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Advanced Algorithms

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Lecture 4. Randomized Algorithms

(based on lecture notes of Sabine Storandt)

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Randomized Algorithms

- are faster or use less space than deterministic algorithms in practise,
- have theoretical runtimes beyond deterministic lower bounds,
- are easier to implement/more elegant than deterministic strategies,
- allow for trading runtime against output quality,
- provide a good strategy for games or search in unknown environments.

Some Basics

A (discrete) random variable X maps a (finite) set Ω of possible outcomes of a random experiment to some measurable set Ω' of observations (e.g., $\mathbb{N}$ or $\mathbb{R}$).

Example: dice: {⚀,⚁,⚂,⚃,⚄,⚅} $\rightarrow \{1, 2, 3, 4, 5, 6\}$

The *expected value* of a discrete random variable X is

$$\mathbf{E}[X] = \sum_{i \in \Omega'} i \cdot \mathbf{Pr}[X = i].$$

Example: $\mathbf{E}[\text{fair dice}] = (1 + 2 + 3 + 4 + 5 + 6)/6 = 3.5$

strange dice: {⚀,⚀,⚀,⚅,⚅,⚅} $\rightarrow \{1, 1, 1, 6, 6, 6\}$

$$\mathbf{E}[\text{strange dice}] = (1 + 1 + 1 + 6 + 6 + 6)/6 = 3.5$$

First Success

Let $X\colon \{\text{failure}, \text{success}\} \to \{0, 1\}$ be a random variable.

Let $p = \mathbf{Pr}[X = 1]$ be the success probability.

$\Rightarrow$ $q := \mathbf{Pr}[X = 0] = 1 - p$ is the *failure probability* (or *rate*).

Repeat experiment many times.
Assume that outcomes are independent from each other.

Random variable Y counts the number of rounds until $X = 1$ for the first time.

$\Rightarrow \mathbf{Pr}[Y = j] = \; q^{j-1}p$

$\Rightarrow \mathbf{E}[Y] = \; \sum_{j=1}^{\infty} j \cdot q^{j-1}p = \; p \cdot (\sum_{j=1}^{\infty} q^j)' = p \cdot (\frac{1}{1-q})' =$

$= p \cdot (\frac{1}{p})' = \; p \cdot \frac{1}{p^2} = \; 1/p$ $\square$

Linearity of Expectation

Let X and Y be two random variables and $\lambda \in \mathbb{R}$. Then
$\mathbf{E}[X + \lambda \cdot Y] = \mathbf{E}[X] + \lambda \cdot \mathbf{E}[Y]$

Indicator Random Variables

Example I: Guessing cards (without memory). Deck of n cards.

Let $X_i \colon \{\text{guessed}, \text{not guessed}\} \to \{0, 1\}$ be a random variable that indicates whether card i was guessed or not ($i = 1, \ldots, n$).

$X_1, \ldots, X_n$ are *indicator* random variables.

$\Rightarrow \mathbf{E}[X_i] = \mathbf{Pr}[X_i = 1] =_{\text{here}} 1/n$

Let X count the number of correct guesses.

$\Rightarrow \quad X = X_1 + \cdots + X_n$

$\Rightarrow \mathbf{E}[X] = \mathbf{E}[X_1 + \cdots + X_n] = \mathbf{E}[X_1] + \cdots + \mathbf{E}[X_n] = n \cdot \frac{1}{n} = 1.$

Note that this is independent of n!

Using Indicator Random Variables

Example II: Guessing cards (*with* memory).

Now $\mathbf{Pr}[X_i = 1]$ depends on the current size of the deck.

$\mathbf{E}[X_i] = \mathbf{Pr}[X_i = 1] = 1/(n - i + 1)$

$\Rightarrow \mathbf{E}[X] = \mathbf{E}[X_1 + \cdots + X_n] = \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} + 1 = H_n$

H_n is the n-th harmonic number; $\ln(n+1) \leq H_n \leq \ln(n) + 1$.

$\Rightarrow \mathbf{E}[X] = H_n \in \Theta(\log n)$ *Note that this does depend on n!*

Example III: Collecting goodies

How often do you have to shop to collect all n goodies? (X)

Assume: Each time you get a random goodie. You can't choose.

$X_i :=$ number of times you must shop to get i-th new goodie.

$\mathbf{Pr}[X_i = 1] = (n - i + 1)/n \Rightarrow \mathbf{E}[X_i] = n/(n - i + 1)$

$\Rightarrow \mathbf{E}[X] = \mathbf{E}[X_1 + \cdots + X_n] = n(\frac{1}{n} + \cdots + \frac{1}{2} + 1) = \Theta(n \log n)$

Las Vegas & Monte Carlo

Example IV: Drug detection (n lockers, $n/2$ with drugs)

Deterministic approach:

Need to break $n/2+1$ lockers in the worst case.

If students know your strategy, you must break *exactly* $n/2+1$.

Randomization removes the adversary.

RandA: – Compute random permutation of the lockers.
– Break lockers in this order. **Las Vegas Algorithm**
We break $n/2+1$ lockers in w-c, but expect to break fewer.

RandO: – Compute random permutation of $k \leq \frac{n}{2}+1$ lockers.
– Break lockers in this order. **Monte Carlo Algorithm**
We don't damage so many lockers, but may not find any drugs.

Analysis

RandA: expected number of broken lockers $= 1/(1/2) = 2$

RandO: failure probability for 1 locker $= \quad 1/2$
failure probability for k lockers $= (1/2)^k$
success probability for k lockers $= 1 - 2^{-k}$

Las Vegas Algorithm

Algorithm returns correct result, but resource (runtime) is a random variable.
Examples: RandomizedQuickSort, RandomizedSelect (Median)

Monte Carlo Algorithm

Algorithm errs or fails with certain probability, but runtime does not depend on random choices.
Example: Karger's randomized MinCut algorithm

Monte Carlo Example

Example V: Find large number ($\geq$ median, in array of n ints)

Deterministic approach:

Go through all elements, return maximum.
(Actually, suffices to go through $n/2$ elements.)

runtime $\Theta(n)$

```
MonteCarloFind(int[] A, int k ≥ 1)
    pick a_1, ..., a_k ∈ {1, ..., n} u.a.r.
    m = max{A[a_1], ..., A[a_k]}
    return m
```

[uniformly at random]

The algorithm has error probability $\leq 2^{-k}$.

Set $k := c \log_2 n$ for some constant $c > 1$.

$\Rightarrow$ Error probability $\leq n^{-c}$, runtime $\in O(\log n)$

Las Vegas Example

Example VI: Find repeated element
(array of $n \geq 4$ ints, $n/2$ distinct, $n/2$ identical)

Deterministic approach:

Sort and find repeated element. $\Theta(n \log n)$ time

Faster: Find median. $\Theta(n)$ time

LasVegasFindRepeated(int[] A)
while true **do**
  pick $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n\} \setminus \{i\}$, both u.a.r.
  if $A[i] == A[j]$ **then return** $A[i]$

Algorithm always returns correct result – but may take forever.

Success probability $= \dfrac{n/2}{n} \cdot \dfrac{n/2 - 1}{n - 1} \approx \dfrac{1}{4}$.

$\Rightarrow$ Expected number of iterations $\approx 4 \in O(1)$.

From Las Vegas to Monte Carlo

Theorem. (Markov inequality)
For any non-negative random variable X and $t \geq 1$,

$$\mathbf{Pr}[X > t] \leq \mathbf{E}[X]/t.$$

Equivalently,

$$\mathbf{Pr}[X > t \cdot \mathbf{E}[X]] \leq 1/t.$$

Let X be the running time of a Las Vegas algorithm and $f(n) = \mathbf{E}[X]$ its expected running time and $\alpha > 1$. Then

$$\mathbf{Pr}[X > \alpha \cdot f(n)] \leq 1/\alpha$$

So the probability that the Las Vegas algorithm does not find a solution in the first $\alpha \cdot f(n)$ steps is less than $1/\alpha$, which is the error probability of the respective Monte Carlo algorithm.

Closest Pair

Given a set $P = \{p_1, \dots, p_n\}$ of points in the plane, find a pair in $\binom{P}{2}$ whose Euclidean distance is minimum.

ADS: Deterministic divide-and-conquer algorithm,
worst-case runtime $O(n \log n)$.

Element Uniqueness Problem: Given n numbers, are they unique?
Cannot be solved in $o(n \log n)$ w-c time.

(under some assumption concerning the arithmetic model)

$\Rightarrow$ Closest Pair cannot be solved in $o(n \log n)$ w-c time.

(under the same assumption concerning the arithmetic model)

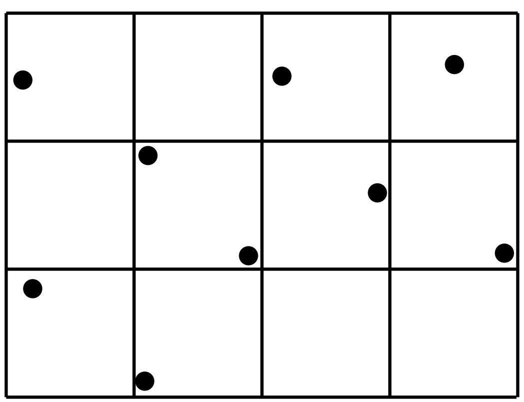
A Randomized Incremental Algorithm

Assume: – Can use the floor function in $O(1)$ time.
– Can use hashing in $O(1)$ time.

Define: $P_i = \{p_1, \ldots, p_i\}$
δ_i = distance of the closest pair in P_i.

Problem: Given δ_{i-1}, how can we compute δ_i?

Idea: Consider a square grid with cells of size $\delta_{i-1} \times \delta_{i-1}$:



How many points in P_{i-1} can lie in the same grid cell?

At most 4 (in the corners).

After finding p_i's cell, need to check only $O(1)$ points in vicinity.

Cases:
- $\delta_i < \delta_{i-1}$: Need to recompute grid in $O(i)$ time.
- $\delta_i = \delta_{i-1}$: Need to store p_i in its cell in $O(1)$ time.

Backwards Analysis

What is the w-c running time of the algorithm? $\Theta(n^2)$
How do we randomize? Randomly permute points at beginning.

How many points p in P_i have the property
that the minimum distance in $P_i \setminus \{p\}$ is larger than in P_i?

- The closest distance in P_i is unique: 2 points.
- One point has the same smallest distance to several points: 1 point.
- There are at least two disjoint closest pairs: 0 points.

Let X_i be the work for adding p_i.
$\Rightarrow \mathbf{E}[X_i] \leq 2/i \cdot O(i) + (i-2)/i \cdot O(1) = O(1)$

Let X be the total work done by the algorithm.
$\Rightarrow \mathbf{E}[X] = \mathbf{E}[X_1 + \cdots + X_n] \in O(n)$ $\square$