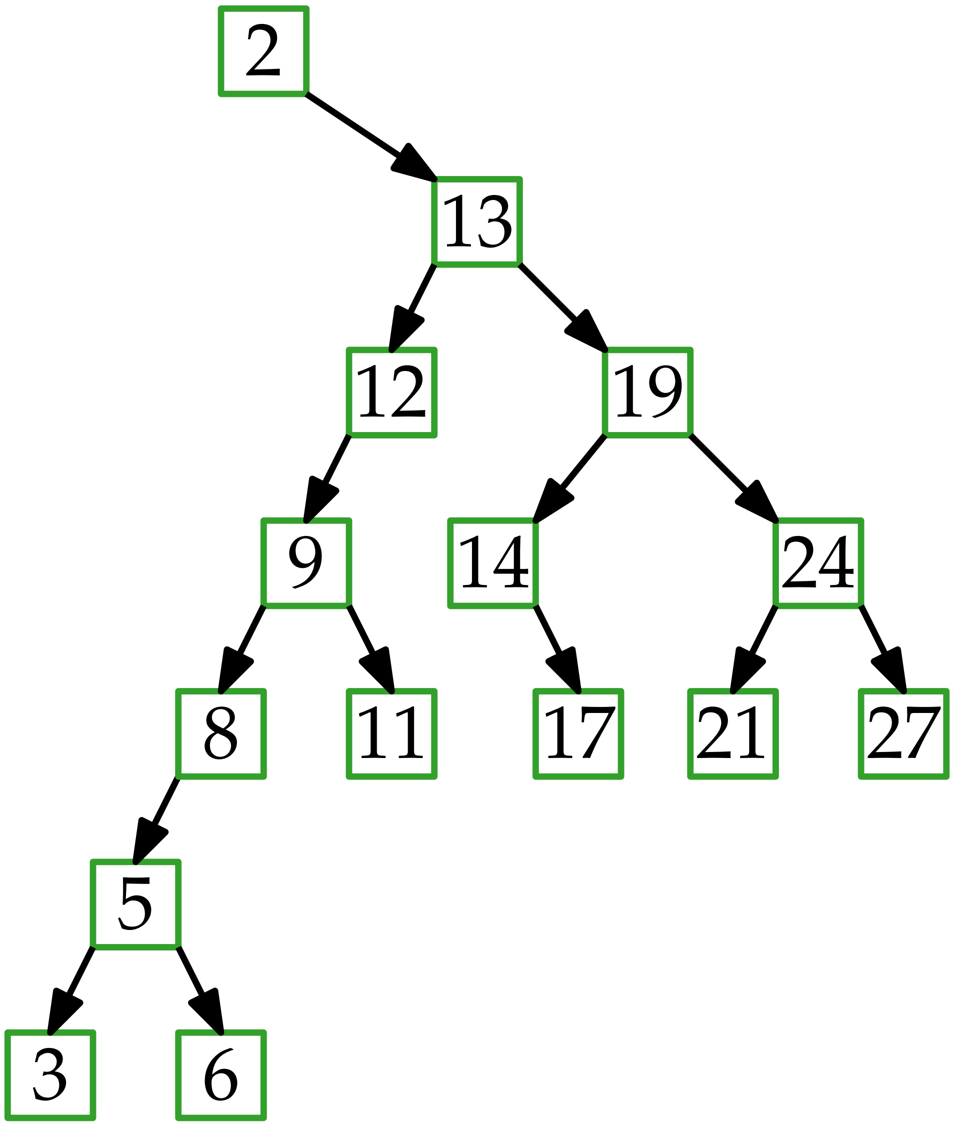
Call Splay(x):
- after Search(x)
- after Insert(x)
- before Delete(x)

Why is Splay fast?



Why is Splay fast?

$w(x)$: weight of x (here 1), $W = \sum w(x)$ (here n)



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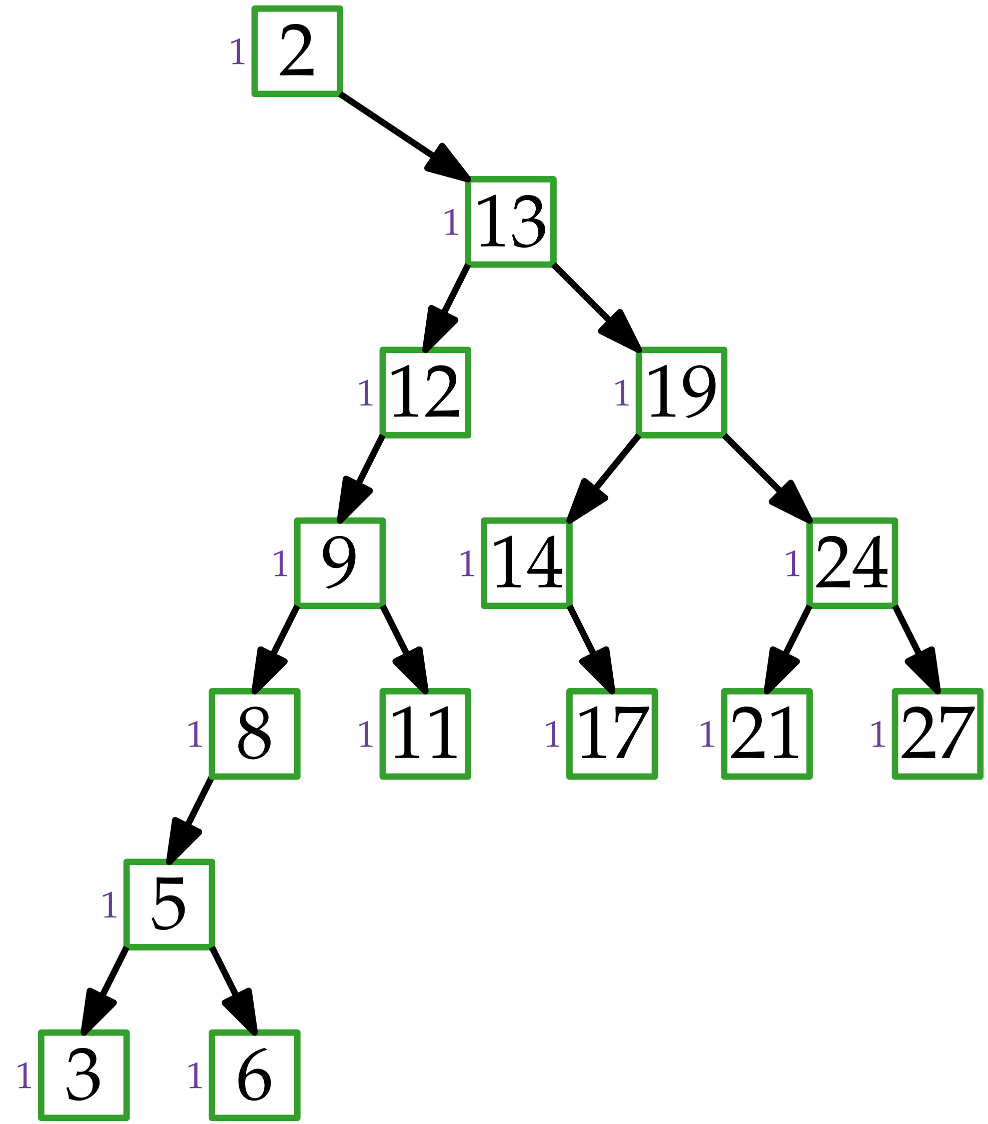
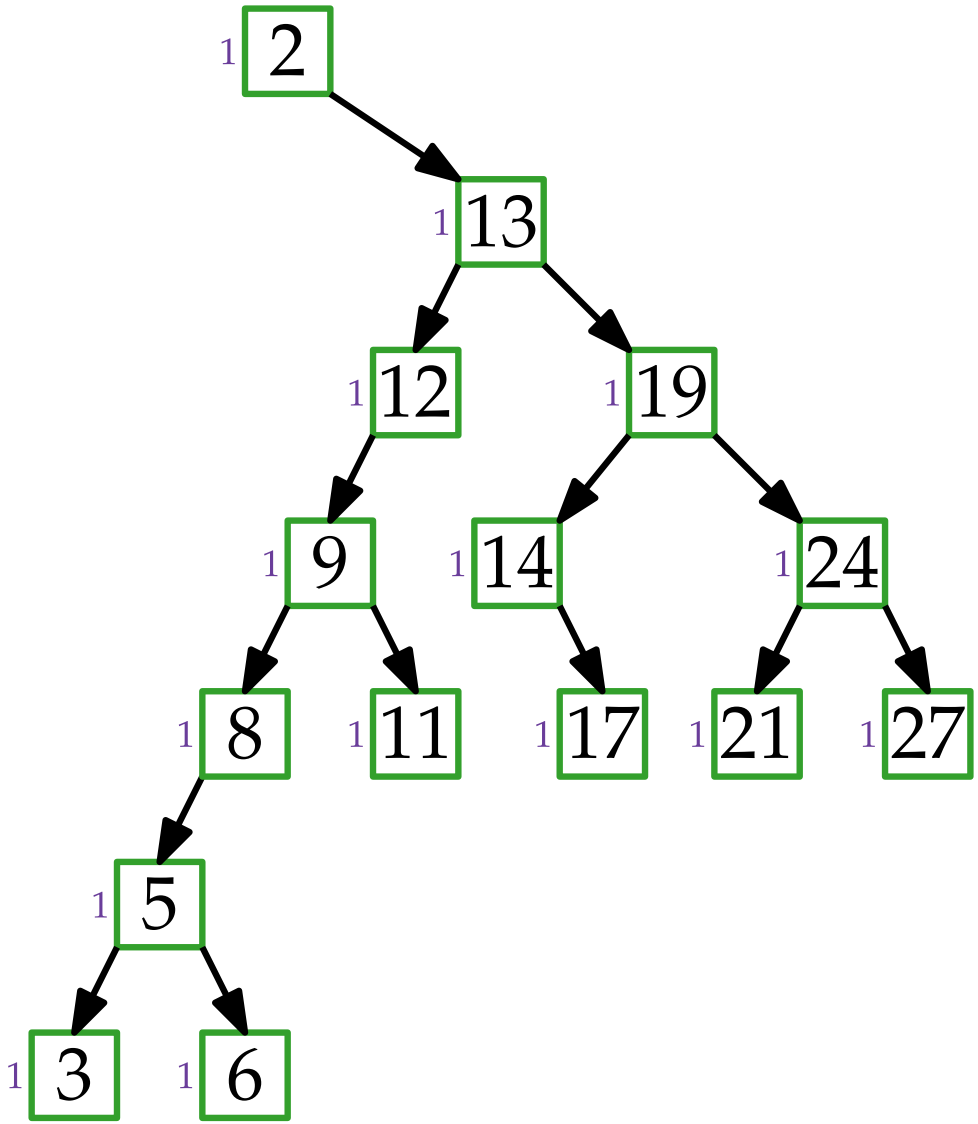


Why is Splay fast?

$w(x)$: weight of x (here 1), $W = \sum w(x)$ (here n)

$s(x)$: sum of all $w(x)$ in subtree of x_i



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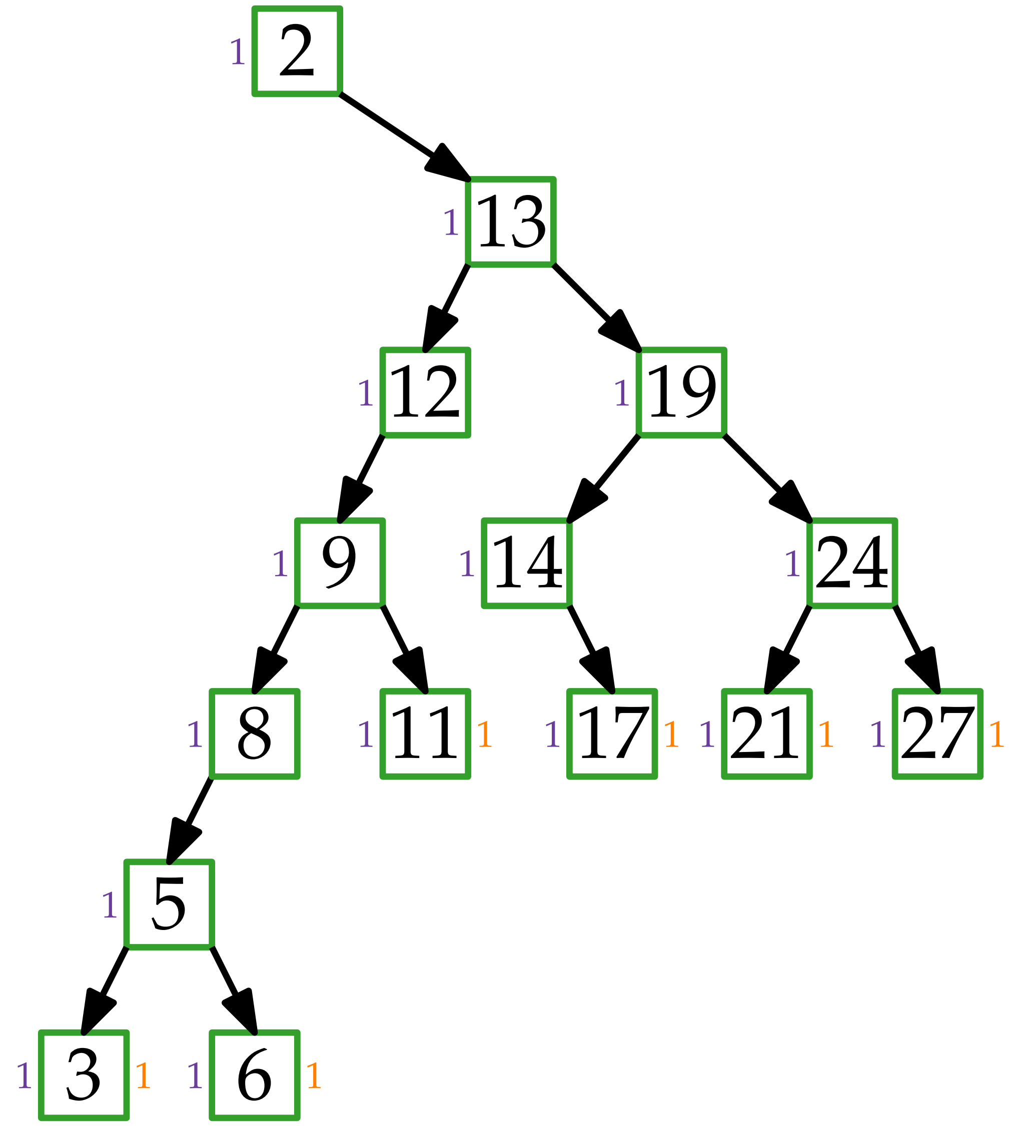

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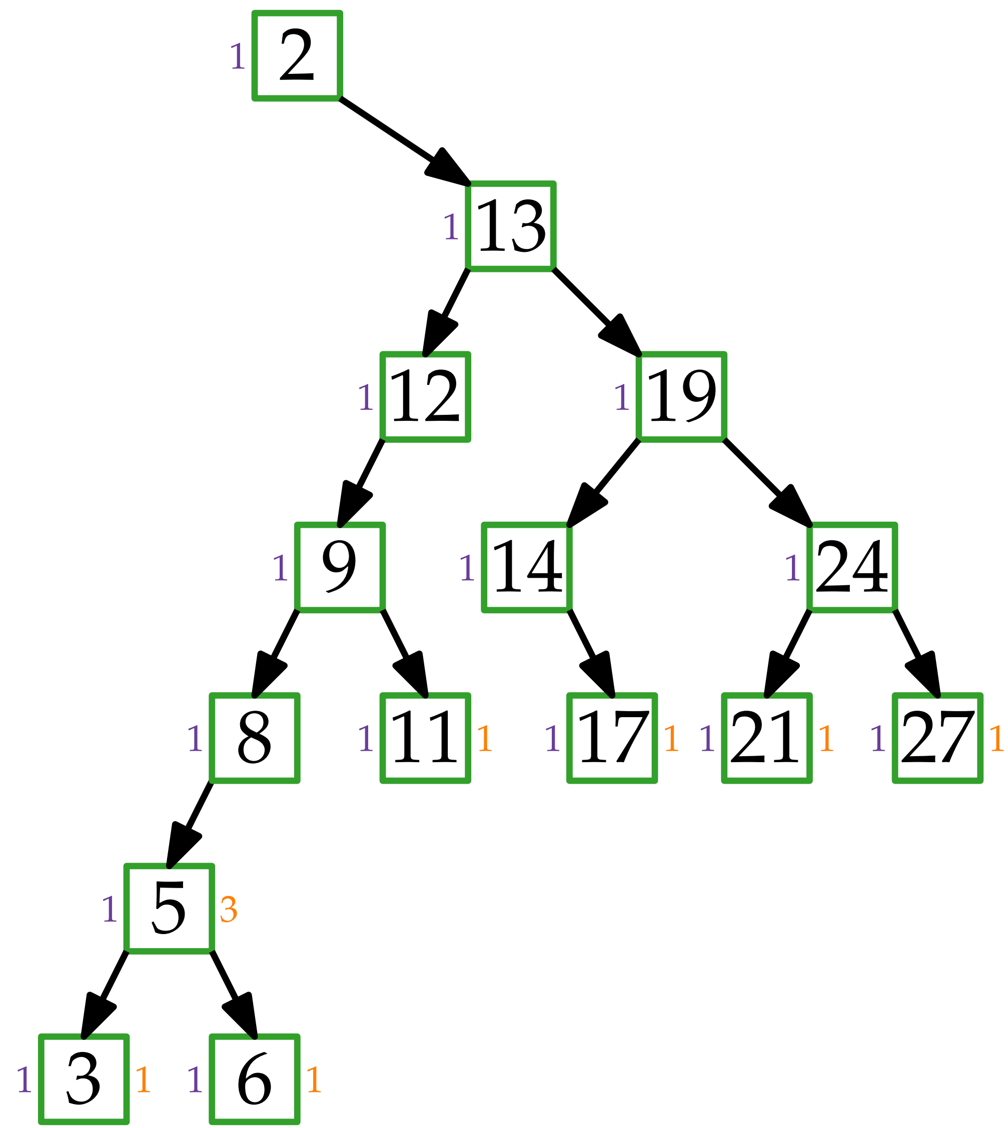

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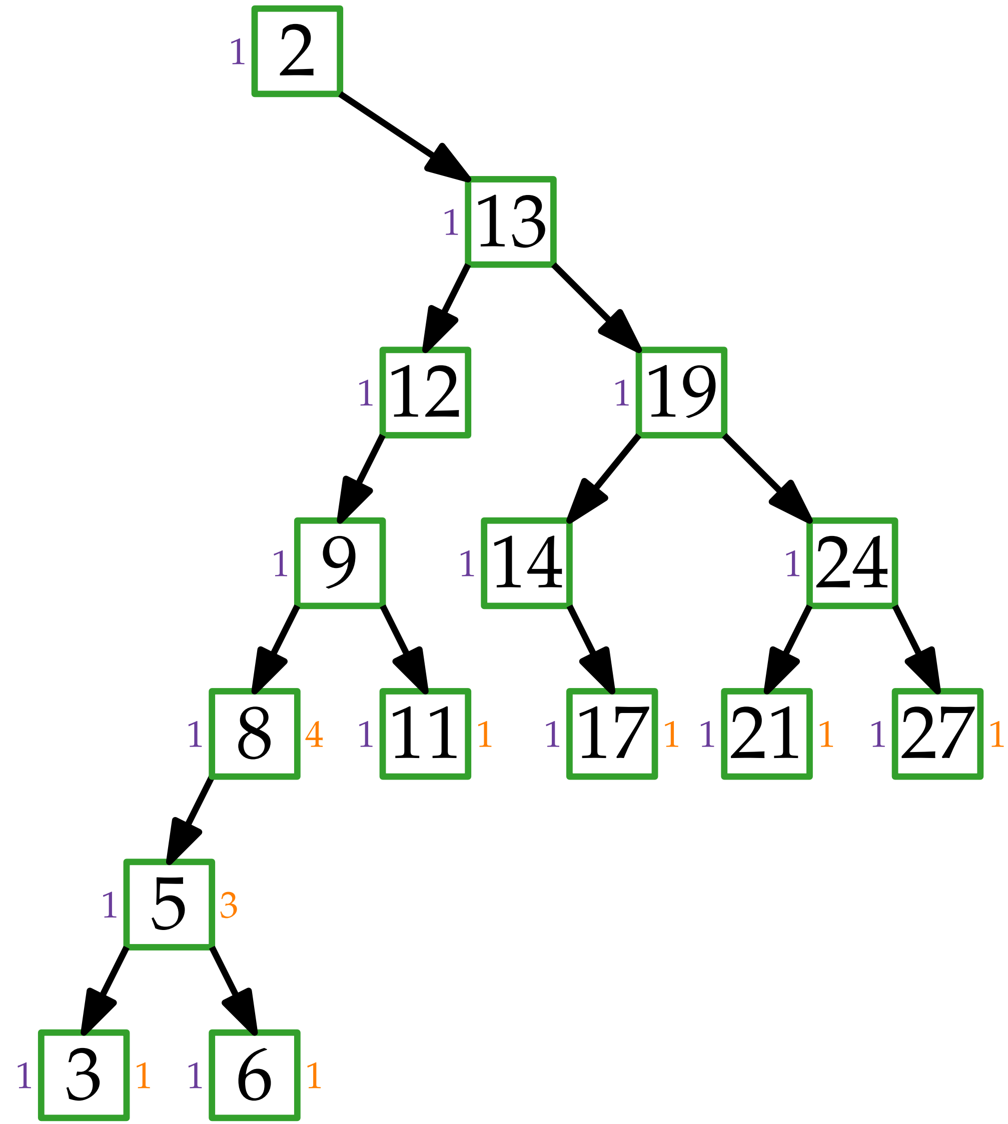

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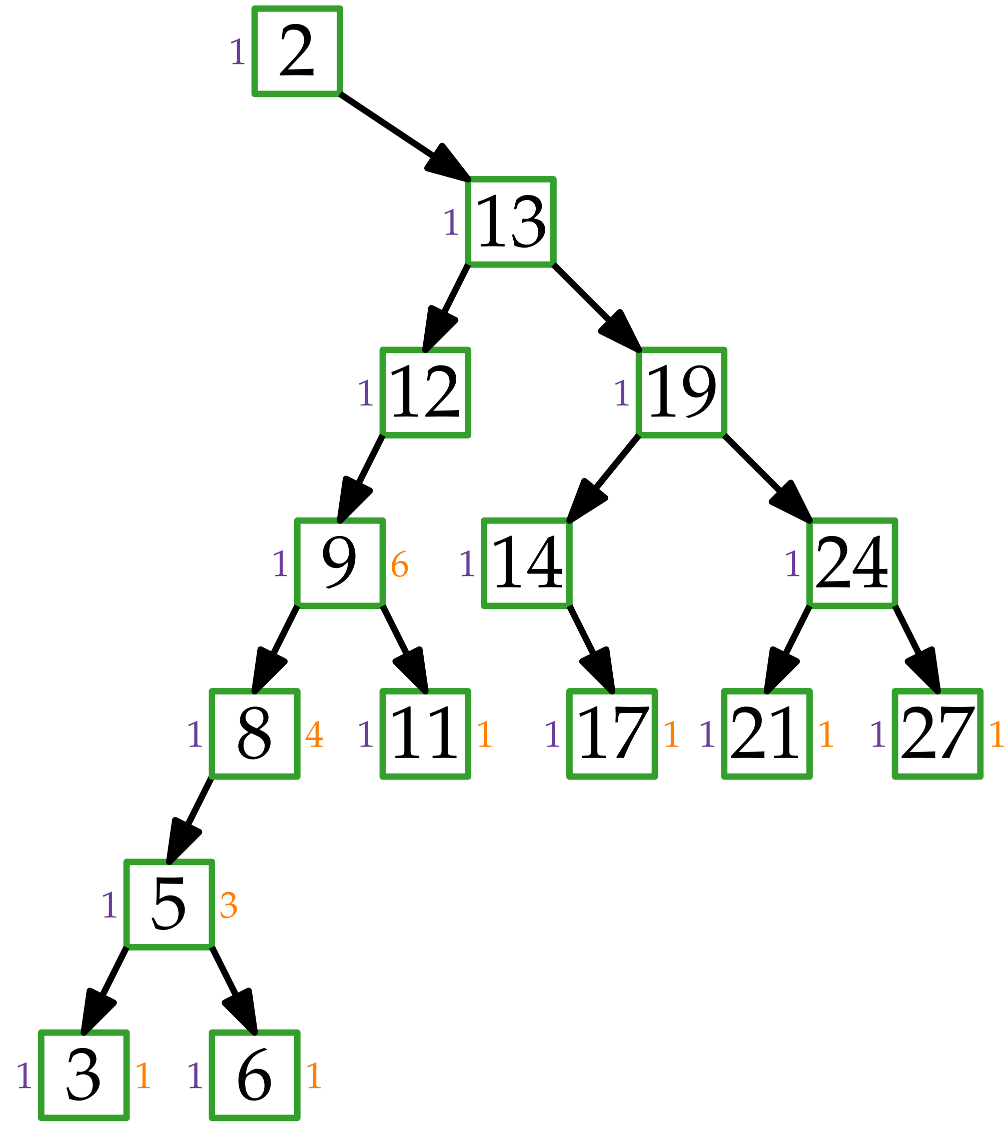

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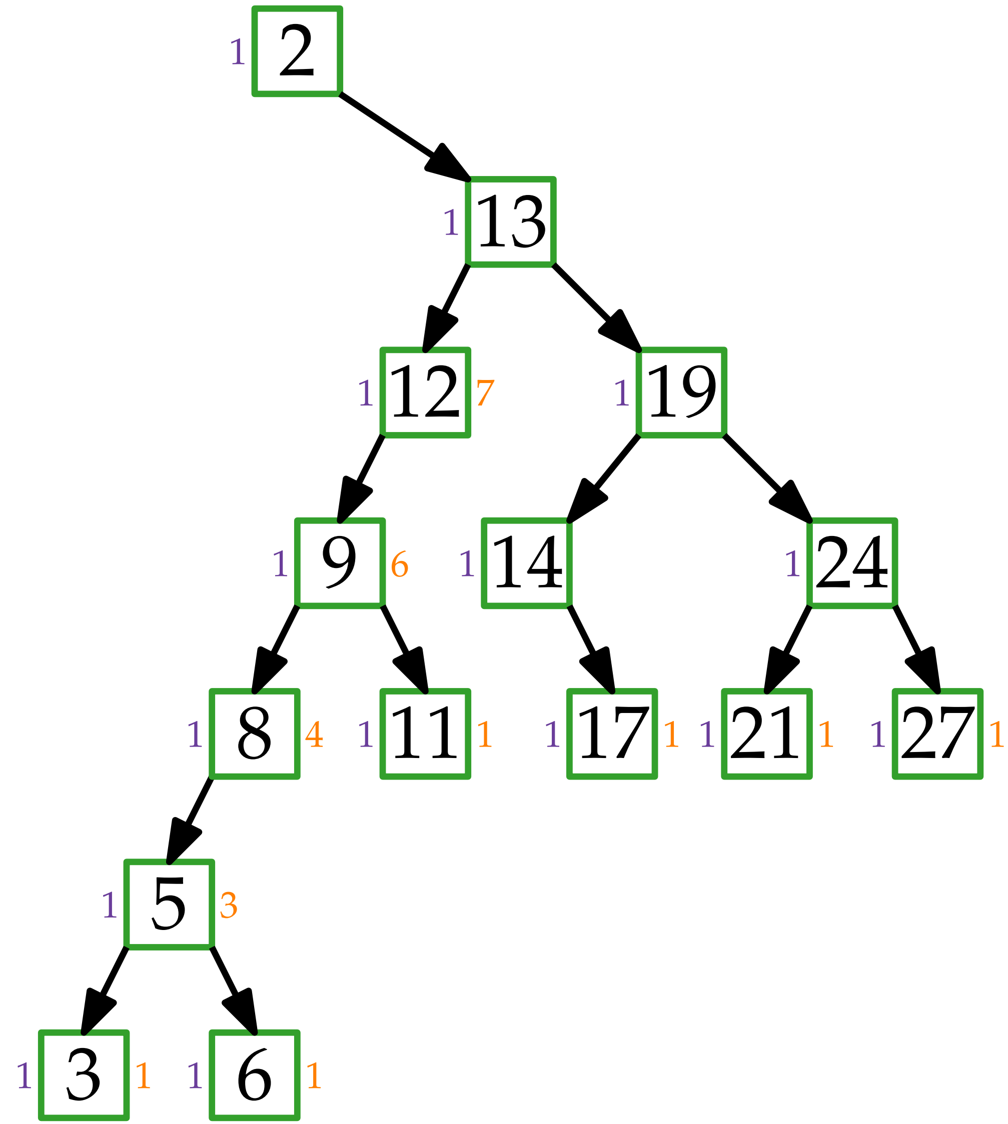

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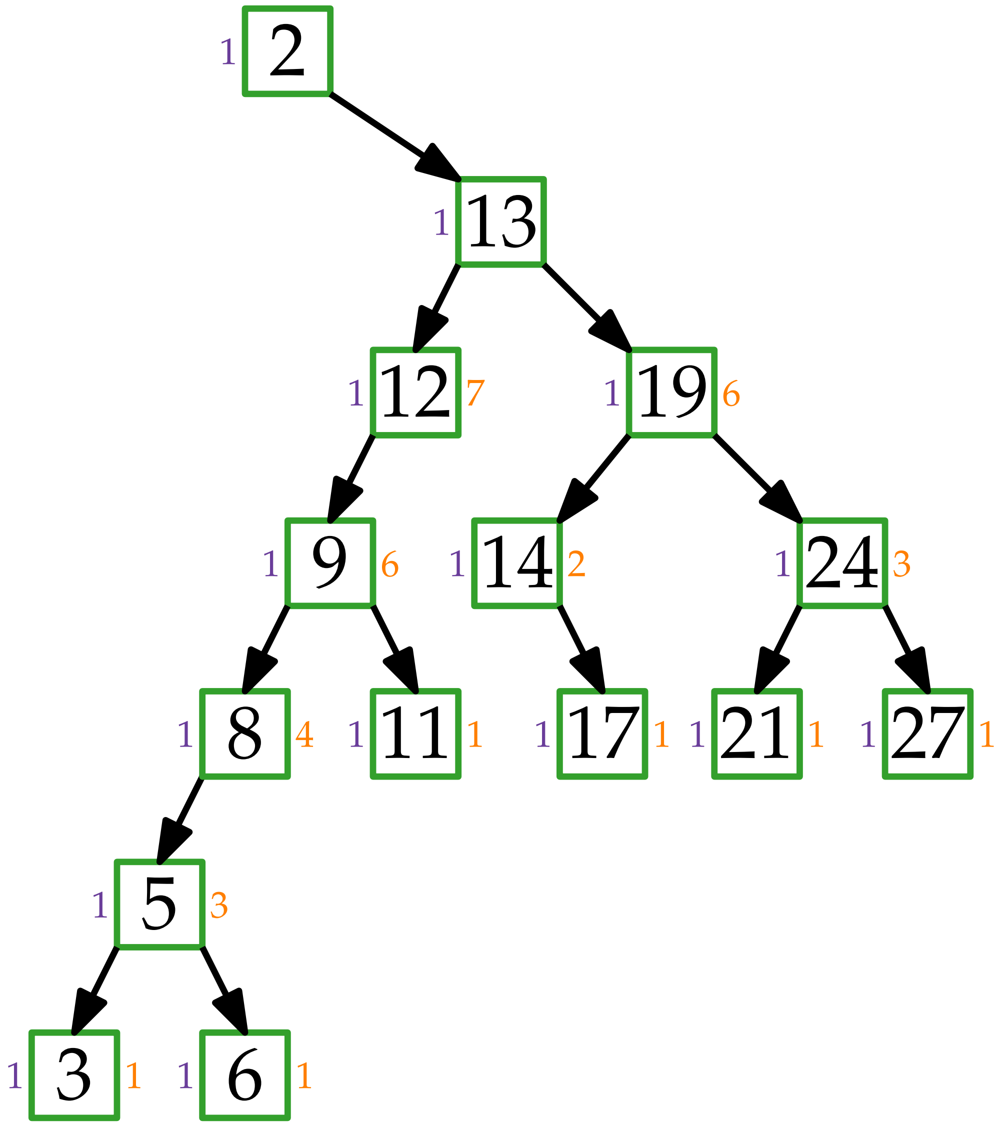

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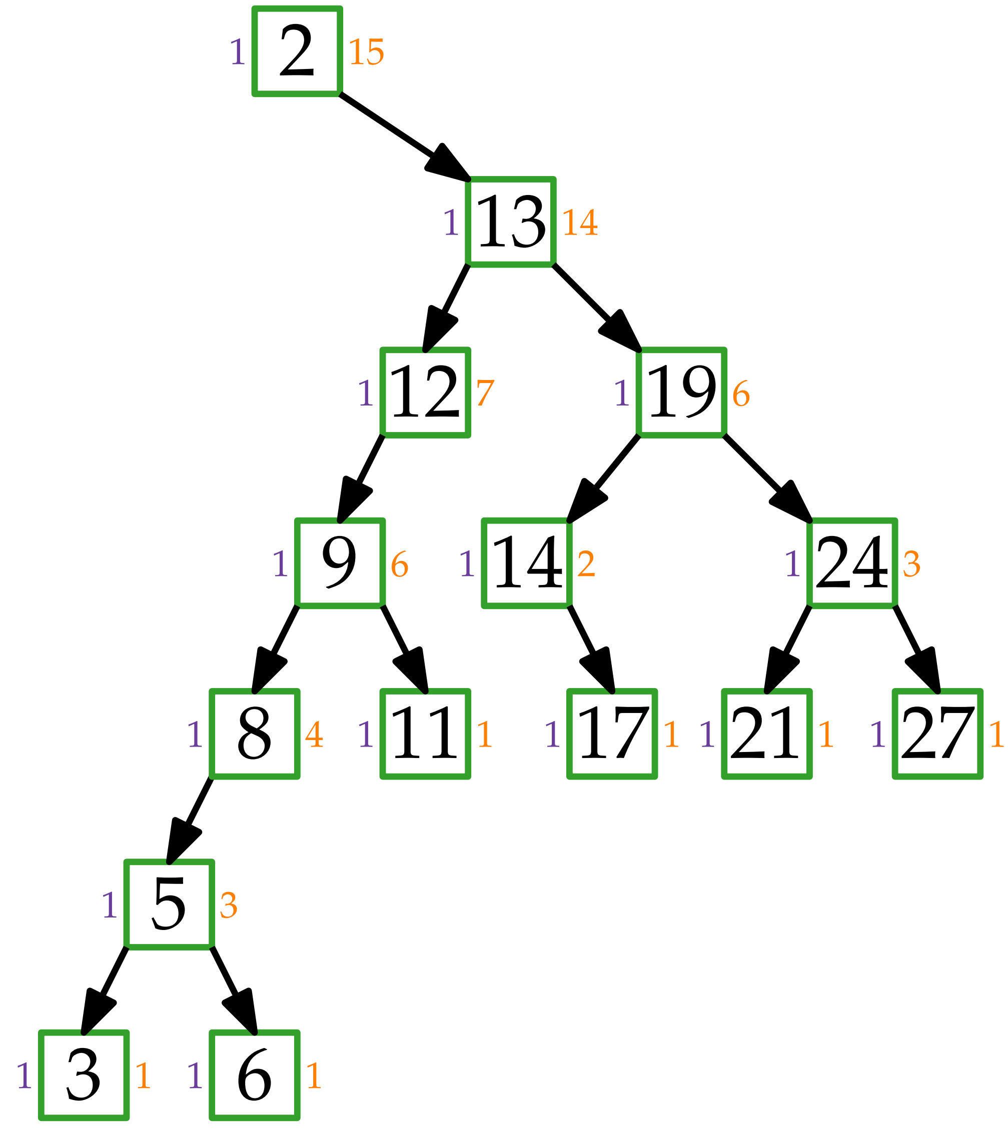

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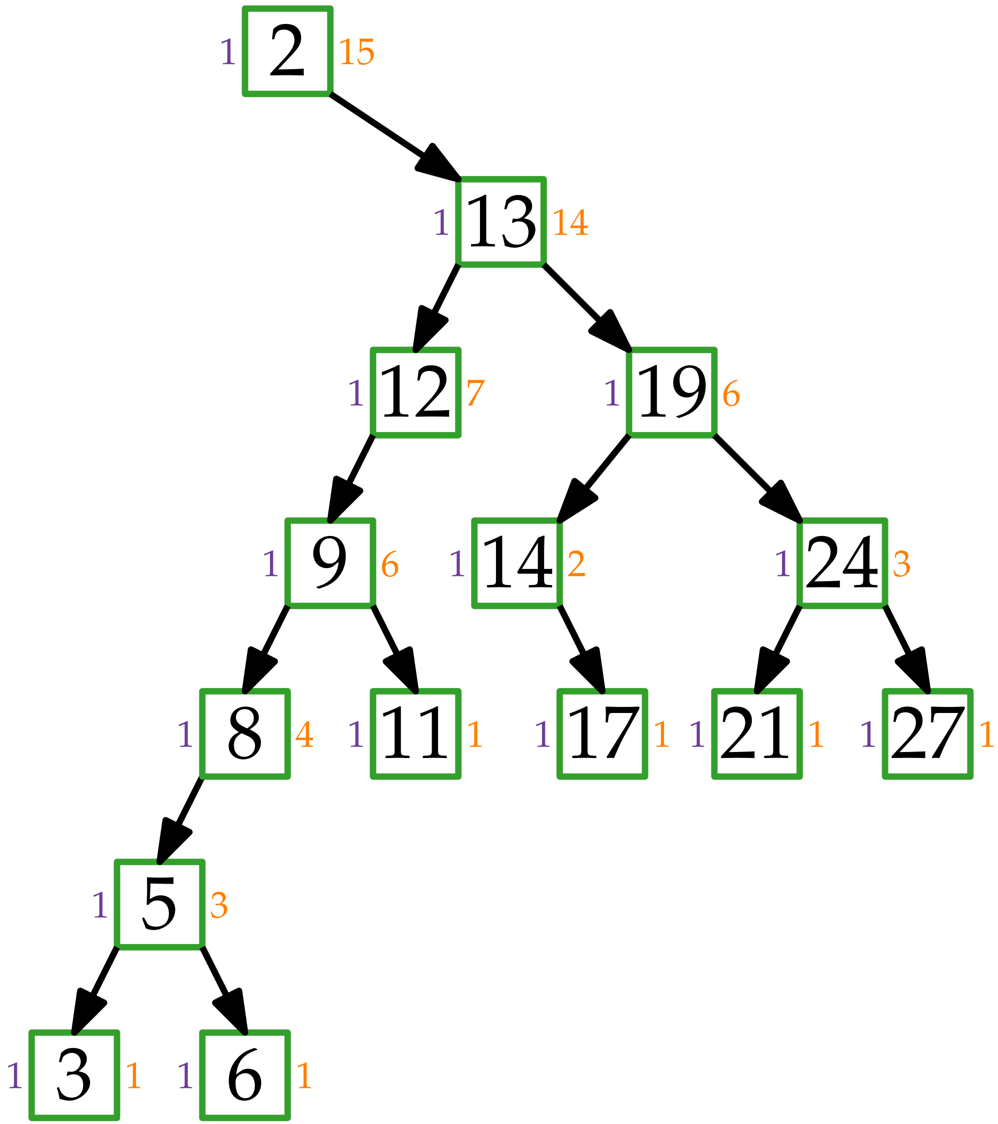
mark edges:



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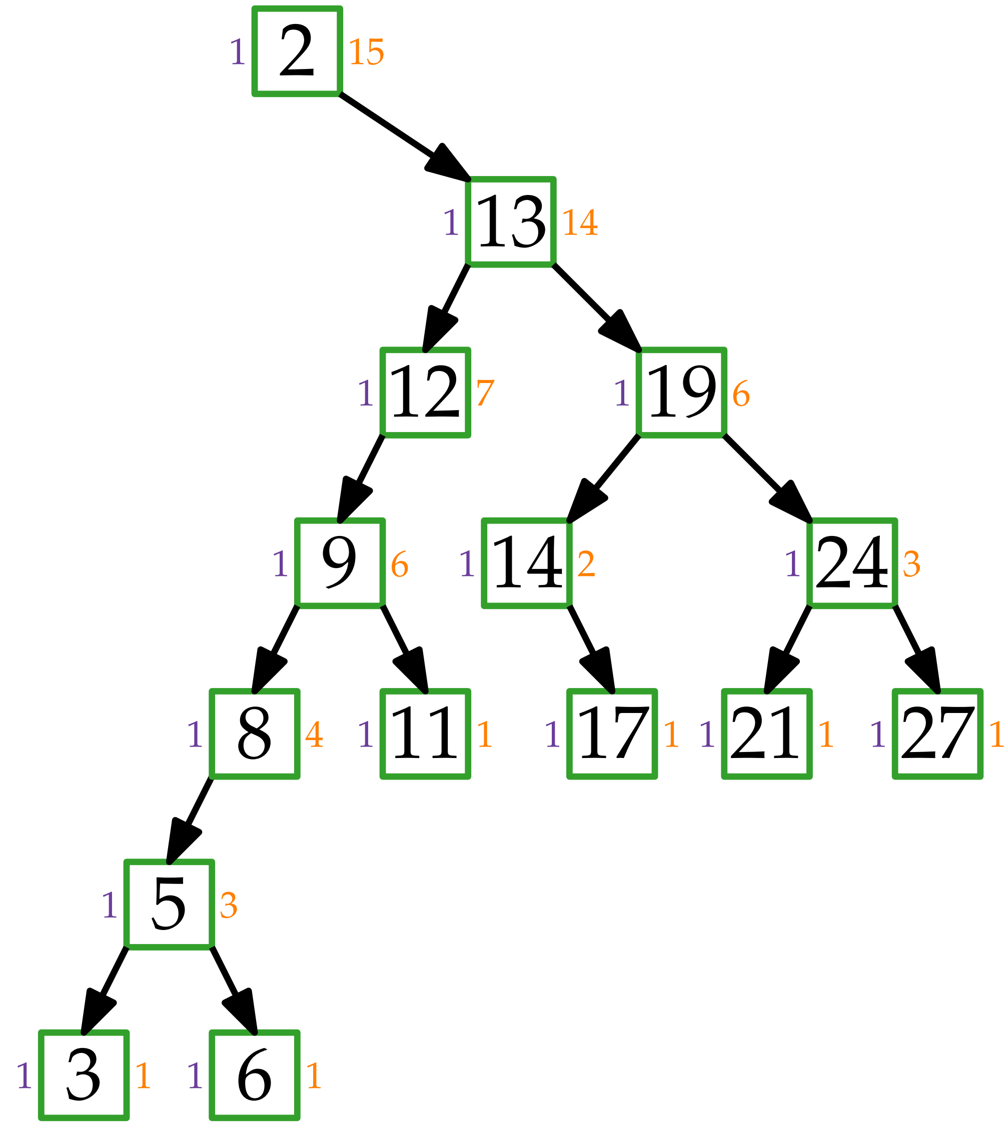
mark edges:

$\longrightarrow$ $s(\text{child}) \leq s(\text{parent})/2$



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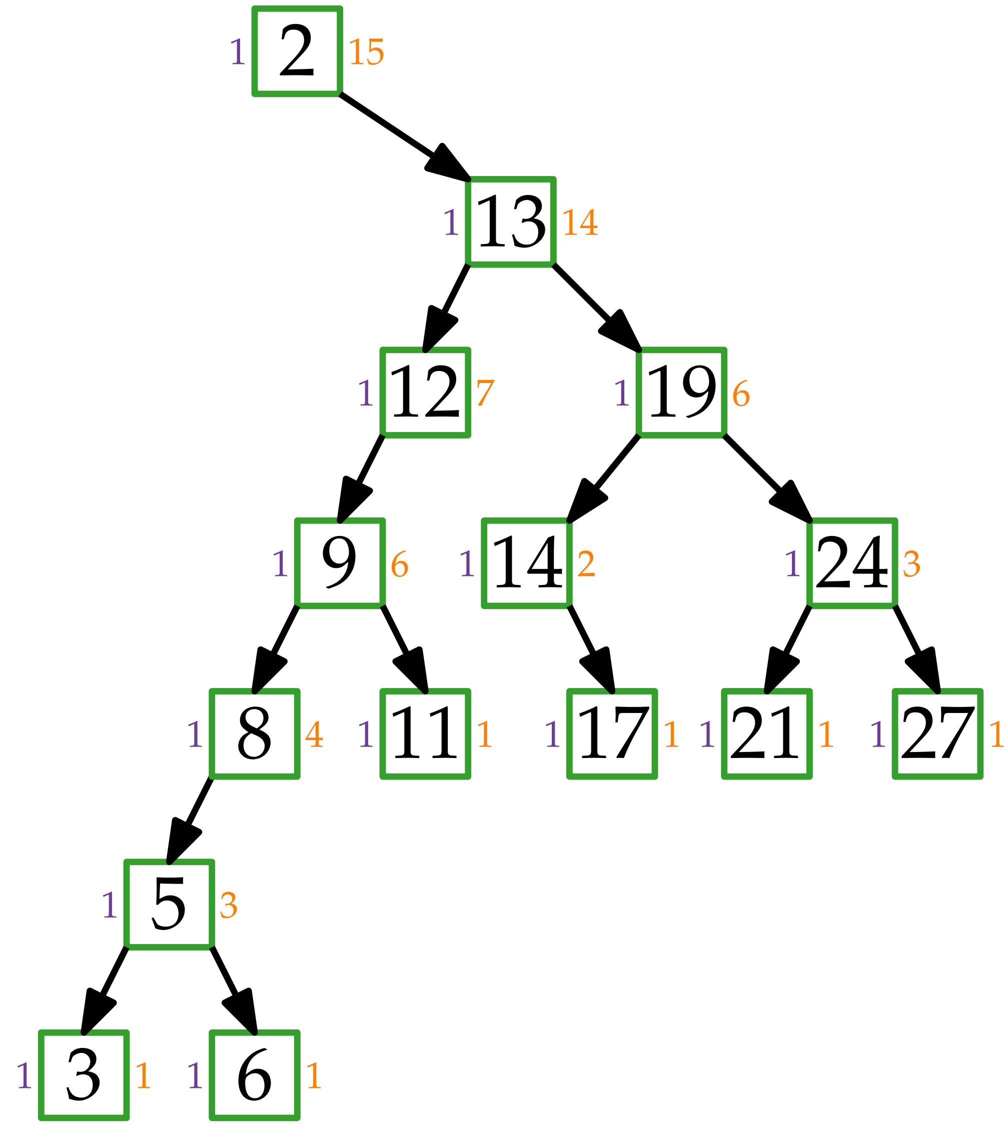

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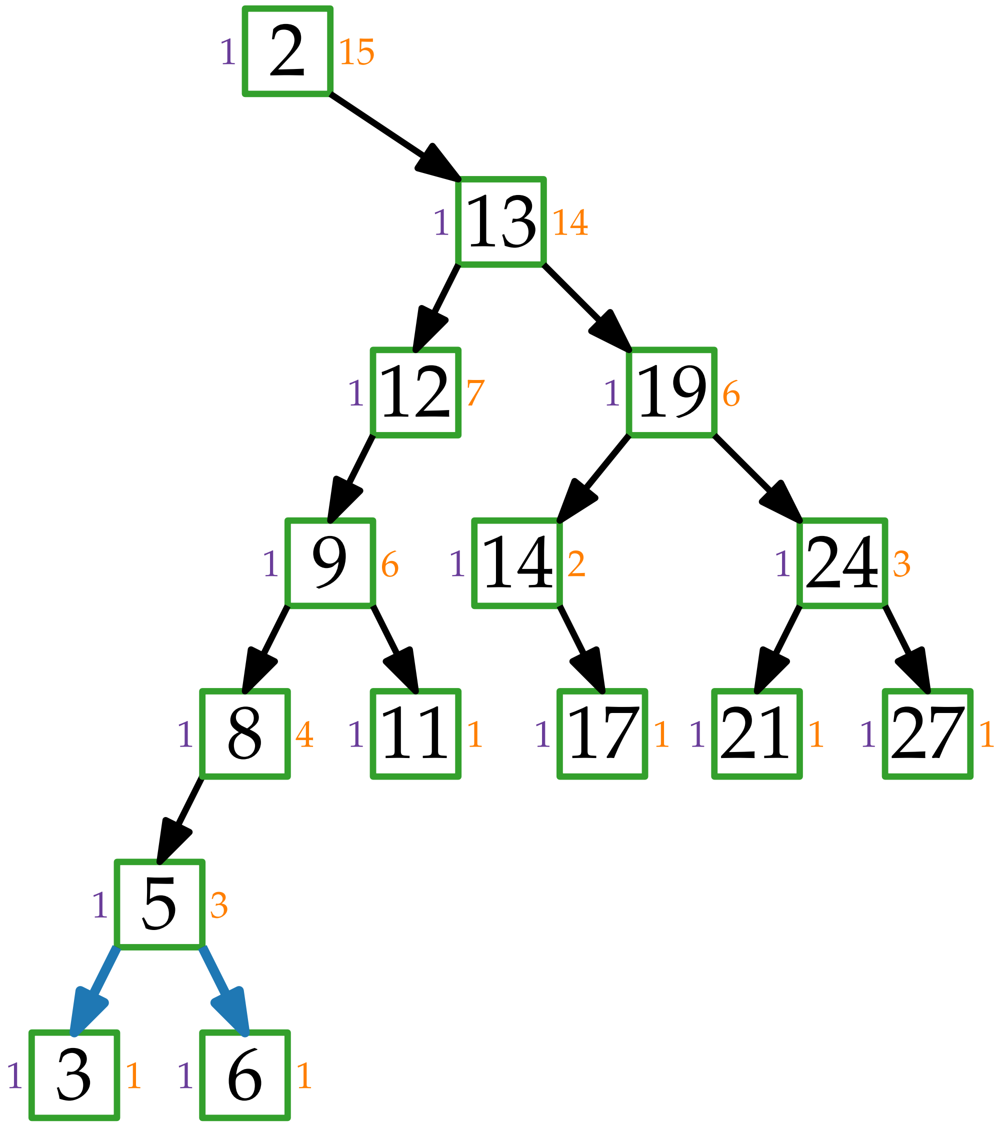

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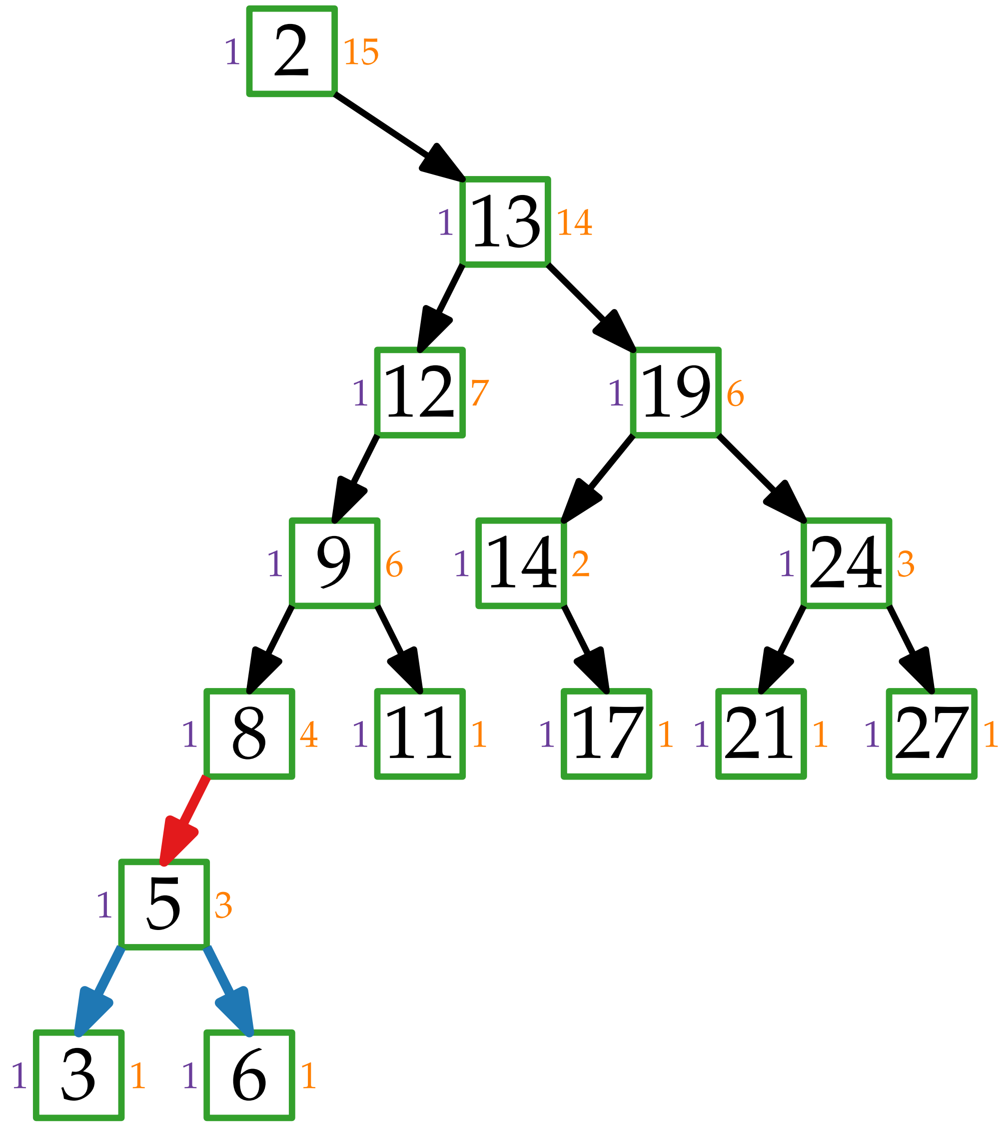

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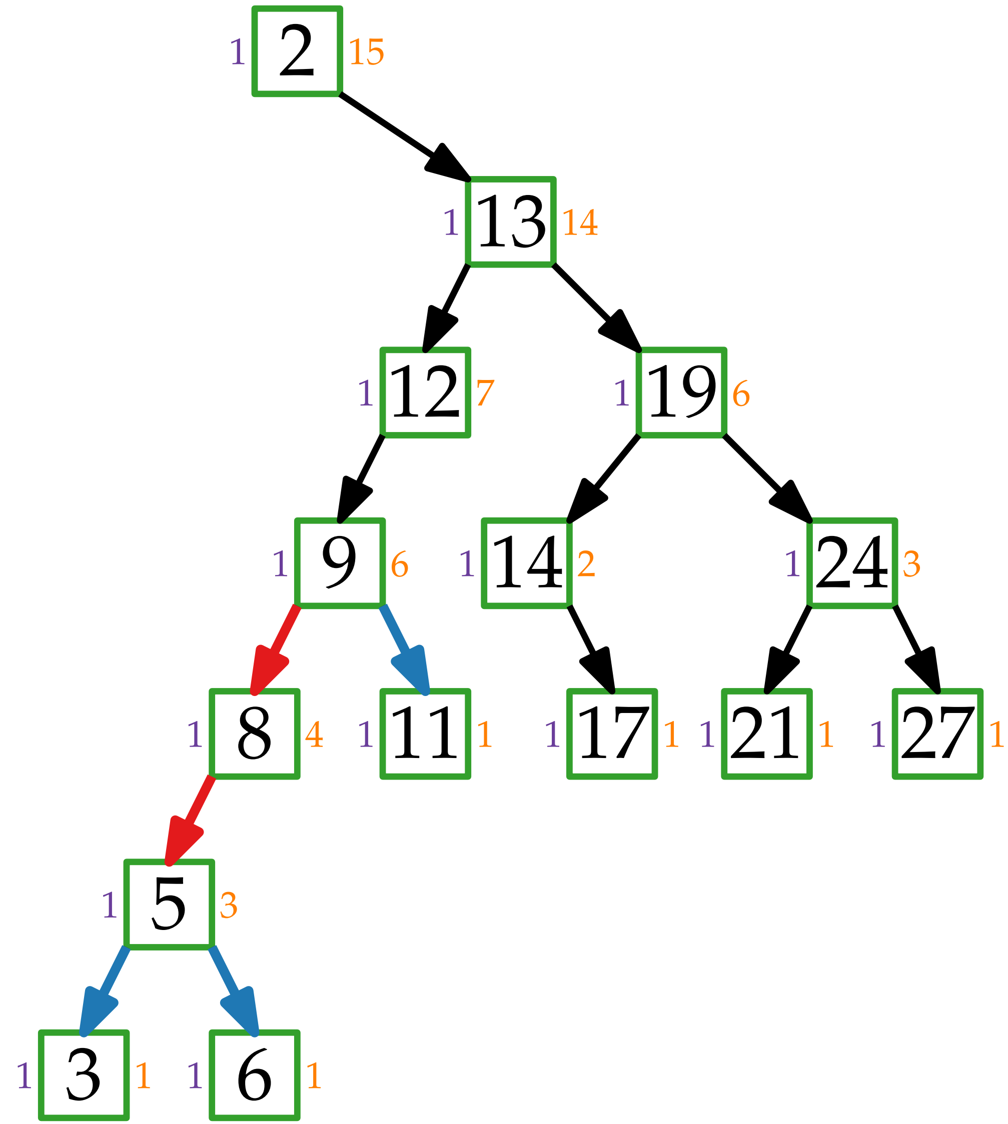

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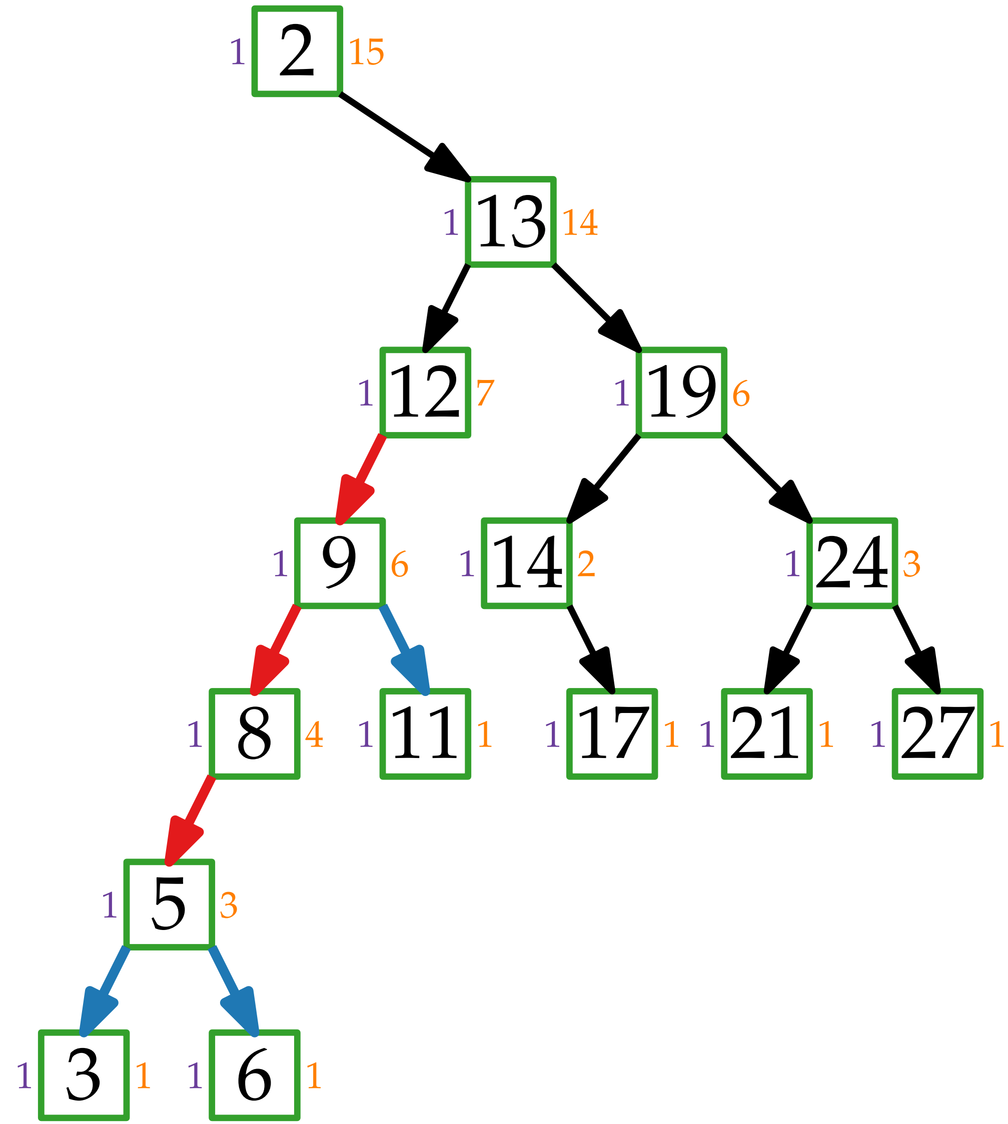

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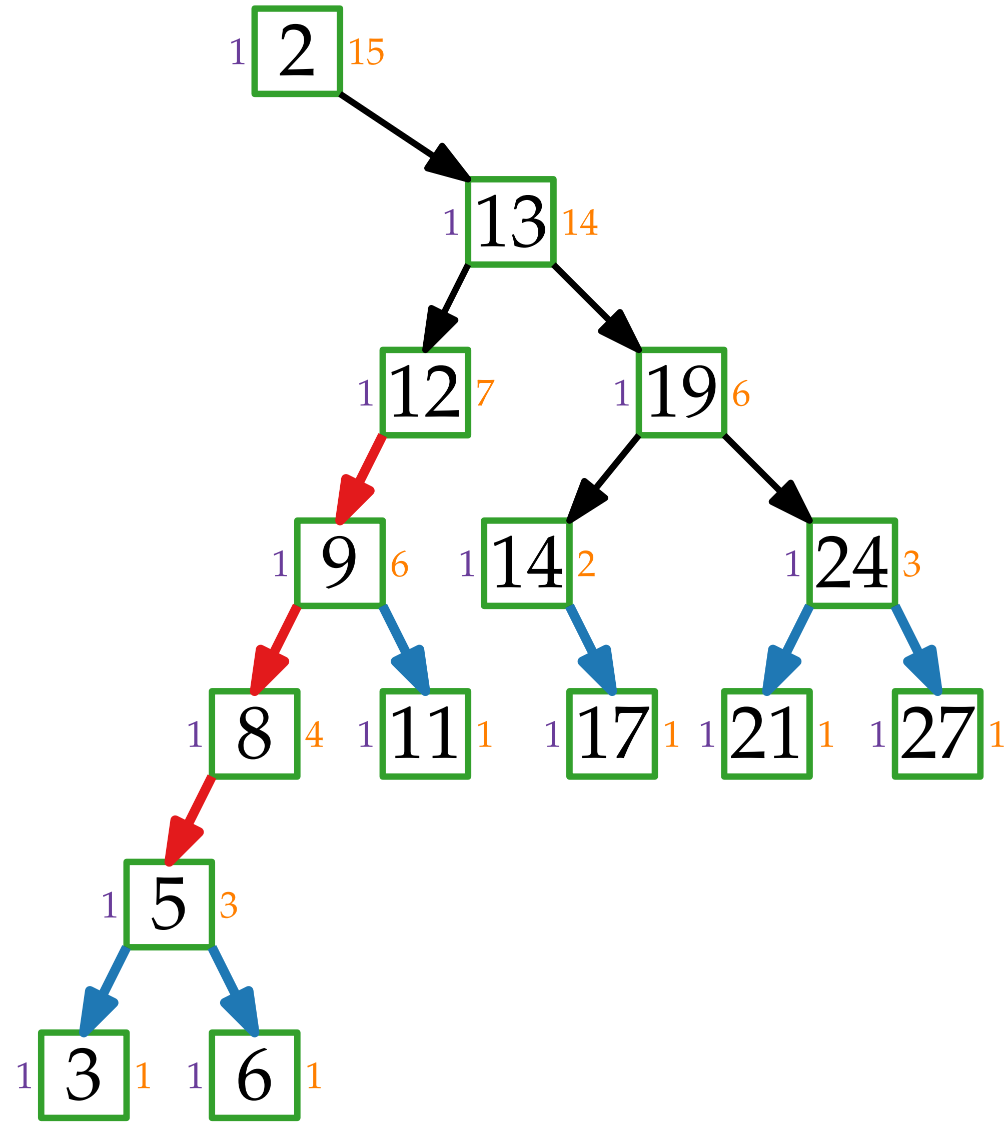

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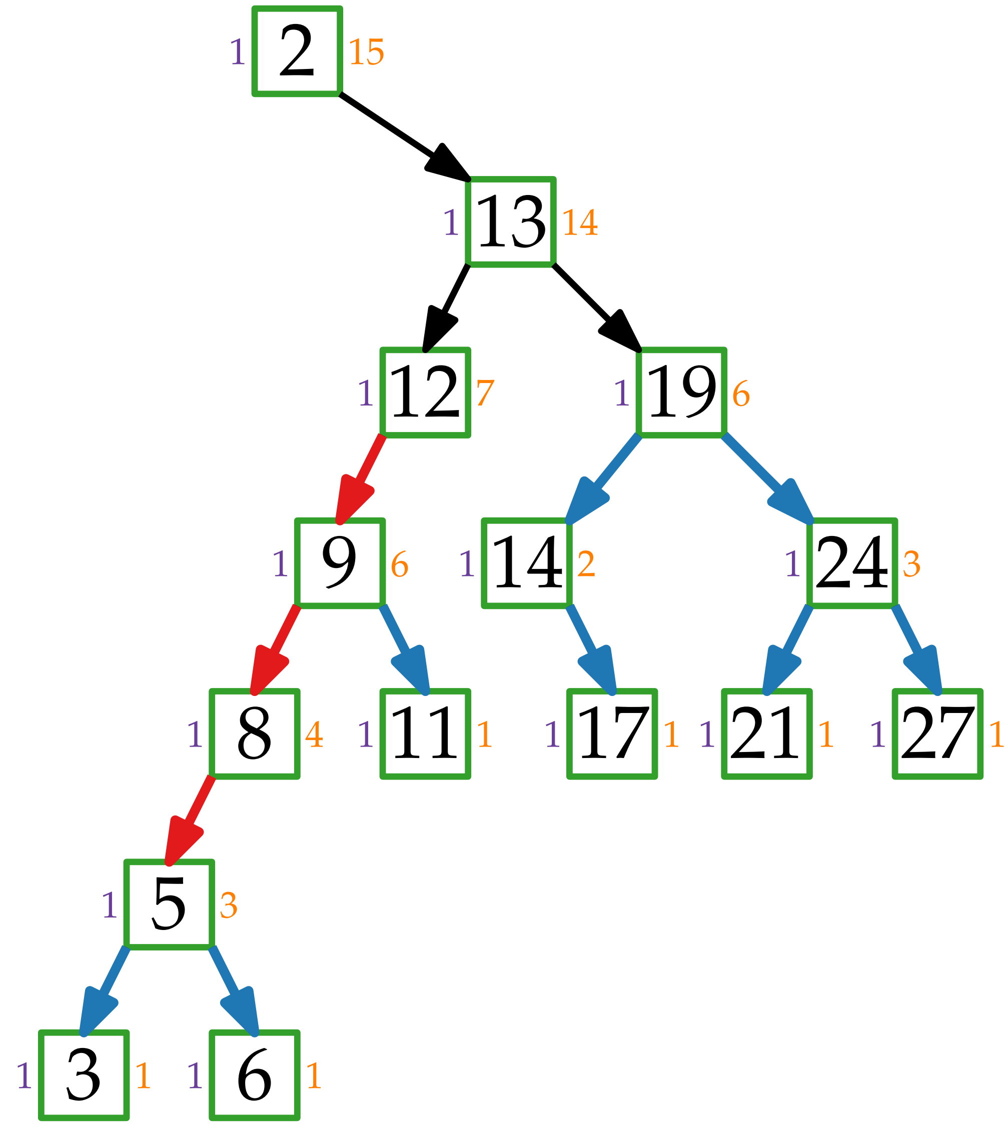

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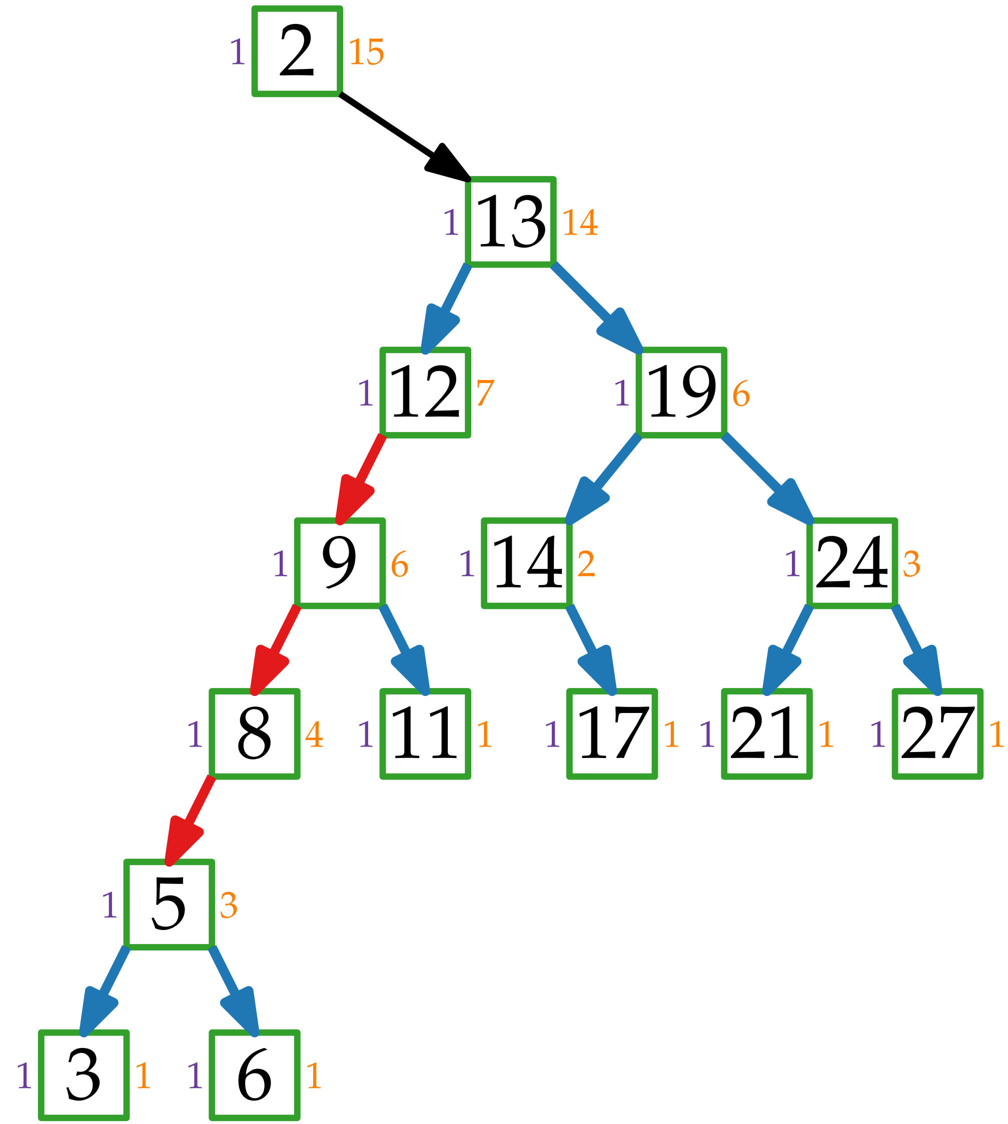


Why is Splay fast?

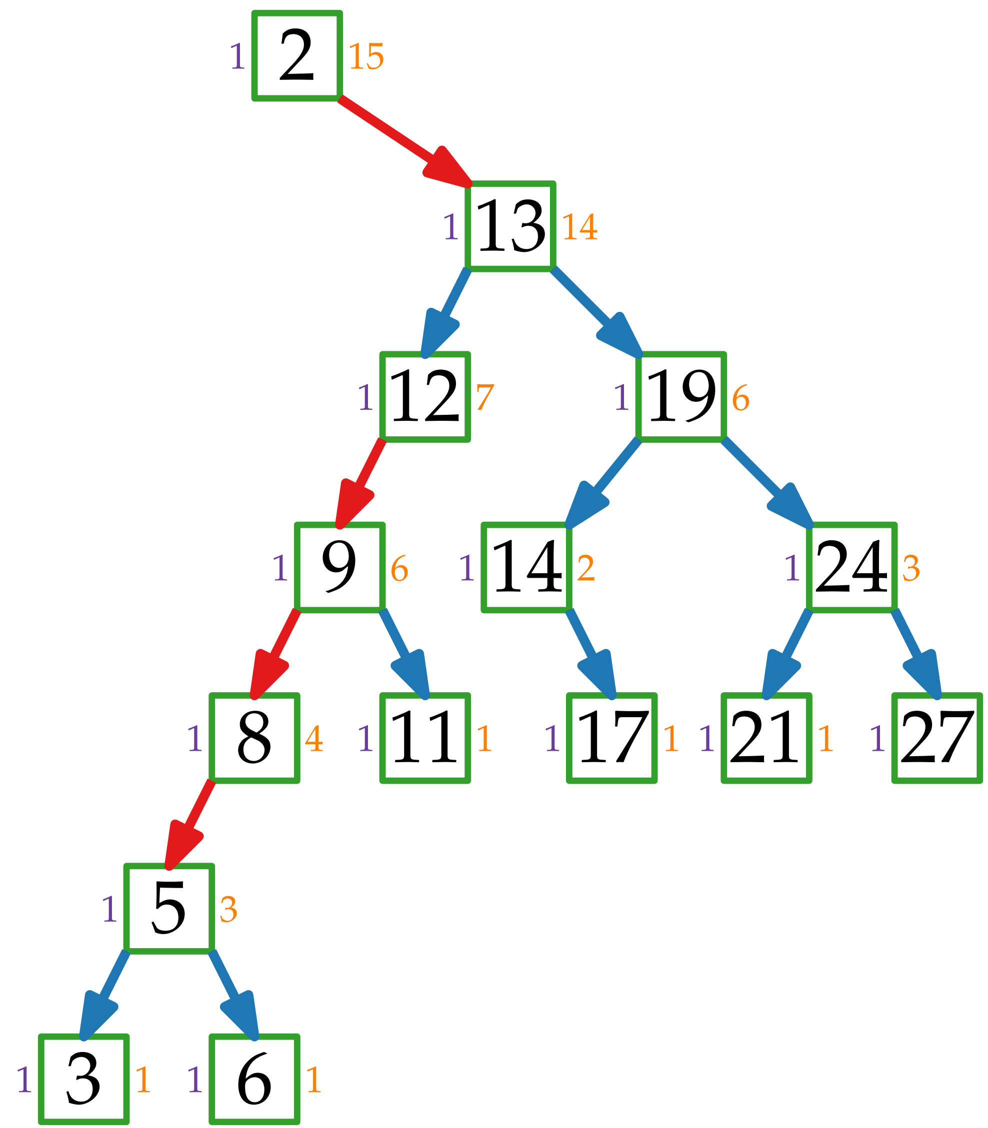
$w(x)$: weight of x (here 1), $W = \sum w(x)$ (here n)

$s(x)$: sum of all $w(x)$ in subtree of x_i

mark edges:

$s(\text{child}) \leq s(\text{parent})/2$

$s(\text{child}) > s(\text{parent})/2$



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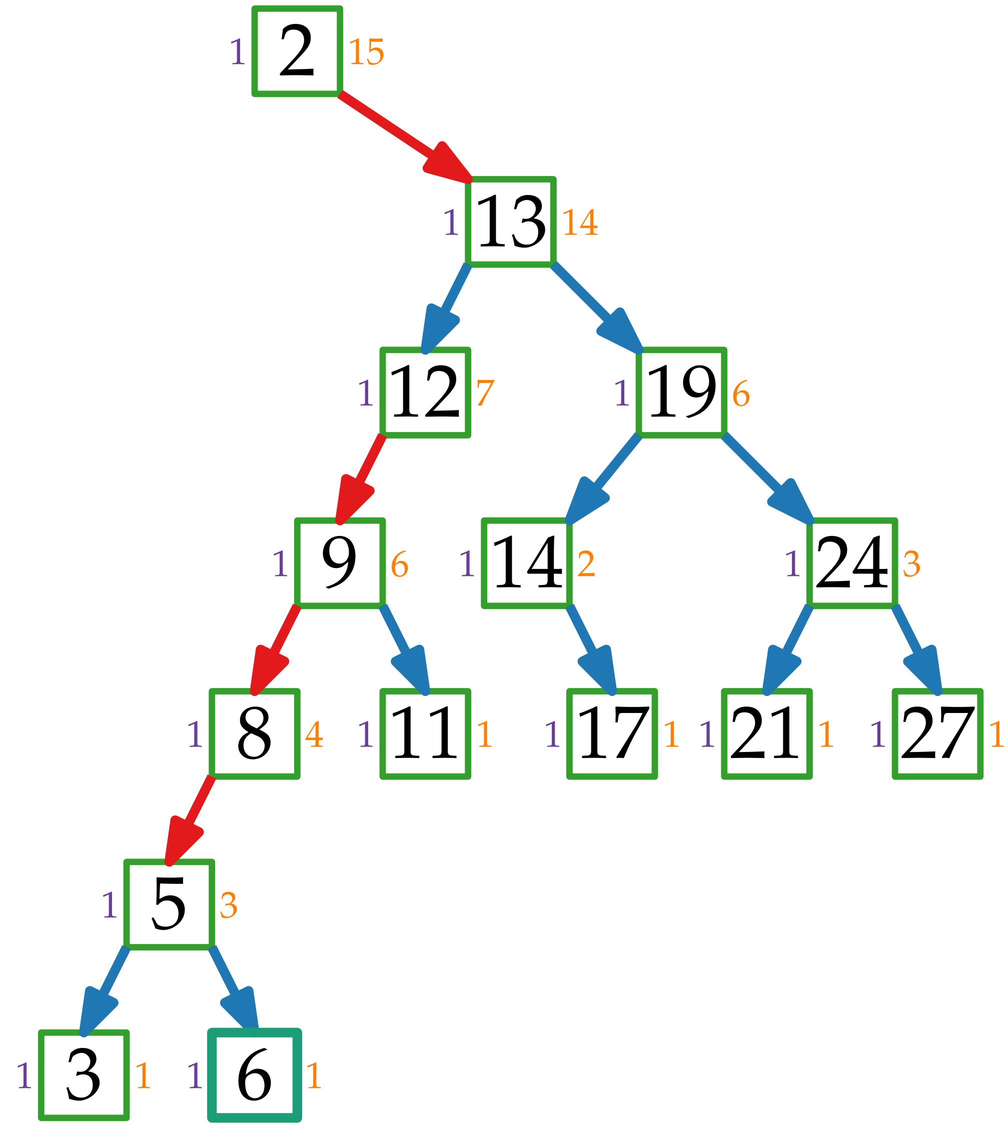
Cost to query x_i:



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$w(x)$: weight of x (here 1), $W = \sum w(x)$ (here n)

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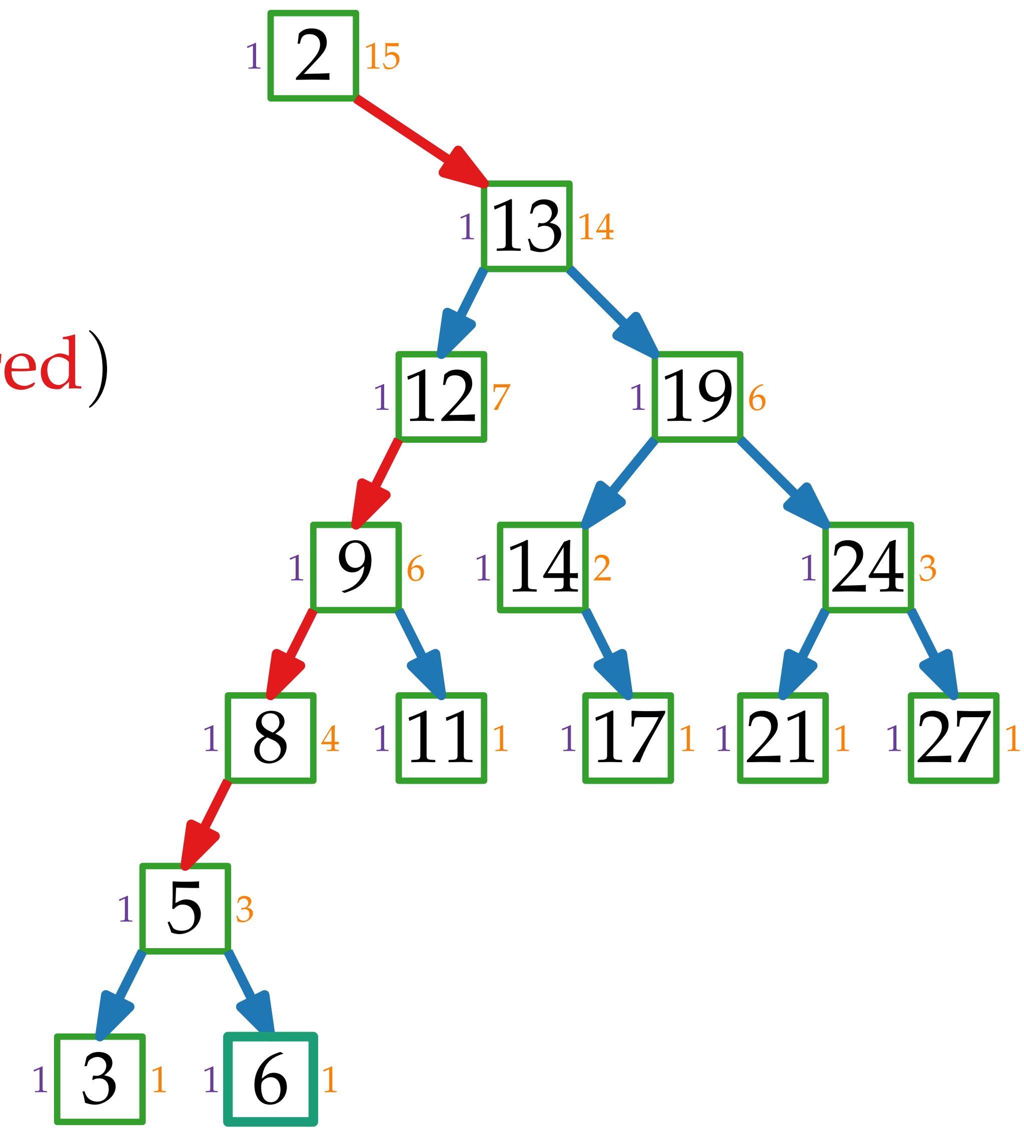
mark edges:

blue arrow: $s(\text{child}) \leq s(\text{parent})/2$

red arrow: $s(\text{child}) > s(\text{parent})/2$

Cost to query x_i: $O(\#\text{blue} + \#\text{red})$



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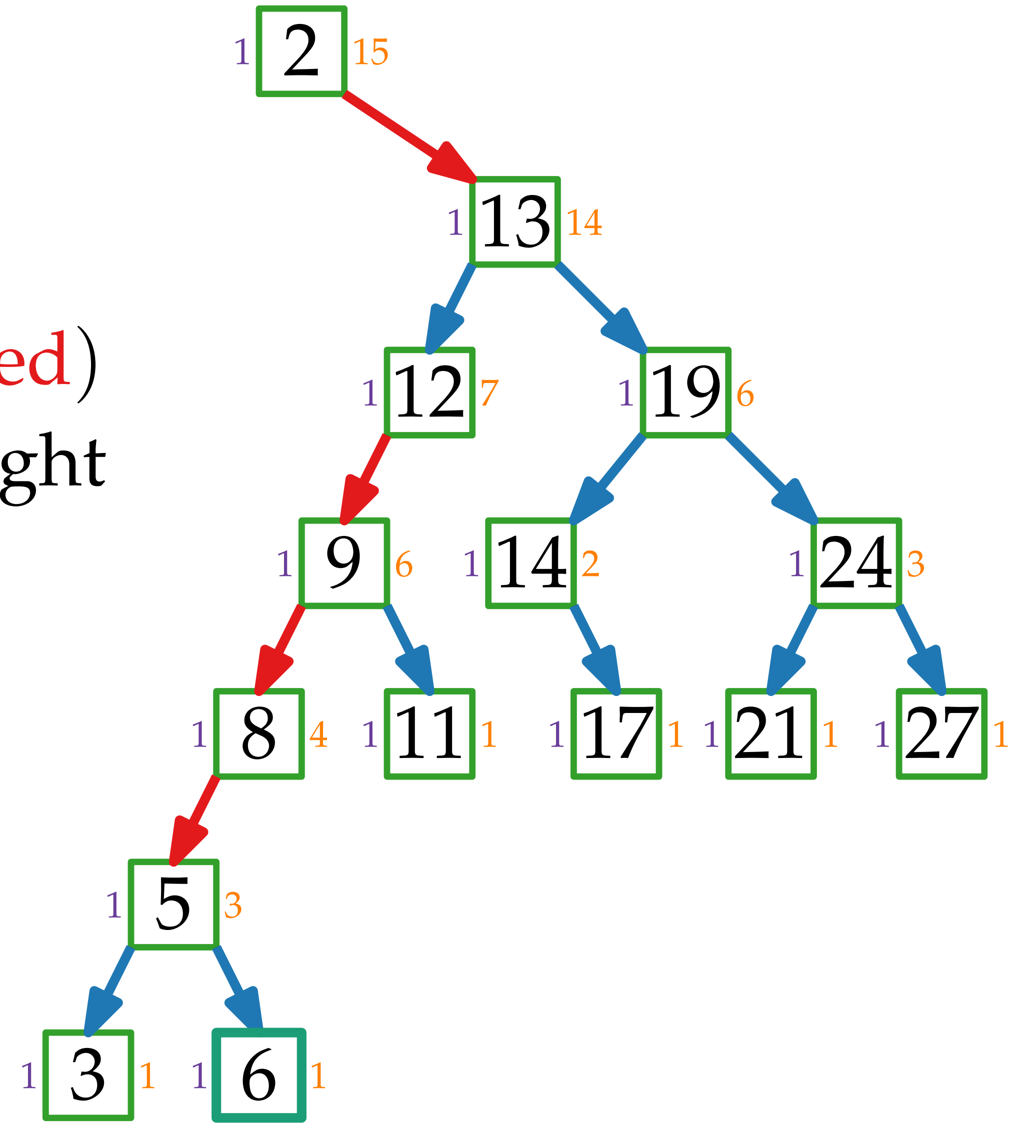
Idea: blue edges halve the weight



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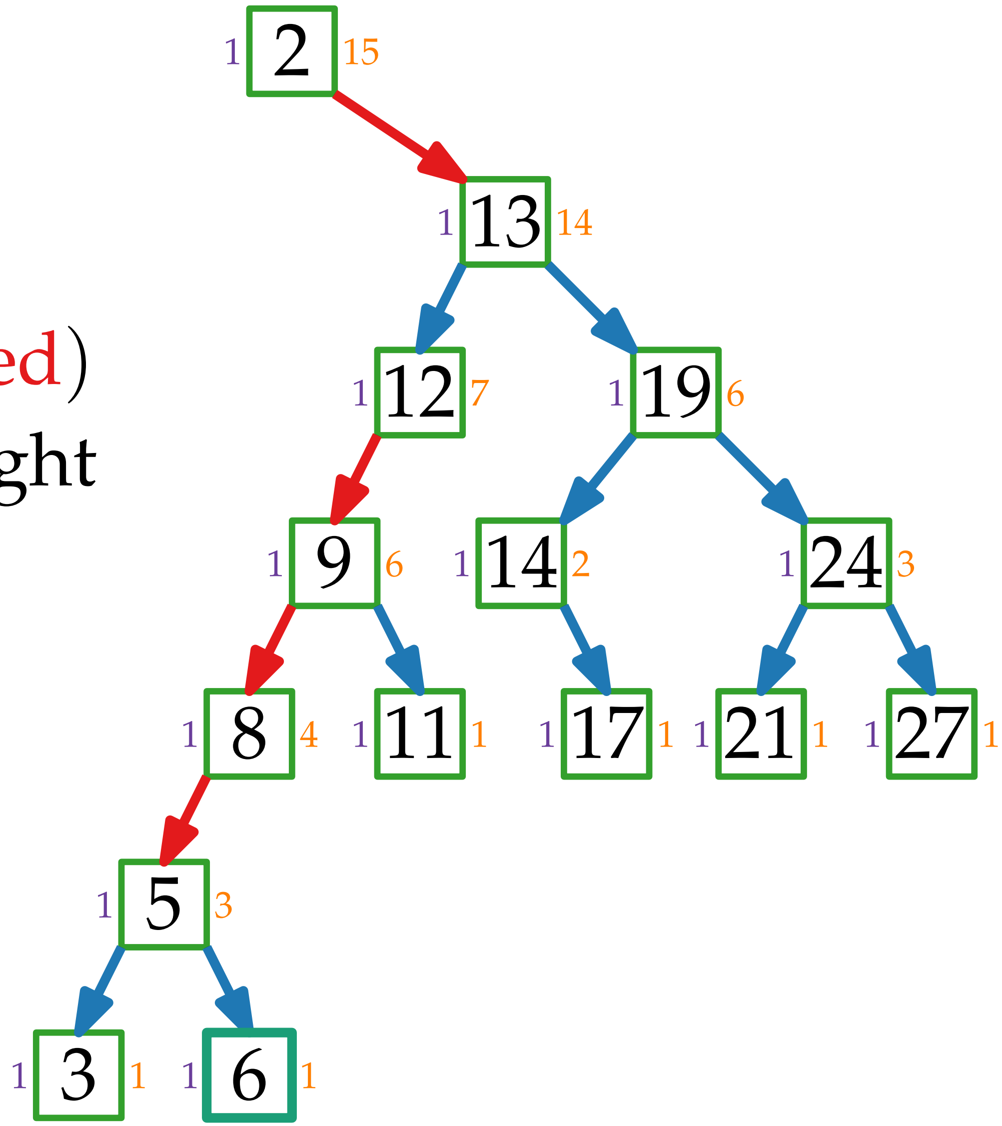
$\Rightarrow \#\text{blue} \in O(\log W)$



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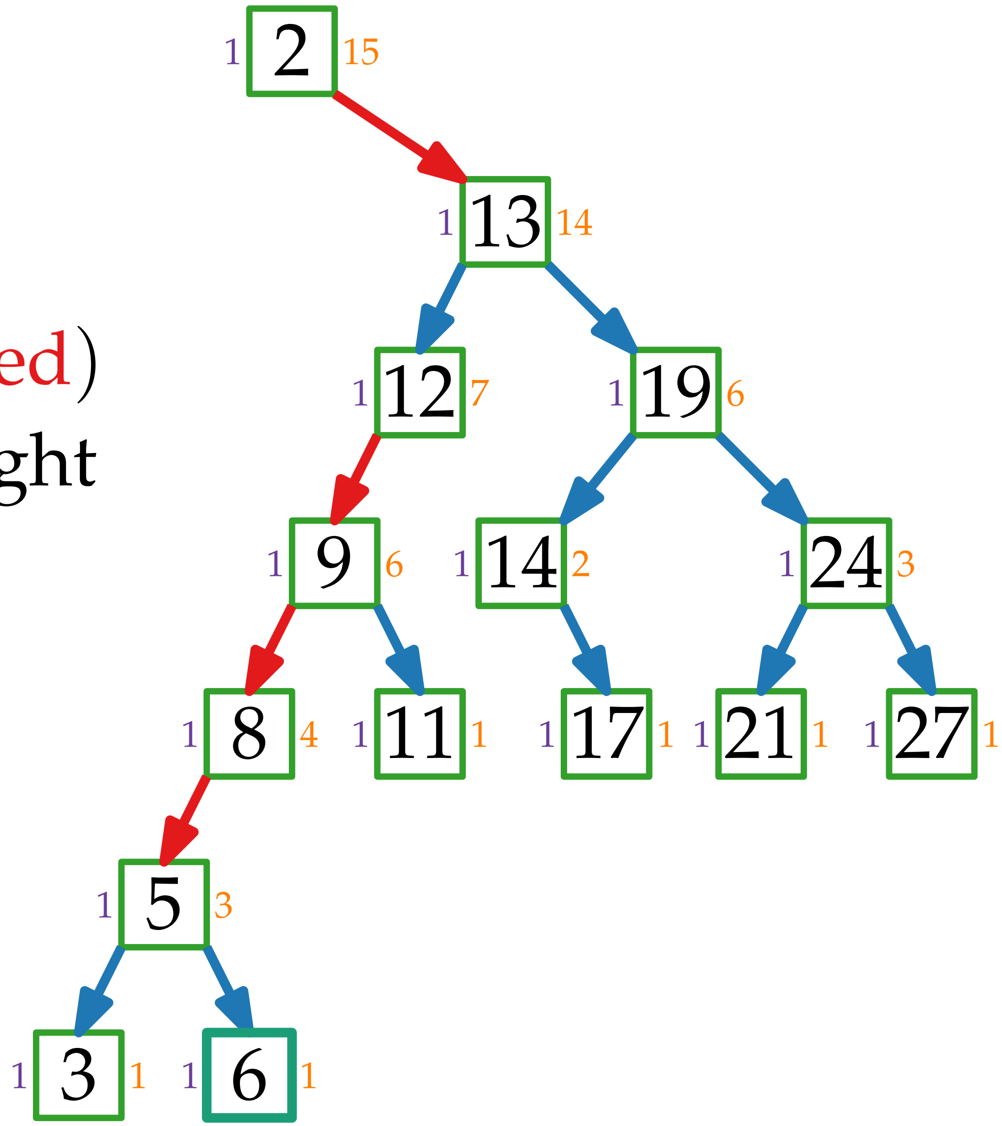

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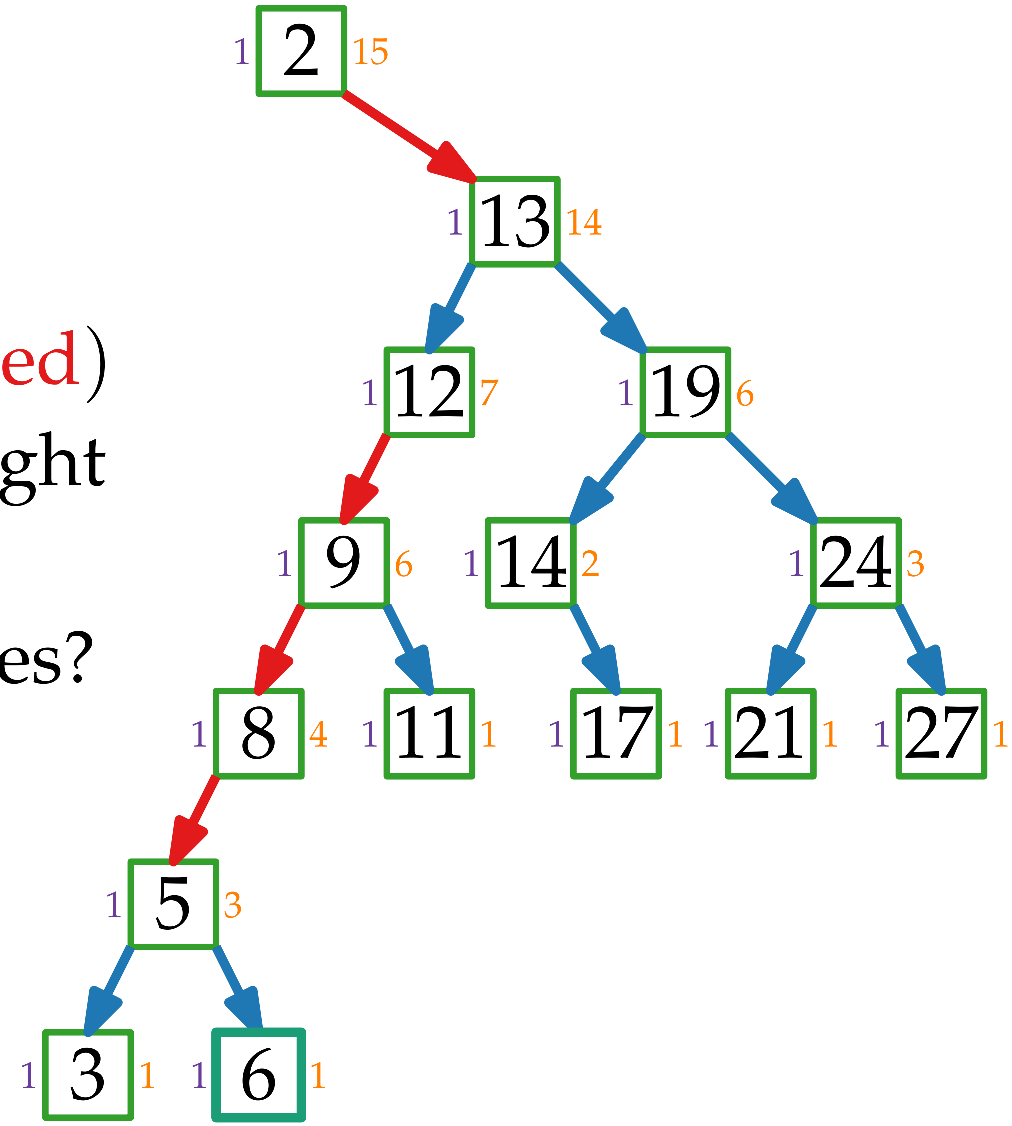
How can we amortize red edges?



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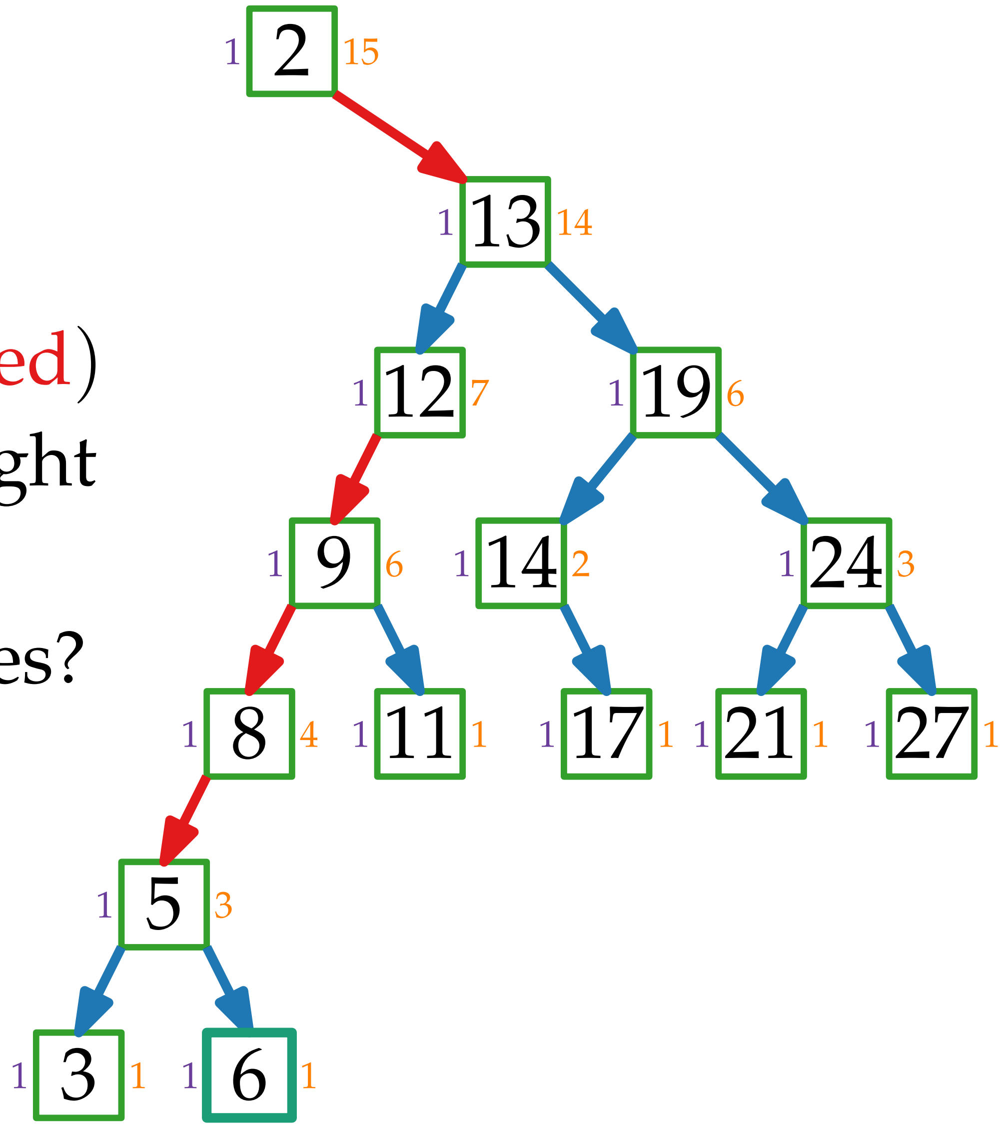
Use sum-of-logs potential

$\Phi = \sum \log s(x)$



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How can we amortize red edges?

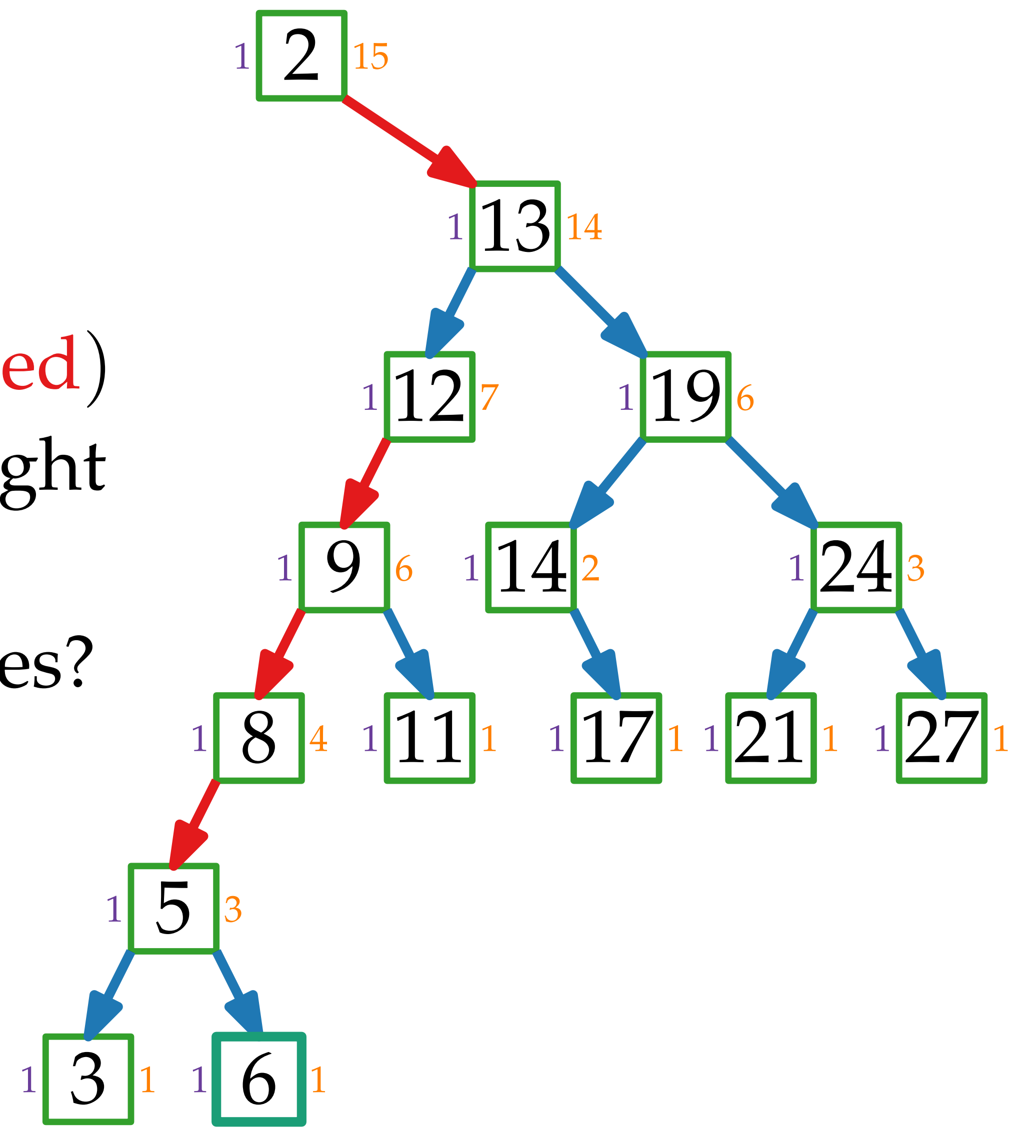
Use sum-of-logs potential

$\Phi = \sum \log s(x)$

Amortized cost:

real cost + $\Phi_+ - \Phi$

(potential after splay)



Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

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Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a single rotation, the potential increases by $\leq 3\left(\log s_+(x) - \log s(x)\right)$.

Proof. $\text{Right}(x)$

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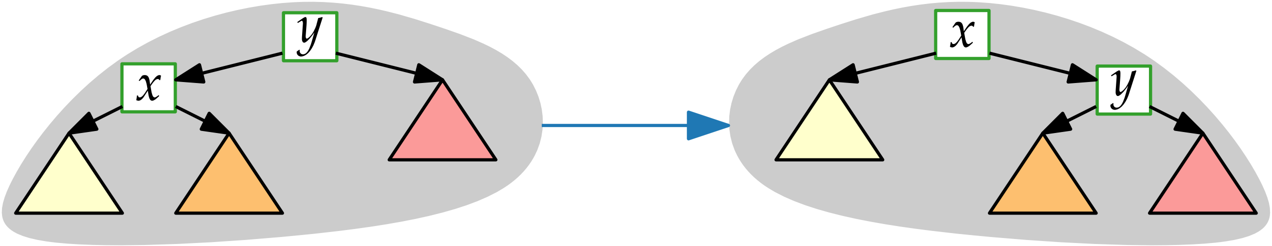

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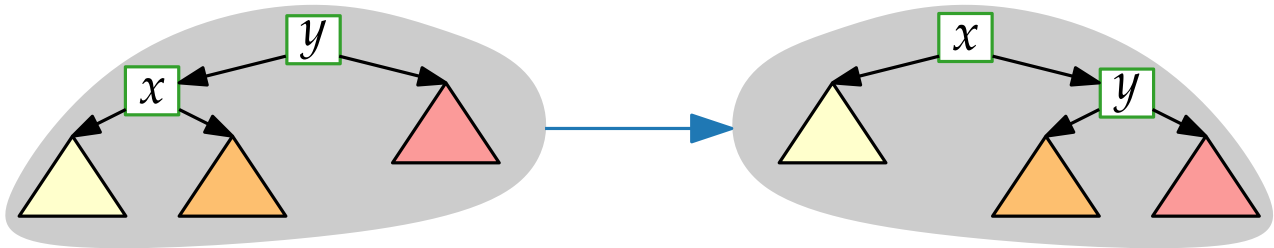

Observe: Only $s(x)$ and $s(y)$ change.

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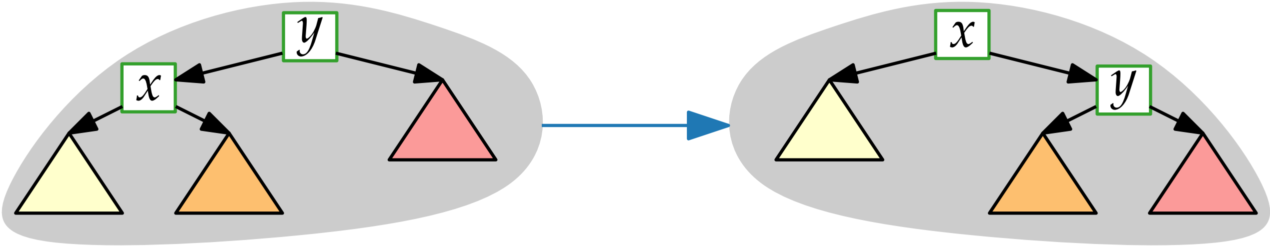

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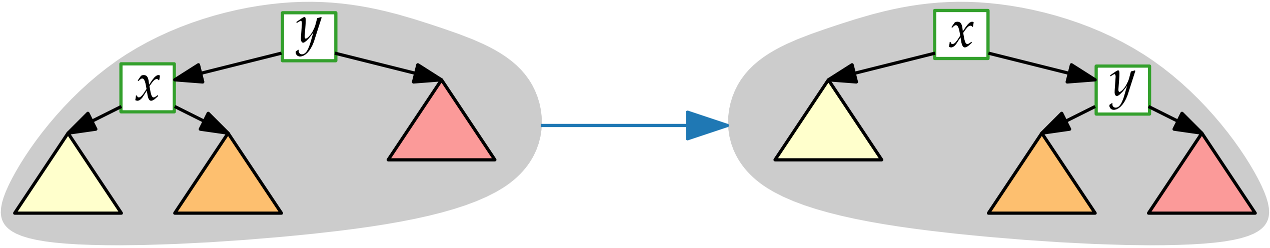

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$(s_+(y) \leq s(y))$

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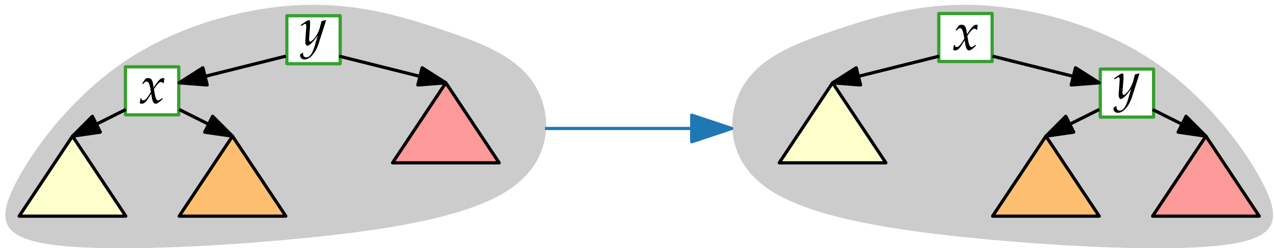

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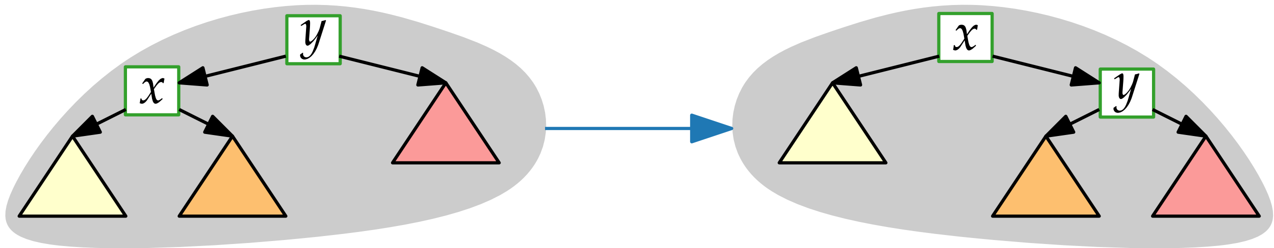

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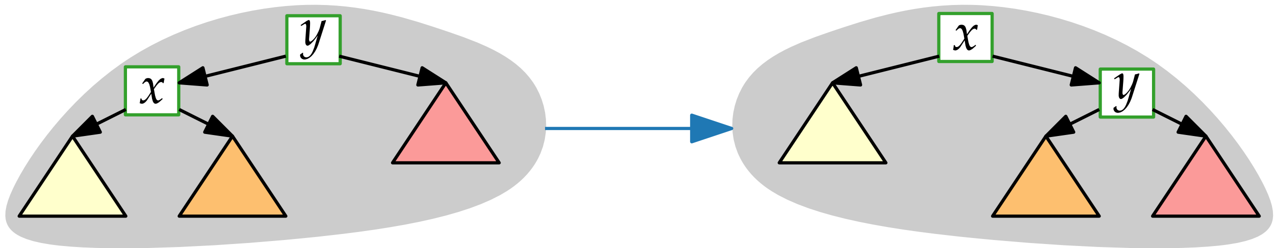

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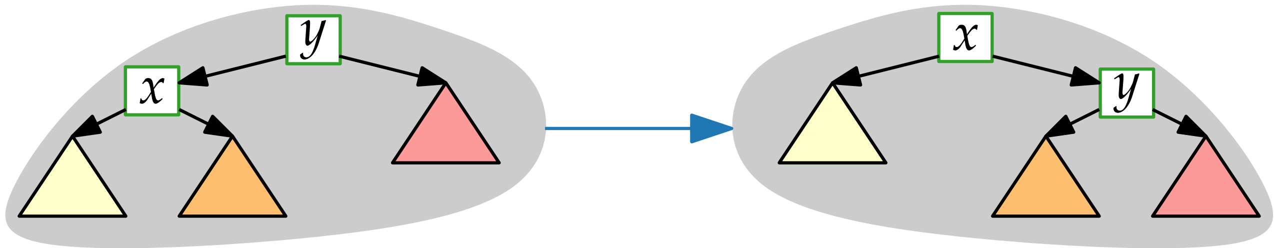

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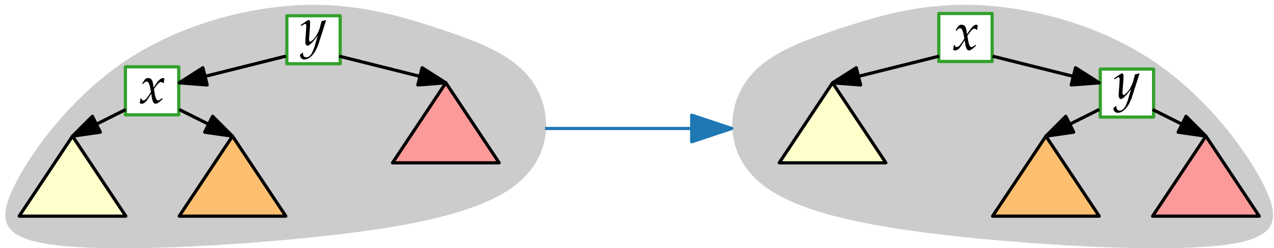

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\end{aligned}$$

Left(x) analogue ✓

Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\left(\log s_+(x) - \log s(x)\right) - 2$.

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Case 1. Right-Right(x)

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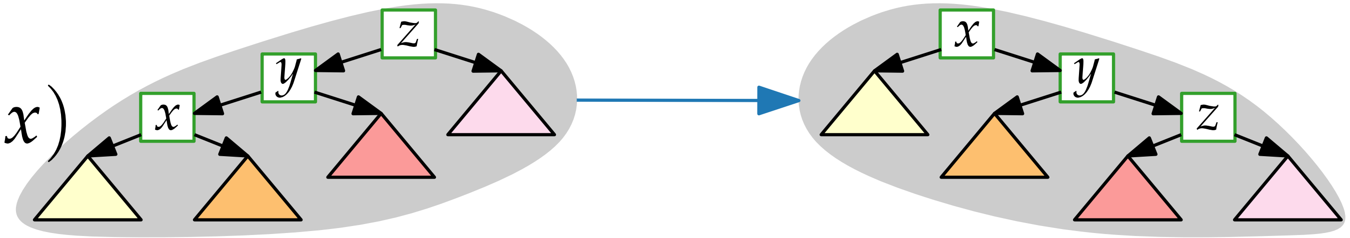

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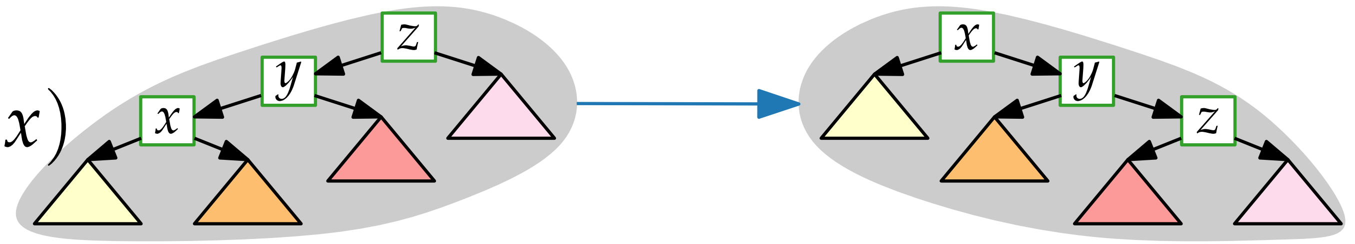

pot. change $\displaystyle = \log s_+(x) + \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) - \log s(z)$

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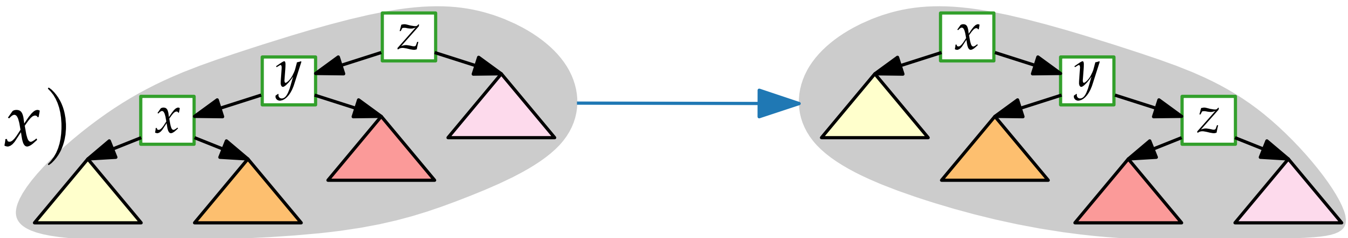

$$\text{pot. change} = \log s_+(x) + \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) - \log s(z)$$

$(s_+(x) = s(z))$

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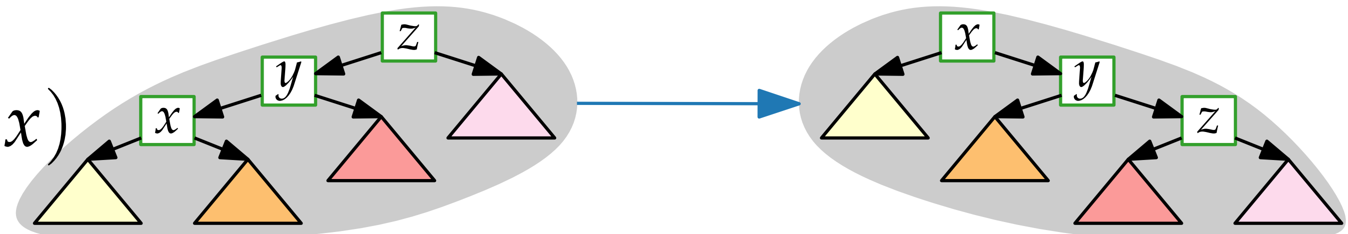

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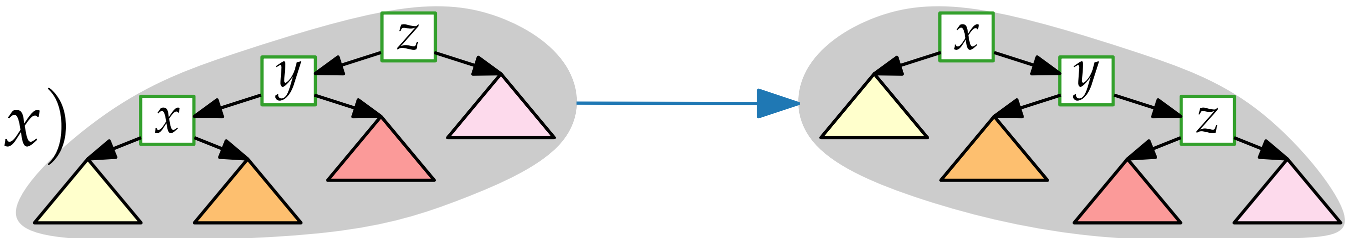

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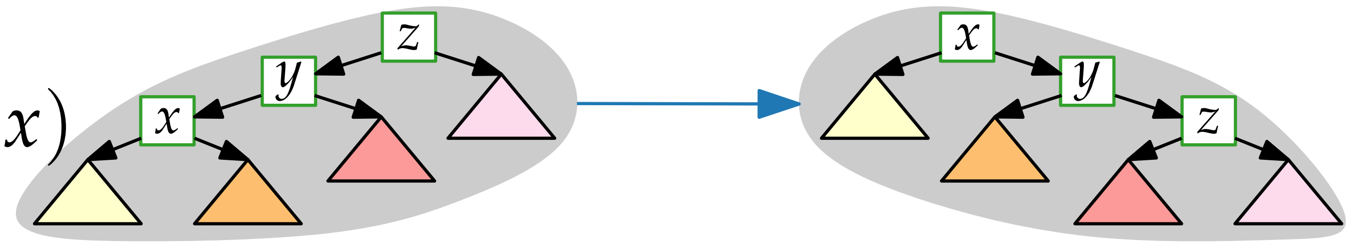

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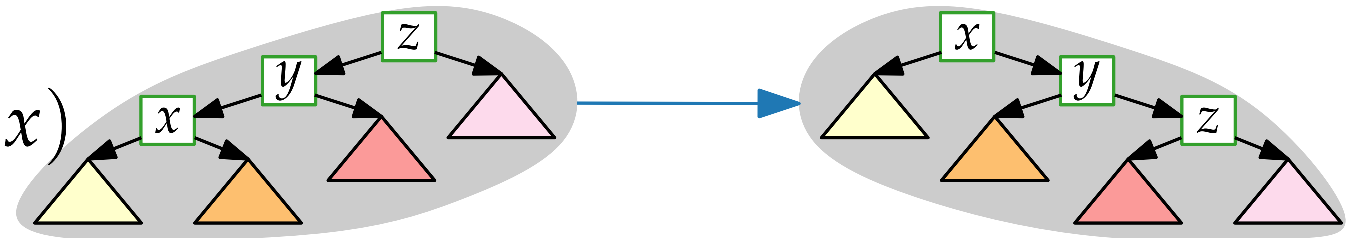

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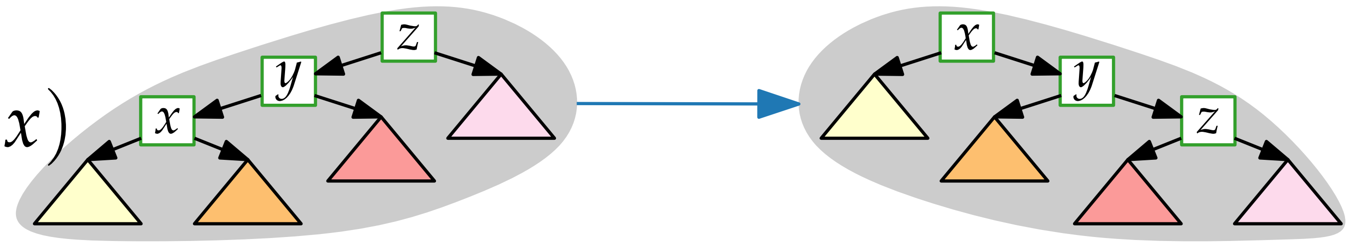

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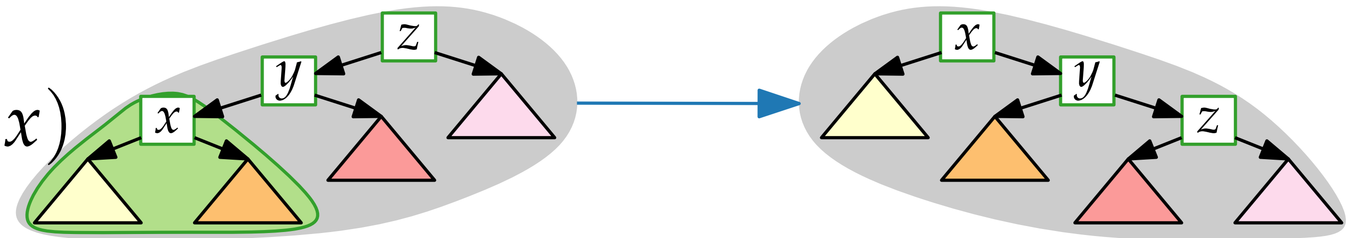

$$
\begin{aligned}
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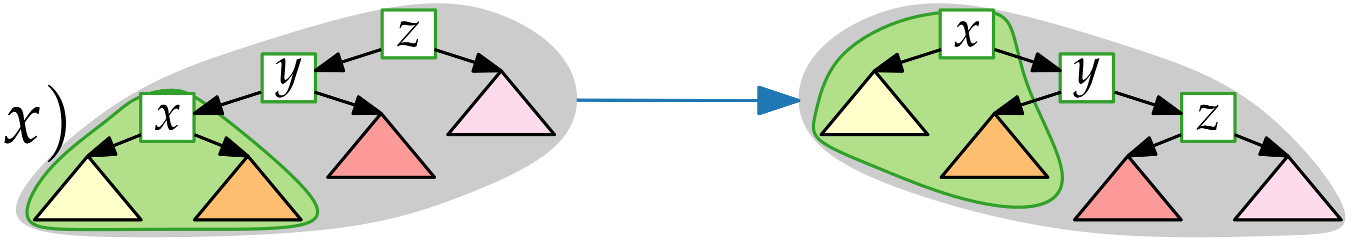

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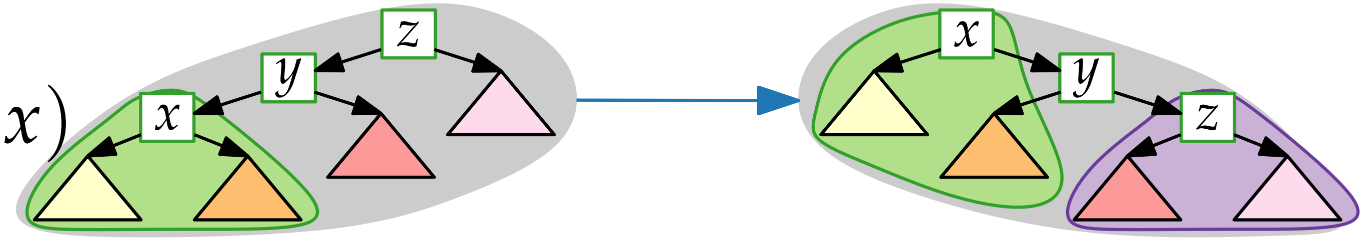
Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Proof.

Case 1. Right-Right(x)



$$\begin{aligned}
\text{pot. change} \quad &= \log s_+(x) + \log s_+(y) + \log s_+(z) \\
&\quad - \log s(x) - \log s(y) - \log s(z) \\
(s_+(x) = s(z)) \quad &= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) \\
(s(x) \leq s(y)) \quad &\leq \log s_+(y) + \log s_+(z) - 2\log s(x) \\
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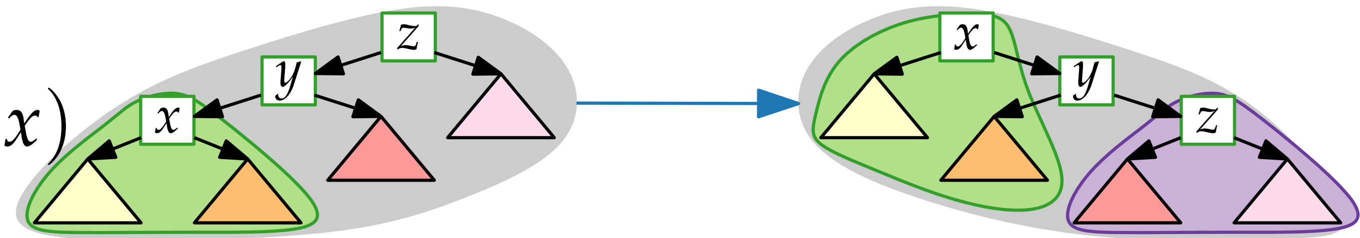

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(s_+(y) \leq s_+(x)) \quad &\leq \log s_+(x) + \log s_+(z) - 2\log s(x)
\end{aligned}$$

$s(x) + s_+(z) \leq s_+(x)$

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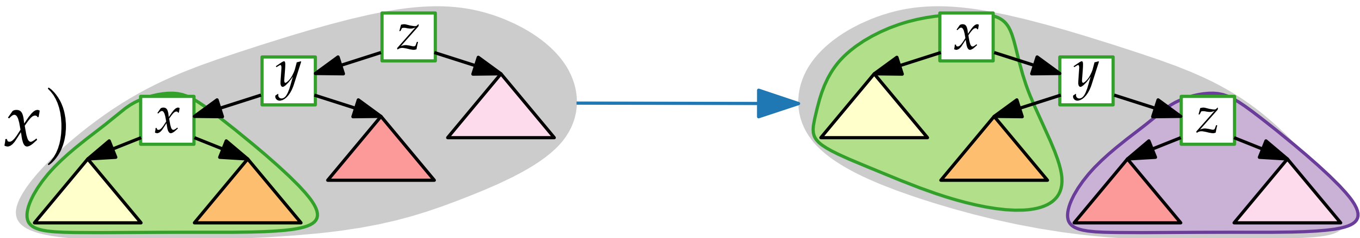

$$
\begin{aligned}
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(s(x) \leq s(y)) \quad & \leq \log s_+(y) + \log s_+(z) - 2\log s(x) \\
(s_+(y) \leq s_+(x)) \quad & \leq \log s_+(x) + \log s_+(z) - 2\log s(x)
\end{aligned}
$$

$s(x) + s_+(z) \leq s_+(x) \Rightarrow \log s(x) + \log s_+(z)$

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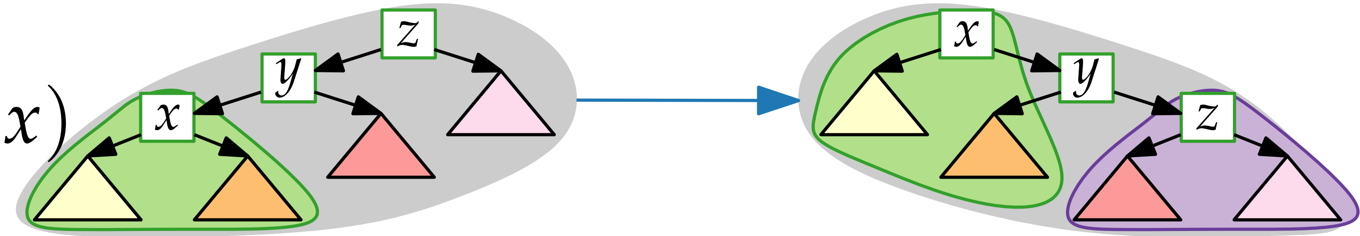

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\end{aligned}$$

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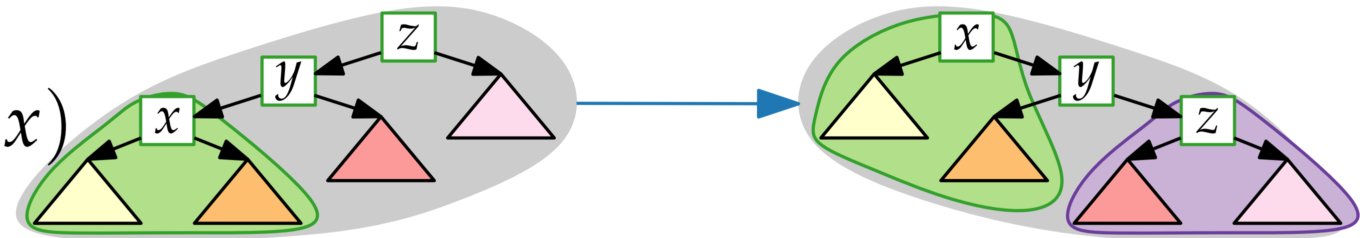

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\end{aligned}$$

$$s(x) + s_+(z) \leq s_+(x) \Rightarrow \log s(x) + \log s_+(z) = \log s(x)s_+(z) \underset{\text{(AM-GM)}}{\leq} \log\left(s_+(x)/2\right)^2$$

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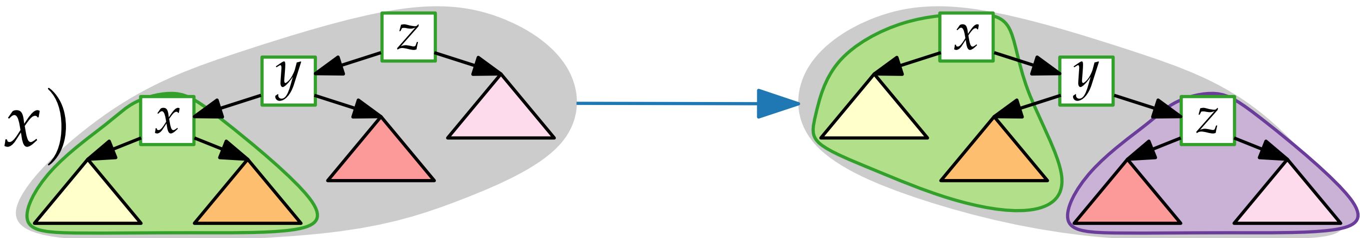

$$
\begin{aligned}
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(s(x) \leq s(y)) \quad & \leq \log s_+(y) + \log s_+(z) - 2\log s(x) \\
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Proof.
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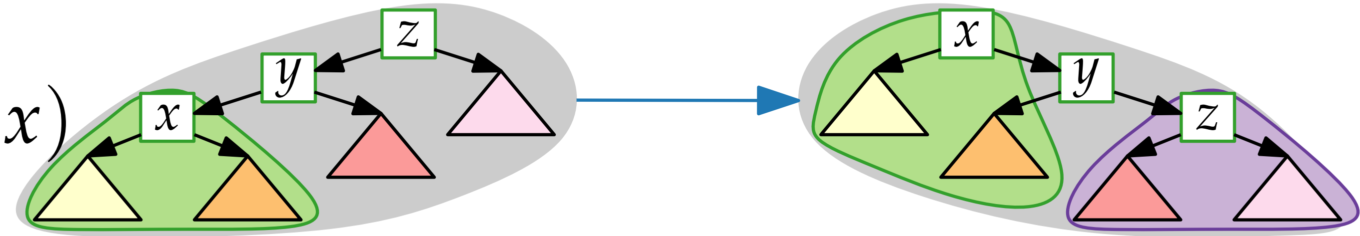

$$
\begin{aligned}
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& \quad - \log s(x) - \log s(y) - \log s(z) \\
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(s(x) \leq s(y)) \quad & \leq \log s_+(y) + \log s_+(z) - 2\log s(x) \\
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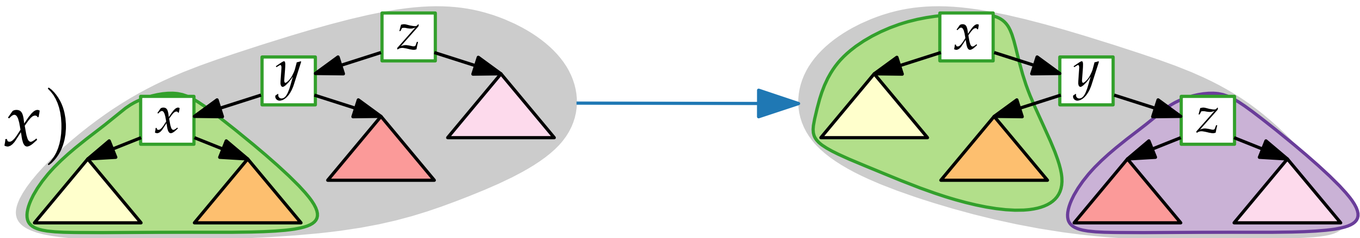

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\begin{aligned}
\text{pot. change} \quad & = \log s_+(x) + \log s_+(y) + \log s_+(z) \\
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(s(x) \leq s(y)) \quad & \leq \log s_+(y) + \log s_+(z) - 2\log s(x) \\
(s_+(y) \leq s_+(x)) \quad & \leq \log s_+(x) + \log s_+(z) - 2\log s(x) \\
& \leq 3\log s_+(x) - 3\log s(x) - 2
\end{aligned}
$$

$$
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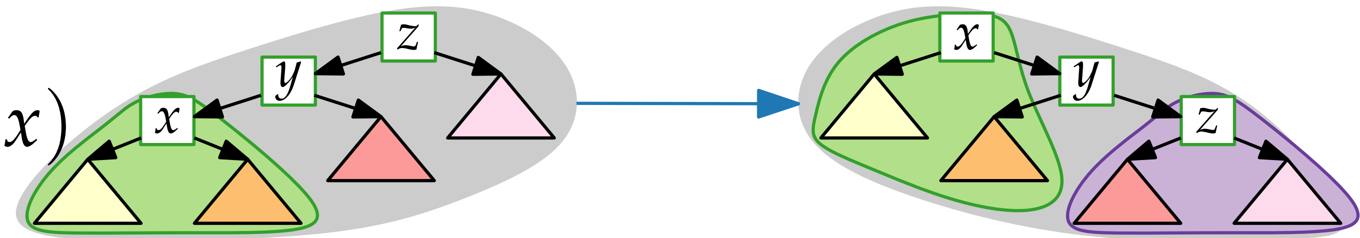

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(s_+(y) \leq s_+(x)) \quad & \leq \log s_+(x) + \log s_+(z) - 2 \log s(x) \\
& \leq 3 \log s_+(x) - 3 \log s(x) - 2 \quad \checkmark
\end{aligned}$$

$$s(x) + s_+(z) \leq s_+(x) \Rightarrow \log s(x) + \log s_+(z) = \log s(x) s_+(z)$$
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Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

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Proof. / Left-Left(x)

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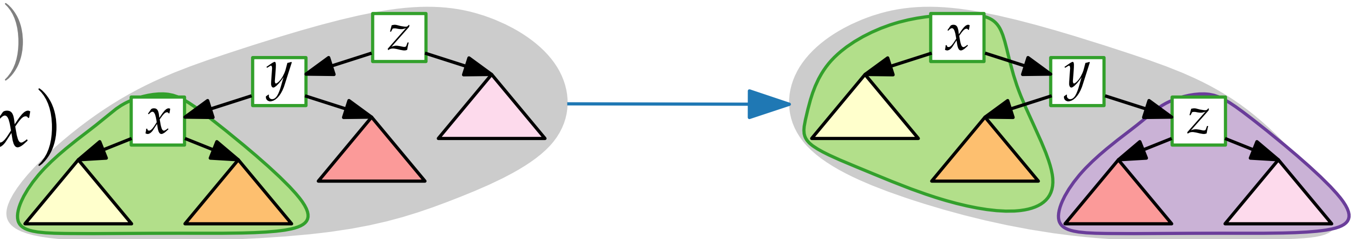

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(s_+(y) \leq s_+(x)) \quad &\leq \log s_+(x) + \log s_+(z) - 2 \log s(x) \\
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\end{aligned}
$$

$$
\begin{aligned}
s(x) + s_+(z) \leq s_+(x) \Rightarrow \log s(x) + \log s_+(z) &= \log s(x) s_+(z) \\
&\underset{\text{(AM-GM)}}{\leq} \log\,(s_+(x)/2)^2 \leq 2 \log s_+(x) - 2
\end{aligned}
$$

Potential after Rotation

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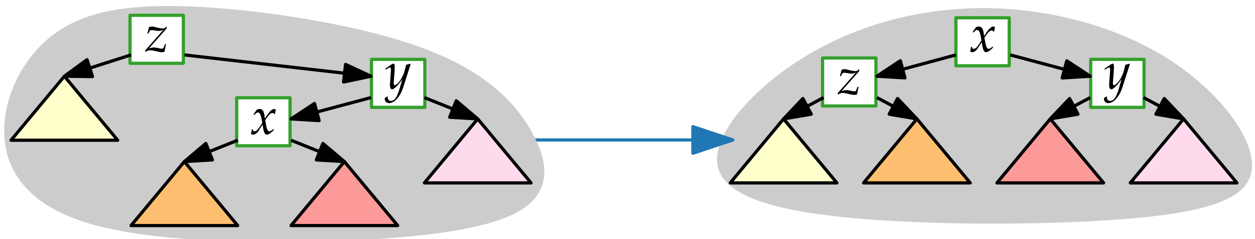
Proof.

Case 2. Right-Left(x)



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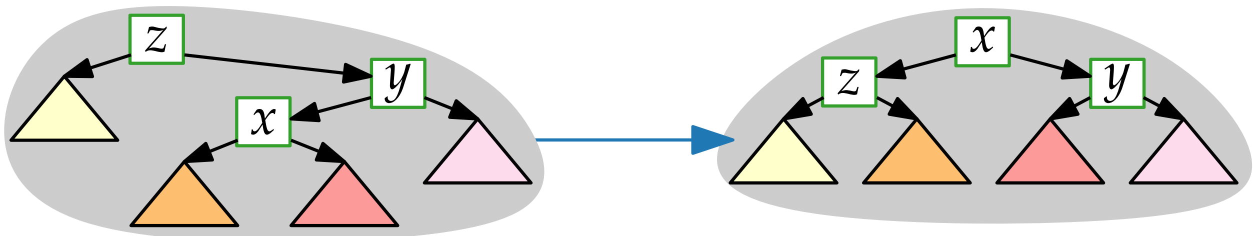

$$\text{pot. change} = \log s_+(x) + \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) - \log s(z)$$

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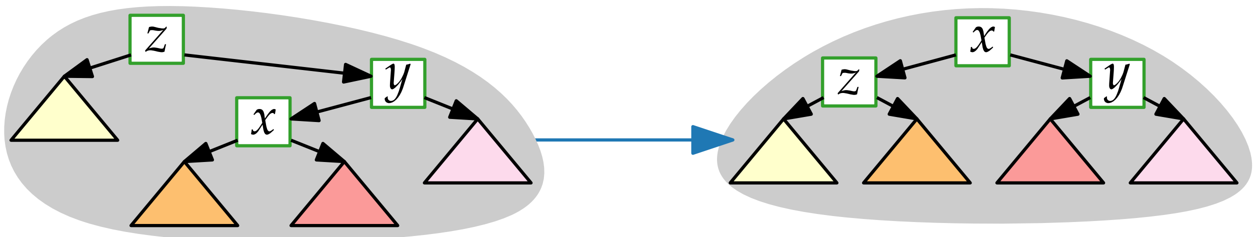

$$\begin{aligned}
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&\quad - \log s(x) - \log s(y) - \log s(z) \\
(s_+(x) = s(z)) \quad &= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y)
\end{aligned}$$

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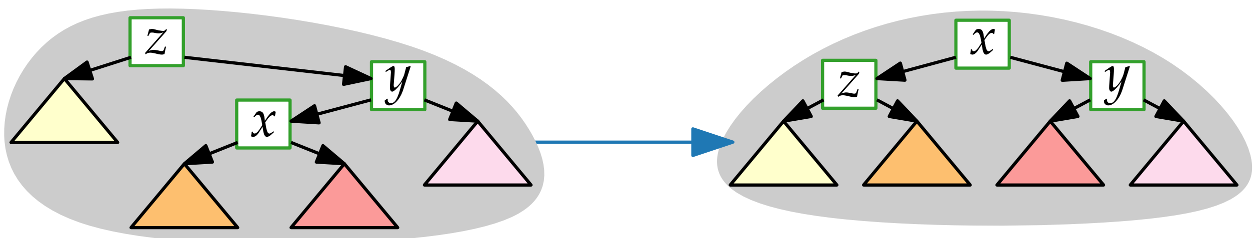

$$
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& \quad - \log s(x) - \log s(y) - \log s(z) \\
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(s(x) \leq s(y)) \quad & \leq \log s_+(y) + \log s_+(z) - 2\log s(x)
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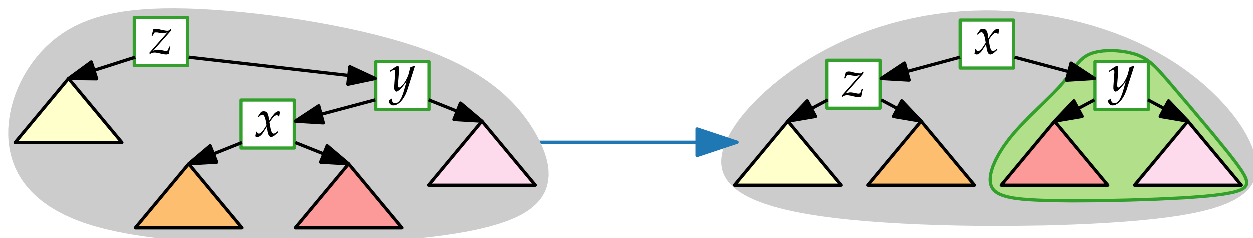

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\end{aligned}$$

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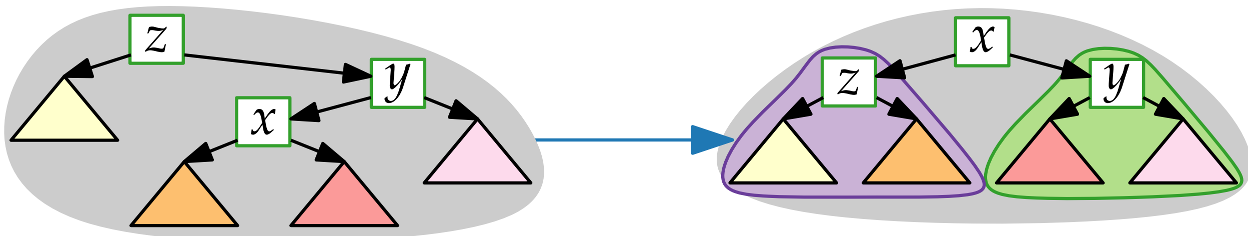

pot. change $= \log s_+(x) + \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) - \log s(z)$

$(s_+(x) = s(z))$ $= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y)$

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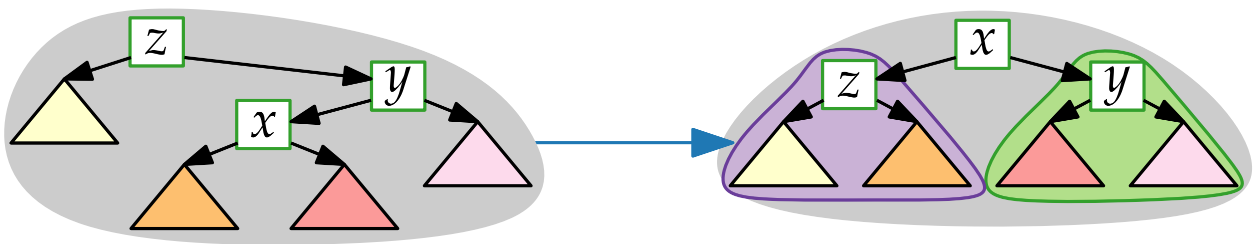

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\end{aligned}$$

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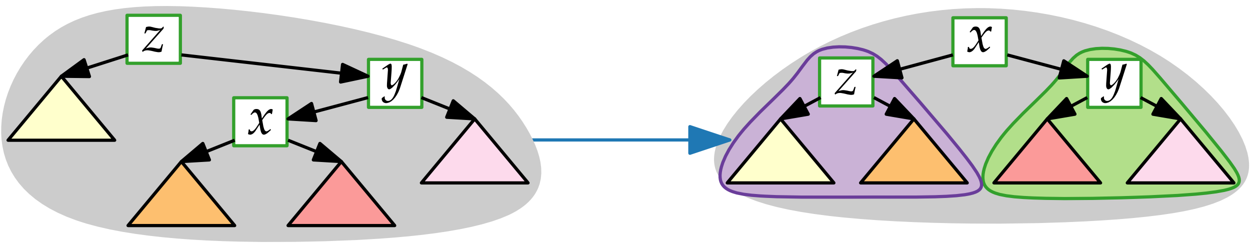

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\end{aligned}$$

$s_+(y) + s_+(z) \leq s_+(x) \quad \Rightarrow \log s_+(y) + \log s_+(z) \leq 2\log s_+(x) - 2$

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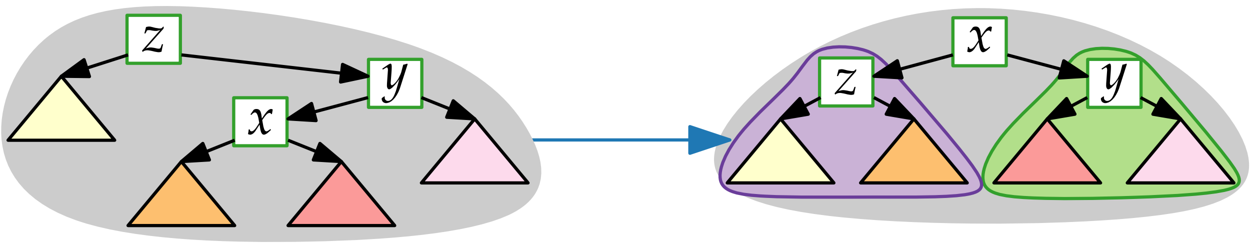

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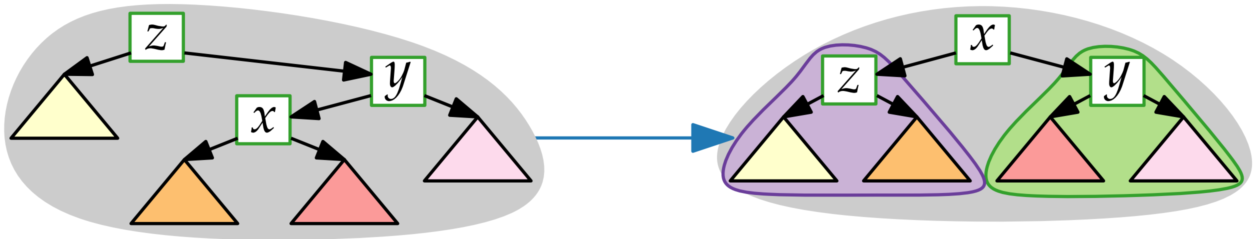

$$\begin{aligned}
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&\quad - \log s(x) - \log s(y) - \log s(z) \\
(s_+(x) = s(z)) \quad &= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) \\
(s(x) \leq s(y)) \quad &\leq \log s_+(y) + \log s_+(z) - 2\log s(x) \\
&\leq 2\log s_+(x) - 2\log s(x) - 2
\end{aligned}$$

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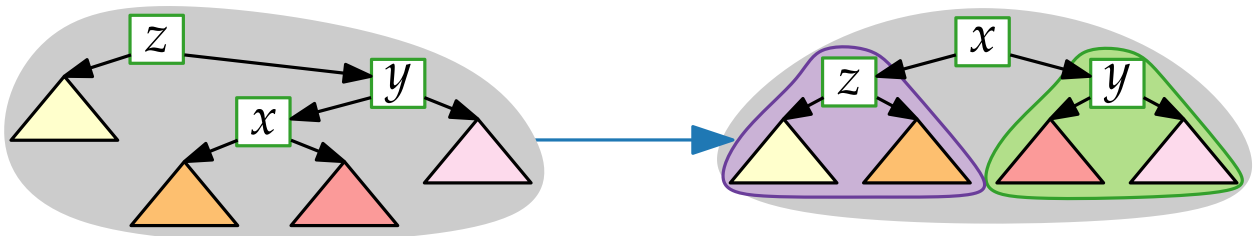
Case 2. Right-Left(x)



pot. change $= \log s_+(x) + \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) - \log s(z)$

$(s_+(x) = s(z))$ $\quad = \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y)$

$(s(x) \leq s(y))$ $\quad \leq \log s_+(y) + \log s_+(z) - 2\log s(x)$

$\leq 2\log s_+(x) - 2\log s(x) - 2$

$(s_+(x) > s(x))$

$s_+(y) + s_+(z) \leq s_+(x) \quad \Rightarrow \log s_+(y) + \log s_+(z) \leq 2\log s_+(x) - 2$

Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3(\log s_+(x) - \log s(x)) - 2$.

Proof.

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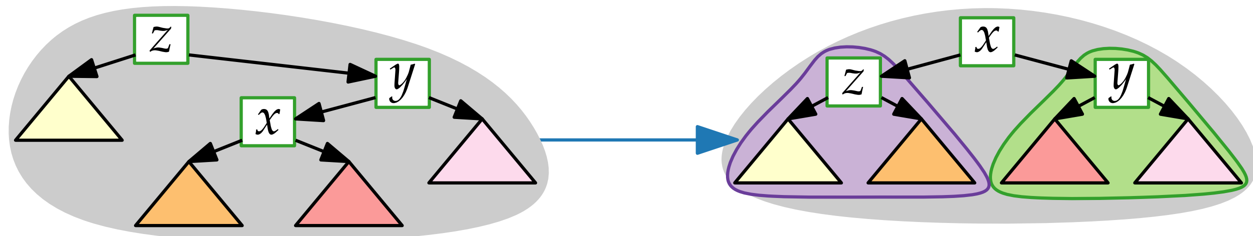

$$\begin{aligned}
\text{pot. change} \quad &= \log s_+(x) + \log s_+(y) + \log s_+(z) \\
&\quad - \log s(x) - \log s(y) - \log s(z) \\
(s_+(x) = s(z)) \quad &= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) \\
(s(x) \leq s(y)) \quad &\leq \log s_+(y) + \log s_+(z) - 2\log s(x) \\
&\leq 2\log s_+(x) - 2\log s(x) - 2 \\
(s_+(x) > s(x)) \quad &\leq 3\log s_+(x) - 3\log s(x) - 2
\end{aligned}$$

$$s_+(y) + s_+(z) \leq s_+(x) \;\Rightarrow\; \log s_+(y) + \log s_+(z) \leq 2\log s_+(x) - 2$$

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Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

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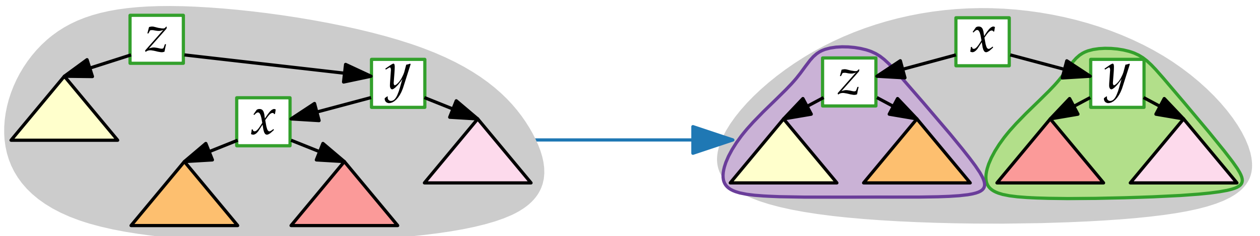

$$
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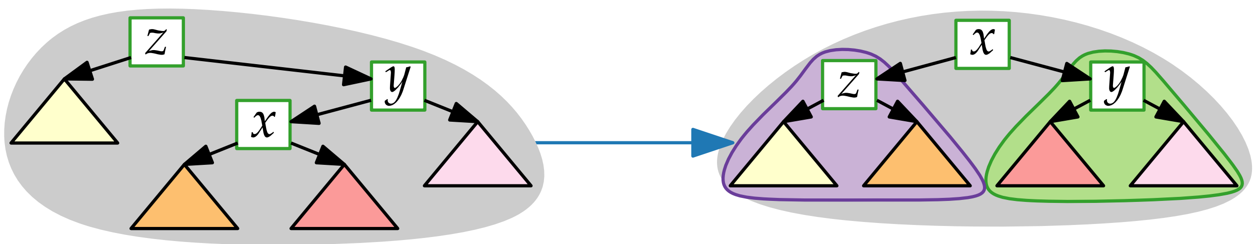
Lemma. After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Proof. / Left-Right(x)

Case 2. Right-Left(x)



$$
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Access Lemma

Lemma. After a single rotation, the potential increases by $\leq 3\left(\log s_{+}(x) - \log s(x)\right)$.
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Lemma. The (amortized) cost of Splay(x) is $\leq 1 + 3\log(W/w(x))$

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Let $s_i(x)$ be $s(x)$ after i single/double rotations.
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$\sum_{i=1}^{k} (3\,(\log s_i(x) - \log s_{i-1}(x)) - 2)$

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$$+3\left(\log s_{k+1}(x) - \log s_k(x)\right)$$
$$= 3\left(\log s_{k+1}(x) - \log s(x)\right) - 2k$$

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root!

$$= 3\,(\log s_{k+1}(x) - \log s(x)) - 2k$$

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$(s(x) \leq w(x))$

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$$(s(x) \leq w(x)) \quad \leq 3\,(\log W - \log w(x)) - 2k$$

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$$
\begin{aligned}
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\text{root!} \quad & = 3\left(\log s_{k+1}(x) - \log s(x)\right) - 2k \\
& = 3\left(\log W - \log s(x)\right) - 2k \\
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$2k+1$ rotations $\Rightarrow$ (amort.) cost

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$= 3\,(\log s_{k+1}(x) - \log s(x)) - 2k$
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$2k+1$ rotations $\Rightarrow$ (amort.) cost $\leq 1 + 3\log(W/w(x))$ $\square$

Balance

Lemma. The (amortized) cost of Splay(x) is $\leq 1 + 3\log(W/w(x))$

Definition. A BST is **balanced** if the (amortized) cost of *any* query is $O(\log n)$.

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Splay(x) costs at least as much as finding x

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Entropy

Lemma. The (amortized) cost of Splay(x) is $\leq 1 + 3\log(W/w(x))$

Definition. A BST has the **entropy property** if queries take expected $O(1 - \sum_{i=1}^{n} p_i \log p_i)$ time.

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Time to query x_i:

$$\leq 1 + 3\log(W/w(x_i))$$
$$= 1 + 3\log(1/p_i)$$
$$= 1 - 3\log p_i$$

$\Rightarrow$ expected query time:

$$O(\textstyle\sum_{i=1}^{n} p_i(1 - 3\log p_i))$$
$$= O(1 + \textstyle\sum_{i=1}^{n} -p_i \log p_i)$$

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Lemma. The (amortized) cost of Splay(x) is $\leq 1 + 3\log(W/w(x))$

Definition. A BST has the **entropy property** if queries take expected $O(1 - \sum_{i=1}^{n} p_i \log p_i)$ time.

Theorem. Splay Trees have the entropy property.

Proof. Choose $w(x_i) = p_i \quad \Rightarrow W = 1$

Time to query x_i:

$$\begin{aligned} &\leq 1 + 3\log(W/w(x_i)) \\ &= 1 + 3\log(1/p_i) \\ &= 1 - 3\log p_i \end{aligned}$$

$\Rightarrow$ expected query time:

$$\begin{aligned} &O(\textstyle\sum_{i=1}^{n} p_i(1 - 3\log p_i)) \\ &= O(1 + \textstyle\sum_{i=1}^{n} -p_i \log p_i) \end{aligned}$$

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$\Rightarrow$ as long as every key is queried at least once, it doesn't change the running time.

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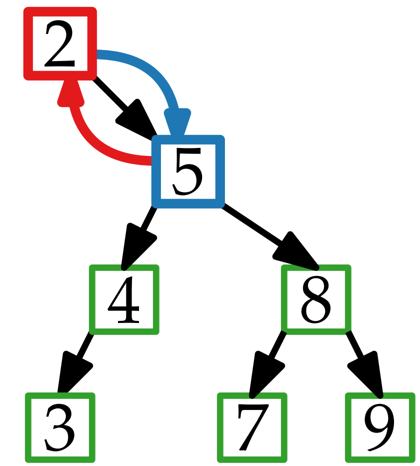

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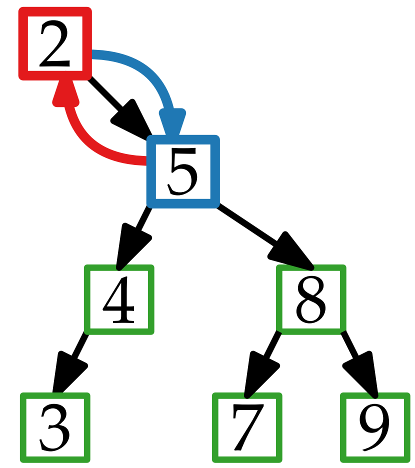

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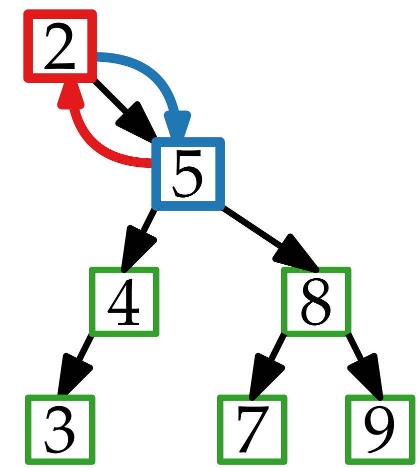

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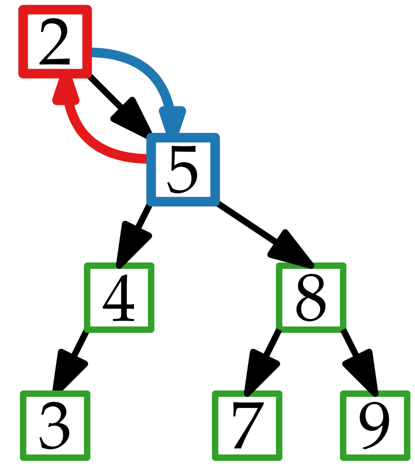

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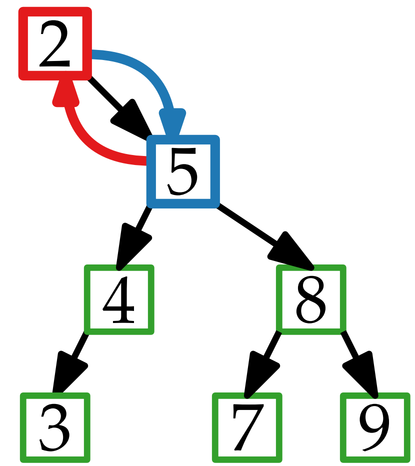

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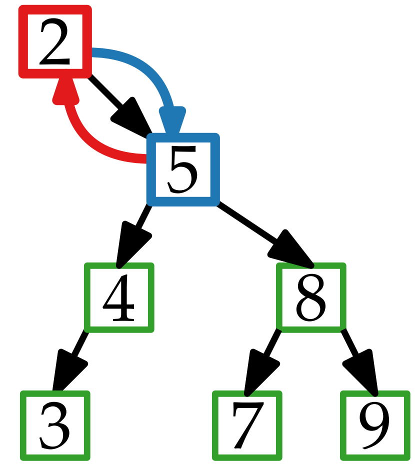

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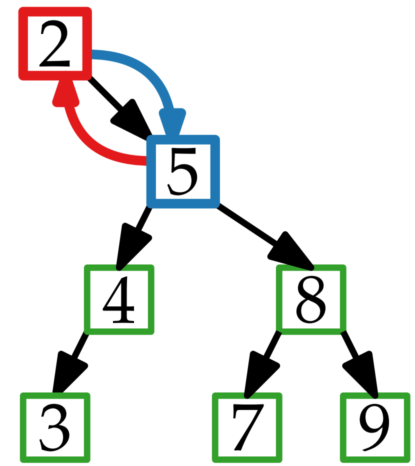

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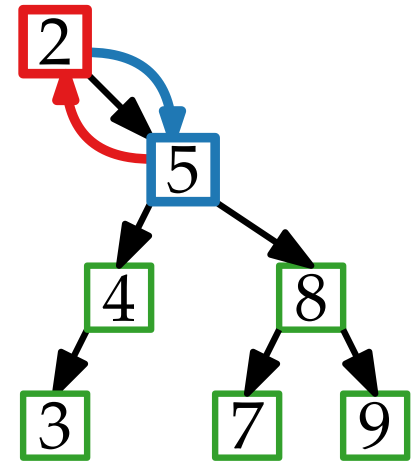

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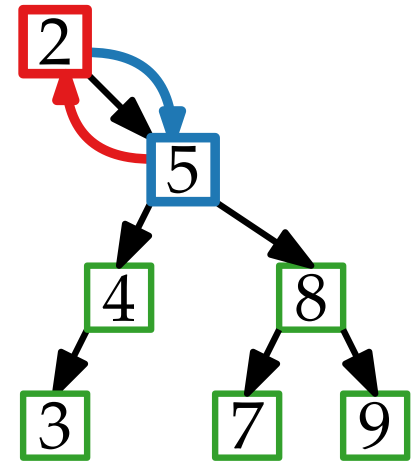

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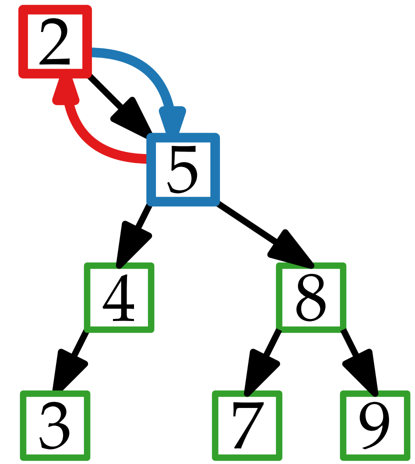

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Conjecture. Splay Trees are dynamically optimal.