Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Approximationsalgorithmen

k-Center via Parametric Pruning

6. Vorlesung

Steven Chaplick

Wintersemester 2019/20

The metric k-Center-Problem

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satisfying the triangle inequality

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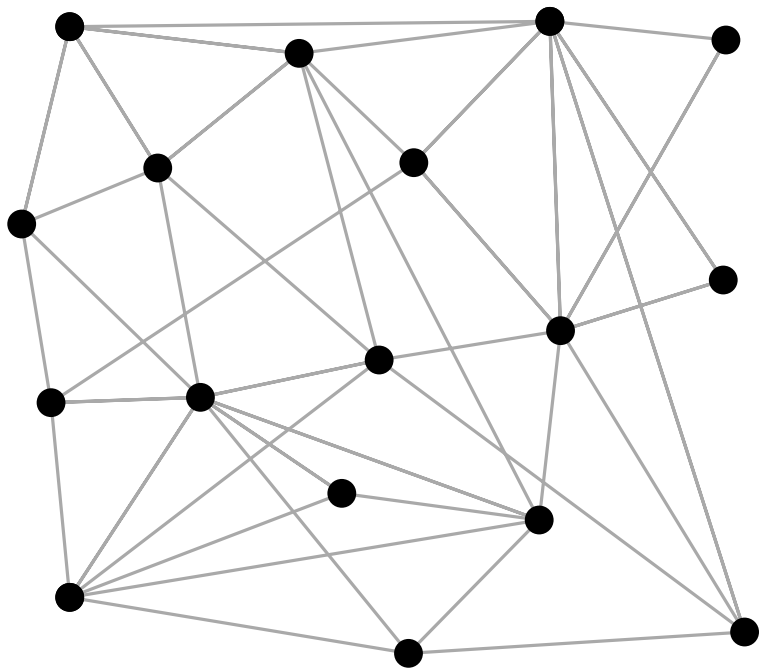

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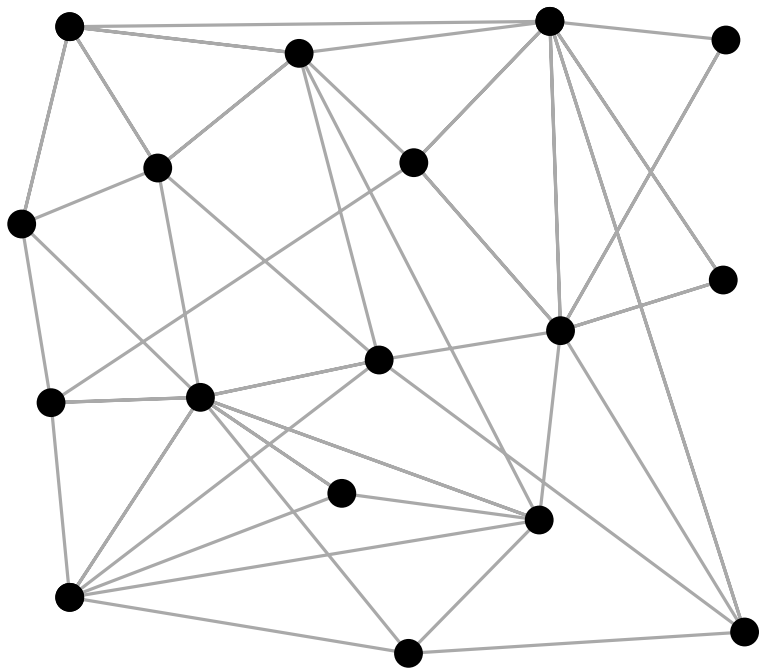

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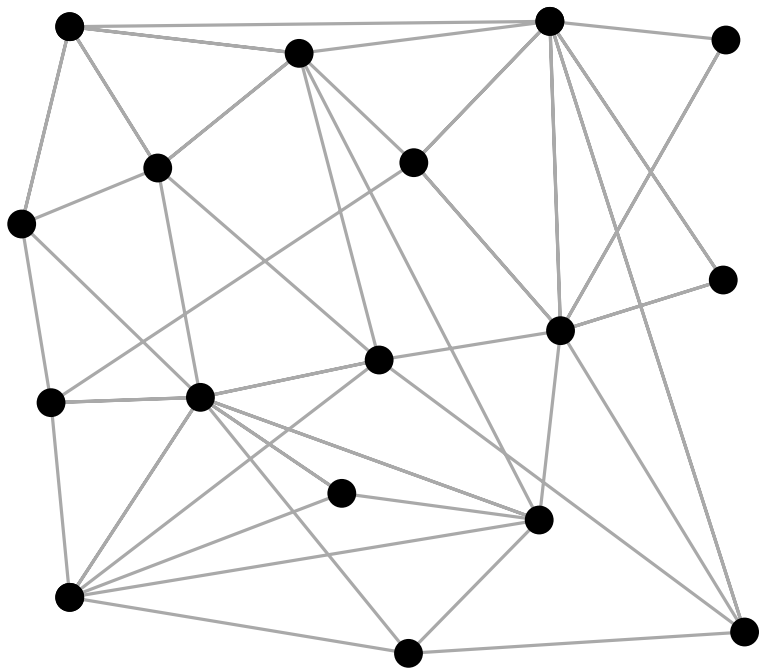

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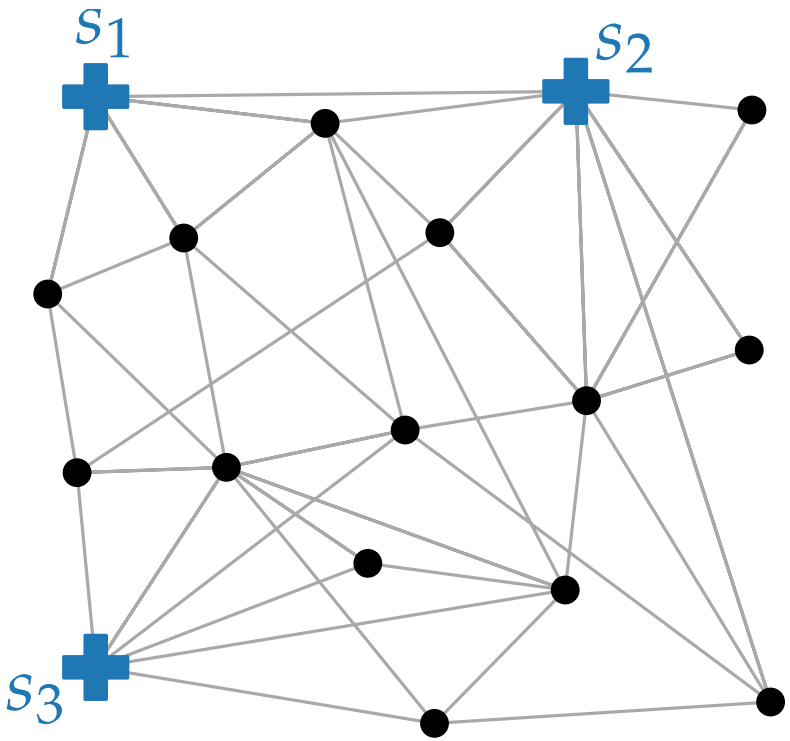

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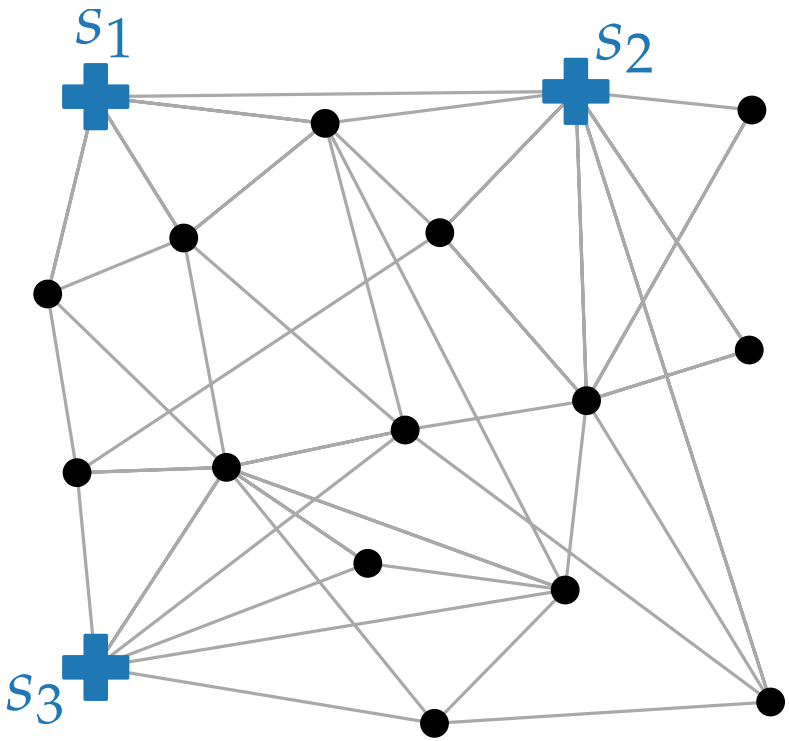

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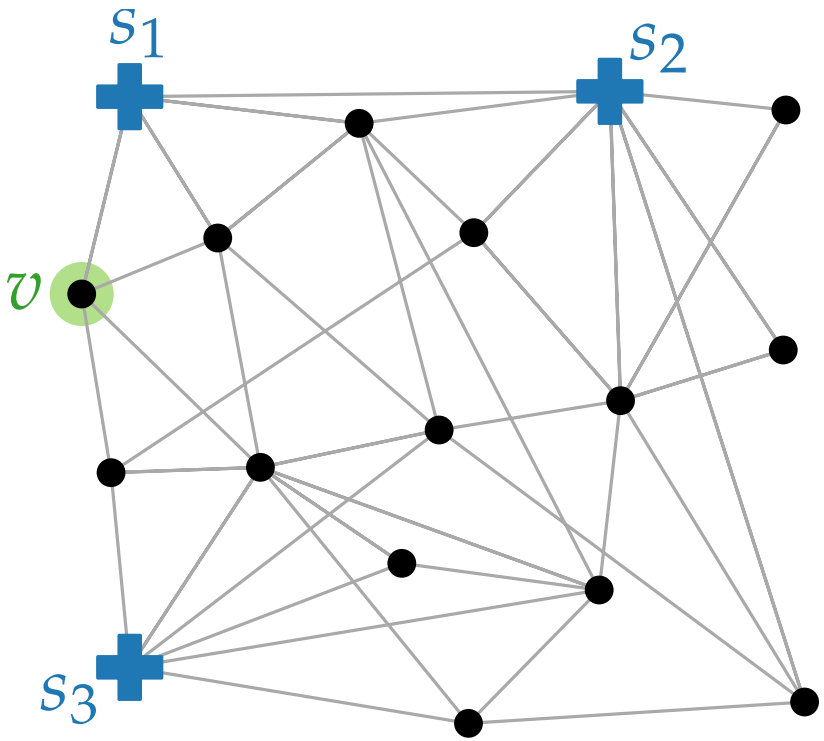

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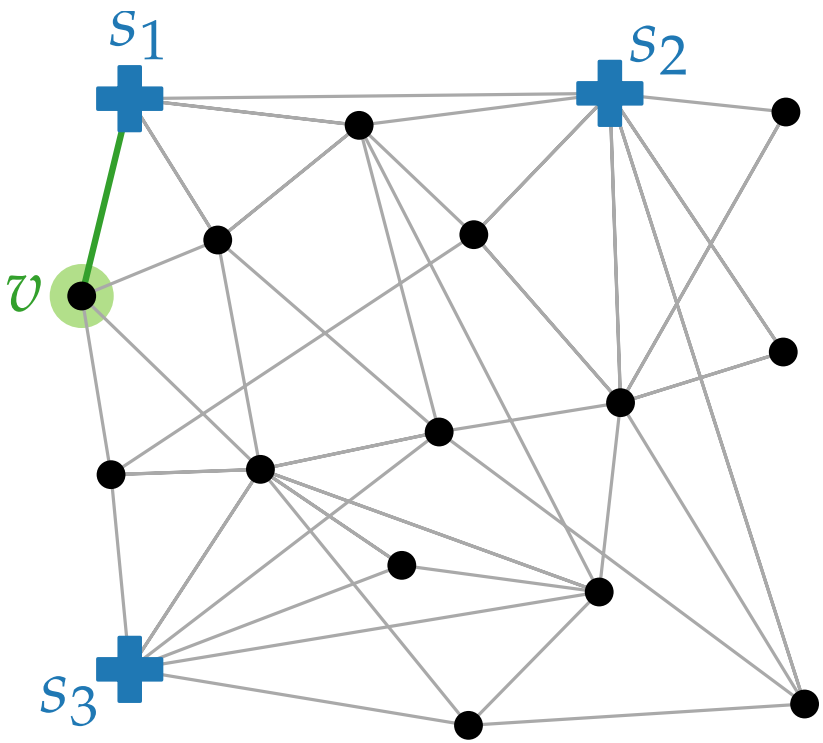

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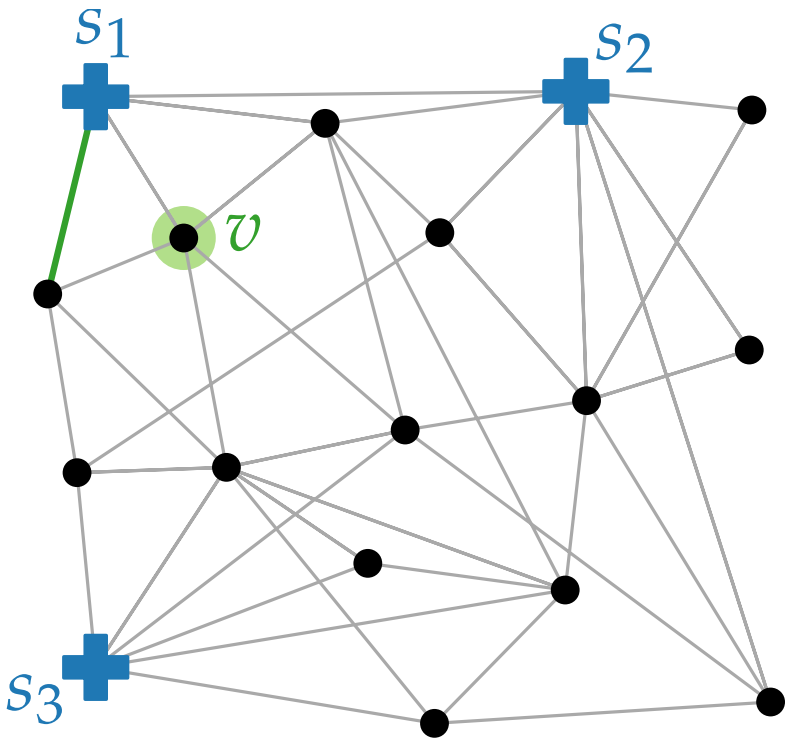

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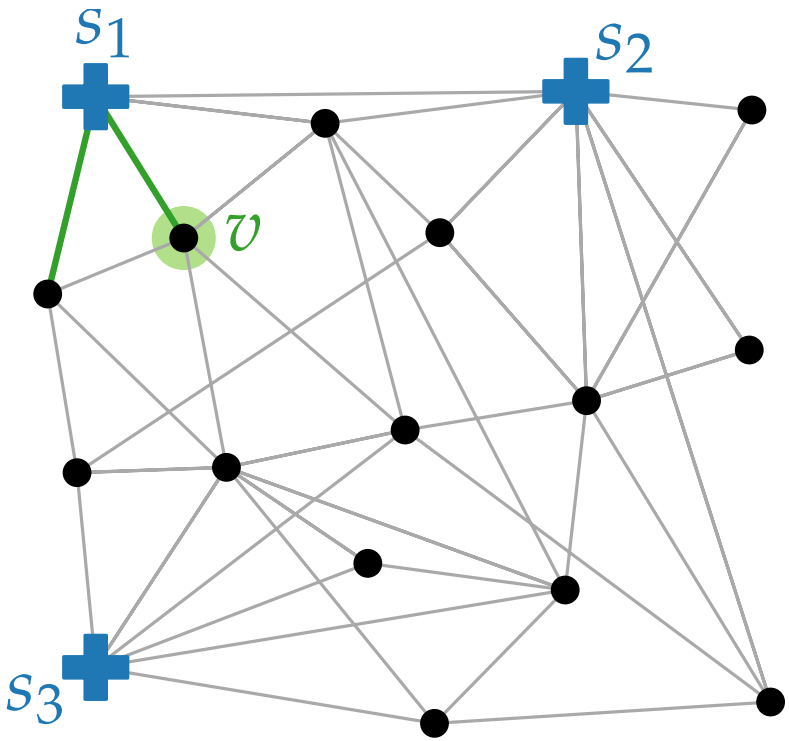

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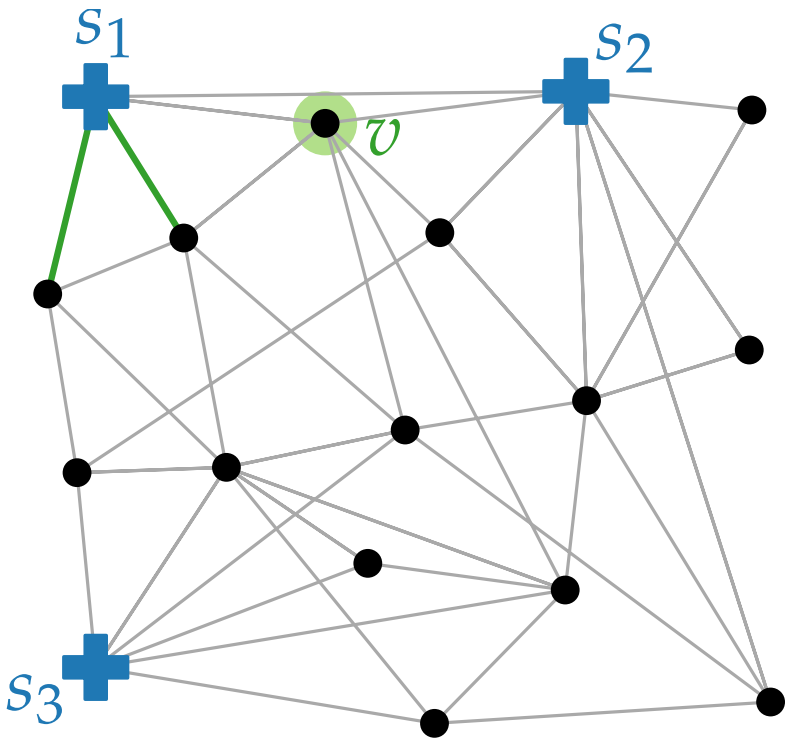

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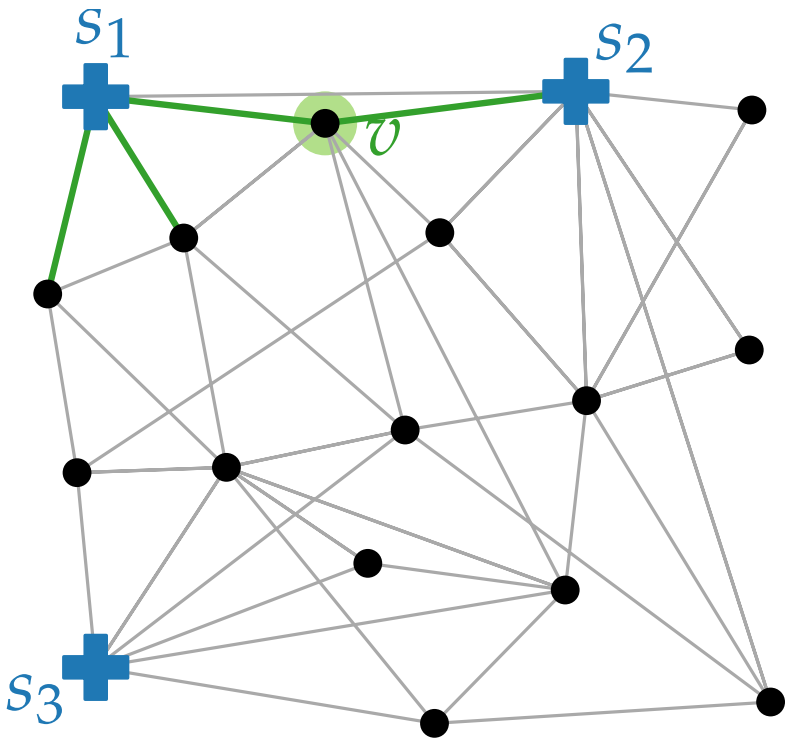

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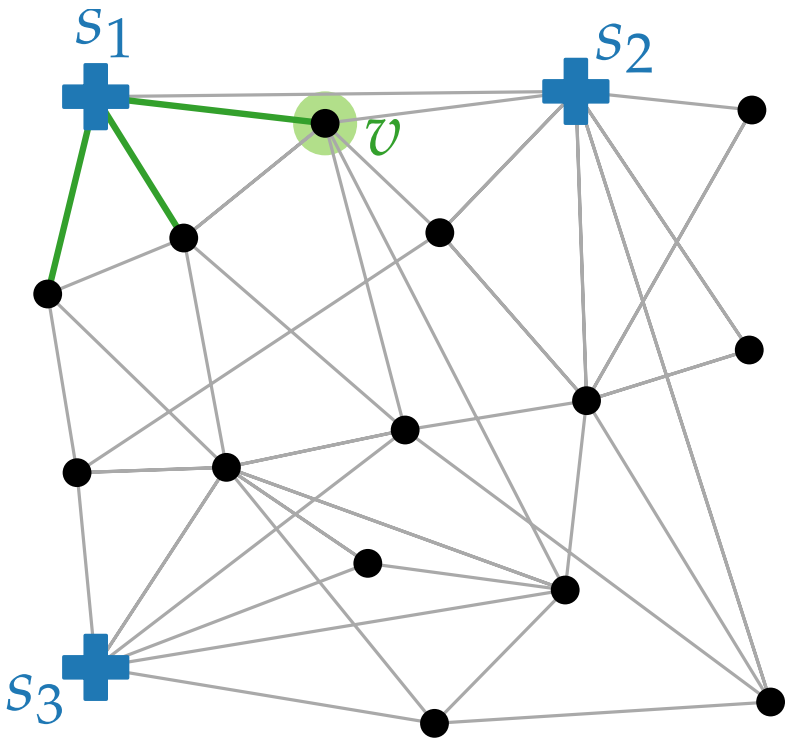

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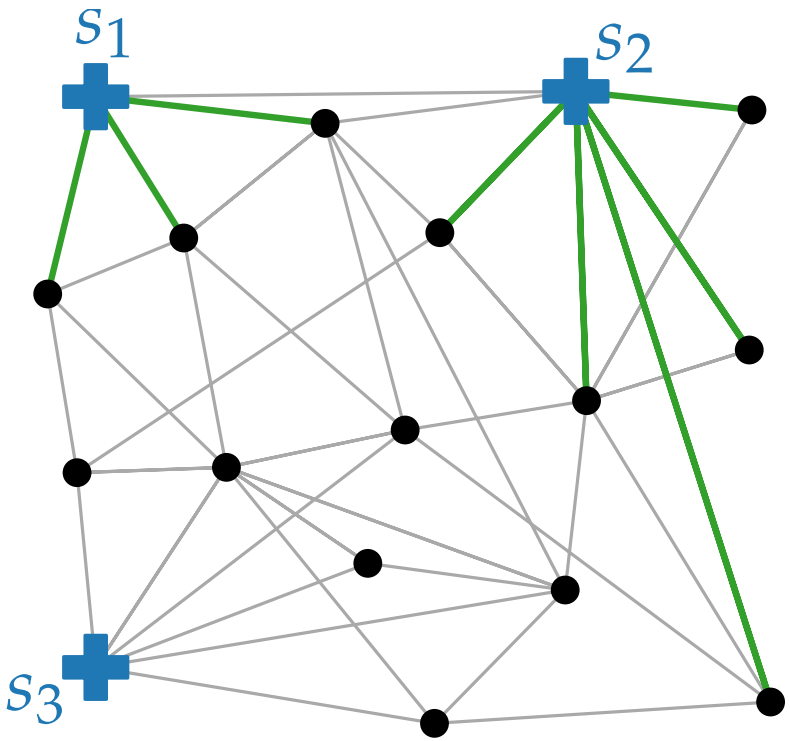

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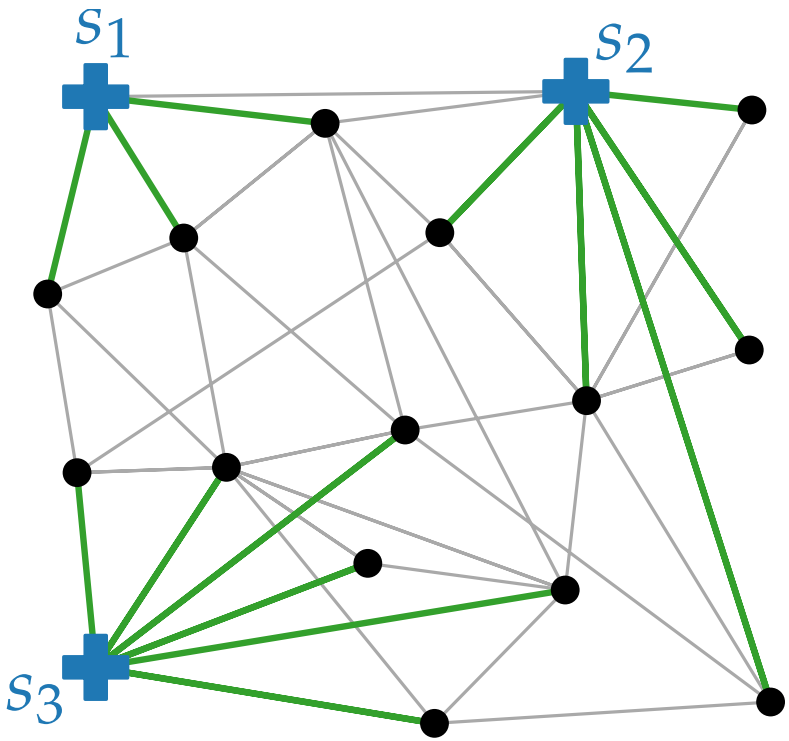

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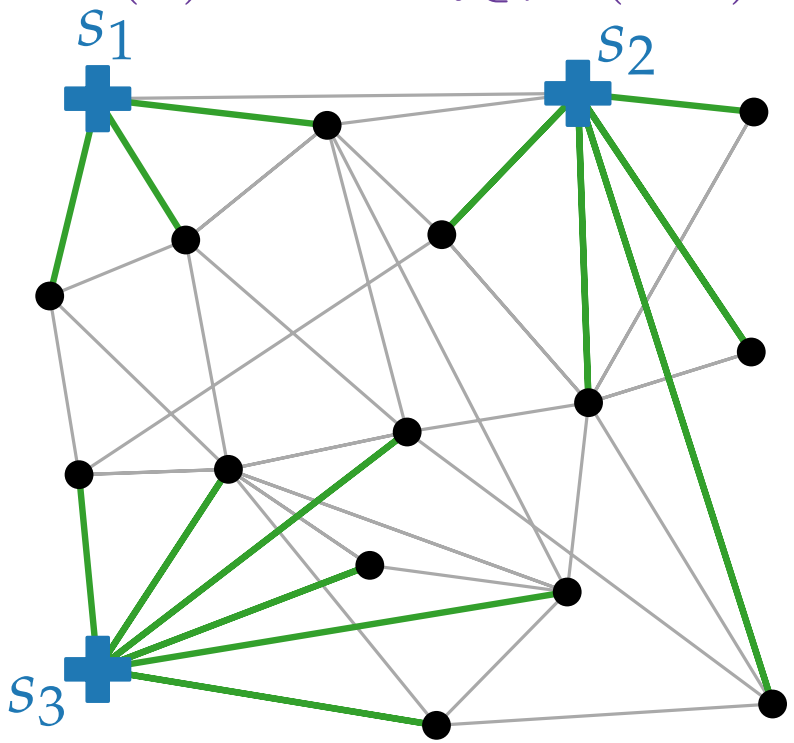

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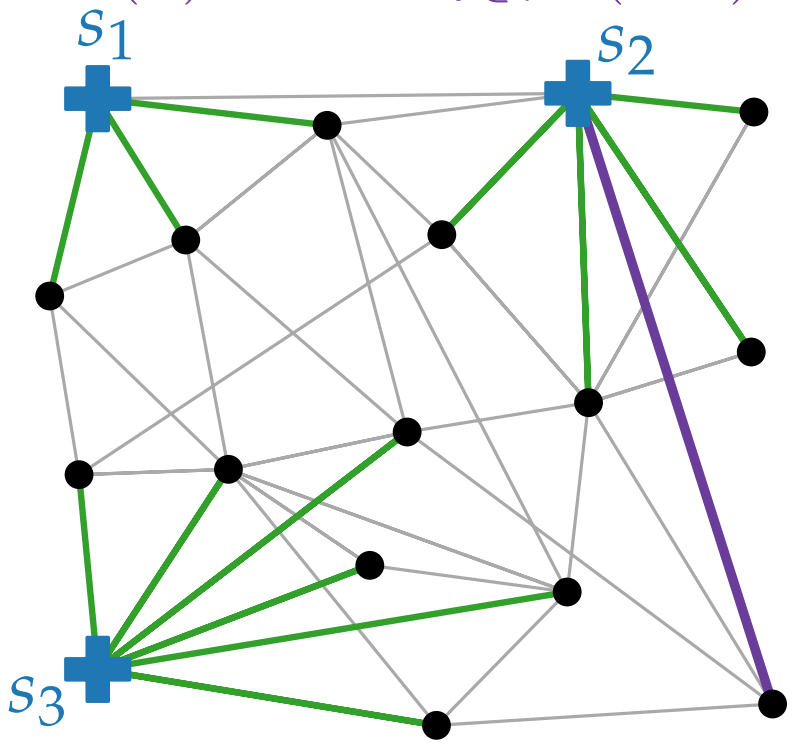

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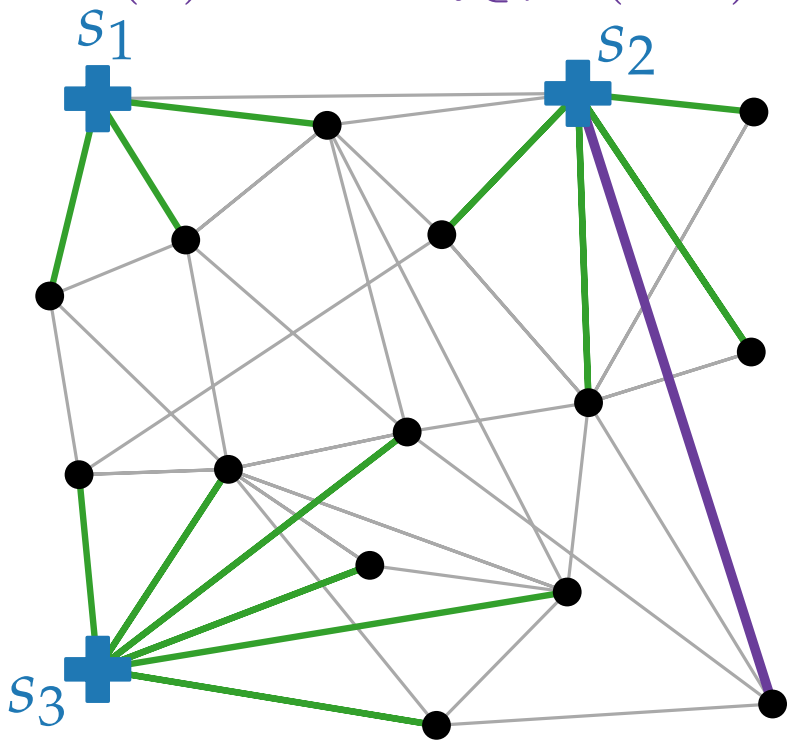

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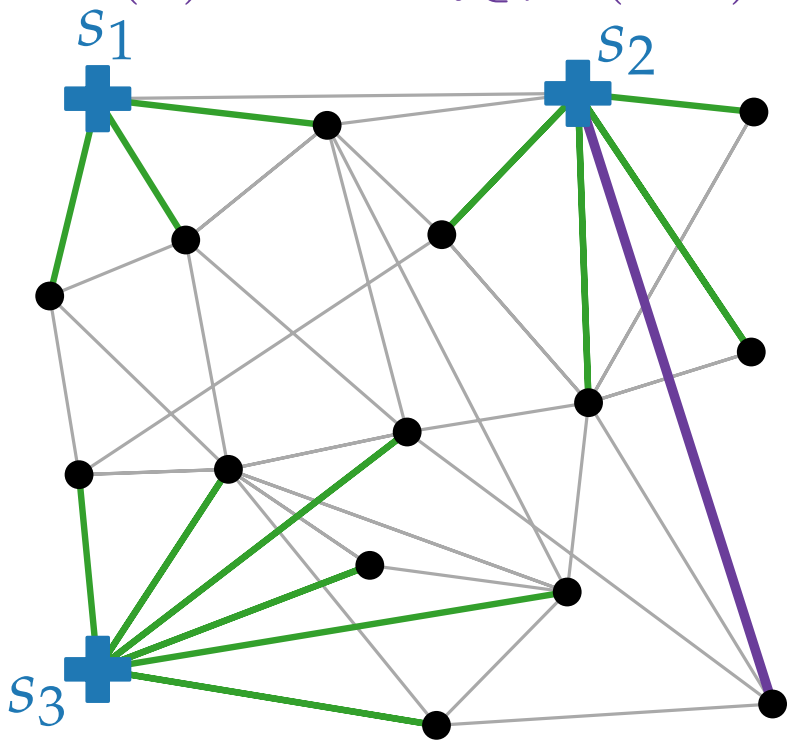

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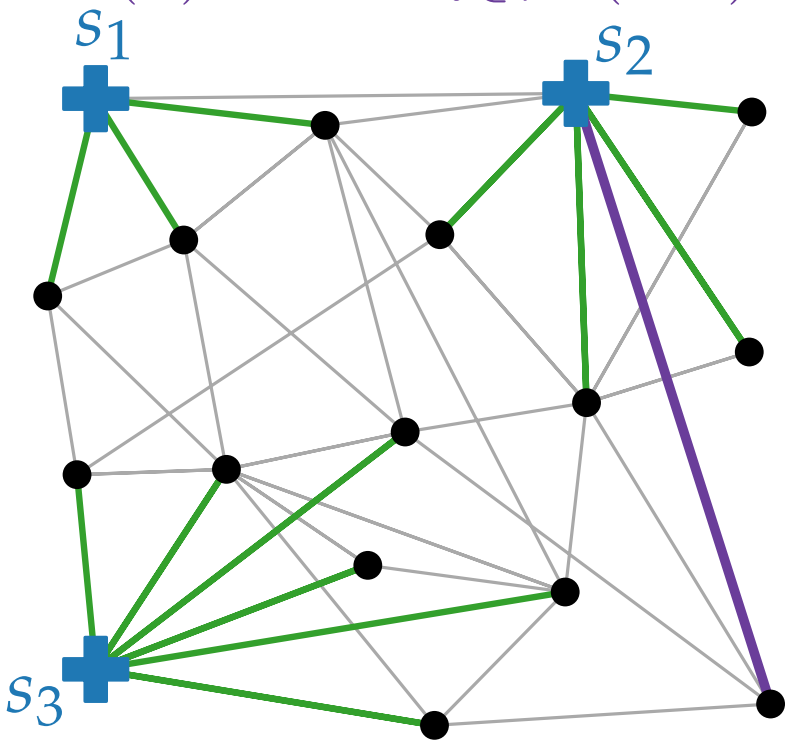

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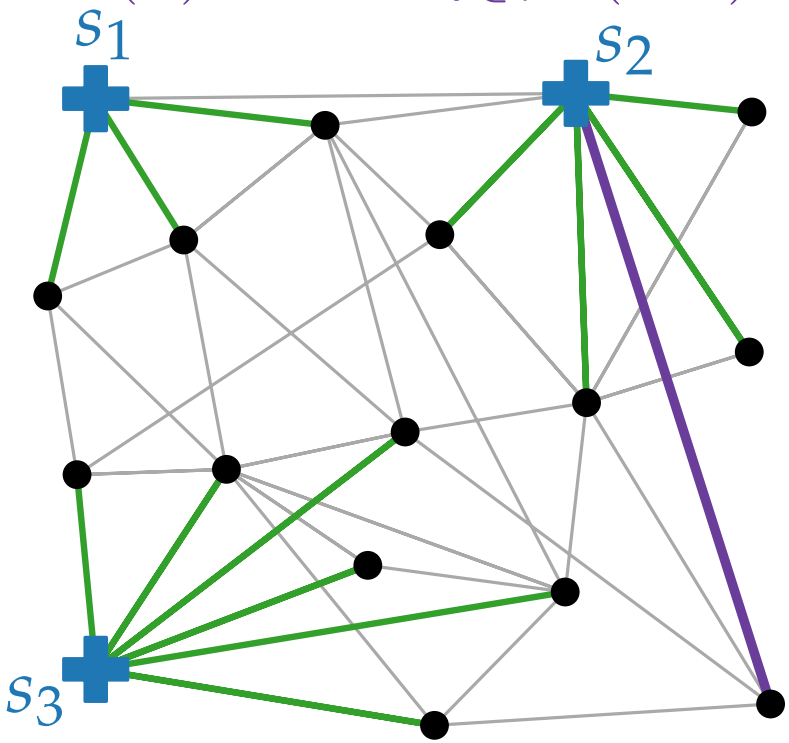

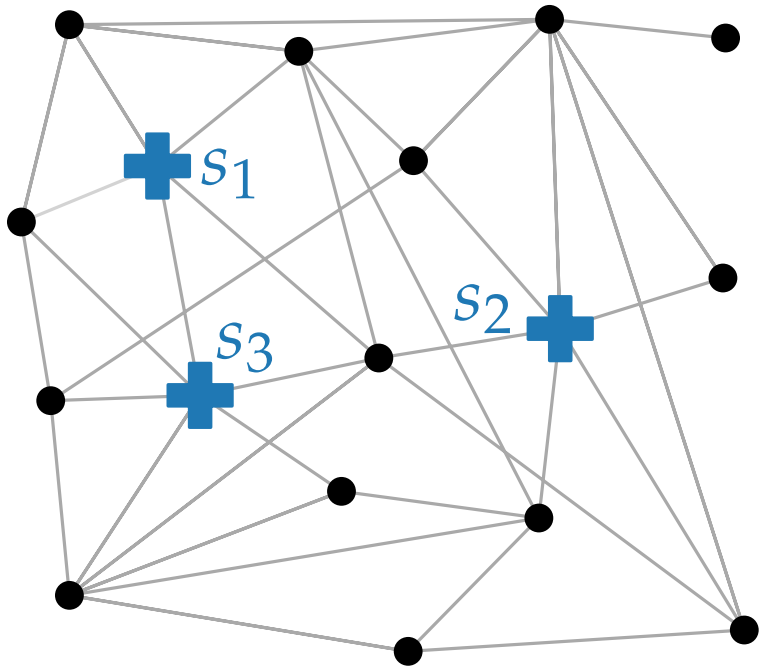

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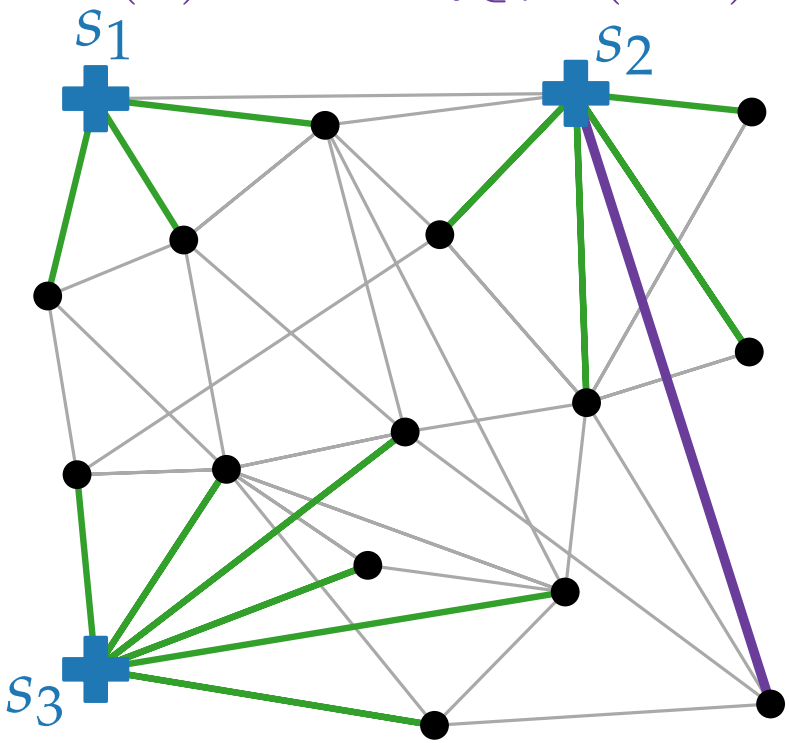

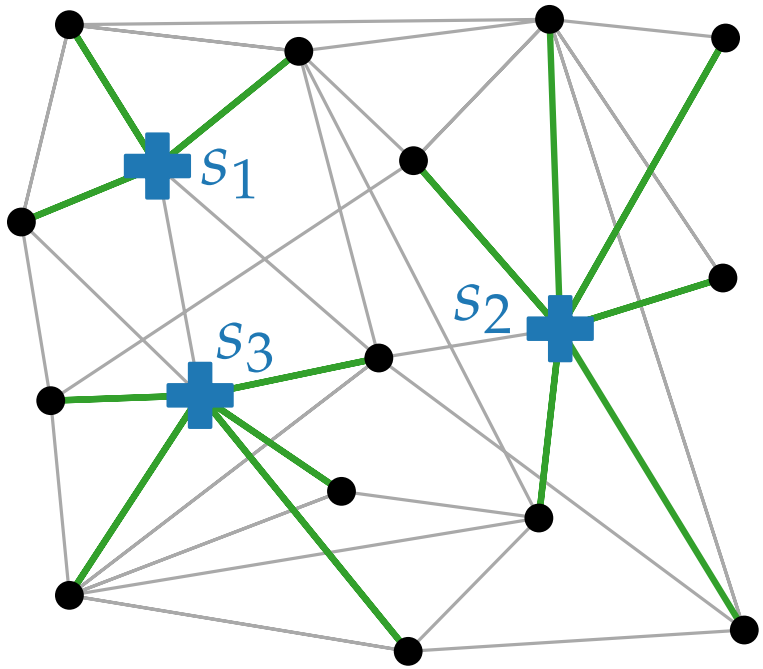

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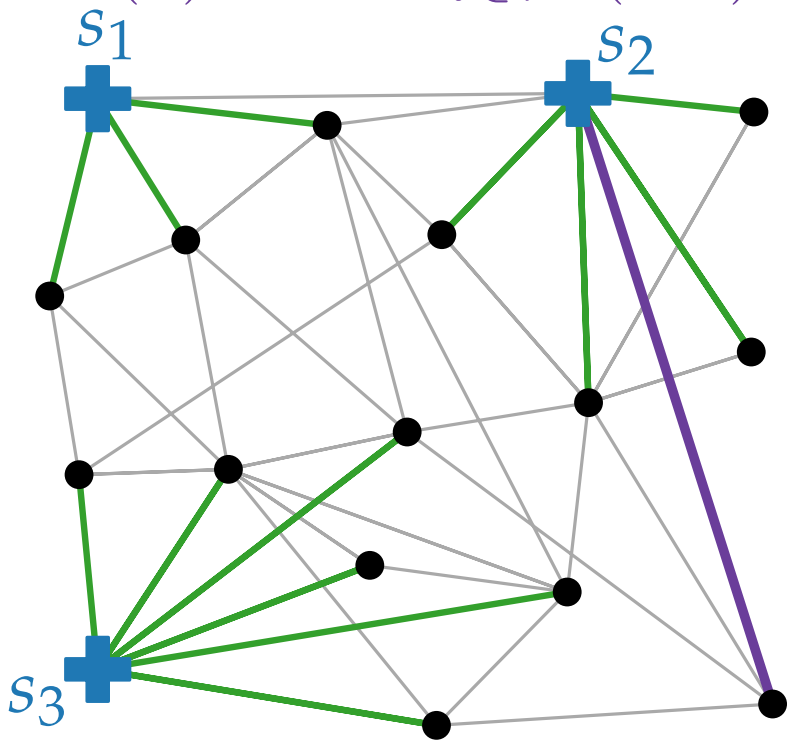

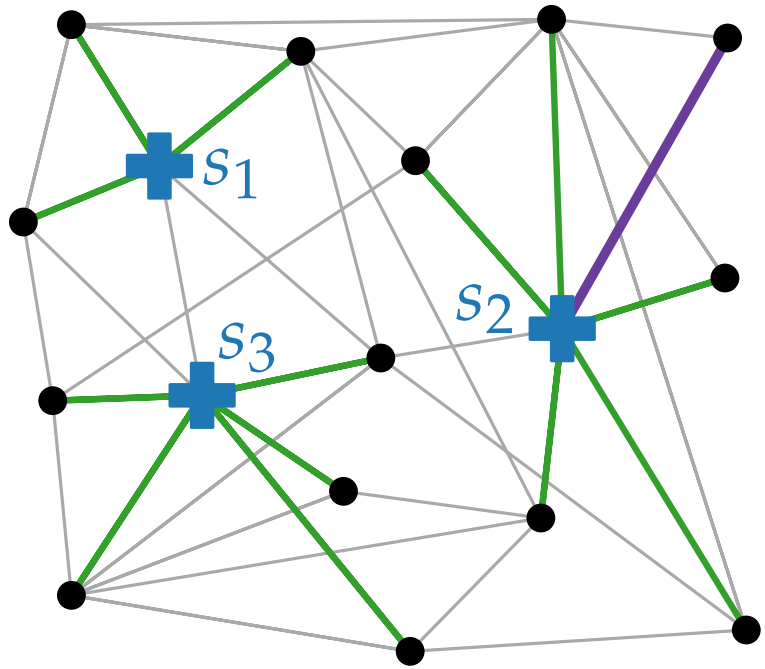

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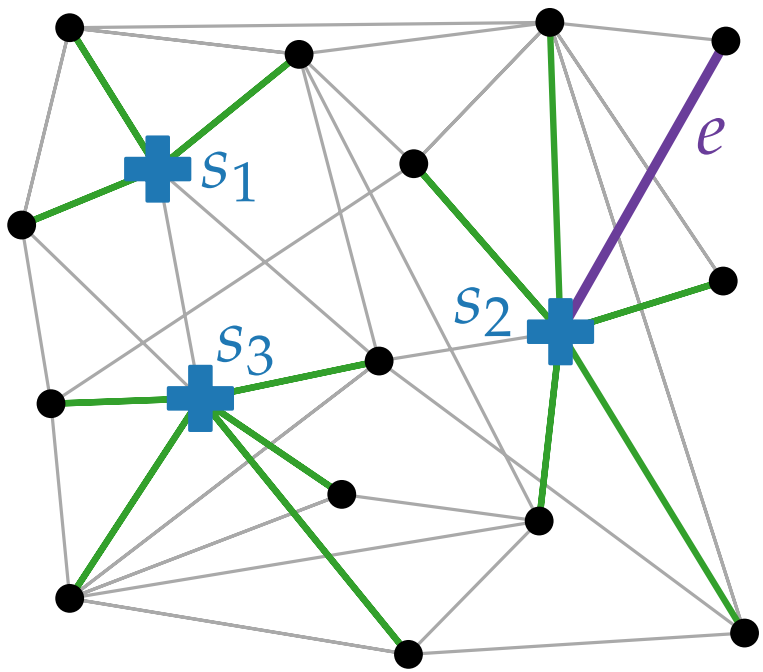

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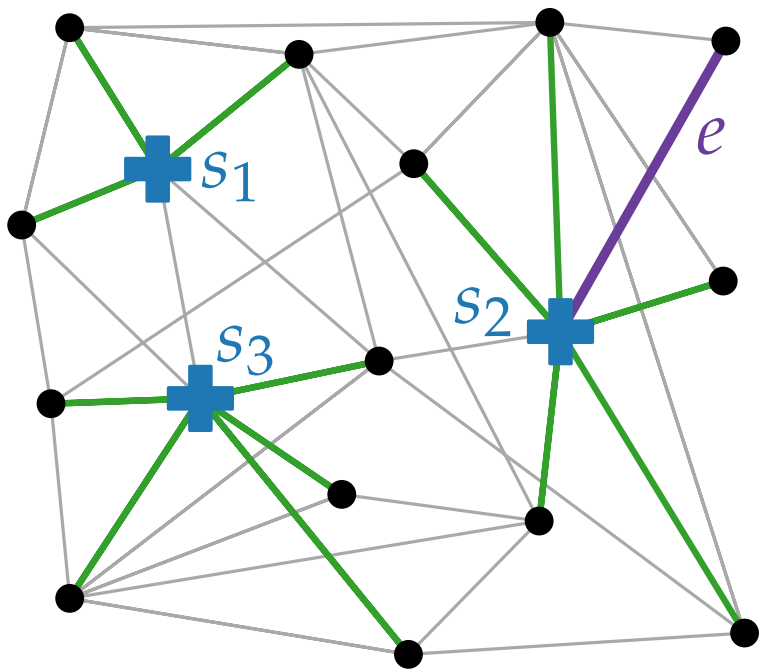

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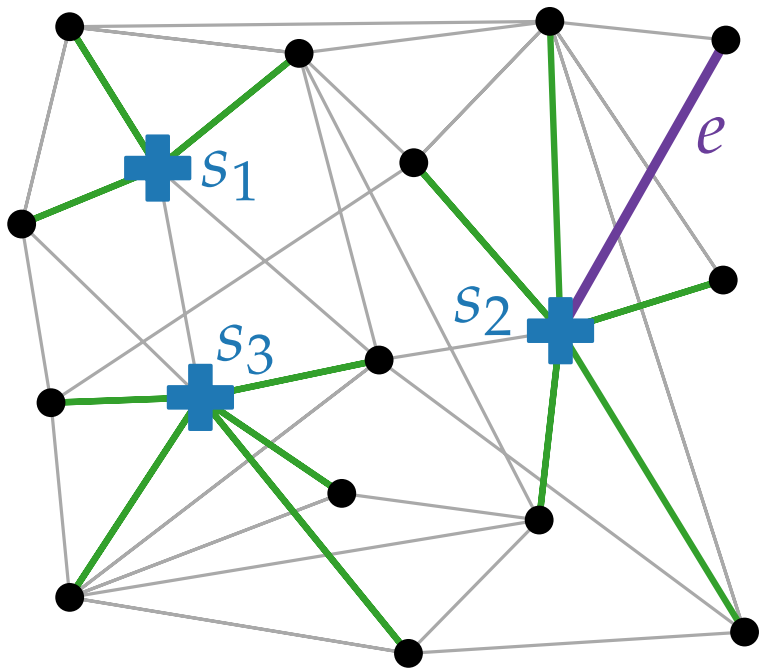

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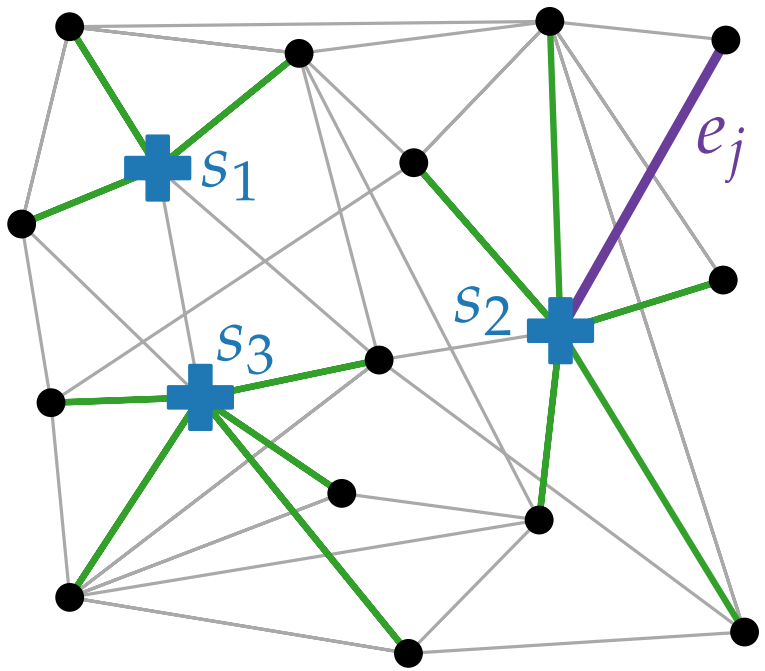

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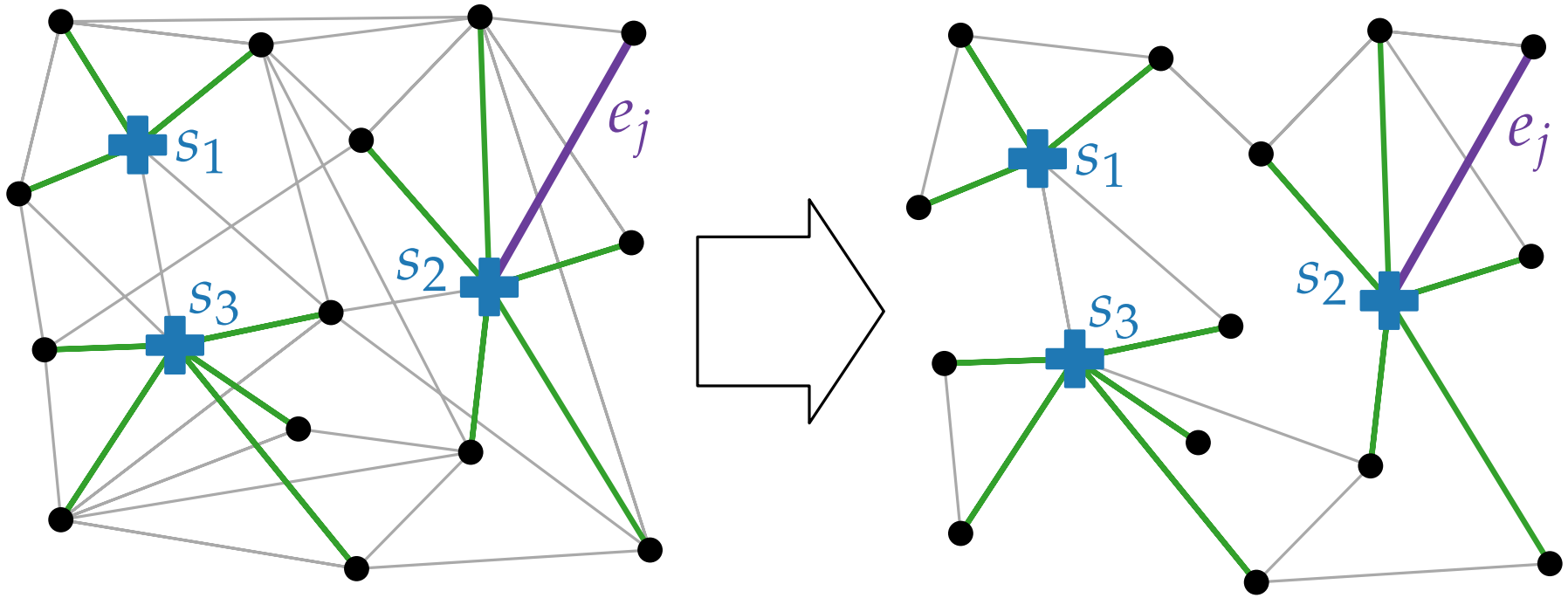

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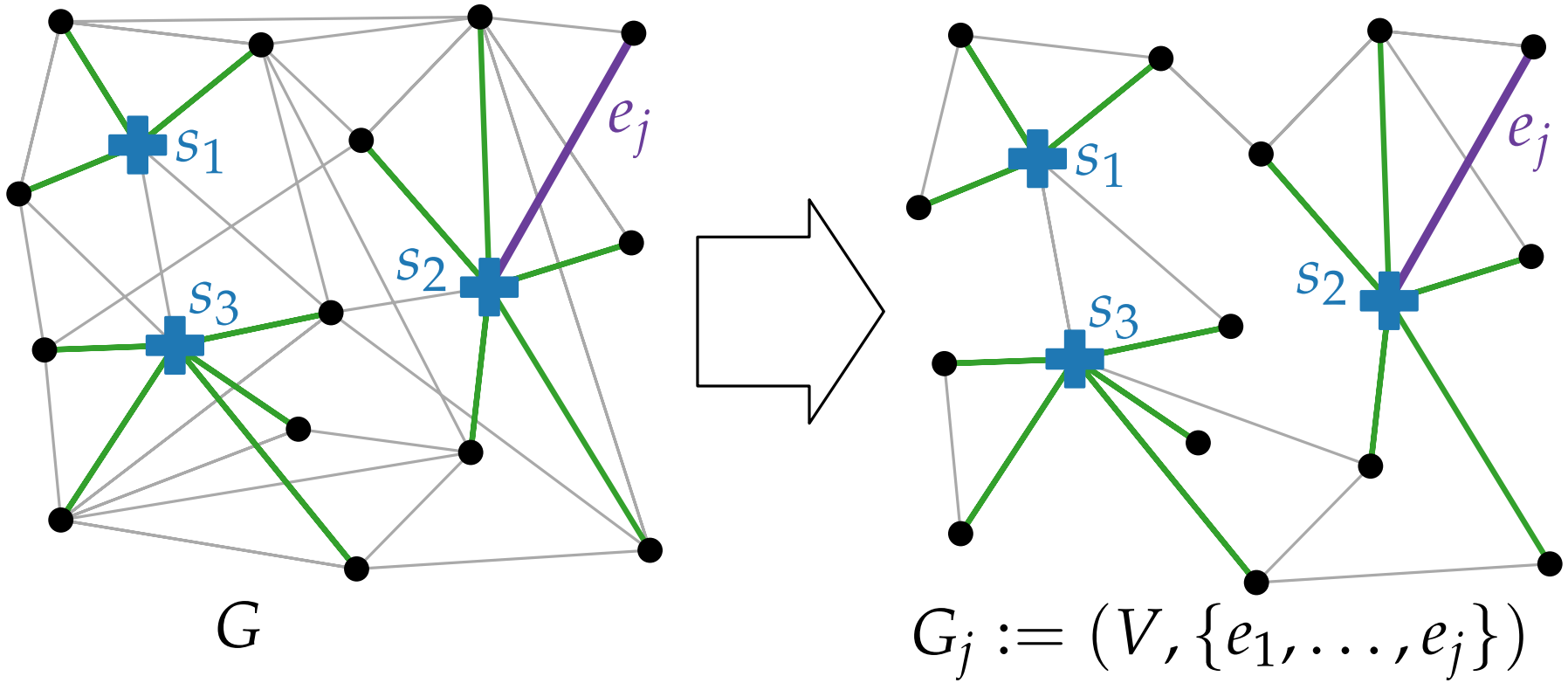

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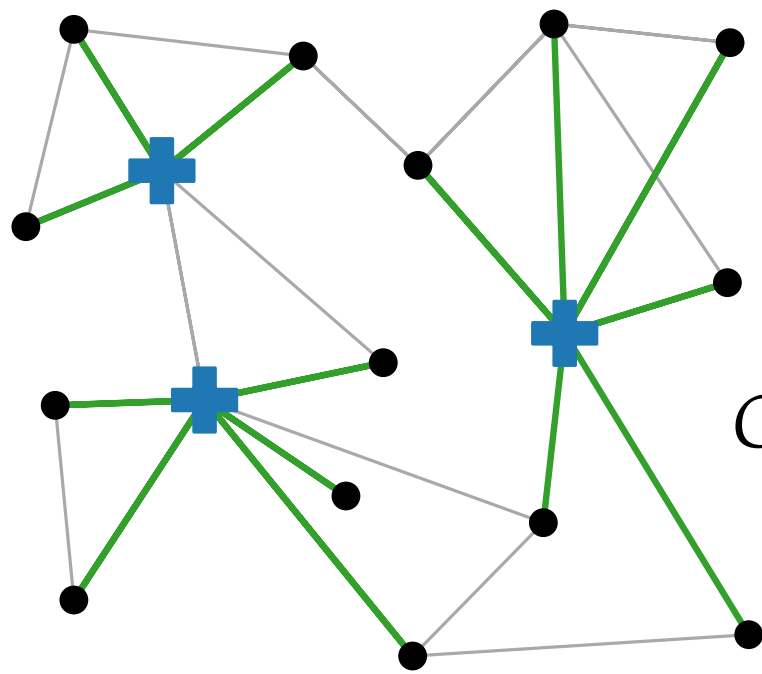

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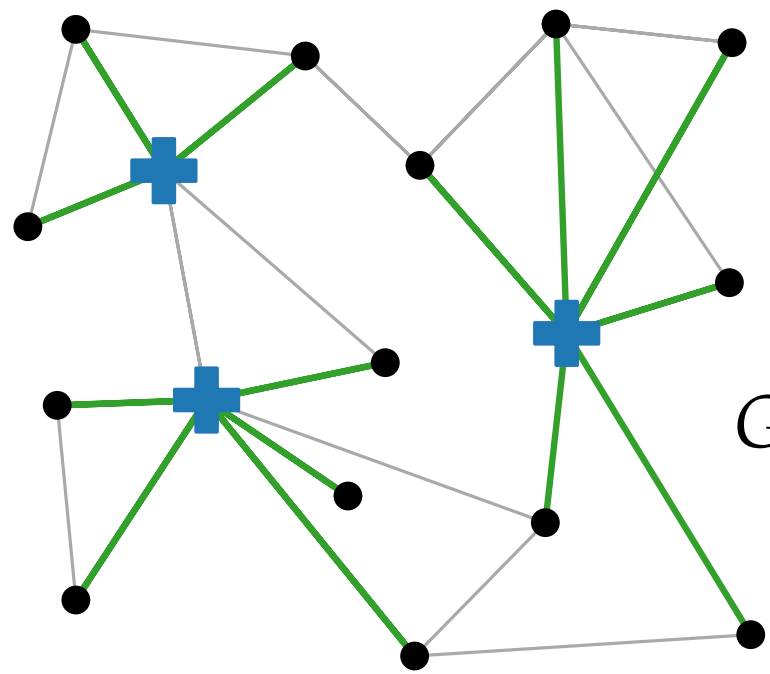
Def. A vertex set D of a graph H is **dominated**, when each vertex is either in D or adjacent to a vertex in D.



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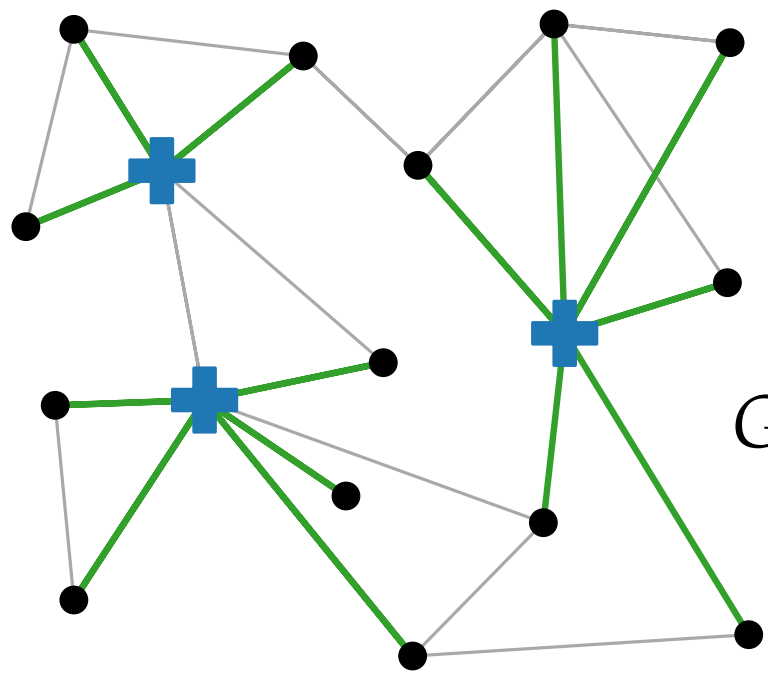

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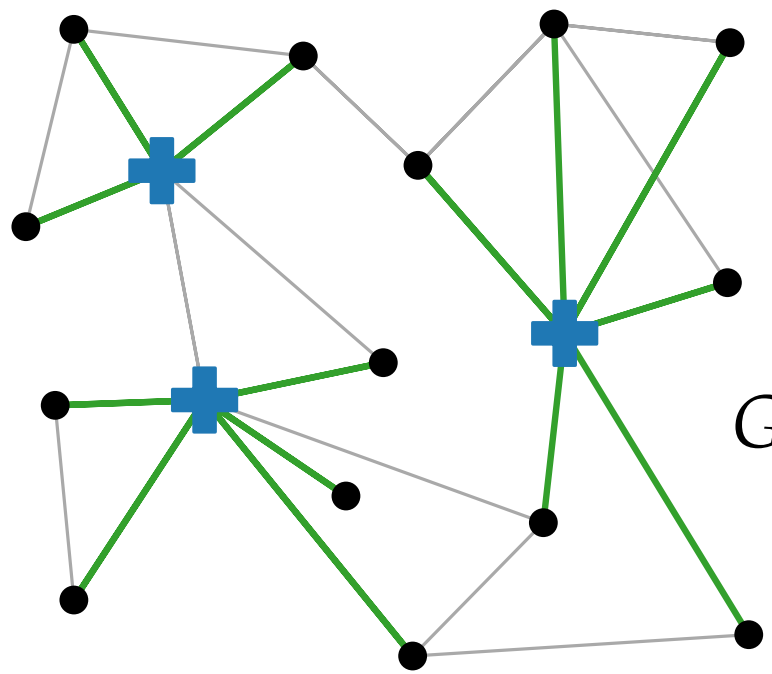

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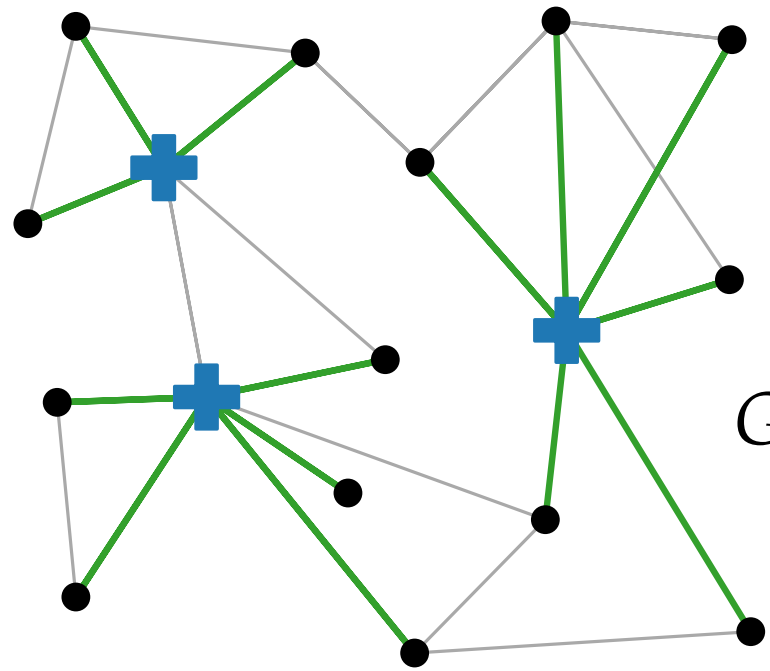

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. . . but computing $\mathrm{dom}(H)$ is NP-hard.

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Idea: Find a small dominating set in a "coarsened" G_j

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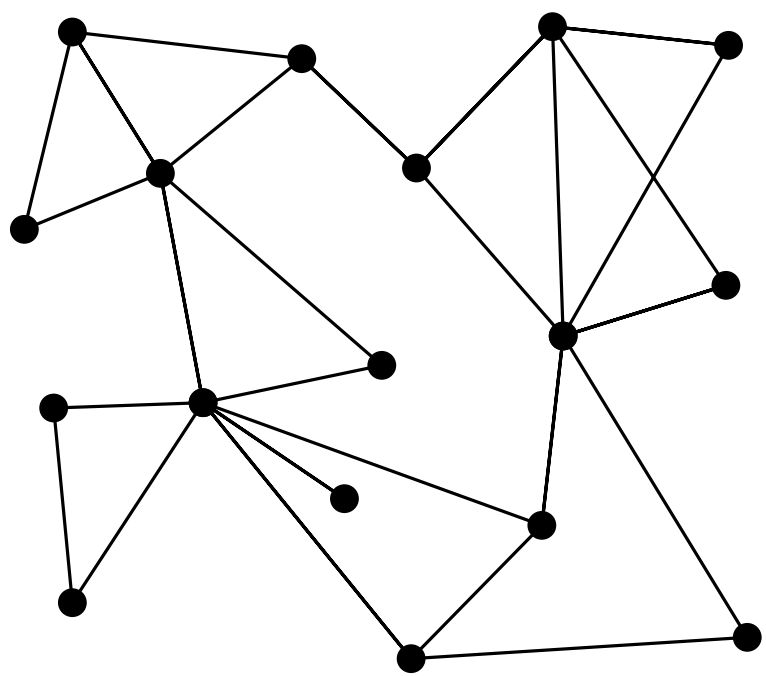

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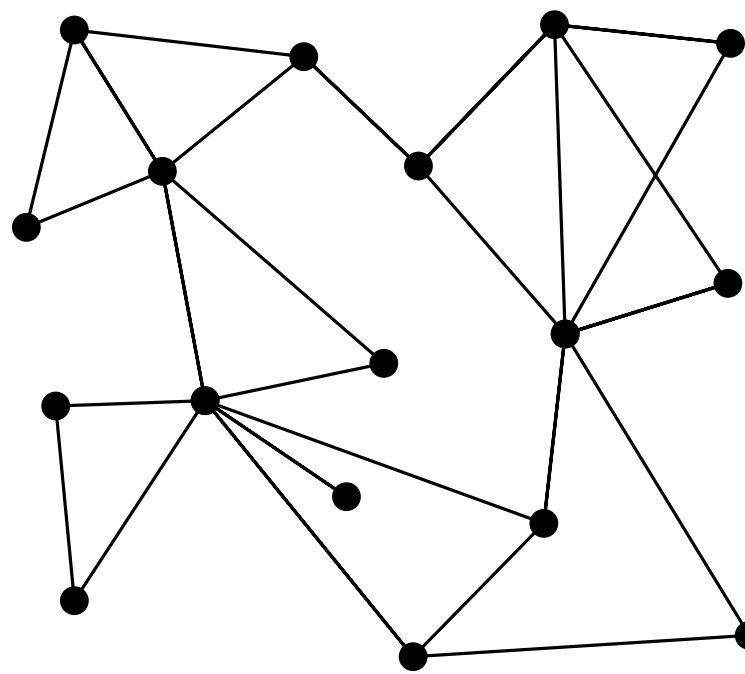

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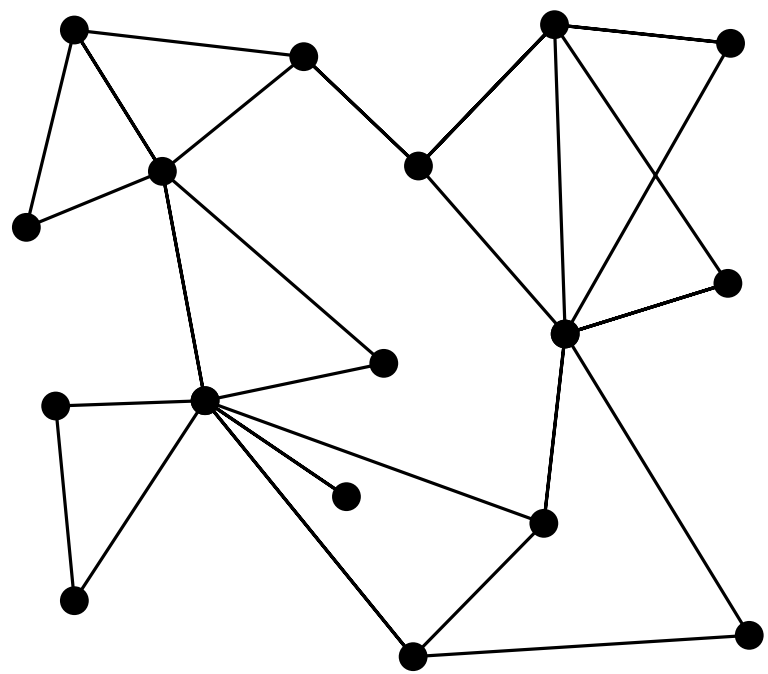

G_j

G_j^2

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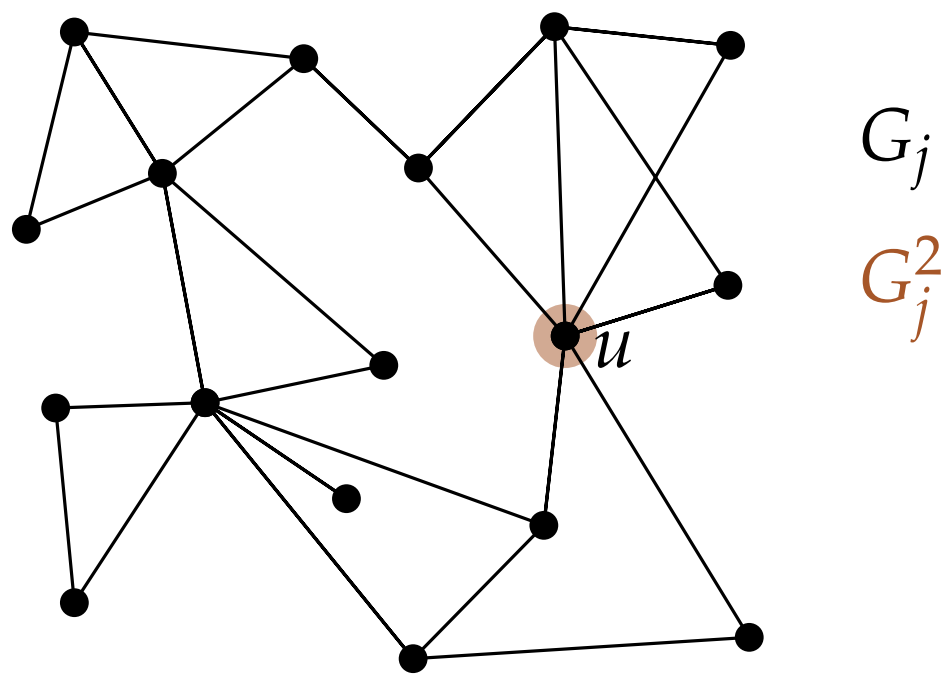

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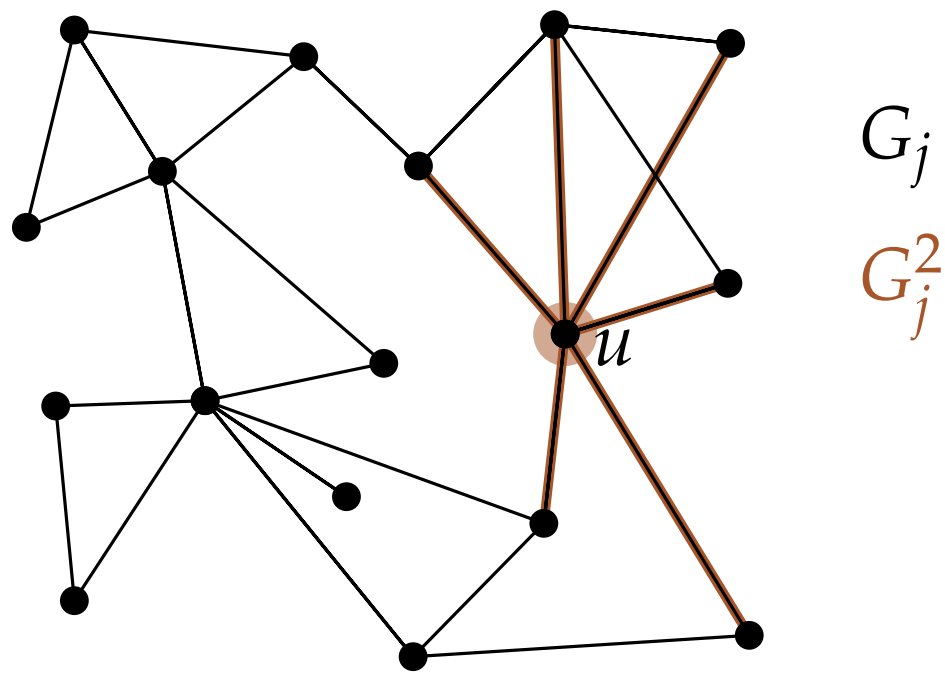

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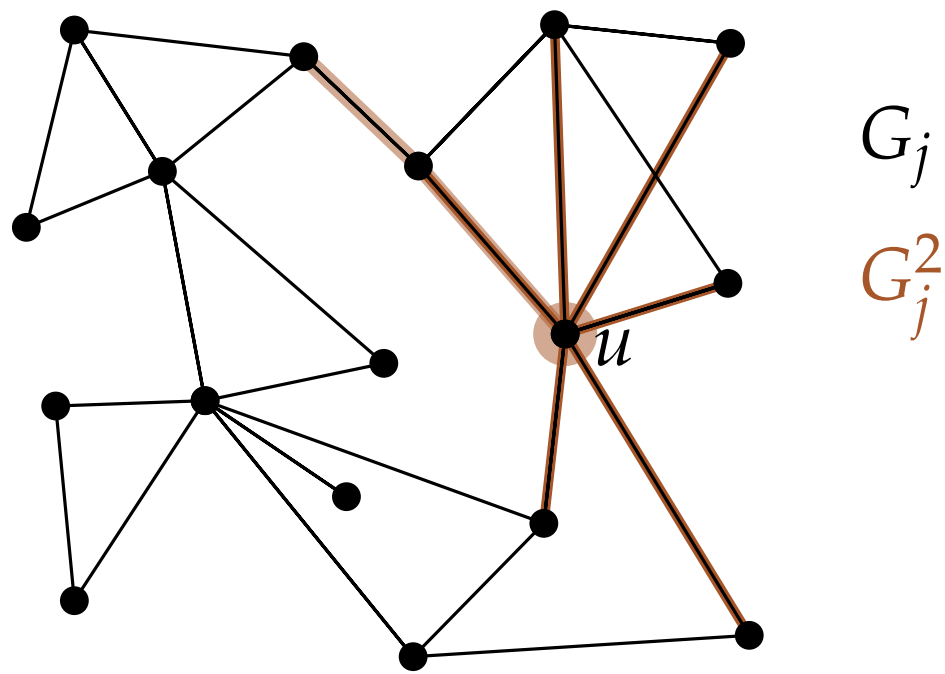

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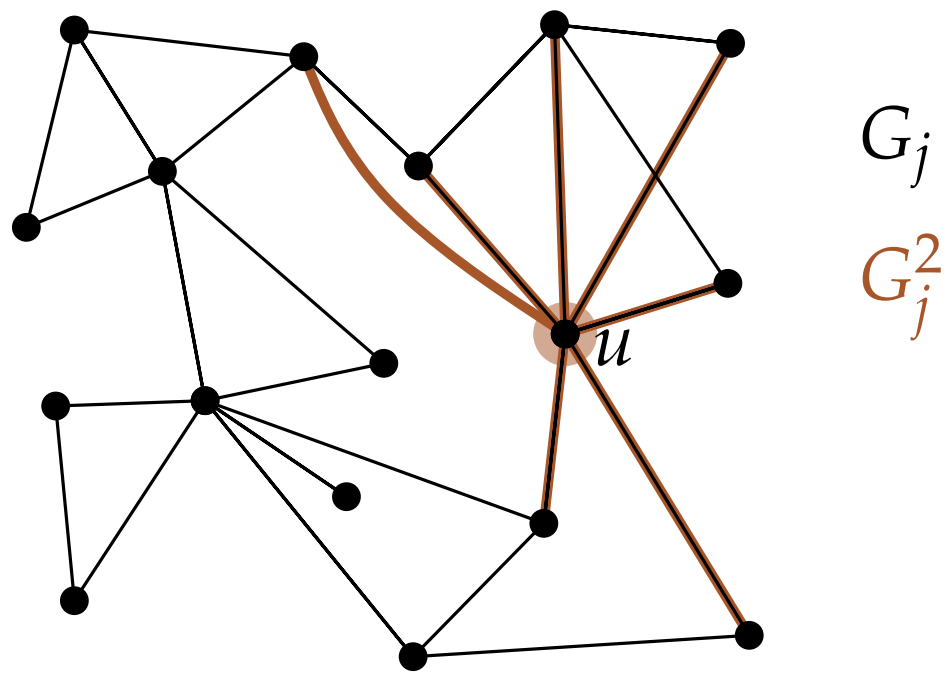

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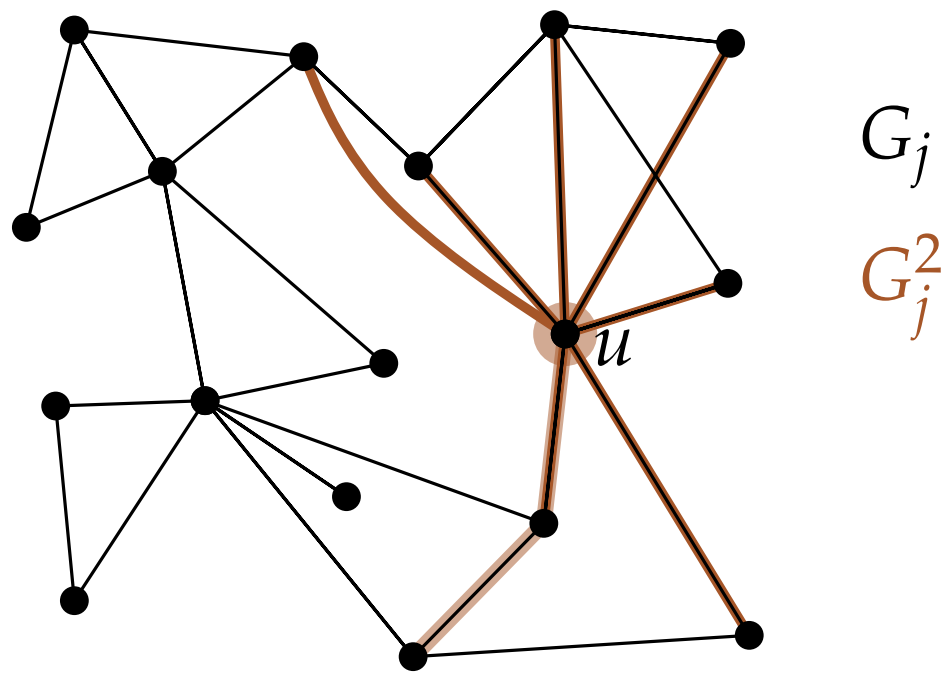

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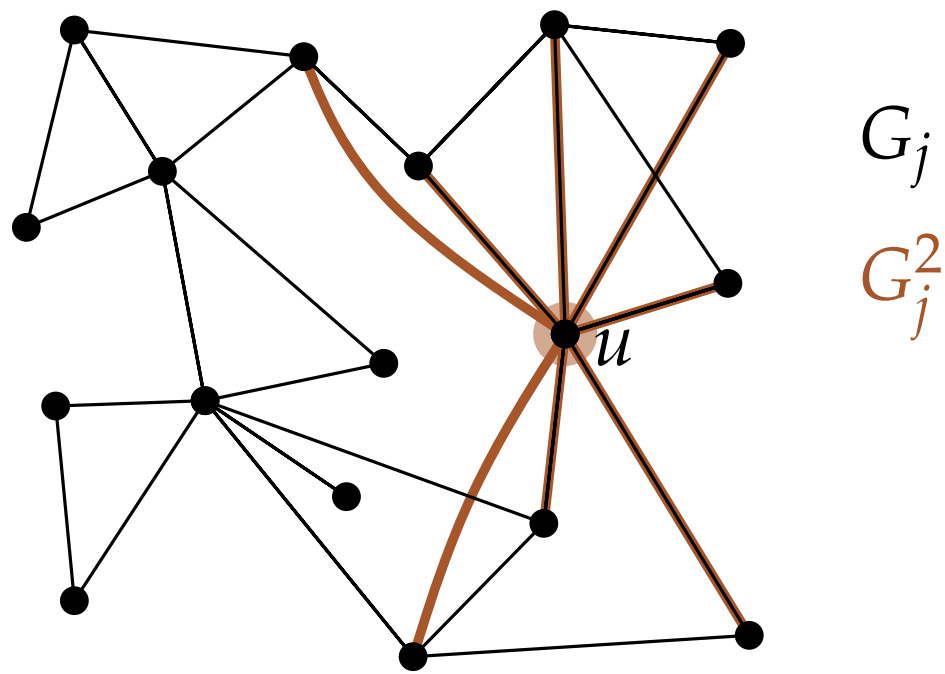

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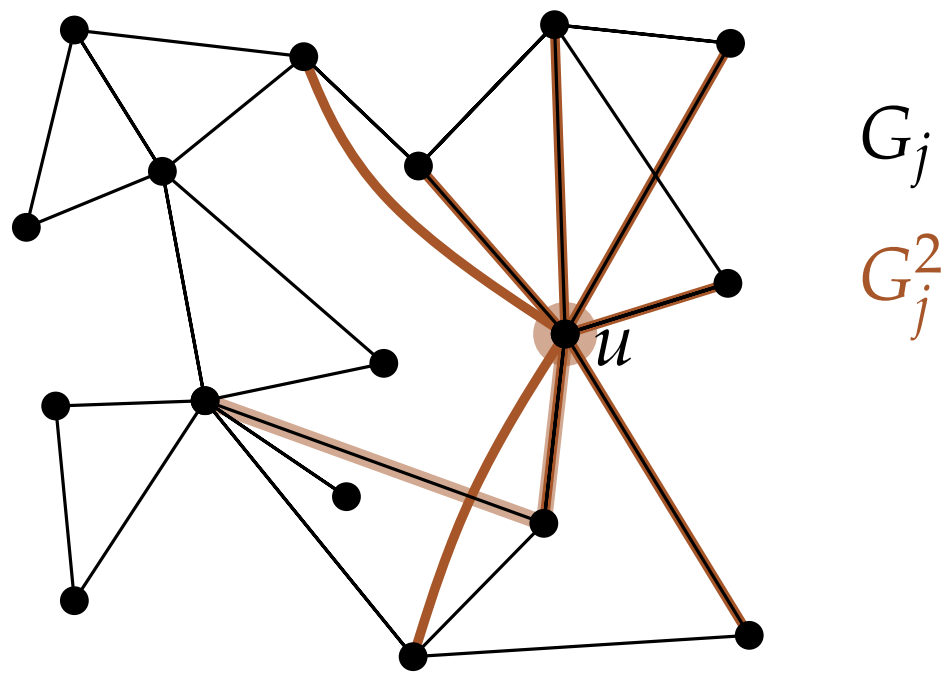

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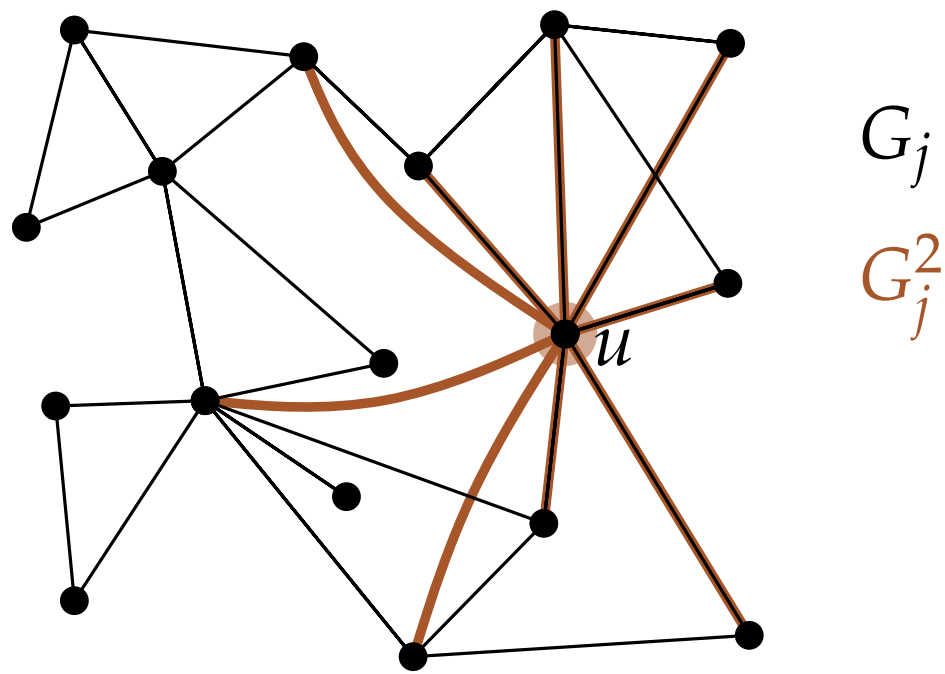

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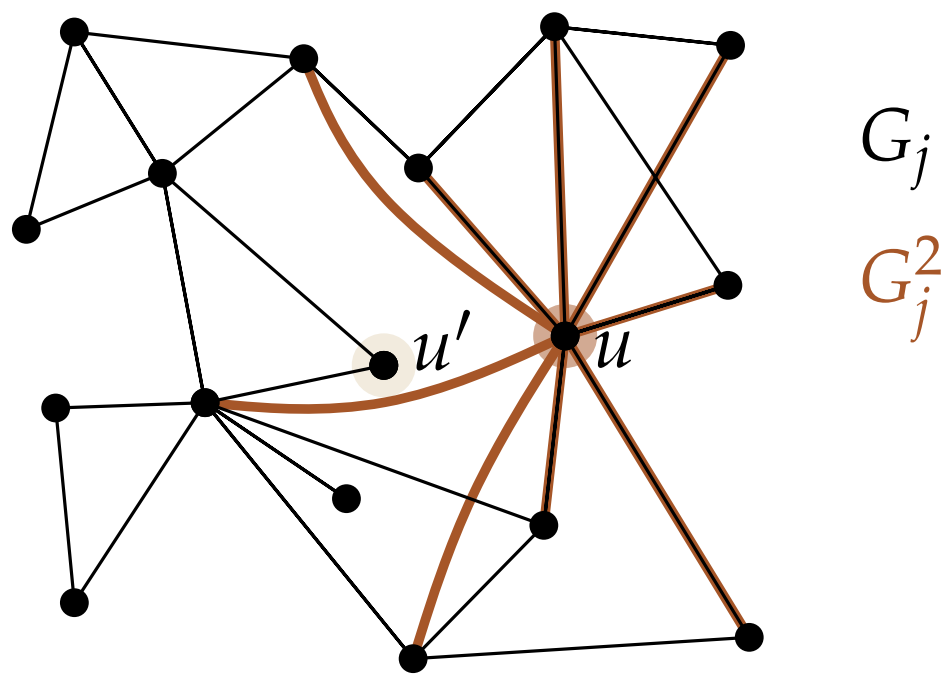

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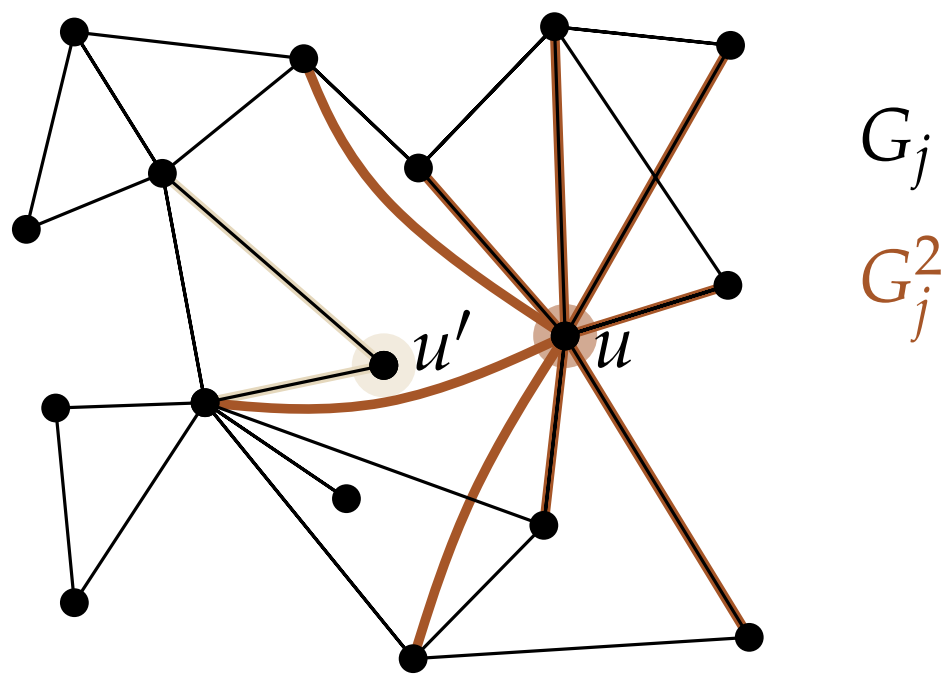

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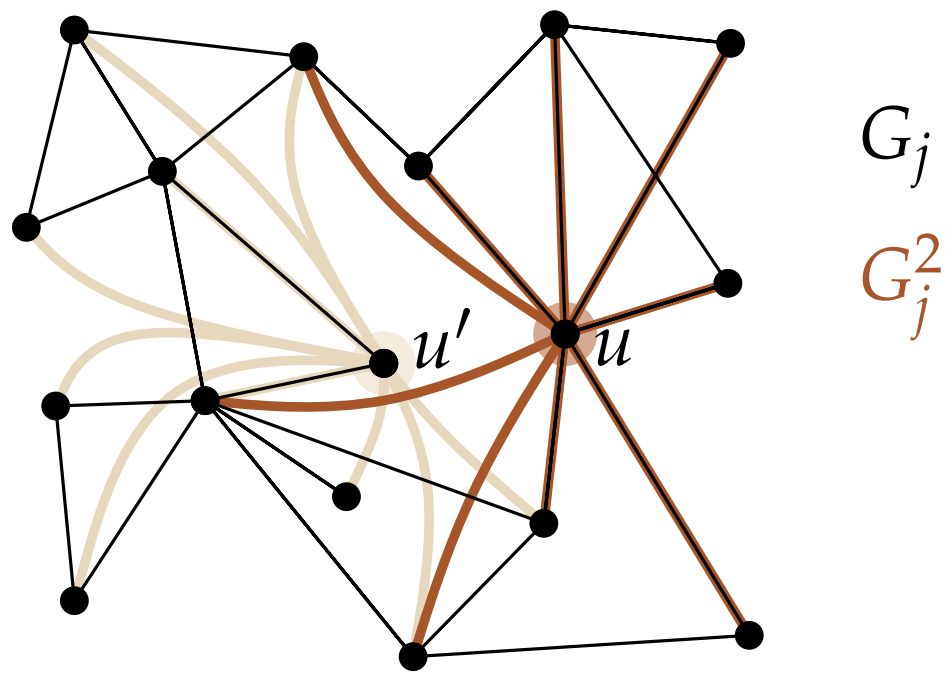

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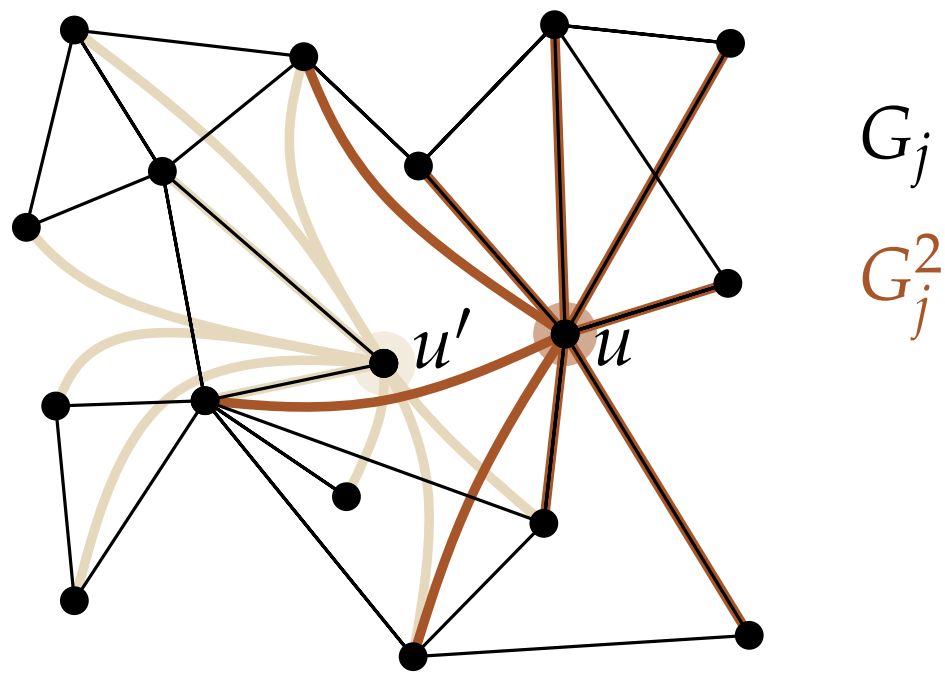

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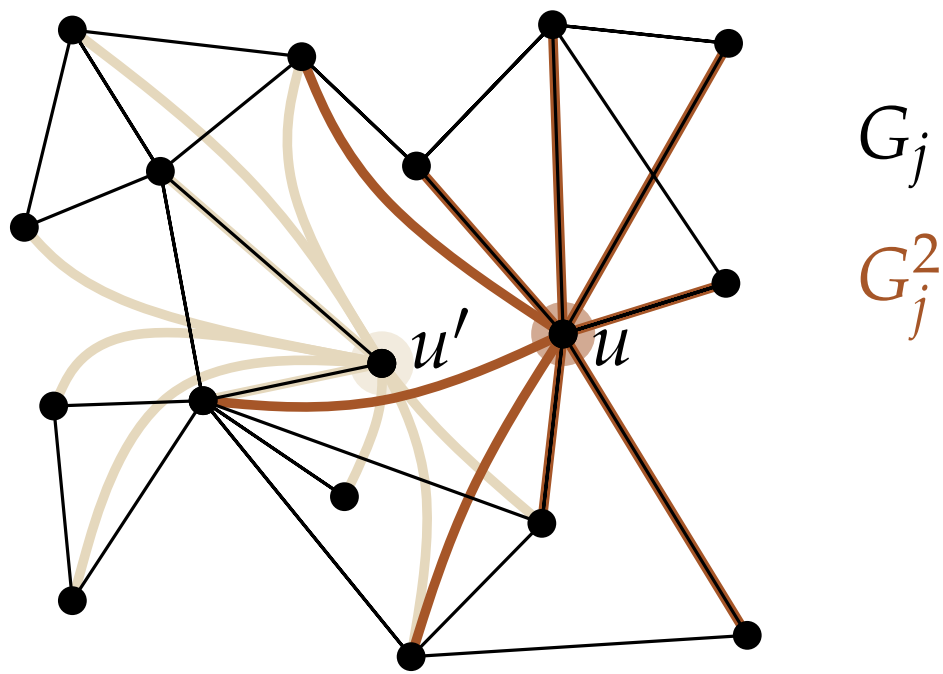

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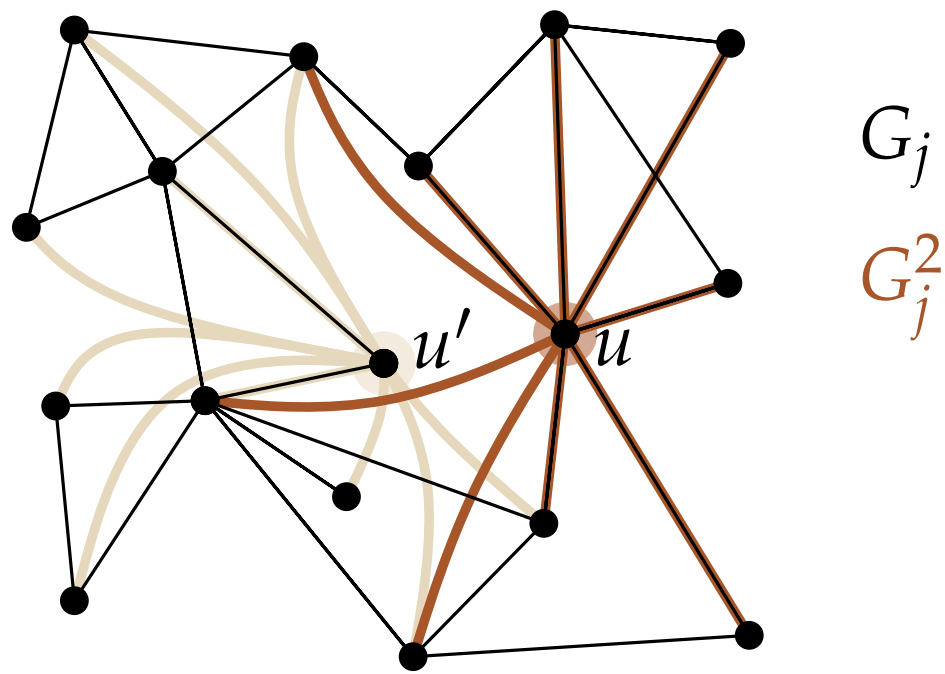

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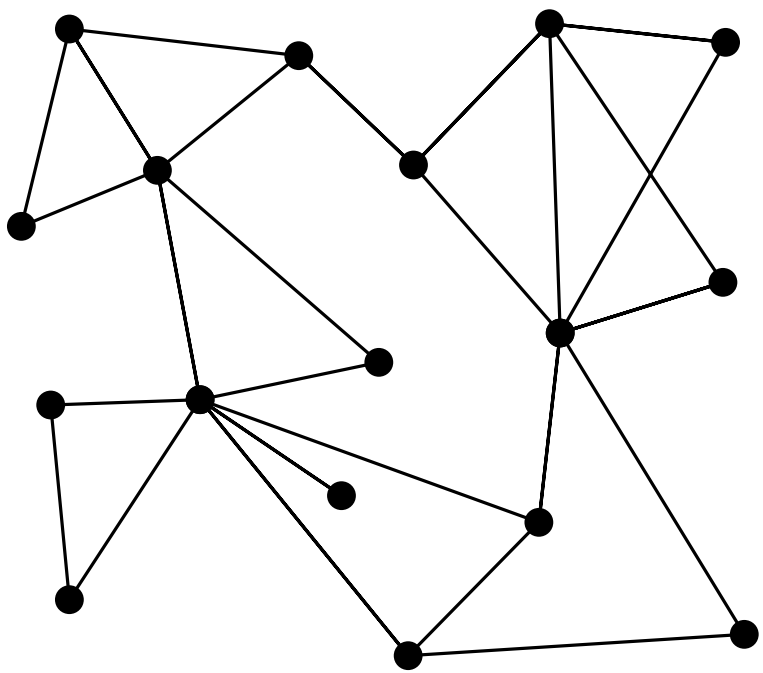

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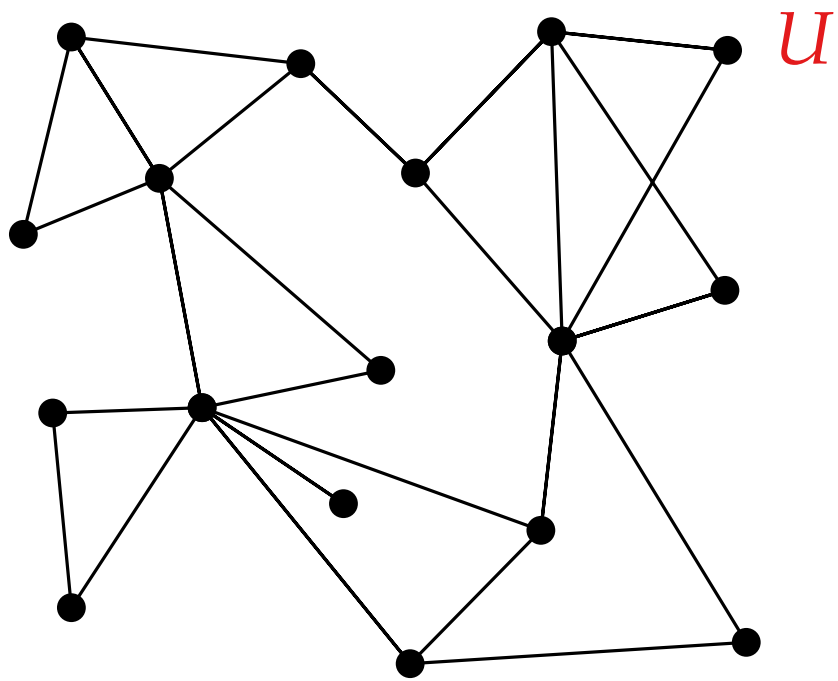

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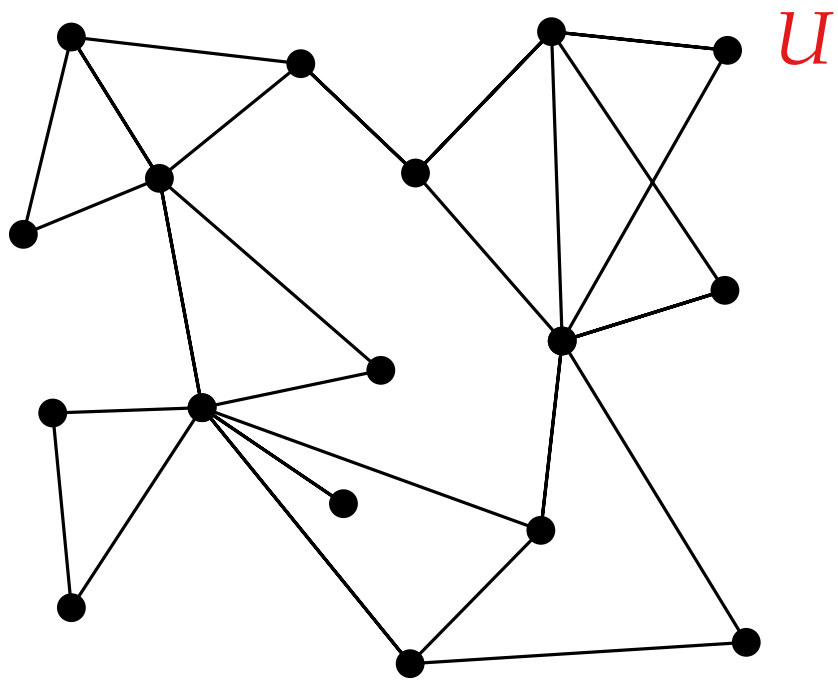

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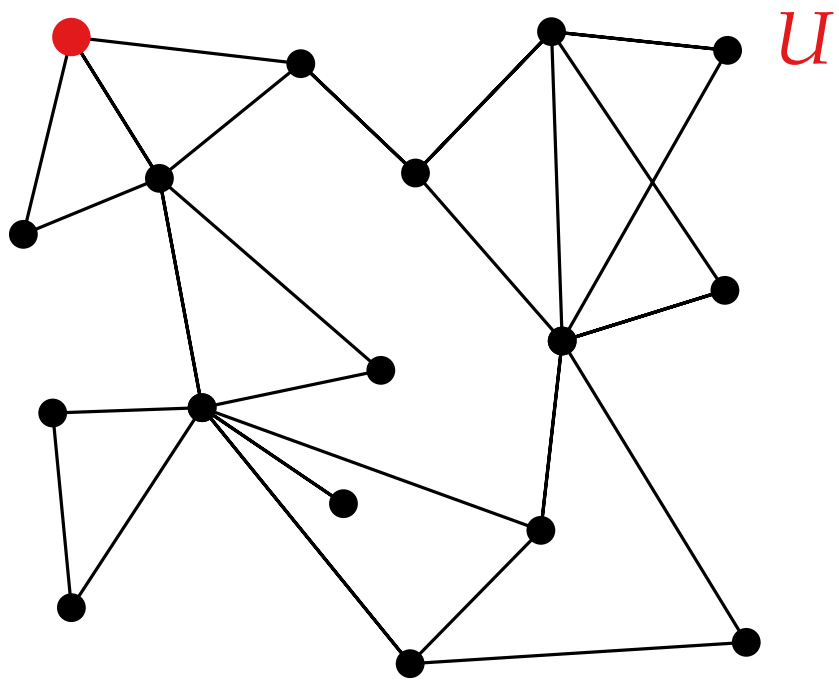

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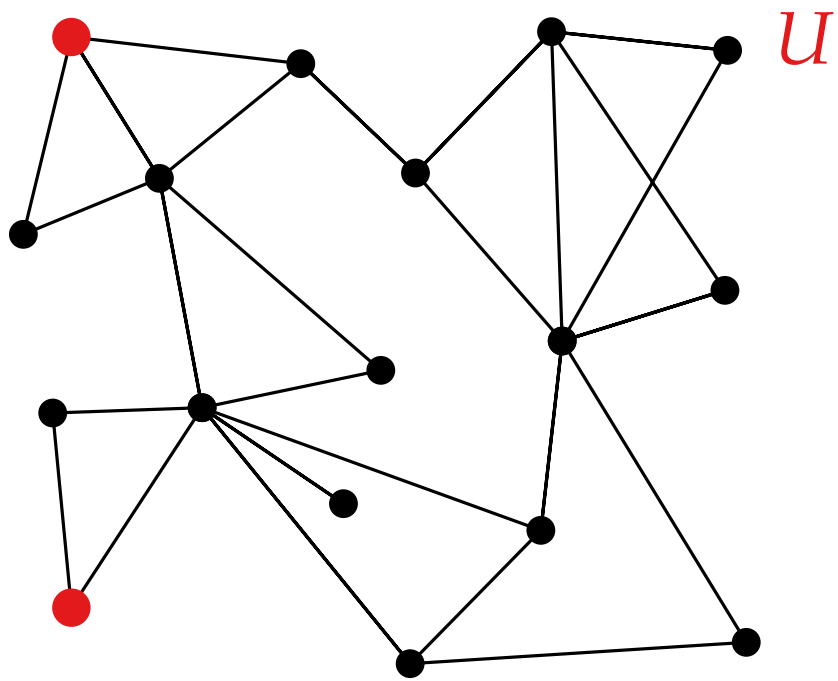

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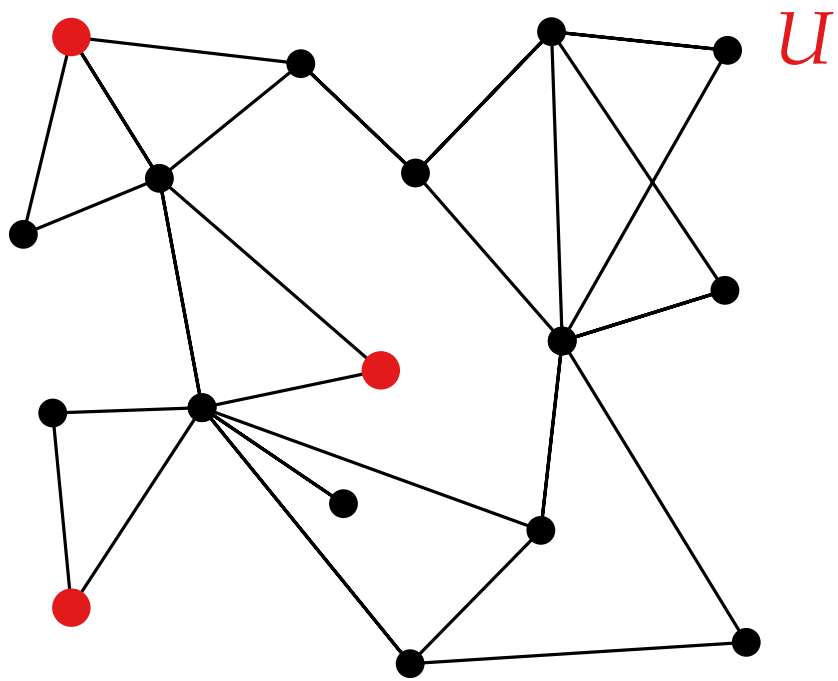

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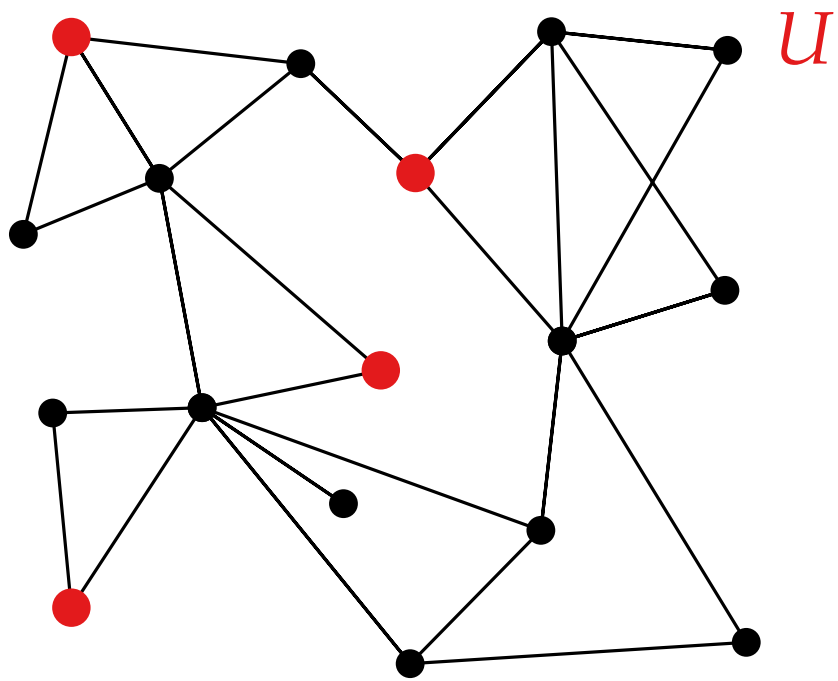

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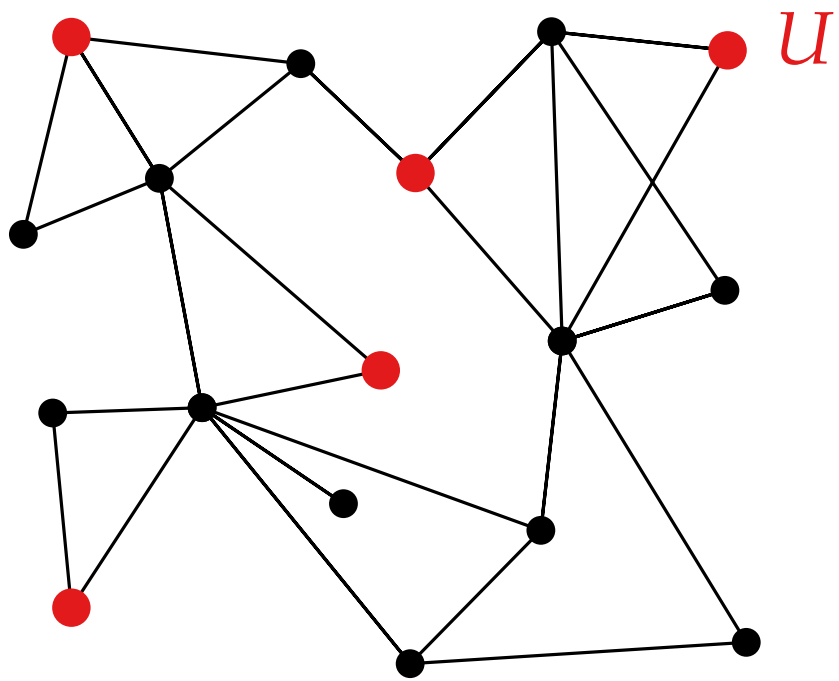

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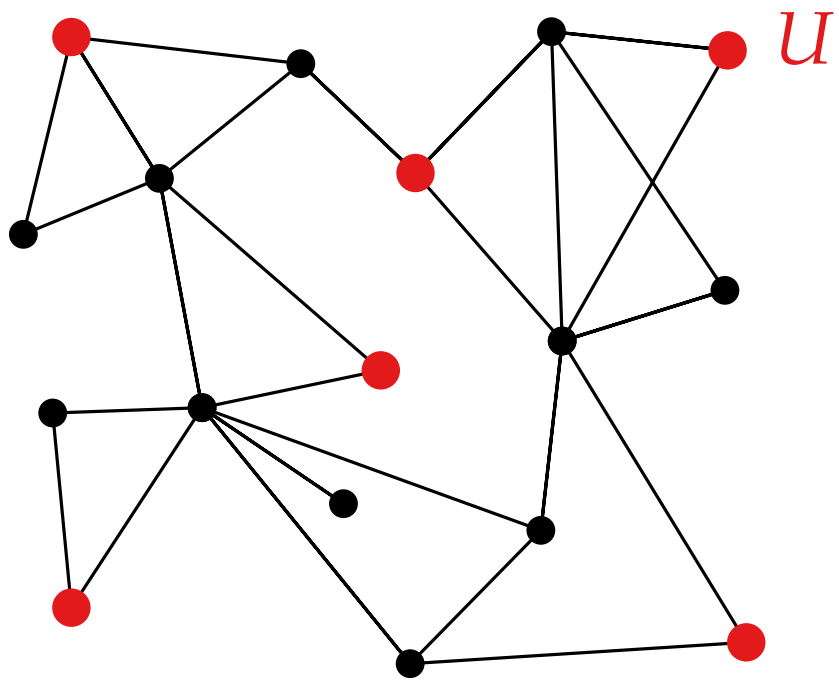

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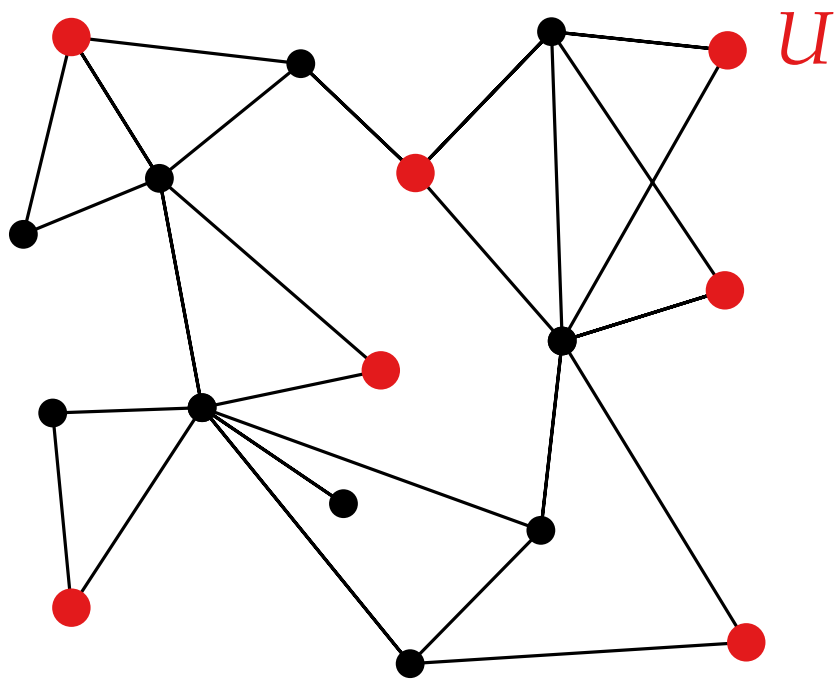

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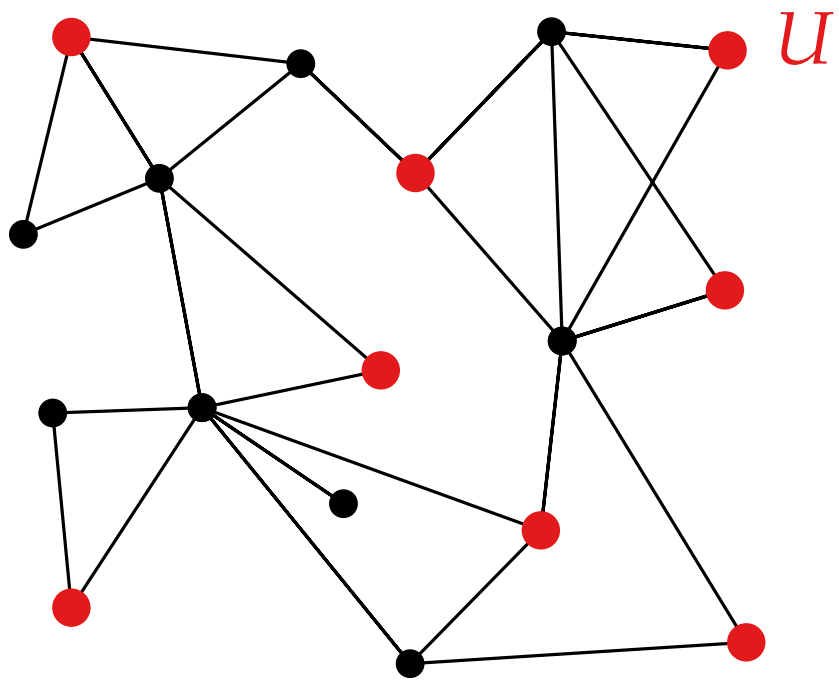

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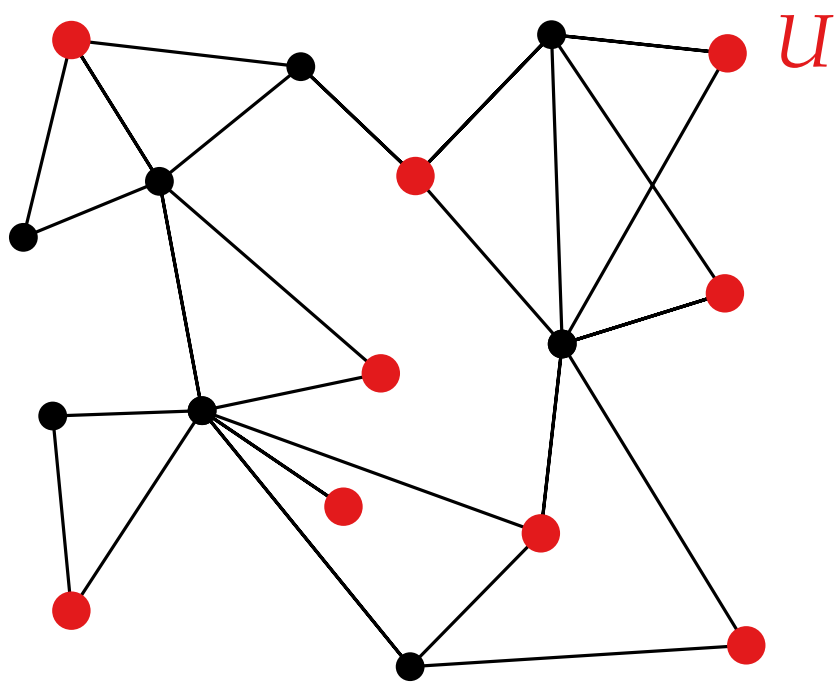

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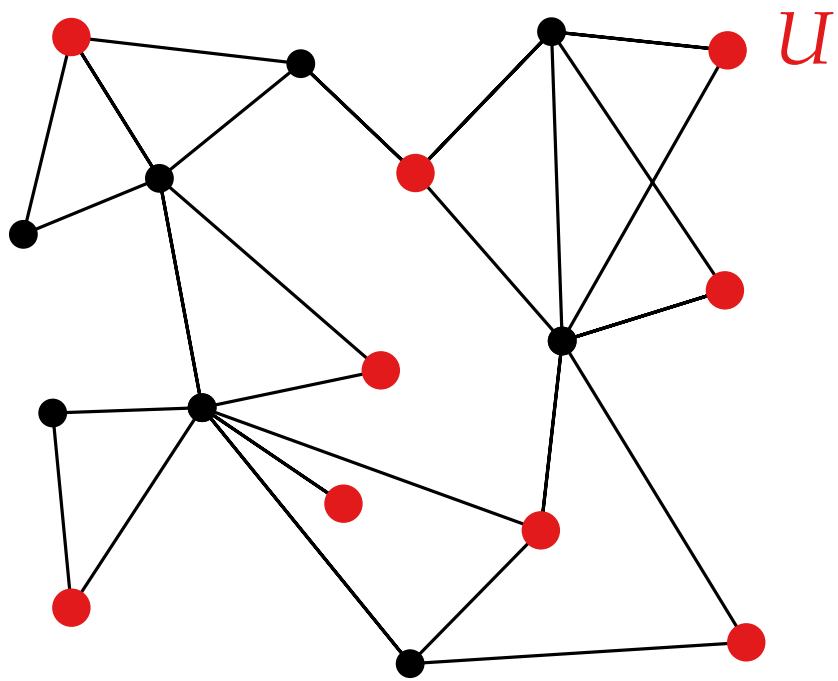

Obs. Maximal independent sets are dominating sets :-)

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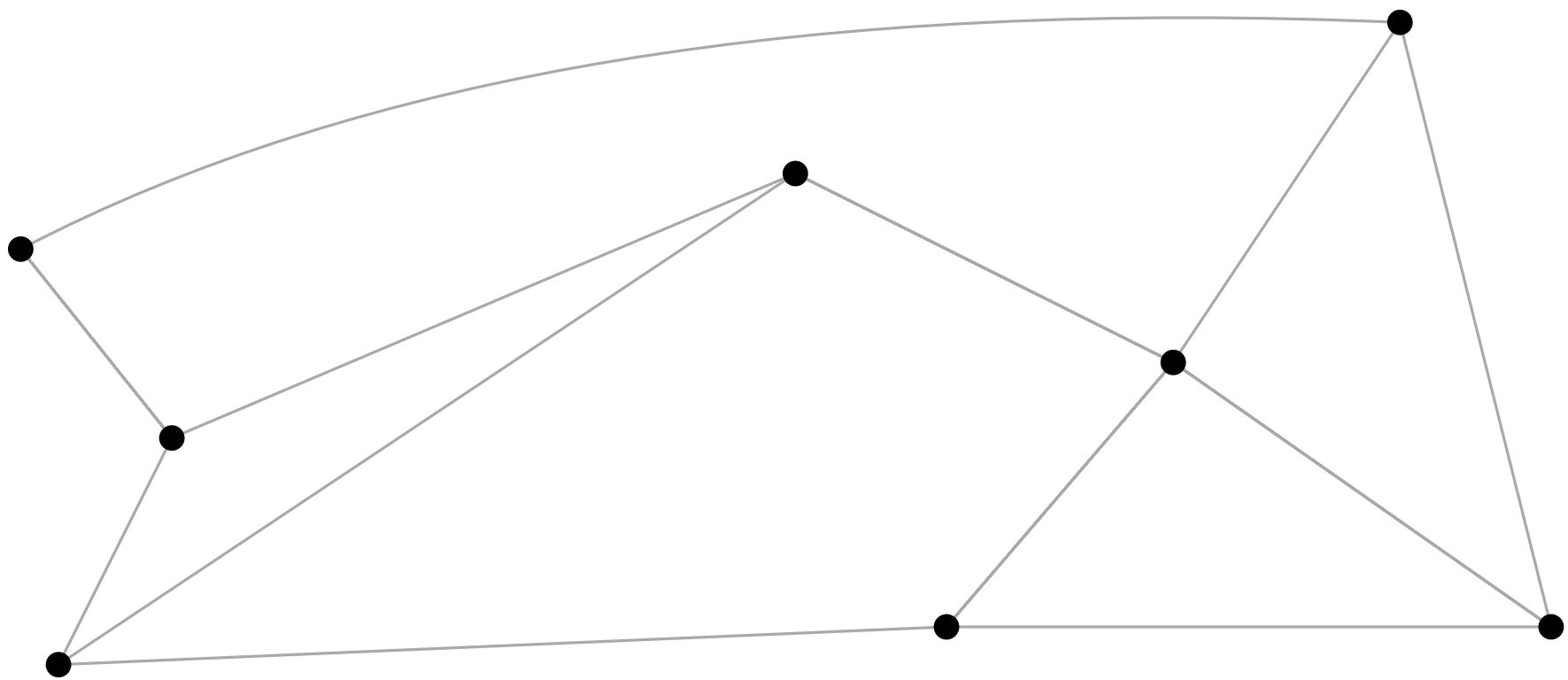

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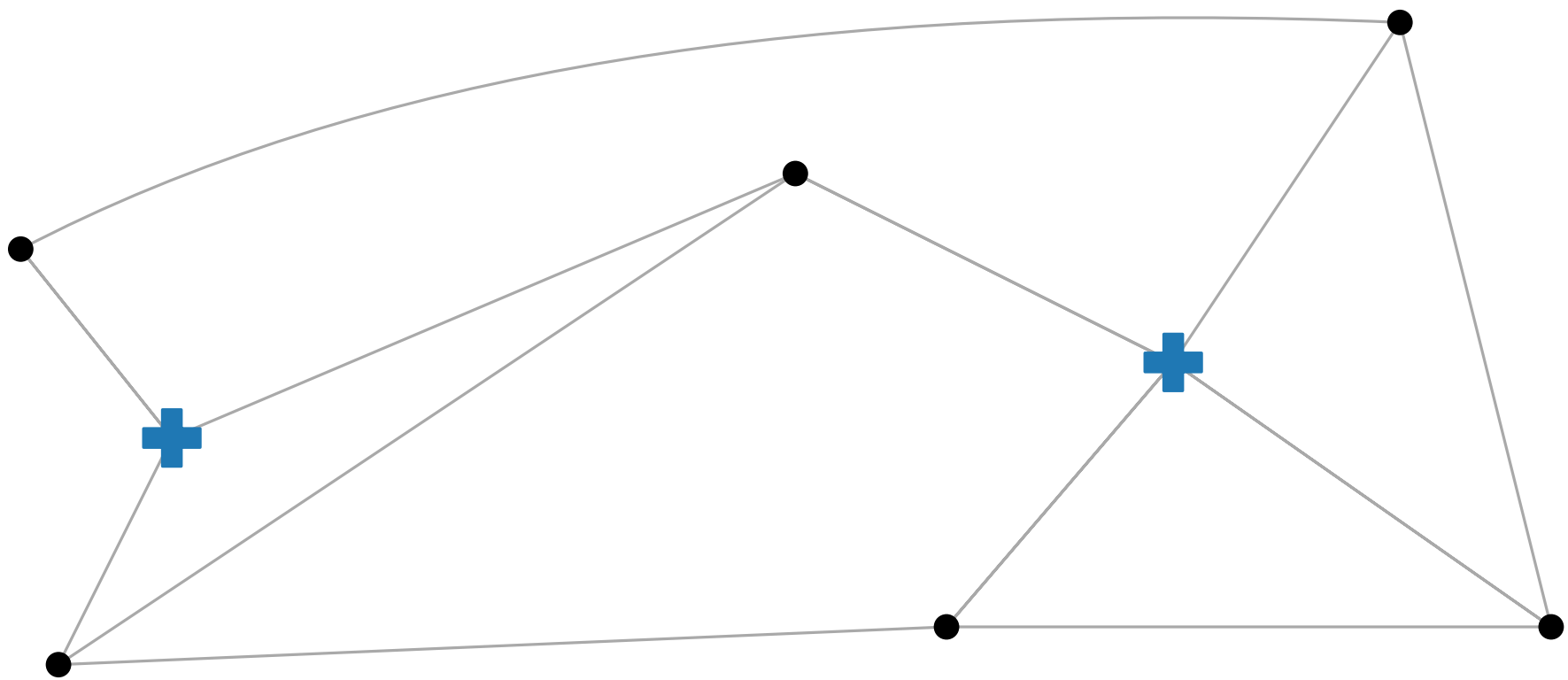

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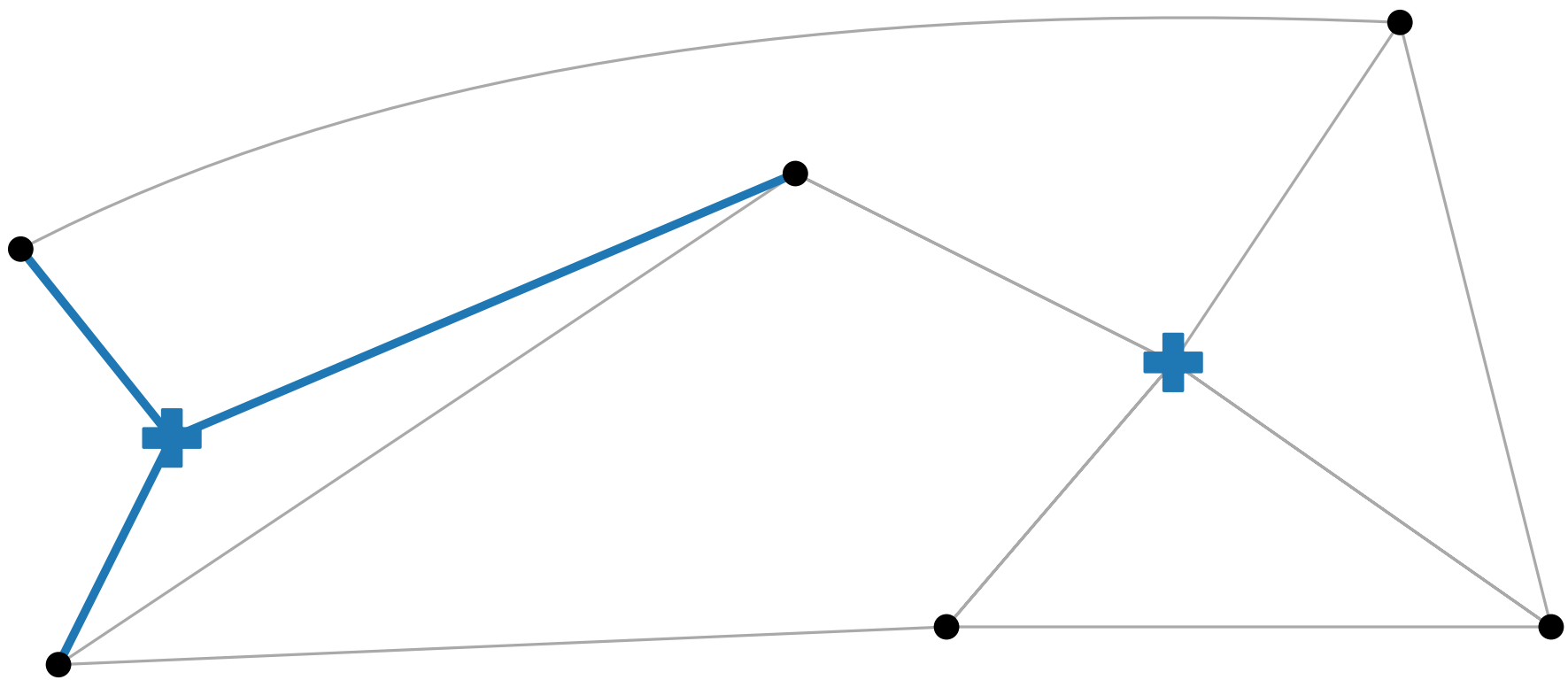

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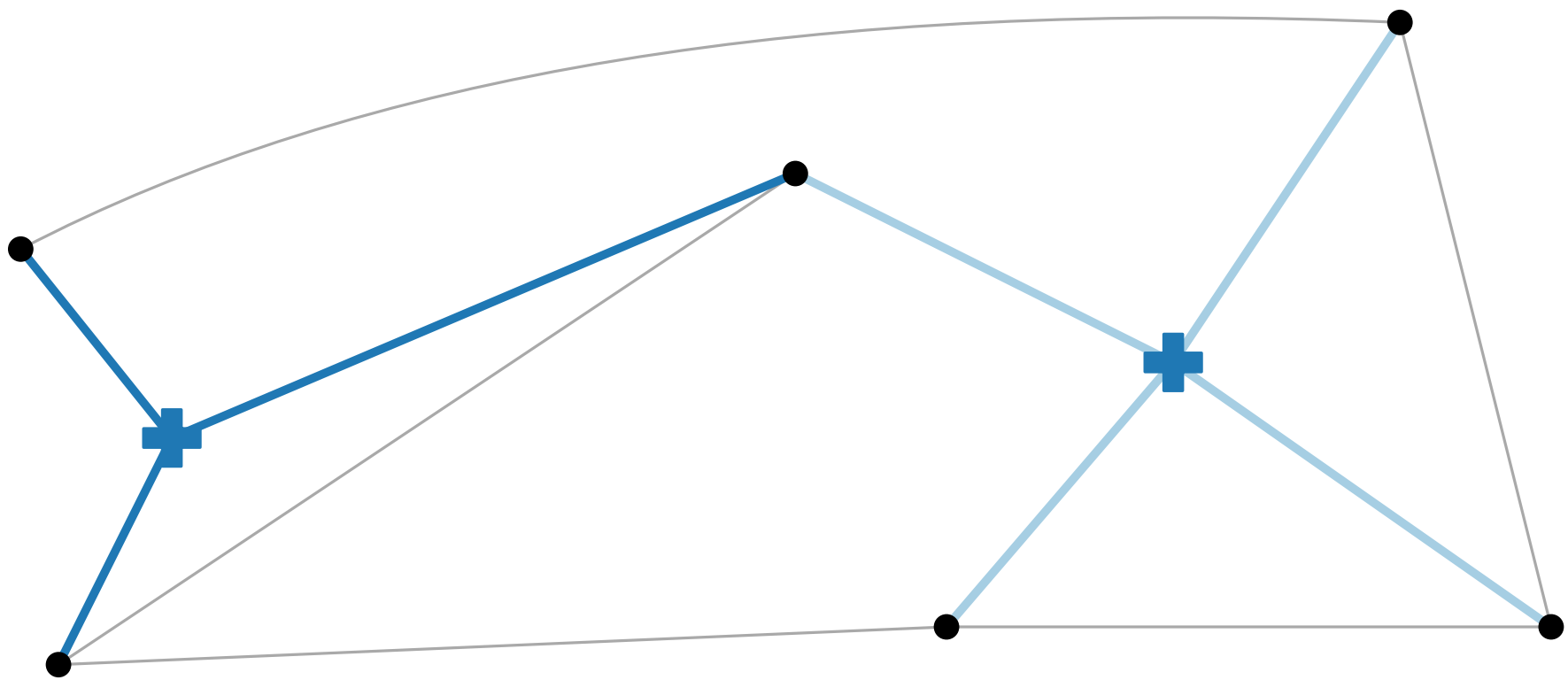

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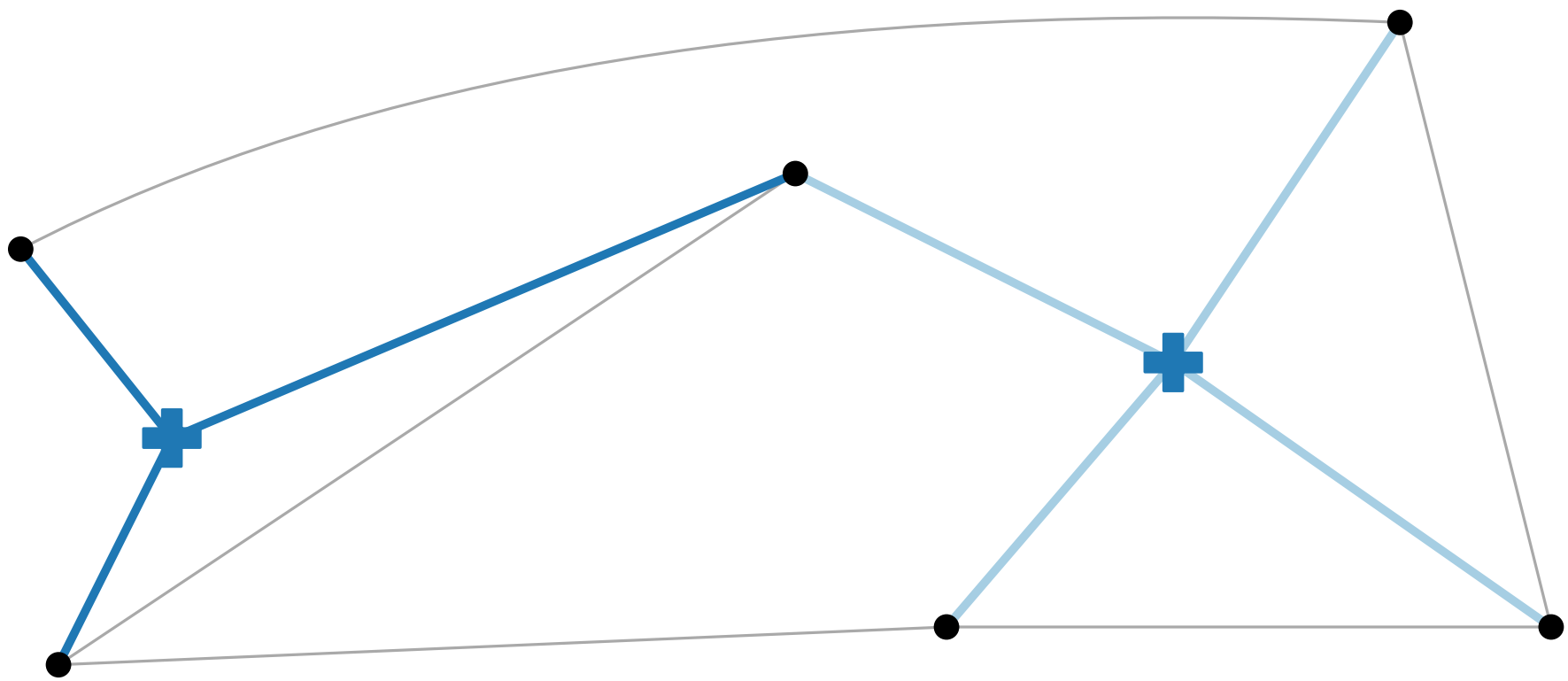

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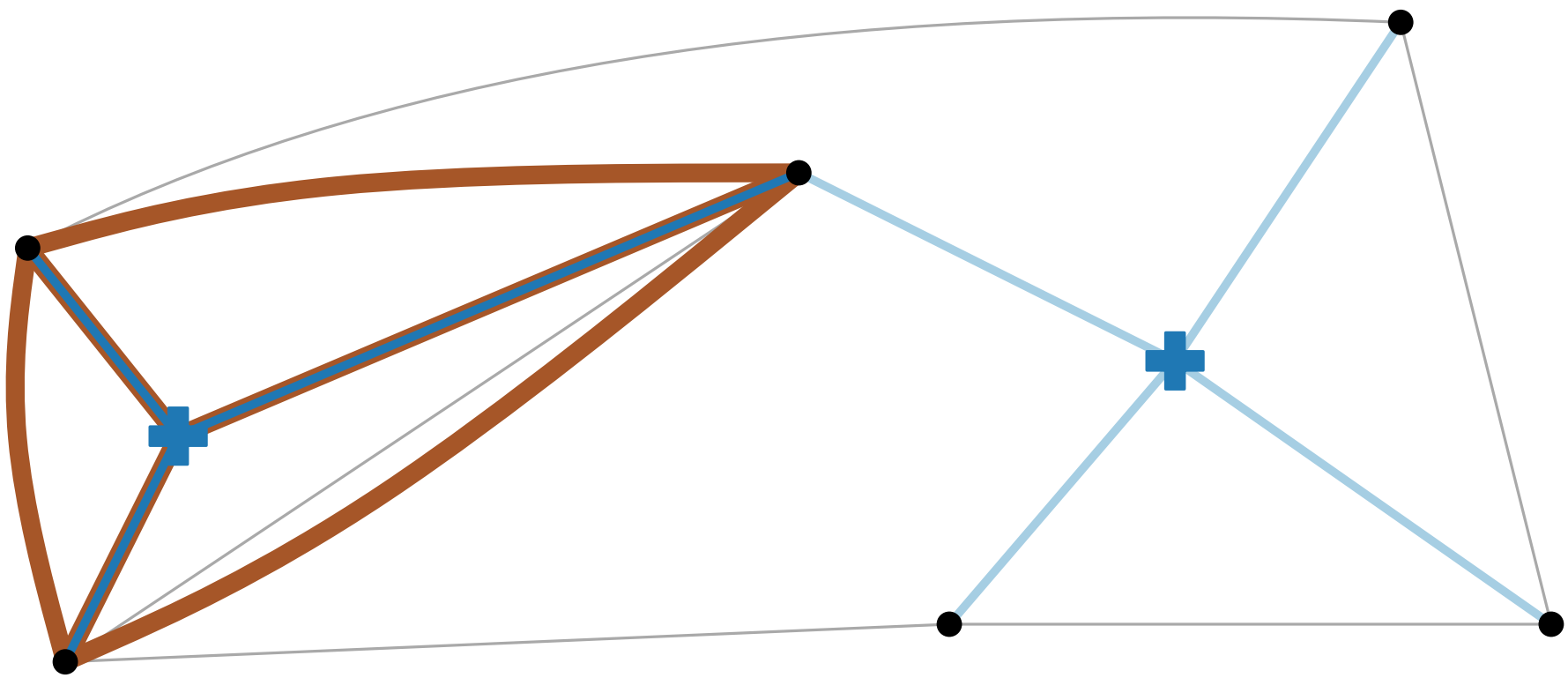

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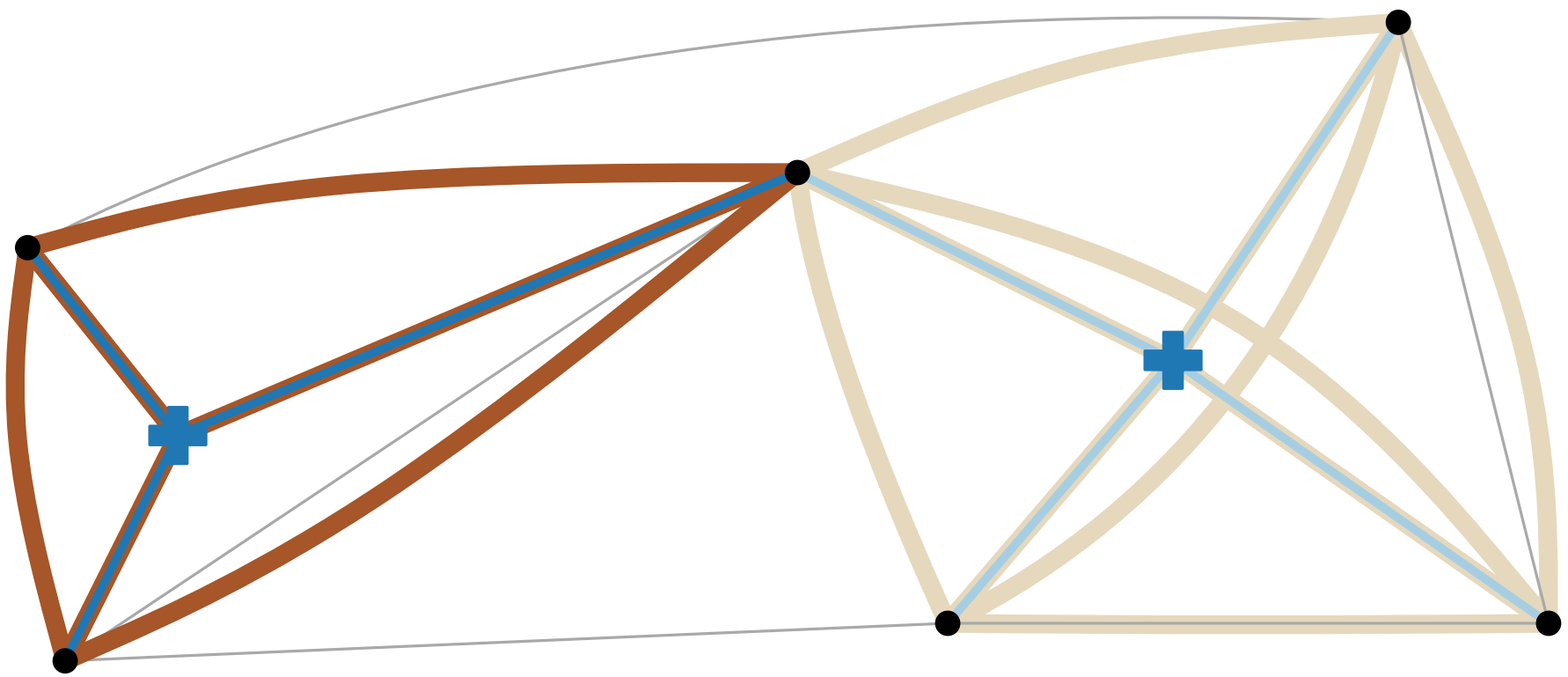

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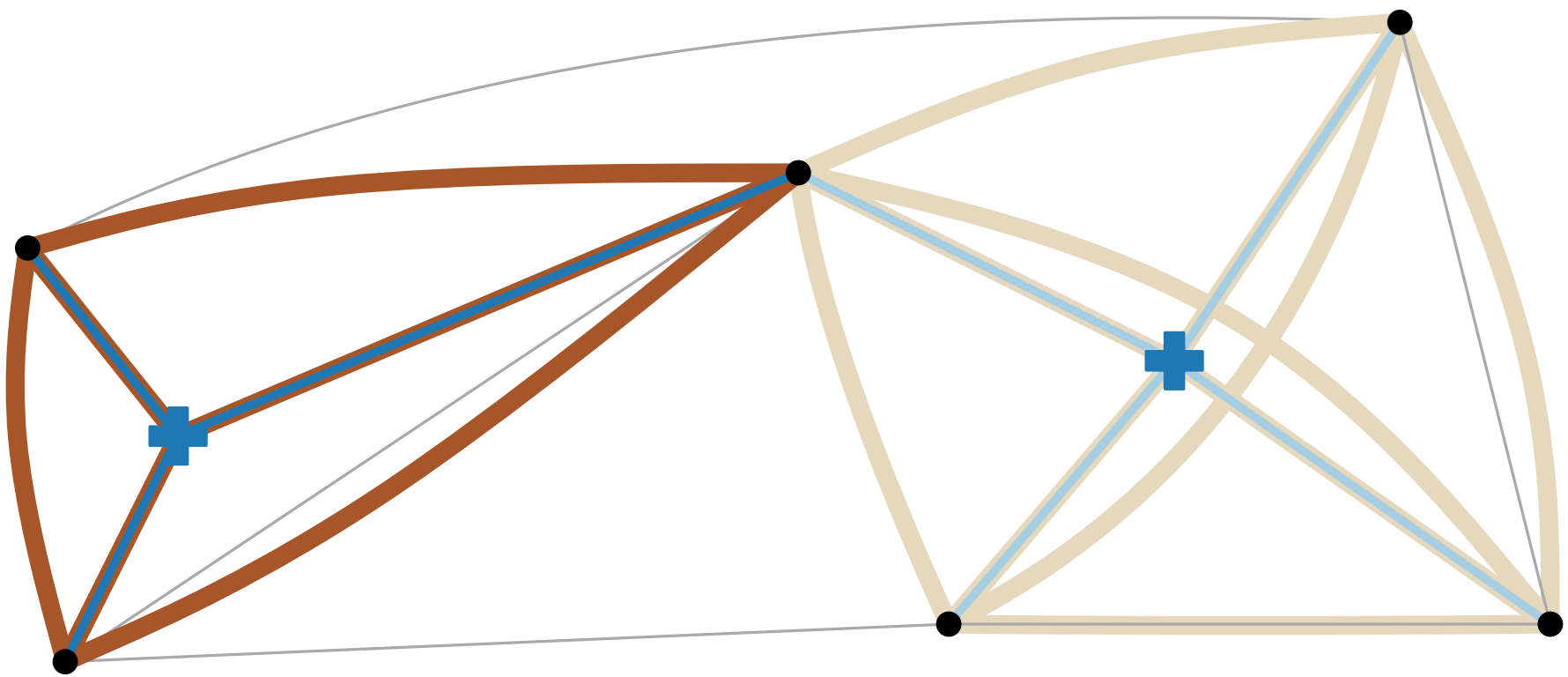

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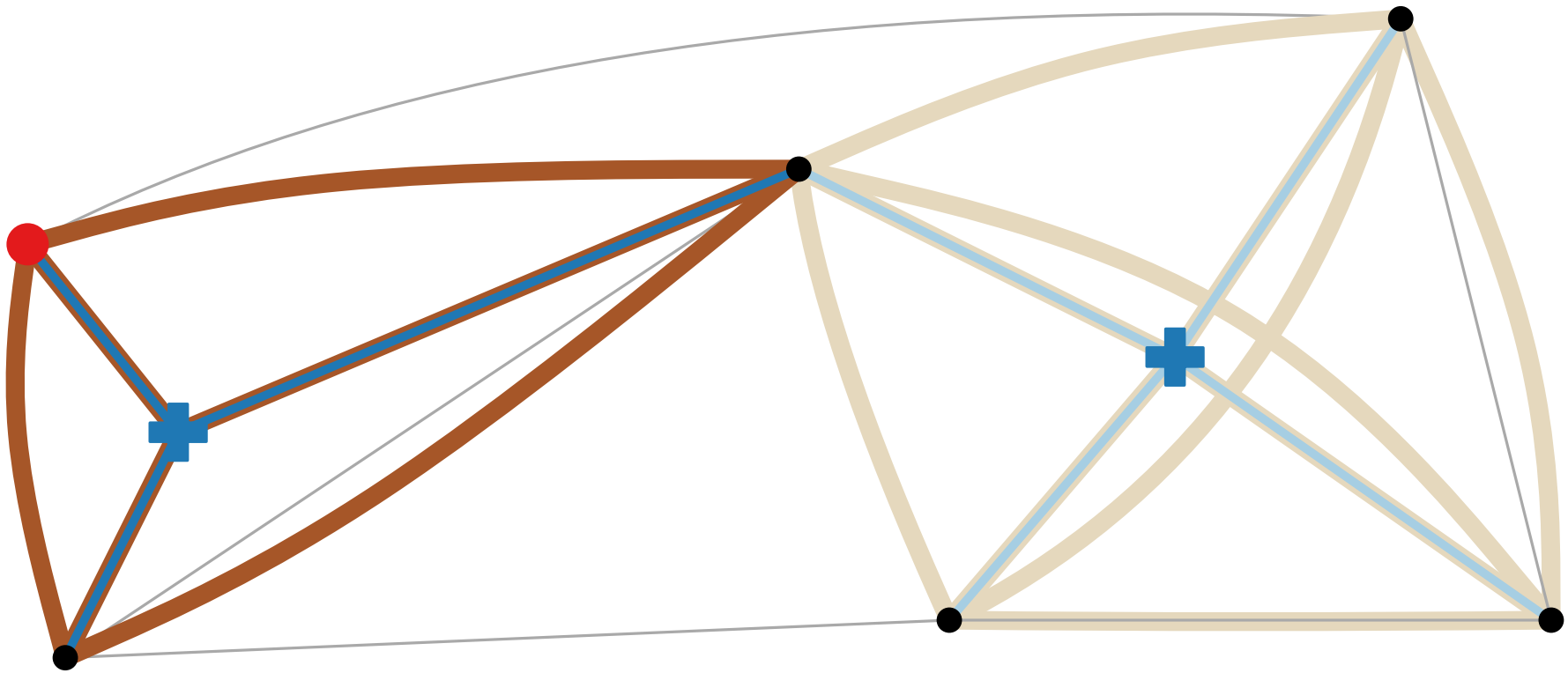

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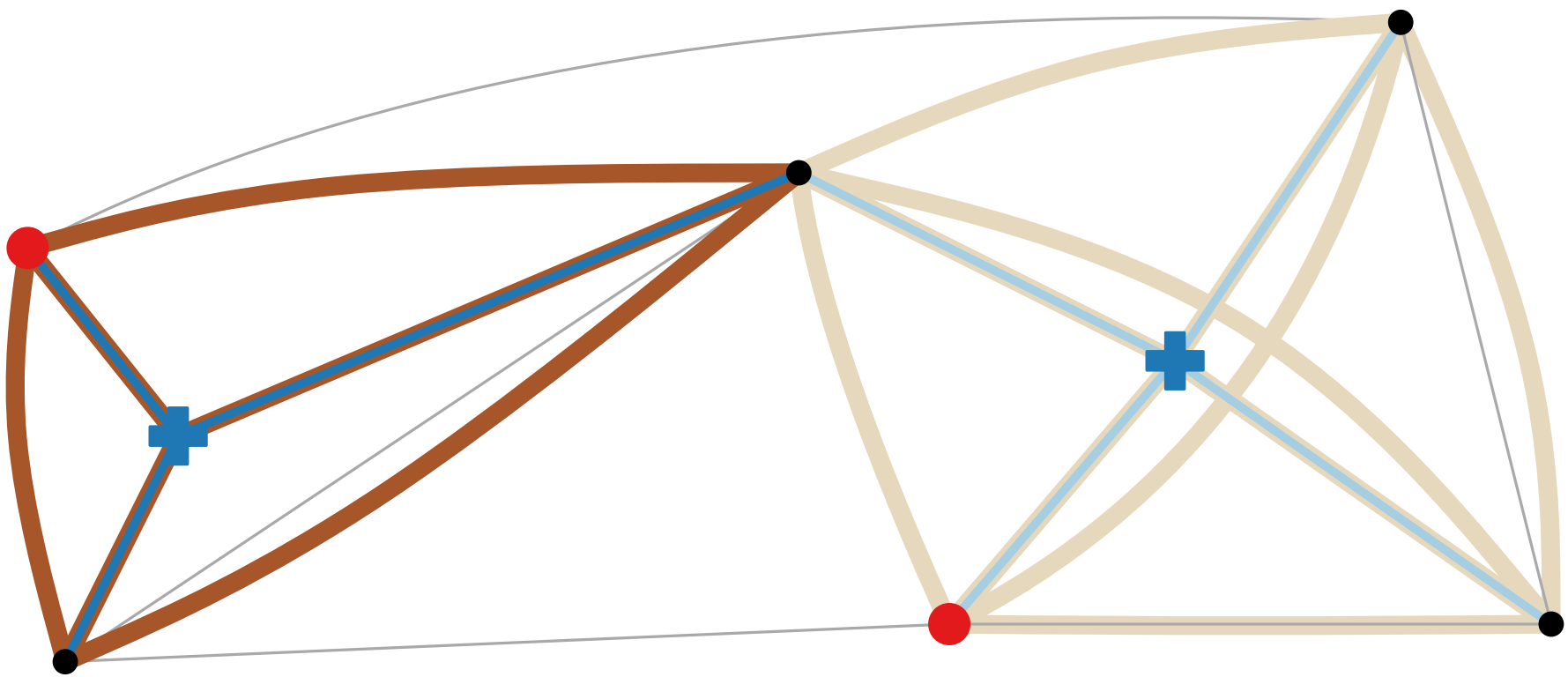

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Factor-2 approx. for metric k-Center

Algorithm Metric-k-Center

Sort the edges of G by cost: $c(e_1) \leq \ldots \leq c(e_m)$

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Lemma. For j provided by the Algorithm, we have $c(e_j) \leq \mathrm{OPT}$.

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for $j = 1, \ldots, m$ **do**
  Construct G_j^2
  Find a maximal independent set U_j in G_j^2
  if $|U_j| \leq k$ **then**
    return U_j

Lemma. For j provided by the Algorithm, we have $c(e_j) \leq \mathrm{OPT}$.

Theorem. The above algorithm is a factor-2 approximation algorithm for the metric k-Center problem.

Can we do better . . . ?

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What about a tight example?

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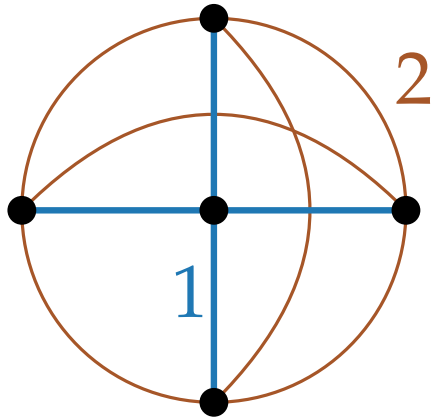

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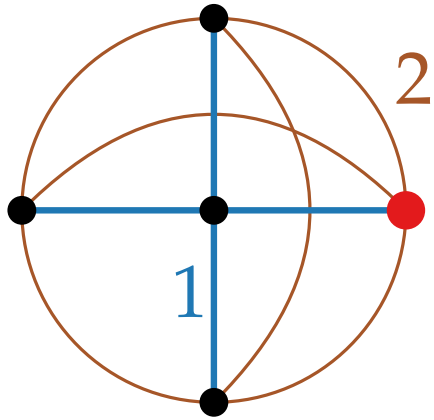

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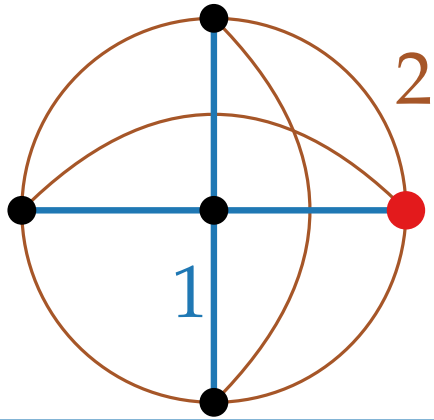

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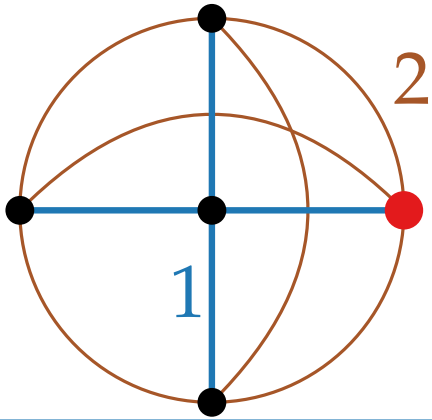

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Proof. Reduce from dominating set to metric k-Center.

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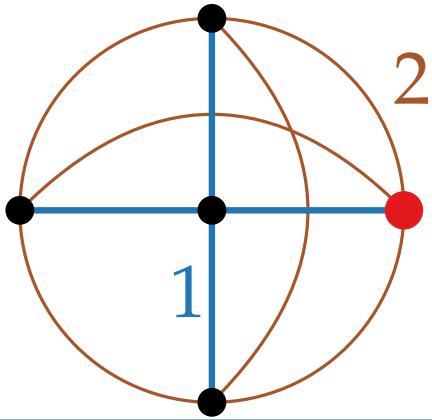

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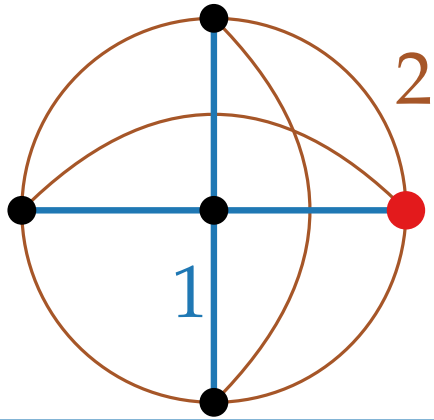

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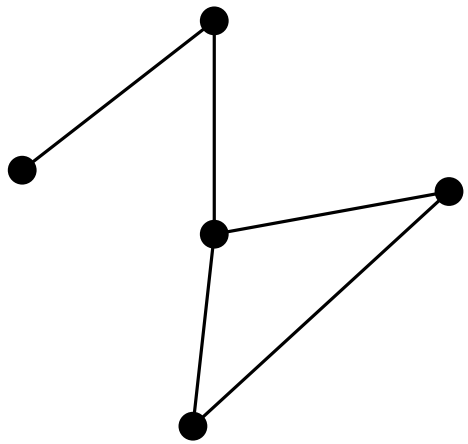

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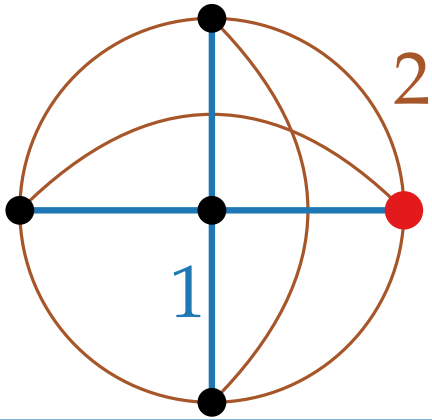

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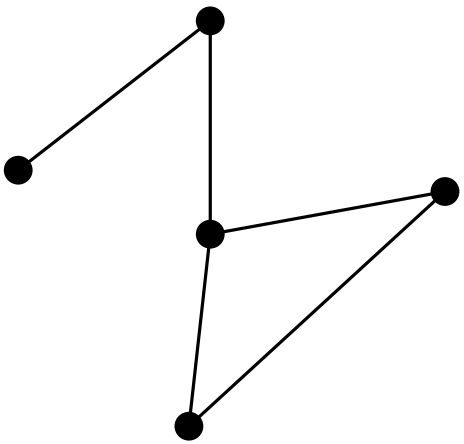

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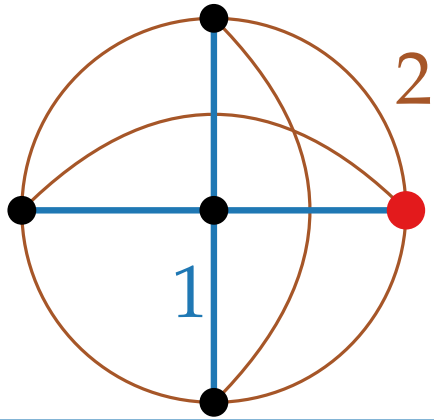

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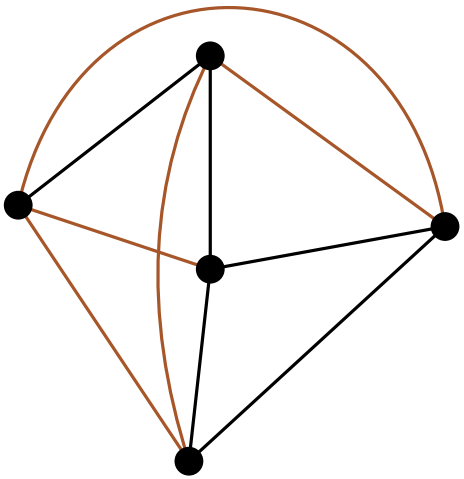

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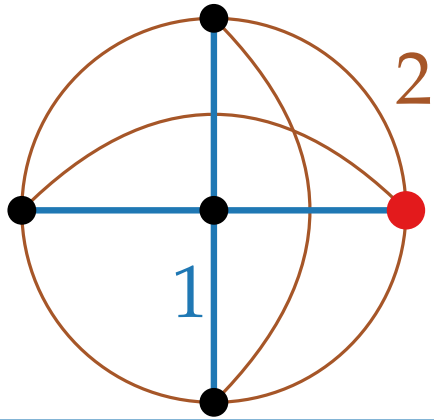

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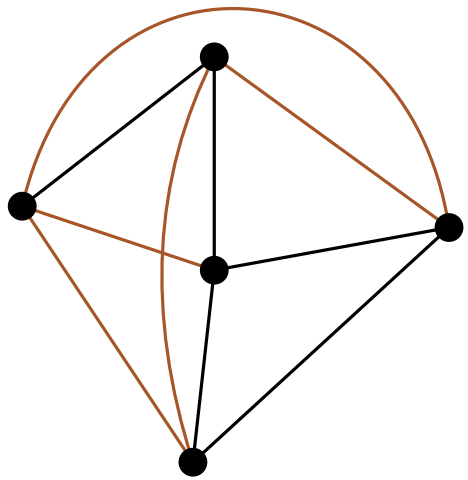

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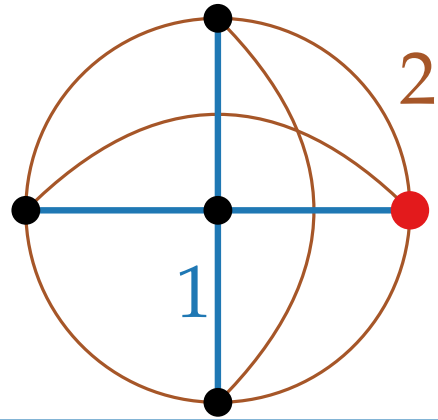

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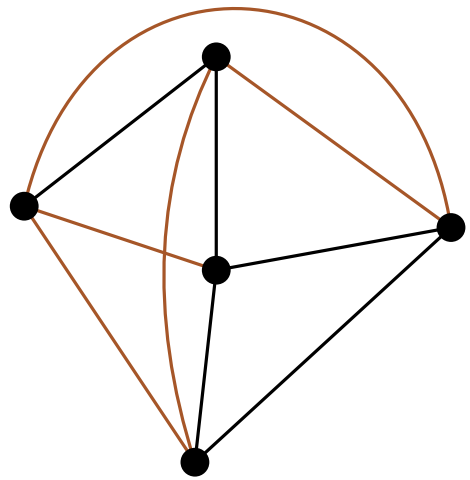

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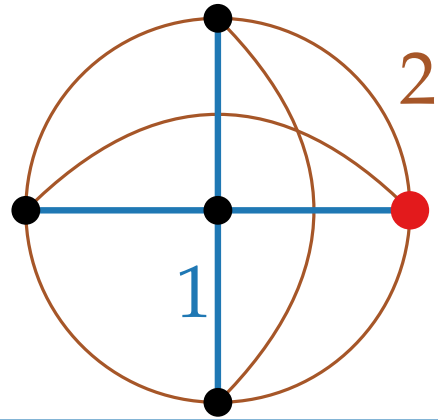

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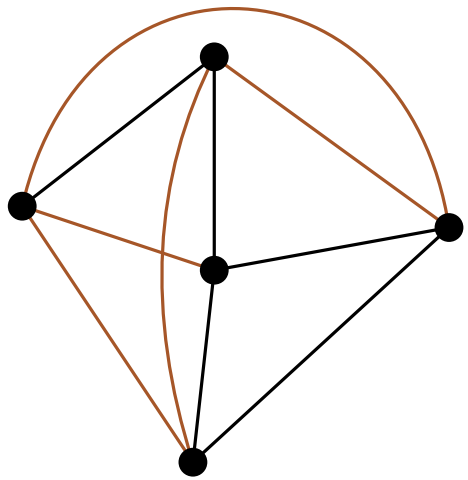

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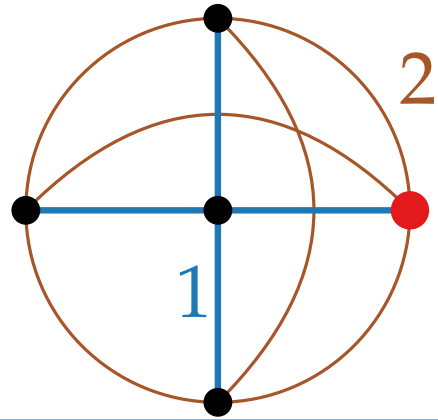

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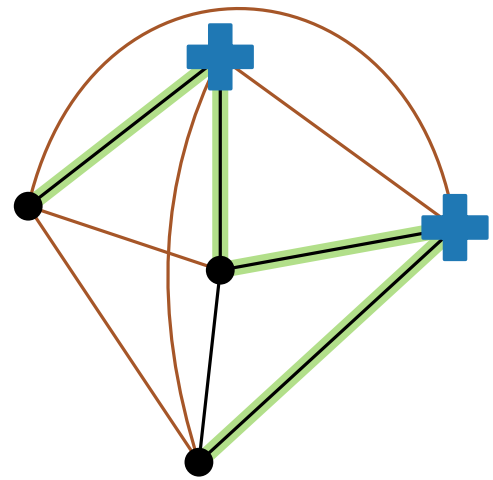

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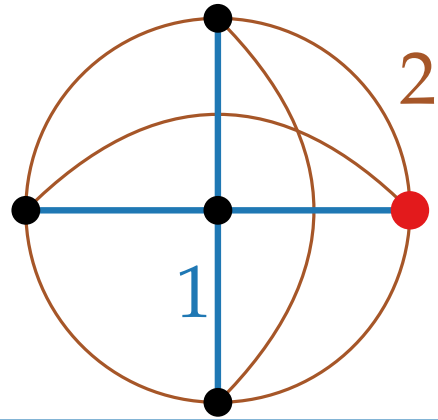

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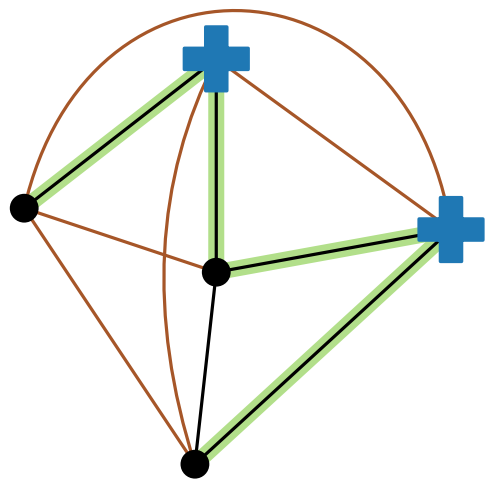

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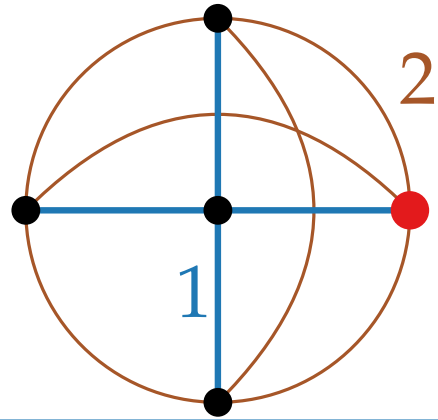

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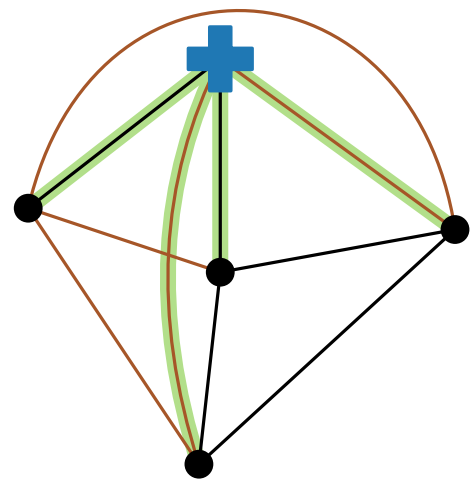

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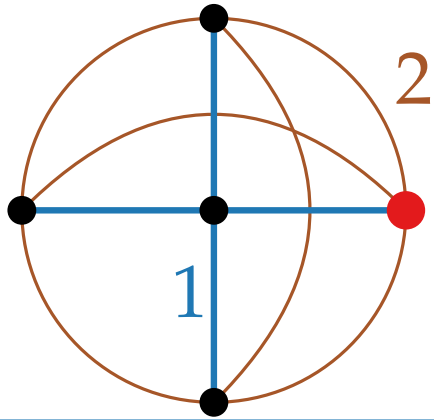

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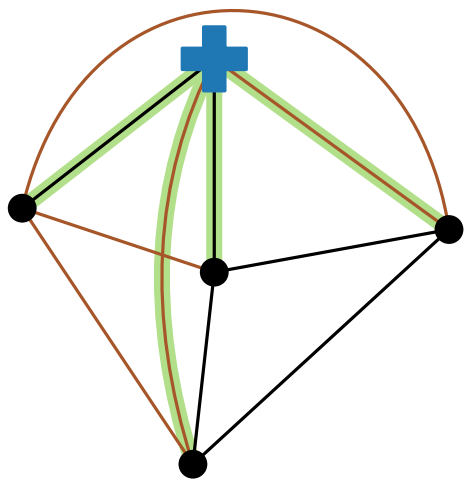

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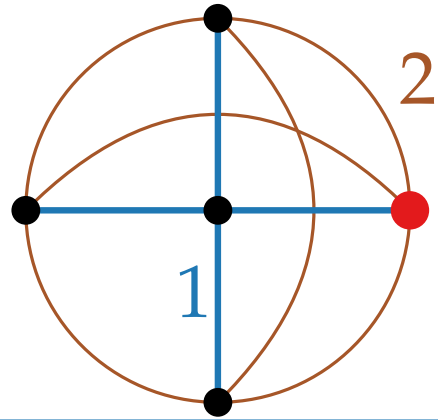

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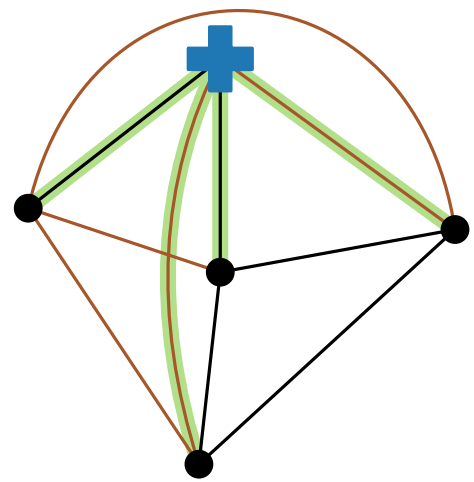

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Metric k-Center problem

Given: A complete graph $G = (V, E)$ with metric edge costs $c\colon E \to \mathbb{Q}_{\geq 0}$ and a natural number $k \leq |V|$.

For each vertex set $S \subseteq V$, $c(v, S)$ is the cost of the cheapest edge from v to the a vertex in S.

Find: A k-element vertex set S, such that $\text{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.

Metric k-CENTER problem

weighted

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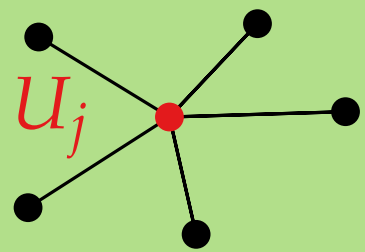

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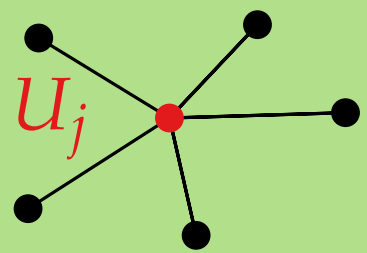

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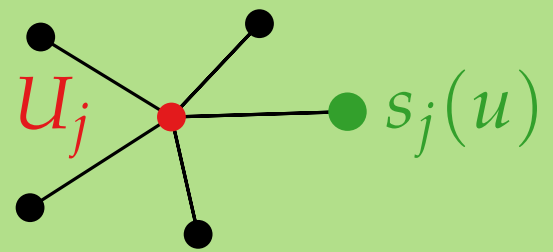

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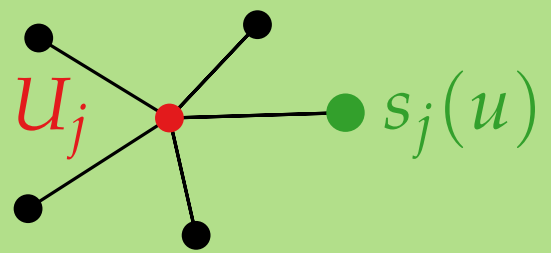

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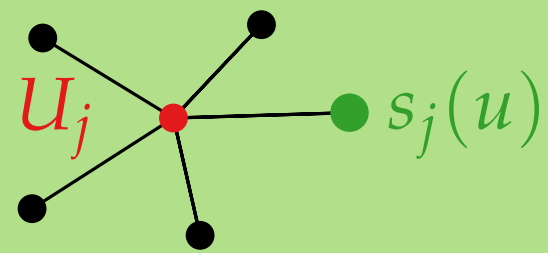

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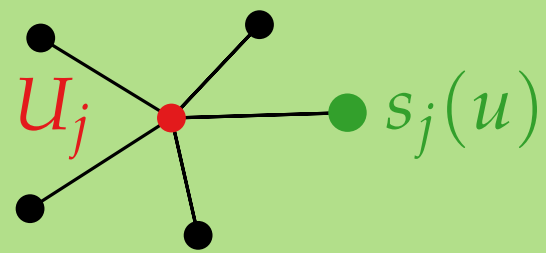

$s_j(u)$:= lightest node in $N_{G_j}(u) \cup \{u\}$

The weighted version

Algorithm metric-weighted-CENTER

Sort the edges of G by cost : $c(e_1) \leq \ldots \leq c(e_m)$

for $j = 1, \ldots, m$ **do**

  Construct G_j^2

  Find a maximal independent set U_j in G_j^2

  Compute $S_j := \{ s_j(u) \mid u \in U_j \}$

  if ~~$|U_j| \leq k$~~ **then** $w(S_j) \leq W$

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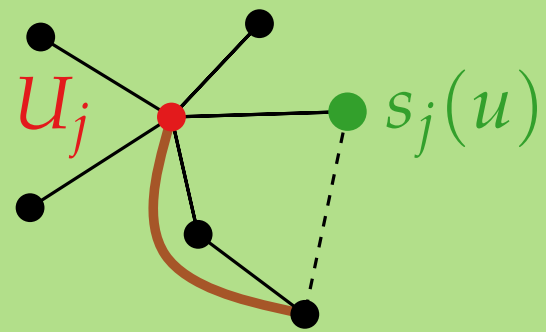

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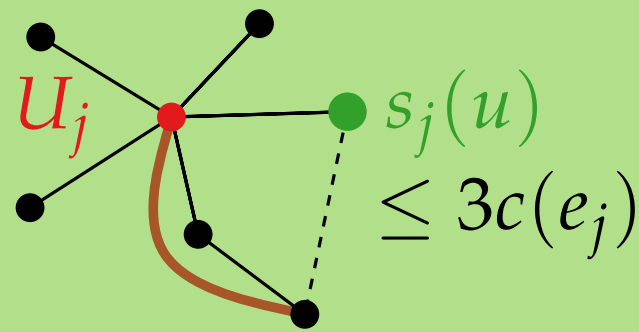

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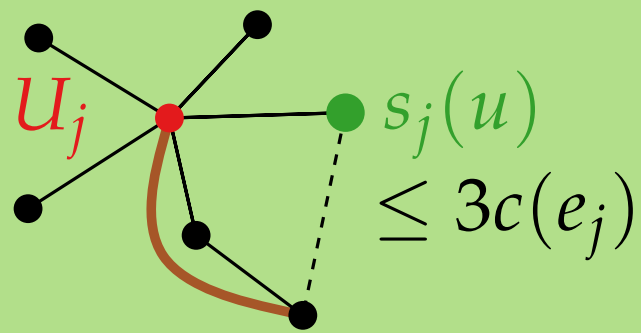

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Theorem. The above is a factor-3 approximation algorithm for the metric weighted-CENTER problem.