Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Approximationsalgorithmen

k-Center via Parametric Pruning

6. Vorlesung

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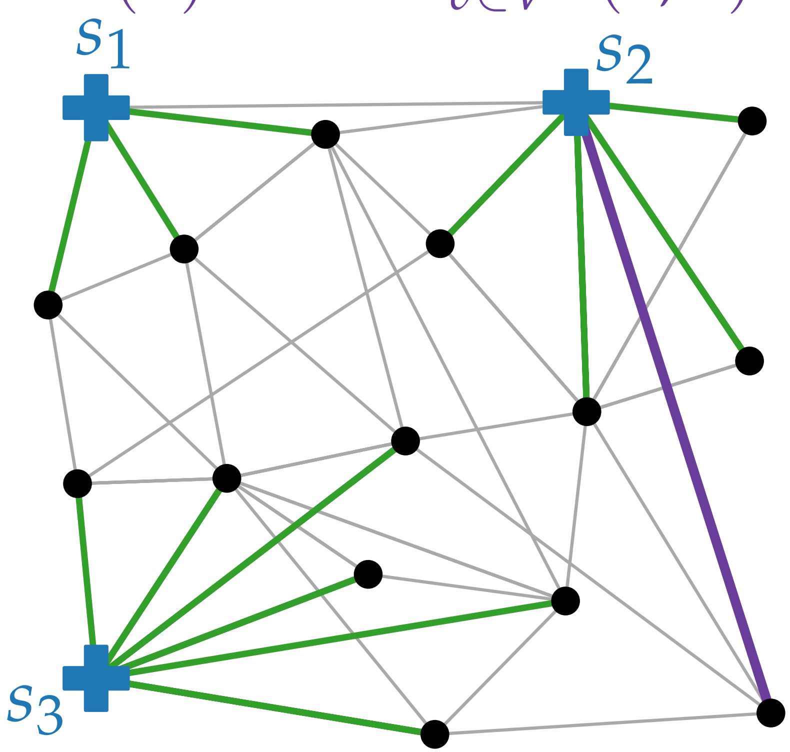
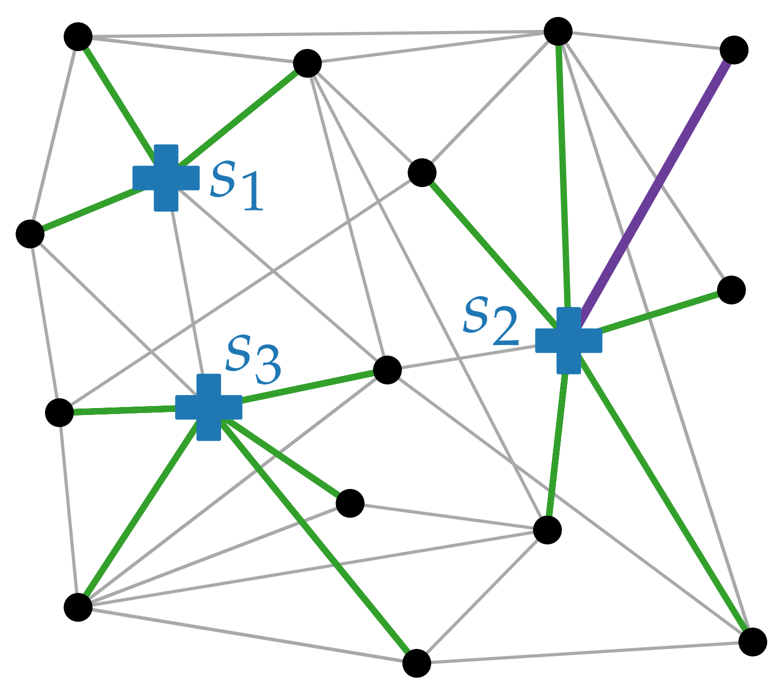
The metric k-Center-Problem

Given: A complete graph $G = (V, E)$ with edge costs $c \colon E \to \mathbb{Q}_{\geq 0}$ satisfying the triangle inequality and a natural number $k \leq |V|$.

For each vertex set $S \subseteq V$, $c(v, S)$ is the cost of the cheapest edge from v to the a vertex in S.

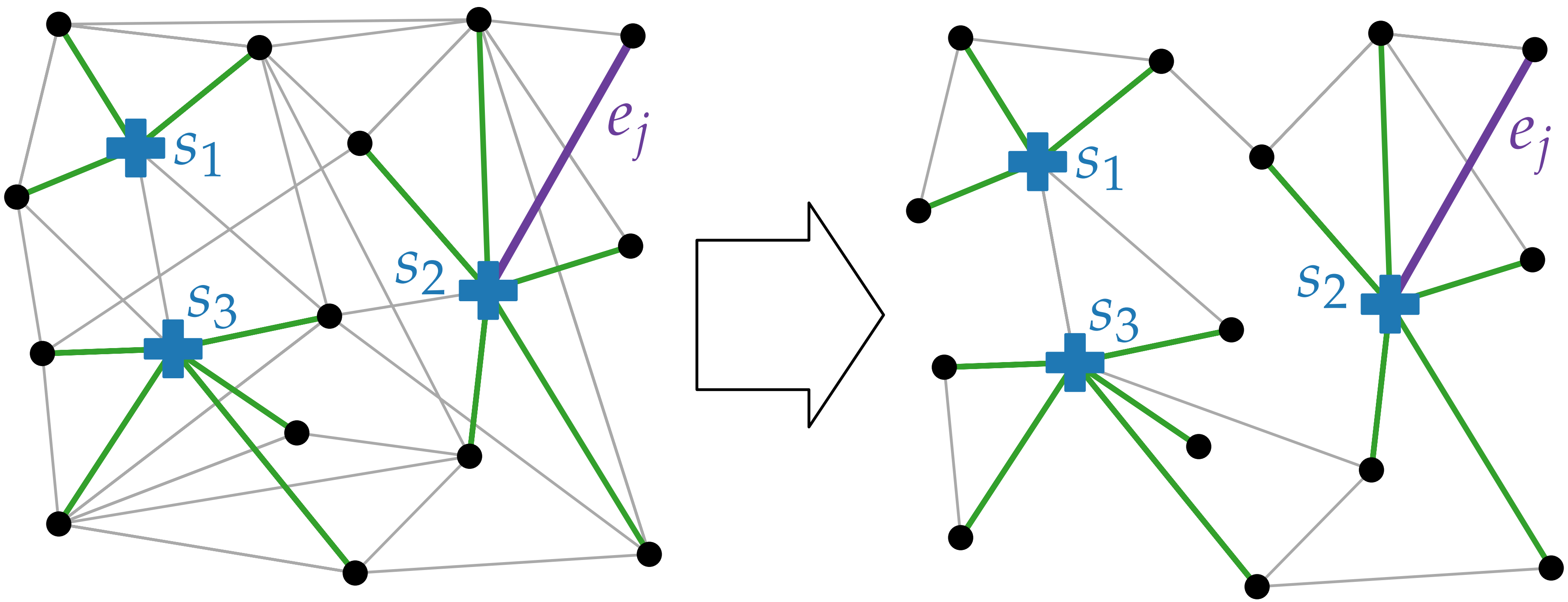
Find: A k-element vertex set S, such that $\mathrm{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.





Parametric Pruning

Let $E = \{e_1, \ldots, e_m\}$ with $c(e_1) \leq \ldots \leq c(e_m)$.
Suppose we know that $\mathrm{OPT} = c(e_j)$.

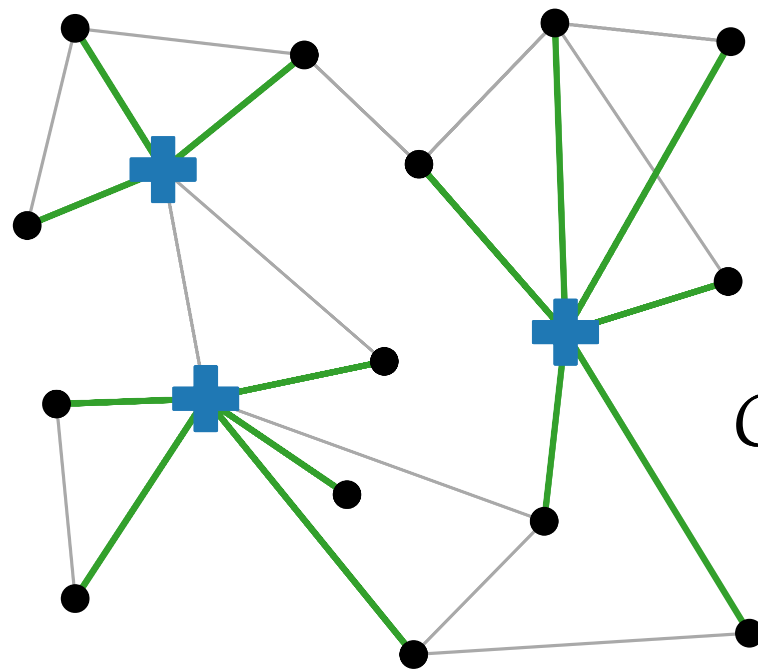


G $\quad$ $G_j := (V, \{e_1, \ldots, e_j\})$

…try each G_i.

… try each G_i.

Def. A vertex set D of a graph H is **dominated**, when each vertex is either in D or adjacent to a vertex in D. The cardinality of a smallest dominating set in H is denoted by $\mathrm{dom}(H)$.



$\mathrm{dom}(G_j) \leq k$

$G_j := (V, \{e_1, \ldots, e_j\})$

… but computing $\mathrm{dom}(H)$ is NP-hard.

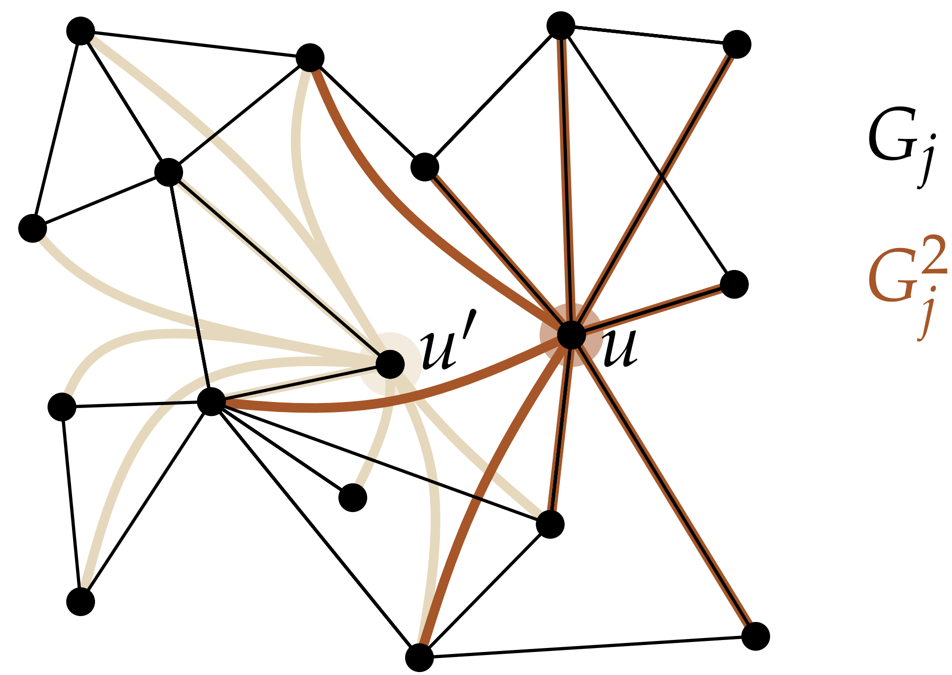
Square of a Graph

Idea: Find a small dominating set in a "coarsened" G_j

Def. The **square** H^2 of a graph H has the same vertex set as H. Additionally, two vertices $u \neq v$ are adjacent in H^2 when they are within distance **two** in H.

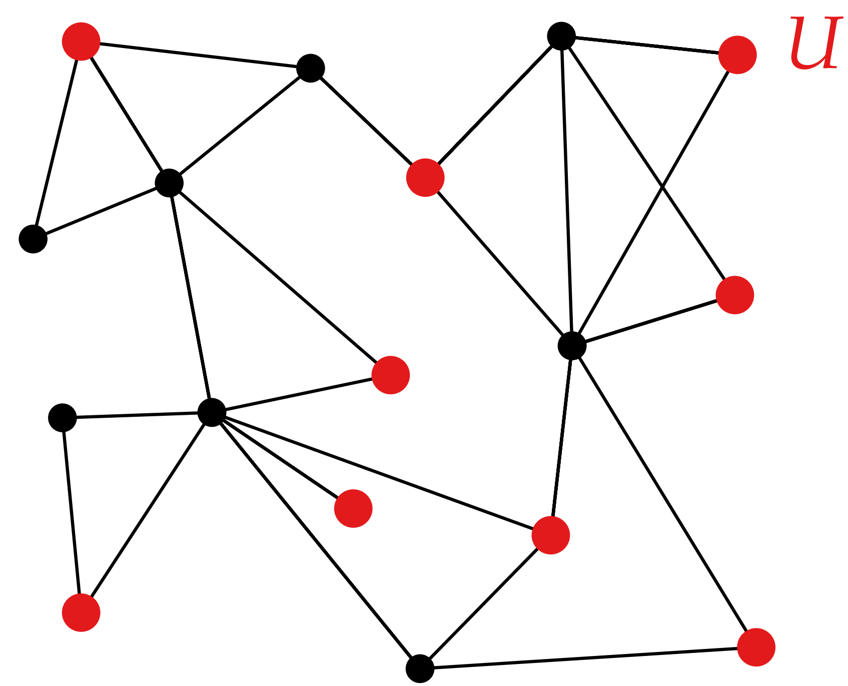
Obs. A dominating set in G_j^2 with $\leq k$ elements is already a 2-Approximation.

Why? $\max_{e \in E(G_j)} = e_j$!



Independent Sets

Def. A vertex set U in a graph is called **independent** (or **stable**), if no pair of vertices in U form an edge. An independent set is called **maximal** when no superset of it is an independent set.

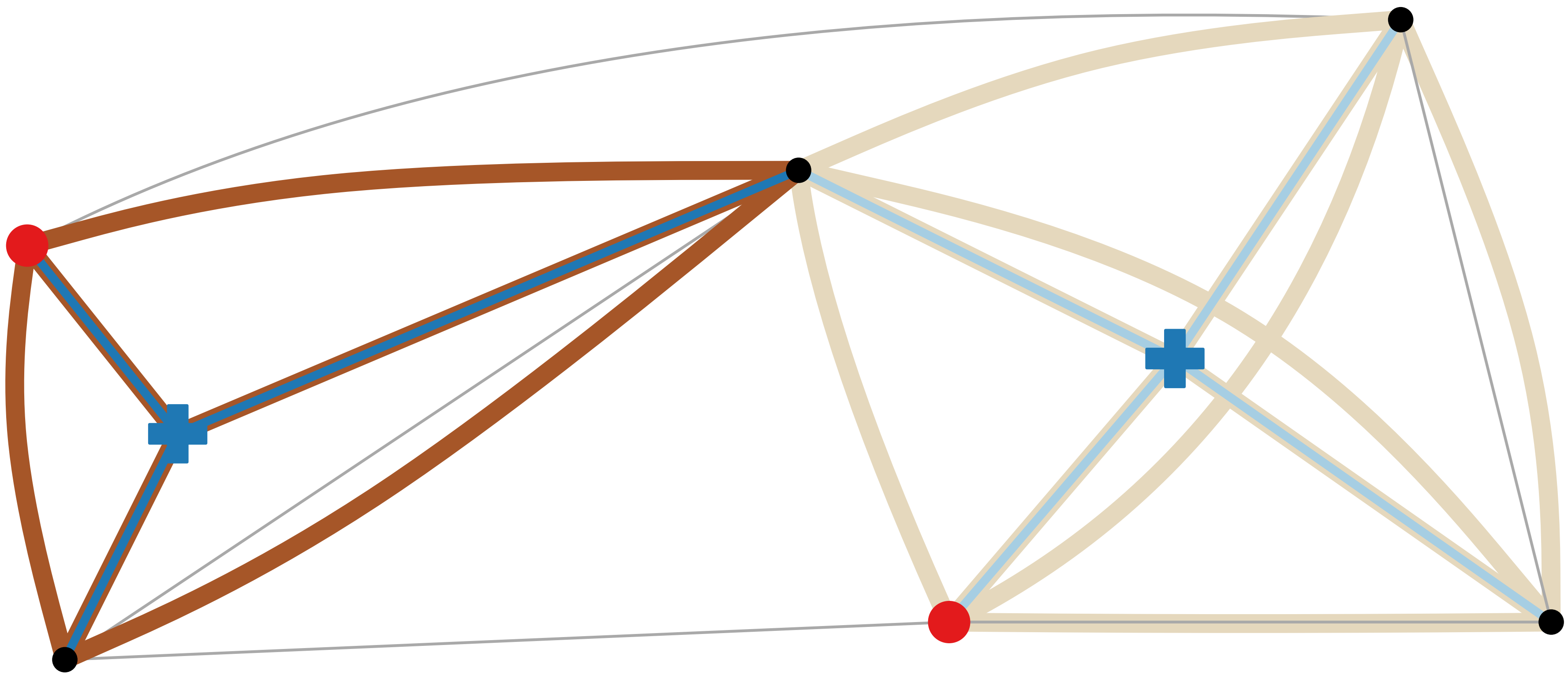


Obs. Maximal independent sets are dominating sets :-)

Independent Sets in H^2

Lemma. For a graph H and an independent set U in H^2, $|U| \leq \mathrm{dom}(H)$.

What does a dominating set of H look like in H^2?



Star

Clique

Factor-2 approx. for metric k-Center

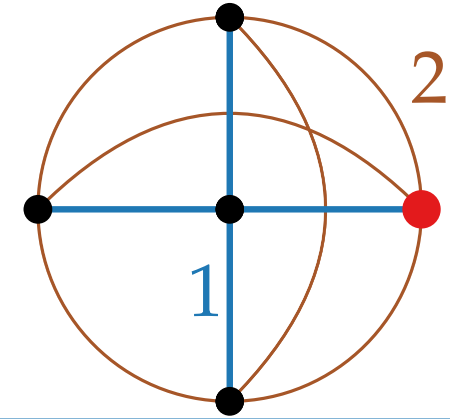
Algorithm Metric-k-Center

Sort the edges of G by cost: $c(e_1) \leq \ldots \leq c(e_m)$
for $j = 1, \ldots, m$ **do**
  Construct G_j^2
  Find a maximal independent set U_j in G_j^2
  if $|U_j| \leq k$ **then**
    return U_j

Lemma. For j provided by the Algorithm, we have $c(e_j) \leq \mathrm{OPT}$.

Theorem. The above algorithm is a factor-2 approximation algorithm for the metric k-Center problem.

Can we do better . . . ?



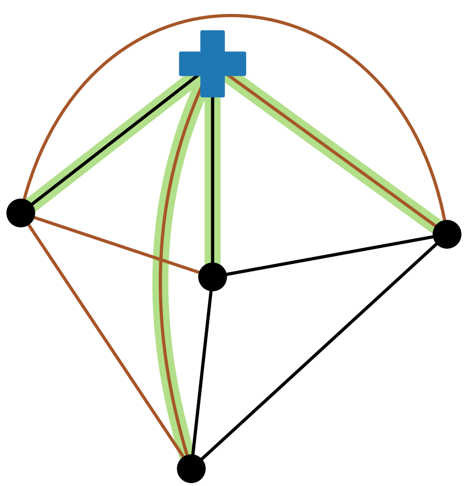
What about a tight example?

Theorem. Assuming P $\neq$ NP, there is no factor-$(2-\epsilon)$ approximation algorithm for the metric k-CENTER problem, for any $\epsilon > 0$.

Proof. Reduce from dominating set to metric k-CENTER.
Given.: $G = (V, E), k$
Constr. complete graph $G' = (V, E \cup E')$

with $c(e) = \begin{cases} 1, & \text{if } e \in E \\ 2, & \text{if } e \in E' \end{cases}$

$\triangle$-inequality holds



S: metric k-Center
If $\mathrm{dom}(G) \leq k$, then $\mathrm{cost}(S) = 1$
If $\mathrm{dom}(G) > k$, then $\mathrm{cost}(S) = 2$

Metric ~~k~~-Center problem

weighted

Given: A complete graph $G = (V, E)$ with metric edge costs $c \colon E \to \mathbb{Q}_{\geq 0}$ and ~~a natural number $k \leq |V|$.~~ , vertex weights $w \colon V \to \mathbb{Q}_{\geq 0}$ and a weight limit $W \in \mathbb{Q}_{+}$

For each vertex set $S \subseteq V$, $c(v, S)$ is the cost of the cheapest edge from v to the a vertex in S.

vertex set S of weight at most W

Find: A ~~k-element vertex set S~~, such that $\mathrm{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.

The weighted version

Algorithm metric-weighted-CENTER

Sort the edges of G by cost : $c(e_1) \leq \ldots \leq c(e_m)$

for $j = 1, \ldots, m$ **do**

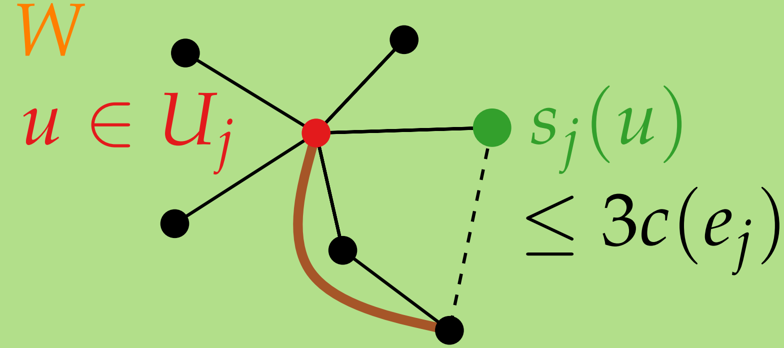
  Construct G_j^2

  Find a maximal independent set U_j in G_j^2

  Compute $S_j := \{ s_j(u) \mid u \in U_j \}$

  if $\cancel{|U_j| \leq k}$ $w(S_j) \leq W$ **then**

    return $\cancel{U_j}$ S_j



$s_j(u) :=$ lightest node in $N_{G_j}(u) \cup \{u\}$

Theorem. The above is a factor-3 approximation algorithm for the metric weighted-CENTER problem.