Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Lehrstuhl für
INFORMATIK I
Effiziente Algorithmen und
wissensbasierte Systeme

Institut für Informatik

Advanced Algorithms

Winter term 2019/20

Lecture 1. Introduction & Held-Karp-algorithm for TSP

(slides by Joachim Spoerhase, Thomas van Dijk, & Alexander Wolff)

Steven Chaplick & Alexander Wolff *Chair for Computer Science I*

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Evaluation:

- oral exam at the end of the semester
- 0,3 bonus for 50% on the exercises

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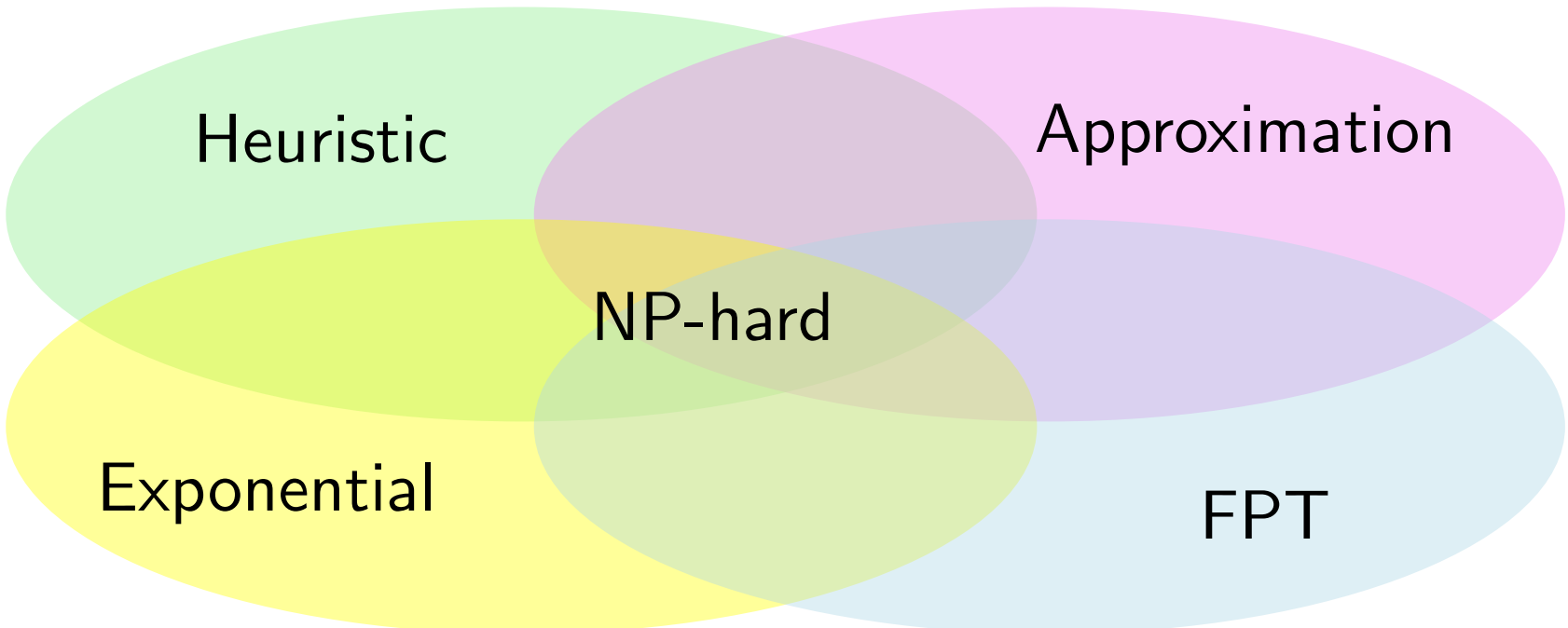

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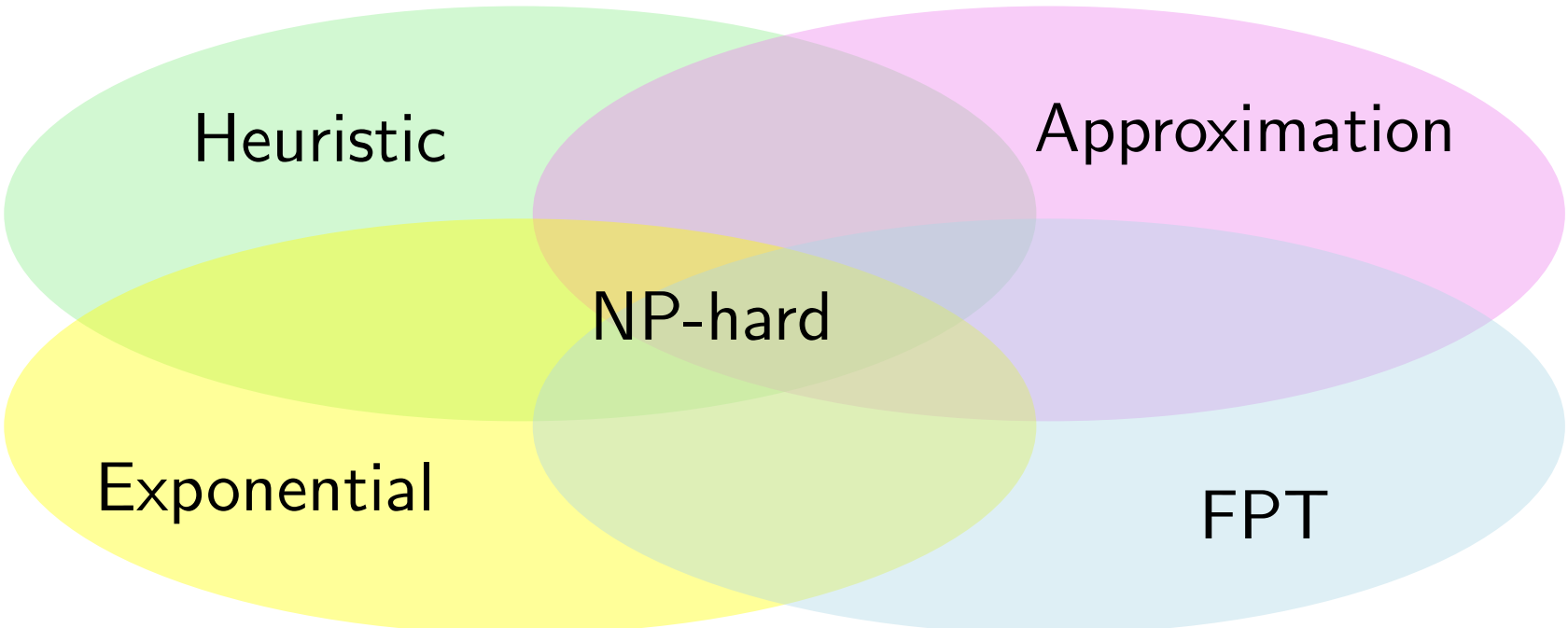
Many important (practical) problems are NP-hard

- Sacrifice optimality for speed
  - Heuristics (sim. Annealing, Tabu-Search)
  - Approximation Algorithms (Christofides-Algorithm)
- Optimal Solutions
  - Exact (exponential) time algorithms
  - Fine-grained analysis (parameterized) algorithms



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Also, more on polytime solvable problems

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- More graph algorithms (shortest paths w/ neg. weights)
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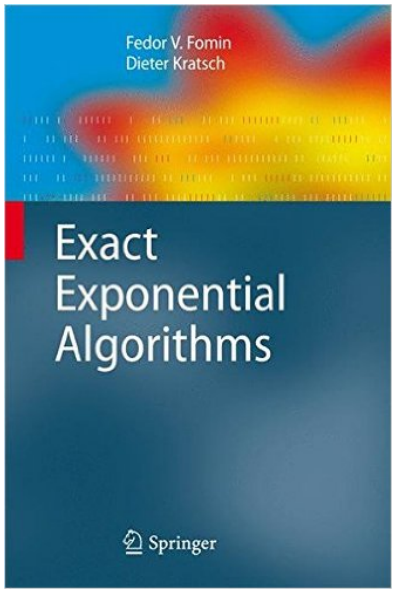
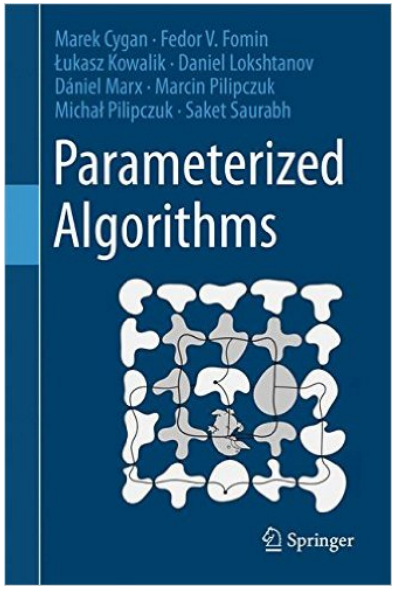
Textbooks



F. Fomin & D. Kratsch: Exact Exponential Algorithms, Springer 2010
abbrev: **EEA**



Marek Cygan et al.: Parameterized Algorithms, Springer 2015
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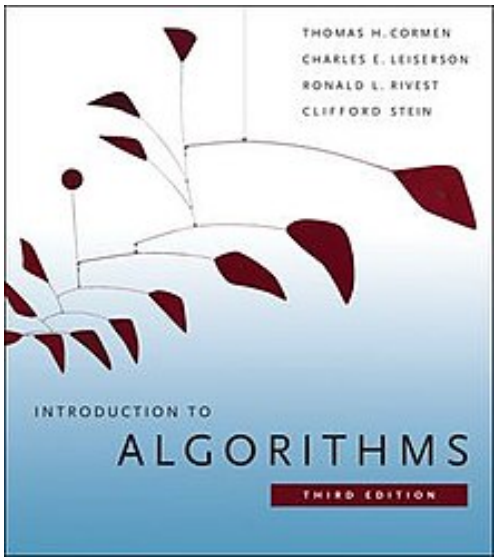

C.L.R.S.: Intro. to Algorithms MIT Press 2009.
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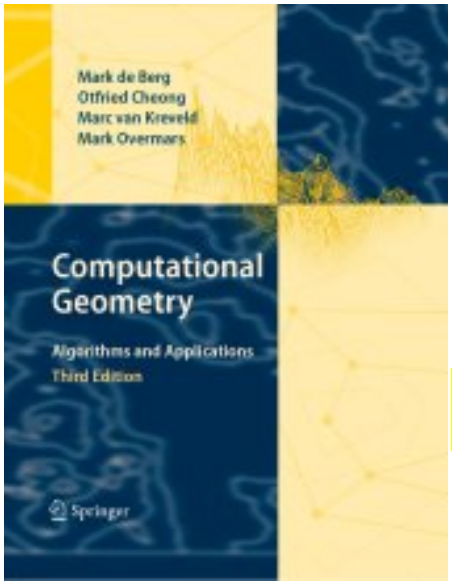
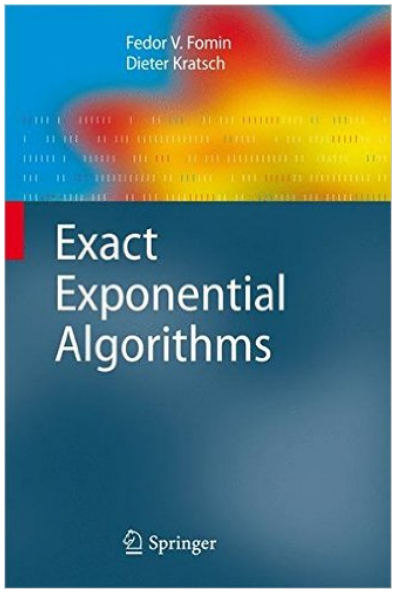


M. de Berg et al: Computational Geometry: Algorithms & Applications Springer 2008, 3rd edition.
abbrev: **CG: A&A**

Textbooks

This Lecture: Chapter 1



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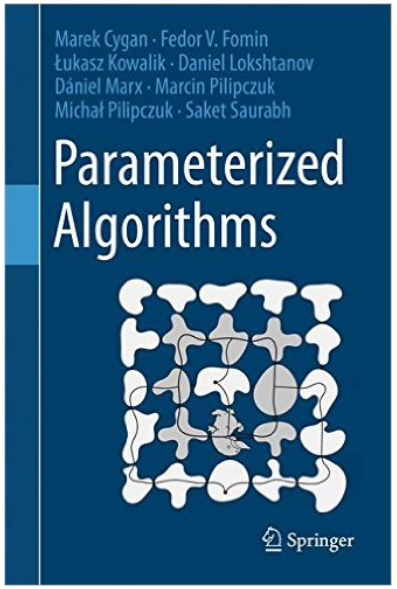

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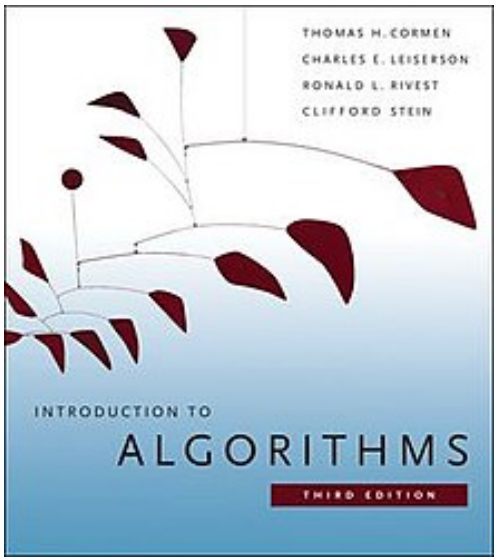

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Background

- efficient vs. inefficient algorithms

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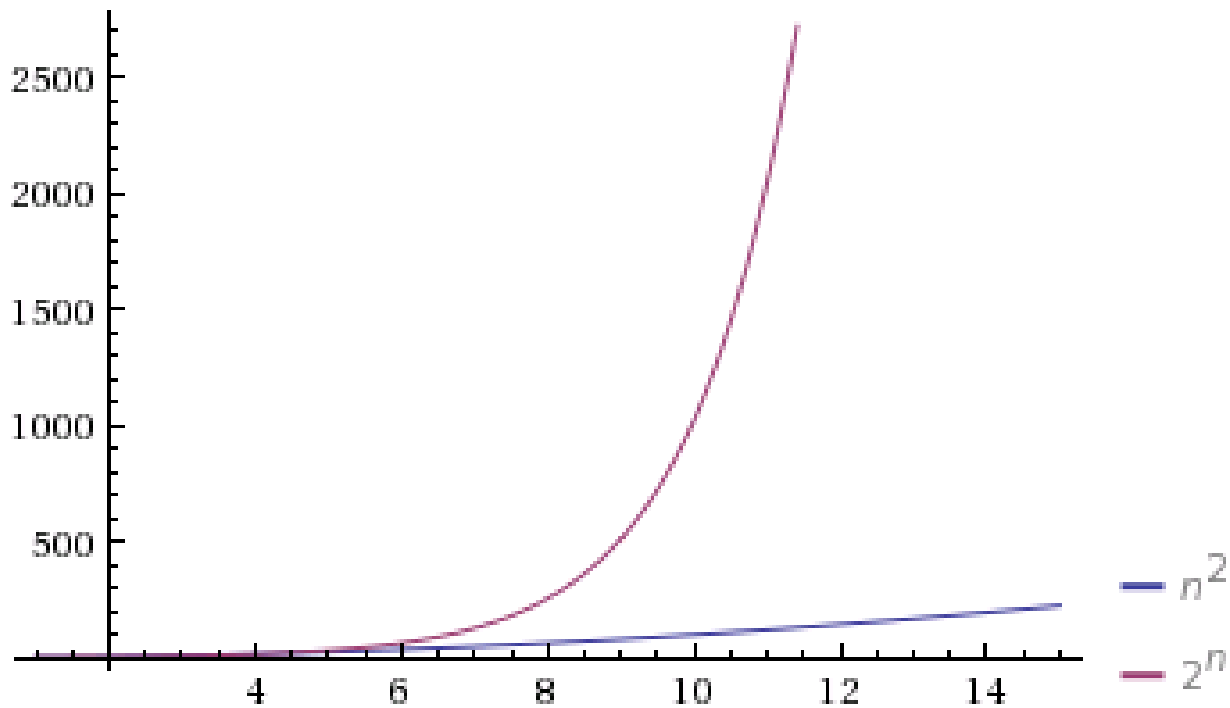
$\rightsquigarrow$ polynomial vs. super-polynomial algorithms



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- can be "fast" for **medium-sized** instances

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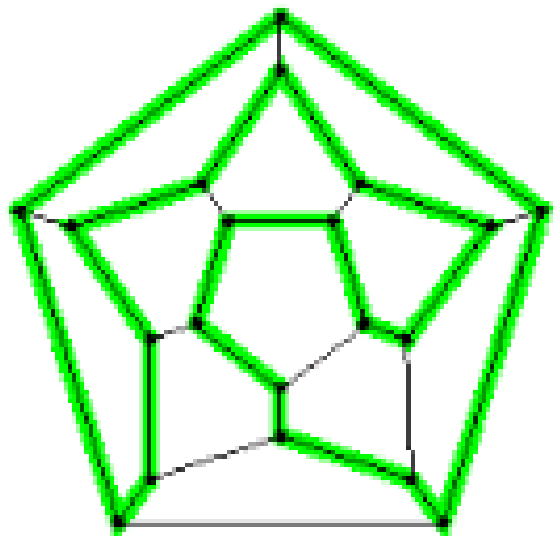

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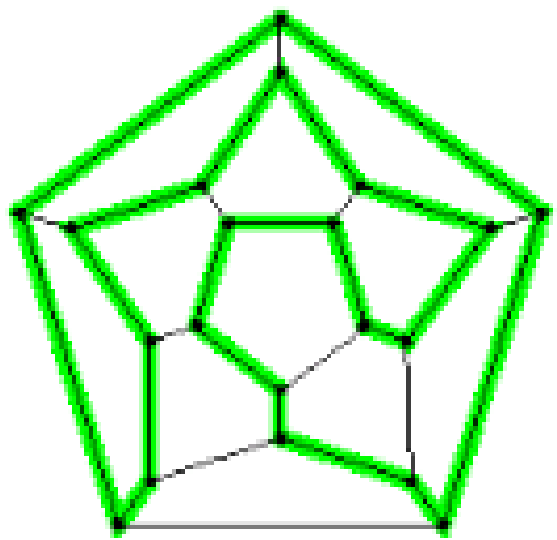

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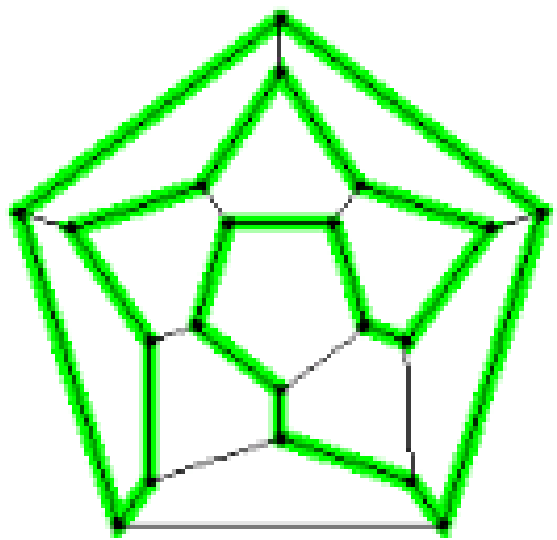
- theoretical interest



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$f(n) \in O^*(g(n)) \Leftrightarrow \exists$ polynomial $p(n)$ w/ $f(n) \in O(g(n)p(n))$

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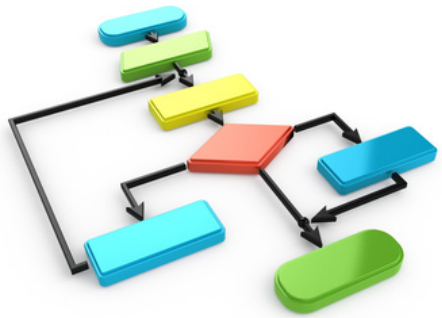
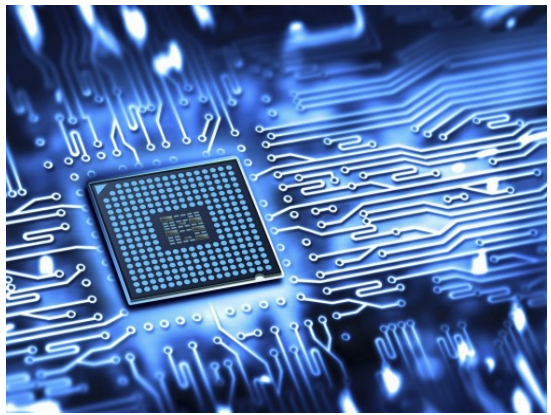
Suppose an algorithm uses a^n steps.





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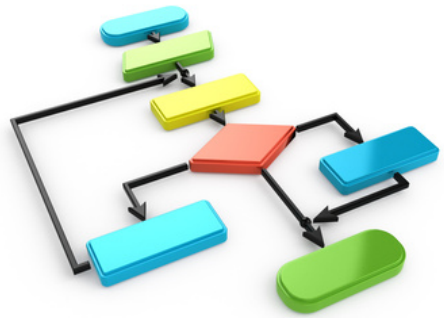

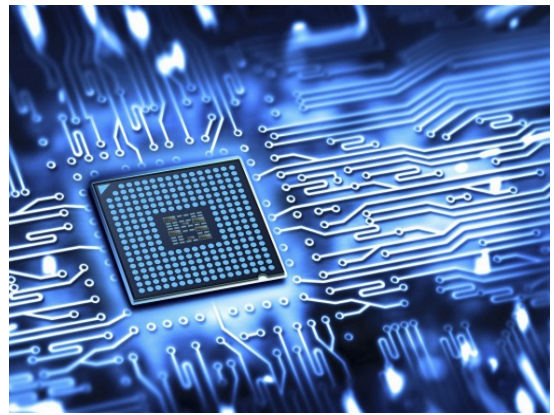

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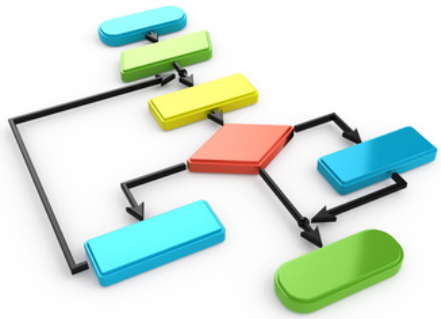

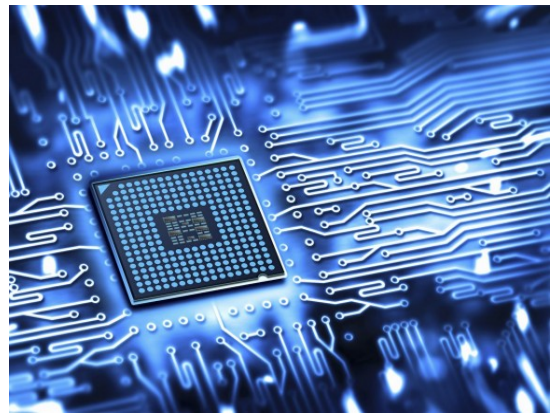

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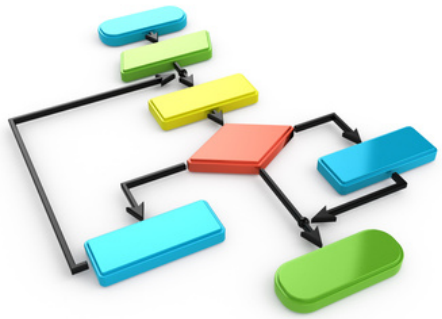

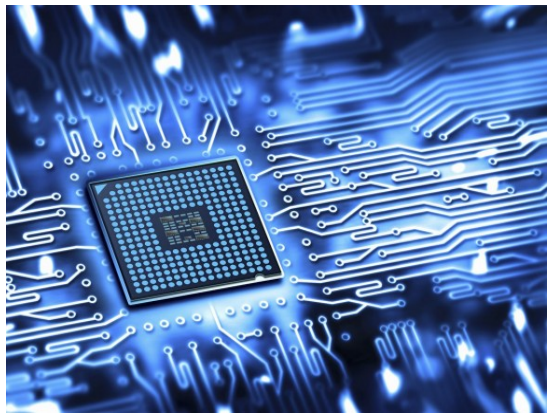

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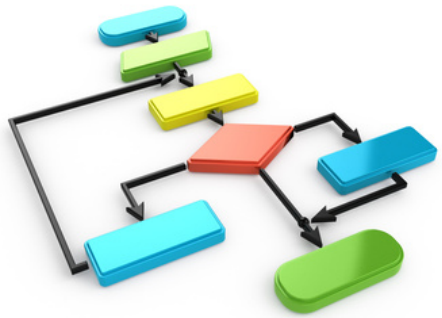

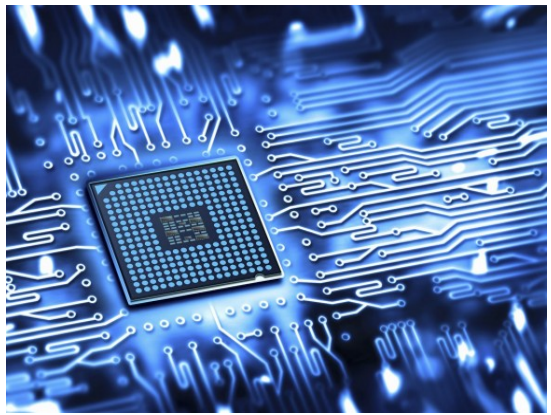

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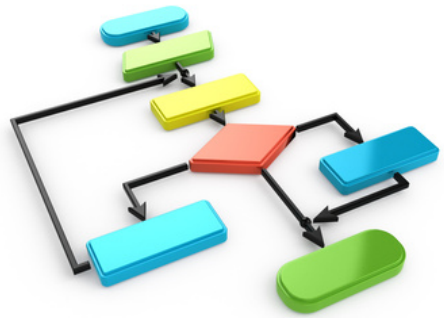
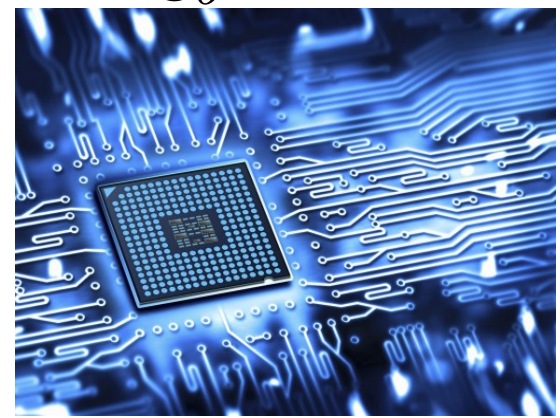
Base reduction: $b^{n_0'} = a^{n_0} \rightsquigarrow n_0' = n_0 \cdot \log_b a$





Traveling Salesperson Problem (TSP)

Input Complete directed graph $G = (V, E)$ with n vertices and edge weights $c\colon E \to \mathbb{Q}_{\geq 0}$

Output Hamiltonian cycle $(v_{\pi(1)}, \ldots, v_{\pi}(n), v_{\pi(n+1)} = v_{\pi(1)})$ of G, of minimum weight $\sum_{i=1}^{n} c(v_i, v_{i+1})$, permutation π.

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Runtime: $\Theta(n! \cdot n) = n \cdot 2^{\Theta(n \log n)}$

Bellman-Held-Karp-Algorithm

Richard M. Karp

Richard E. Bellman

Bellman-Held-Karp-Algorithm

Technique: Dynamic Programming!

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Bellman-Held-Karp-Algorithm

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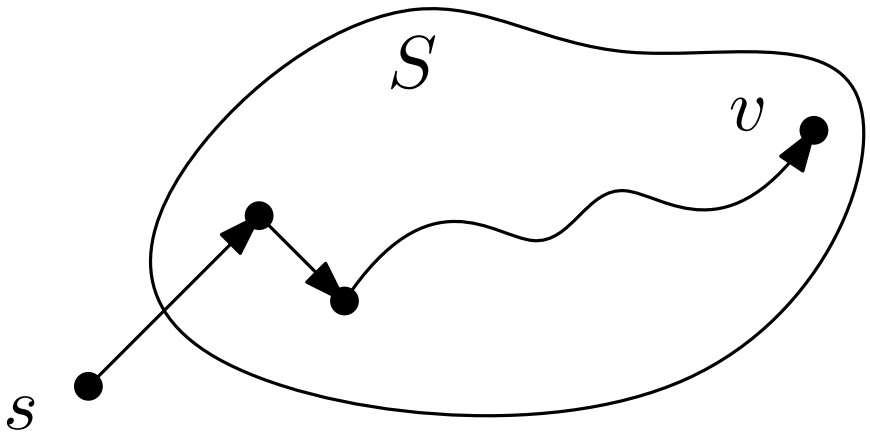
For each $S \subseteq V - s$ and $v \in S$, let:

OPT$[S, v]$ = length of a shortest s-v-path that visits precisely the vertices of $S \cup \{s\}$.



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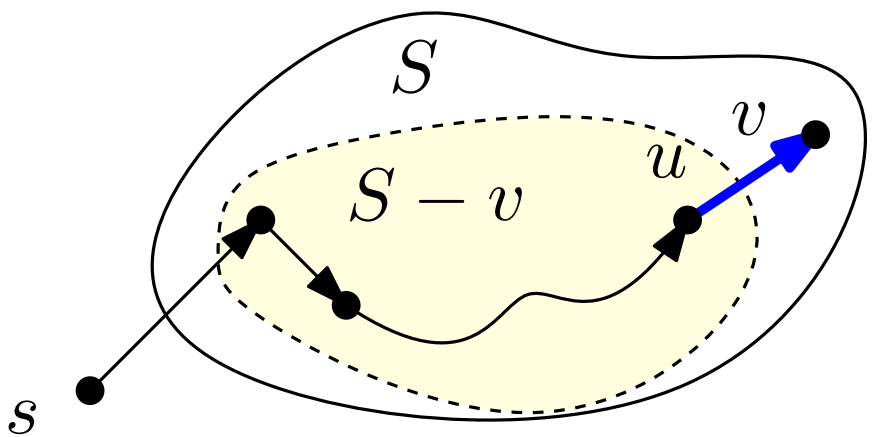
$$\mathrm{OPT}[S, v] = \min\{\ \mathrm{OPT}[S - v, u] + c(u, v)\ \mid u \in S - v\ \}$$



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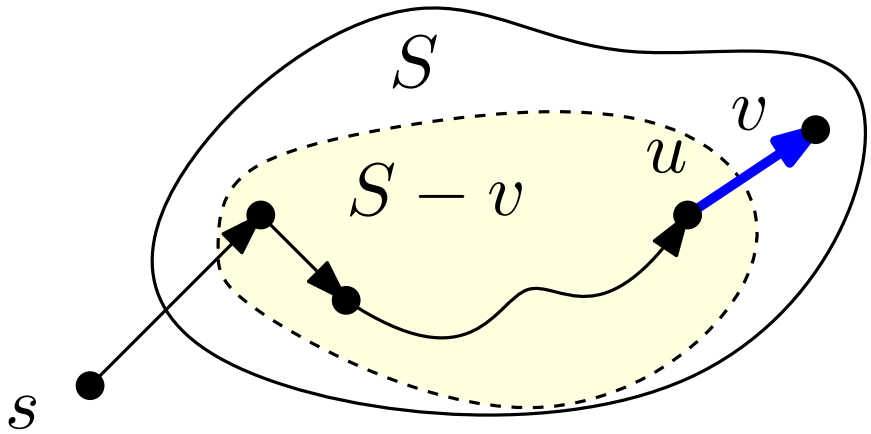
$$\mathrm{OPT}[S, v] = \min\{\ \mathrm{OPT}[S - v, u] + c(u, v)\ \mid u \in S - v\ \}$$



After computing $\mathrm{OPT}[S, v]$ for each $S \subseteq V - s$, the optimal solution is easily obtained as follows:

Bellmann-Held-Karp-Algorithm

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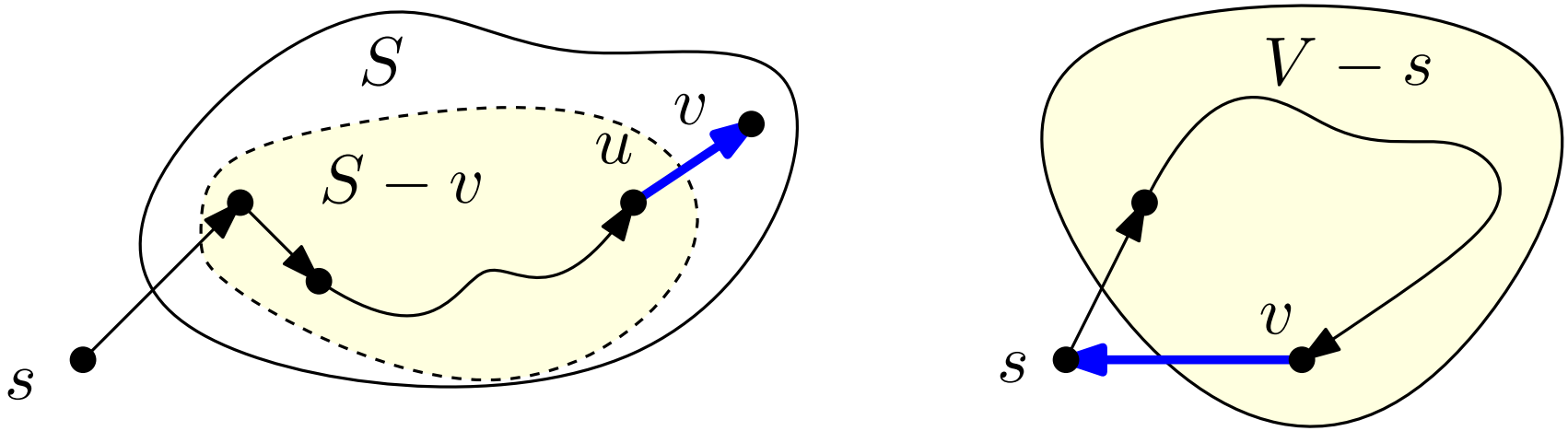

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Pseudocode for the dynamic program

Algorithm Bellmann-Held-Karp(G, c)

foreach $v \in V - s$ **do**

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for $j = 2$ **to** $n - 1$ **do**

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Only use table-values for $j - 1$ to compute j, less space?

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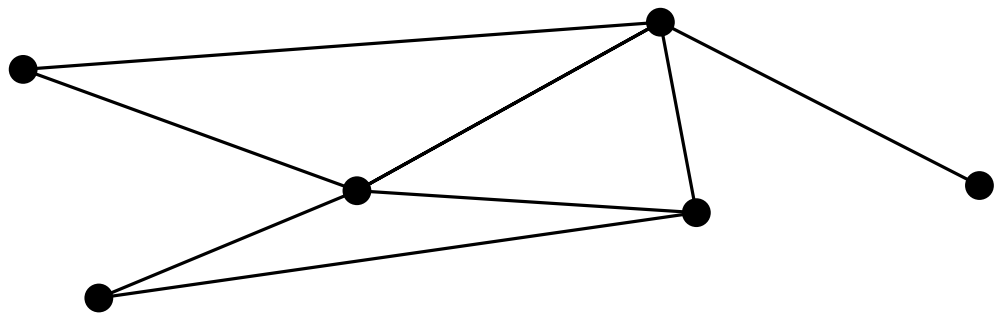
Compare: $O^*(2^n)$ with $2^{O(n \log n)}$ for Brute-Force

Maximum Independent Set

Input Graph $G = (V, E)$ with n vertices.

Output Maximum size *independent* set, i.e., a largest set $U \subseteq V$, such that no pair of vertices in U are adjacent in G.



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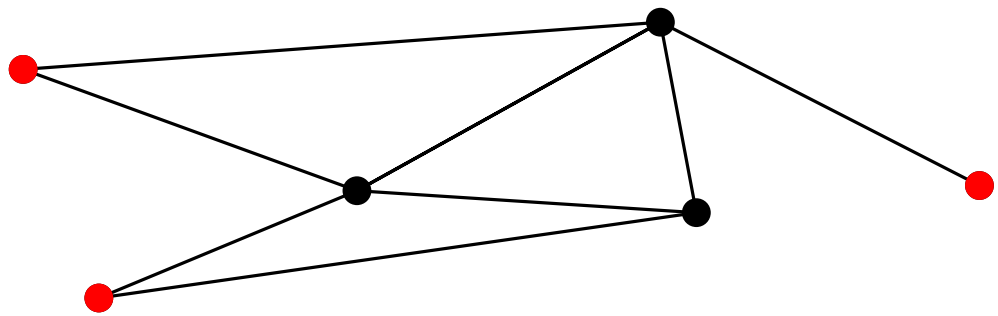

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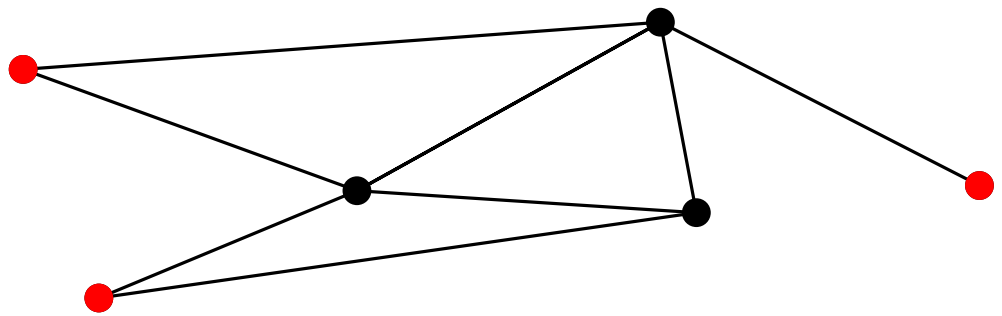

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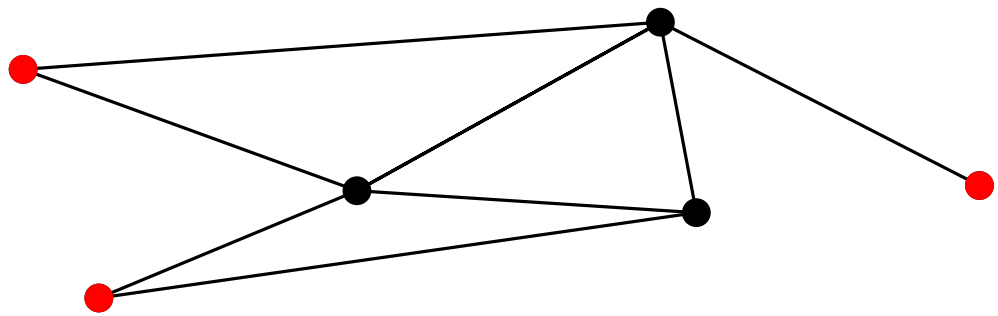

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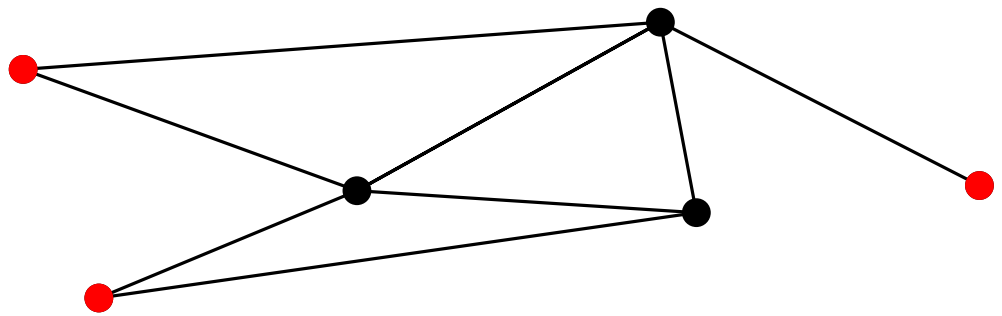
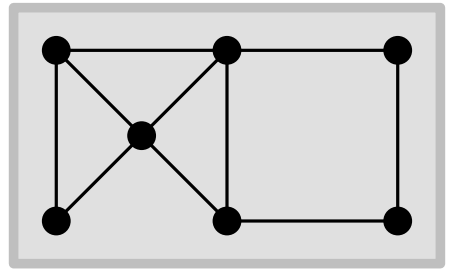
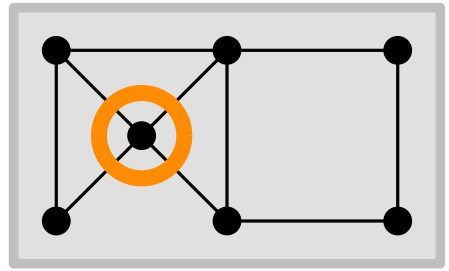


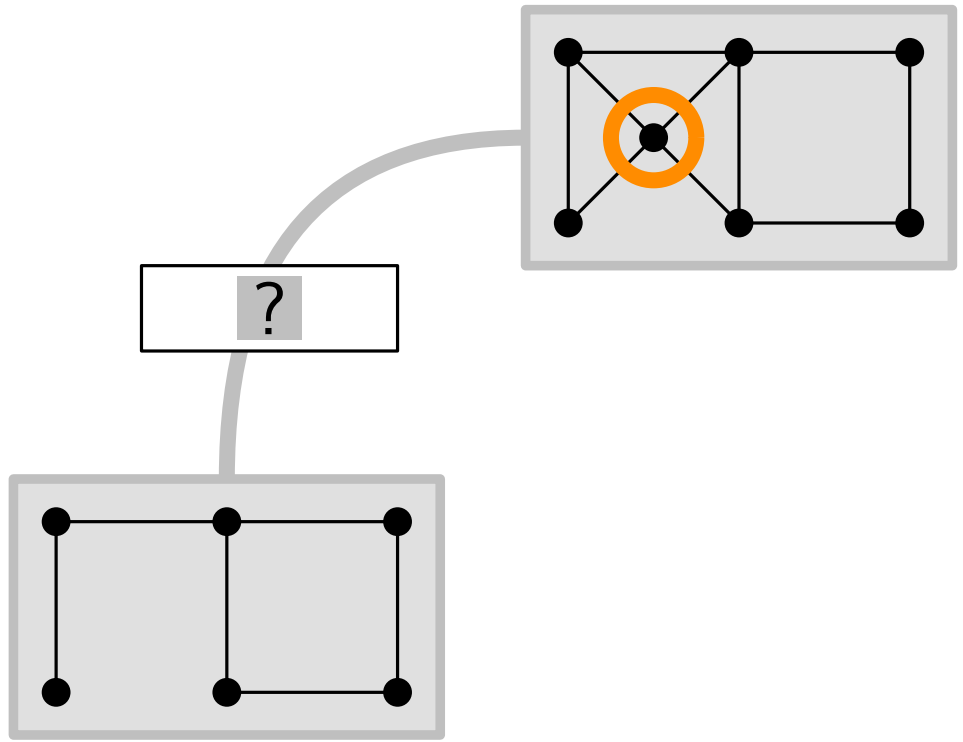
Brute Force?

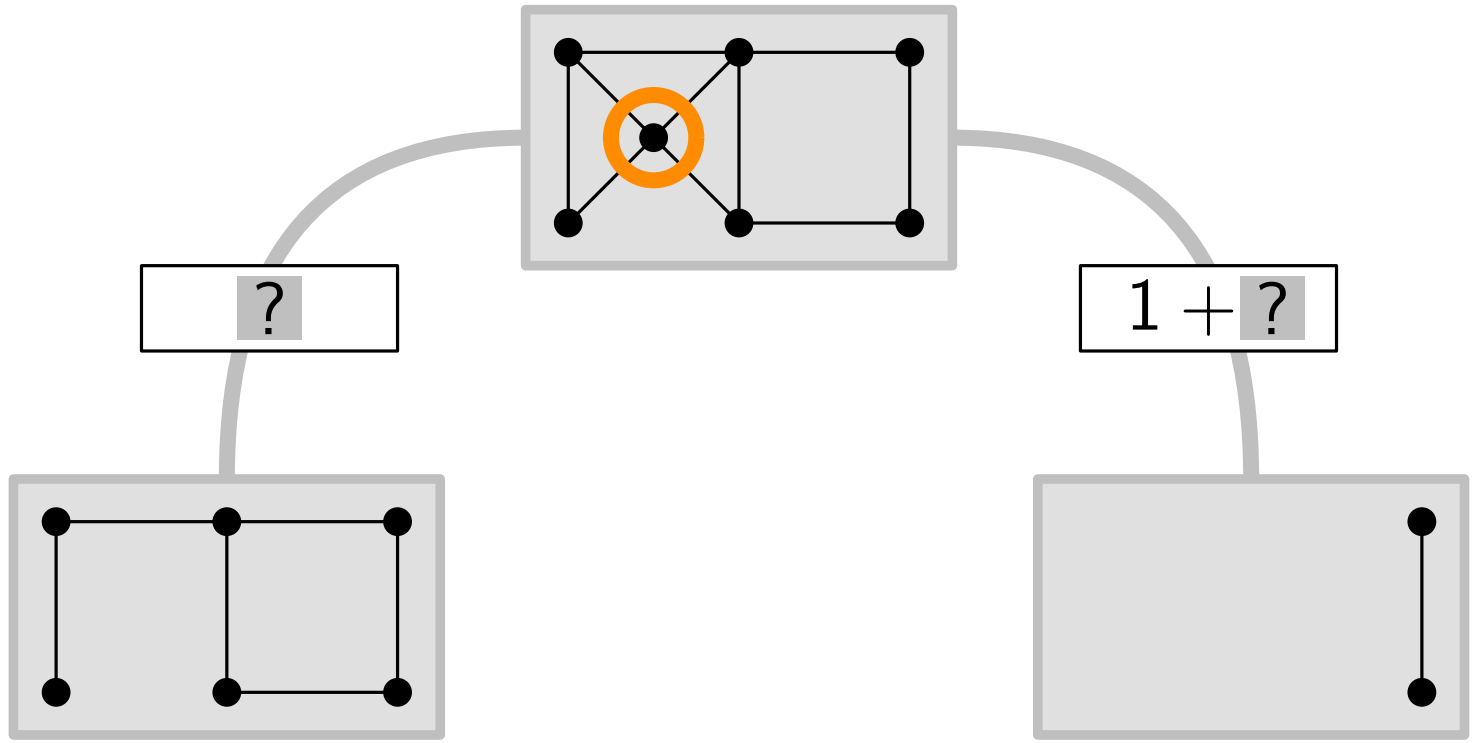
- Try all subsets of $V \rightsquigarrow O(2^n \cdot n)$ runtime.

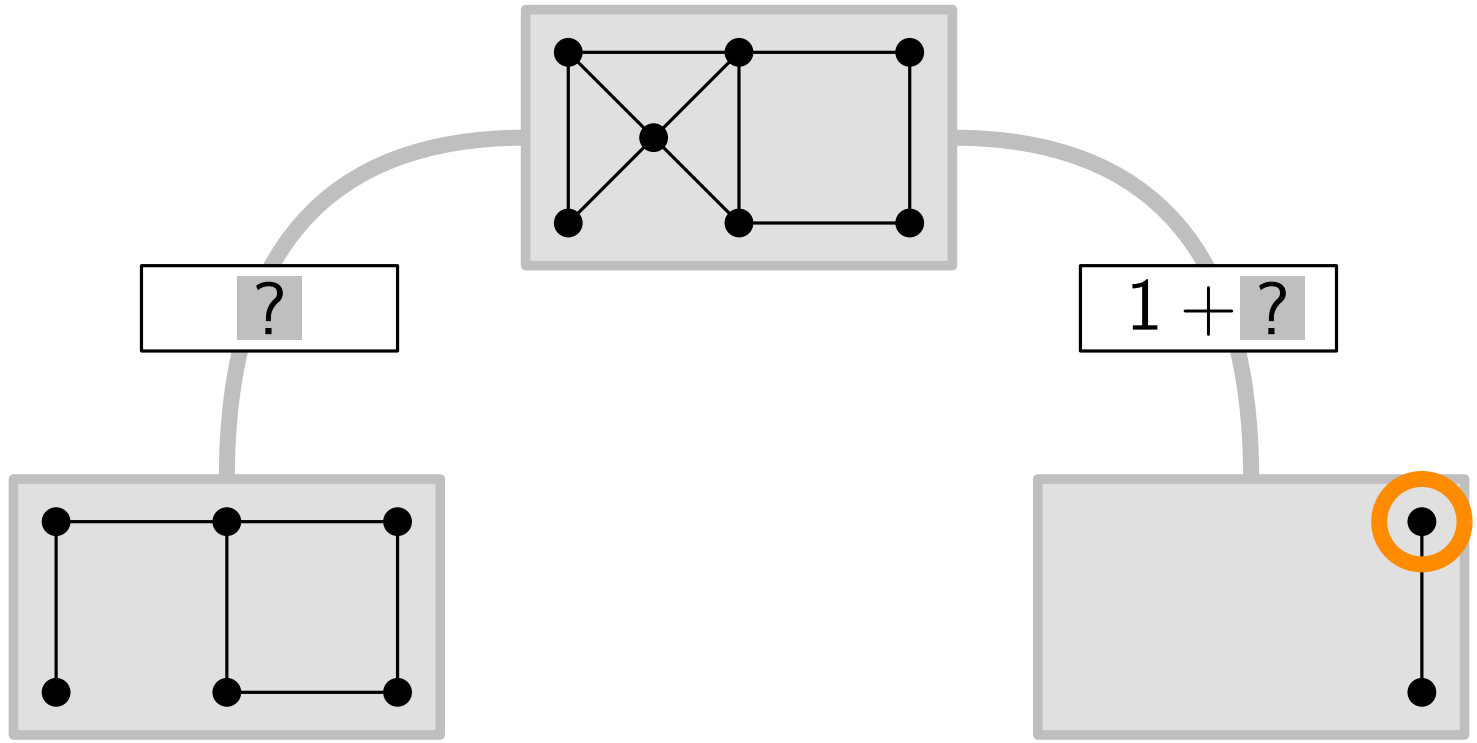
```
Algorithm NaiveMIS(G)
  if V = ∅ then
    return 0
  v ← arbitrary vertex in V(G)
  return max{1+ NaiveMIS(G − N(v) − {v}), NaiveMIS(G − {v})}
```








?
1 + ?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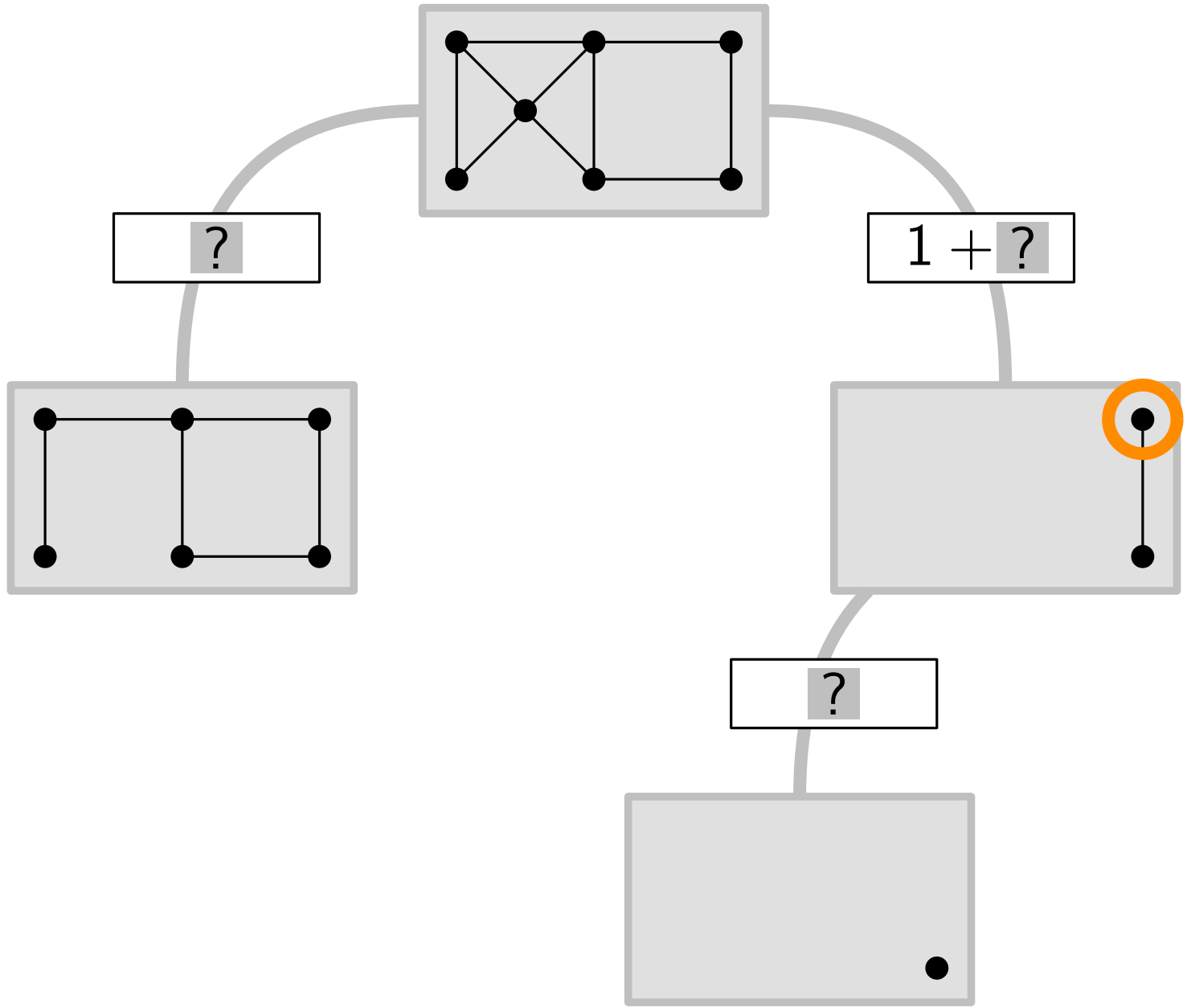
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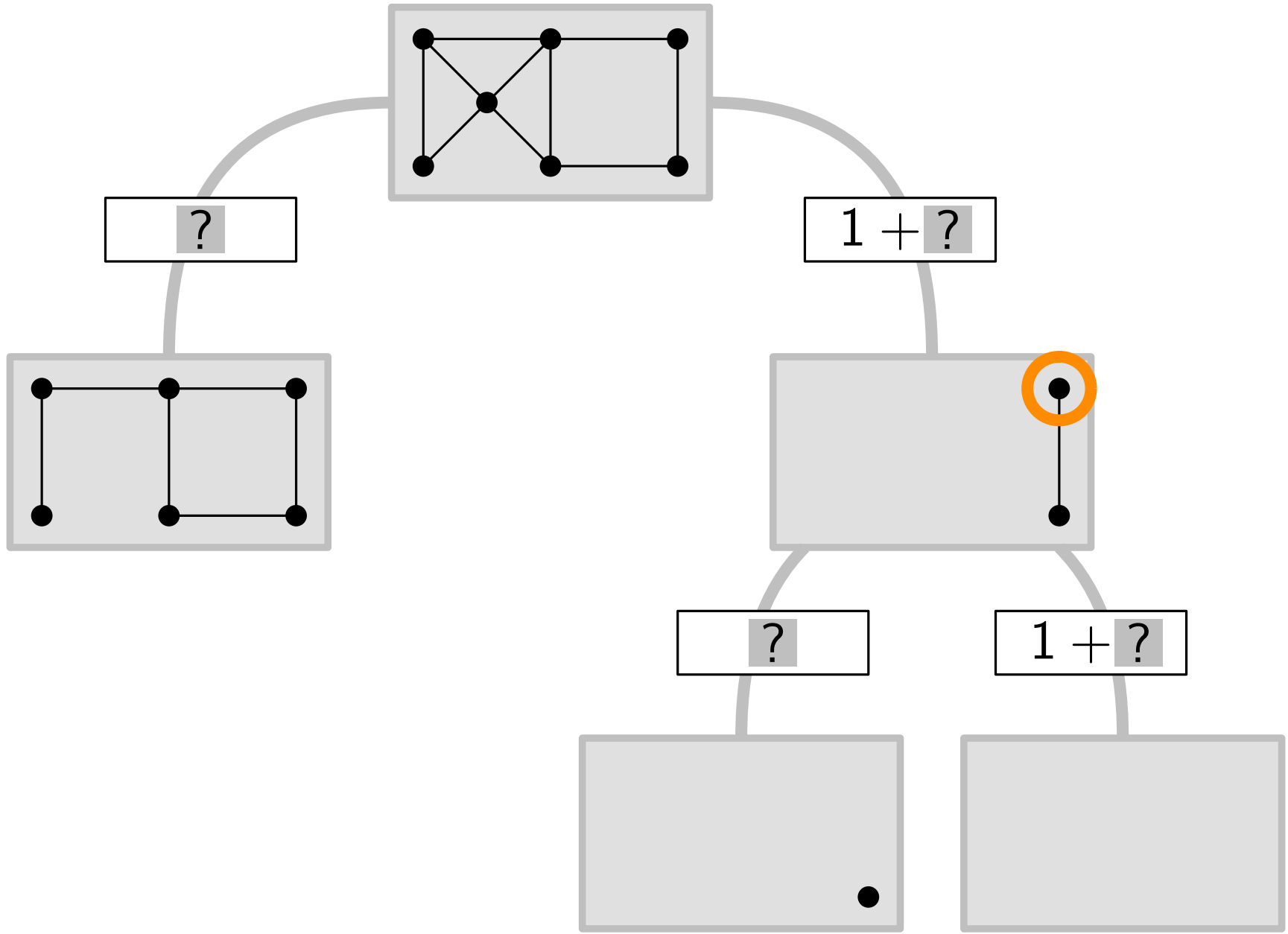
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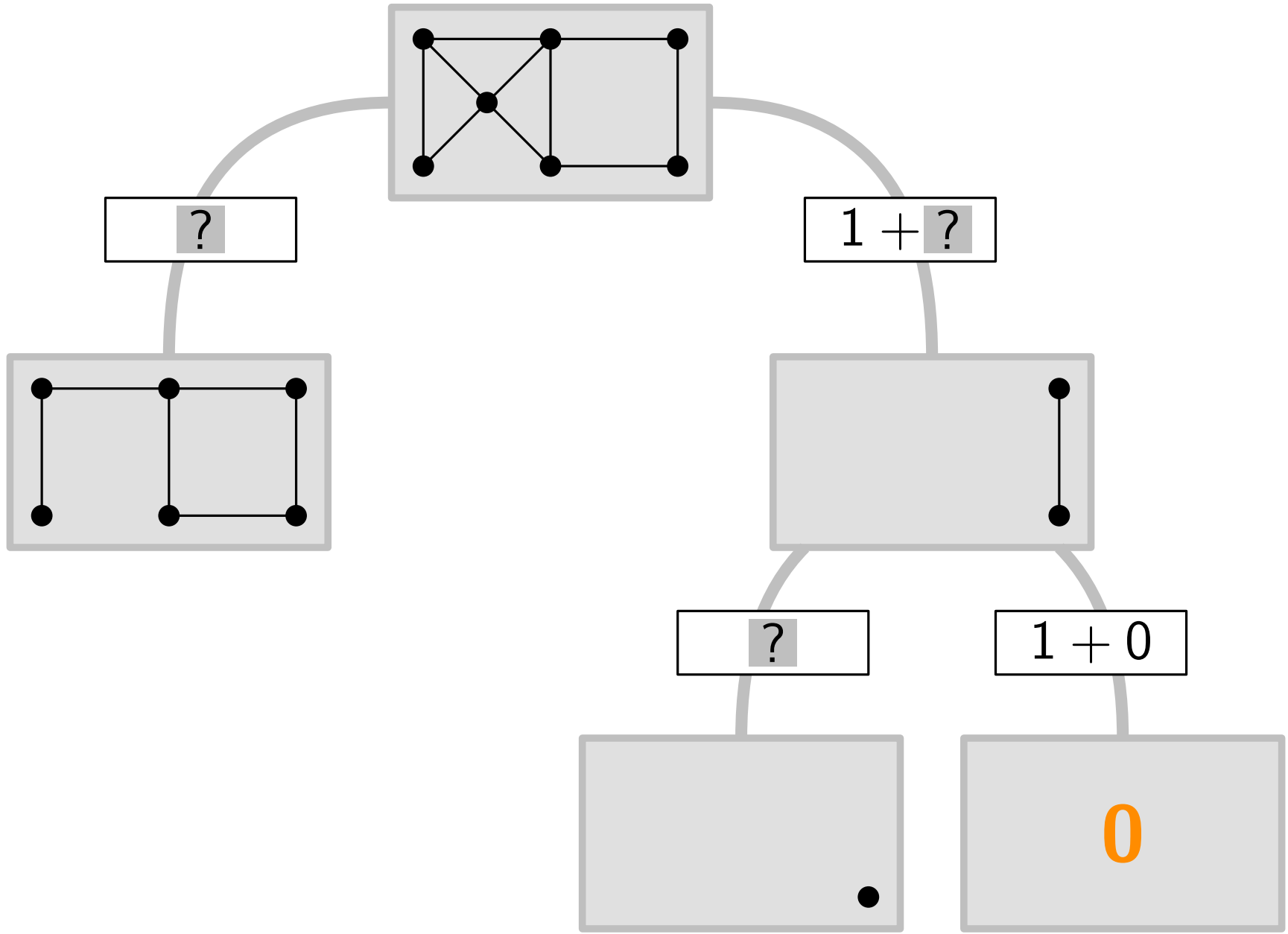
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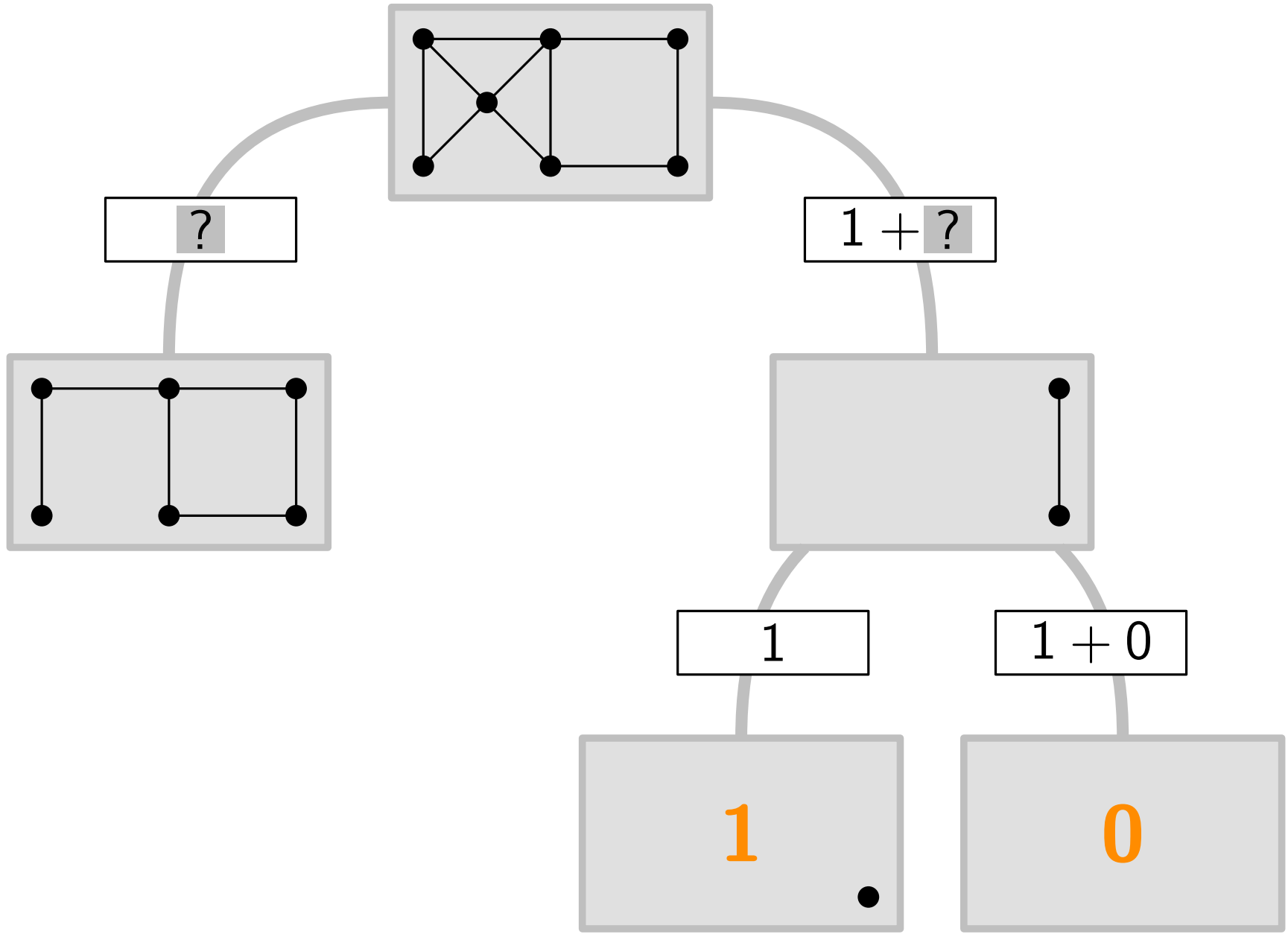
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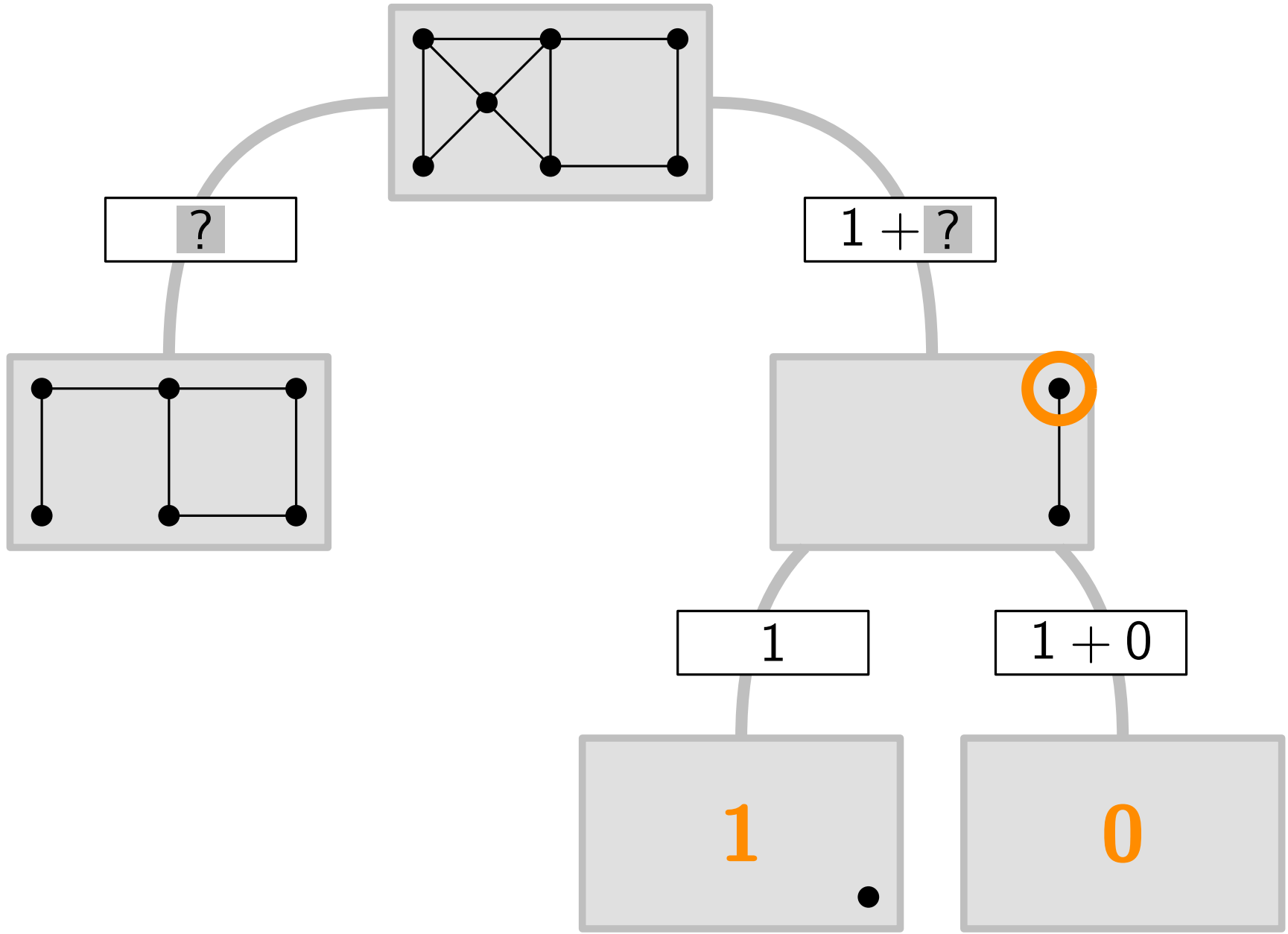
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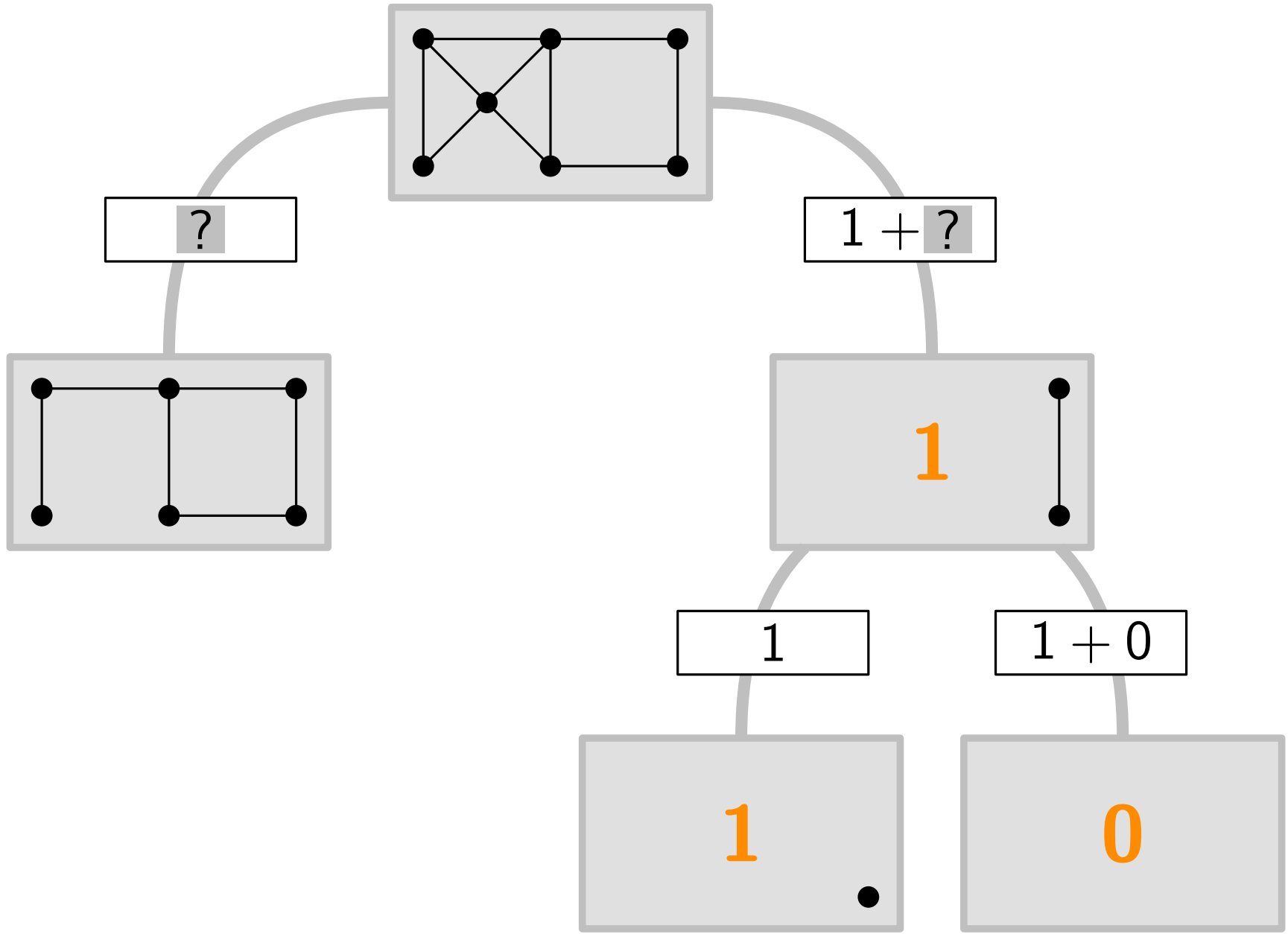
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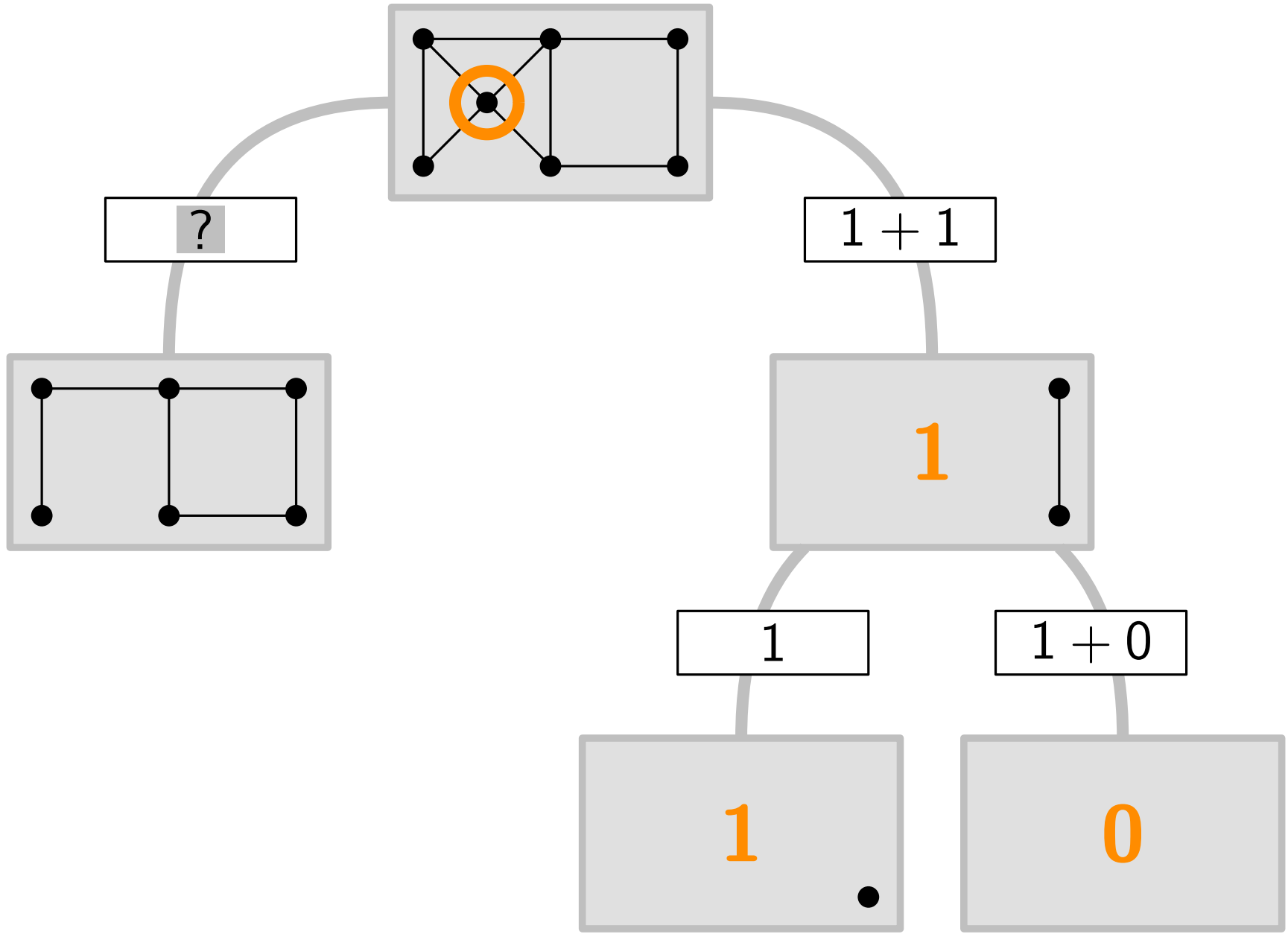
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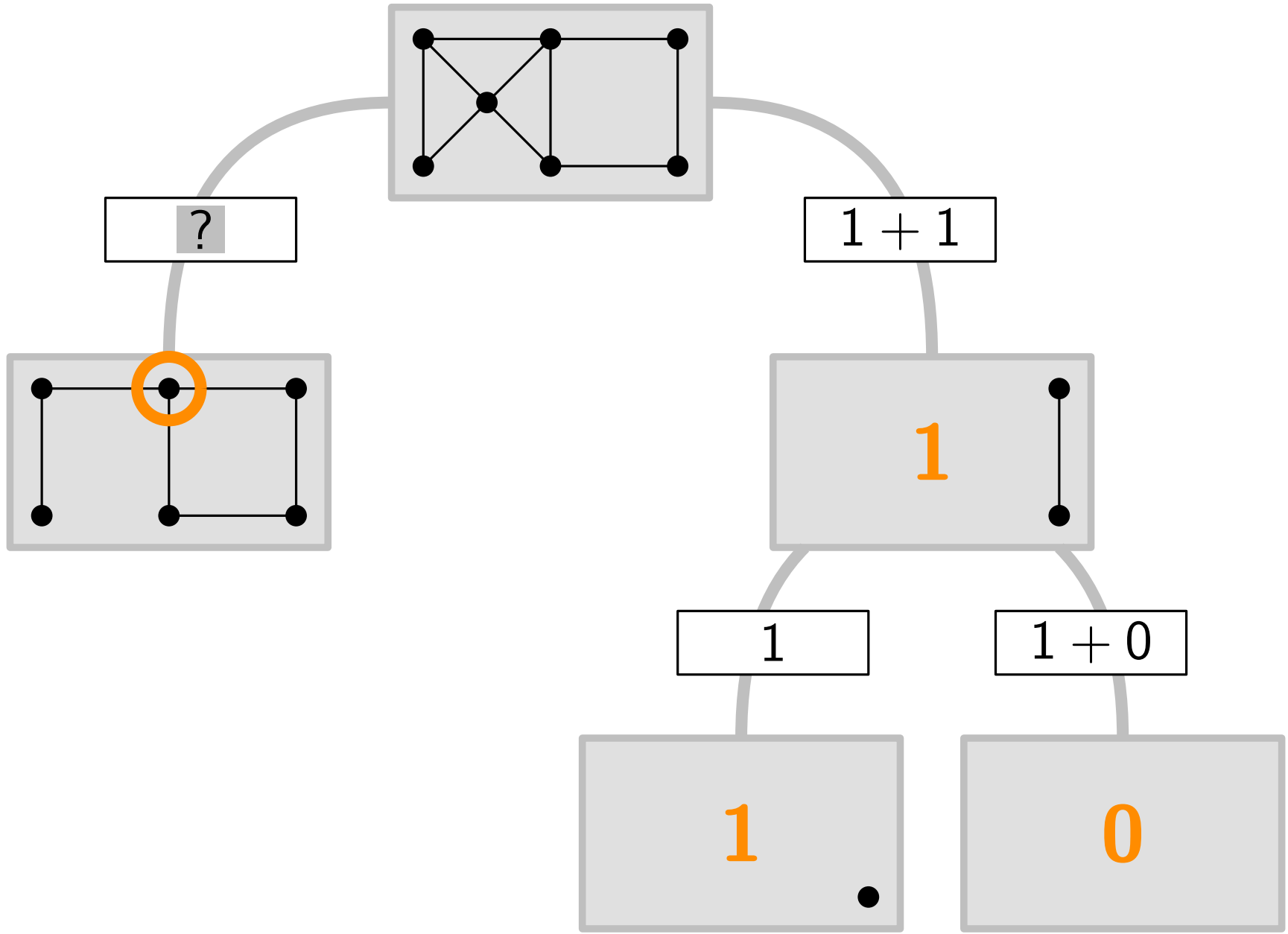
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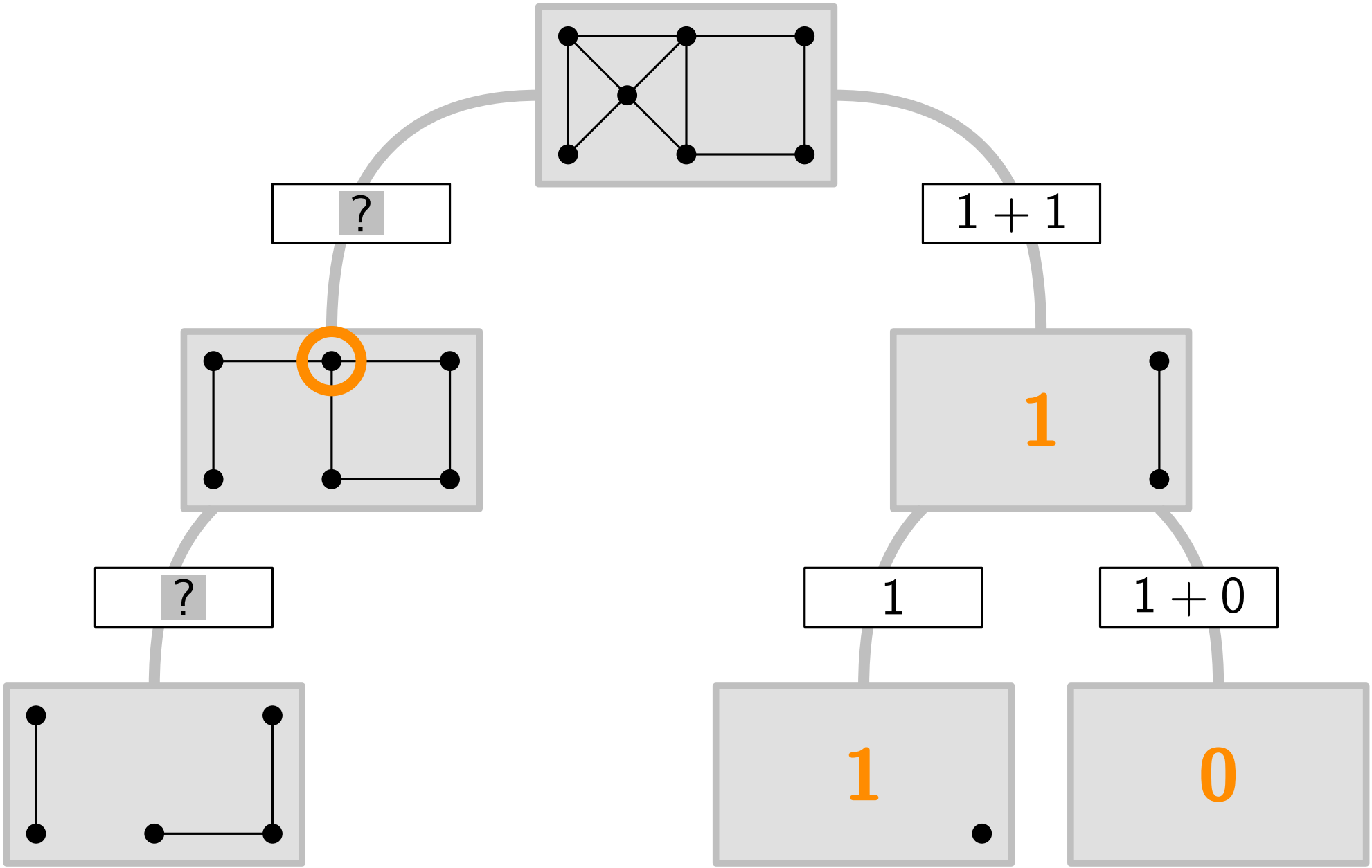
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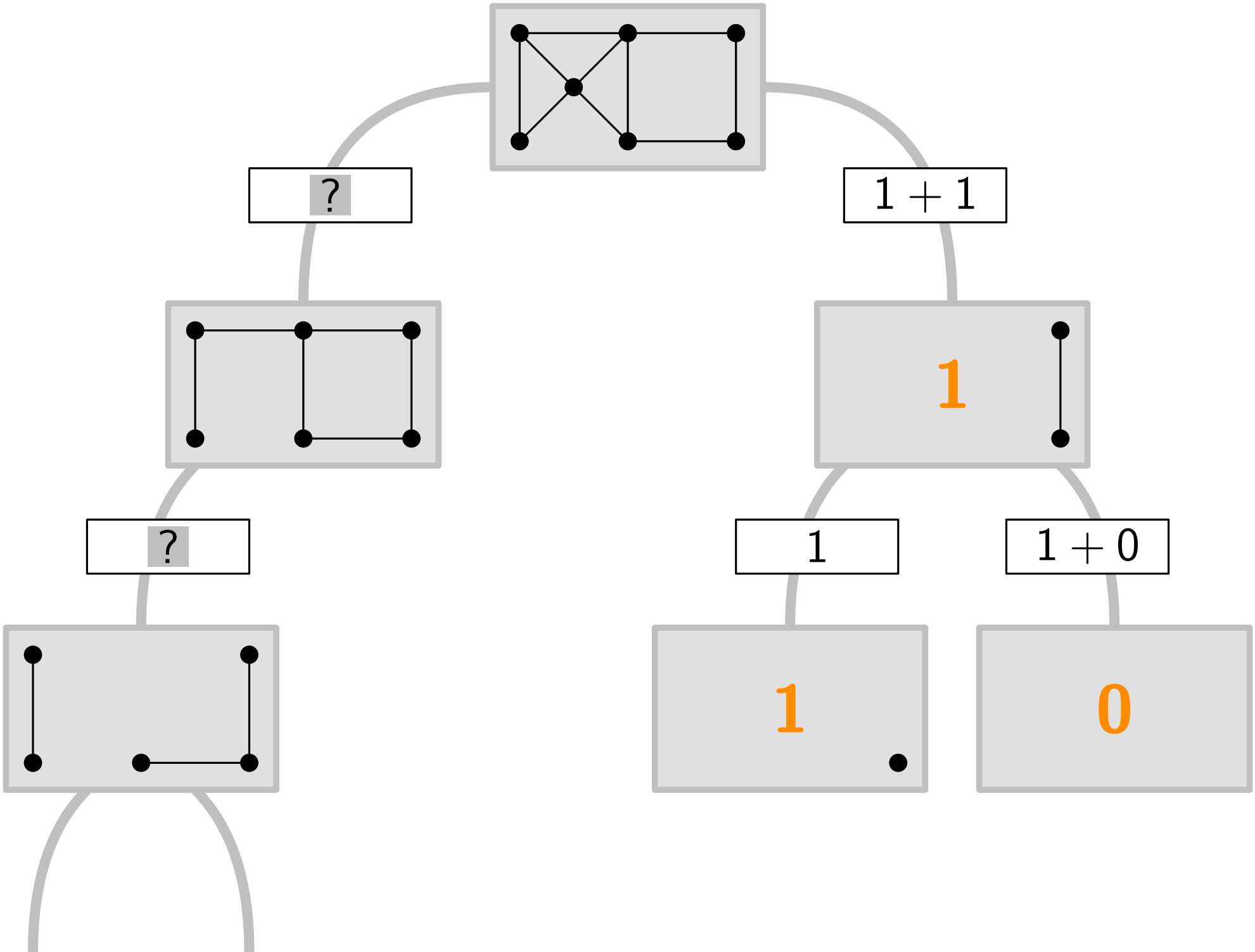
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1 + 0
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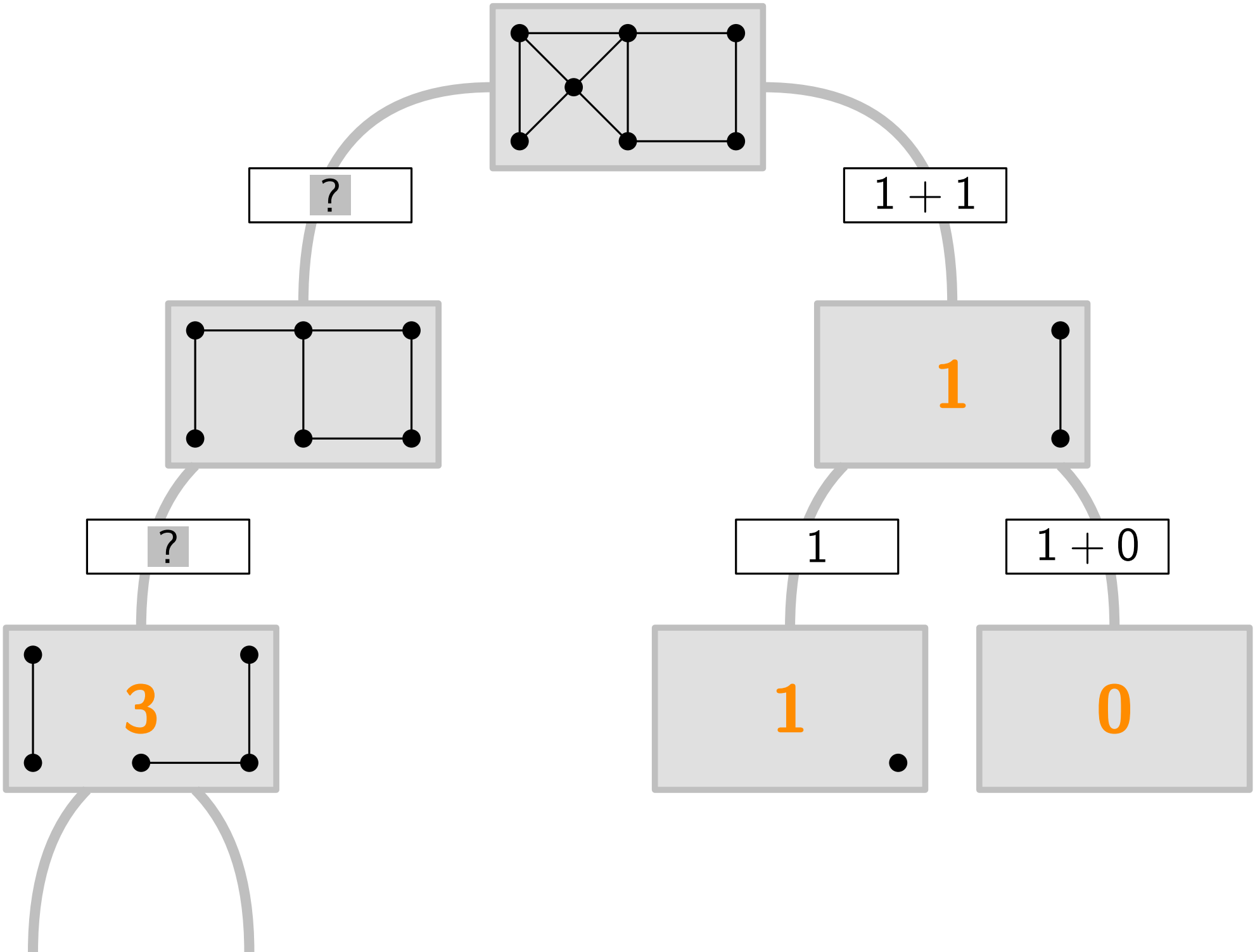
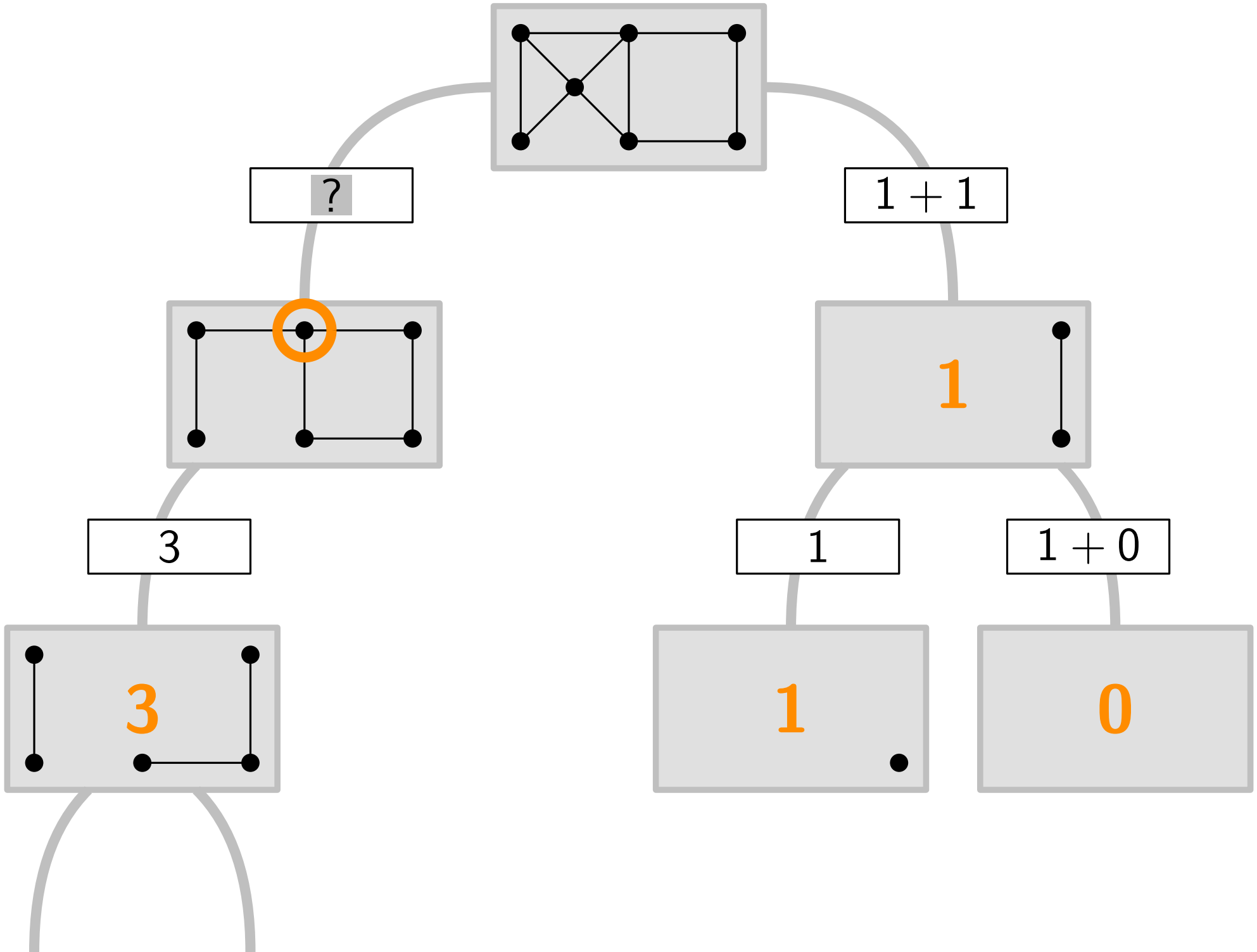

?
1 + 1
?
1
1
1 + 0
3
1
0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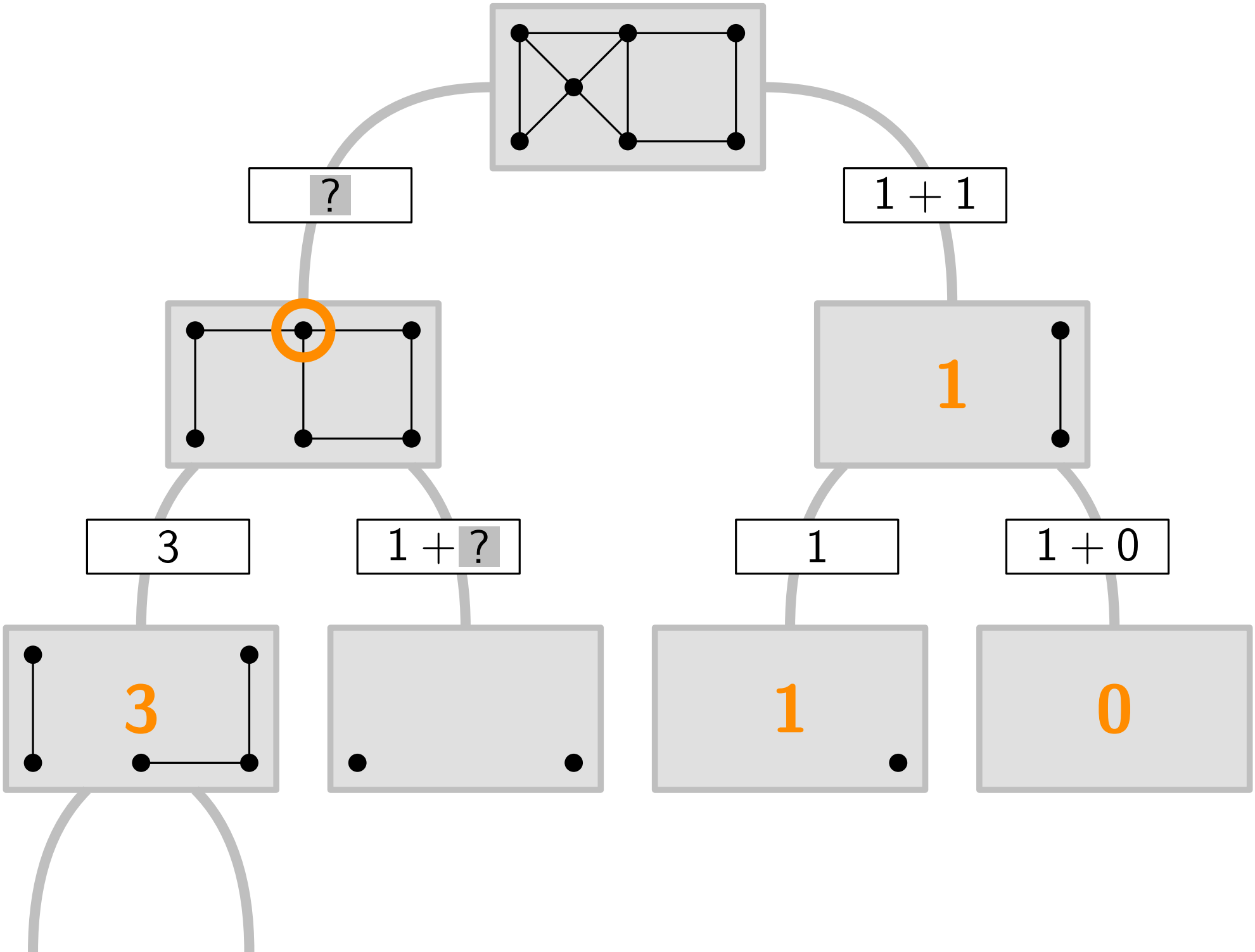
1 + 0
3
1
0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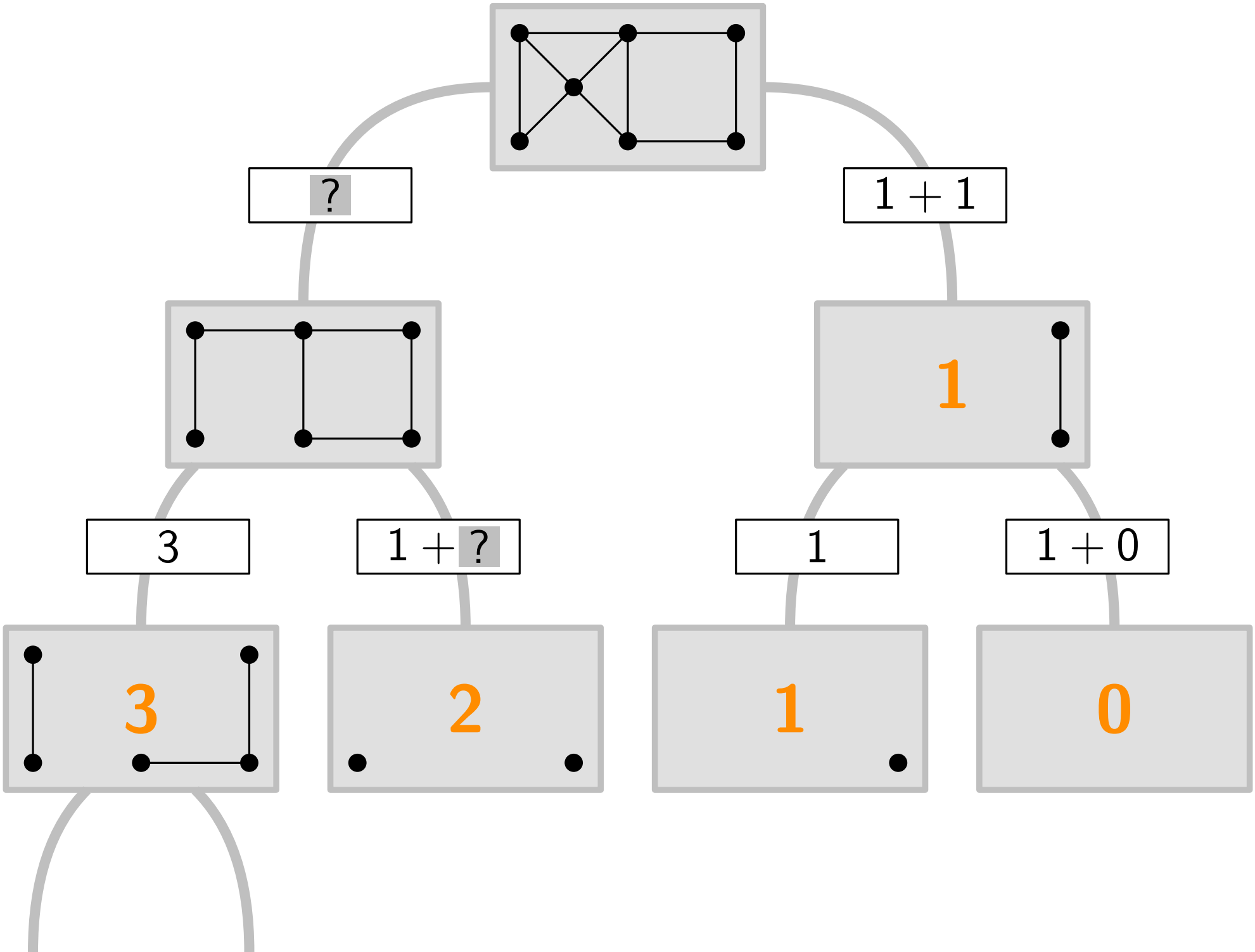
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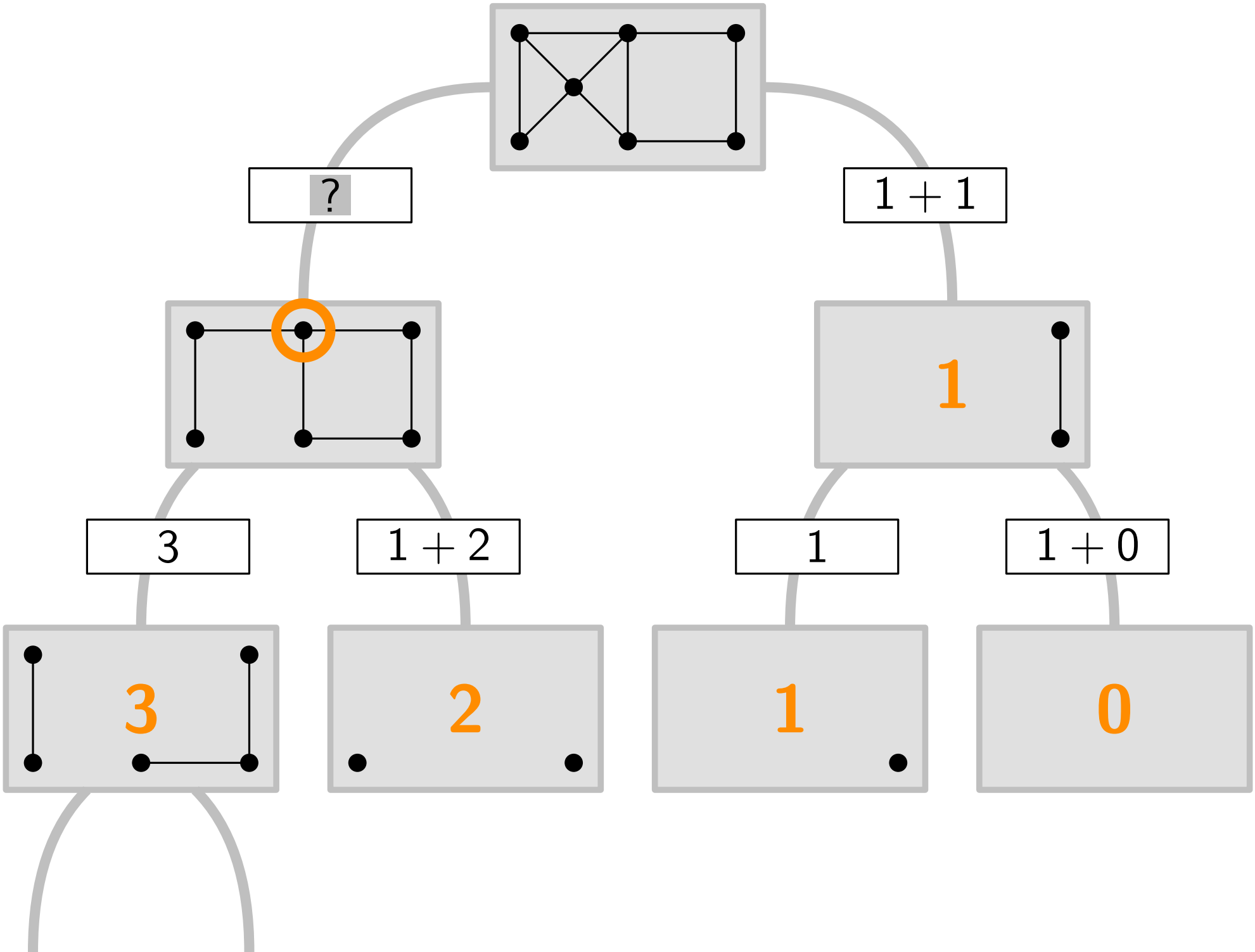
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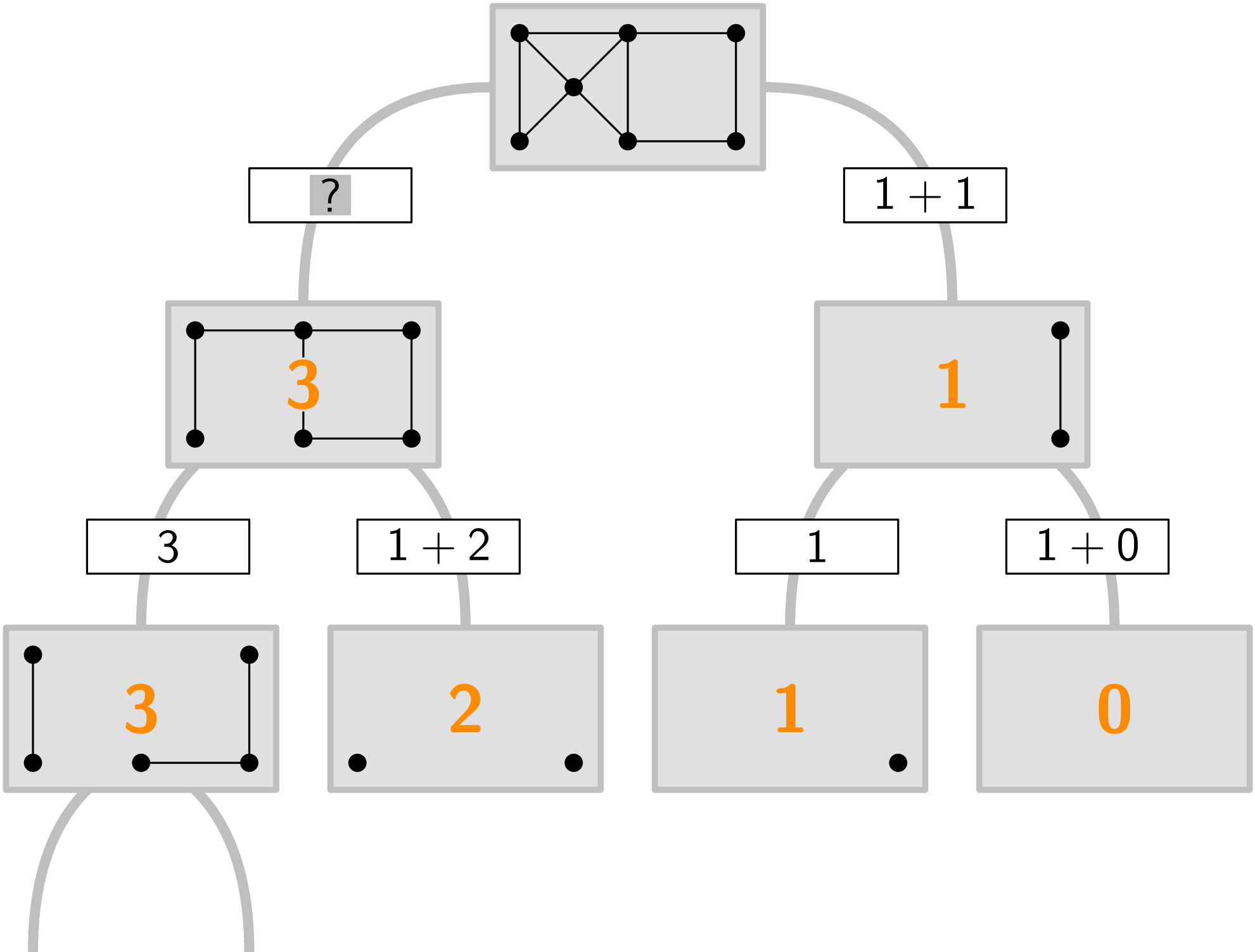
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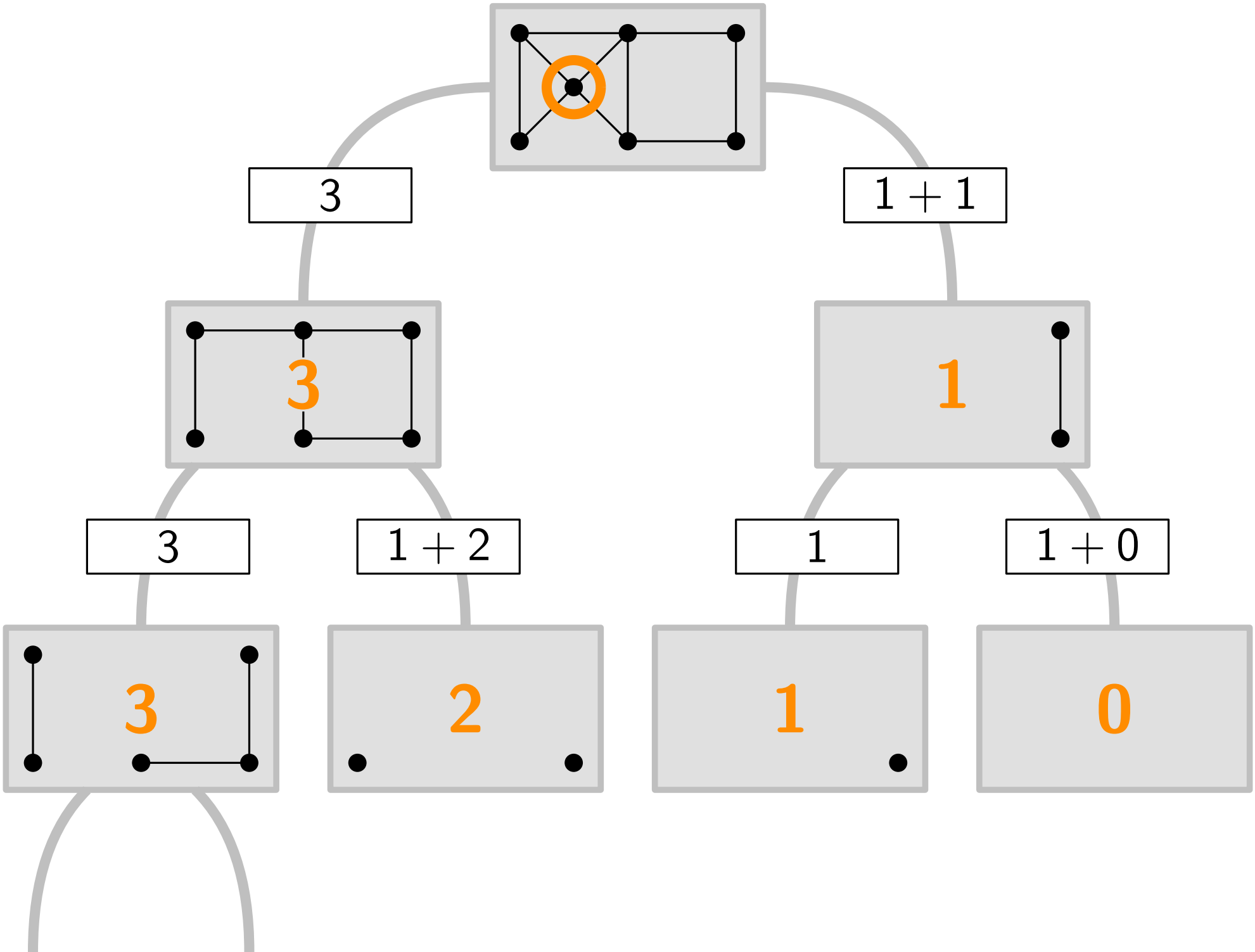
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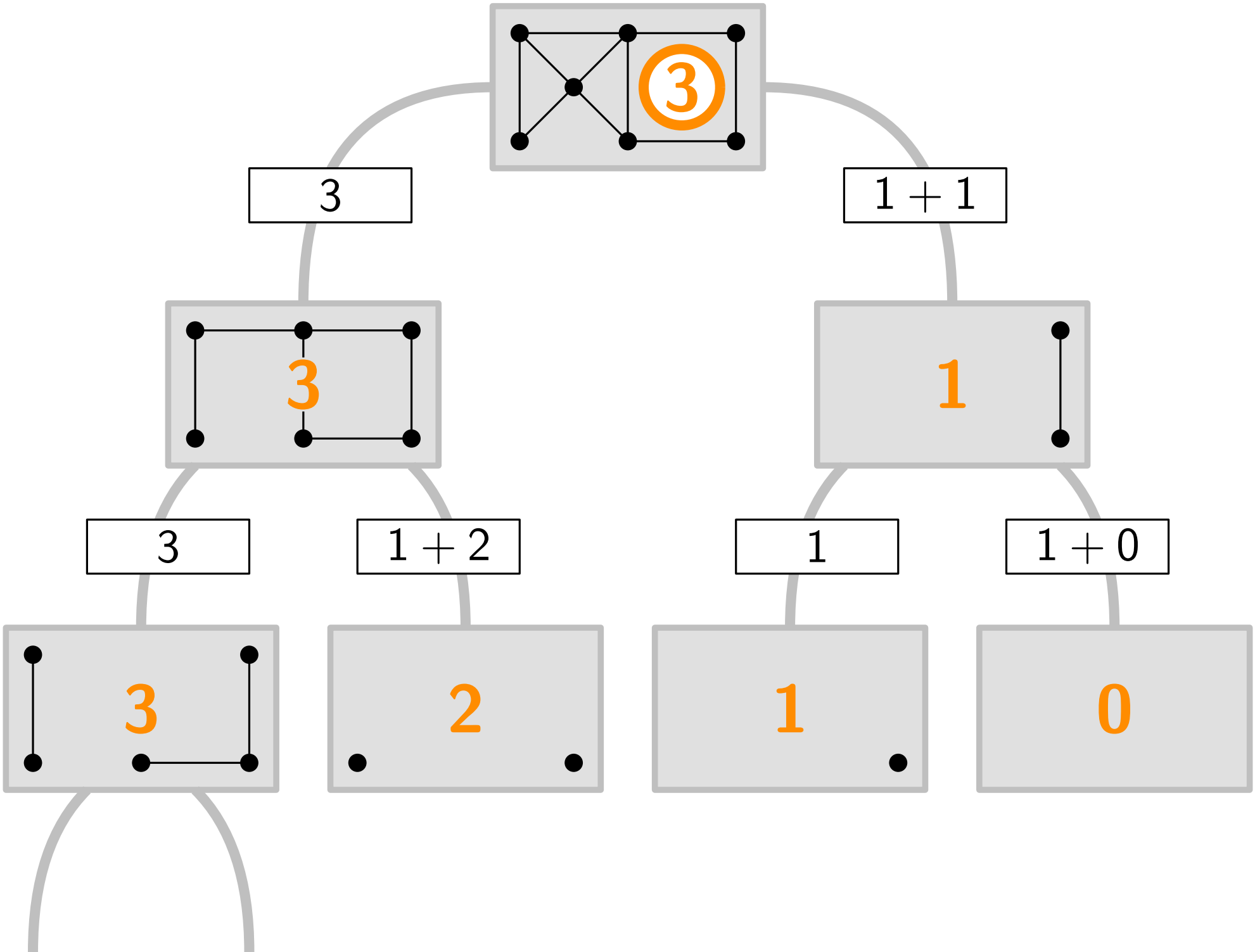
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0

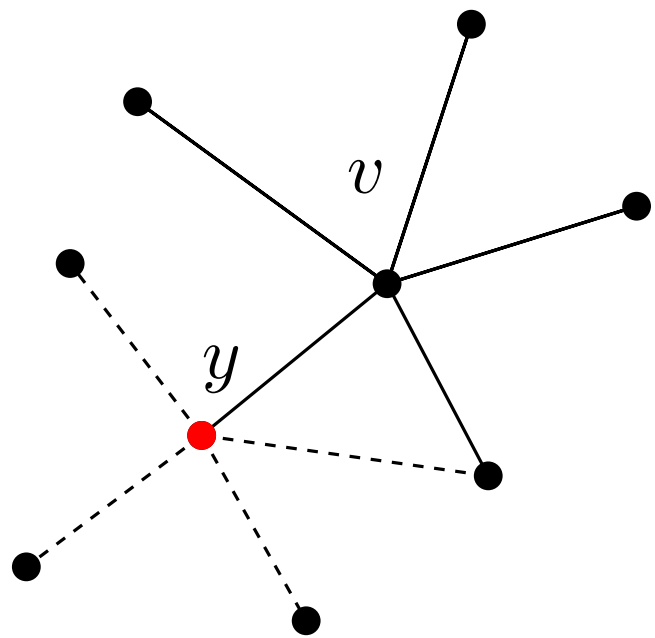
Observations

Lemma Let U be a *maximum* independent set in G. Then, for each vertex $v \in V$:

(i) $v \in U \rightsquigarrow N(v) \cap U = \emptyset$

(ii) $v \notin U \rightsquigarrow |N(v) \cap U| \geq 1$

Thus, $N[v] := N(v) \cup \{v\}$ contains some $y \in U$ and no other vertex of $N[y]$ is in U



Smarter Branching-Algorithm

Algorithm MIS(G)
  if $V = \emptyset$ **then**
    return 0
  $v \leftarrow$ vertex of minimum degree in $V(G)$
  return $1 + \max\{\text{MIS}(G - N[y]) \mid y \in N[v]\}$

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Correctness follows from the previous Lemma.

Smarter Branching-Algorithm

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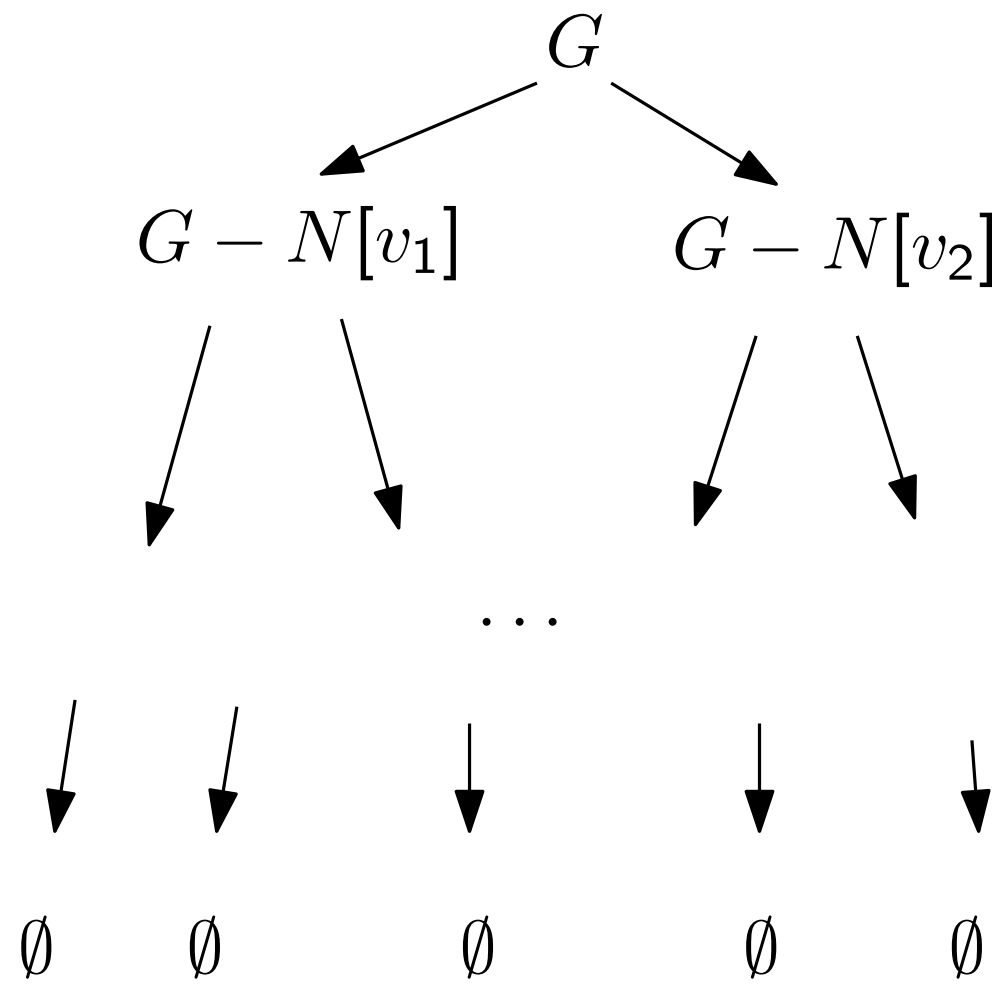
  v ← vertex of minimum degree in V(G)
  return 1 + max{MIS(G − N[y]) | y ∈ N[v]}
```

Correctness follows from the previous Lemma.

We will now prove a runtime of $O^*(3^{n/3}) = O^*(1.4423^n)$

Runtime

Execution corresponds to a *search tree* whose nodes are labeled with the input of the respective recursive call.



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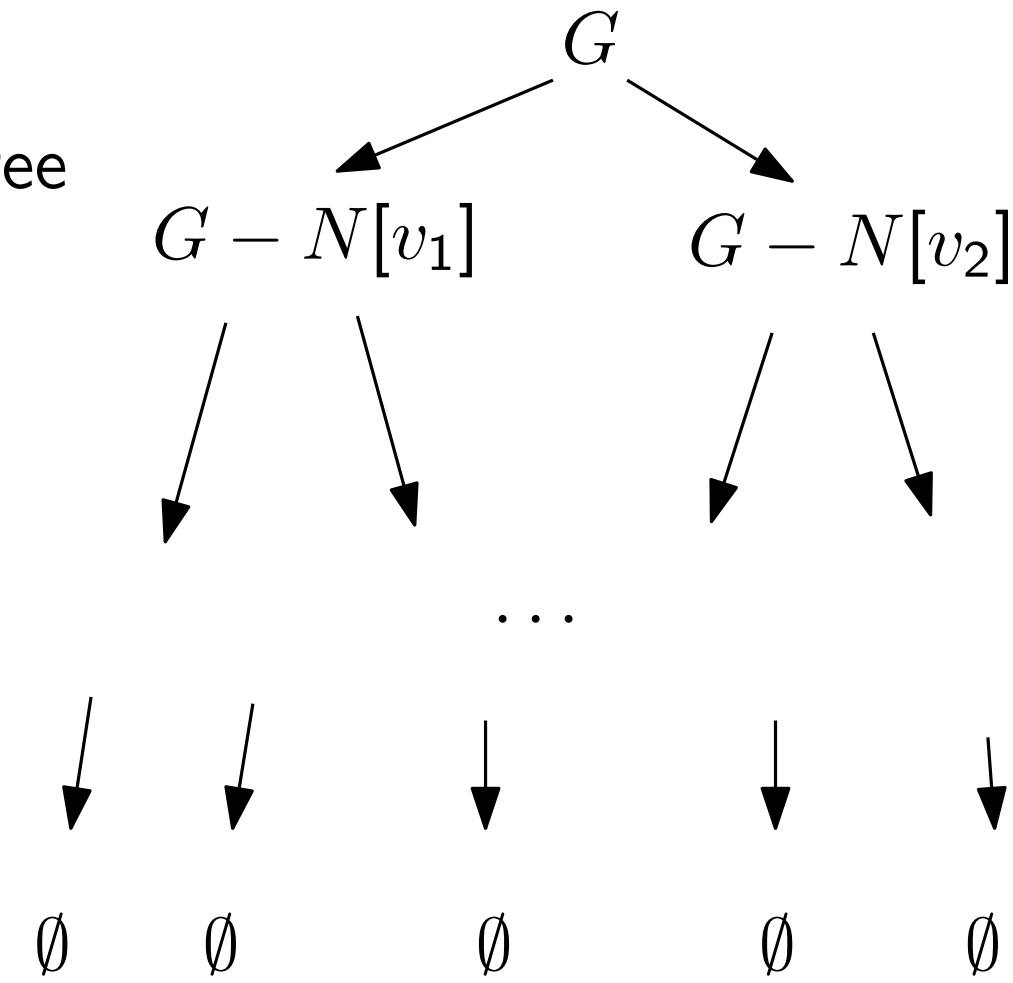
Let $B(n)$ be the maximum number of leaves of a search tree for a graph with n vertices.



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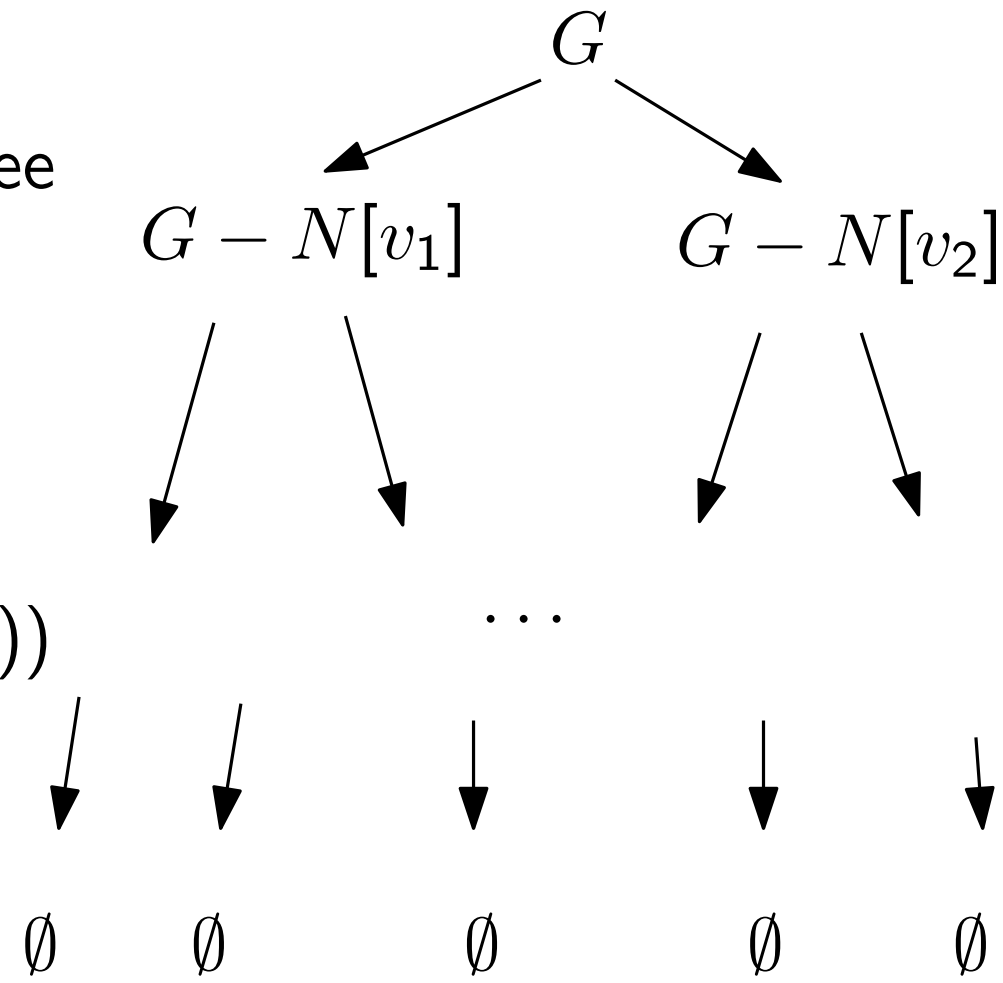
Search-tree has height $\leq n$,
$\rightsquigarrow$ the algorithm's runtime is
$T(n) \in O^*(nB(n)) = O^*(B(n))$



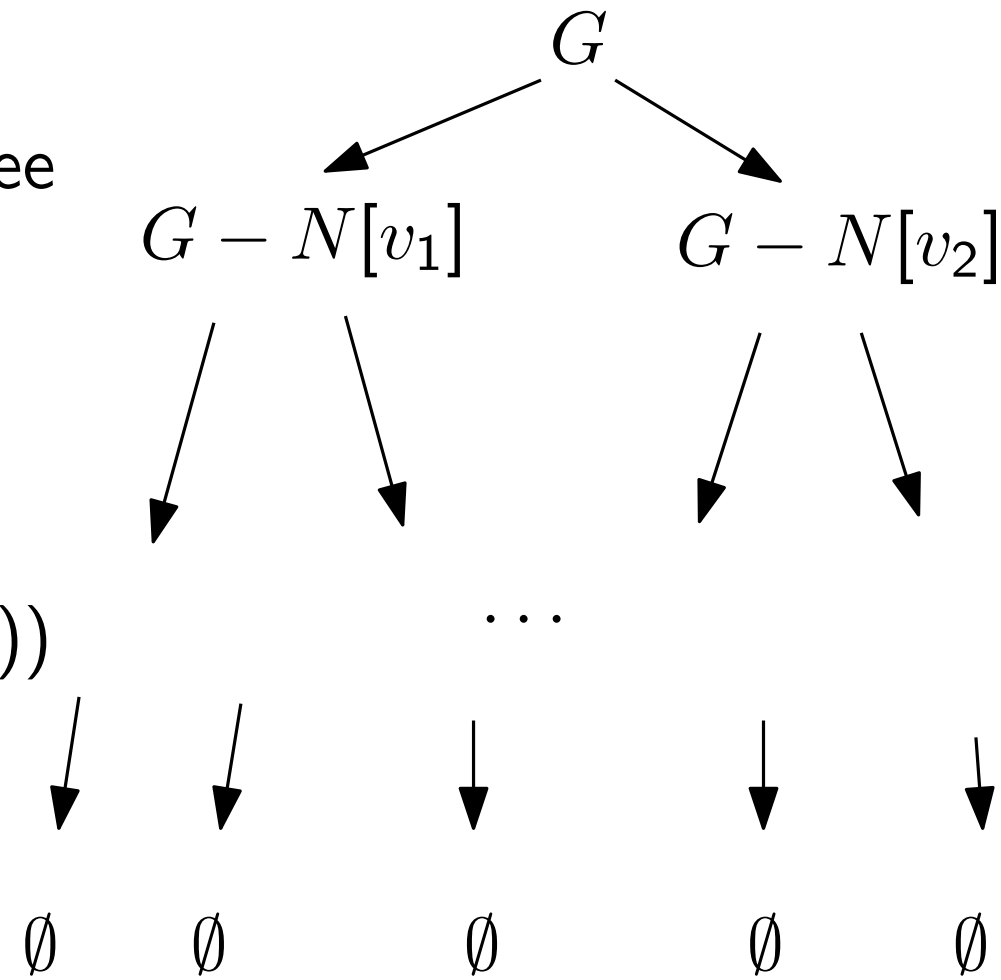
Runtime

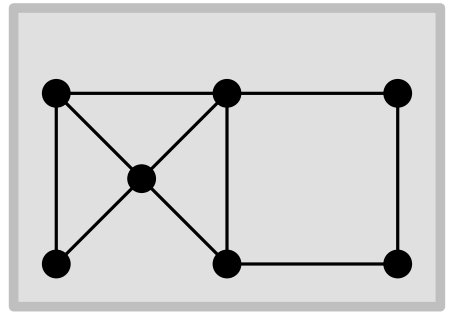
Execution corresponds to a *search tree* whose nodes are labeled with the input of the respective recursive call.

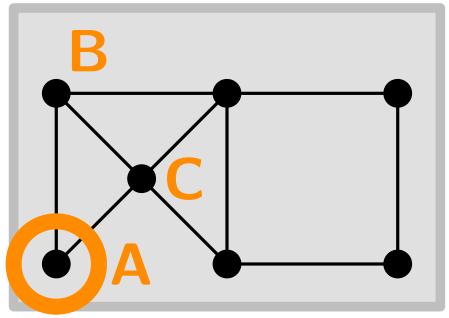
Let $B(n)$ be the maximum number of leaves of a search tree for a graph with n vertices.

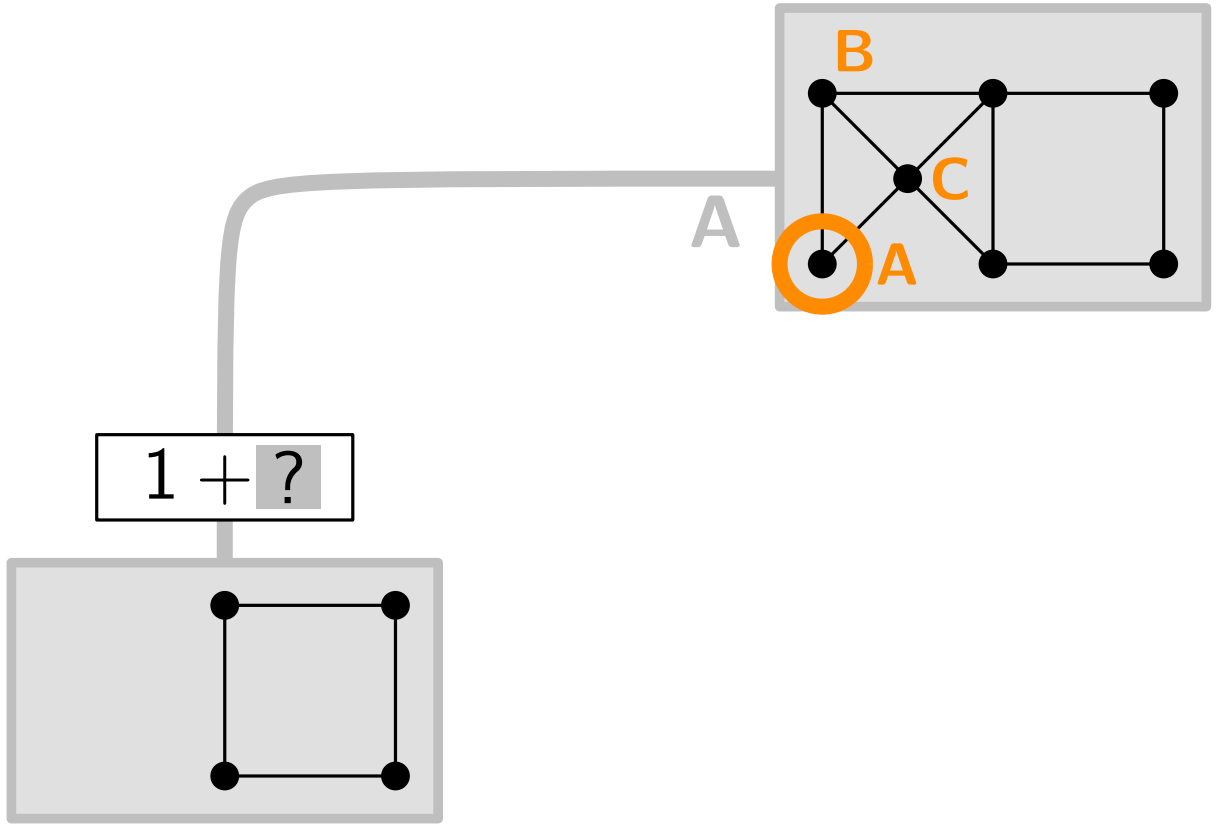
Search-tree has height $\leq n$,
$\rightsquigarrow$ the algorithm's runtime is
$T(n) \in O^*(nB(n)) = O^*(B(n))$

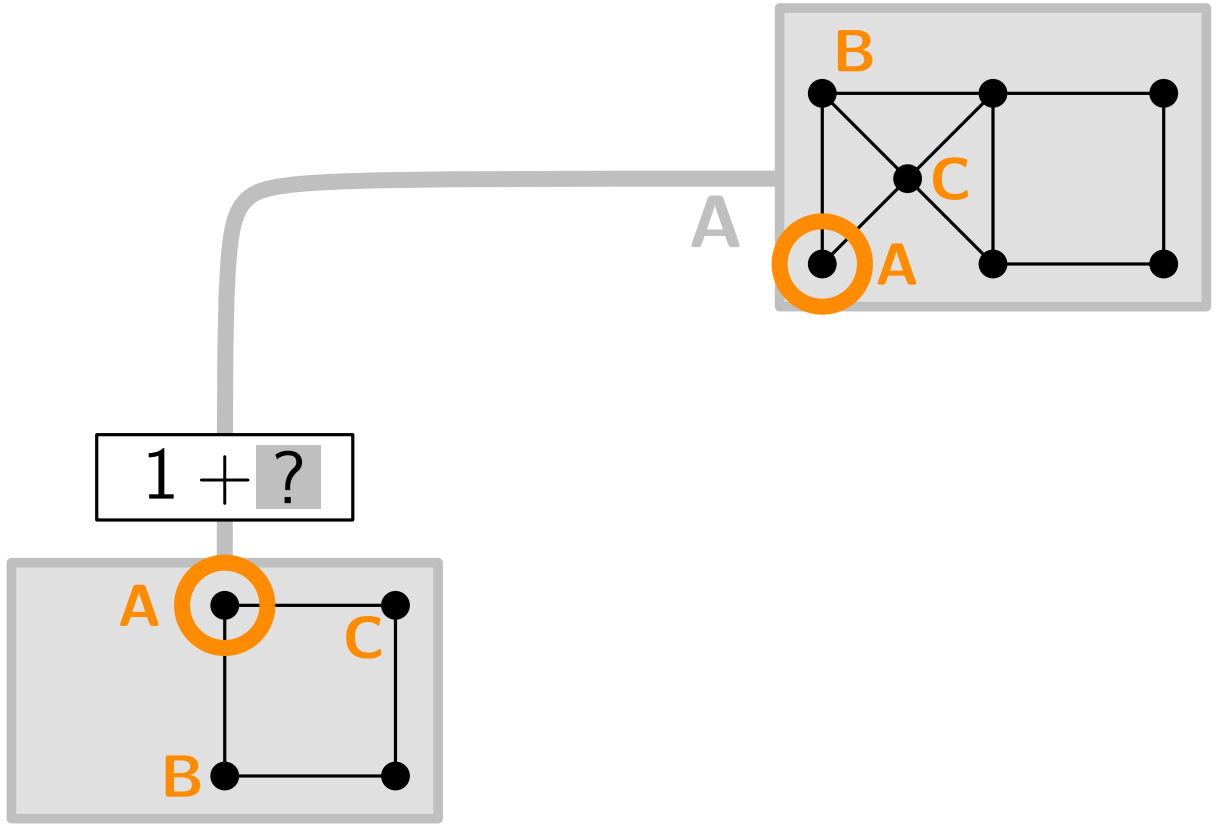
Let's consider an example run.

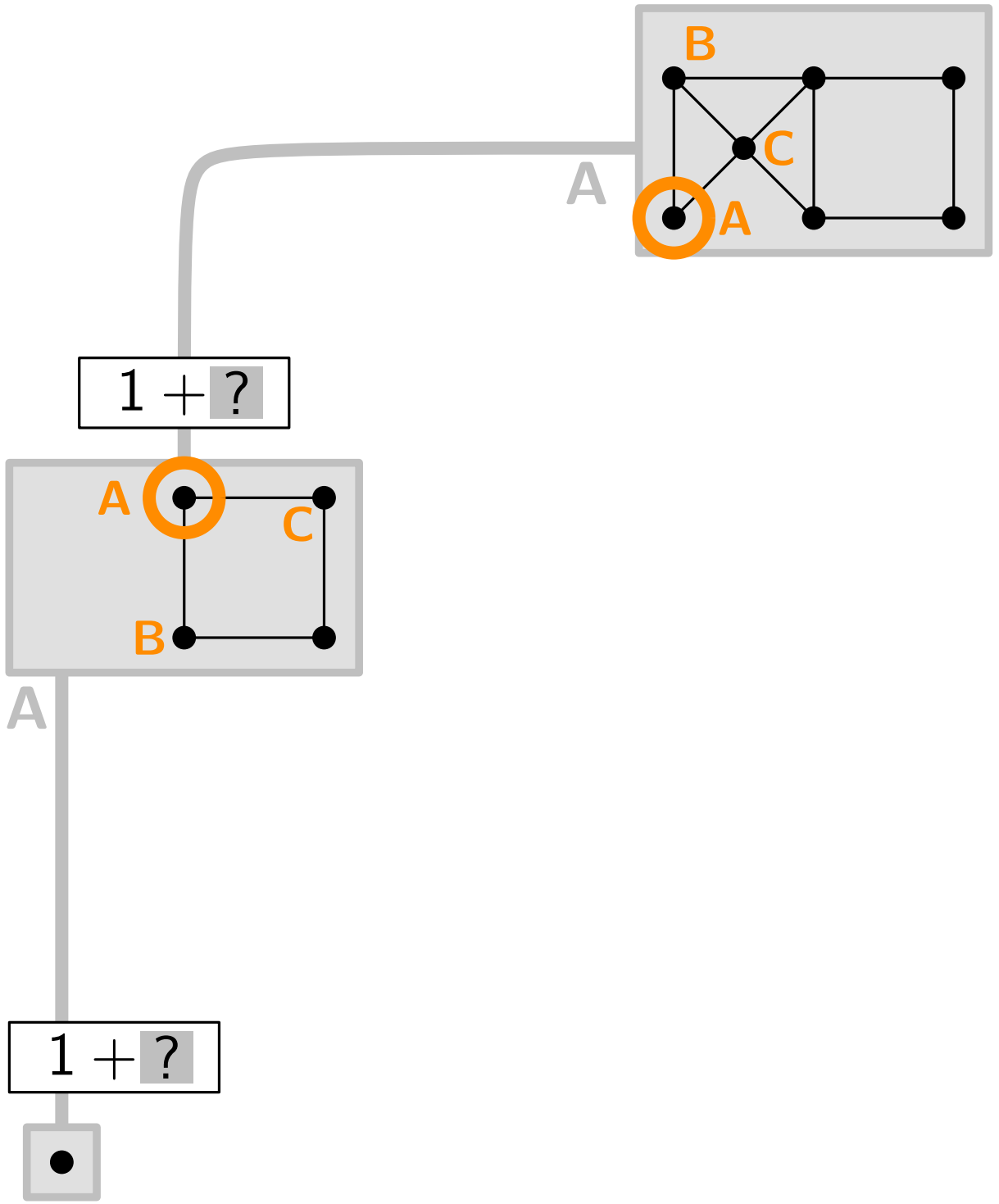


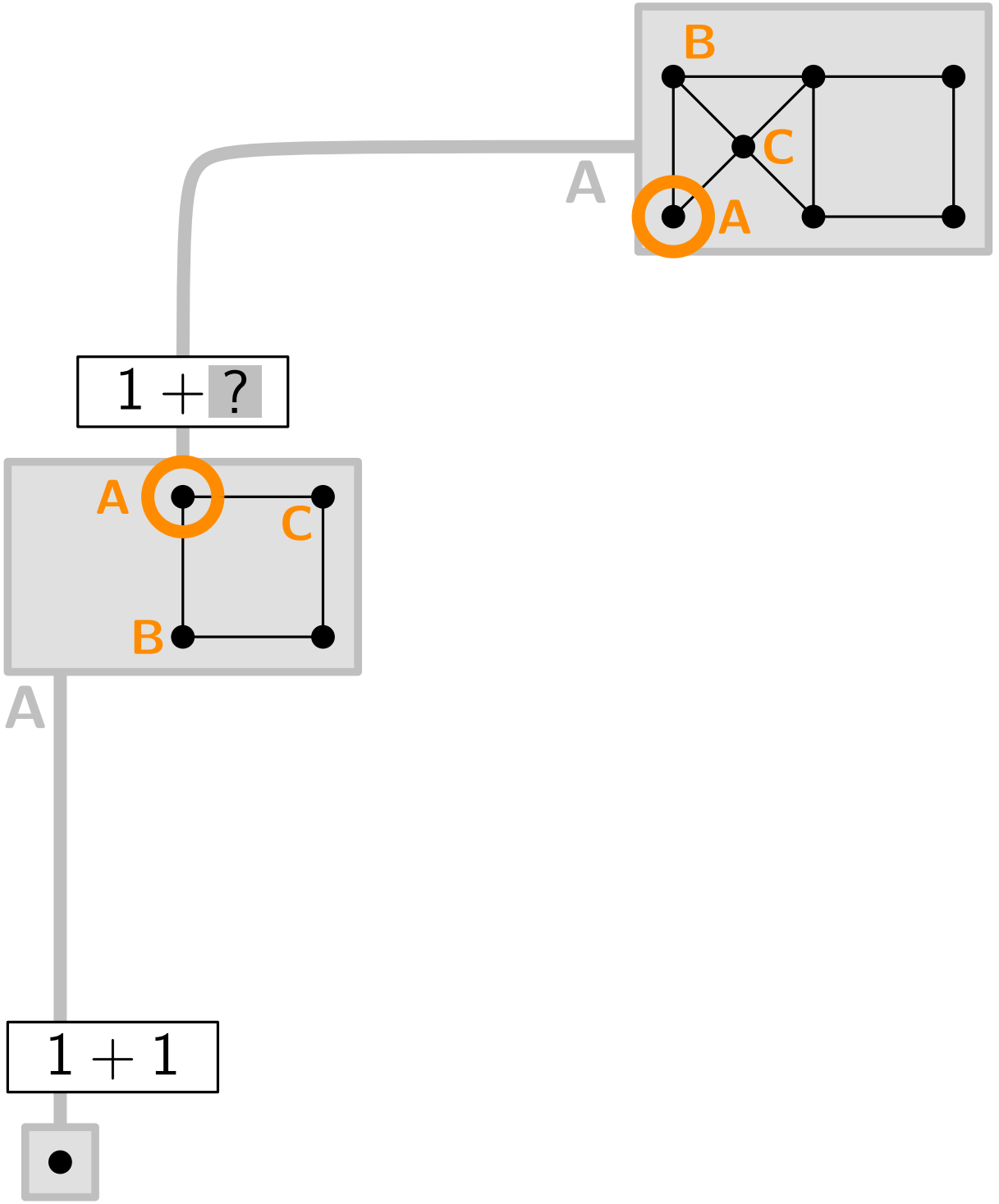


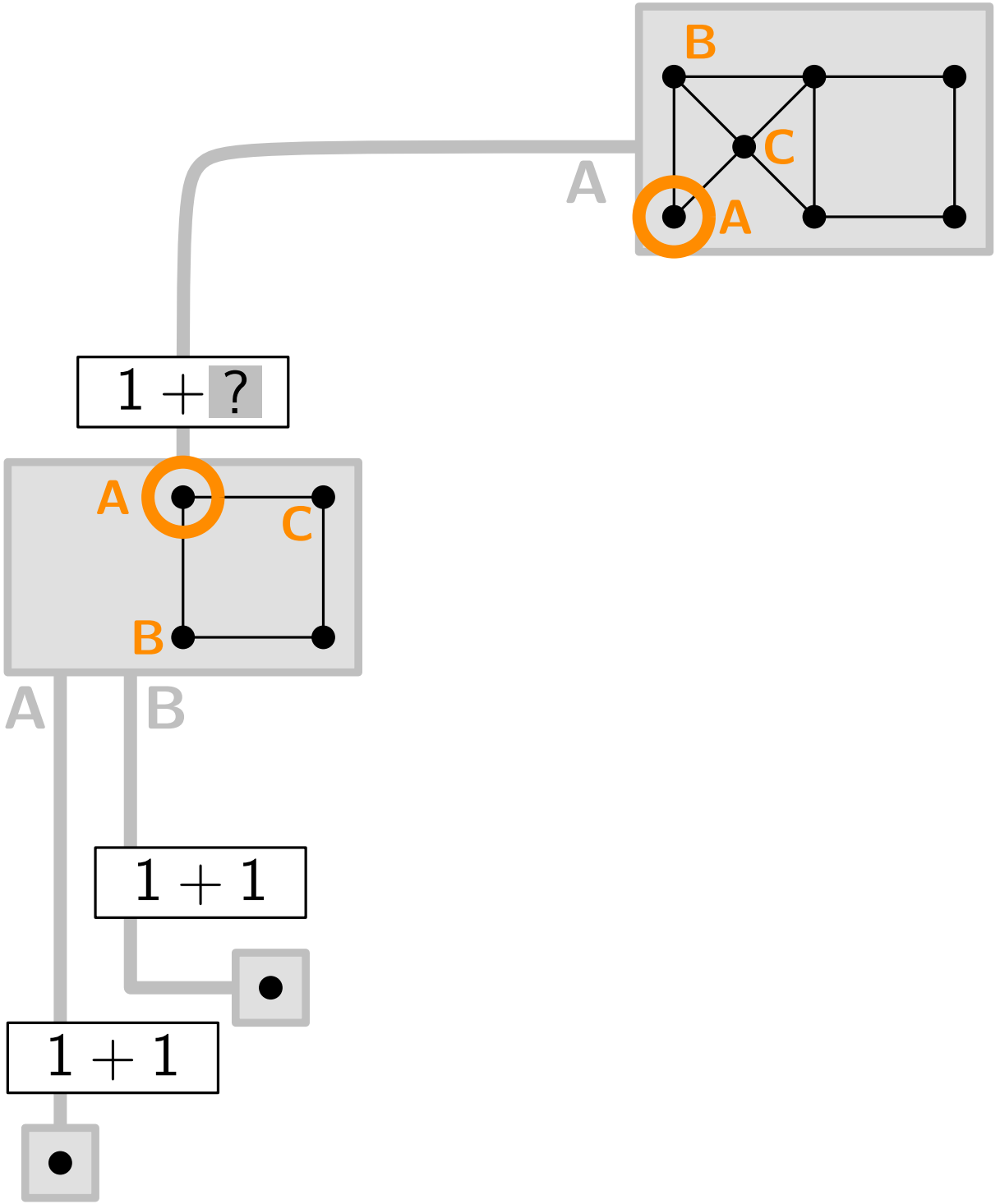

B
C
A

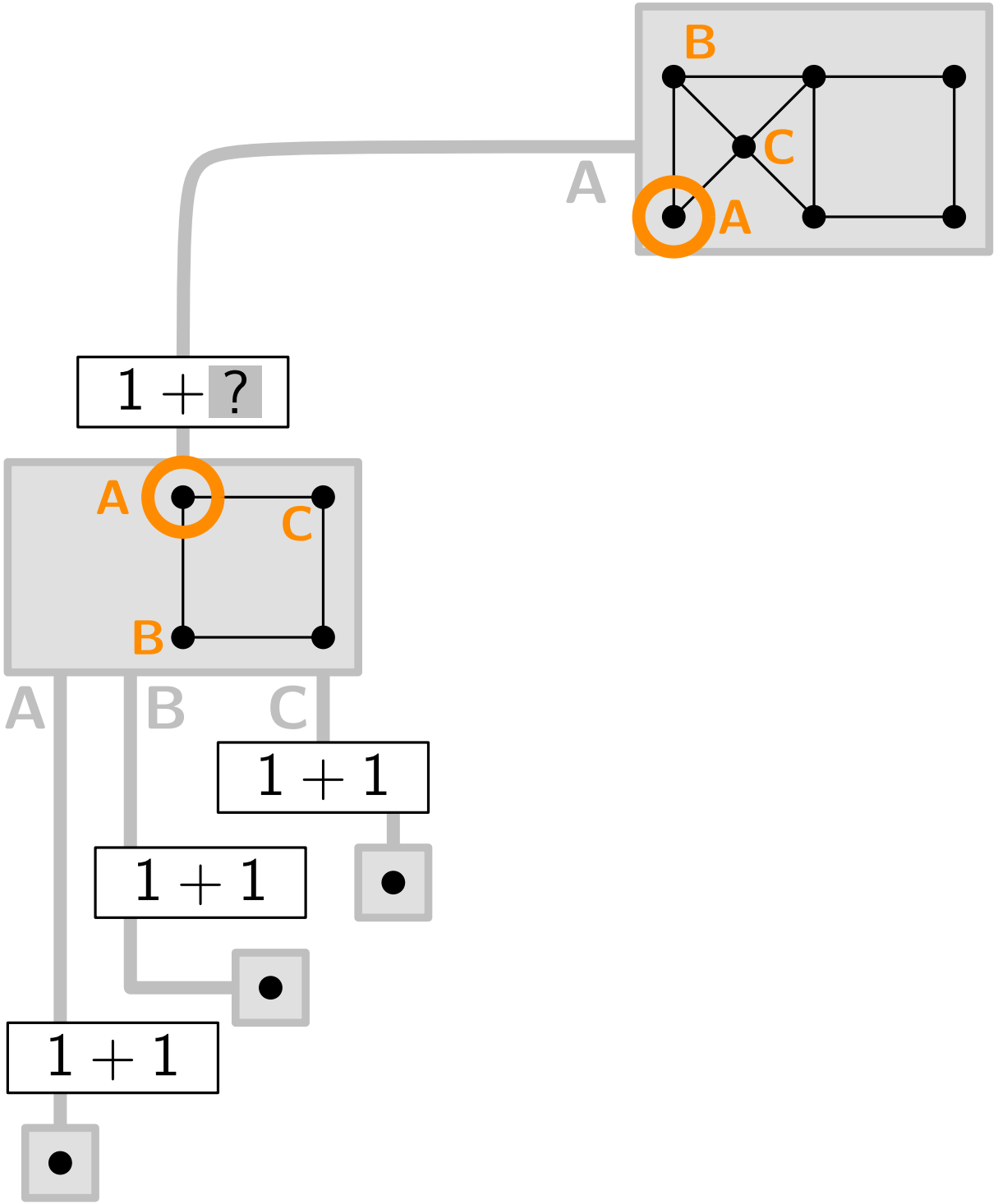

B
C
A
A
1 + ?

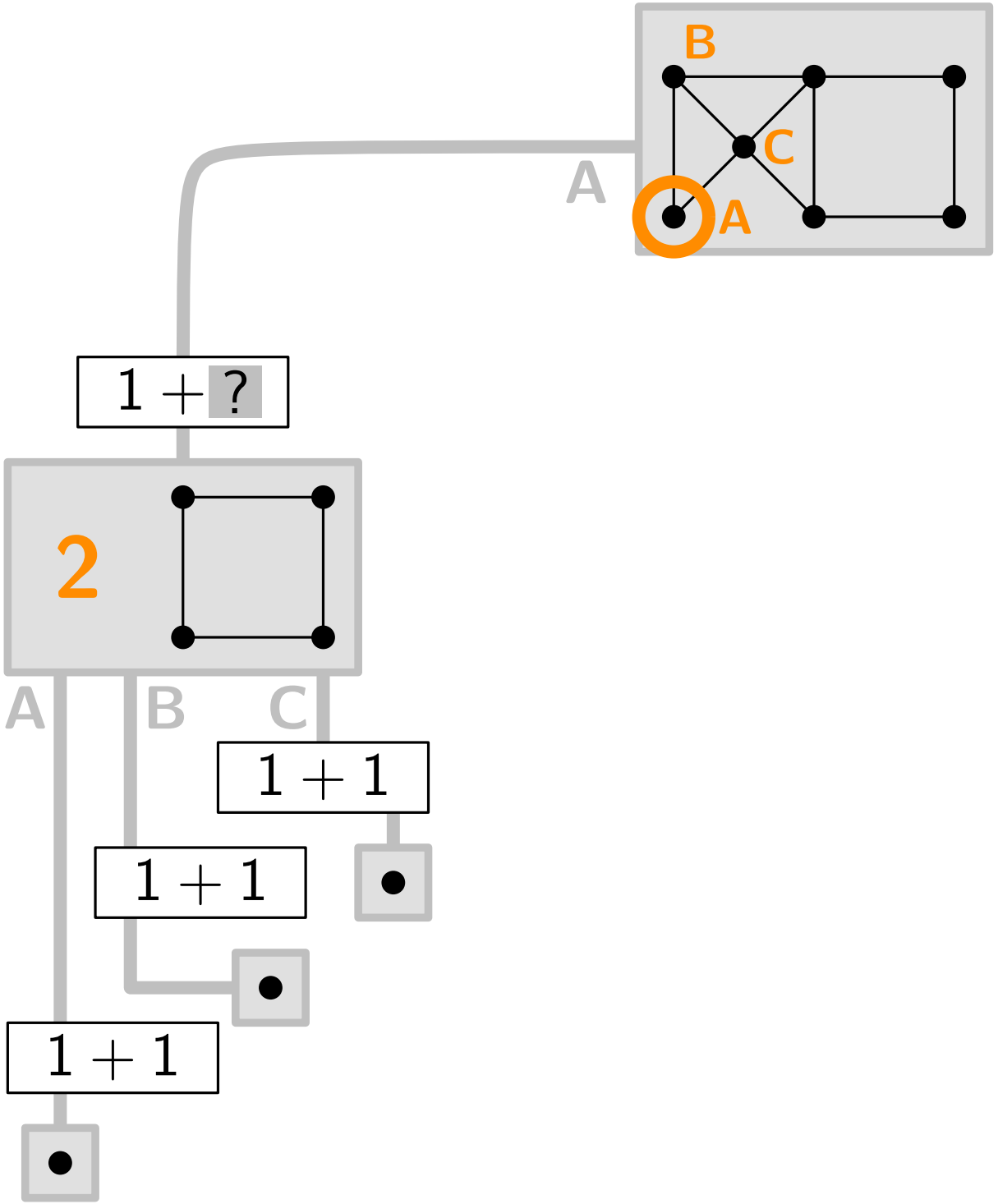

B
C
A
A
1 + ?
A
C
B

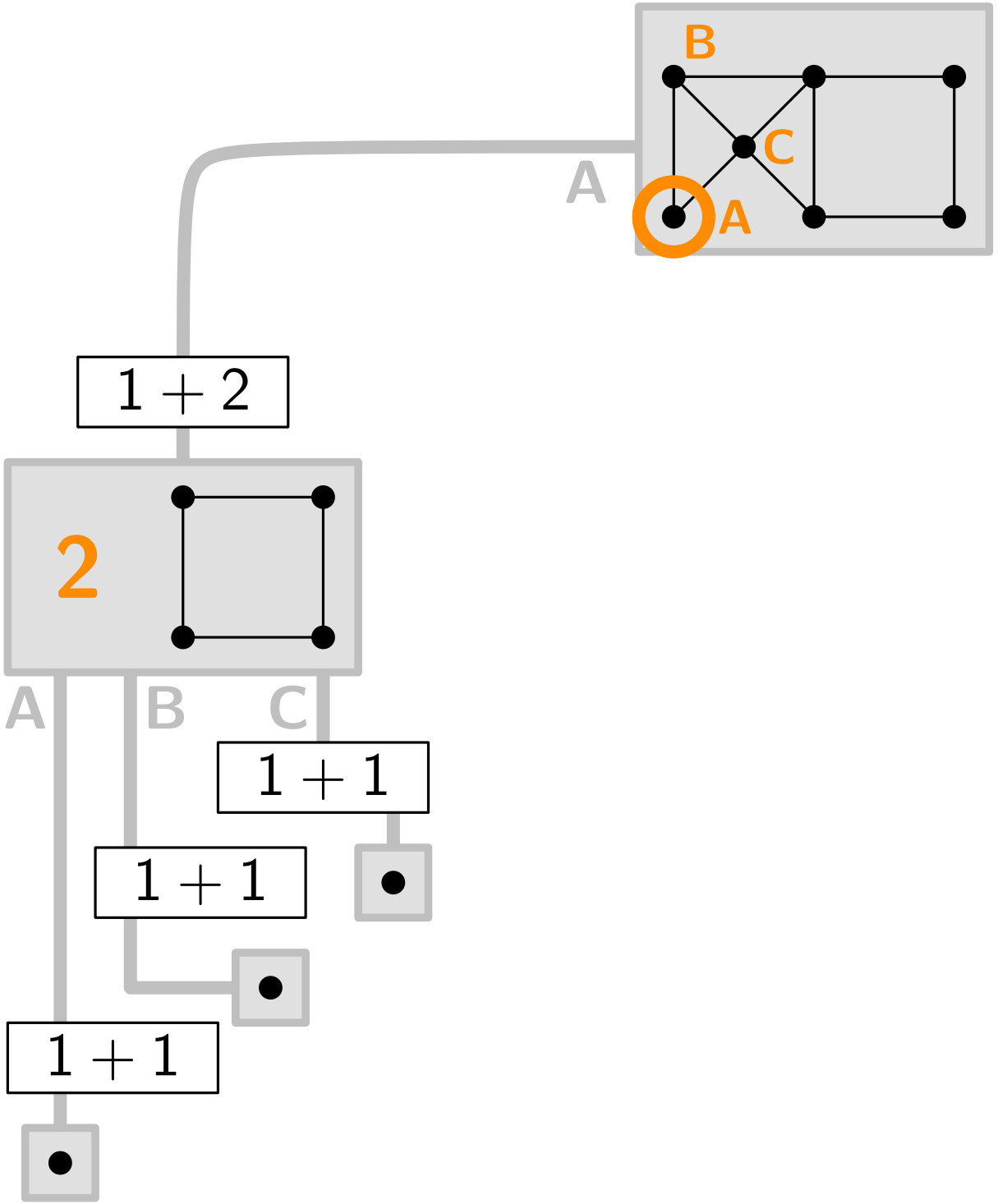

B
C
A
A
1 + ?
A
C
B
A
1 + ?

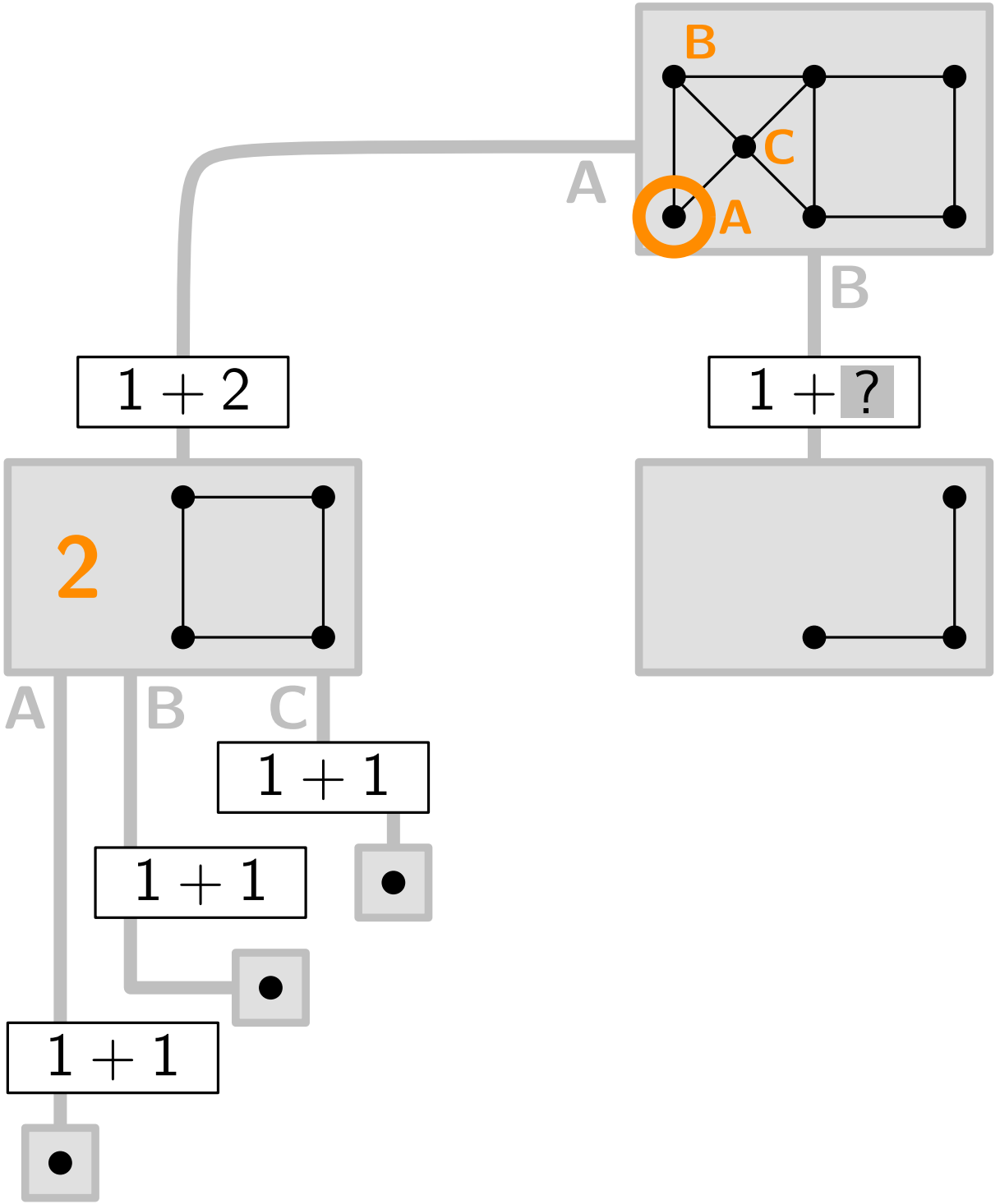

B
C
A
A
A
1 + ?
A
C
B
A
1 + 1

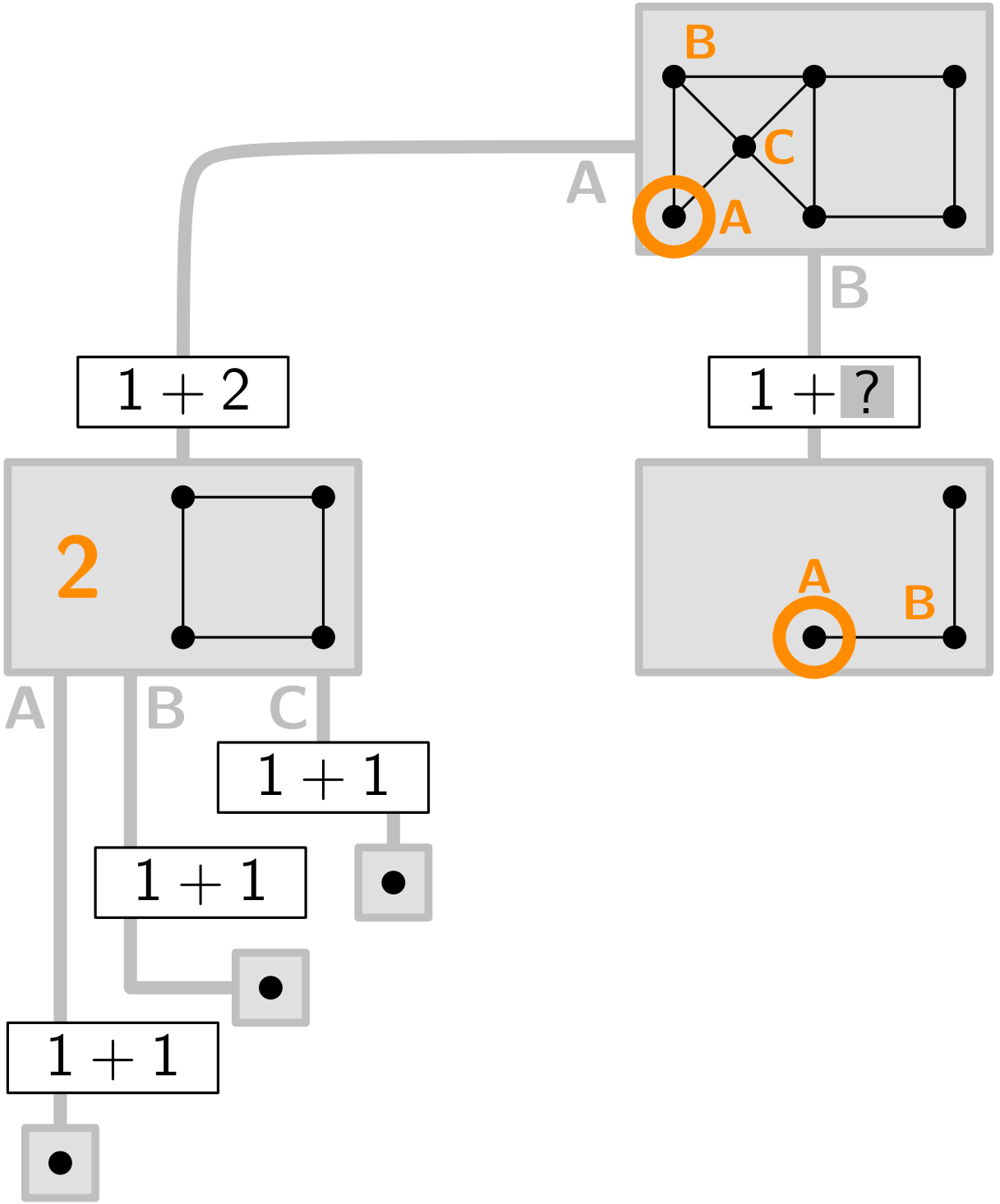

B
C
A
A
1 + ?
A
C
B
A
B
1 + 1
1 + 1

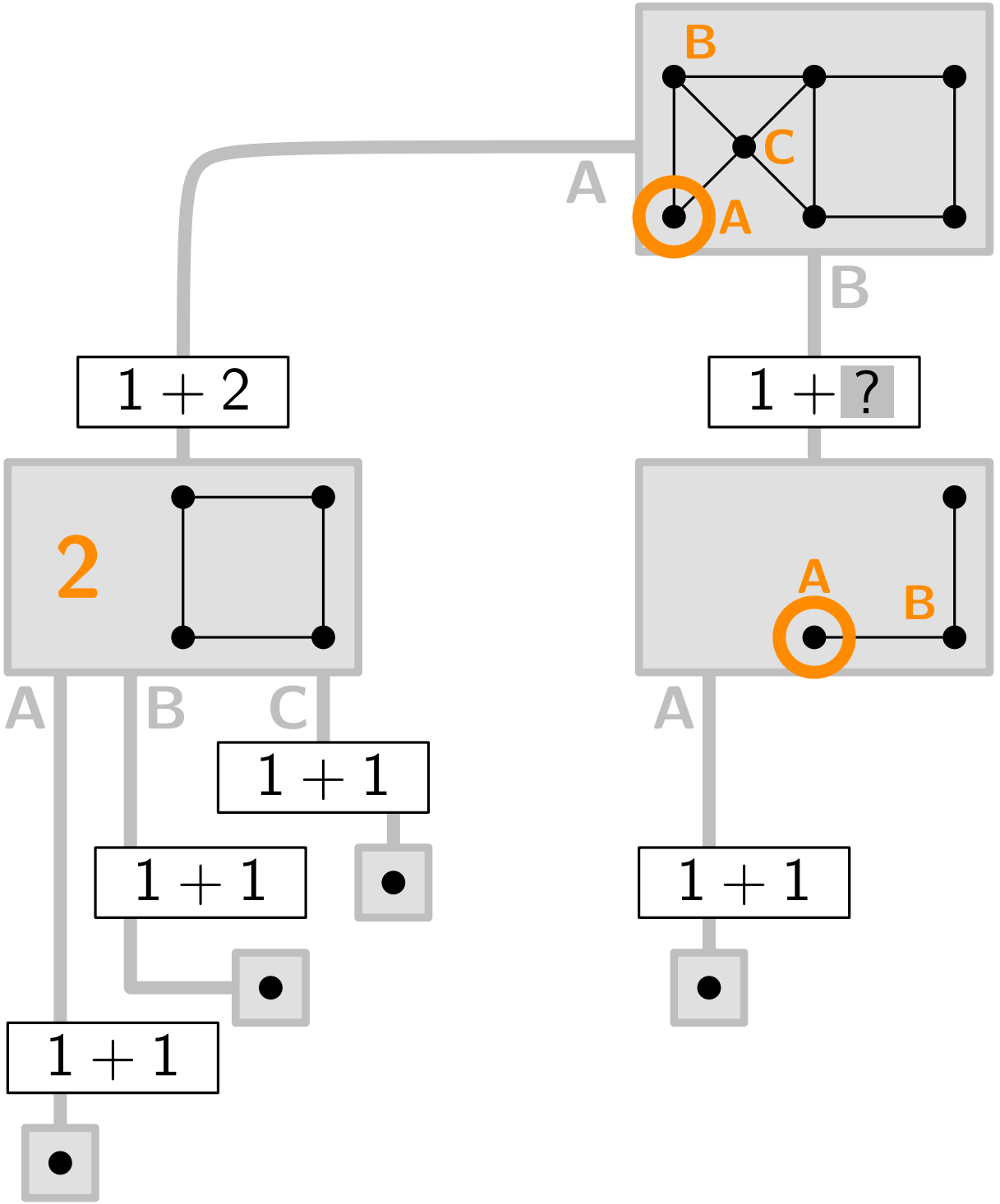

B
C
A
A
A
1 + ?
A
C
B
A
B
C
1 + 1
1 + 1
1 + 1

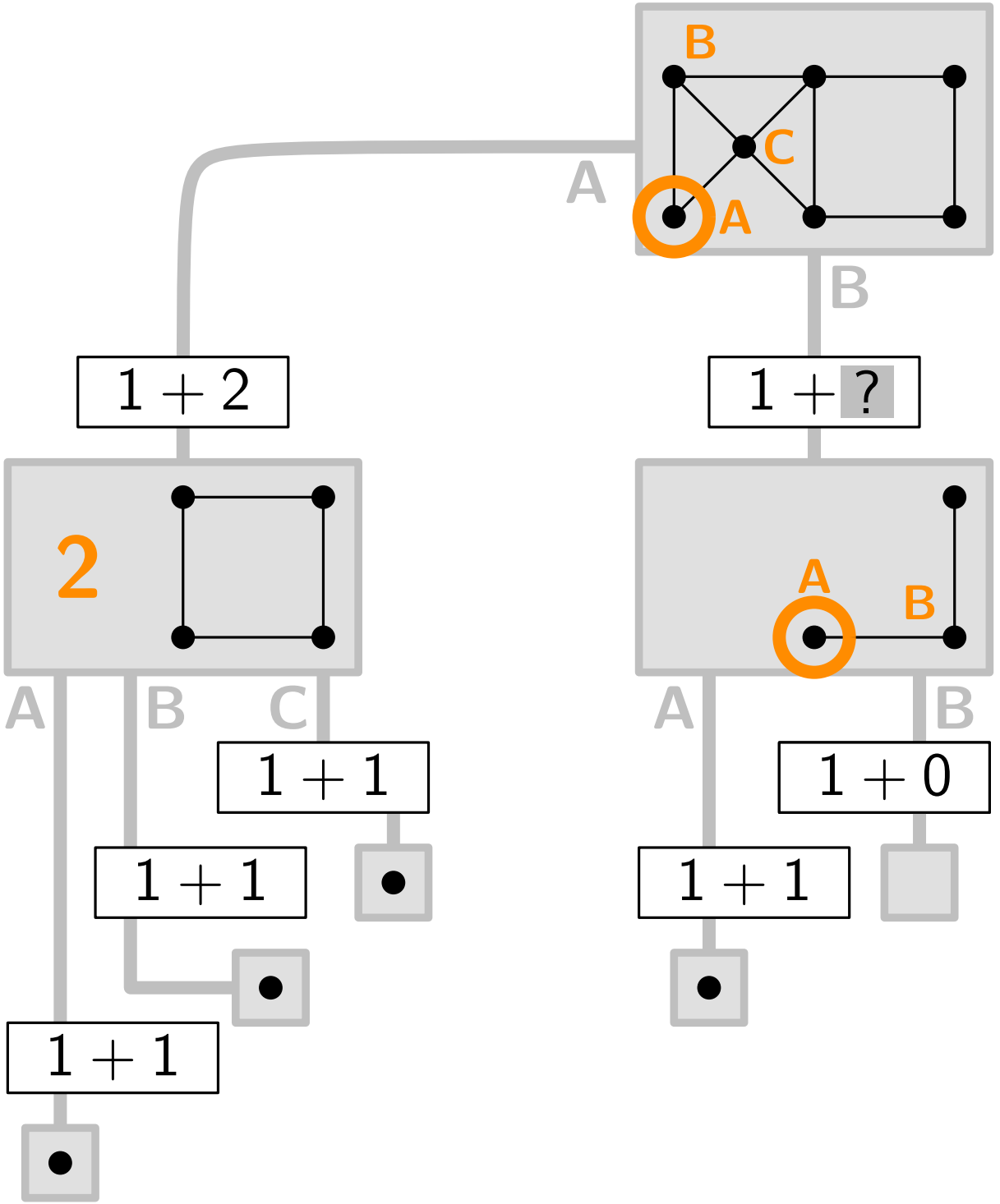

B
C
A
A
1 + ?
2
A
B
C
1 + 1
1 + 1
1 + 1

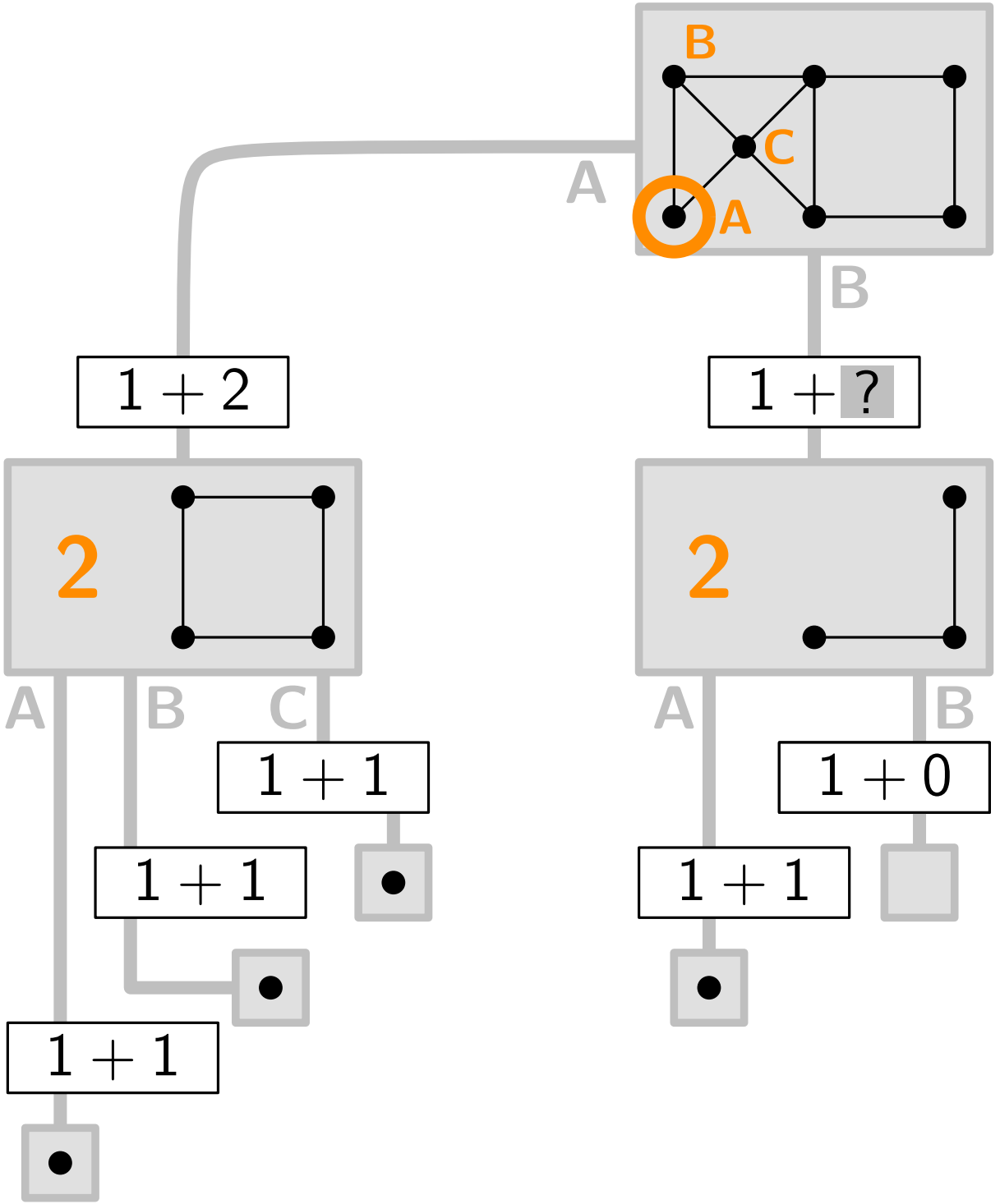

B
C
A
A
1 + 2
2
A
B
C
1 + 1
1 + 1
1 + 1

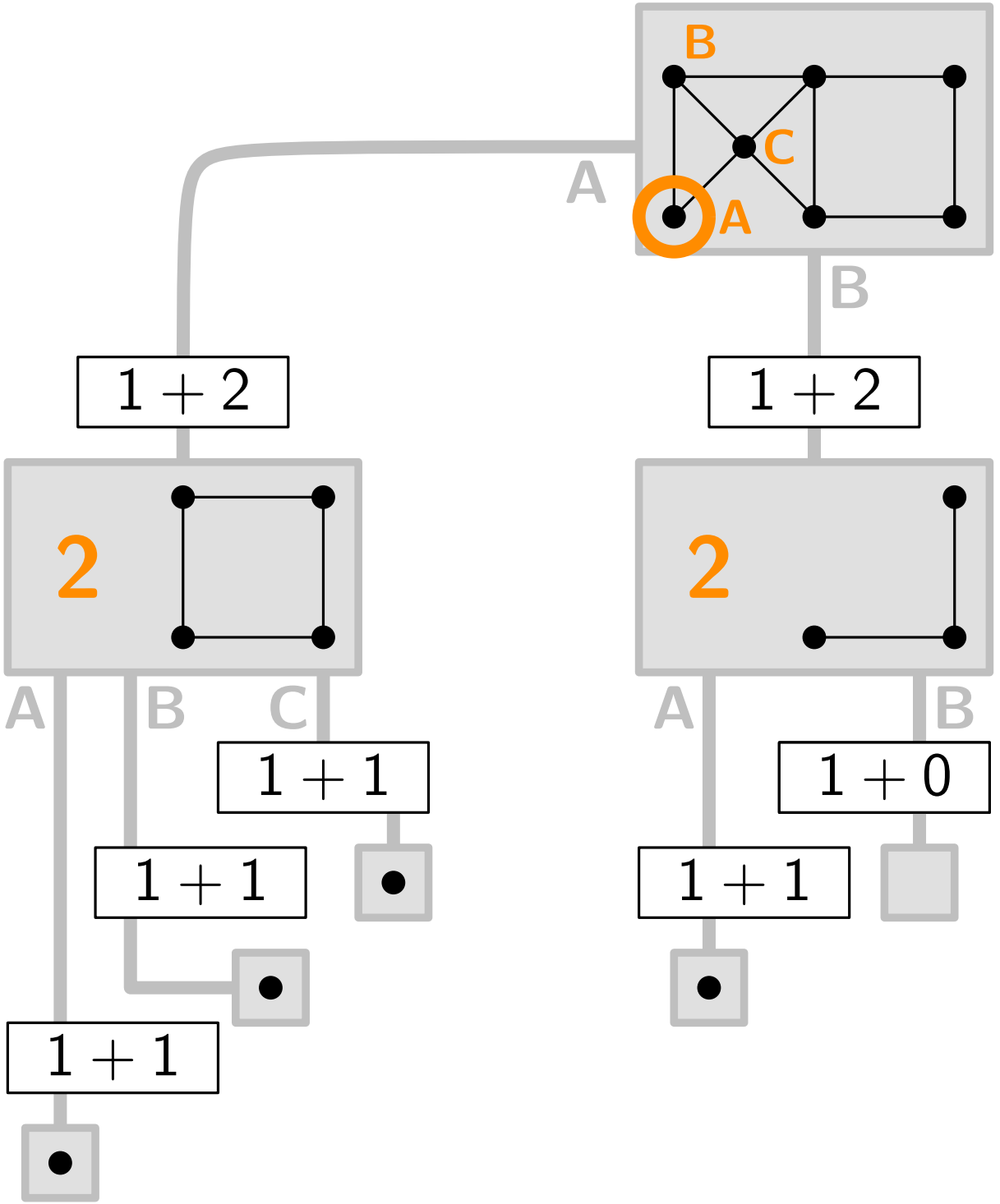

B
C
A
A
B
1 + 2
1 + ?
2
A
B
C
1 + 1
1 + 1
1 + 1

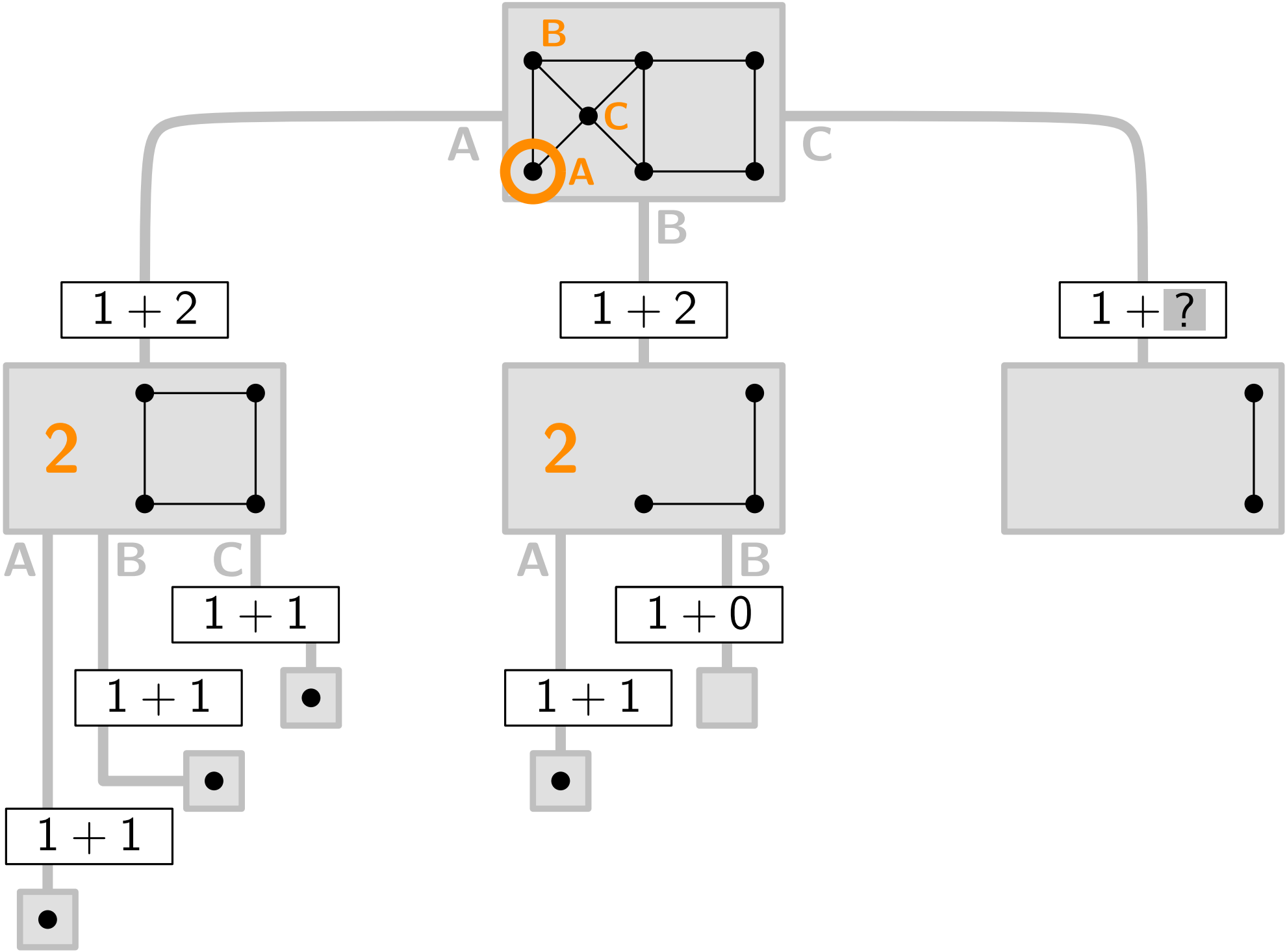

B
C
A
A
B
1 + 2
1 + ?
2
A
B
A
B
C
1 + 1
1 + 1
1 + 1

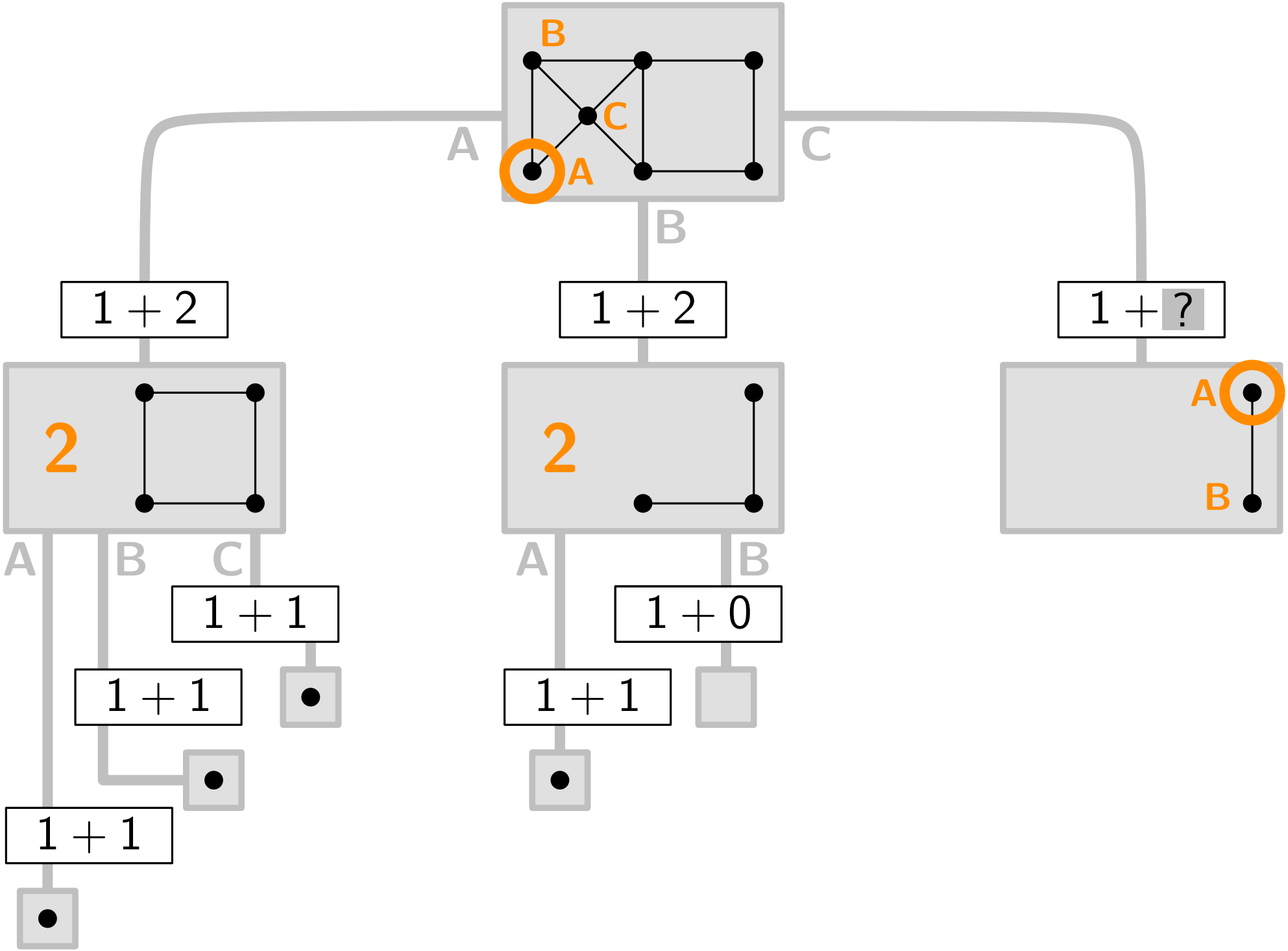

B
C
A
A
B
1 + 2
1 + ?
2
A
B
A
B
C
A
1 + 1
1 + 1
1 + 1


B
C
A
A
B
1 + 2
1 + ?
2
A
B
A
B
C
A
B
1 + 1
1 + 0
1 + 1
1 + 1


B
C
A
A
B
1 + 2
1 + ?
2
2
A
B
C
A
B
1 + 1
1 + 0
1 + 1
1 + 1


B
C
A
A
B
1 + 2
1 + 2
2
2
A
B
C
A
B
1 + 1
1 + 0
1 + 1
1 + 1


B
C
A
A
C
B
1 + 2
1 + 2
1 + ?
2
2
A
B
C
A
B
1 + 1
1 + 0
1 + 1
1 + 1


B
C
A
A
C
B
1 + 2
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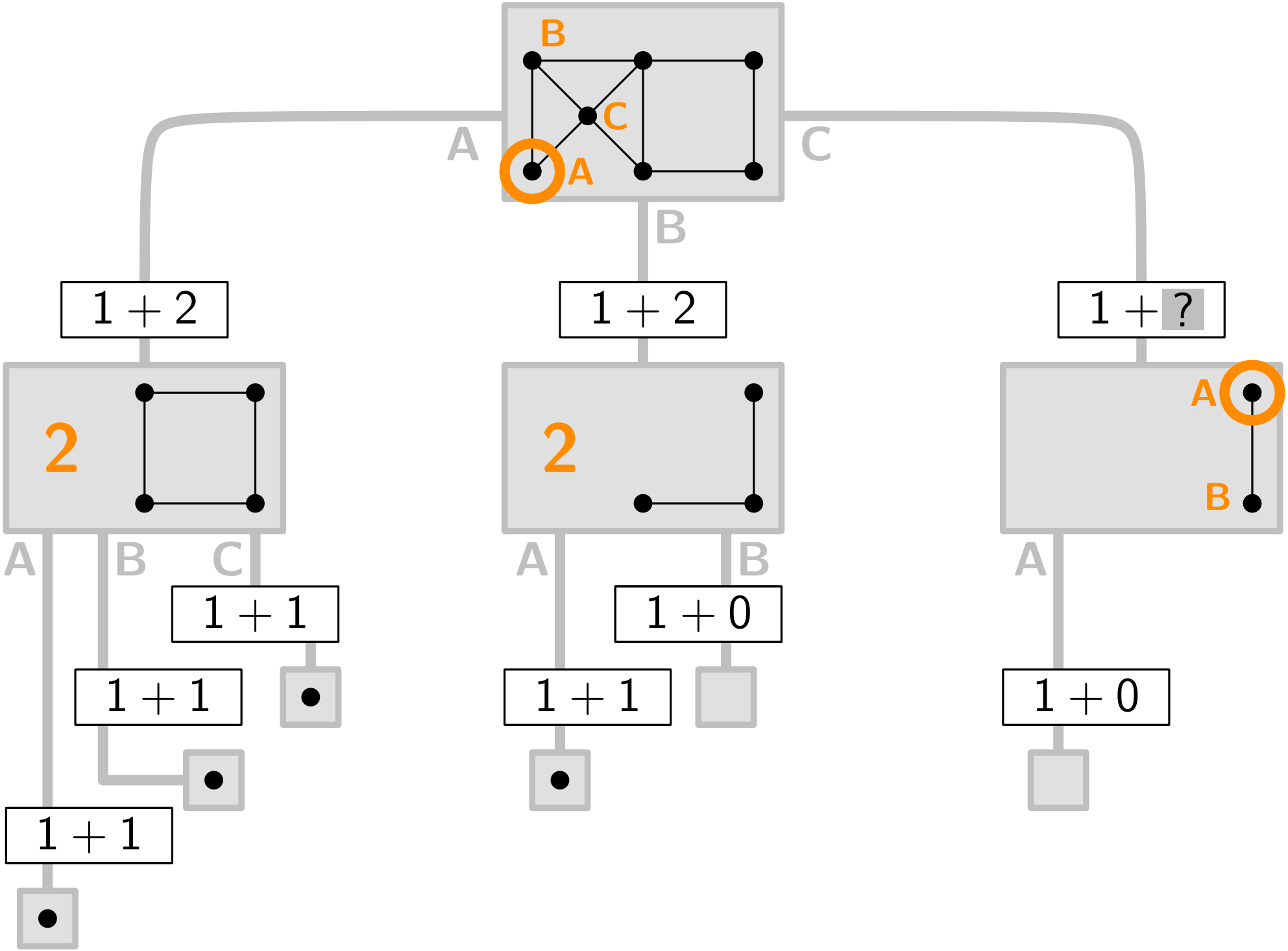
1 + ?
2
2
A
B
A
B
C
A
B
1 + 1
1 + 0
1 + 1
1 + 1


B
C
A
A
C
B
1 + 2
1 + 2
1 + ?
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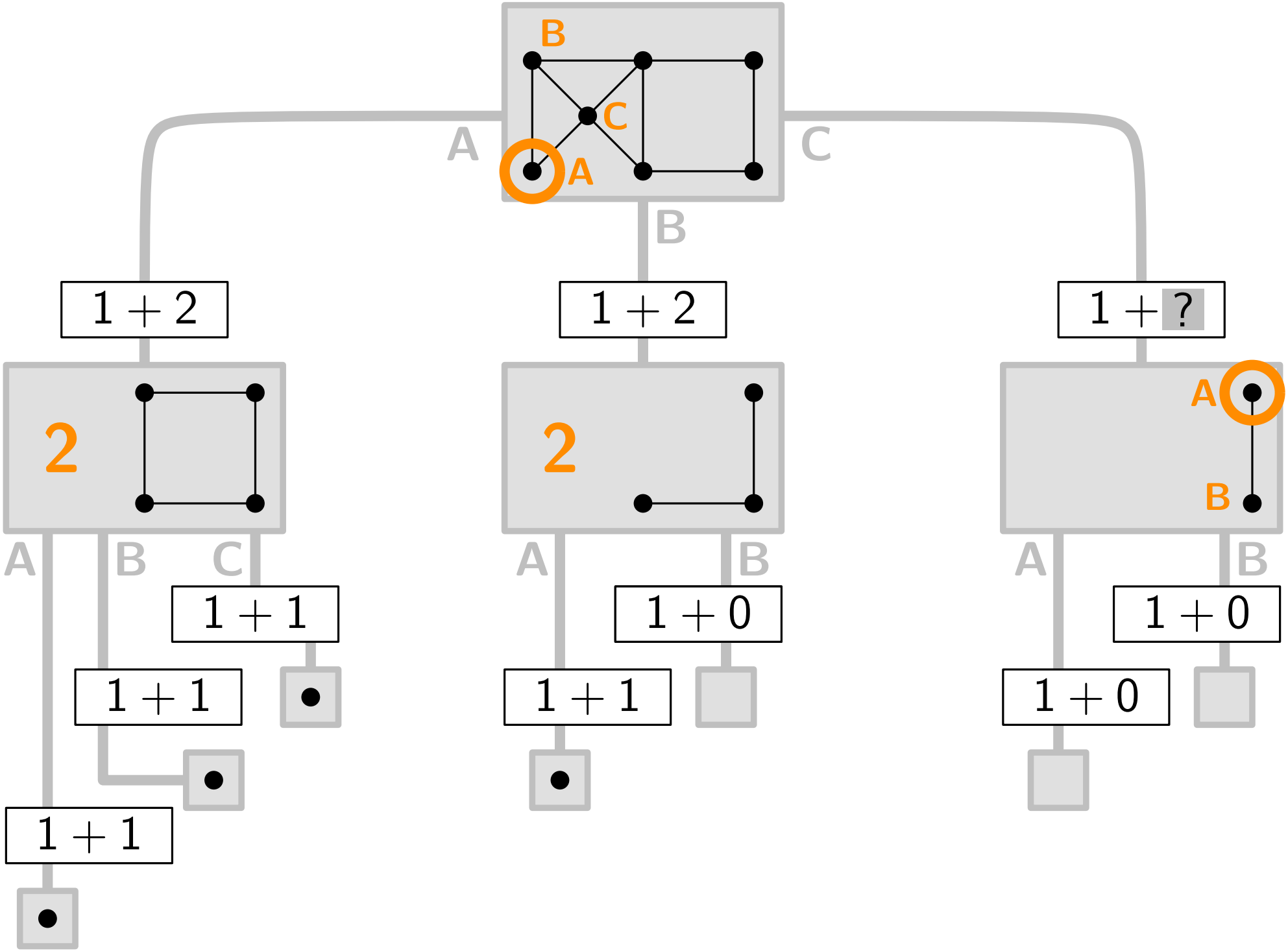
2
A
B
A
B
C
A
B
1 + 1
1 + 0
1 + 1
1 + 1
1 + 0


B
C
A
A
C
B
1 + 2
1 + 2
1 + ?
2
2
A
B
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B
C
A
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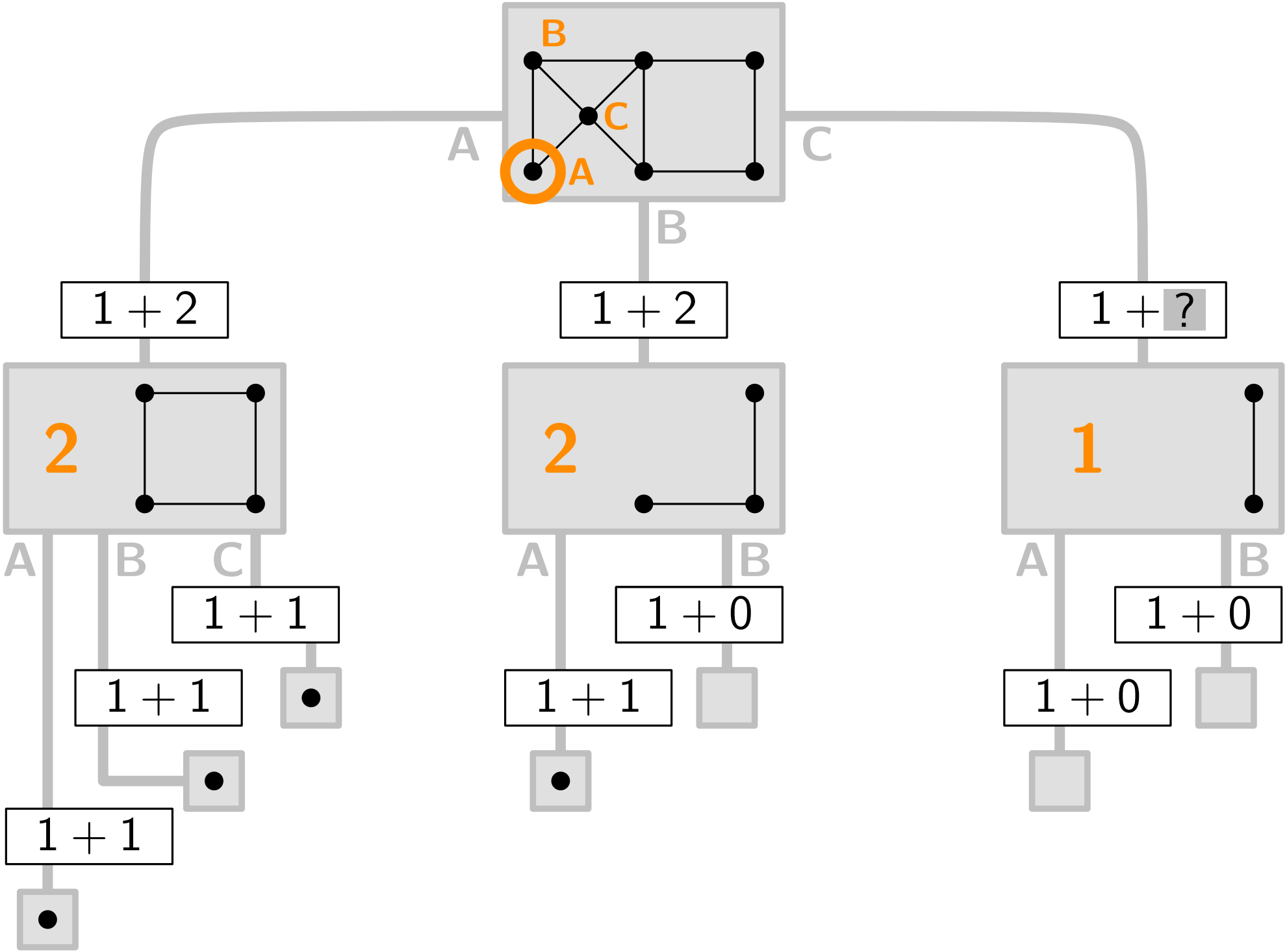
1 + 1
1 + 0
1 + 0
1 + 1
1 + 1
1 + 1


B
C
A
A
C
B
1 + 2
1 + 2
1 + ?
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2
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C
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1 + 1
1 + 0
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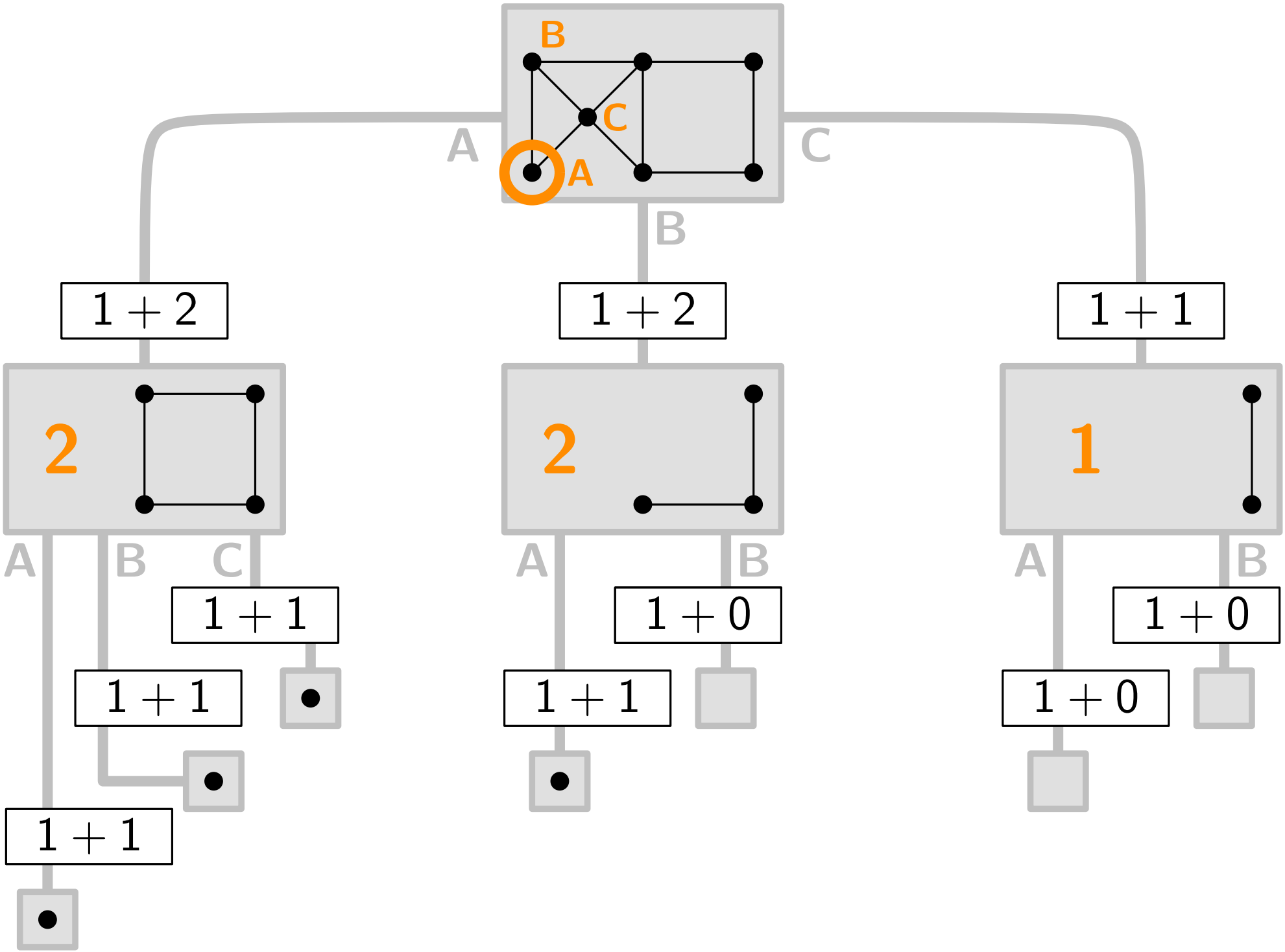
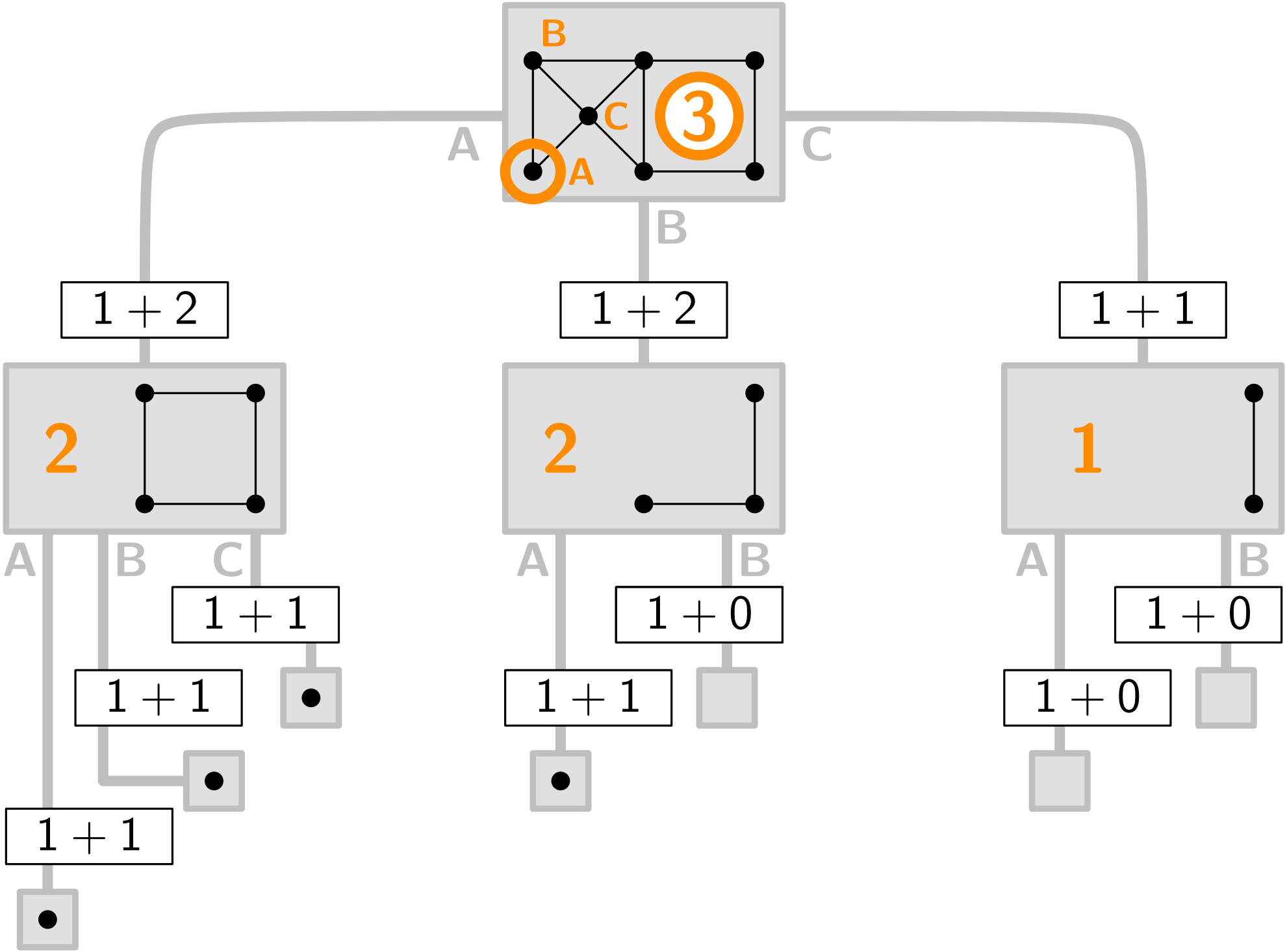
1 + 1
1 + 1
1 + 1


B
C
A
A
C
B
1 + 2
1 + 2
1 + 1
2
2
1
A
B
C
A
B
A
B
1 + 1
1 + 0
1 + 0
1 + 1
1 + 1
1 + 0
1 + 1


B
C
3
A
A
A
B
C
1 + 2
1 + 2
1 + 1
2
2
1
A
B
C
A
B
A
B
1 + 1
1 + 0
1 + 0
1 + 1
1 + 1
1 + 0

Runtime Analysis

For a worst-case n-vertex graph G ($n \geq 1$):

$$\begin{aligned} B(n) &\leq \sum_{y \in N[v]} B(n - (\deg(y) + 1)) \\ &\leq (\deg(v) + 1) \cdot B(\ n - (\deg(v) + 1)\ )\,, \end{aligned}$$

where v is a minimum degree vertex of G, and we note that $B(n') \leq B(n)$ for any $n' \leq n$.

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Runtime Analysis (cont)

$B(n) \leq (\deg(v) + 1) \cdot B(\ n - (\deg(v) + 1)\ )$

We proceed by induction to show $B(n) \leq 3^{n/3}$

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Hypothesis: for $n \geq 1$, set $s = \deg(v) + 1$ in the above inequality

$B(n) \leq s \cdot B(n - s)$

Runtime Analysis (cont)

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Base case: $B(0) = 1 \leq 3^{0/3}$

Hypothesis: for $n \geq 1$, set $s = \deg(v) + 1$ in the above inequality

$B(n) \leq s \cdot B(n - s) \leq s \cdot 3^{(n-s)/3}$

Runtime Analysis (cont)

$B(n) \leq (\deg(v) + 1) \cdot B(\ n - (\deg(v) + 1)\ )$

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Base case: $B(0) = 1 \leq 3^{0/3}$

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$B(n) \leq s \cdot B(n - s) \leq s \cdot 3^{(n-s)/3} = \frac{s}{3^{s/3}} \cdot 3^{n/3}$

Runtime Analysis (cont)

$B(n) \leq (\deg(v) + 1) \cdot B(\ n - (\deg(v) + 1)\ )$

We proceed by induction to show $B(n) \leq 3^{n/3}$

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$$B(n) \leq s \cdot B(n-s) \leq s \cdot 3^{(n-s)/3} = \frac{s}{3^{s/3}} \cdot 3^{n/3} \overset{?}{\leq} 3^{n/3}$$

Runtime Analysis (cont)

$B(n) \leq (\deg(v) + 1) \cdot B(\ n - (\deg(v) + 1)\ )$

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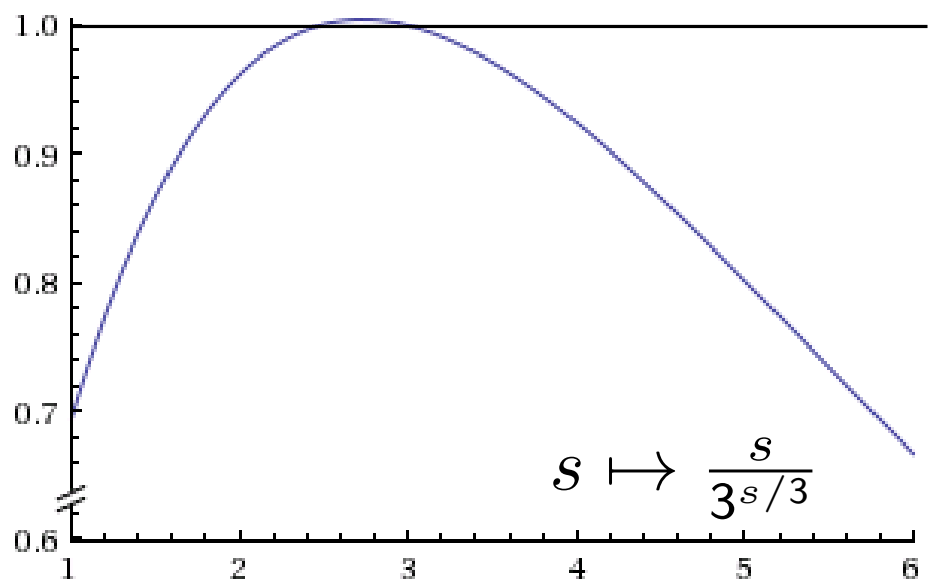
$$B(n) \leq s \cdot B(n-s) \leq s \cdot 3^{(n-s)/3} = \frac{s}{3^{s/3}} \cdot 3^{n/3} \overset{?}{\leq} 3^{n/3}$$



Runtime Analysis (cont)

$B(n) \leq (\deg(v) + 1) \cdot B(\ n - (\deg(v) + 1)\ )$

We proceed by induction to show $B(n) \leq 3^{n/3}$

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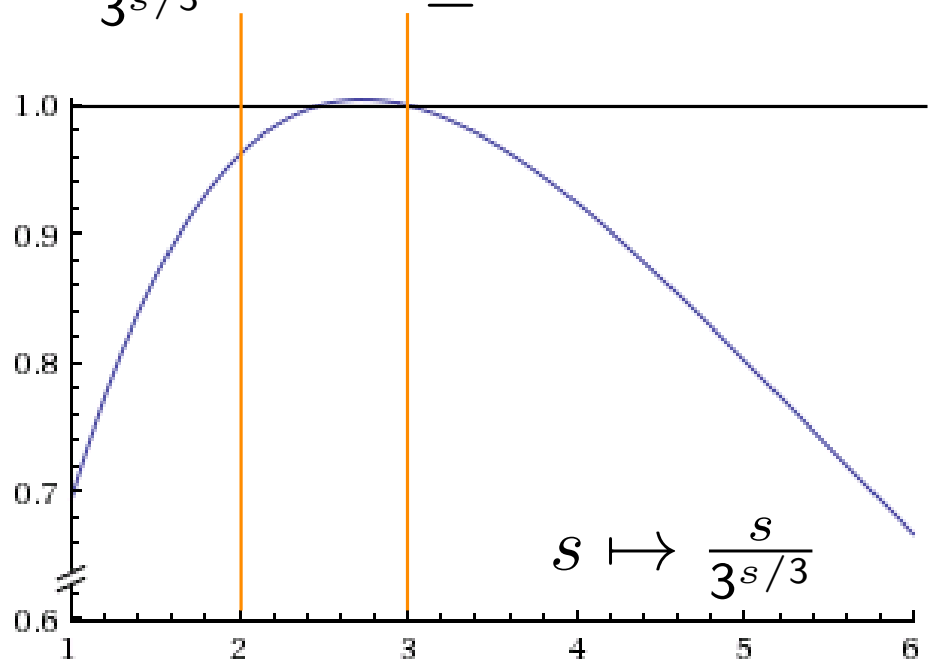

Runtime Analysis (cont)

$B(n) \leq (\deg(v) + 1) \cdot B(\ n - (\deg(v) + 1)\ )$

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$$B(n) \in O^*(\sqrt[3]{3}^n) \subset O^*(1.44225^n)$$