Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Lehrstuhl für
INFORMATIK I
Effiziente Algorithmen und
wissensbasierte Systeme

Institut für Informatik

Advanced Algorithms

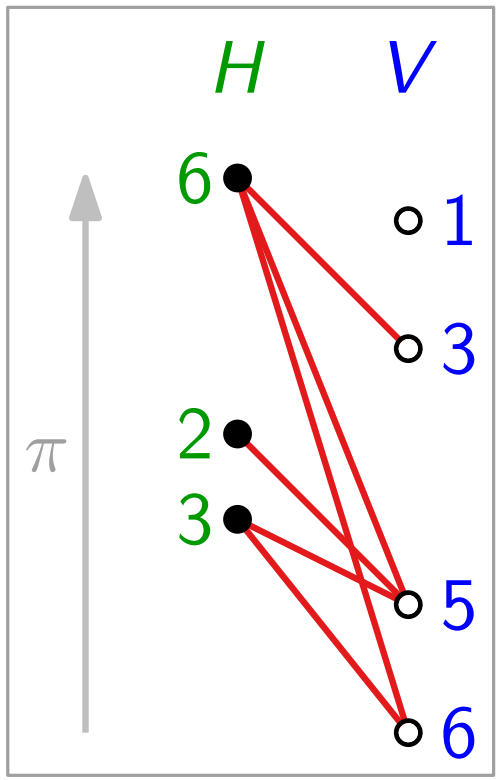
Winter term 2019/20

Lecture 7. Shortest Paths in Graphs with Negative Weights

Steven Chaplick & Alexander Wolff *Chair for Computer Science I*

Motivation

Problem. Let $G = (H \cup V, E)$ be a bipartite graph with positive vertex weights $\ell\colon H \cup V \to \mathbb{N}$ and a permutation π of $H \cup V$.



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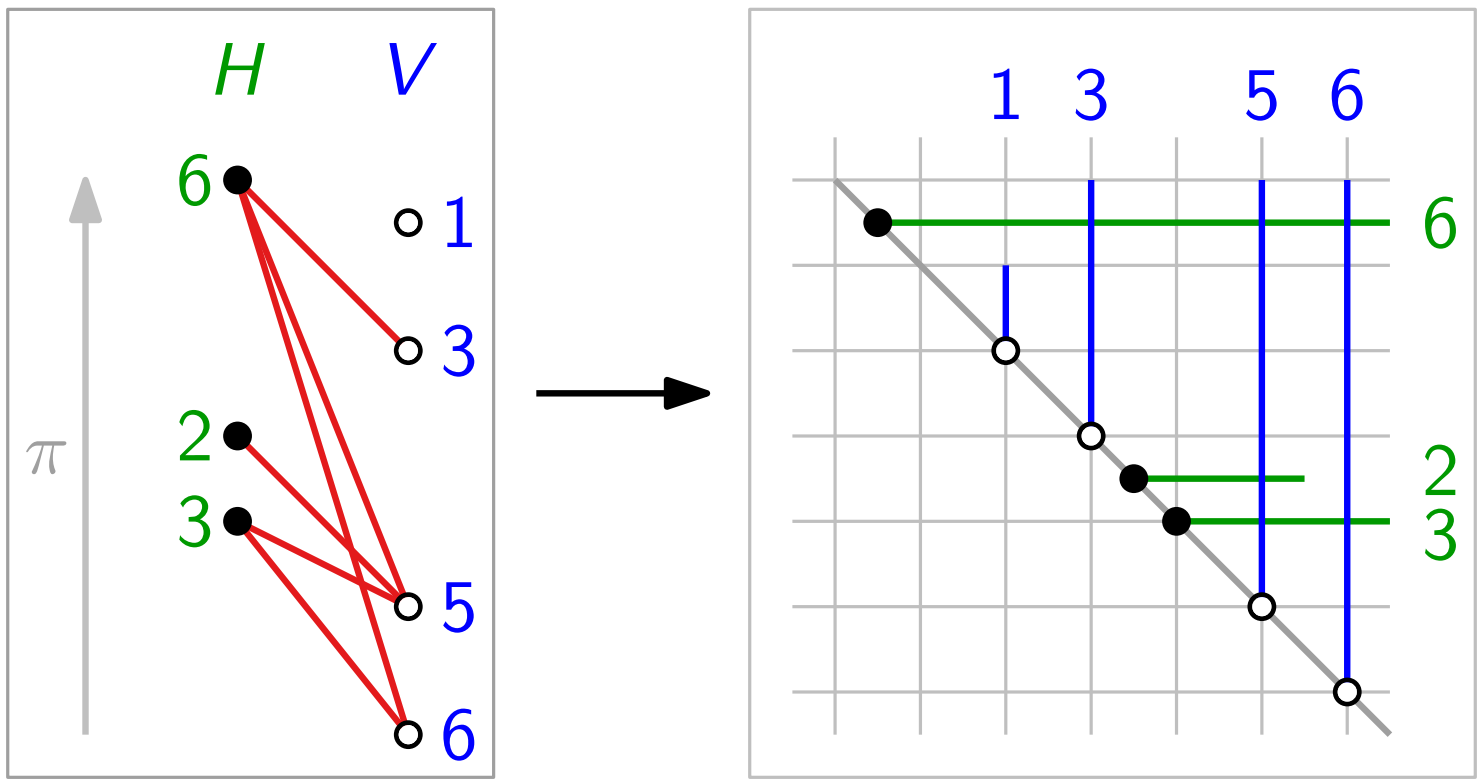
Does G admit a *stick representation* that respects the given stick order π and stick lengths ℓ?



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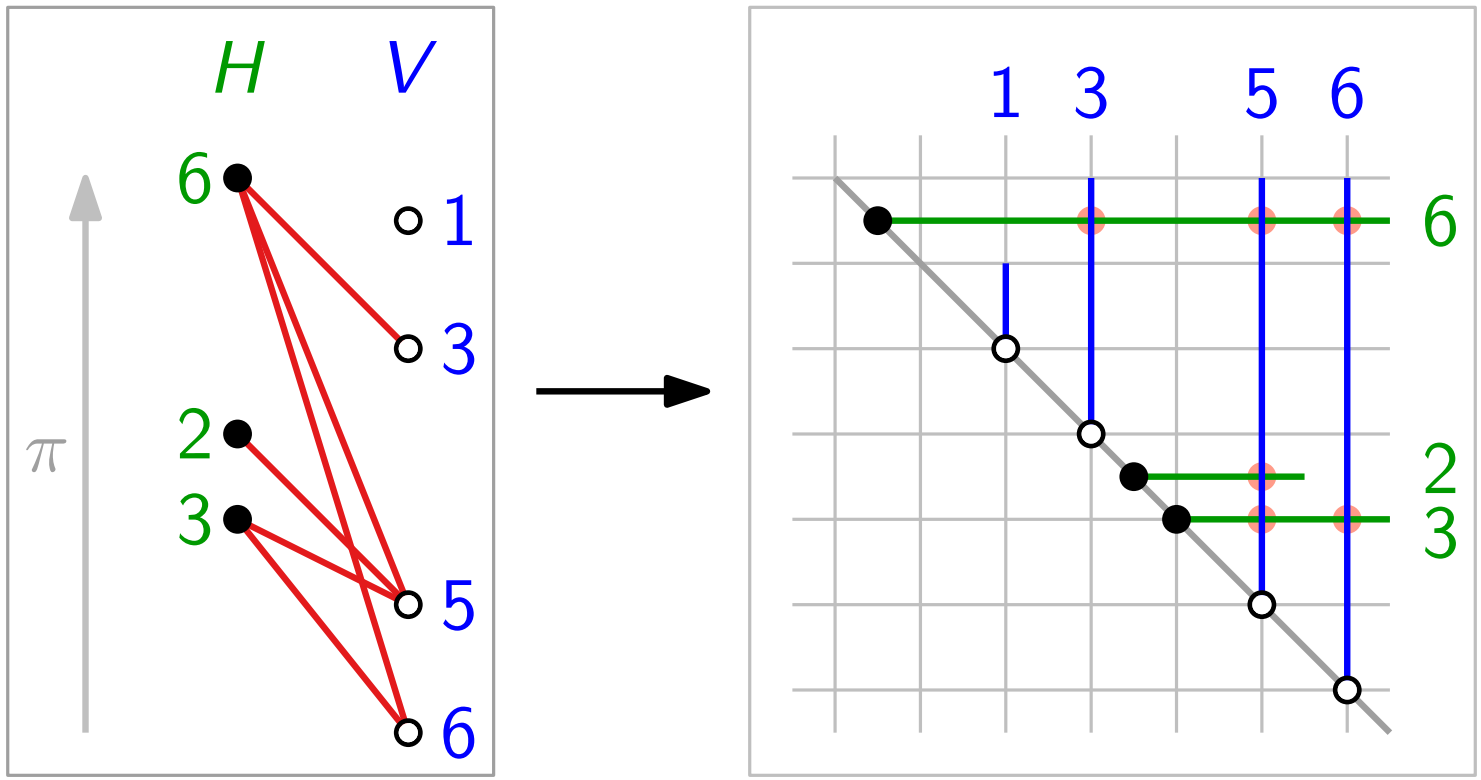

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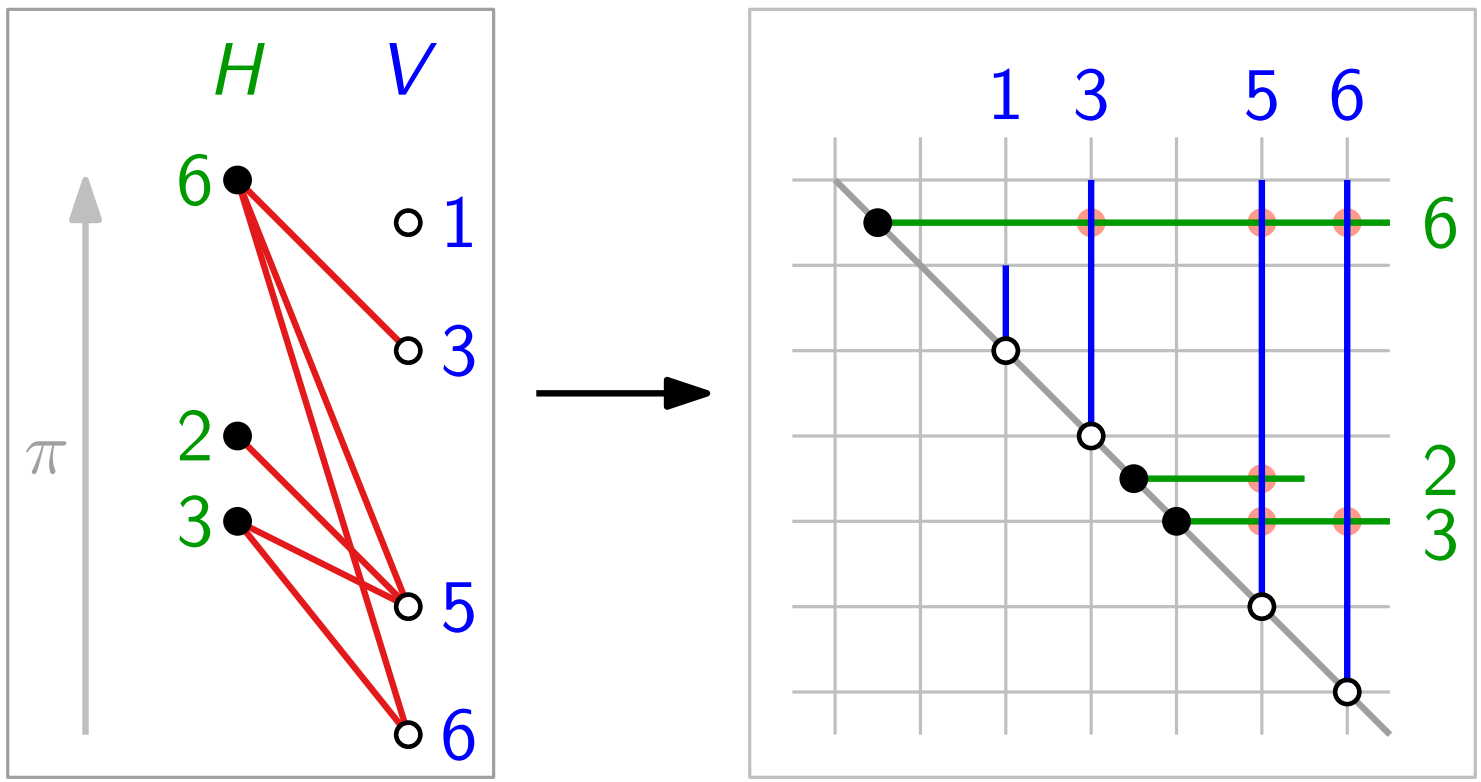
Algorithm.



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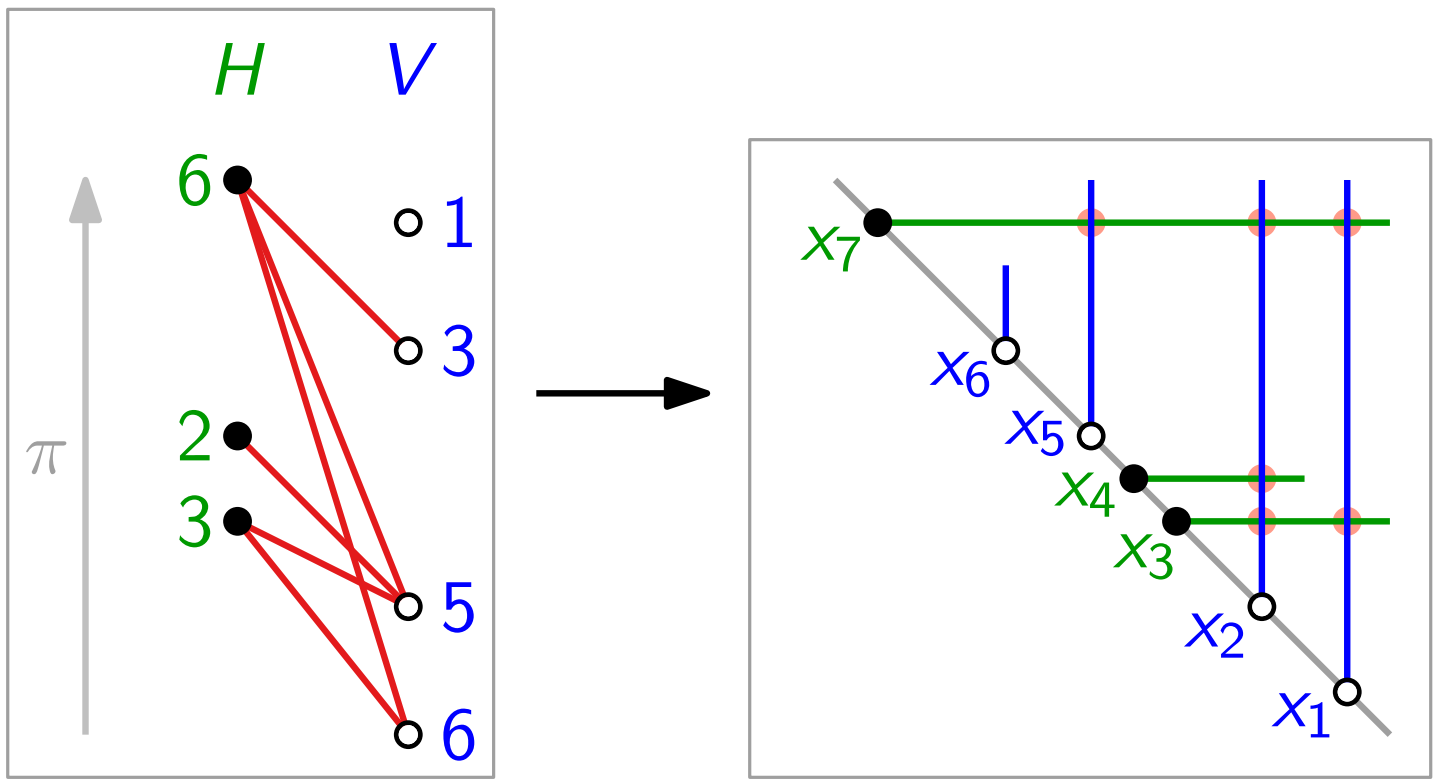

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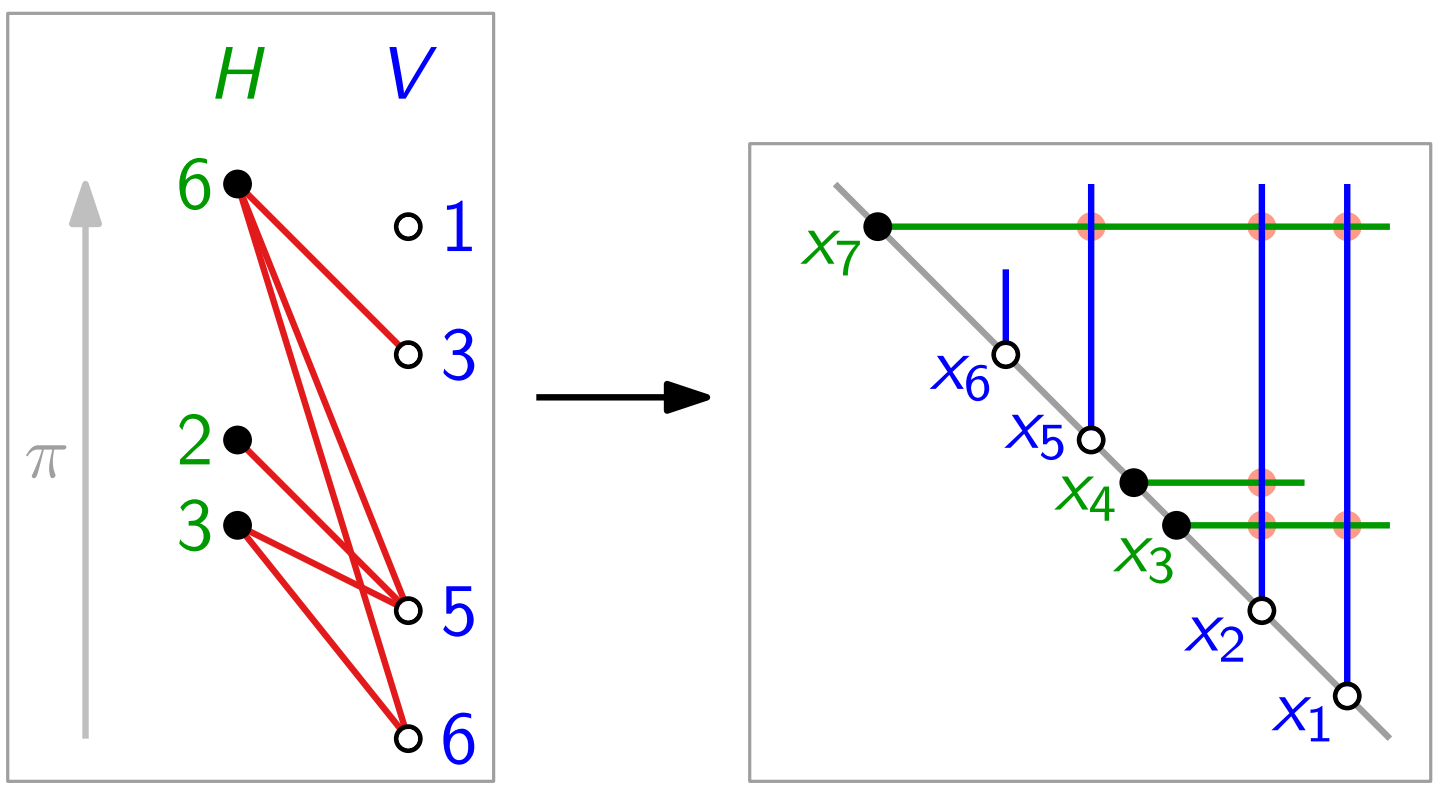
$n = |H| + |V|$

Algorithm. $x_n < \cdots < x_2 < x_1 = 0$



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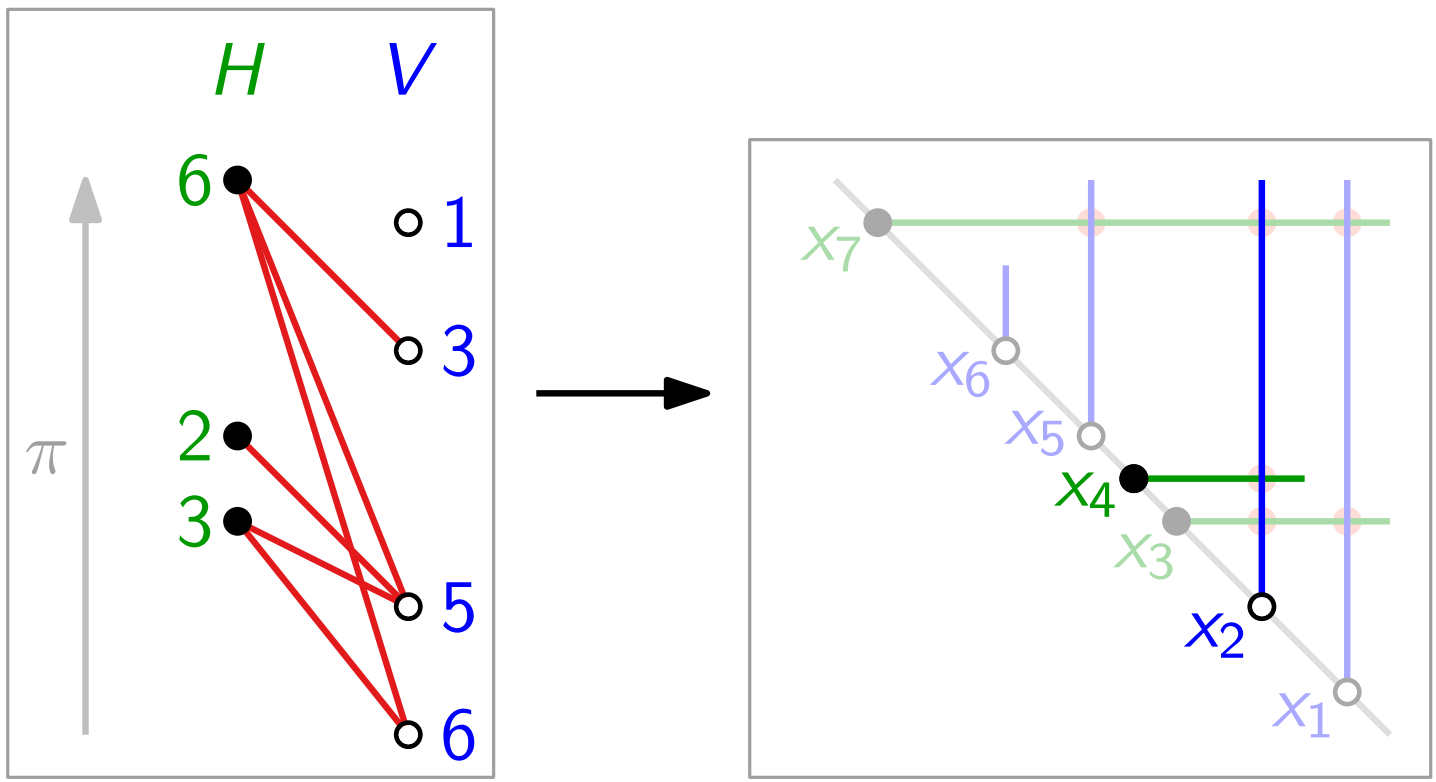
Algorithm. $x_n < \cdots < x_2 < x_1 = 0$. Let $n \geq h > v \geq 1$.



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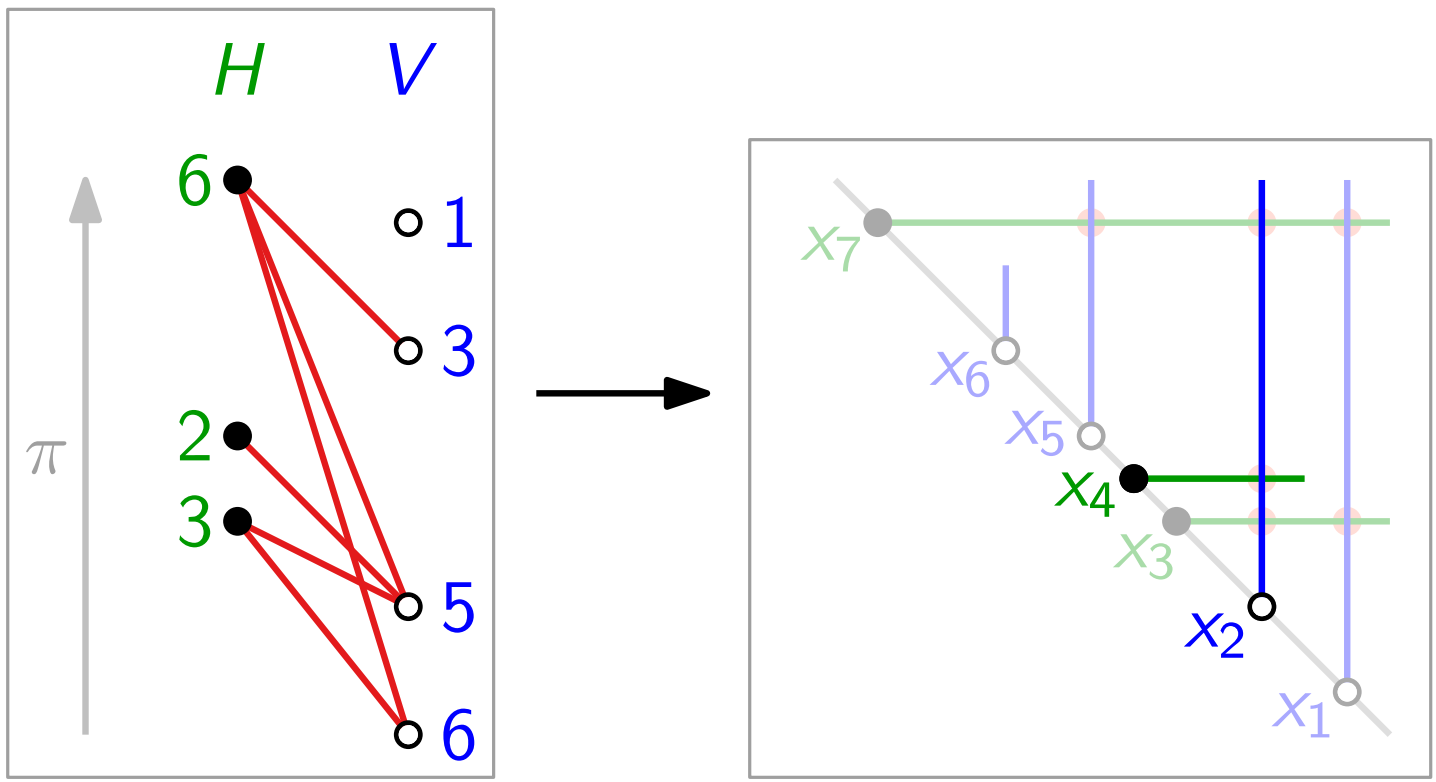

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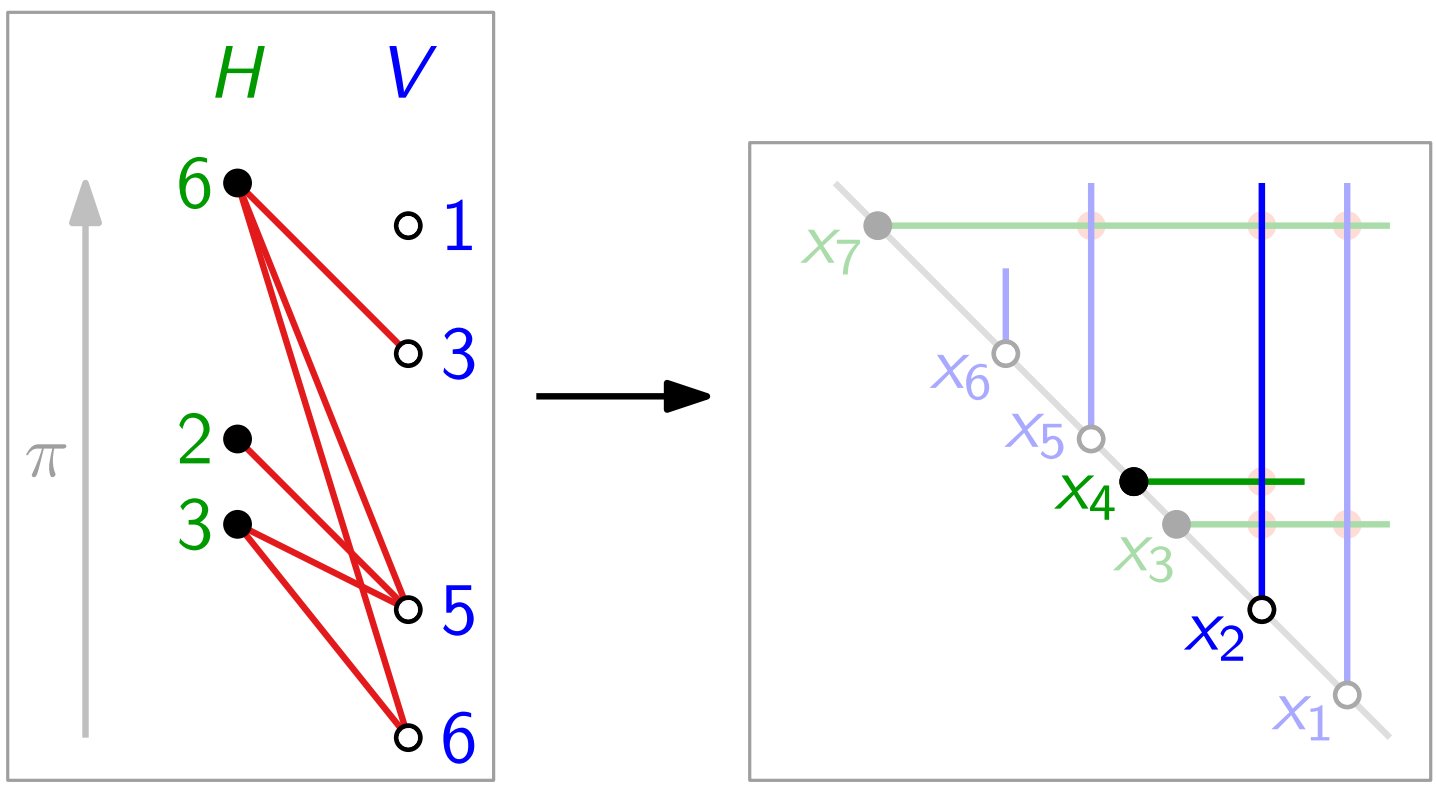

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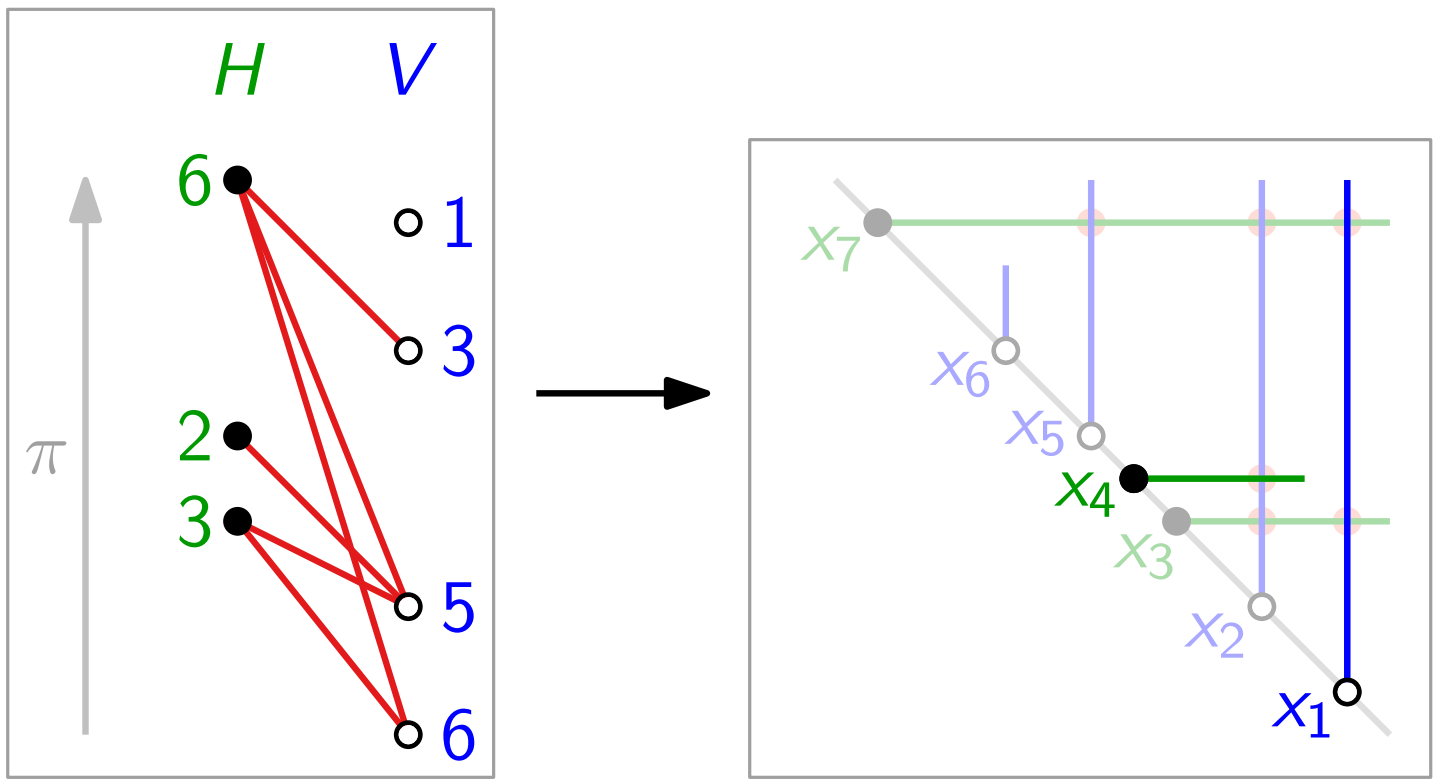

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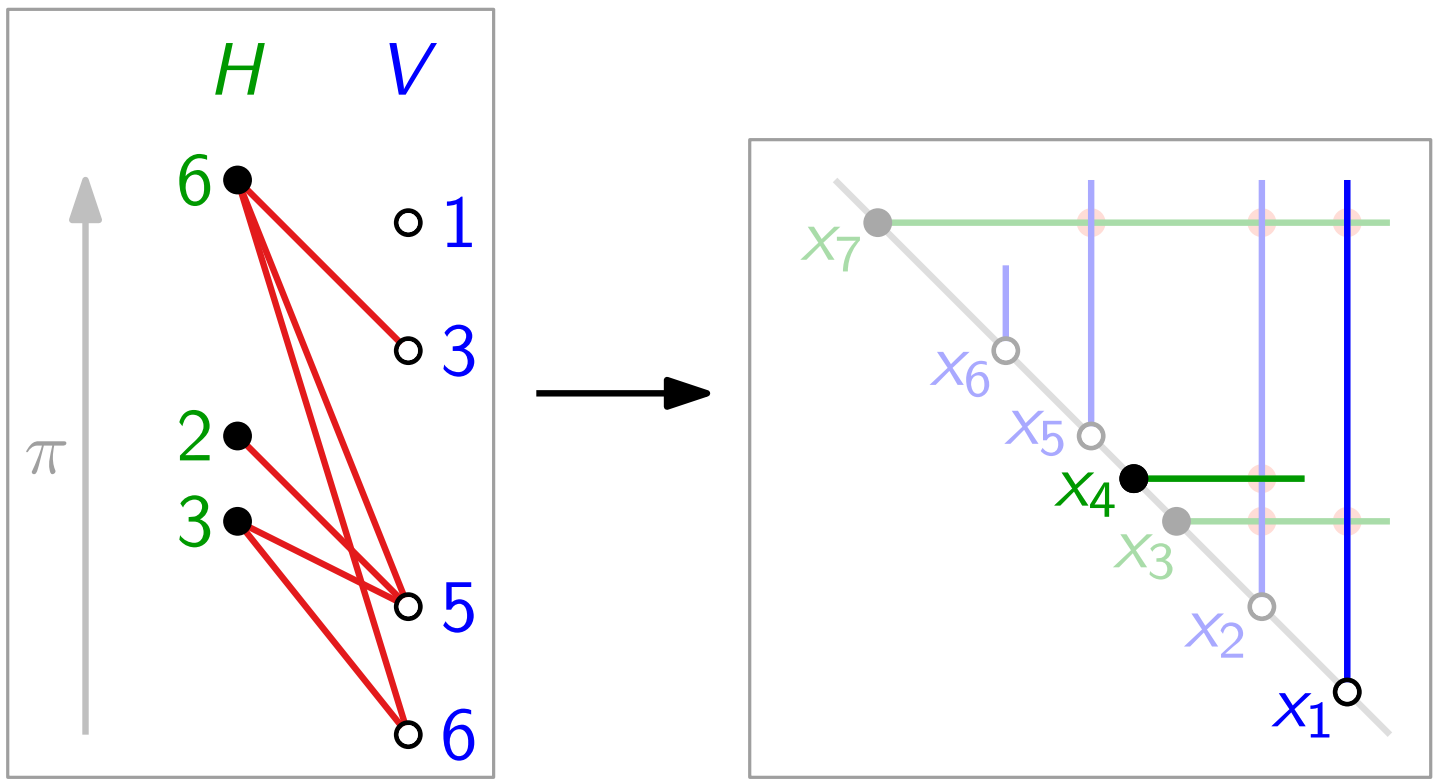

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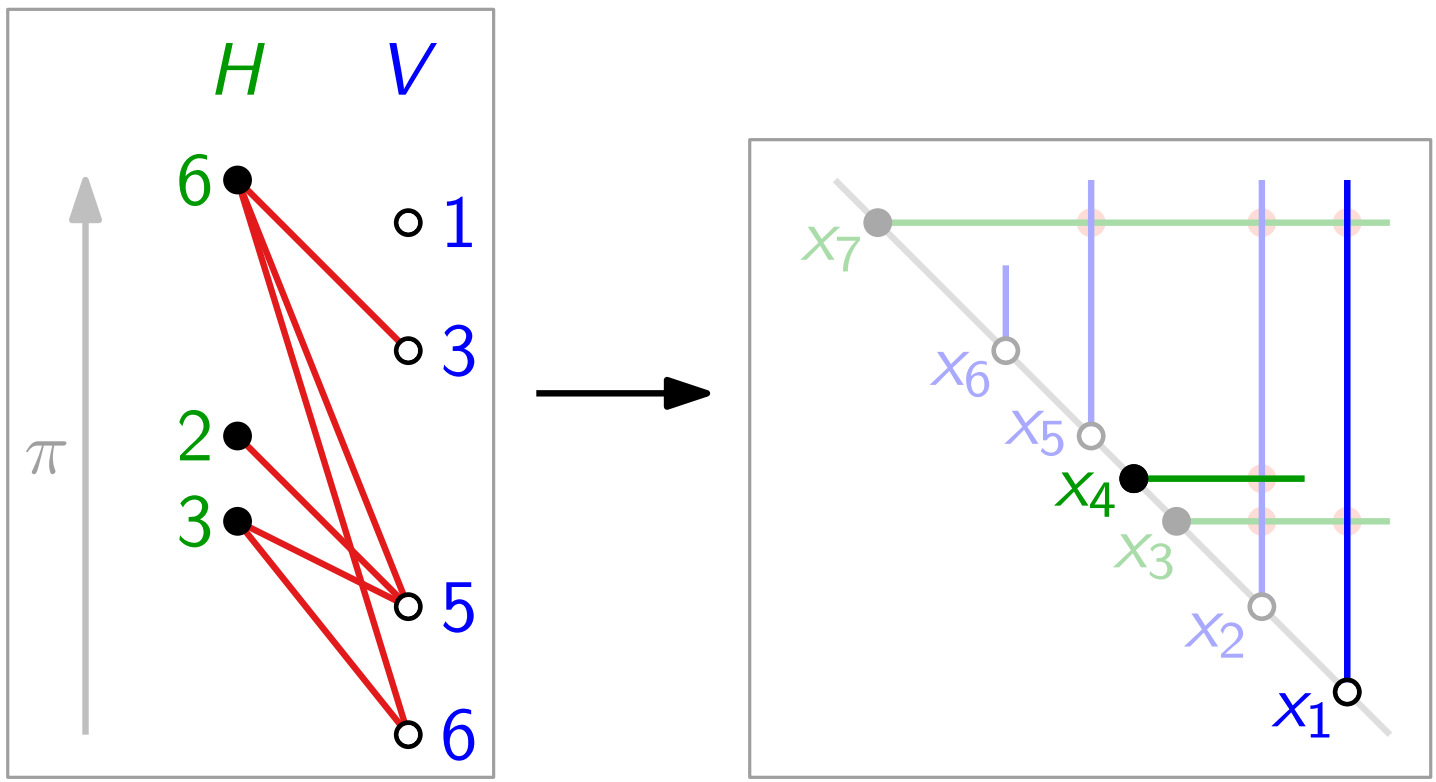
Solve LP



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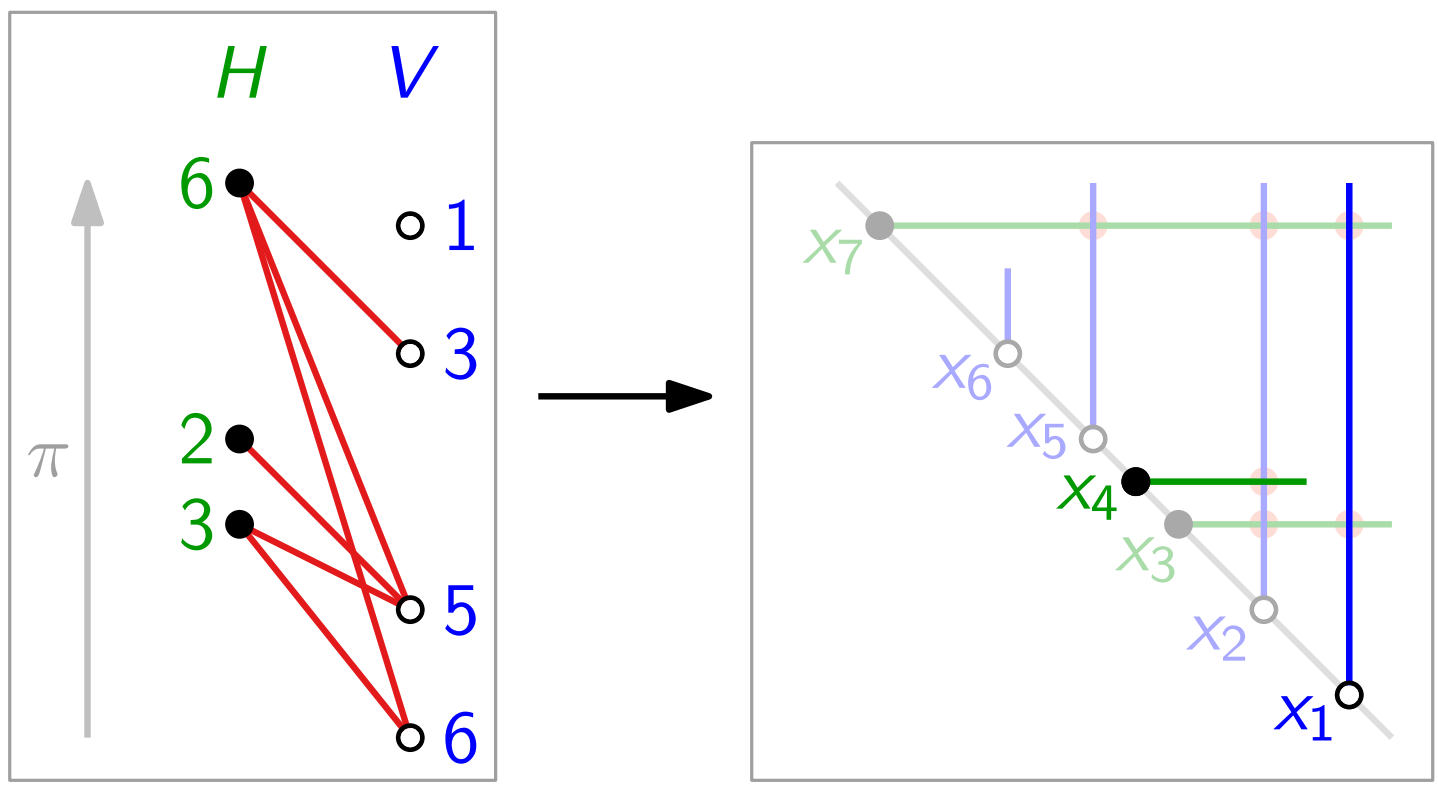

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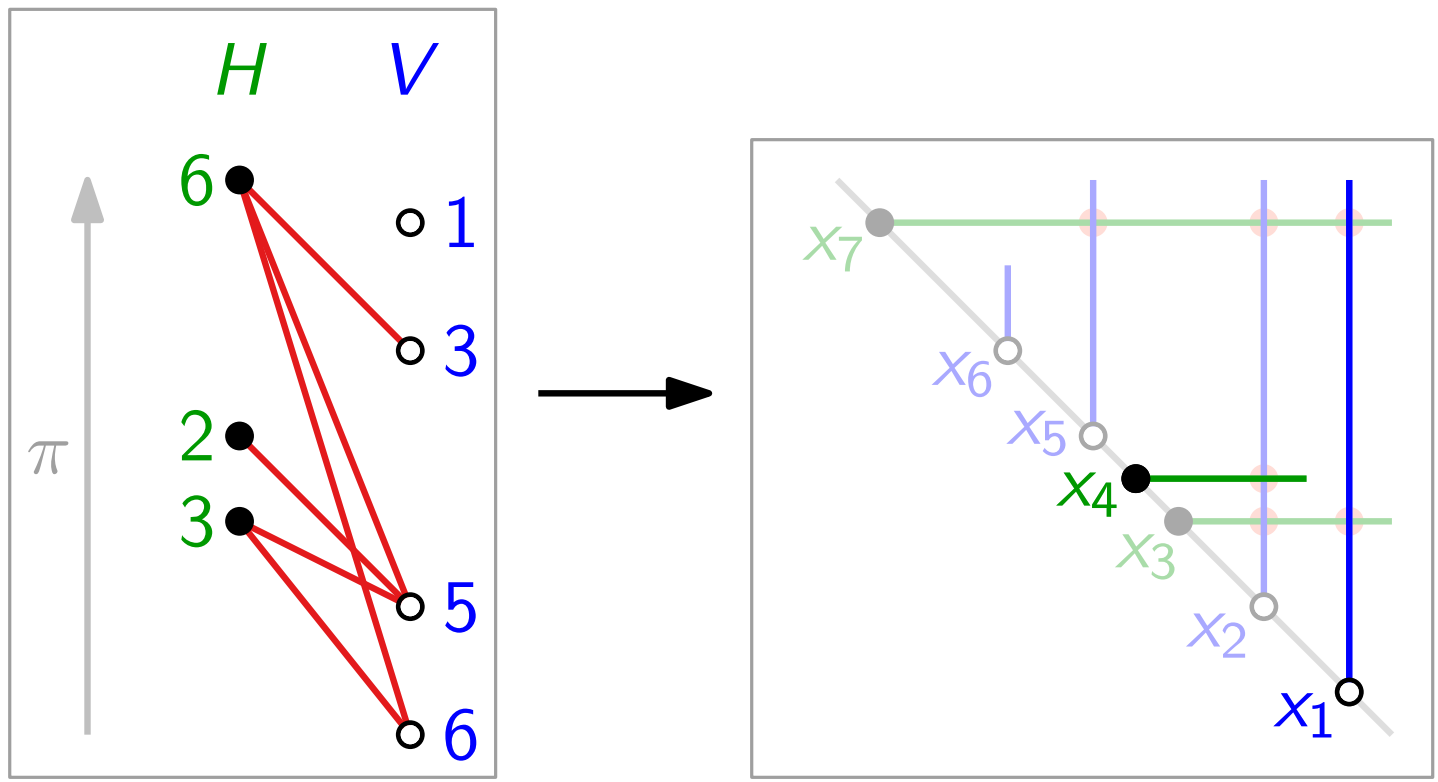

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Solve LP (without objective function):

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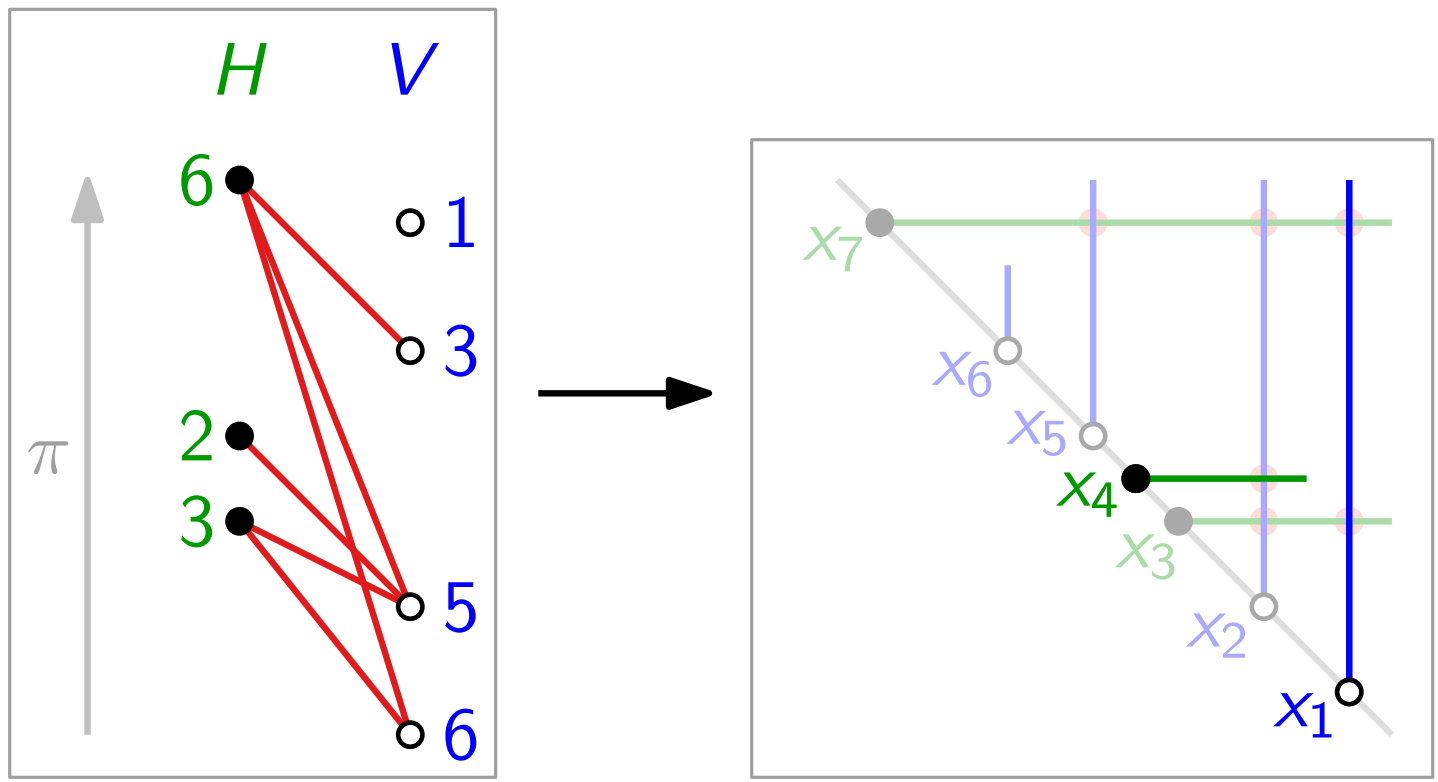

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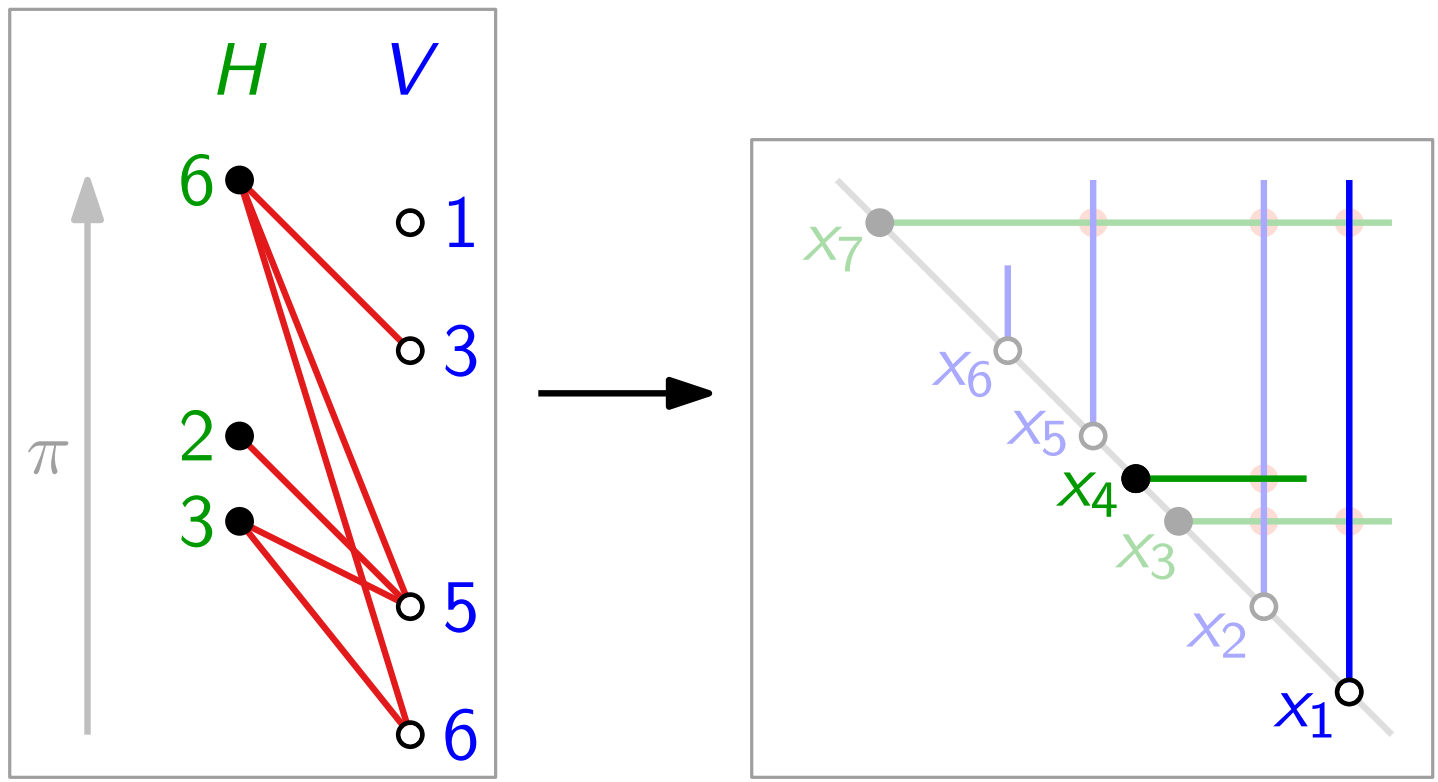

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Exercise: Improve this to $O(|E| + n)$!

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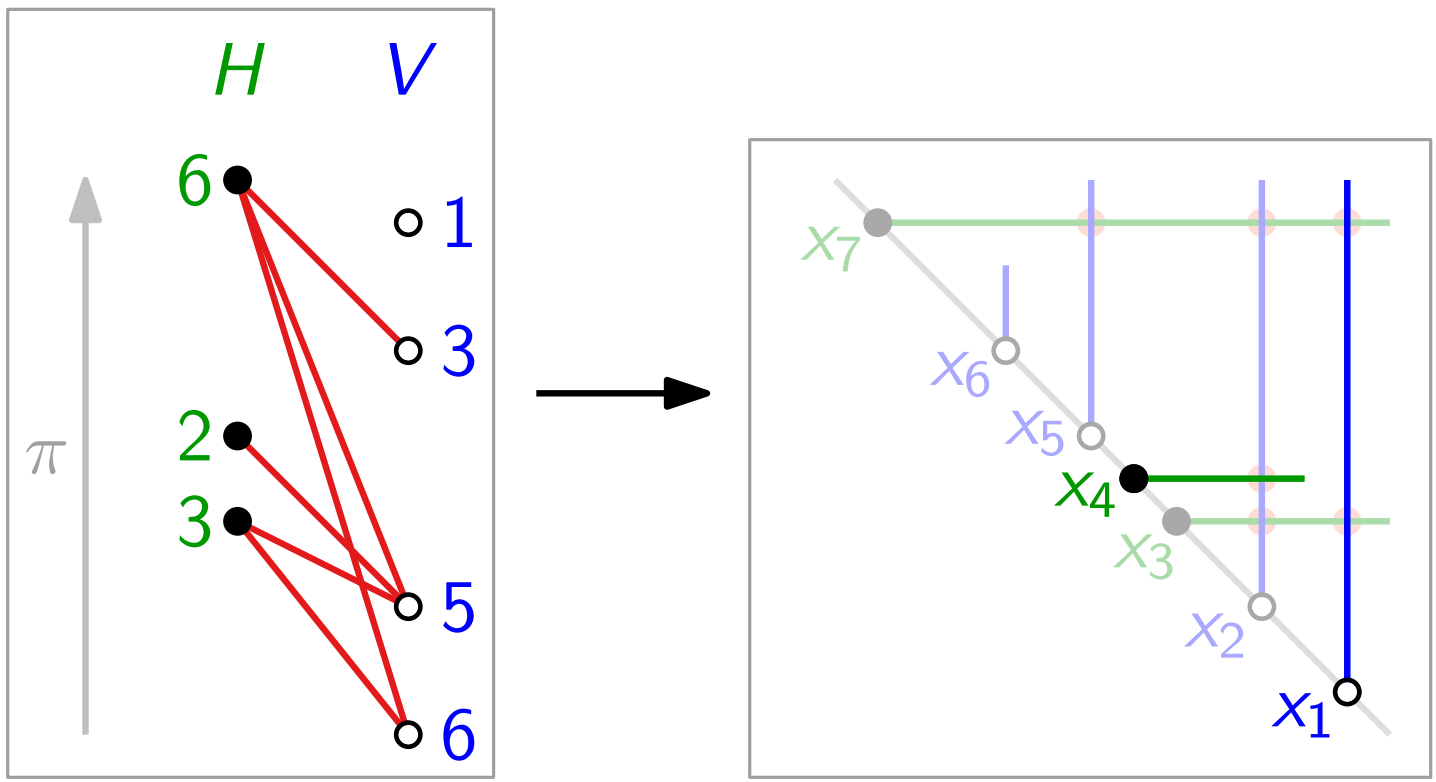

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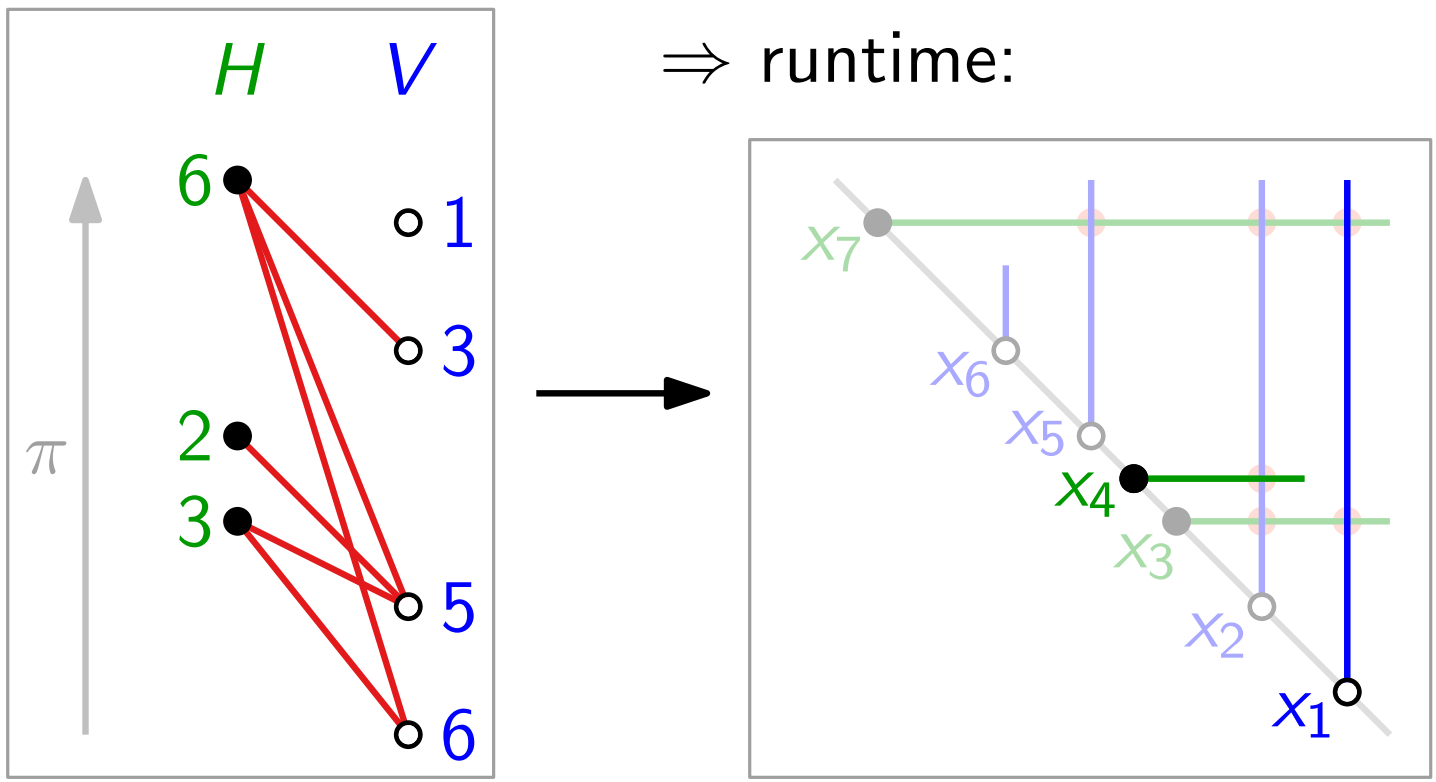
$\Rightarrow$ runtime:



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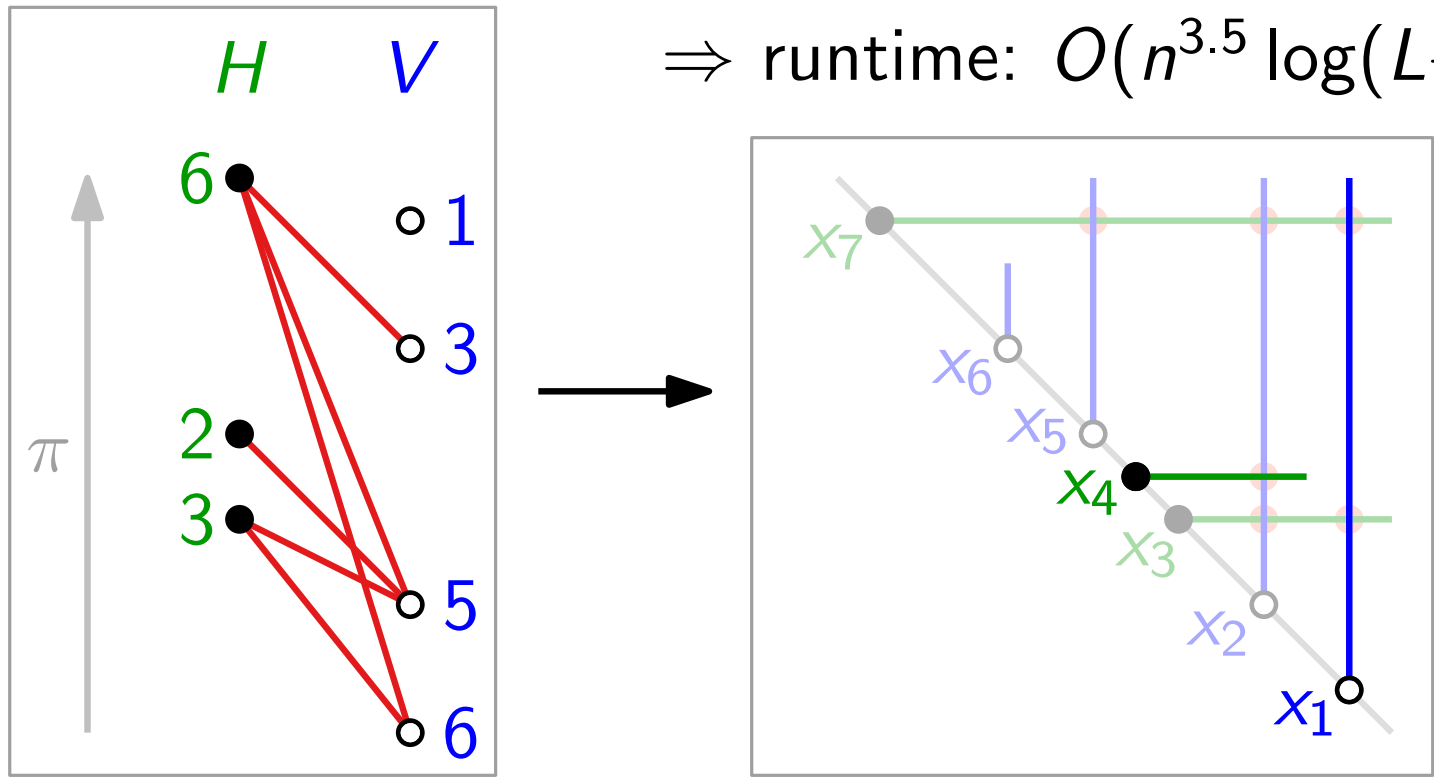
$\Rightarrow$ runtime: $O(n^{3.5}\log(L+n))$



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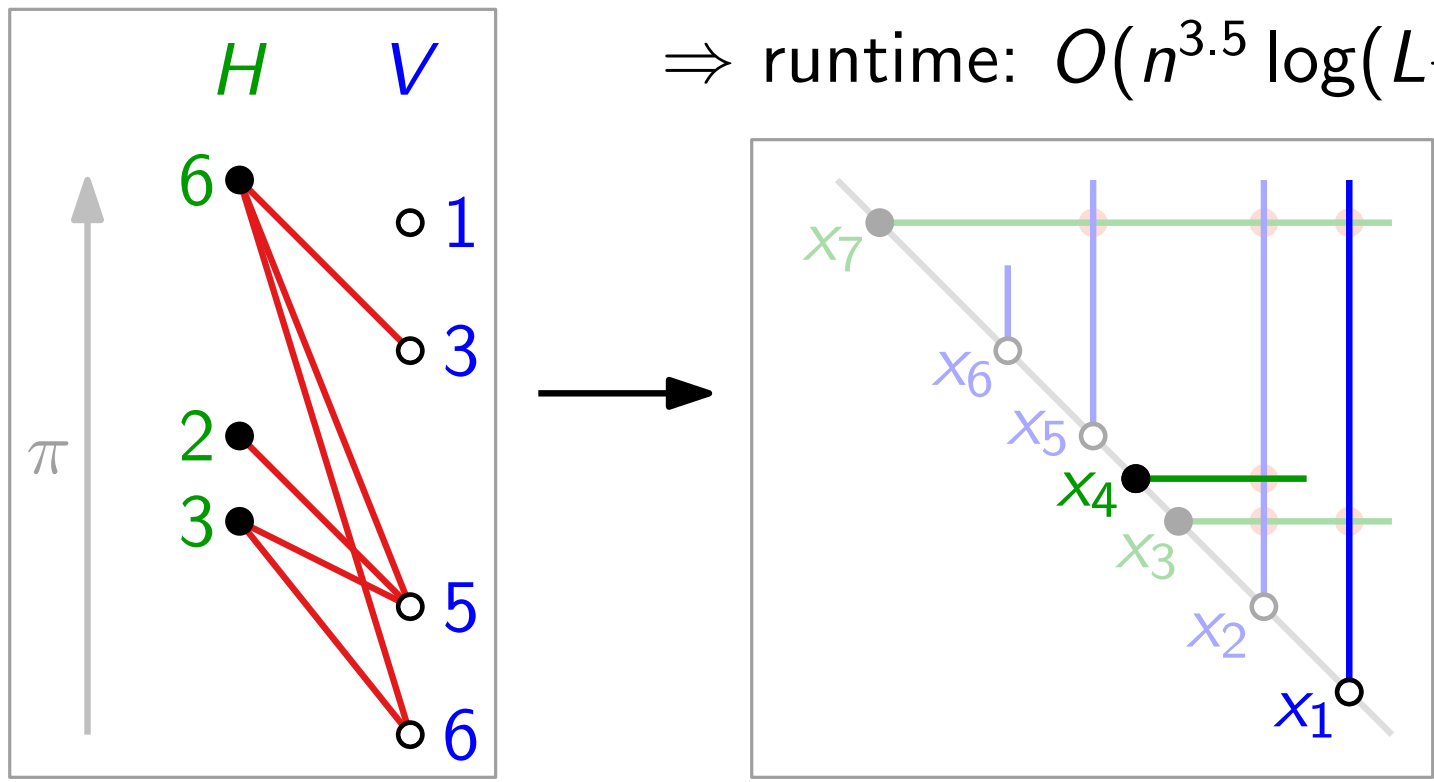

Motivation

difference constraints

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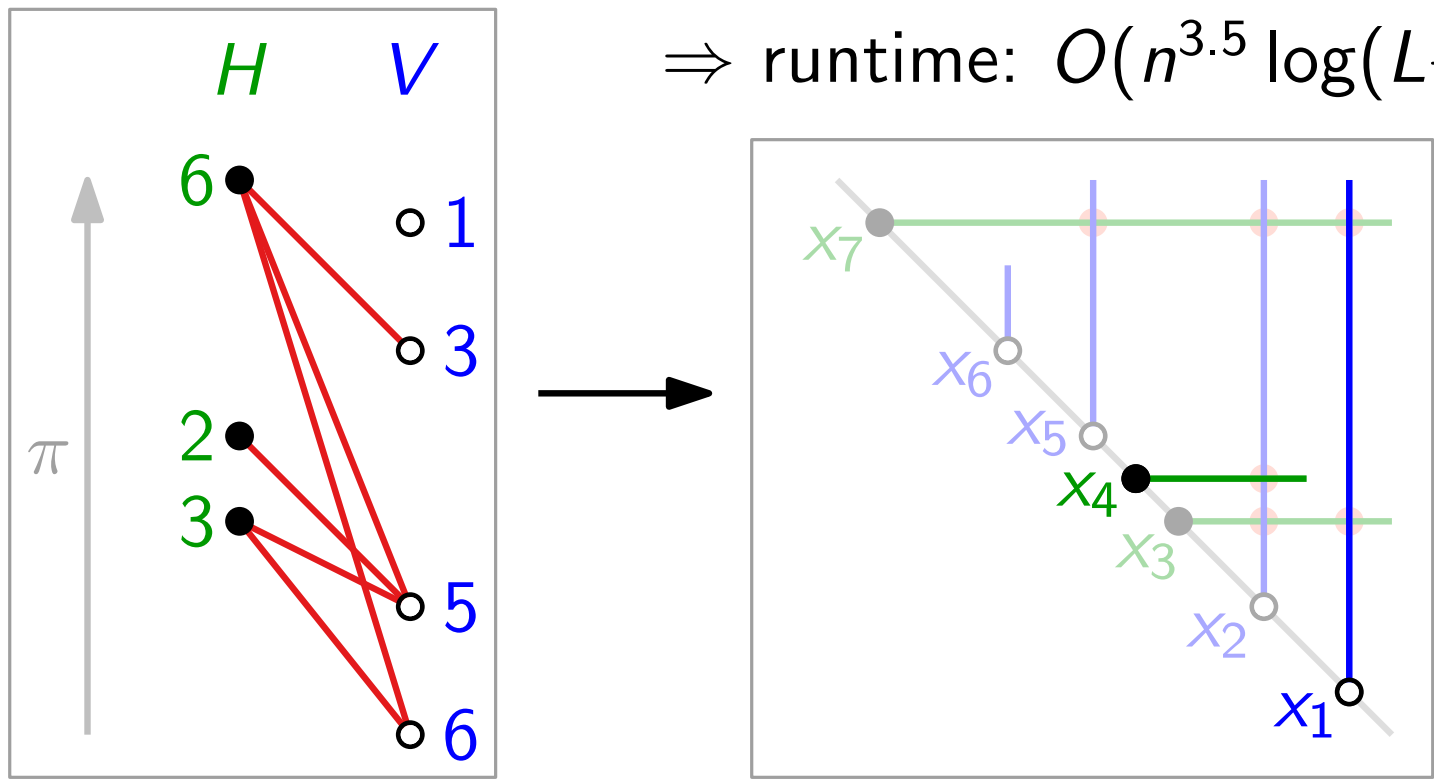

Solving Systems of Difference Constraints

Is this system feasible?

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\begin{aligned}
x_1 - x_2 &\leq 0 \\
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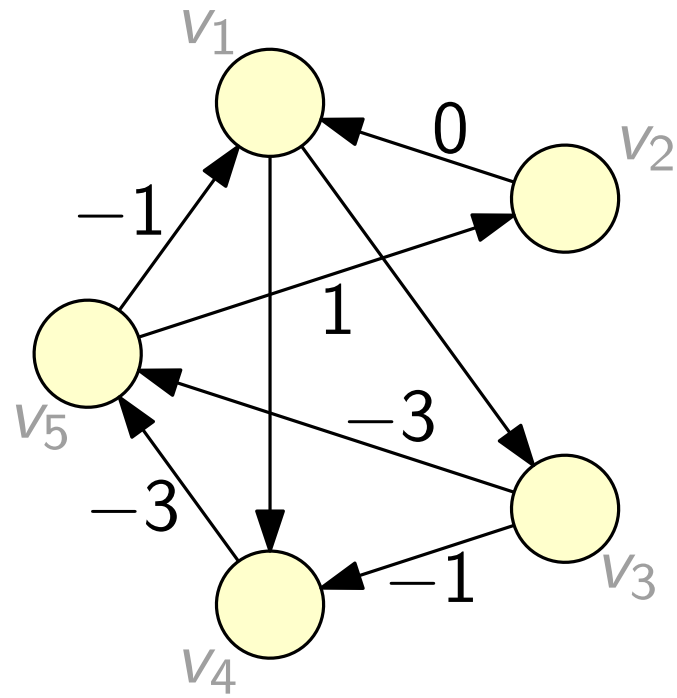

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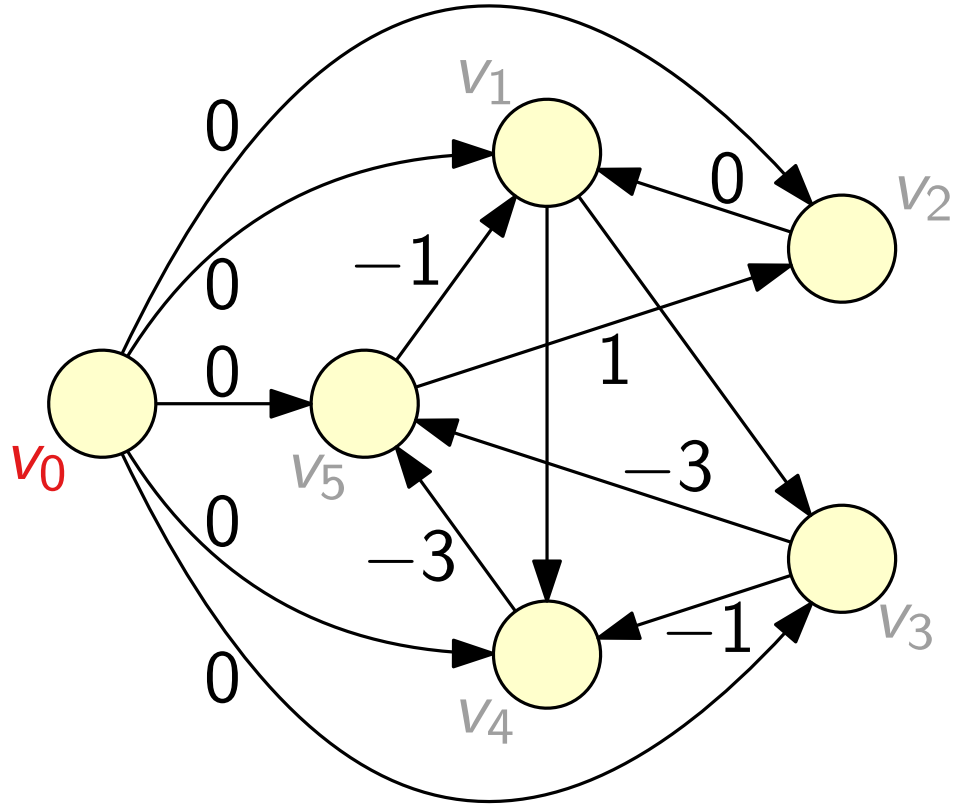

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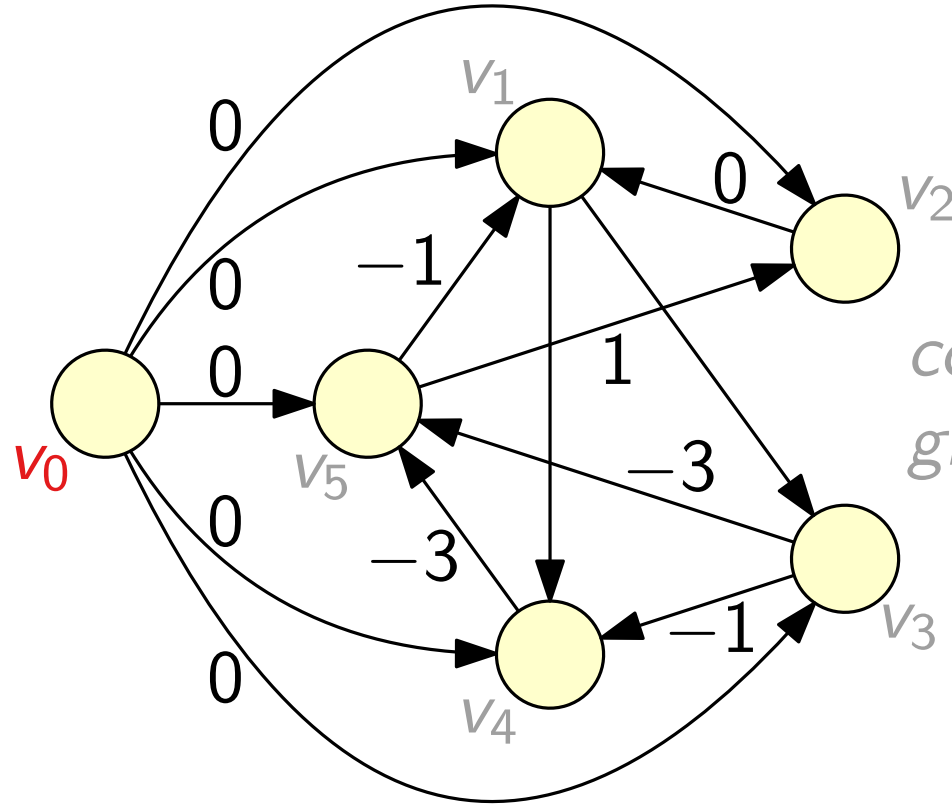

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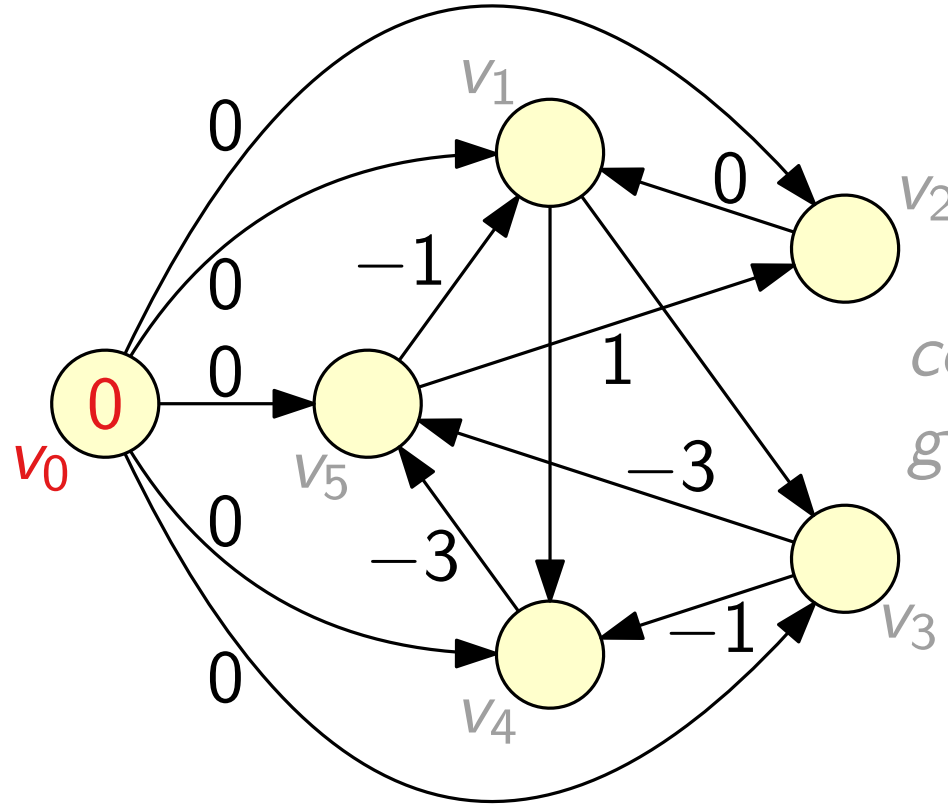

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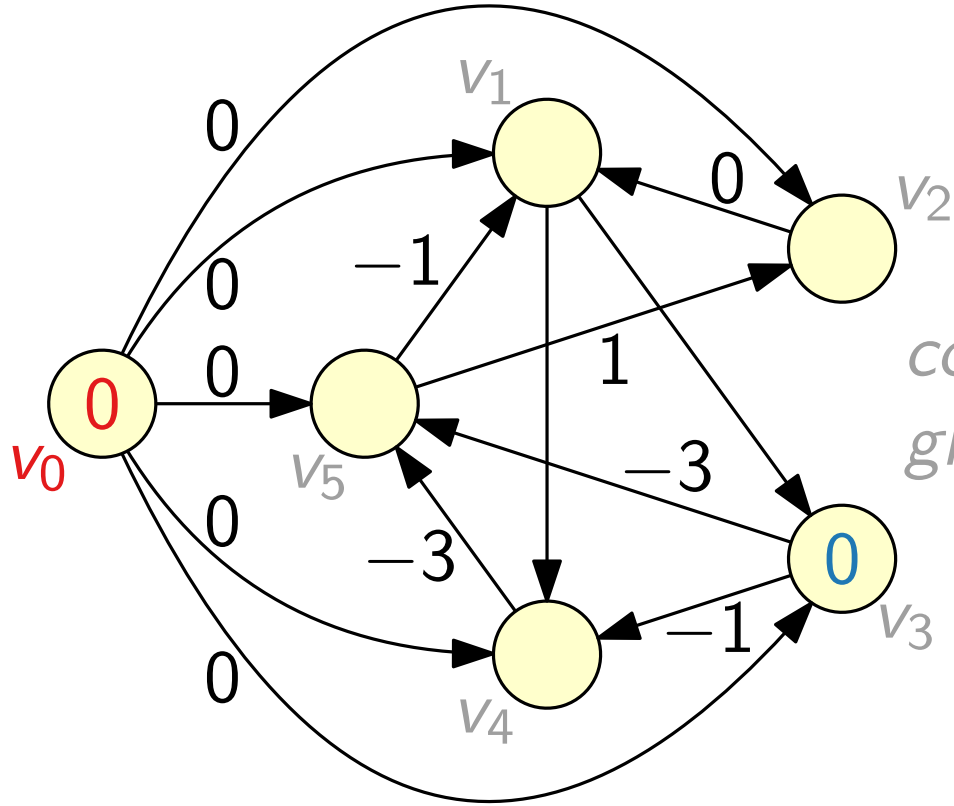

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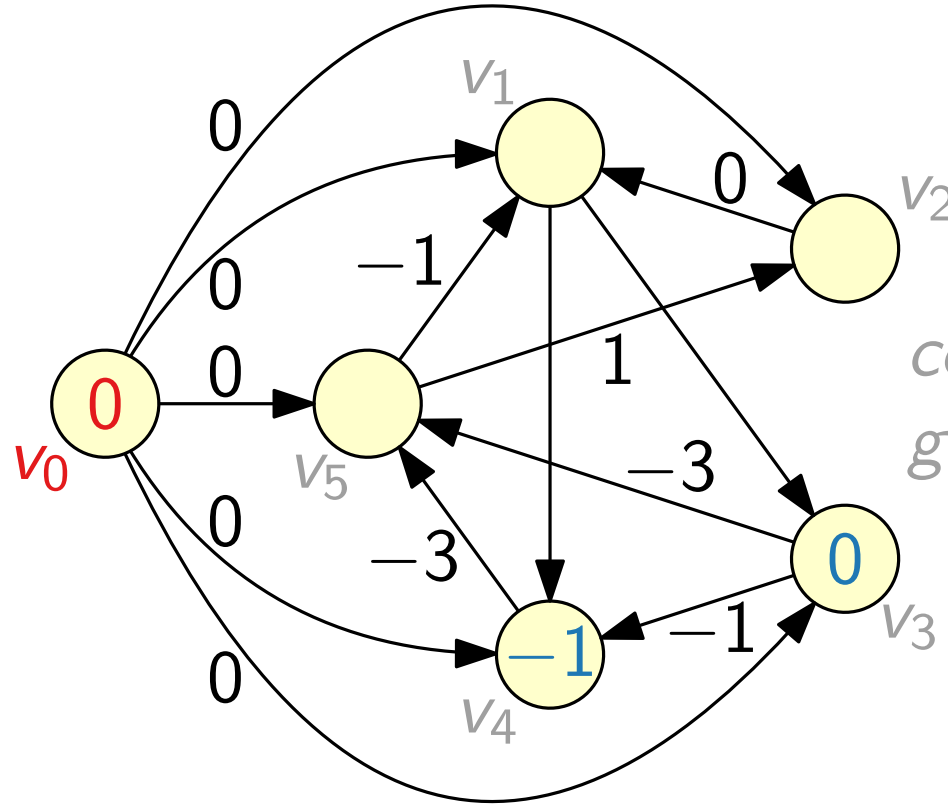

constraint graph $G_{A,b}$

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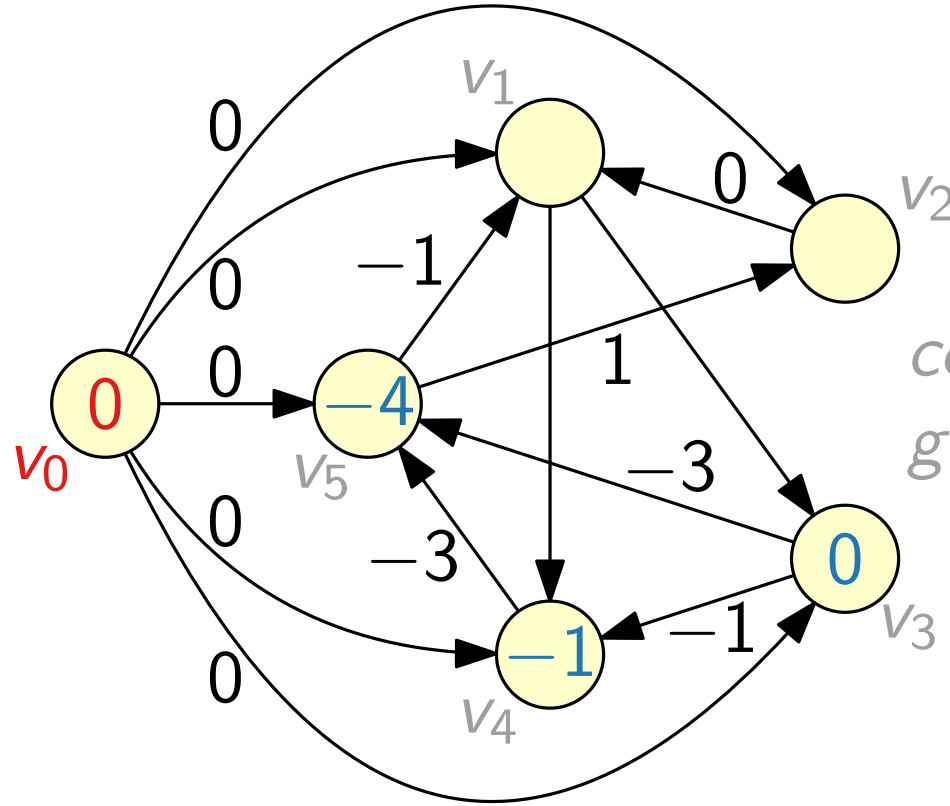
x_2 - x_5 &\leq 1 \\
x_3 - x_1 &\leq 5 \\
x_4 - x_1 &\leq 4 \\
x_4 - x_3 &\leq -1 \\
x_5 - x_3 &\leq -3 \\
x_5 - x_4 &\leq -3
\end{aligned}
$$



constraint graph $G_{A,b}$

Yes: Take $x = (-5, -3, 0, -1, 4)$ or $x' = (0, 2, 5, 4, 9)$.

Lemma. Let x be a solution to a system $Ax \leq b$ of difference constraints, and let $d \in \mathbb{R}$. Then $x + d = (x_1 + d, \ldots, x_n + d)$ is a solution, too.

Solving Systems of Difference Constraints

Is this system feasible?

$$
\begin{aligned}
x_1 - x_2 &\le 0 \\
x_1 - x_5 &\le -1 \\
x_2 - x_5 &\le 1 \\
x_3 - x_1 &\le 5 \\
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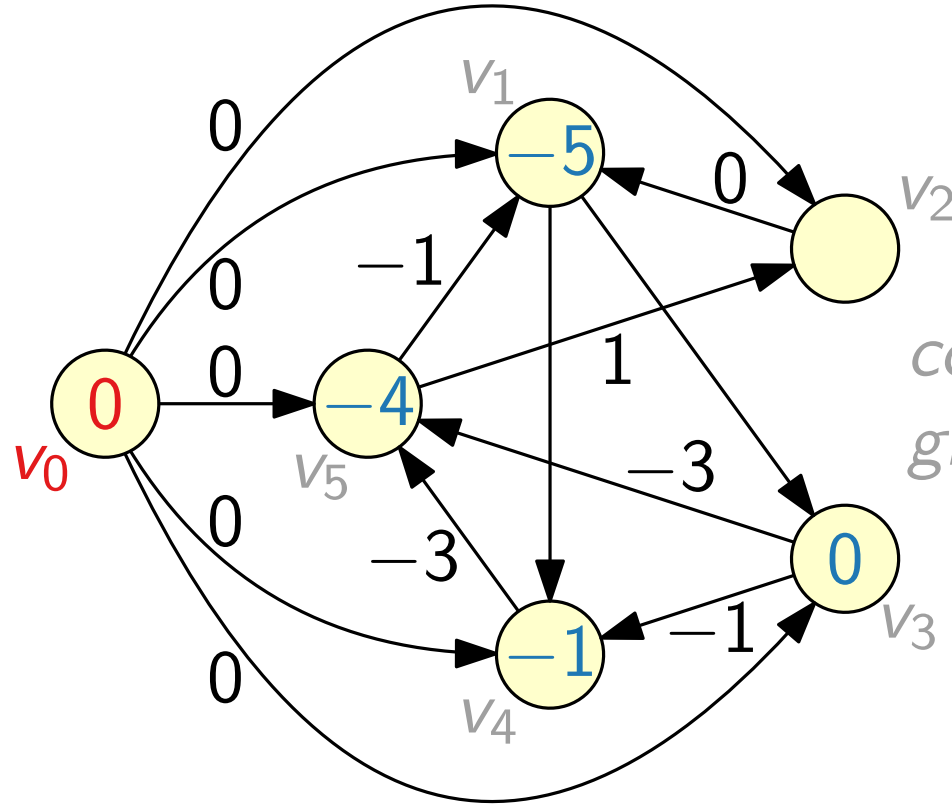

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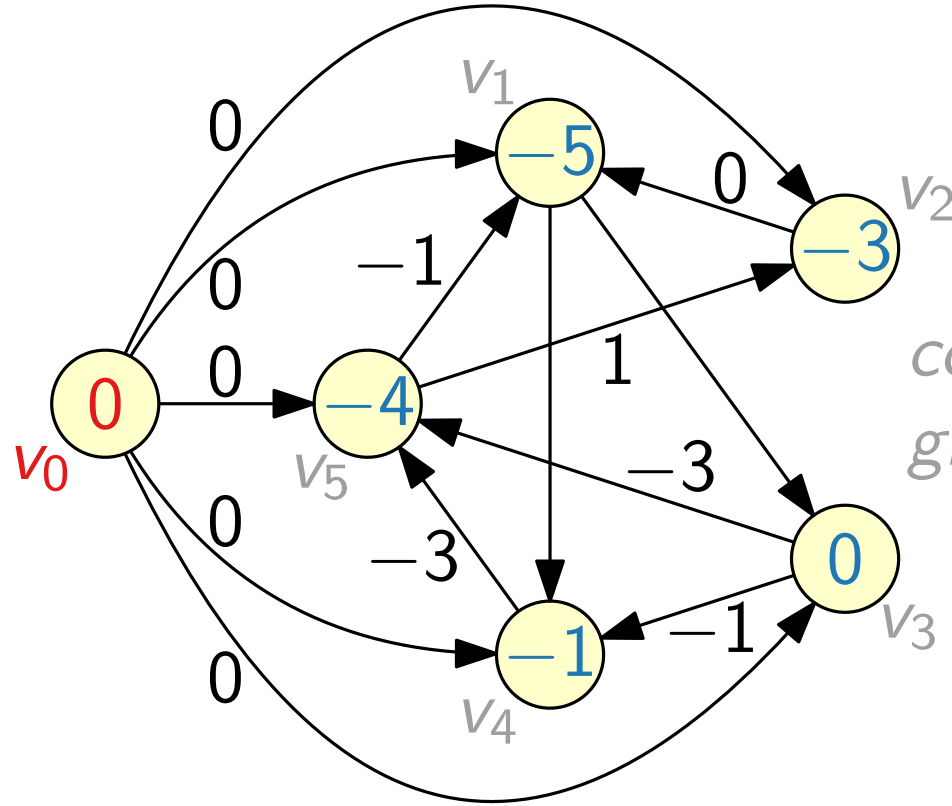

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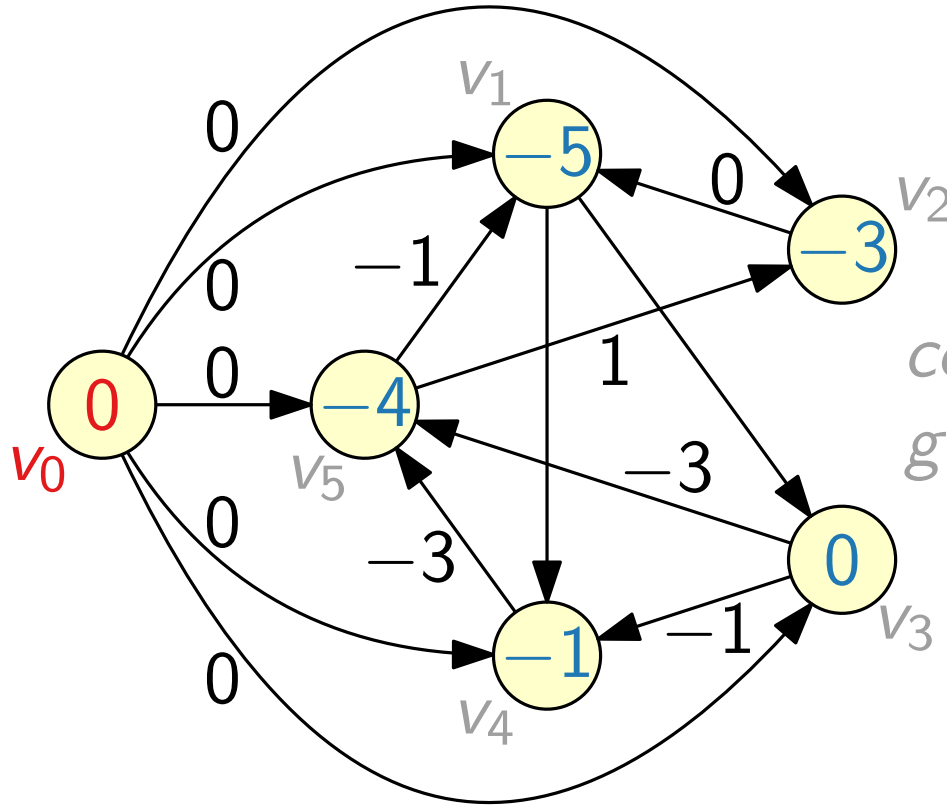

Definition. The constraint graph $G_{A,b}$ is a weighted digraph with vertex set $V_A = \{v_0, v_1, \ldots, v_n\}$ and edge set

$$E_A = \{v_i v_j : x_j - x_i \le b_{ij} \text{ is a constraint}\} \cup \{v_0 v_k : 1 \le k \le n\}.$$

The weight of $v_i v_j$ is b_{ij} if $i > 0$ and 0 otherwise.

Shortest Paths Do the Job

Theorem. Let $Ax \leq b$ be a system of difference constraints, and let $\delta_k = \delta(v_0, v_k)$ be the length of a shortest v_0–v_k path for $k = 1, \ldots, n$.
If $G_{A,b}$ contains no negative cycles,
then $x = (\delta_1, \ldots, \delta_n)$ is a feasible solution.
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Proof. Assume no neg. cycles.

Now assume $\exists$ neg. cycle and $Ax \leq b$ has a solution x.

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Proof. Assume no neg. cycles. Consider $v_i v_j \in E_A$ with $i > 0$.
Δ-inequality $\Rightarrow$ $\delta_j \leq \delta_i + b_{ij}$, or

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Wlog., let $C = \langle v_1, v_2, \ldots, v_k = v_1 \rangle$ be a neg. cycle.
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Relax(vtx u, vtx v, weights w)
  if $v.d > u.d + w(u, v)$ **then**
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    Q.DecreaseKey($v, v.d$)

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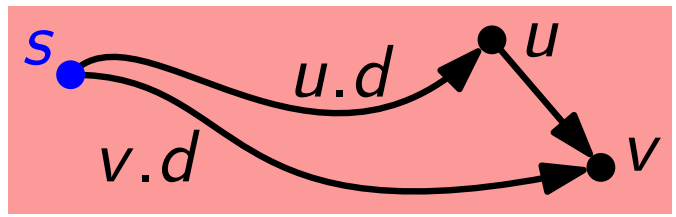

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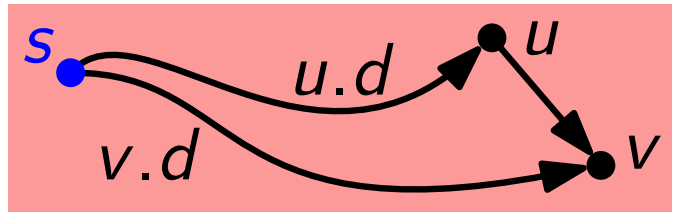

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The Bellman–Ford Algorithm

```
Dijkstra(graph G, weights w, vtx s)
  Initialize(G, s)
  Q = new PriorityQueue(G.V, d)
  while not Q.Empty() do
      u = Q.ExtractMin()
      foreach v ∈ Adj[u] do
          Relax(u, v; w)
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```
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  Initialize(G, s)
  for i = 1 to |G.V| − 1 do
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$O(E + V \log V)$

```
Bellman–Ford(graph G, weights w, vtx s)
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      foreach uv ∈ G.E do
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  foreach uv ∈ G.E do
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The Bellman–Ford Algorithm

```
Dijkstra(graph G, weights w, vtx s)
  Initialize(G, s)
  Q = new PriorityQueue(G.V, d)
  while not Q.Empty() do
      u = Q.ExtractMin()
      foreach v ∈ Adj[u] do
          Relax(u, v; w)
```

runtime?

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Correctness of Bellman–Ford

If G contains no neg. cycle reachable from s, after the for-i loop, for every vertex v, $v.d = \delta(s, v)$ and Bellman–Ford returns true.

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$\Rightarrow v_i.d \leq \delta(s, v_{i-1}) + w(v_{i-1}, v_i) = \delta(s, v_i)$

(δ s.p.)

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Suppose v is *not* reachable from s. $\Rightarrow$ Initially, $v.d = \infty$

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$\Rightarrow$ At termination, $v.d = \infty = \delta(s, v)$. $\square$

The Bellman–Ford Algorithm (overview)

```
Initialize(graph G, vtx s)
  foreach u ∈ V do
      u.d = ∞
      u.π = nil
  s.d = 0
```

```
Relax(vtx u, vtx v, weights w)
  if v.d > u.d + w(u, v) then
      v.d = u.d + w(u, v)
      v.π = u
```

```
Bellman–Ford(graph G, weights w, vtx s)
  Initialize(G, s)
  for i = 1 to |G.V| − 1 do
      foreach uv ∈ G.E do
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  foreach uv ∈ G.E do
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If G contains a negative cycle that is reachable from s, then Bellman–Ford returns false.

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For this implication we additionally need that $\sum_i v_i.d < \infty$.

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Improvement: $O(\sqrt{V}E \log W)$, where $W = \max_{uv \in E} w(u, v)$.
[Goldberg, SIAM J. Comput. 1995]

All-Pairs Shortest Paths

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$\Rightarrow \delta(i,j) = \ell_{ij}^{(n-1)} = \ell_{ij}^{(n)} = \ell_{ij}^{(n+1)} = \cdots$ since shortest paths are simple (if there are no neg. cycles)!

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Extend-Shortest-Paths(L, W)
 $L' = (\ell'_{ij} = \infty)$ new $n \times n$ matrix
 for $i = 1$ **to** n **do**
  for $j = 1$ **to** n **do**
   for $k = 1$ **to** n **do**
    $\ell'_{ij} = \min\{\ell'_{ij}, \ell_{ik} + w_{kj}\}$
 return L'

All-Pairs Shortest Paths

$w_{jj} = 0$

Assume that the graph is given by a matrix $W = (w_{ij})_{1 \le i,j \le n}$.

Let $\ell_{ij}^{(m)}$ be the length of a shortest i–j path with $\le m$ edges.

Then $\ell_{ij}^{(0)} = \begin{cases} 0 & \text{if } i = j \\ \infty & \text{otherwise} \end{cases}$ and $\ell_{ij}^{(m)} = \min_{1 \le k \le n} \{\ell_{ik}^{(m-1)} + w_{kj}\}$.

$\Rightarrow \delta(i,j) = \ell_{ij}^{(n-1)} = \ell_{ij}^{(n)} = \ell_{ij}^{(n+1)} = \cdots$ since shortest paths are simple (if there are no neg. cycles)!

Extend-Shortest-Paths(L, W)

```
L' = (ℓ'_ij = ∞) new n × n matrix
for i = 1 to n do
    for j = 1 to n do
        for k = 1 to n do
            ℓ'_ij = min{ℓ'_ij, ℓ_ik + w_kj}
return L'
```

Slow-All-Pairs-SP(W)

All-Pairs Shortest Paths

$w_{jj} = 0$

Assume that the graph is given by a matrix $W = (w_{ij})_{1 \le i,j \le n}$.

Let $\ell_{ij}^{(m)}$ be the length of a shortest i–j path with $\le m$ edges.

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            ℓ'_ij = min{ℓ'_ij, ℓ_ik + w_kj}
return L'
```

Slow-All-Pairs-SP(W)
```
L^(1) = W
for m = 2 to n − 1 do
    L^(m) = new matrix
    L^(m) = ESP(L^(m−1), W)
return L^(n−1)
```

All-Pairs Shortest Paths

$w_{jj} = 0$

Assume that the graph is given by a matrix $W = (w_{ij})_{1 \le i,j \le n}$.

Let $\ell_{ij}^{(m)}$ be the length of a shortest i–j path with $\le m$ edges.

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$\Rightarrow \delta(i,j) = \ell_{ij}^{(n-1)} = \ell_{ij}^{(n)} = \ell_{ij}^{(n+1)} = \cdots$ since shortest paths are simple (if there are no neg. cycles)!

Extend-Shortest-Paths(L, W)

$L' = (\ell'_{ij} = \infty)$ new $n \times n$ matrix
for $i = 1$ **to** n **do**
  for $j = 1$ **to** n **do**
    for $k = 1$ **to** n **do**
      $\ell'_{ij} = \min\{\ell'_{ij}, \ell_{ik} + w_{kj}\}$
return L'

Slow-All-Pairs-SP(W)

$L^{(1)} = W$
for $m = 2$ **to** $n - 1$ **do**
  $L^{(m)} =$ new matrix
  $L^{(m)} = \text{ESP}(L^{(m-1)}, W)$
return $L^{(n-1)}$

Runtime:

All-Pairs Shortest Paths

$w_{jj} = 0$

Assume that the graph is given by a matrix $W = (w_{ij})_{1 \le i,j \le n}$.

Let $\ell_{ij}^{(m)}$ be the length of a shortest i–j path with $\le m$ edges.

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Extend-Shortest-Paths(L, W)

$L' = (\ell'_{ij} = \infty)$ new $n \times n$ matrix
for $i = 1$ **to** n **do**
  for $j = 1$ **to** n **do**
    for $k = 1$ **to** n **do**
      $\ell'_{ij} = \min\{\ell'_{ij}, \ell_{ik} + w_{kj}\}$
return L'

Slow-All-Pairs-SP(W)

$L^{(1)} = W$
for $m = 2$ **to** $n - 1$ **do**
  $L^{(m)}$ = new matrix
  $L^{(m)} = \text{ESP}(L^{(m-1)}, W)$
return $L^{(n-1)}$

Runtime: $O(n^4)$

Faster APSP

Faster-All-Pairs-SP($n \times n$ matrix W)

$L^{(1)} = W$

$m = 1$

while $m < n - 1$ **do**

    L = new $n \times n$ matrix

    L = Extend-Shortest-Path( )

    $m =$

return $L^{(m)}$

Runtime: $O(n^3 \log n)$

Faster APSP

Faster-All-Pairs-SP($n \times n$ matrix W)

$L^{(1)} = W$

$m = 1$

while $m < n - 1$ **do**

  $L^{(2m)} =$ new $n \times n$ matrix

  $L^{(2m)} =$ Extend-Shortest-Path($L^{(m)}, L^{(m)}$)

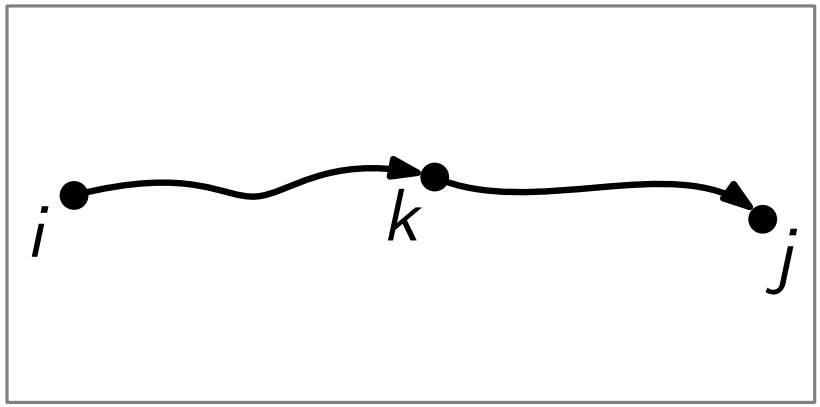
  $m = 2m$

return $L^{(m)}$

Runtime: $O(n^3 \log n)$

The Floyd–Warshall Algorithm

[W., J. ACM 1962]
[F., Comm. ACM 1977]



The Floyd–Warshall Algorithm

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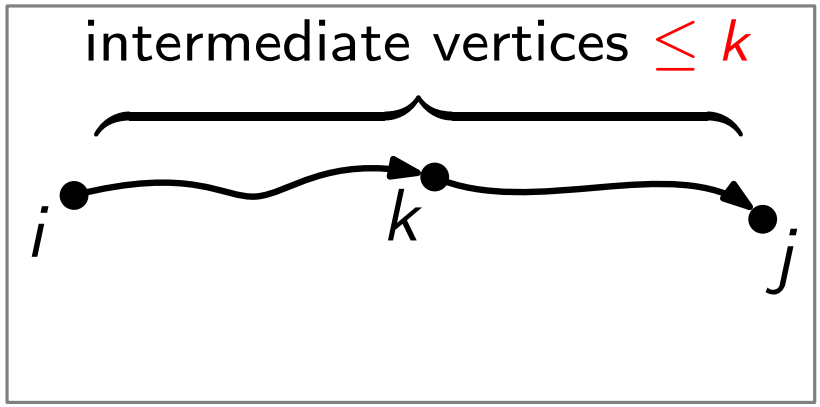

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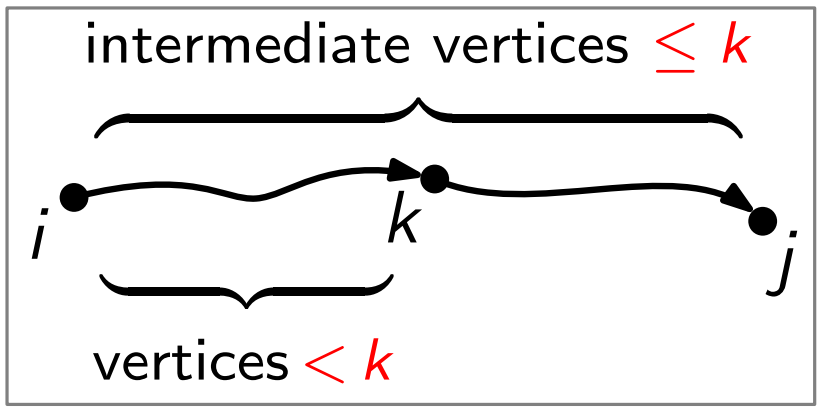

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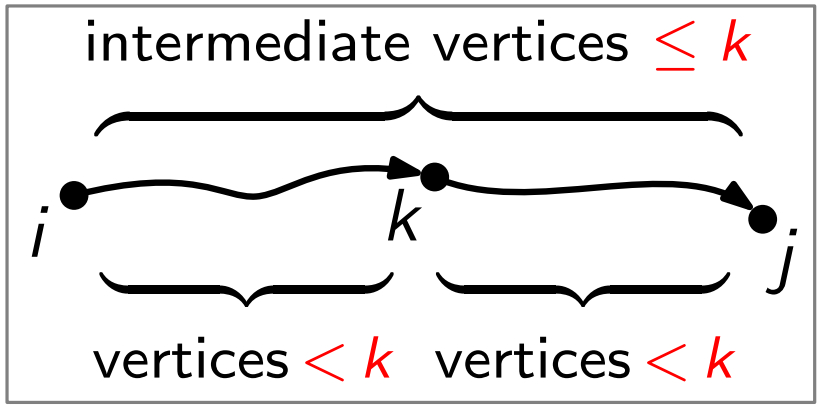

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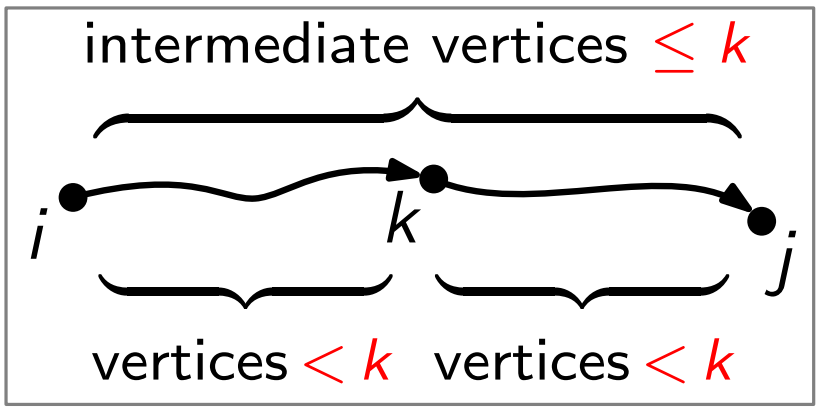

$d_{ij}^{(0)} = w_{ij}$; if $k > 0$ then

The Floyd–Warshall Algorithm

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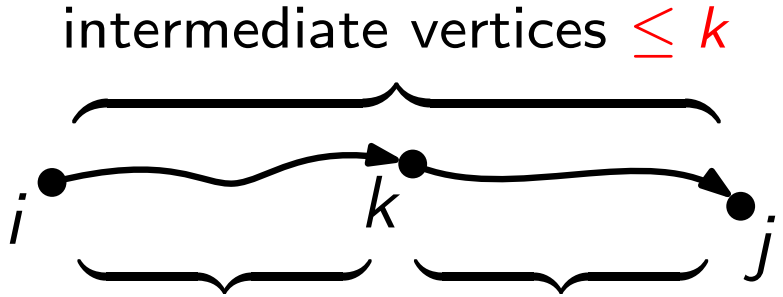
[F., Comm. ACM 1977]



$$d_{ij}^{(0)} = w_{ij}\,; \quad \text{if } k > 0 \text{ then}$$

$$d_{ij}^{(k)} = \min\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}$$

The Floyd–Warshall Algorithm

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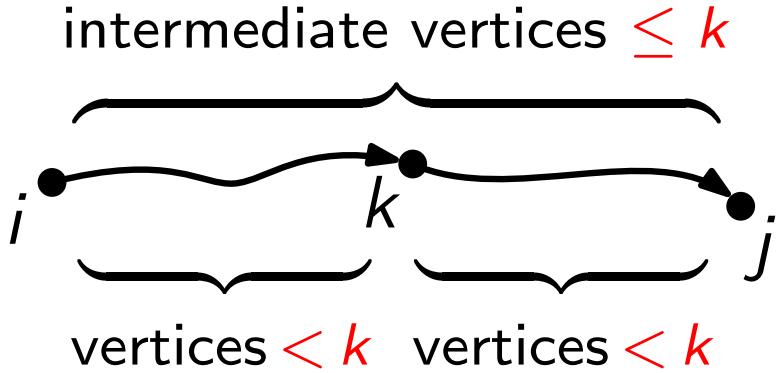

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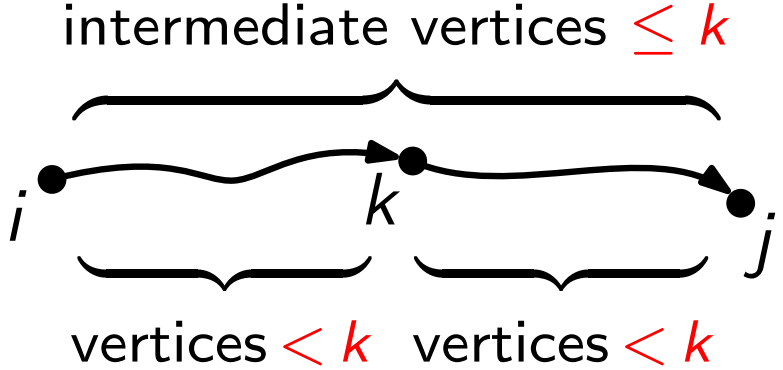
Floyd–Warshall($n \times n$ matrix W)

$D^{(0)} = W$
for $k = 1$ **to** n **do**
  $D^{(k)} =$ new $n \times n$ matrix
  for $i = 1$ **to** n **do**
    for $j = 1$ **to** n **do**
      $d_{ij}^{(k)} = \min\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}$
return $D^{(n)}$

The Floyd–Warshall Algorithm

[W., J. ACM 1962]
[F., Comm. ACM 1977]



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Floyd–Warshall($n \times n$ matrix W)

```
D^(0) = W
for k = 1 to n do
    D^(k) = new n × n matrix
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        for j = 1 to n do
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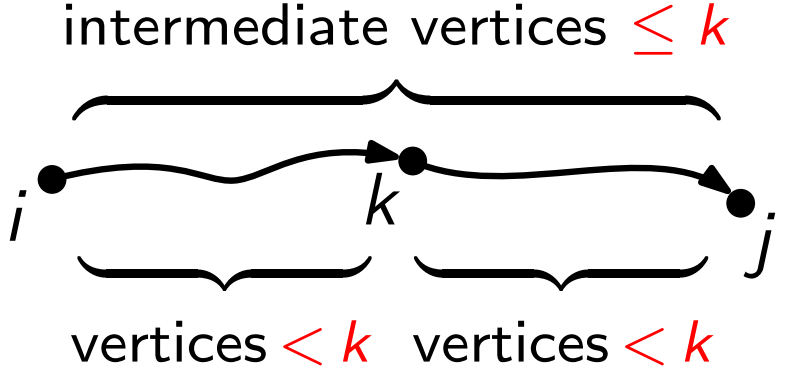
```

Runtime:

The Floyd–Warshall Algorithm

[W., J. ACM 1962]
[F., Comm. ACM 1977]



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Runtime:
$O(n^3)$

The Floyd–Warshall Algorithm

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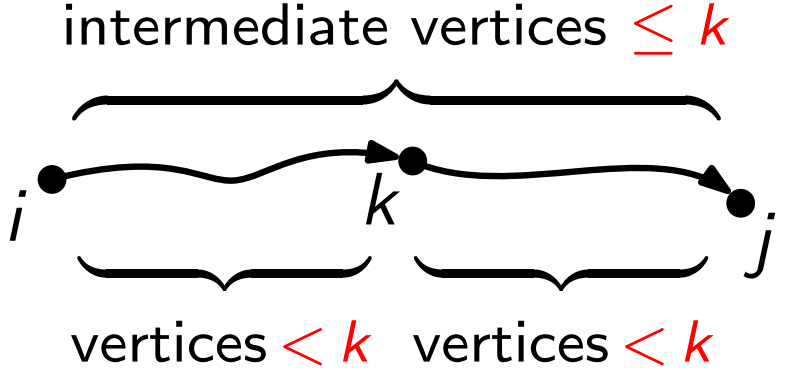

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Floyd–Warshall($n \times n$ matrix W)

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      $d_{ij}^{(k)} = \min\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}$
return $D^{(n)}$

Improvement:
$O(V(V \log V + E))$
Johnson's algorithm
[J. ACM 1977]

Runtime:
$O(n^3)$