Julius-Maximilians-
UNIVERSITÄT WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Advanced Algorithms

Winter term 2019/20

Lecture 6. Approaches using Randomisation

or: Color Coding and "Isolation" Lemmas

Source: **[Parameterized Algorithms: §3.3, 5, 5.1, 5.2, 11.2]**

(slides by Thomas van Dijk & Alexander Wolff)

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In this lecture:

- Coloring $\neq$ **Graph**coloring.
- k-coloring of n elements:
  label each element with one number from $1..k$.

Randomised Algorithms

Probabilistic Polynomial Time ($\mathcal{PP}$)

Runtime: polynomial

Result: $\textsc{Yes}$-instance $\rightarrow \Pr[\textsc{Yes}] > \frac{1}{2}$

$\textsc{No}$-instance $\rightarrow \Pr[\textsc{Yes}] \leq \frac{1}{2}$

Las Vegas ($\mathcal{ZPP}$), zero-error probabilistic polynomial time

Runtime: expected polynomial

Result: correct

Monte Carlo:

Runtime: polynomial

	$\mathcal{BPP}$ (bounded-error prob. poly. time)	$\mathcal{RP}$ (randomised poly. time)	co-$\mathcal{RP}$
Result: $\textsc{Yes}$-instance $\rightarrow \Pr[\textsc{Yes}] >$	$\frac{2}{3}$	$\frac{1}{2}$	1
$\textsc{No}$-instance $\rightarrow \Pr[\textsc{Yes}] \leq$	$\frac{1}{3}$	0	$\frac{1}{2}$

Thm: $\mathcal{ZPP} = \mathcal{RP} \cap \text{co-}\mathcal{RP}$

Amplification

$\mathcal{RP}$: $\textsc{Yes}$-Instance $\to$ $\Pr[\textsc{Yes}] \geq t$
$\textsc{No}$-Instance $\to$ $\Pr[\textsc{Yes}] = 0$

If an $\mathcal{RP}$-algorithm returns $\textsc{Yes}$, it is correct

If an $\mathcal{RP}$-algorithm returns $\textsc{No}$,
it is incorrect with probability $\leq 1 - t$

Algorithm: Run the original algorithm $\lceil 1/t \rceil$ times
Return $\textsc{Yes}$ if every some returns $\textsc{Yes}$
Otherwise $\textsc{No}$ $\quad x := 1/t$

Error Probability : $(1-t)^{1/t} = (1 - \frac{1}{x})^x < \frac{1}{e} < \frac{1}{2}$

(**Obs.:** Repeating $100 \cdot t^{-1}$ times $\rightsquigarrow$ error prob. $< 2^{-100}$.)

Feedback Vertex Set

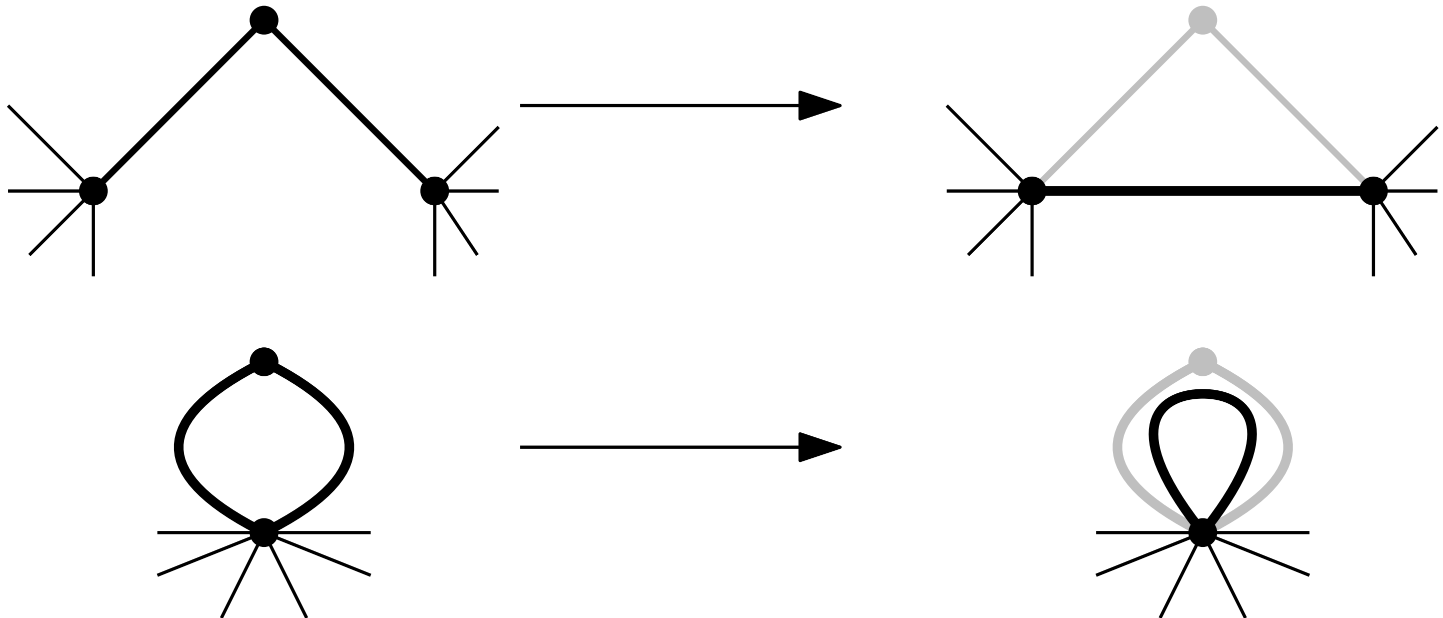
Given: Graph $G = (V, E)$, number k
Question: $\exists\, S \subseteq V$ such that $|S| \leq k$ and $G[V \setminus S]$ is a forest?

Reduction Rule: Delete vertices of degree < 2

Reduction Rule: "Bypass" each degree two vertex.

Reduction Rule: Put vertices incident to loops in FVS

Def.: If no rule applies, the graph is called *reduced*.



Feedback Vertex Set

Given:	Graph $G = (V, E)$, number k
Question:	$\exists\ S \subseteq V$ such that $\|S\| \leq k$ and $G[V \setminus S]$ is a forest?

Def.: Let G be reduced, $S \subseteq V$ be an FVS, and $W := V \setminus S$.
Let $E_W :=$ edges connecting vertices in W
Let $E_{S,W} :=$ edges connecting S and W

mindeg 3

$< |W|$ since forest

$$3|W| \leq \sum_{v \in W} \deg(v) = |E_{S,W}| + 2|E_W| < |E_{S,W}| + 2|W|$$

Lemma: If G is reduced, then $|E_{S,W}| \geq |E_W|$

(see also Lemma 5.1 in textbook)

FVS: algorithm given k

0. $S \leftarrow \varnothing$

1. `while` G is not empty: $\leftarrow$ max k times

2. Apply reduction rules

3. pick a vertex v via randomized proc. on last slide

4. $S \leftarrow S \cup \{v\}$; $G \leftarrow G \setminus v$

5. If $|S| > k$: return No

6. Return Yes

Runtime:	$O(n+m)$	$O(k(n+m))$
Prob. of success:	$> 1/4$	$> 4^{-k}$

Thm: Feedback Vertex Set can be solved in $O(4^k \cdot k(n+m))$ time by a randomised algorithm

Longest Path

Given: Graph $G = (V, E)$, number k
Parameter: k
Question: Does G contain a length k path?

(length := # edges)

Thm: Longest Path is NP-complete

Thm: Longest Path can be solved in $O^*(2^n)$ time.

Special Case:
Longest Path in acyclic graphs: Runtime? $O(m)$

1. Topological sort
2. Let $L(v)$:= longest path to v
3. "backwards" dynamic program
4. Look for v with $L(v) = k$

Longest Path

Idea. Longest Path is easy on acyclic graphs

Plan: make G acyclic!

1. pick random permutation π of V

2. orient edges $\{u, v\}$ from u to v when $\pi(u) < \pi(v)$

Result → $\vec{G}$ (random variable!)

Obs.: $\exists$ k-path in $\vec{G}$ $\rightarrow$ $\exists$ k-path in G

Obs.: Converse does not apply however ...

$\exists$ k-path in G $\rightarrow$ $\Pr[\vec{G}$ has k-path$] > 0.$

Now: Randomisied algorithm? **Runtime?**

Randomised Orientation: Success Prob.

1. pick random permutation π of V

2. orient edges $\{u, v\}$ from u to v when $\pi(u) < \pi(v)$

Result → $\vec{G}$ (random variable!)

Lemma: Let p be a k-path in G. Then $\Pr[p \in \vec{G}] = \frac{2}{(k+1)!}$

Proof: Consider perm. π, but ignore the elements of $p \rightsquigarrow \pi_{/p}$

There are $(k+1)!$ ways to complete $\pi_{/p}$ to some π'.

All have equal probability.

For two of them, p is a path in $\vec{G}$ (two correct)

Thus $\Pr[p \in \vec{G} \mid \pi_{/p}] = \frac{2}{(k+1)!}$

$\rightsquigarrow \Pr[p \in \vec{G}] = \frac{2}{(k+1)!}$ (indep. sum over $\pi_{/p}$) $\square$

Randomised Orientation: Algorithm

Algorithm

1. Repeat $(k+1)!/2$ times:

  2. $\vec{G} \leftarrow$ random acyclic orientation of G

  3. $p \leftarrow$ longest path in $\vec{G}$

  4. If $|p| \geq k$, return Yes.

5. Return No

Runtime: $O^*(k!)$ iterations each $O(m)$ time

Thm: A randomised algorithm can solve Longest Path in $O^*(k! \cdot n)$ time

LONGEST PATH : attempt 2

Obs. LONGEST PATH is easy on acyclic graphs.

Color vertices with $(k+1)$ colors

$(k$-path has $k+1$ vertices$)$

$\neq$ **Graph**coloring!

vertices with equal color might not be adjacent

Def.: A path is *colorful*, when each vertex has a different color.

Obs.: Colorful paths are "easy"

Part 1: Finding a colorful path is ~~easy~~ FPT in k.

Part 2: $\exists$ k-path in G $\rightarrow$ good prob. of a colorful path

Random Coloring: Success Prob.

Lemma: Let c be a random k-coloring of V, and p be a $(k-1)$-path.

Then $\Pr[p \text{ is coloful}] > \; e^{-k}$

Proof: Fix the colors of the nodes outside of p

We get k^k different colorings of p

Each with equal probability

Of these, $k!$ are colorful

Thus $\Pr[p \text{ is colorful}] = \frac{k!}{k^k} \; > \left(\frac{k}{e}\right)^k / k^k \; = e^{-k}$

□

Stirling:
$k! \geq \sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k}$

Finding Colorful Paths

Approach 1: dynamic program

Given c colored graph G

Table entries:

For a subset S of our colors, and vertex u:
$\textbf{Path}(S, u)$ = true if and only if there is an S-colorful path ending at u

Recurrence:

$$\textbf{Path}(S, u) = \begin{cases} \bigvee_{uv \in E(G)} \textbf{Path}(S \setminus c(u), v), & \text{if } c(u) \in S \\ \text{false}, & \text{otherwise} \end{cases}$$

Runtime?

LONGEST PATH: colorful algorithm deterministic

Algorithm

What property of $\mathcal{C}$ do we need?

1. repeat for each coloring $c \in \mathcal{C}$:

sufficient: $\forall S \subseteq V$ with $|S| = k$: $\exists c \in \mathcal{C}$: S is colorful

3. If there is a colorful path, return YES

4. Return NO

Thm [§5.6]: There is $\mathcal{C}$ with this property and $|\mathcal{C}| \in 2^{O(k)} \log n$ so that $\mathcal{C}$ can be produced in $O(|\mathcal{C}|)$ time.

Runtime: ~~e^k~~ $|\mathcal{C}|$ iterations each $O(2^k \cdot m)$ time

total: $O(\alpha^k \cdot m \log n)$ time

Thm: There is an ~~randomised~~ algorithm that solves LONGEST PATH in $O^*(\alpha^k \cdot m \log n)$ time

Color Coding: User's Guide

Given: Graph G
Question: Is H a(n induced) subgraph of G (graph H, $|H| =$

1. Randomly color vertices
2. Show that Pr[copy of H is colorful] $\geq 1/f(k)$
3. Find colorful copy of H in FPT-time
4. repeat $O(\ f(k)\ )$ times

(**5.** Derandomise)

Longest Path: Approach 3 (sketch)

#Multilabeled Walks

Given: Graph $G = (V, E)$, vertex $v \in V$, number k, edge labels $\lambda(e) \in \Lambda = \{1..k\}$

Question: How many walks in G start at v, have length k, but don't use an edge label twice?

Algorithm: dynamic program

Recurrence: $A(S, u)$ with $S \subseteq \Lambda, u \in V$

Runtime: $O(2^k n^2)$

Counting Multilabeled Walks: Algorithm

1. Copy each edge k times and apply labels $1..k$

2. $a \leftarrow$ number of multilabeled k-walks

3. If a is odd, return $\textsc{Yes}$ $\leftarrow$ correct!

4. Otherwise: return $\textsc{No}$

Lemma: Non-simple walks are counted evenly.

Problem: What happens if the number of k-paths is even?

An "Isolation" Lemma

Parameterized Algorithms
Lemma 11.5

Lemma: Let $\mathcal{F}$ be a family of subsets of U, with $|U| = n$.

Indep. at random, assign each $x \in U$ weight $\omega(x)$ from $\{1..N\}$ 1..2n

with probabilty at least $1 - n/N$ 1/2 we have:

$$\text{argmin}_{S \in \mathcal{F}} \sum_{v \in S} \omega(v) \text{ is unique.}$$

not exam material

Proof: Let $\alpha(x) = \min_{S \in \mathcal{F}, x \notin S} \omega(S) - \min_{S \in \mathcal{F}, x \in S} \omega(S \setminus \{x\})$

since $\alpha(x)$ does not depend on $\omega(x)$: $\Pr[\alpha(x) = \omega(x)] \leq 1/N$.

Thus: $\Pr[\exists x \in U \colon \alpha(x) = \omega(x)] \leq n/N$.

Suppose that $A \neq B \in \mathcal{F}$ are both minimum.

Now $\exists x \in U \colon \alpha(x) = \omega(B) - (\omega(A) - \omega(x)) = \omega(x)$.

This has probability at most n/N. □

Counting Multilabeled Walks: Algorithm

1. Copy each edge k times and apply labels $1..k$

2. $a \leftarrow$ number of multilabeled k-walks

3. If a is odd, return YES $\leftarrow$ correct!

4. Otherwise: return NO

Lemma: Non-simple walks are counted evenly.

Problem: What happens if the number of k-paths is even?

Solution: Isolation Lemma gives edge weights (with $\mathcal{F} =$ k-paths in G), such that a *k-path of minimum weight* is unique. Then we just expand DP to count weighted multilabled walks.