Julius-Maximilians-UNIVERSITÄT WÜRZBURG
Lehrstuhl für INFORMATIK I
Effiziente Algorithmen und wissensbasierte Systeme
Institut für Informatik

Advanced Algorithms

Winter term 2019/20

Lecture 1. Introduction & Held-Karp-algorithm for TSP

(slides by Joachim Spoerhase, Thomas van Dijk, & Alexander Wolff)

Steven Chaplick & Alexander Wolff *Chair for Computer Science I*

Advanced Algorithms

Learning goals: At the end of this lecture you will

- have an overview of advanced algorithmic topics (i.e., exact, approximate, geometric, and randomized computations), and advanced data structures,
- be able to analyze (and design algorithms for) new problems via the concepts of the lecture.

Requirements:

- Big-Oh notation (Landau); e.g., $O(n \log n)$
- Some *Algorithms & Data Structures*
  (Balanced) binary search tree, priority queue
- Some *Algorithmic Graph Theory*
  Breadth-first search, Dijkstra's algorithm
- Basic Theoretical Computer Science (P vs. NP)

Evaluation:

- oral exam at the end of the semester
- 0,3 bonus for 50% on the exercises

What is this course about?

Many important (practical) problems are NP-hard

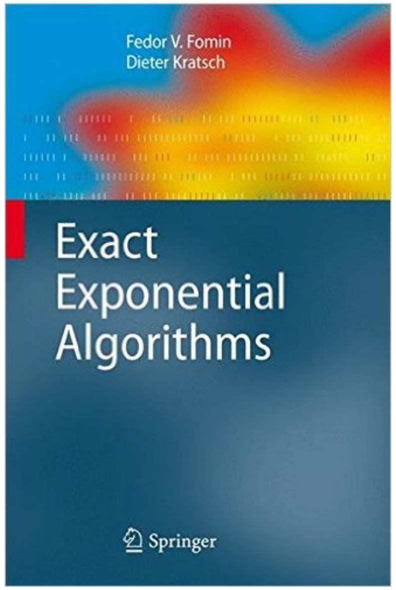
- Sacrifice optimality for speed
  - Heuristics (sim. Annealing, Tabu-Search)
  - Approximation Algorithms (Christofides-Algorithm)
- Optimal Solutions
  - Exact (exponential) time algorithms — Today's Lecture
  - Fine-grained analysis (parameterized) algorithms

Also, more on polytime solvable problems

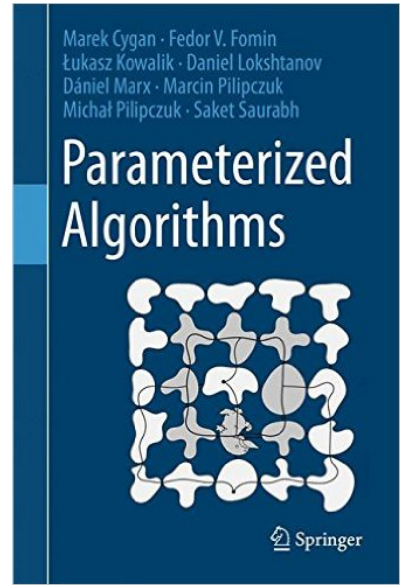
- Geometric algorithms (sweep-line approach)
- More graph algorithms (shortest paths w/ neg. weights)
- Advanded data structures (splay trees)
- Randomized algorithms

Textbooks

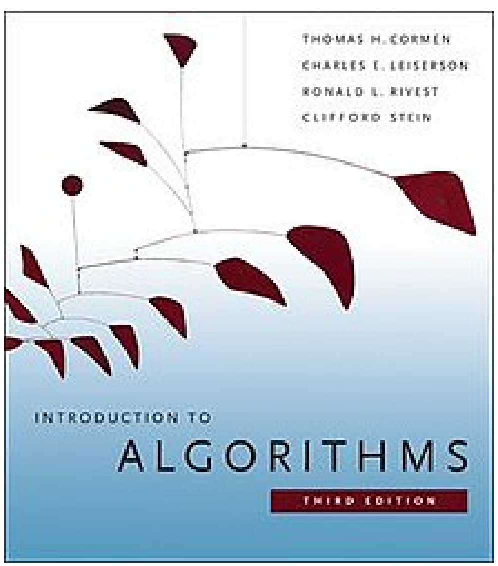
This Lecture: Chapter 1



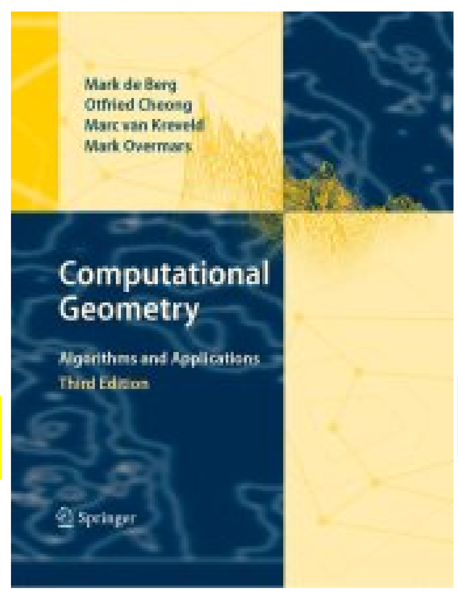
F. Fomin & D. Kratsch:
Exact Exponential Algorithms,
Springer 2010
abbrev: **EEA**



Marek Cygan et al.:
Parameterized Algorithms,
Springer 2015
abbrev: **PA**



C.L.R.S.:
Intro. to Algorithms
MIT Press
2009.
abbrev: **CLRS**

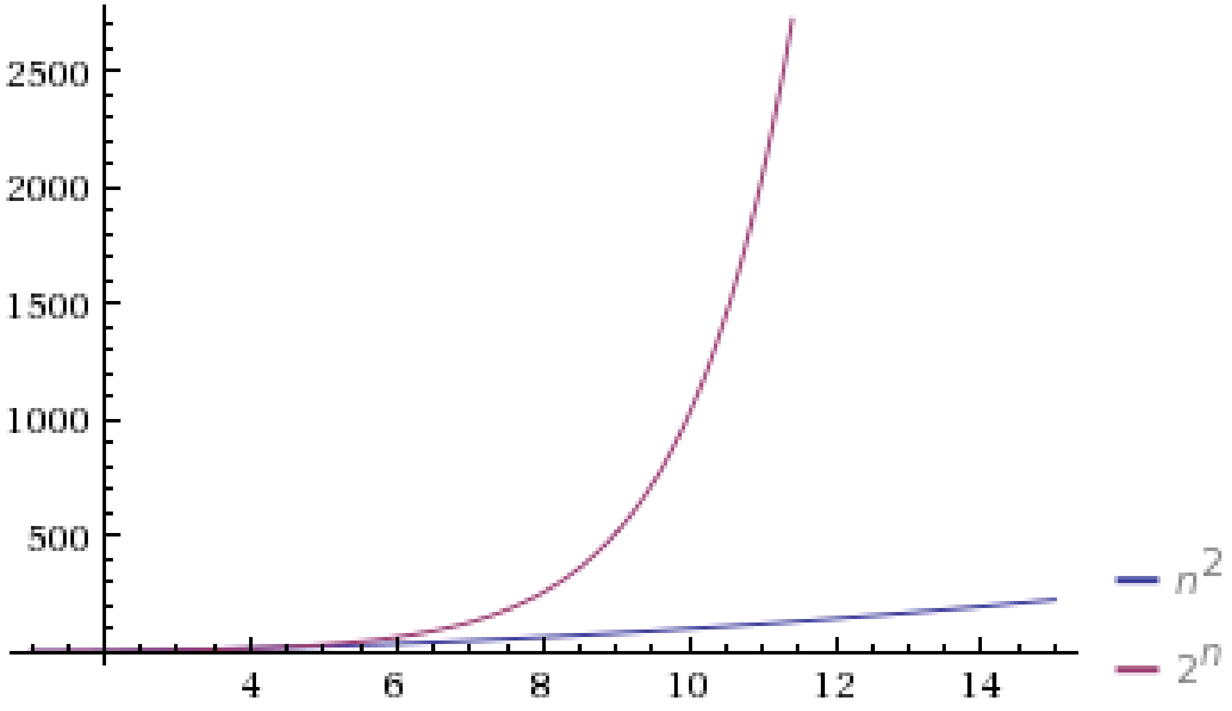


M. de Berg et al:
Computational Geometry:
Algorithms & Applications
Springer 2008,
3rd edition.
abbrev: **CG: A&A**

Background

- efficient vs. inefficient algorithms

$\rightsquigarrow$ polynomial vs. super-polynomial algorithms



Motivation: exact exponential algorithms

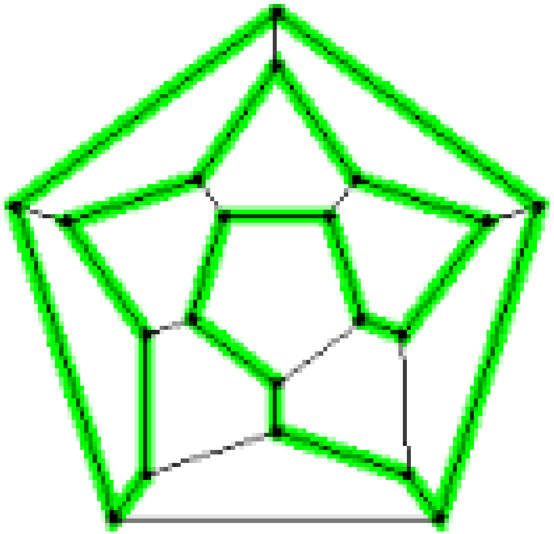
- can be “fast” for **medium-sized** instances

$\rightsquigarrow$ e.g.: $n^4 > 1.2^n$ for $n \leq 100$

$\rightsquigarrow$ e.g.: TSP solvable exactly for $n \leq 2000$ and specialized instances with $n \leq 85900$

$\rightsquigarrow$ “hidden” constants in polynomial time algorithms: $2^{100} \cdot n > 2^n$ for $n \leq 100$

- theoretical interest



Typical Results

- Idea (simplified): find exact algorithms which are faster than *brute force* (trivial) approaches.
- Typically results for a (hypothetical) NP-hard problem

Approach	Runtime in O-Notation	O^*-Notation
Brute-Force	$O(2^n)$	$O^*(2^n)$
Algorithm A	$O(1.5^n \cdot n)$	$O^*(1.5^n)$
Algorithm B	$O(1.4^n \cdot n^2)$	$O^*(1.4^n)$

$O(1.4^n \cdot n^2) \subsetneq O(1.5^n \cdot n) \subsetneq O(2^n)$

$\leadsto$ negligible polynomial factors (exp. dominates)

$f(n) \in O^*(g(n)) \Leftrightarrow \exists$ polynomial $p(n)$ w/ $f(n) \in O(g(n)p(n))$

Better Algorithms vs. Faster Hardware

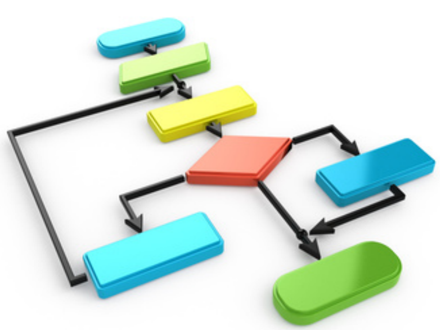
Suppose an algorithm uses a^n steps.

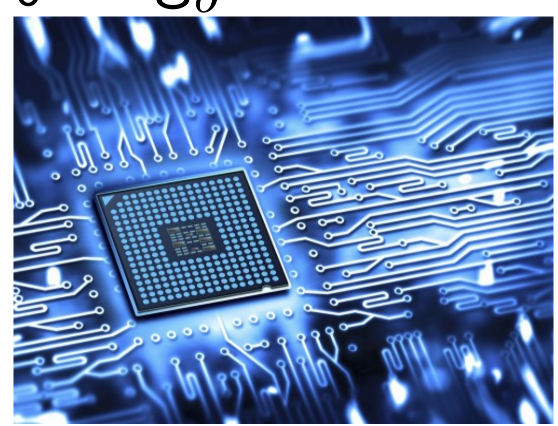
- For a fixed amount of time t, improving hardware by a constant **factor** c only **adds** a (relative to c) constant to the max. size of solvable instances (in time t).
- Whereas reducing the base of the runtime to $b < a$ results in a **multiplicative** increase!

Why?

Hardware speedup: $a^{n_0'} = c \cdot a^{n_0} \rightsquigarrow n_0' = \log_a c + n_0$

Base reduction: $b^{n_0'} = a^{n_0} \rightsquigarrow n_0' = n_0 \cdot \log_b a$





Traveling Salesperson Problem (TSP)

Input Complete directed graph $G = (V, E)$ with n vertices and edge weights $c\colon E \to \mathbb{Q}_{\geq 0}$

Output Hamiltonian cycle $(v_{\pi(1)}, \ldots, v_{\pi}(n), v_{\pi(n+1)} = v_{\pi(1)})$ of G, of minimum weight $\sum_{i=1}^{n} c(v_i, v_{i+1})$, permutation π.

Brute-Force?

- Each tour is a permutation π of the vertices.
- Pick a permutation with the smallest weight.

Runtime: $\Theta(n! \cdot n) = n \cdot 2^{\Theta(n \log n)}$

Bellman-Held-Karp-Algorithm

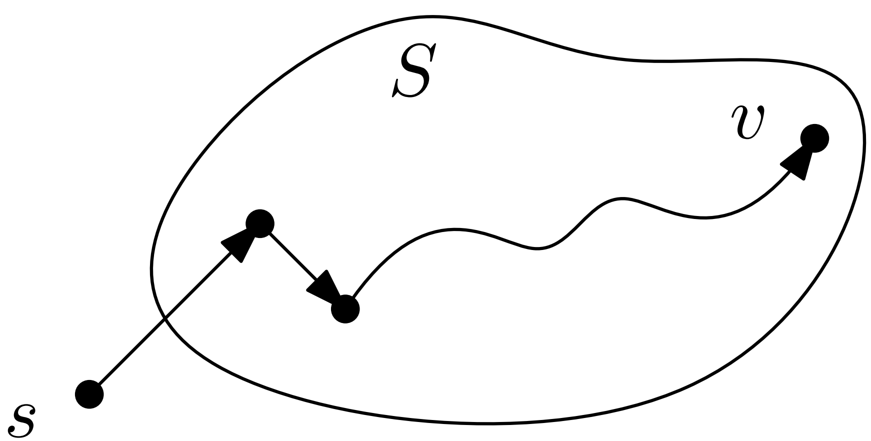
Technique: Dynamic Programming!

Reuse optimal substructures!

Select any starting vertex $s \in V$.

For each $S \subseteq V - s$ and $v \in S$, let:

$\text{OPT}[S, v]$ = length of a shortest s-v-path that visits precisely the vertices of $S \cup \{s\}$.



Richard M. Karp

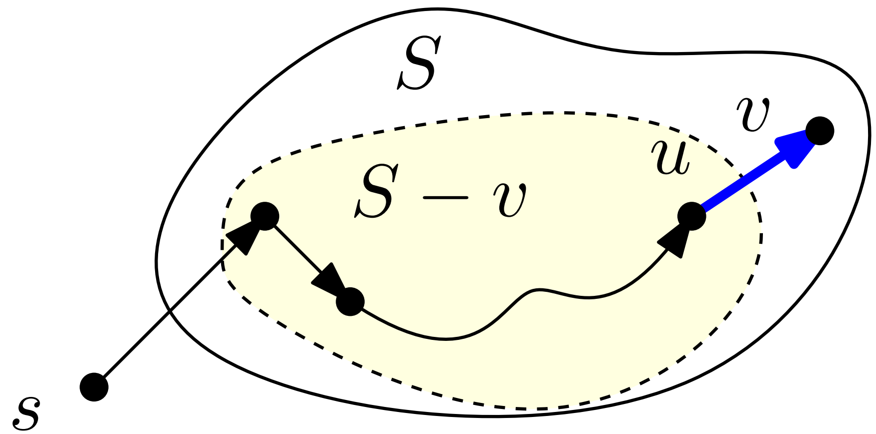
Richard E. Bellman

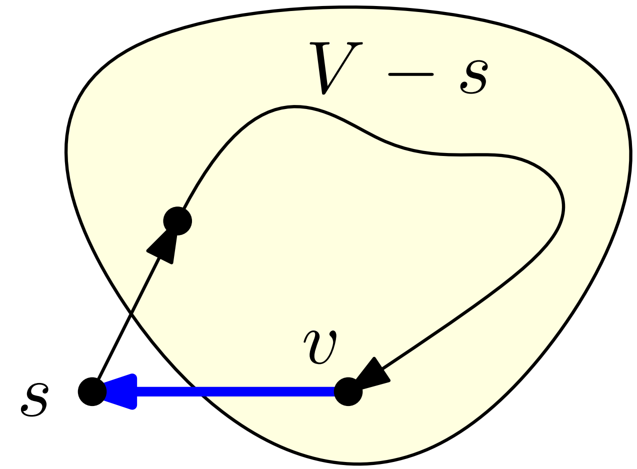
Bellmann-Held-Karp-Algorithm

The base case: $S = \{v\}$, is easy: OPT$[\{v\}, v] = c(s, v)$.

When $|S| \geq 2$, we compute OPT$[S, v]$ recursively:

$$\text{OPT}[S, v] = \min\{\ \text{OPT}[S - v, u] + c(u, v)\ \mid u \in S - v\ \}$$





After computing OPT$[S, v]$ for each $S \subseteq V - s$, the optimal solution is easily obtained as follows:

$$\text{OPT} = \min\{\ \text{OPT}[V - s, v] + c(v, s)\ \mid v \in V - s\ \}$$

Pseudocode for the dynamic program

```
Algorithm Bellmann-Held-Karp(G, c)
  foreach v ∈ V − s do
    OPT[{v}, v] = c(s, v)
  for j = 2 to n − 1 do                                    } O(2^n)
    foreach S ⊆ V − s with |S| = j do
      foreach v ∈ S do                                      } O(n)
        OPT[S, v] = min{ OPT[S − v, u] + c(u, v) | u ∈ S − v }
  return min{ OPT[V − s, v] + c(v, s) | v ∈ V − s }
```

Runtime: the innermost loop executes $O(2^n \cdot n)$ iterations where each one takes $O(n)$ time. Thus, in total, we have $O(2^n \cdot n^2) = O^*(2^n)$.

Space (memory) usage: $\Theta(2^n \cdot n)$

Only use table-values for $j-1$ to compute j, less space?

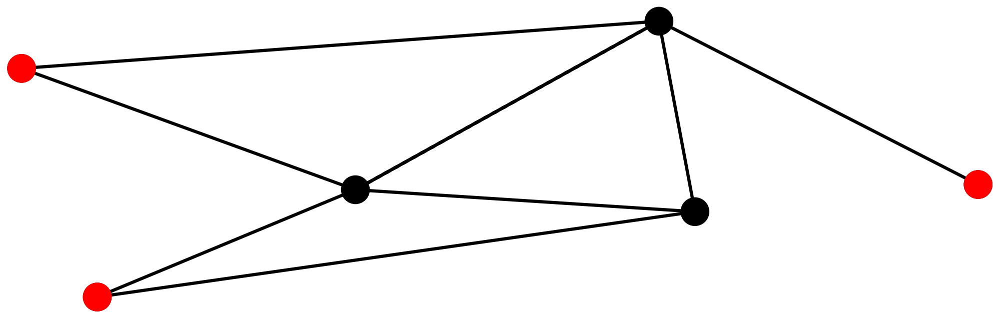
A shortest tour can be produced by backtracking the DP table (as usual).

Compare: $O^*(2^n)$ with $2^{O(n \log n)}$ for Brute-Force

Maximum Independent Set

Input Graph $G = (V, E)$ with n vertices.

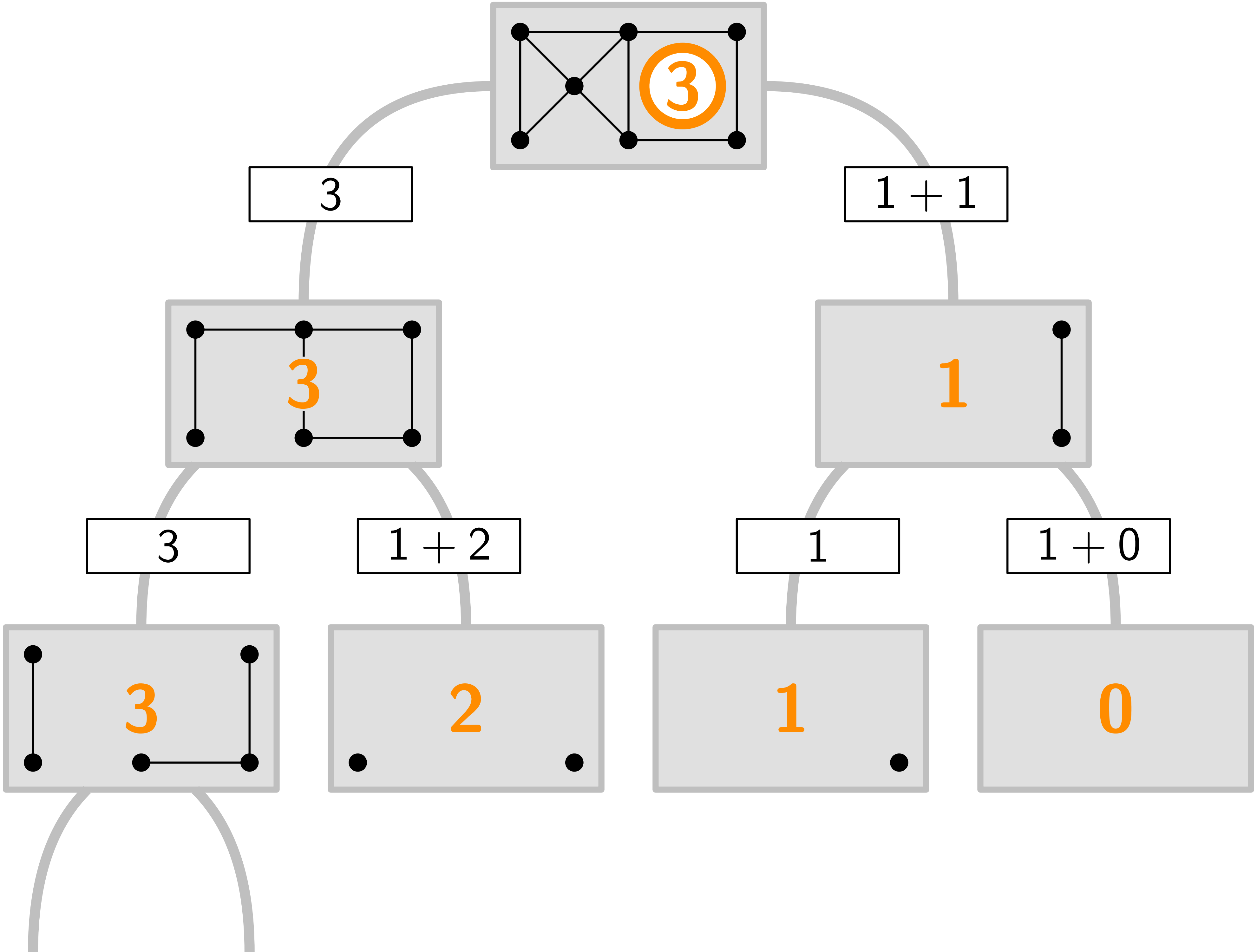
Output Maximum size *independent* set, i.e., a largest set $U \subseteq V$, such that no pair of vertices in U are adjacent in G.



Brute Force?

- Try all subsets of $V \rightsquigarrow O(2^n \cdot n)$ runtime.

Algorithm NaiveMIS(G)
  if $V = \emptyset$ **then**
    return 0
  $v \leftarrow$ arbitrary vertex in $V(G)$
  return max$\{1+$ NaiveMIS$(G - N(v) - \{v\})$, NaiveMIS$(G - \{v\})\}$



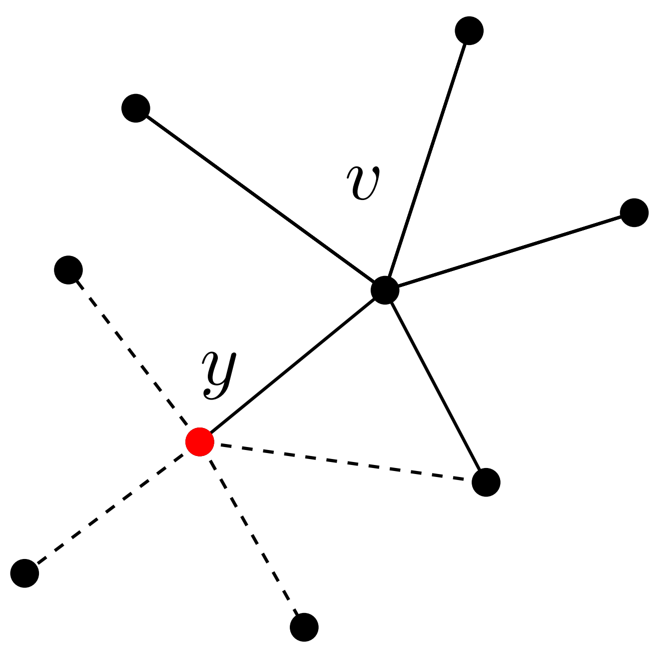
3
3
1 + 1
3
1
3
1 + 2
1
1 + 0
3
2
1
0

Observations

Lemma Let U be a *maximum* independent set in G. Then, for each vertex $v \in V$:

(i) $v \in U \rightsquigarrow N(v) \cap U = \emptyset$
(ii) $v \notin U \rightsquigarrow |N(v) \cap U| \geq 1$

Thus, $N[v] := N(v) \cup \{v\}$ contains some $y \in U$ and no other vertex of $N[y]$ is in U



Smarter Branching-Algorithm

Algorithm MIS(G)
  if $V = \emptyset$ **then**
    return 0
  $v \leftarrow$ vertex of minimum degree in $V(G)$
  return $1 + \max\{\text{MIS}(G - N[y]) \mid y \in N[v]\}$

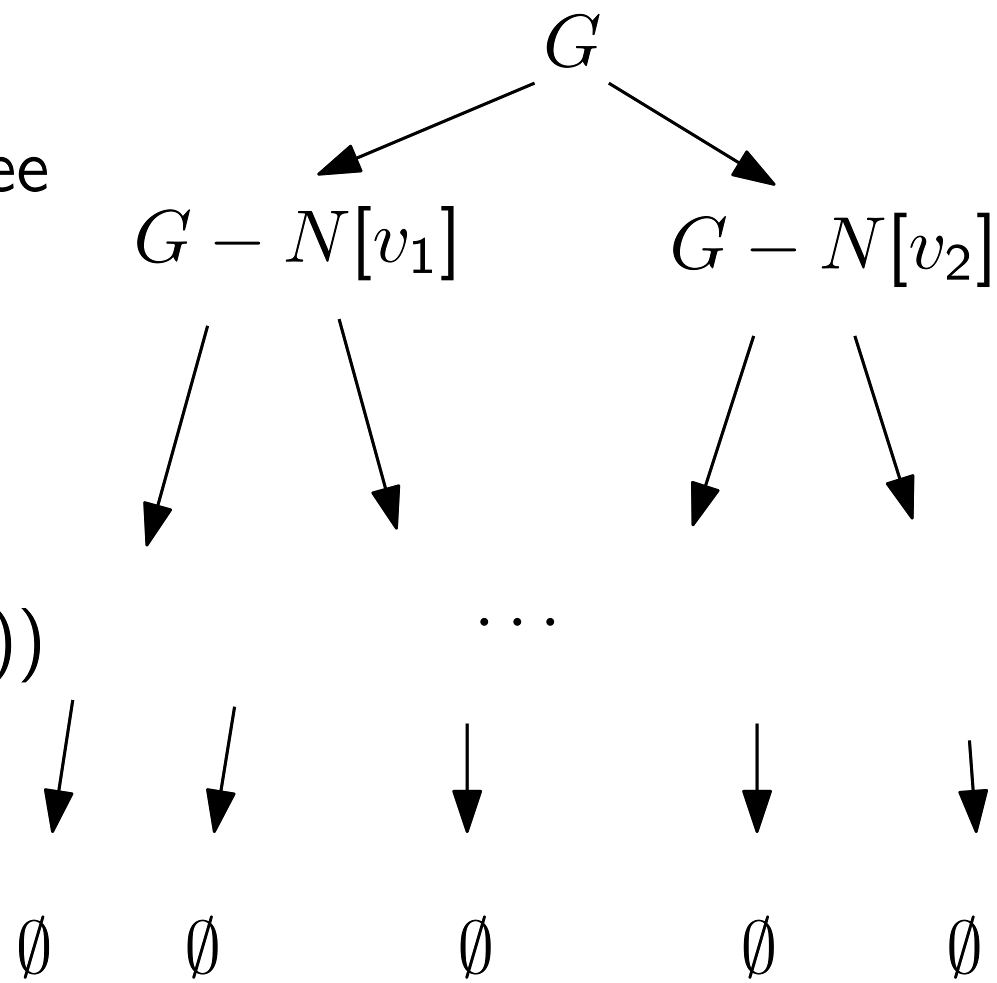
Correctness follows from the previous Lemma.

We will now prove a runtime of $O^*(3^{n/3}) = O^*(1.4423^n)$

Runtime

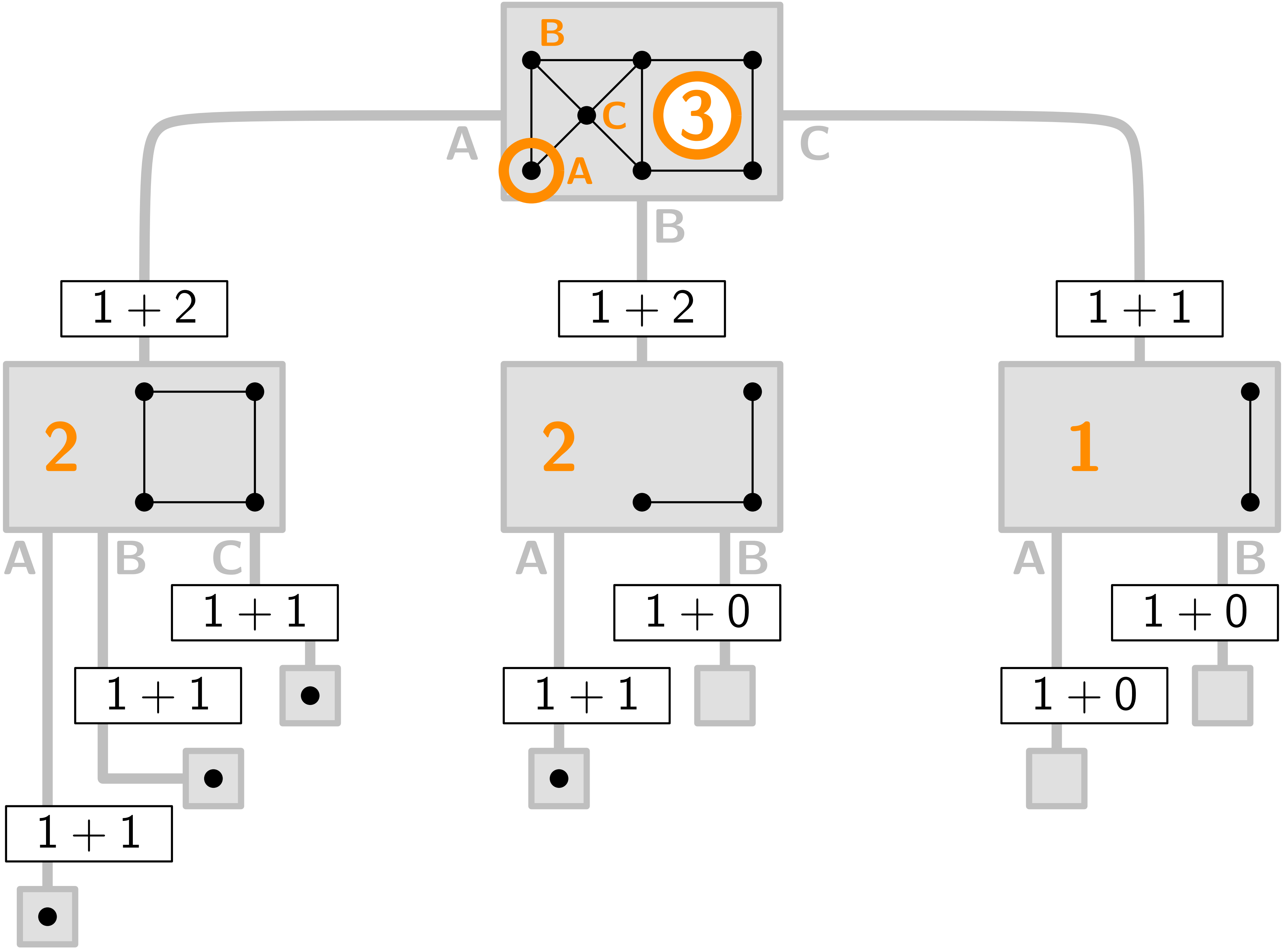
Execution corresponds to a *search tree* whose nodes are labeled with the input of the respective recursive call.

Let $B(n)$ be the maximum number of leaves of a search tree for a graph with n vertices.



Search-tree has height $\leq n$,
$\rightsquigarrow$ the algorithm's runtime is
$T(n) \in O^*(nB(n)) = O^*(B(n))$

Let's consider an example run.



B
C
3
A
A
B
C
1 + 2
1 + 2
1 + 1
2
2
1
A
B
C
A
B
A
B
1 + 1
1 + 0
1 + 0
1 + 1
1 + 1
1 + 0
1 + 1

Runtime Analysis

For a worst-case n-vertex graph G ($n \geq 1$):

$$\begin{aligned} B(n) &\leq \sum_{y \in N[v]} B(n - (\mathrm{deg}(y) + 1)) \\ &\leq (\mathrm{deg}(v) + 1) \cdot B(\, n - (\mathrm{deg}(v) + 1) \,)\,, \end{aligned}$$

where v is a minimum degree vertex of G, and we note that $B(n') \leq B(n)$ for any $n' \leq n$.

Runtime Analysis (cont)

$B(n) \leq (\text{deg}(v) + 1) \cdot B(\ n - (\text{deg}(v) + 1)\ )$

We proceed by induction to show $B(n) \leq 3^{n/3}$

Base case: $B(0) = 1 \leq 3^{0/3}$

Hypothesis: for $n \geq 1$, set $s = \text{deg}(v) + 1$ in the above inequality

$$B(n) \leq s \cdot B(n - s) \leq s \cdot 3^{(n-s)/3} = \frac{s}{3^{s/3}} \cdot 3^{n/3} \leq 3^{n/3}$$

$$B(n) \in O^*(\sqrt[3]{3}^n) \subset O^*(1.44225^n)$$