Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG
Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems
Institute for Informatics

Advanced Algorithms

Winter term 2019/20

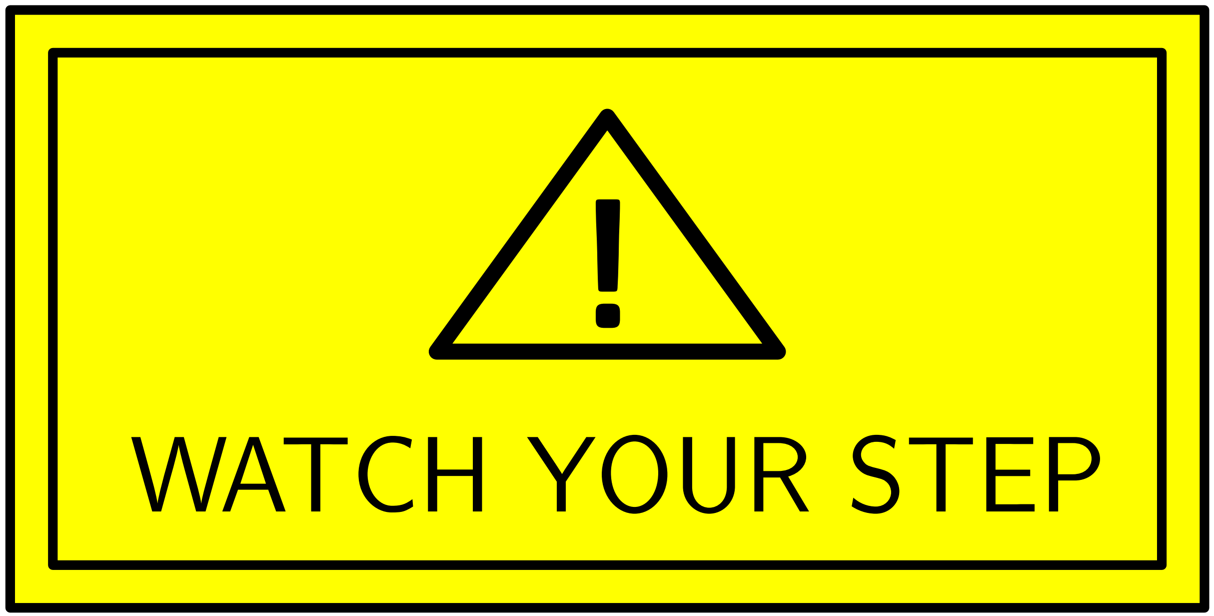
Lecture 6. Approaches using Randomisation

or: Color Coding and “Isolation” Lemmas

Source: **[Parameterized Algorithms: §3.3, 5, 5.1, 5.2, 11.2]**

(slides by Thomas van Dijk & Alexander Wolff)

Steven Chaplick and Alexander Wolff *Chair for Computer Science I*



In this lecture:

- Coloring $\neq$ **Graph**coloring.
- k-coloring of n elements:
  label each element with one number from $1..k$.

Randomised Algorithms

Probabilistic Polynomial Time ($\mathcal{PP}$)

Runtime: polynomial

Result: $\textsc{Yes}$-instance $\rightarrow$ $\Pr[\textsc{Yes}] > \frac{1}{2}$

$\textsc{No}$-instance $\rightarrow$ $\Pr[\textsc{Yes}] \leq \frac{1}{2}$

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$$\text{SAT} \overset{?}{\in} \mathcal{PP}$$

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$$\text{SAT} \overset{?}{\in} \mathcal{PP}$$

1. Randomly assign binary values to variables.
Return $\textsc{Yes}$ when the formula is satisfied.

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Thm: $NP \subseteq \mathcal{PP} \subseteq PSPACE$

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Las Vegas ($\mathcal{ZPP}$), zero-error probabilistic polynomial time

Runtime: expected polynomial

Result: correct

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Monte Carlo:

bounded-error prob. poly. time

$\mathcal{BPP}$

Runtime: polynomial

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Las Vegas ($\mathcal{ZPP}$), zero-error probabilistic polynomial time

Runtime: expected polynomial

Result: correct

Monte Carlo:

		$\mathcal{BPP}$ (bounded-error prob. poly. time)	$\mathcal{RP}$ (randomised poly. time)
Runtime:	polynomial		
Result:	YES-instance → $\Pr[\text{YES}] >$	$\frac{2}{3}$	$\frac{1}{2}$
	NO-instance → $\Pr[\text{YES}] \leq$	$\frac{1}{3}$	0

Randomised Algorithms

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Runtime: polynomial

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bounded-error prob. poly. time

randomised poly. time

		$\mathcal{BPP}$	$\mathcal{RP}$	co-$\mathcal{RP}$
Runtime:	polynomial			
Result:	YES-instance → $\Pr[\text{YES}] >$	$\frac{2}{3}$	$\frac{1}{2}$	1
	NO-instance → $\Pr[\text{YES}] \leq$	$\frac{1}{3}$	0	$\frac{1}{2}$

Randomised Algorithms

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Monte Carlo:

Runtime: polynomial

Result:		$\mathcal{BPP}$	$\mathcal{RP}$	co-$\mathcal{RP}$
$\textsc{Yes}$-instance $\rightarrow$	$\Pr[\textsc{Yes}] >$	$\frac{2}{3}$	$\frac{1}{2}$	1
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bounded-error prob. poly. time

randomised poly. time

Thm: $\mathcal{ZPP} = \mathcal{RP} \cap$ co-$\mathcal{RP}$

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Las Vegas ($\mathcal{ZPP}$), zero-error probabilistic polynomial time

Runtime: expected polynomial

Result: correct

Monte Carlo:

		bounded-error prob. poly. time	randomised poly. time	
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Runtime:	polynomial			
Result:	$\textsc{Yes}$-instance $\rightarrow \Pr[\textsc{Yes}] >$	$\frac{2}{3}$	$\frac{1}{2}$	1
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Thm: $\mathcal{ZPP} = \mathcal{RP} \cap \text{co-}\mathcal{RP}$

Amplification

$$\mathcal{RP}: \quad \begin{array}{l} \textsc{Yes}\text{-Instance} \rightarrow \Pr[\textsc{Yes}] \geq t \\ \textsc{No}\text{-Instance} \rightarrow \Pr[\textsc{Yes}] = 0 \end{array}$$

Amplification

$$\mathcal{RP}: \quad \begin{array}{l} \text{Yes-Instance} \rightarrow \Pr[\text{Yes}] \geq t \\ \text{No-Instance} \rightarrow \Pr[\text{Yes}] = 0 \end{array}$$

If an $\mathcal{RP}$-algorithm returns Yes, it is correct

Amplification

$\mathcal{RP}$: $\quad$ $\textsc{Yes}$-Instance $\rightarrow$ $\Pr[\textsc{Yes}] \geq t$
$\quad\quad\quad$ $\textsc{No}$-Instance $\rightarrow$ $\Pr[\textsc{Yes}] = 0$

If an $\mathcal{RP}$-algorithm returns $\textsc{Yes}$, it is correct

If an $\mathcal{RP}$-algorithm returns $\textsc{No}$,
it is incorrect with probability $\leq 1 - t$

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Algorithm: Run the original algorithm $\lceil 1/t \rceil$ times
Return $\textsc{Yes}$ if every some returns $\textsc{Yes}$
Otherwise $\textsc{No}$

Amplification

$\mathcal{RP}$: $\textsc{Yes}$-Instance $\rightarrow$ Pr[$\textsc{Yes}$] $\geq t$
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Error Probability :

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Error Probability : $(1 - t)^{1/t}$

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Otherwise $\textsc{No}$ $\quad\quad$ $x := 1/t$

Error Probability : $\quad (1-t)^{1/t} = (1 - \frac{1}{x})^x$

Amplification

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(**Obs.:** Repeating $100 \cdot t^{-1}$ times $\rightsquigarrow$ error prob. $< 2^{-100}$.)

Feedback Vertex Set

Given:	Graph $G = (V, E)$, number k
Question:	$\exists\ S \subseteq V$ such that $\lvert S \rvert \leq k$ and $G[V \setminus S]$ is a forest?

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Reduction Rule: Delete vertices of degree < 2

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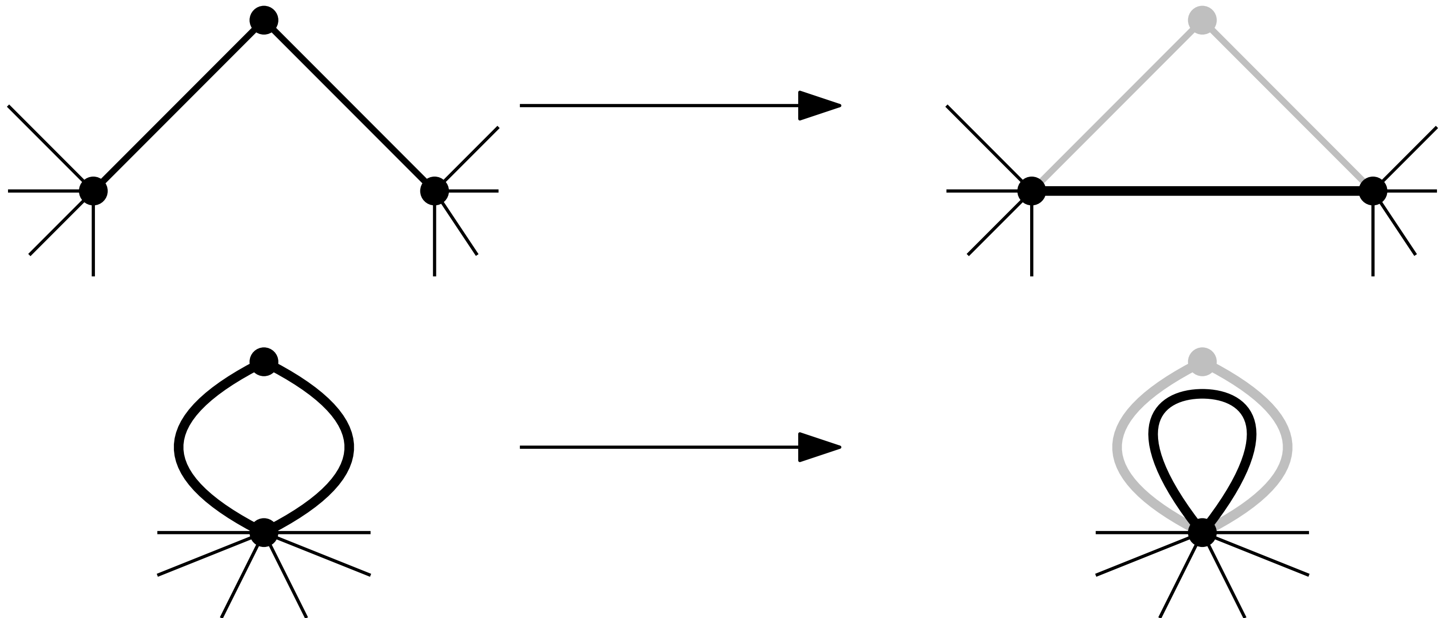
Reduction Rule: Delete vertices of degree < 2

Reduction Rule: "Bypass" each degree two vertex.



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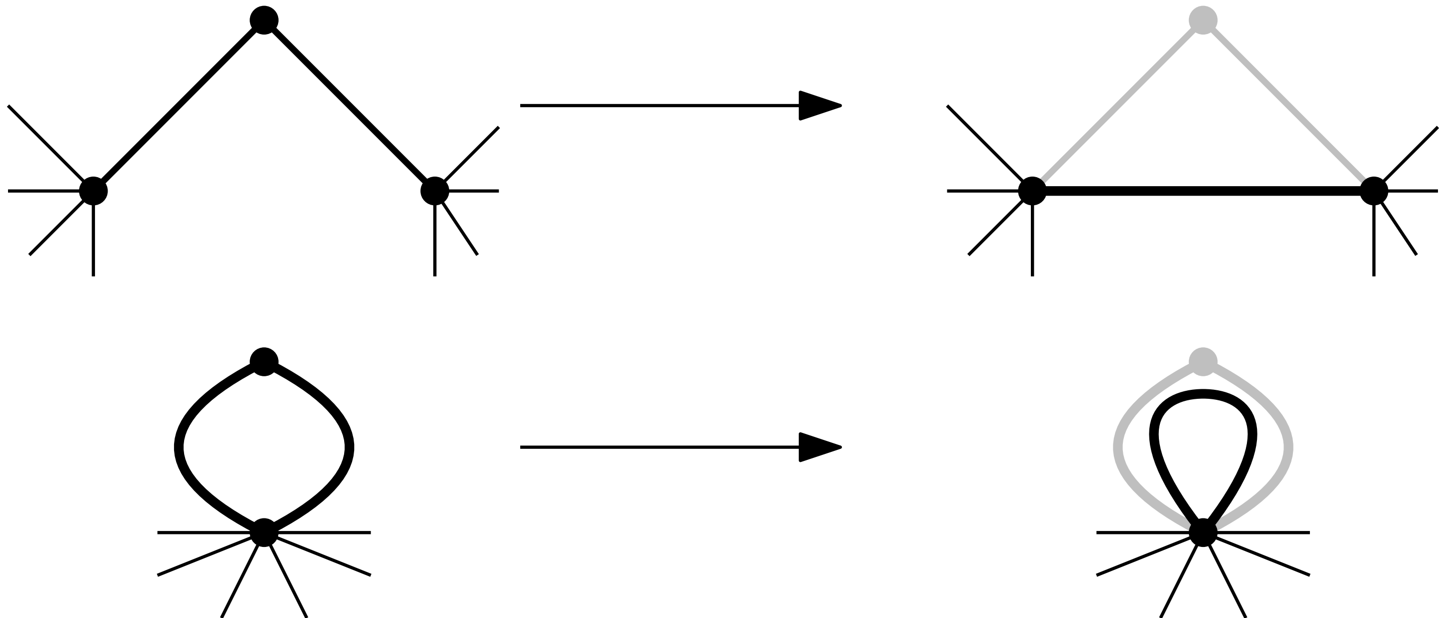
Reduction Rule: Put vertices incident to loops in FVS



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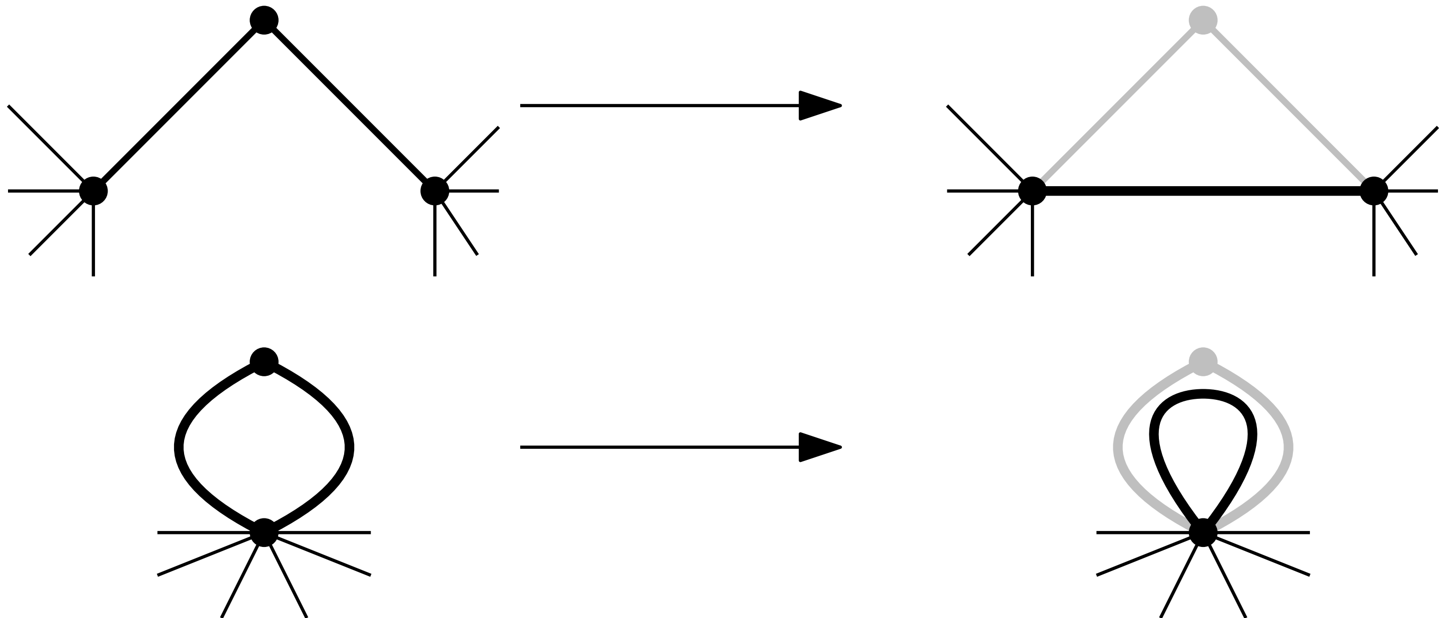
Def.: If no rule applies, the graph is called *reduced*.



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Let $E_W :=$ edges connecting vertices in W
Let $E_{S,W} :=$ edges connecting S and W

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$$\leq \sum_{v \in W} \deg(v)$$

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mindeg 3

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$$3|W| \leq \sum_{v \in W} \deg(v) = |E_{S,W}| + 2|E_W|$$

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$|E_W| < |W|$ since forest

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mindeg 3

$< \lvert W\rvert$ since forest

$$3\lvert W\rvert \leq \sum_{v \in W} \deg(v) = \lvert E_{S,W}\rvert + 2\lvert E_W\rvert < \lvert E_{S,W}\rvert + 2\lvert W\rvert$$

Lemma: If G is reduced, then $\lvert E_{S,W}\rvert \geq \lvert E_W\rvert$

(see also Lemma 5.1 in textbook)

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Idea: Find some $v \in S$

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Algorithm:

1. pick each $e \in E$ with equal prob.

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E_S**?**

Idea: Find some $v \in S$

Algorithm:

1. pick each $e \in E$ with equal prob. $> 1/2$

2. pick $v \in e$ with equal prob.

Success probability: at least $1/2$

Feedback Vertex Set

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Question: $\exists\ S \subseteq V$ such that $|S| \leq k$ and $G[V \setminus S]$ is a forest?

Def.: Let G be reduced, $S \subseteq V$ be an FVS, and $W := V \setminus S$.

Lemma: If G is reduced, then $|E_{S,W}| \geq |E_W|$

E_S**?**

Idea: Find some $v \in S$

Algorithm:

1. pick each $e \in E$ with equal prob.	$> 1/2$
2. pick $v \in e$ with equal prob.	$> 1/2$

Success probability: at least $1/2 \cdot 1/2 = 1/4$

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2. pick $v \in e$ with equal prob.	$> 1/2$

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Obs.: With prob. $\geq 1/4$, we find a node from an (unknown) optimal FVS

FVS: algorithm given k

1. `while` G is not empty:

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  4. $S \leftarrow S \cup \{v\}$; $G \leftarrow G \setminus v$

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5. If $|S| > k$: return $\textsc{No}$

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6. Return Yes

FVS: algorithm given k

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1. `while` G is not empty:

> **2.** Apply reduction rules
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> **3.** pick a vertex v via randomized proc. on last slide
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6. Return YES

Runtime:

Prob. of success:

FVS: algorithm given k

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Runtime:	$O(n+m)$
Prob. of success:	

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5. If $|S| > k$: return No

6. Return Yes

Runtime: $O(n + m)$

Prob. of success: $> 1/4$

FVS: algorithm given k

0. $S \leftarrow \varnothing$

1. `while` G is not empty: $\leftarrow$ max k times

2. Apply reduction rules

3. pick a vertex v via randomized proc. on last slide

4. $S \leftarrow S \cup \{v\}$; $G \leftarrow G \setminus v$

5. If $|S| > k$: return No

6. Return Yes

Runtime:	$O(n+m)$	$O(k(n+m))$
Prob. of success:	$> 1/4$	

FVS: algorithm given k

0. $S \leftarrow \varnothing$

1. `while` G is not empty: $\leftarrow$ max k times

2. Apply reduction rules

3. pick a vertex v via randomized proc. on last slide

4. $S \leftarrow S \cup \{v\}$; $G \leftarrow G \setminus v$

5. If $|S| > k$: return No

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Thm: Feedback Vertex Set can be solved in $O(4^k \cdot k(n+m))$ time by a randomised algorithm

Longest Path

Given:	Graph $G = (V, E)$, number k
Question:	Does G contain a length k path?

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1. Topological sort
2. Let $L(v)$:= longest path to v
3. "backwards" dynamic program
4. Look for v with $L(v) = k$

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Idea. Longest Path is easy on acyclic graphs

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Plan: make G acyclic!

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Proof: Consider perm. π, but ignore the elements of $p \rightsquigarrow \pi_{/p}$

π: 1 4 2 6 8 5 3 7

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$\pi_{/p}$:	1 4 2 $\times$ 8 $\times$ $\times$ 7

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Runtime: $O^*(k!)$ iterations

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Thm: A randomised algorithm can solve Longest Path in $O^*(k! \cdot n)$ time

LONGEST PATH : attempt 2

Obs. LONGEST PATH is easy on acyclic graphs.

Longest Path : attempt 2

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Color vertices with $(k+1)$ colors

$(k$-path has $k+1$ vertices$)$

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Part 2: $\exists$ k-path in G $\rightarrow$ good prob. of a colorful path

Random Coloring: Success Prob.

Lemma: Let c be a random k-coloring of V, and p be a $(k-1)$-path.

Then $\Pr[p \text{ is coloful}] >$

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Stirling:
$k! \geq \sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k}$

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Finding Colorful Paths

Approach 1: dynamic program

Given c colored graph G

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Table entries:

Finding Colorful Paths

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Table entries:

For a subset S of our colors, and vertex u:
Path(S, u) = true if and only if there is an S-colorful path ending at u

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Approach 1: dynamic program

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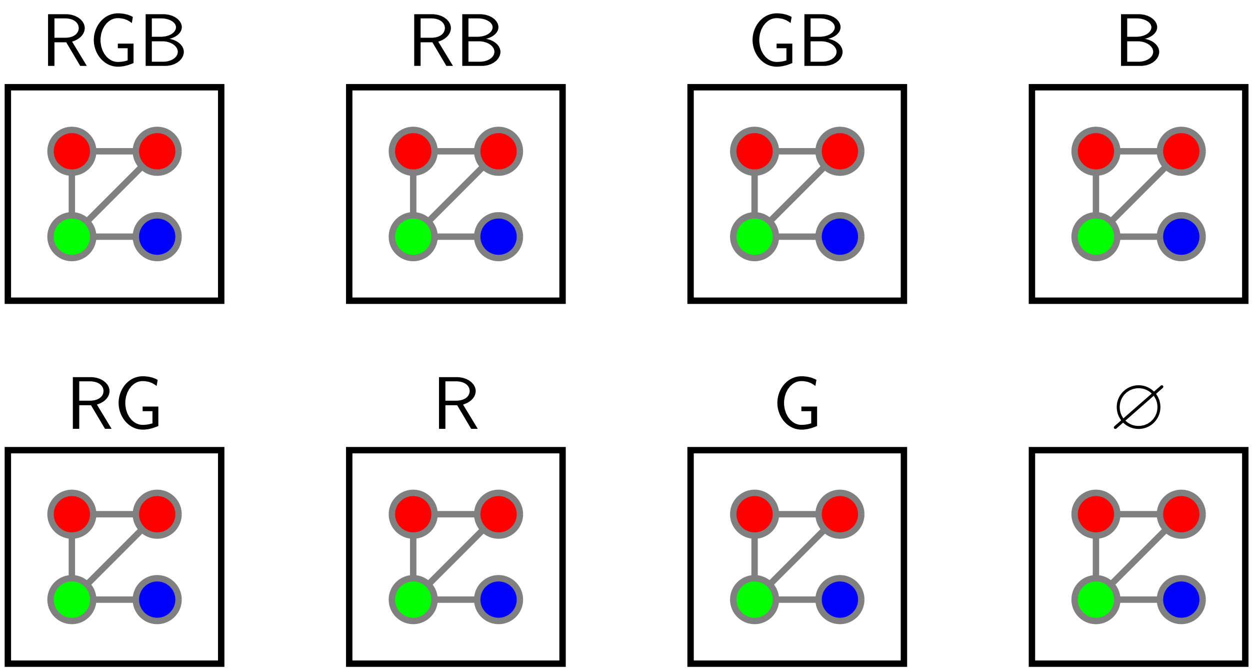
Runtime?

Finding Colorful Paths

Approach 2: For each subset S of the colors, create a copy G_S where G_S contains the vertices colored S and the edges are ...



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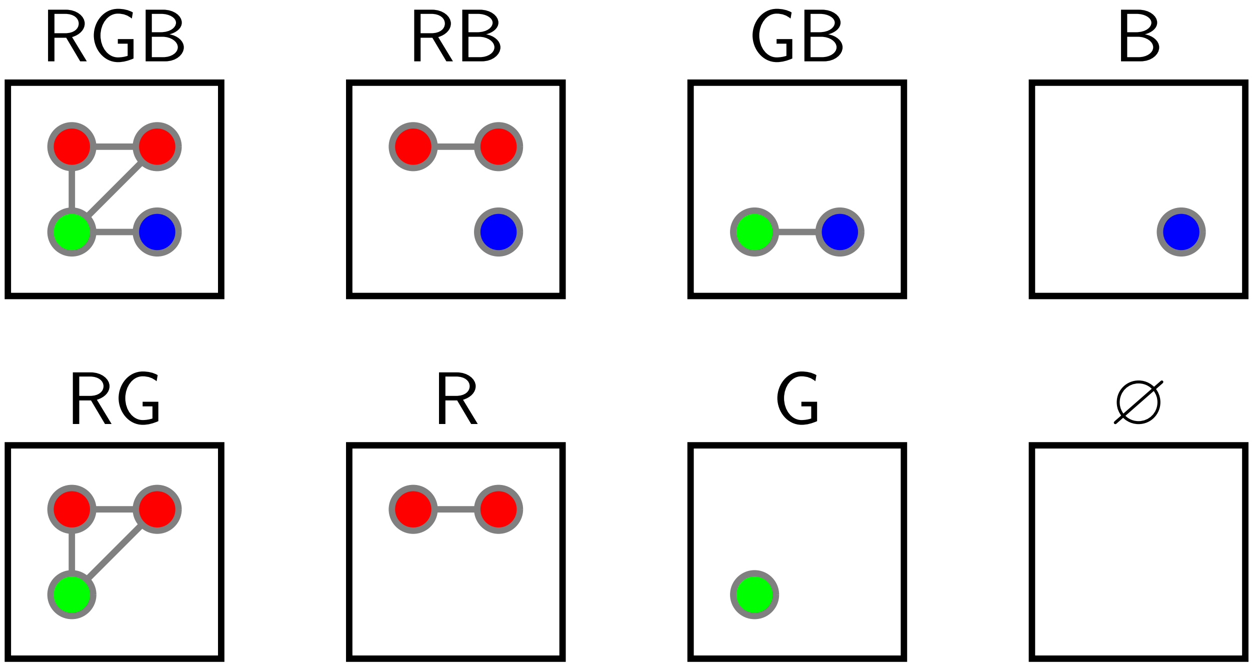

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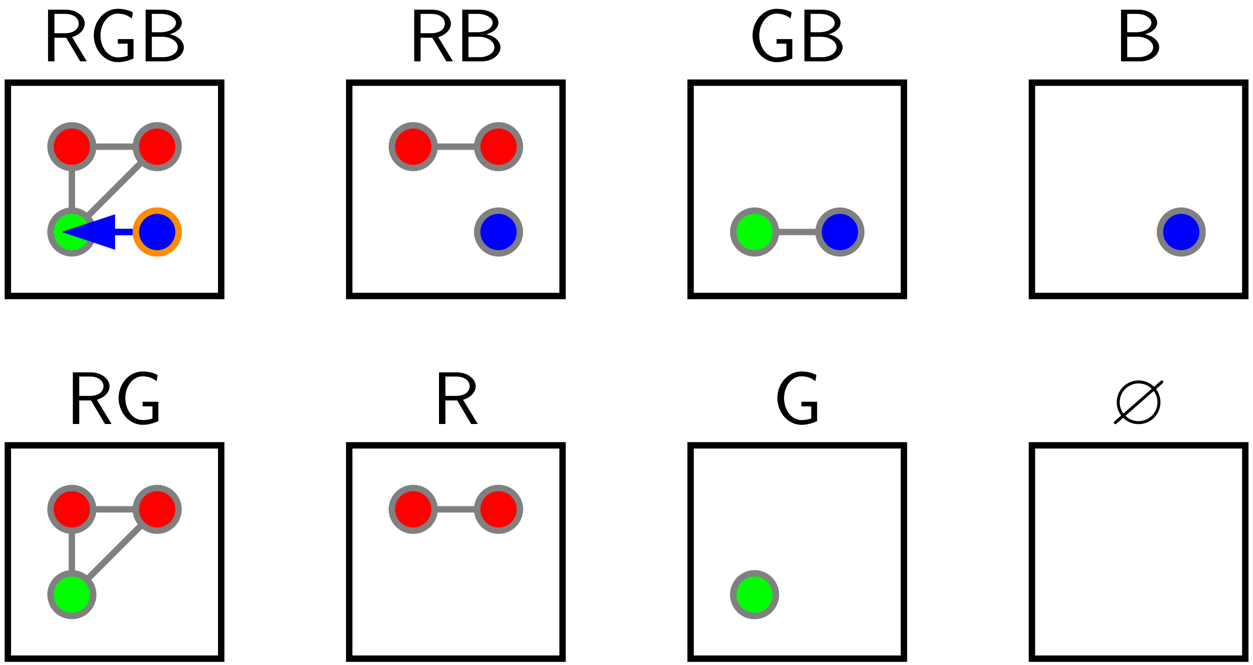

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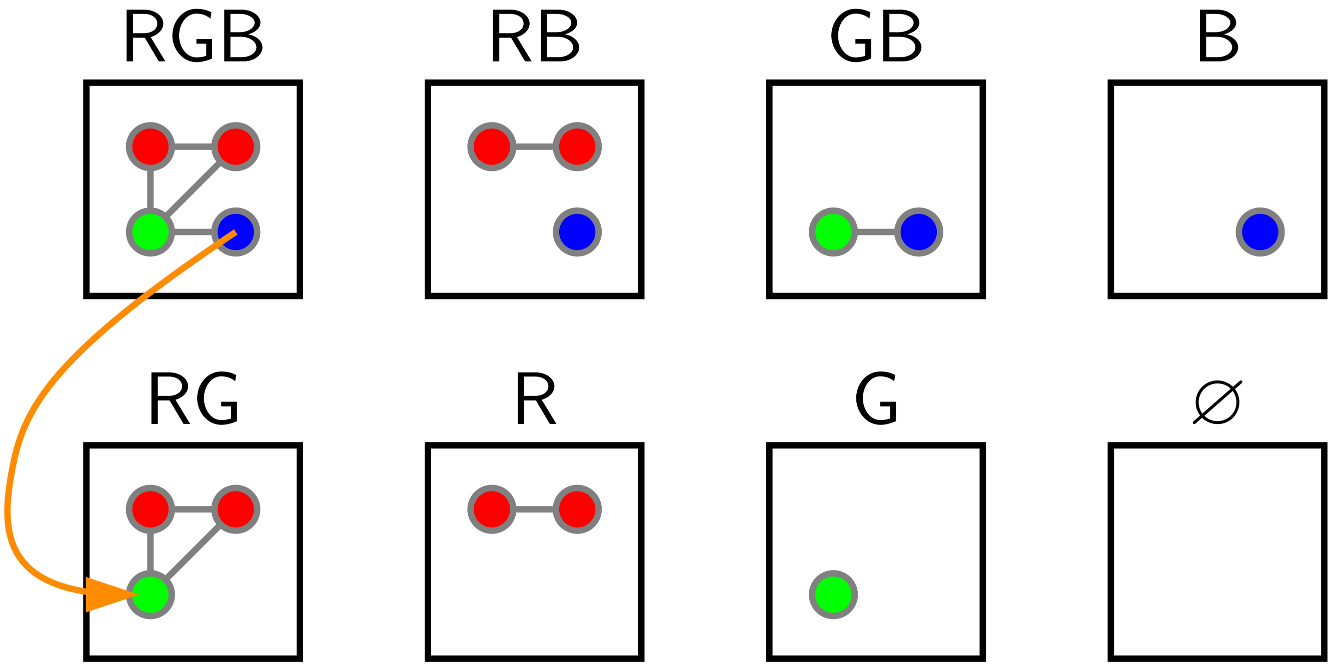

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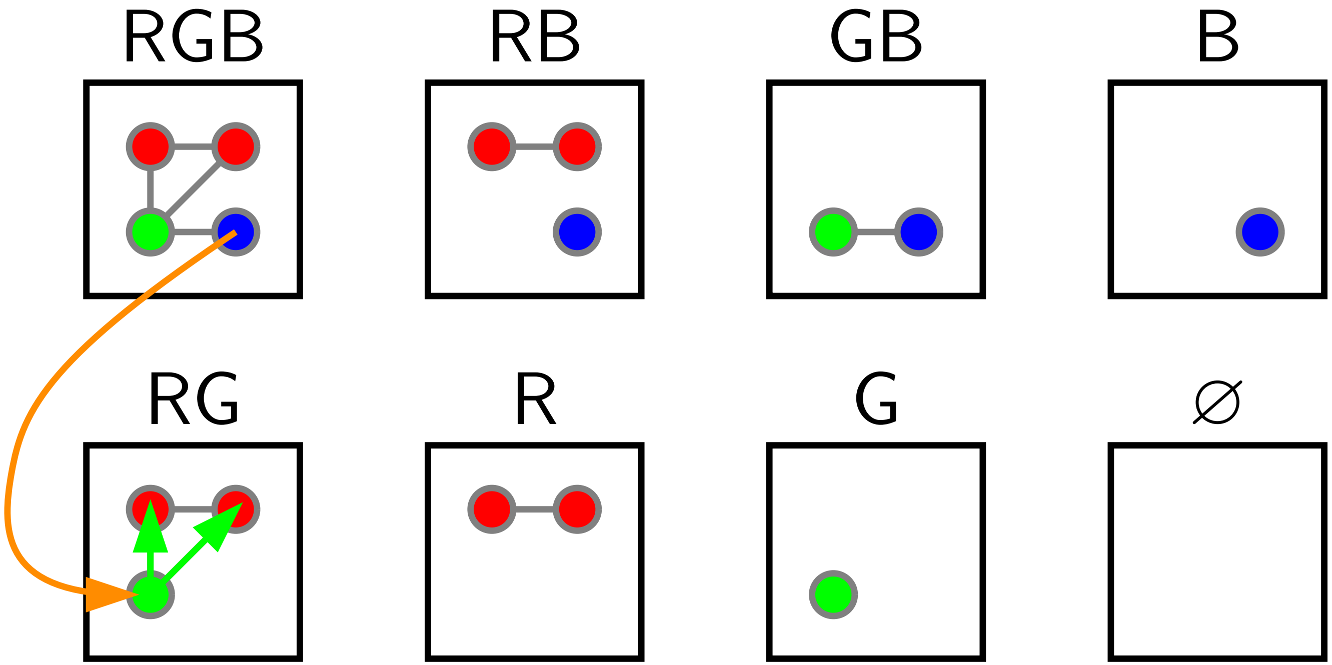

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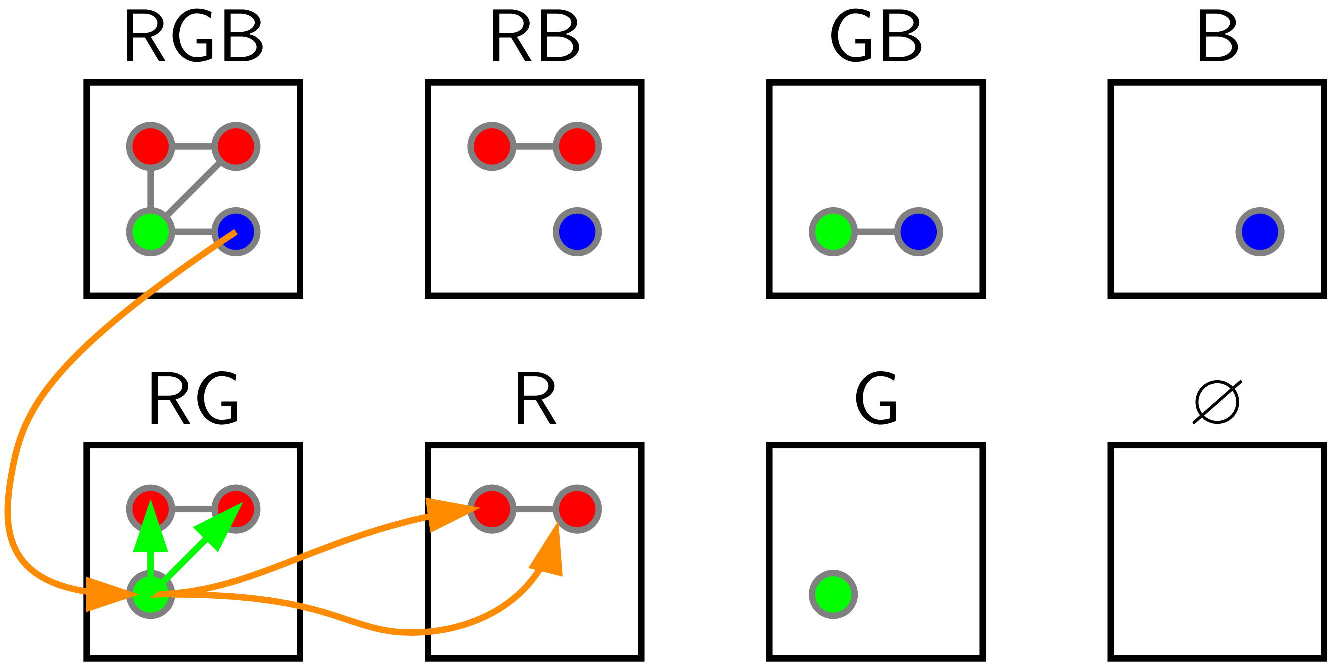

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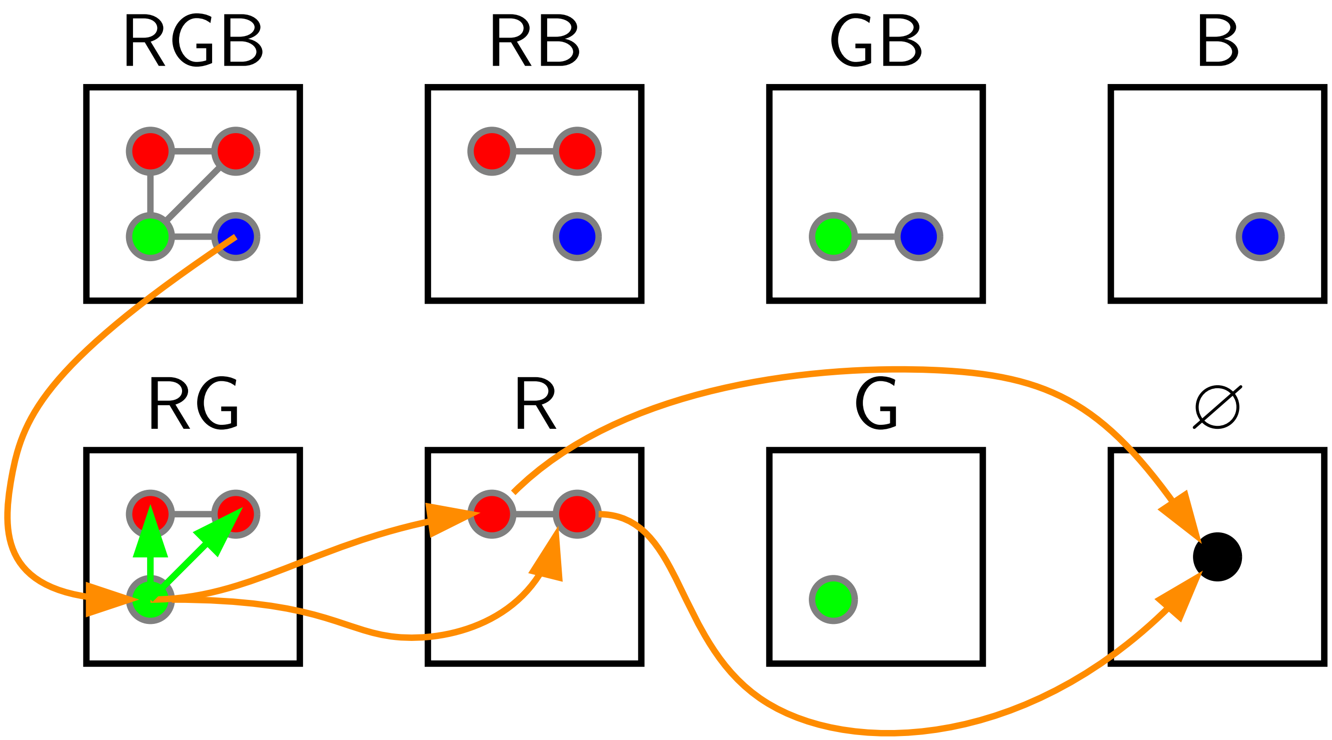

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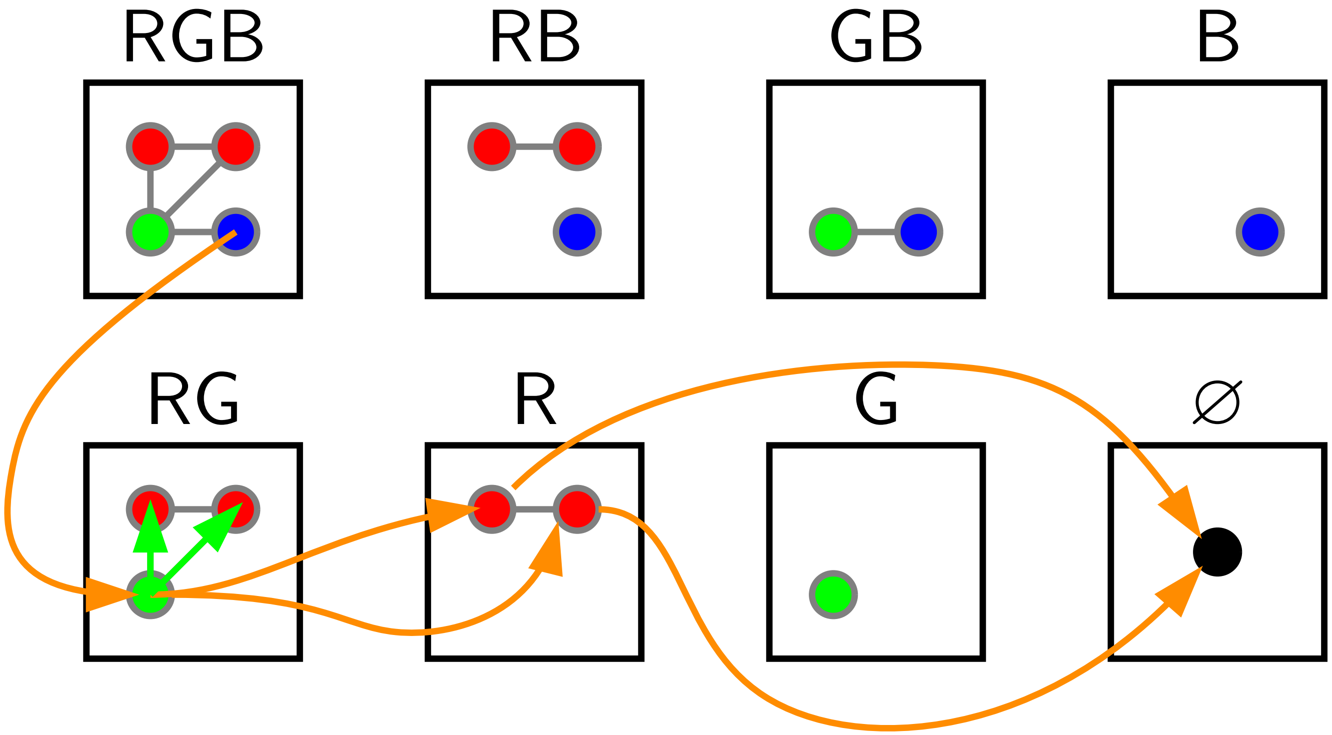

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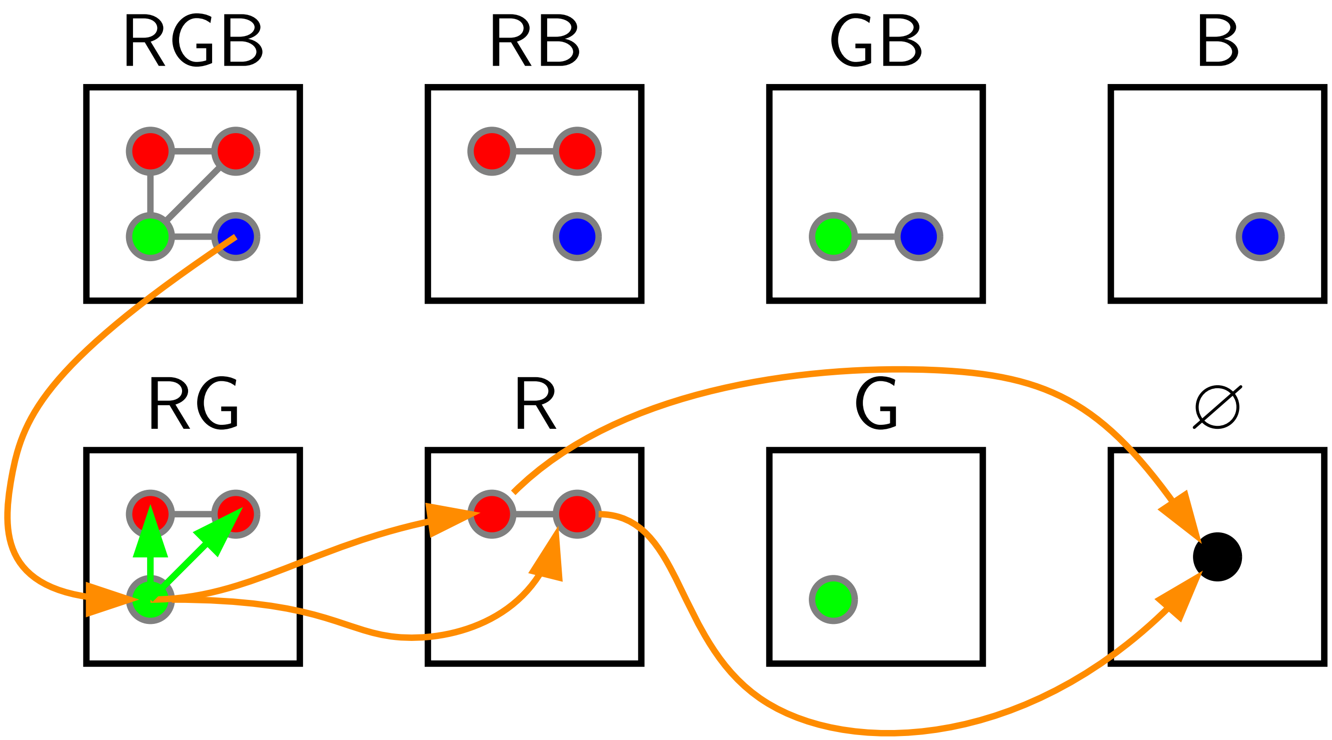

How big is this graph? $O(2^k \cdot n)$ vertices $O(2^k \cdot m)$ edges

Runtime:

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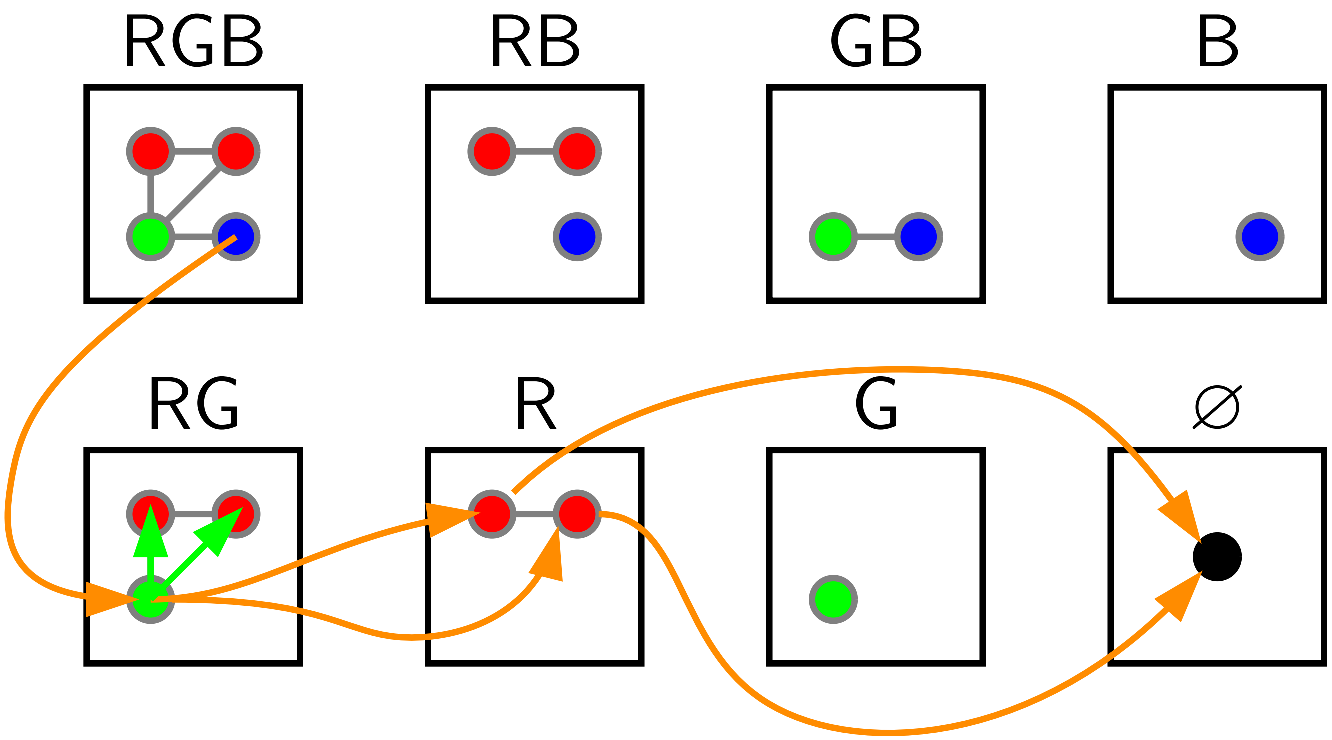

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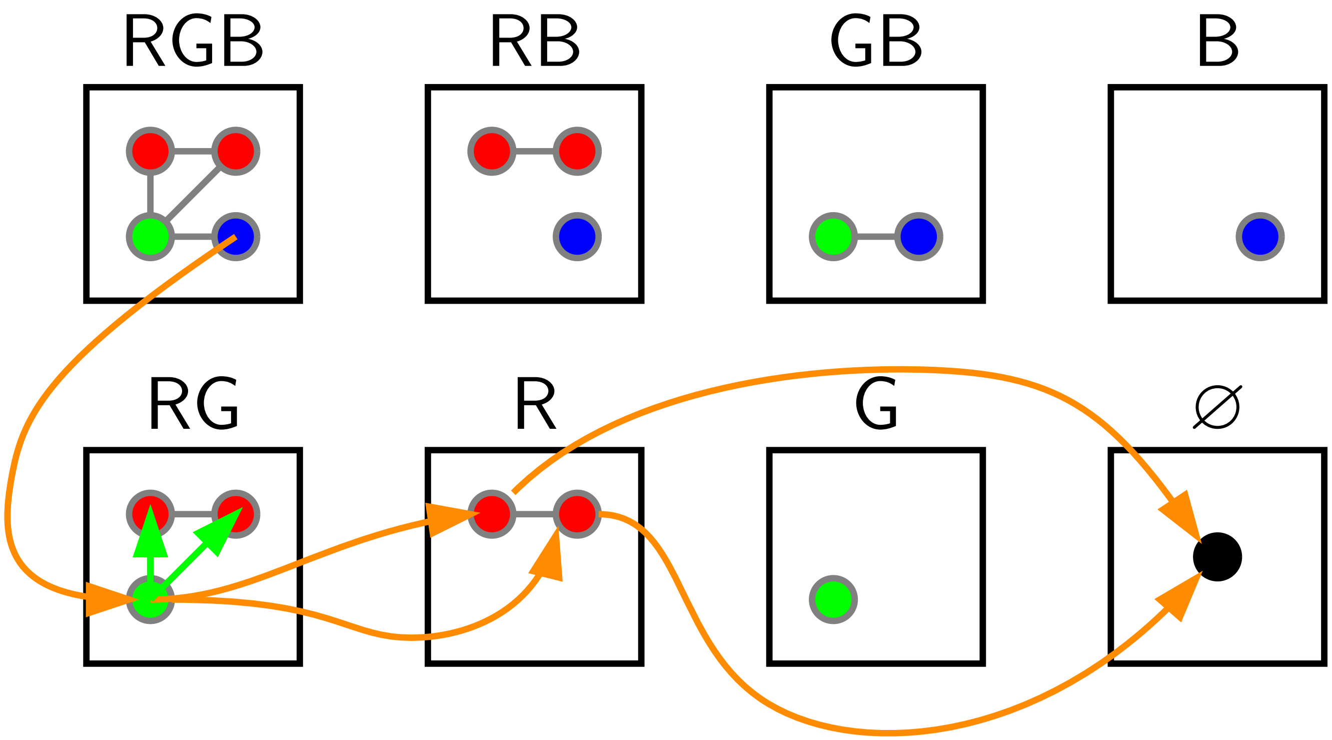

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LONGEST PATH: colorful algorithm

Algorithm

1. repeat e^k times:

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let us recall the purpose of this

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guarantees (randomised):
k-path in G $\rightarrow$ c colorful path

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1. repeat for each coloring $c \in \mathcal{C}$:

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1. repeat for each coloring $c \in \mathcal{C}$:

sufficient: $\forall S \subseteq V$ with $|S| = k$: $\exists c \in \mathcal{C}$: S is colorful

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Question: Is H a(n induced) subgraph of G (graph H, $|H| =$

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(**5.** Derandomise)

Longest Path: Approach 3 (sketch)

#Multilabeled Walks

Given: Graph $G = (V, E)$, vertex $v \in V$, number k, edge labels $\lambda(e) \in \Lambda = \{1..k\}$

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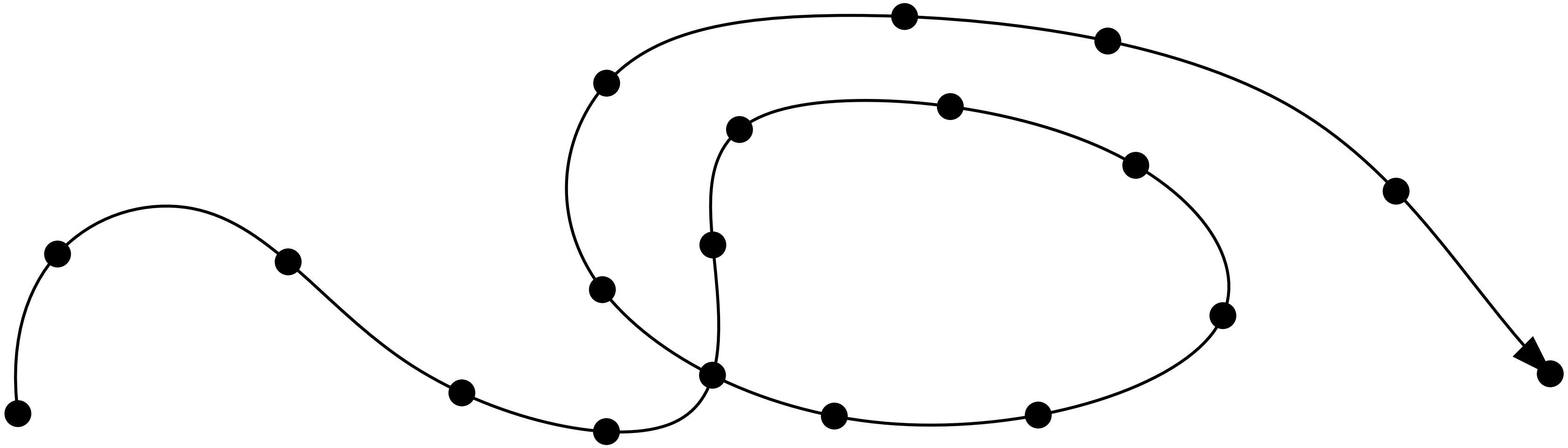

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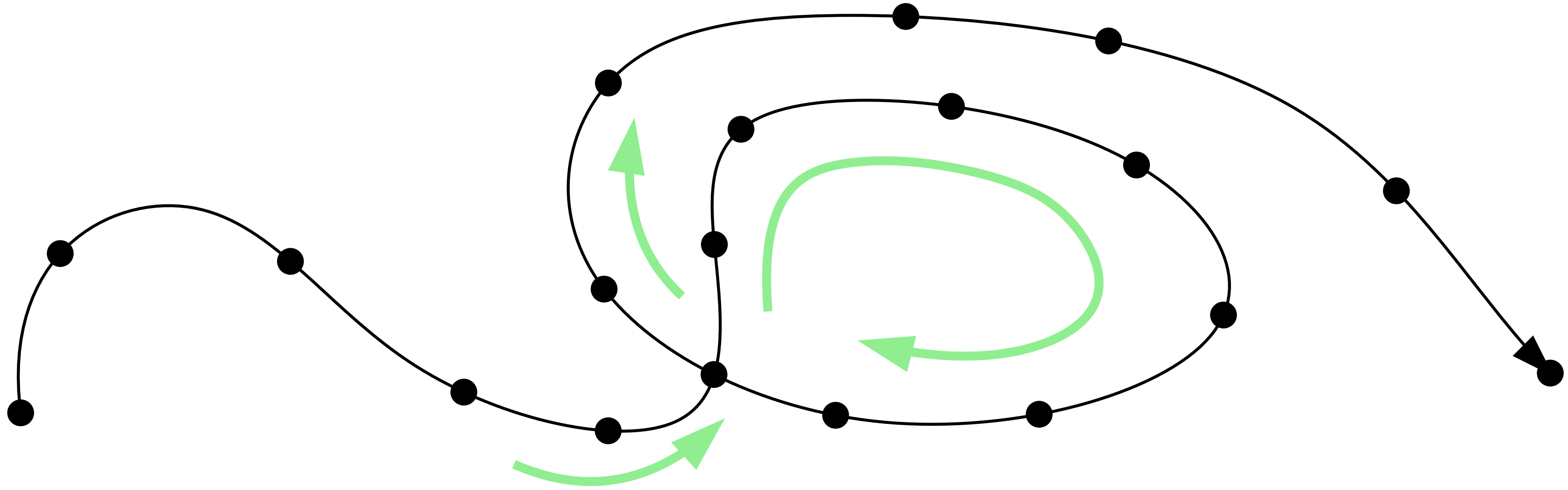

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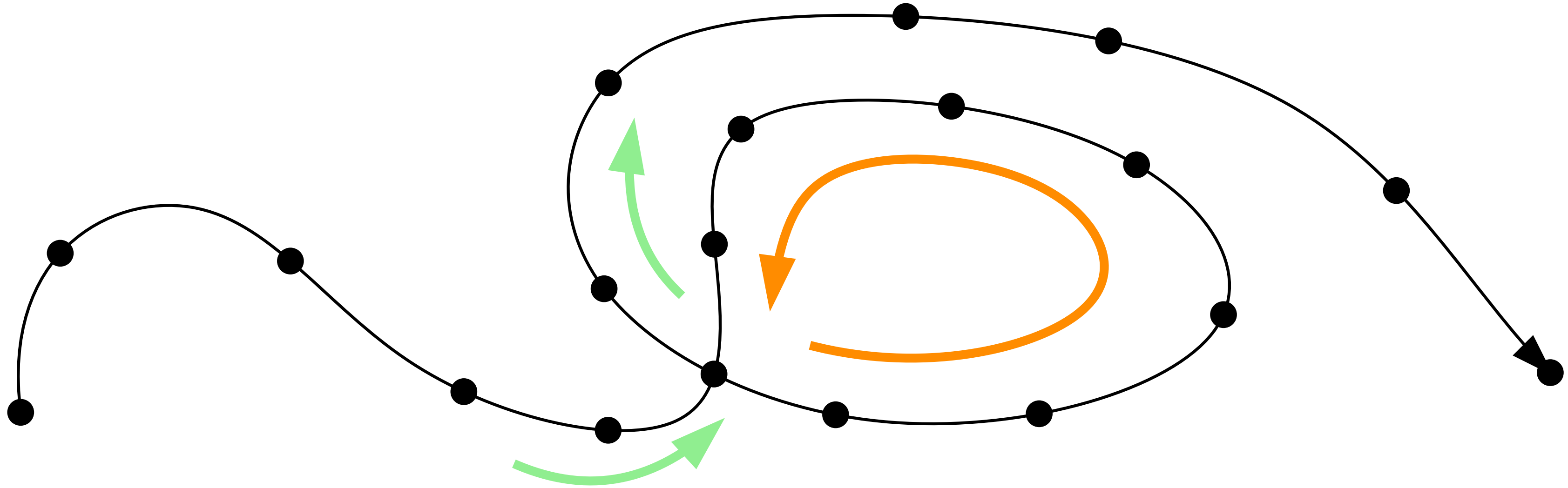

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Suppose that $A \neq B \in \mathcal{F}$ are both minimum.

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This has probability at most n/N. $\square$

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Lemma: Non-simple walks are counted evenly.

Problem: What happens if the number of k-paths is even?

Solution: Isolation Lemma gives edge weights (with $\mathcal{F} =$ k-paths in G), such that a *k-path of minimum weight* is unique. Then we just expand DP to count weighted multilabled walks.