Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Computational Geometry

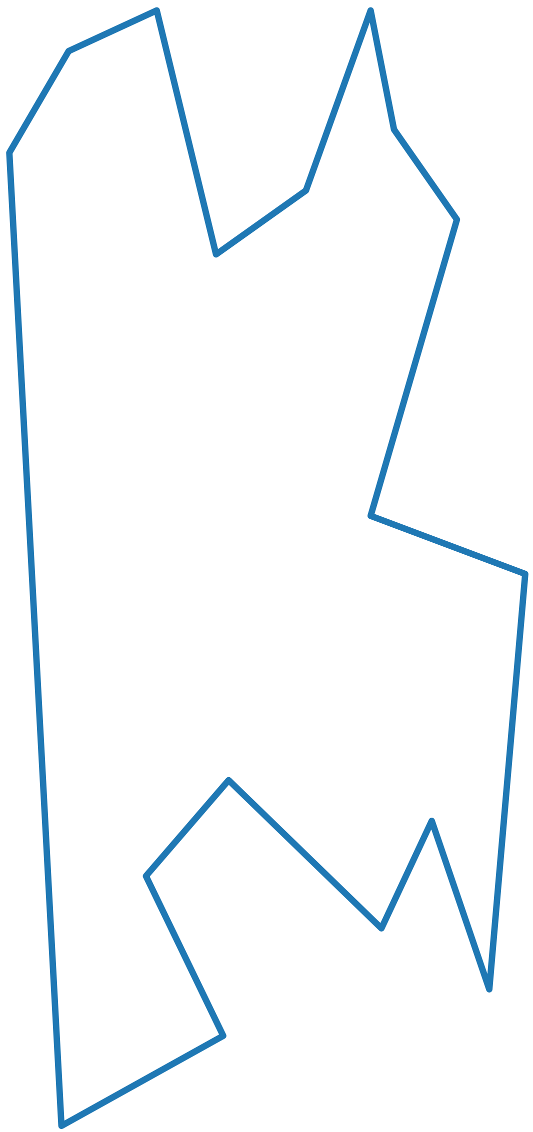
Seidel's Triangulation Algorithm

Lecture #13

Dr. Philipp Kindermann

Winter Semester 2018/19

Triangulating a Polygon



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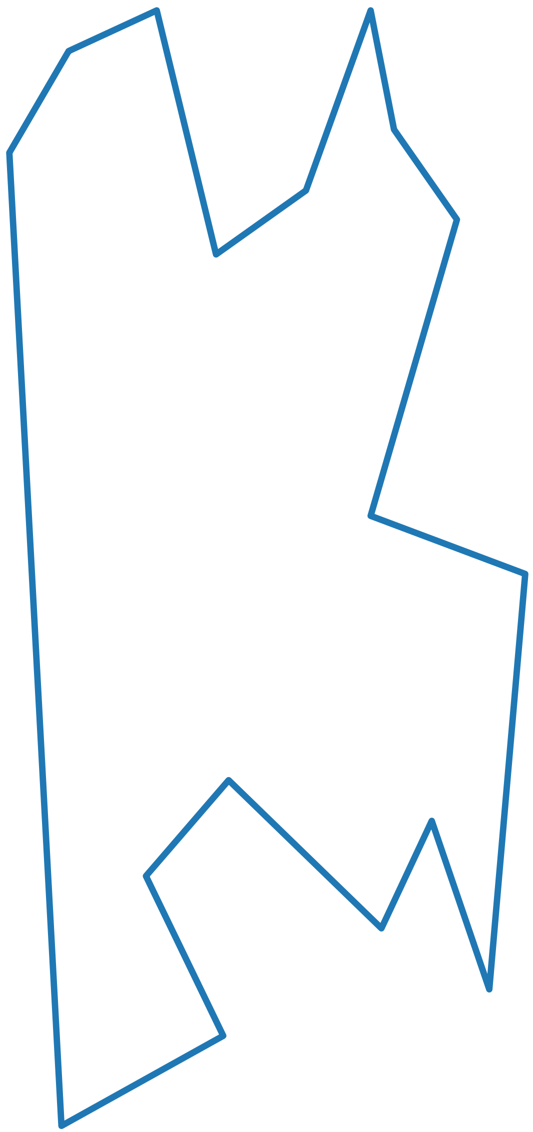

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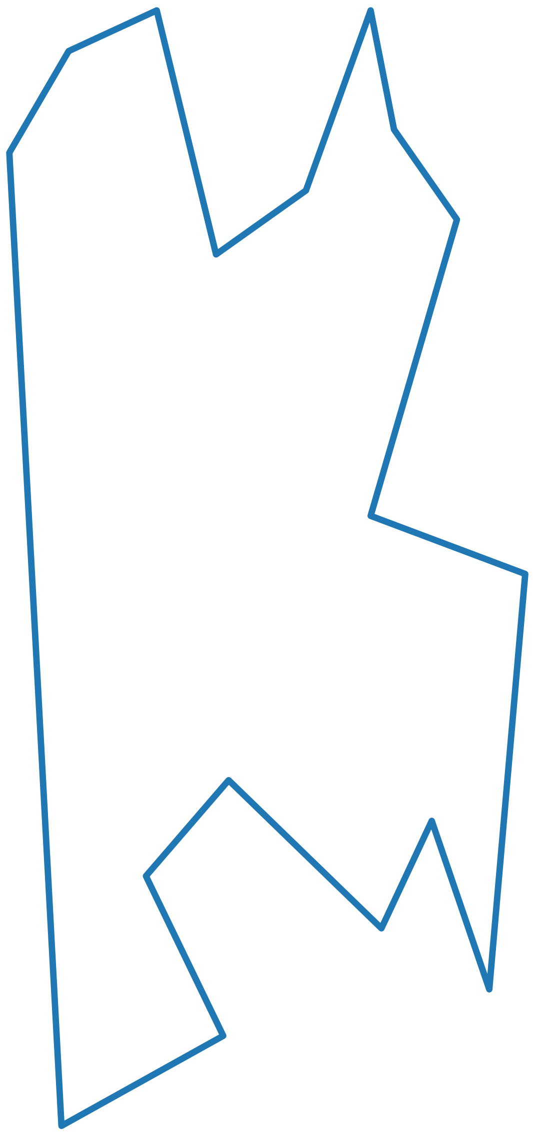

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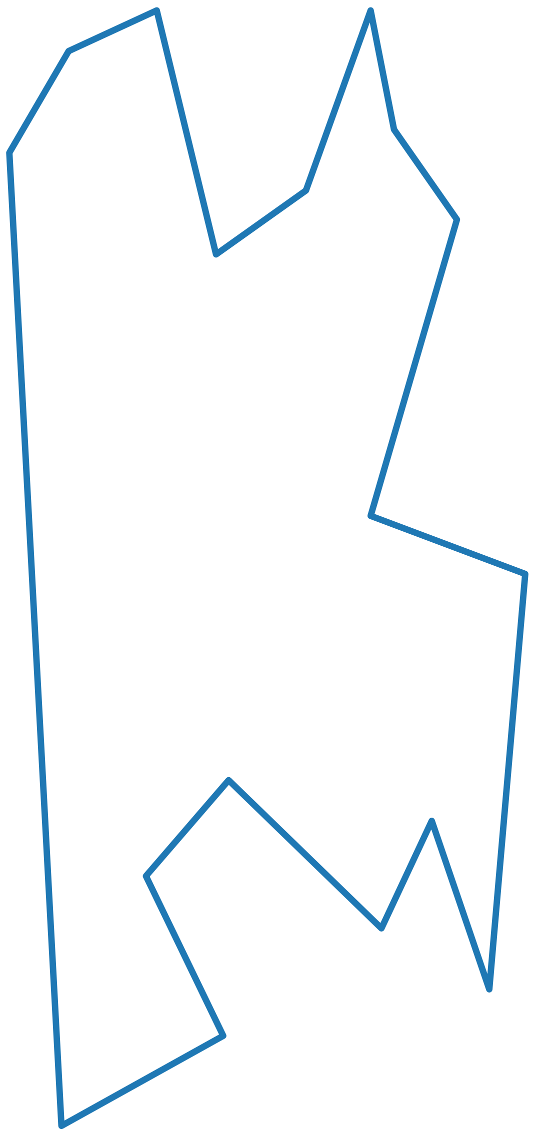

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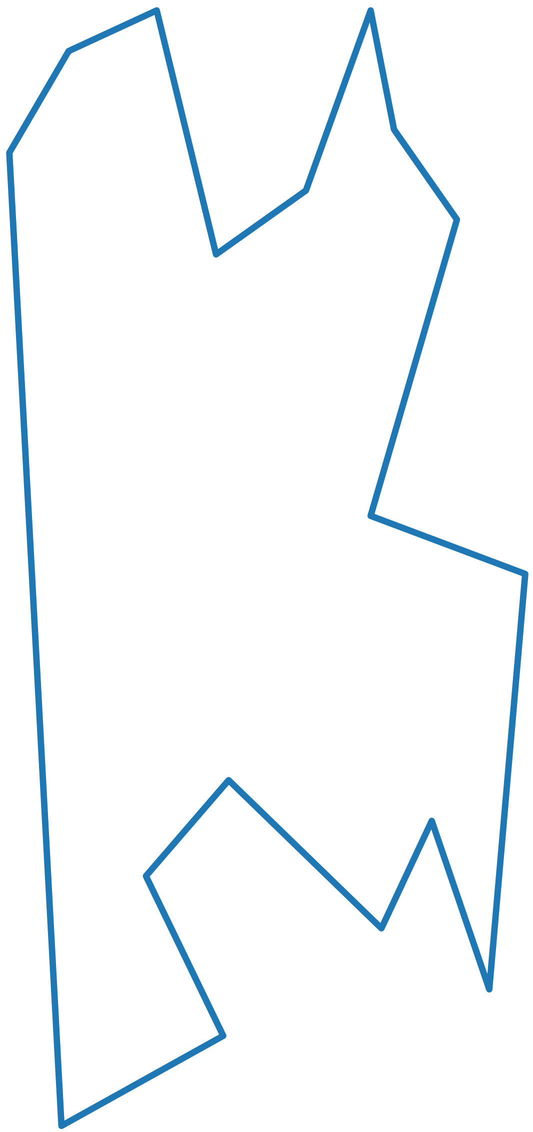

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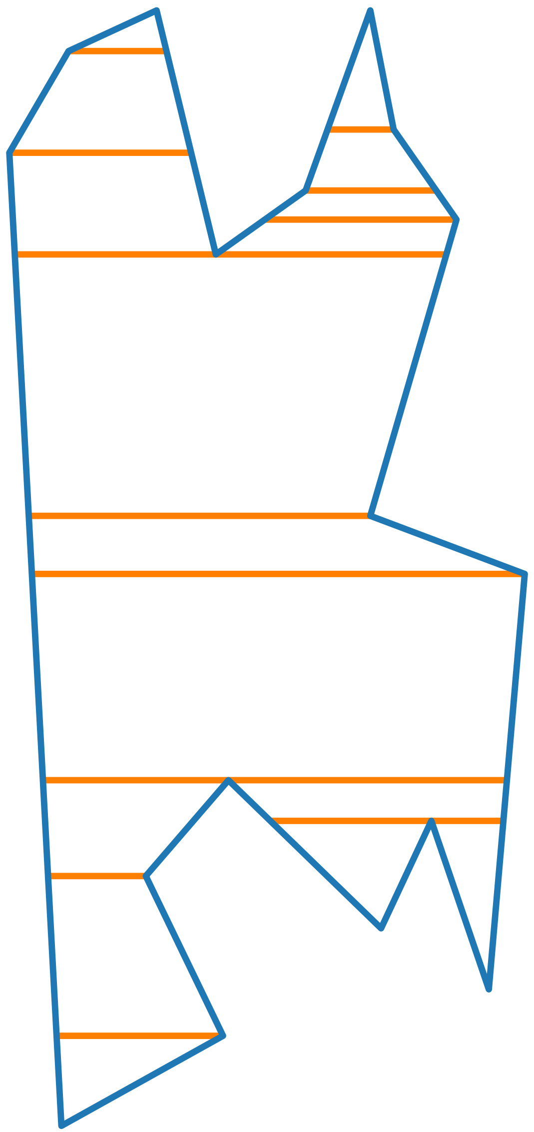

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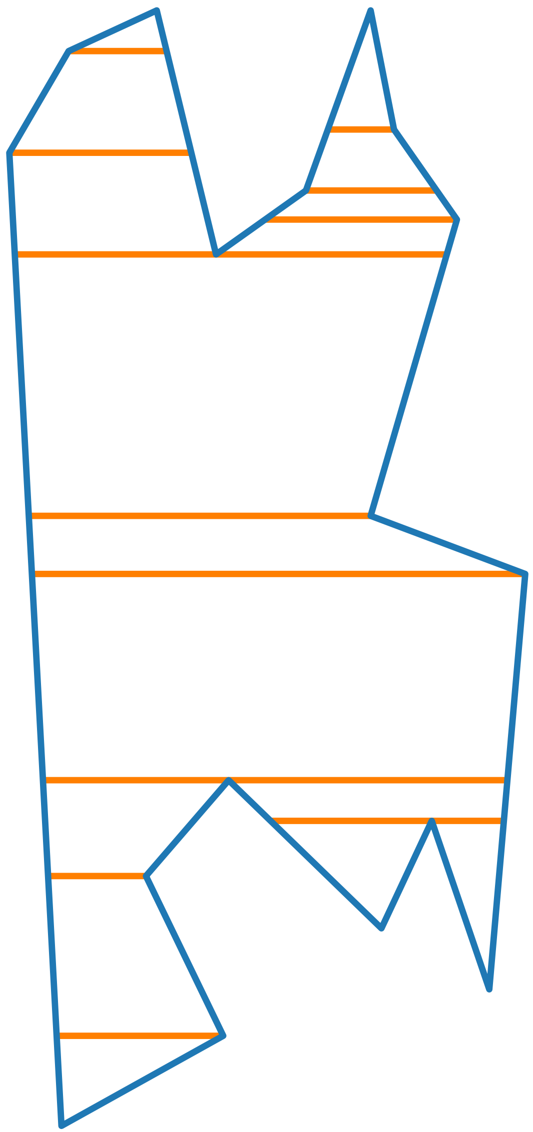

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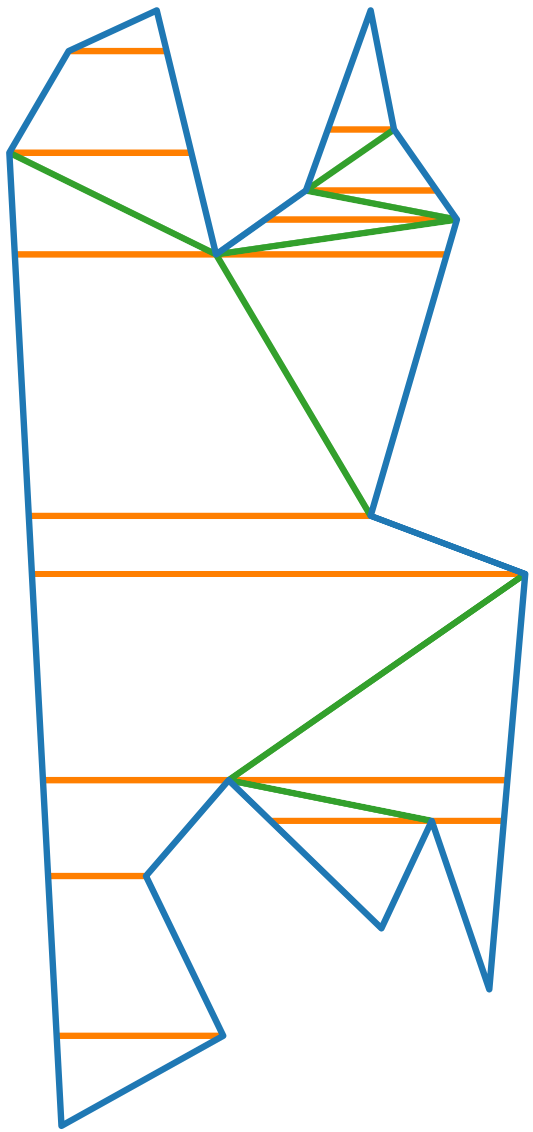

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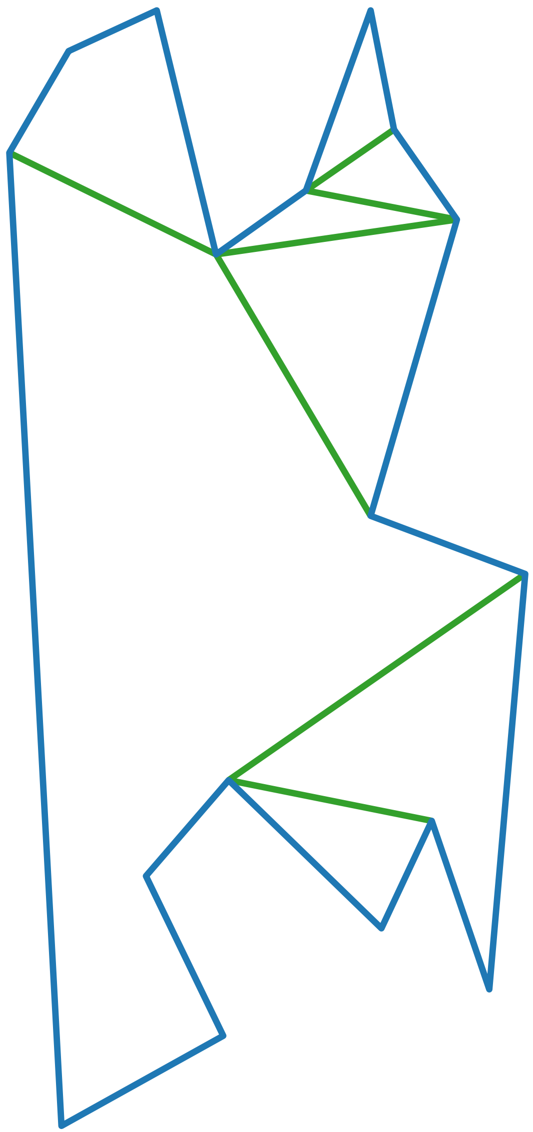

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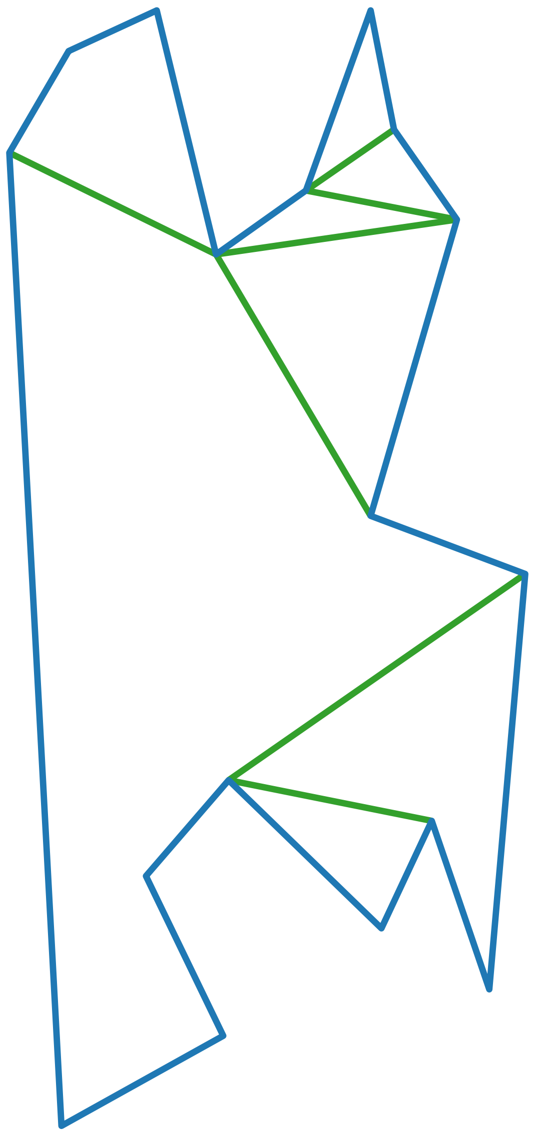

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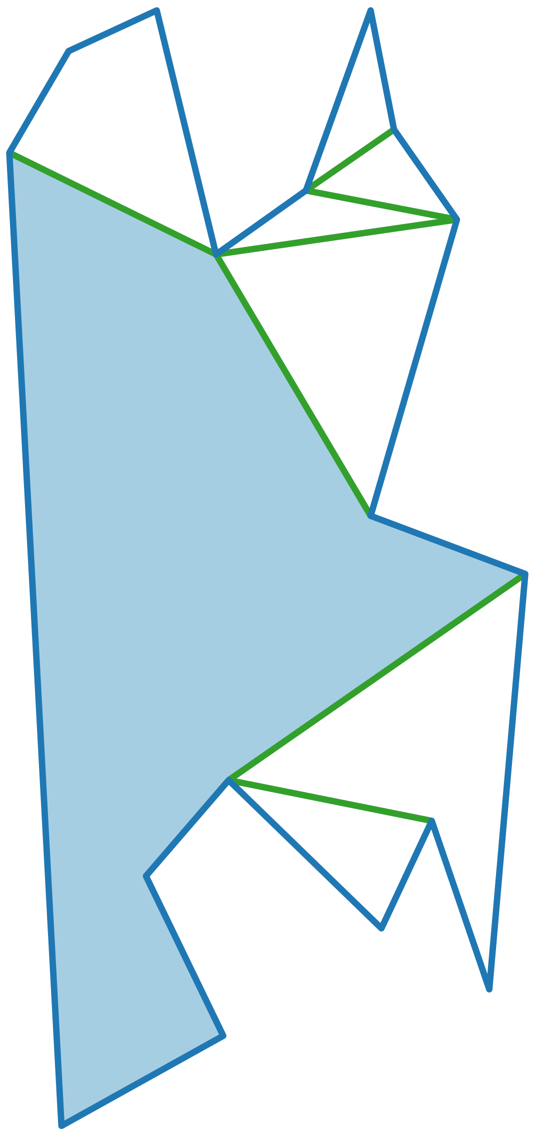

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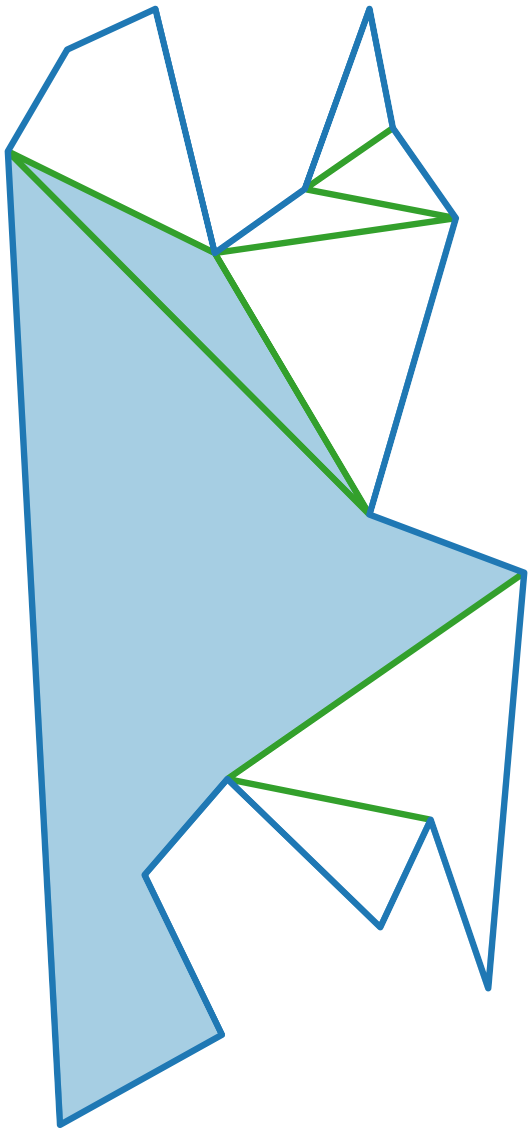

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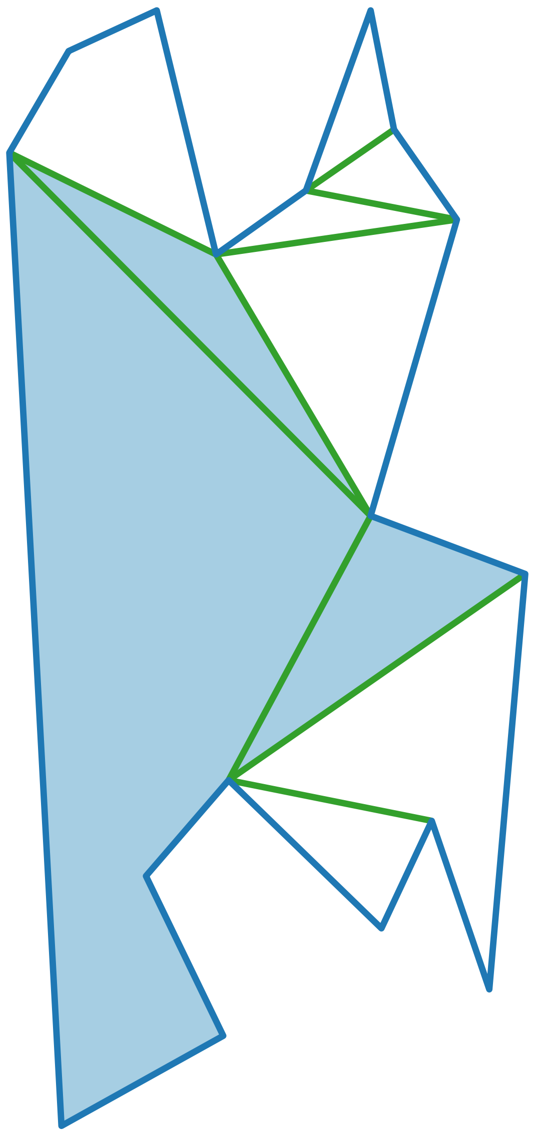

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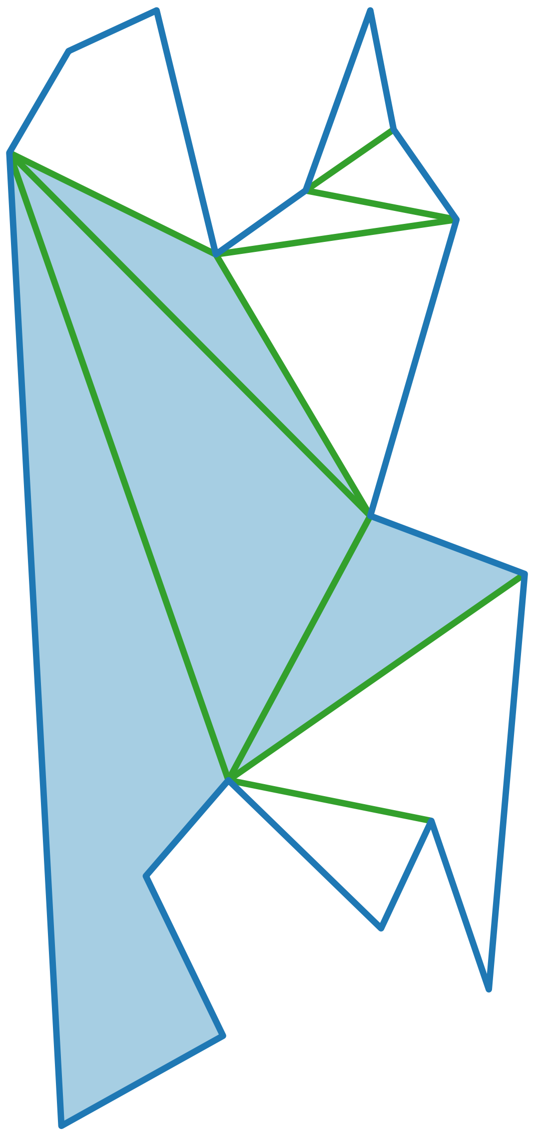

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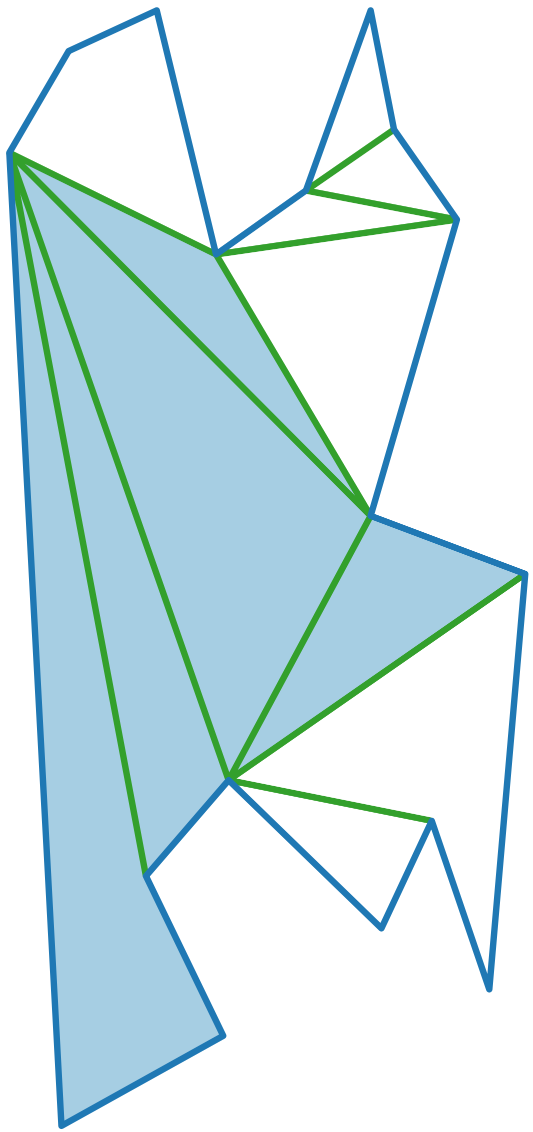

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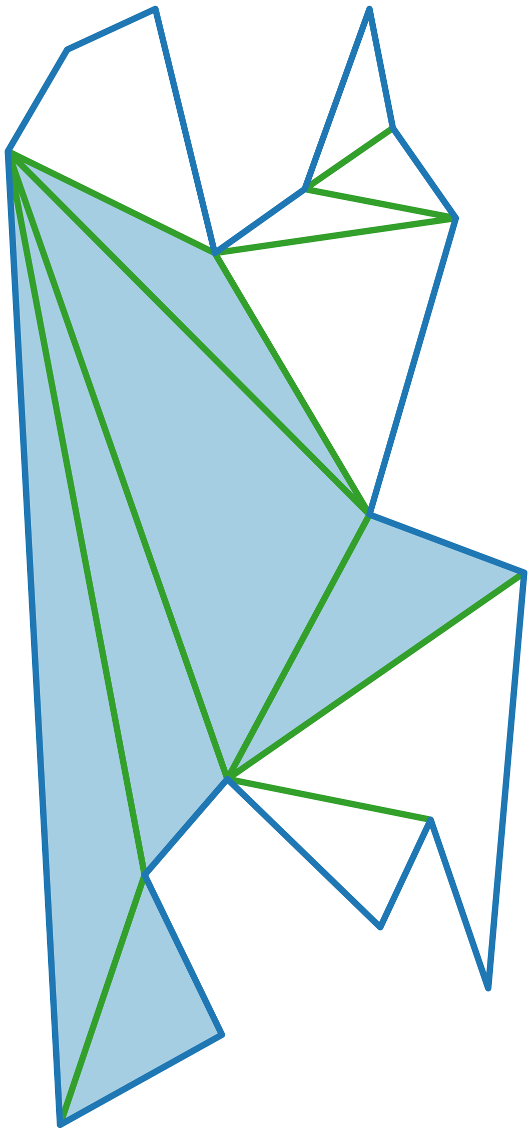

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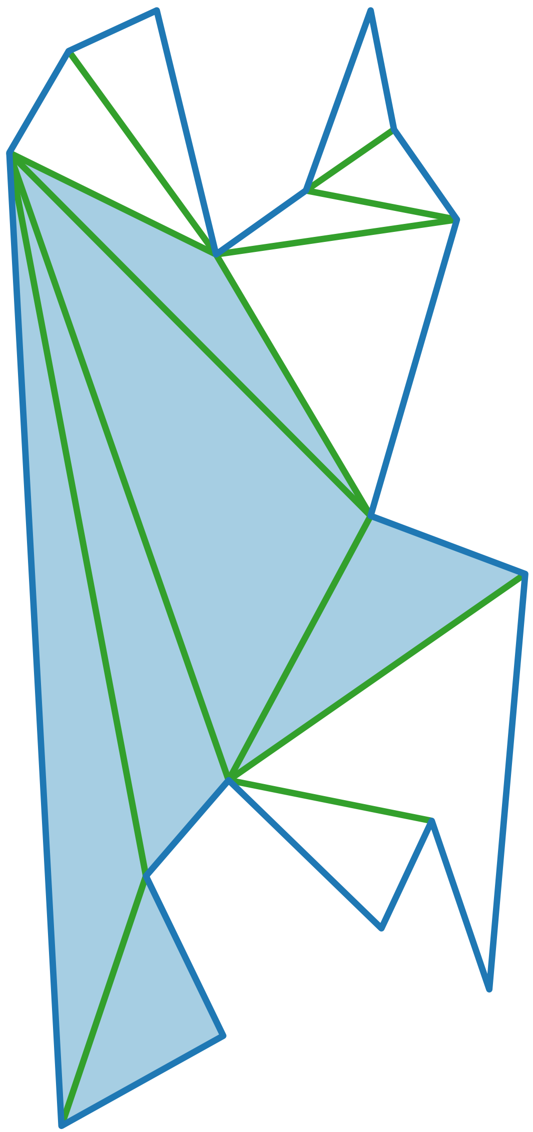

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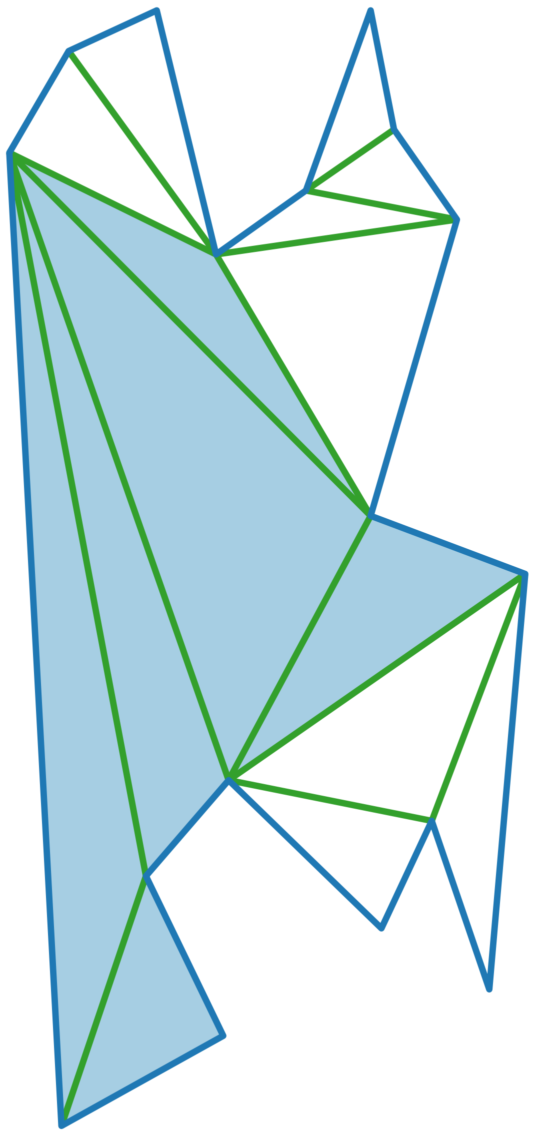

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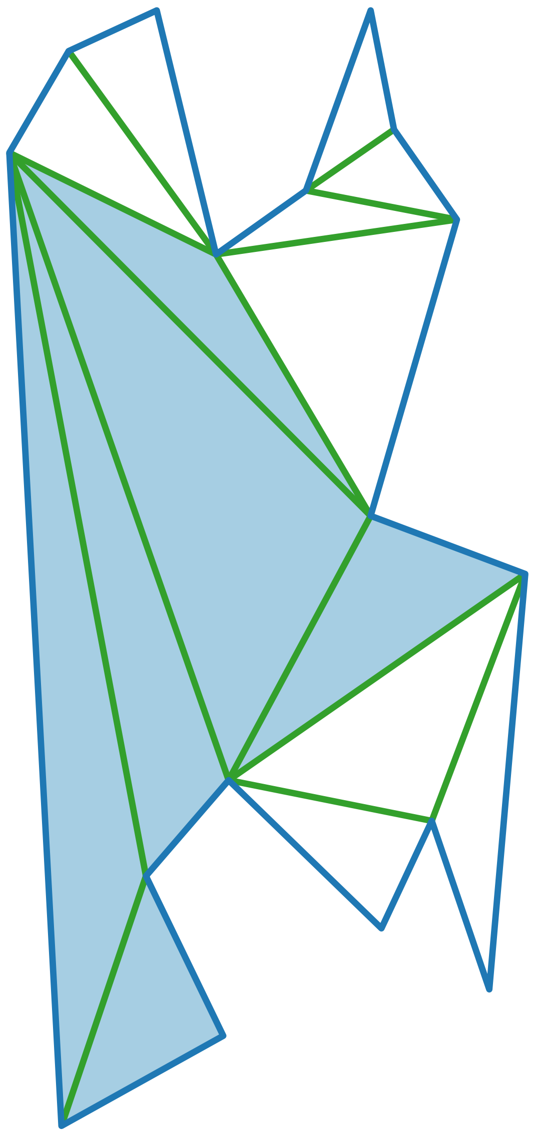

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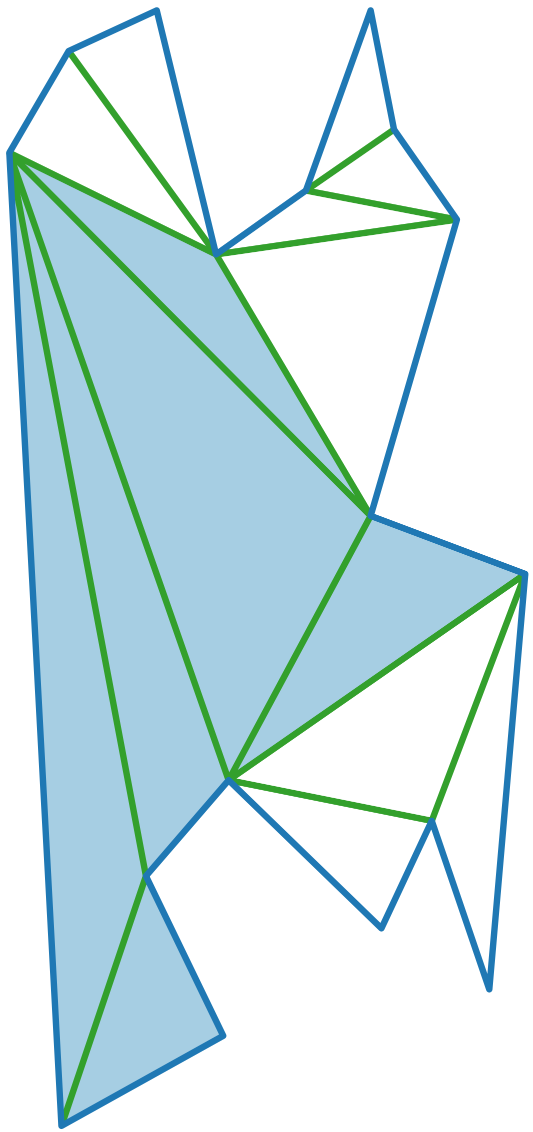

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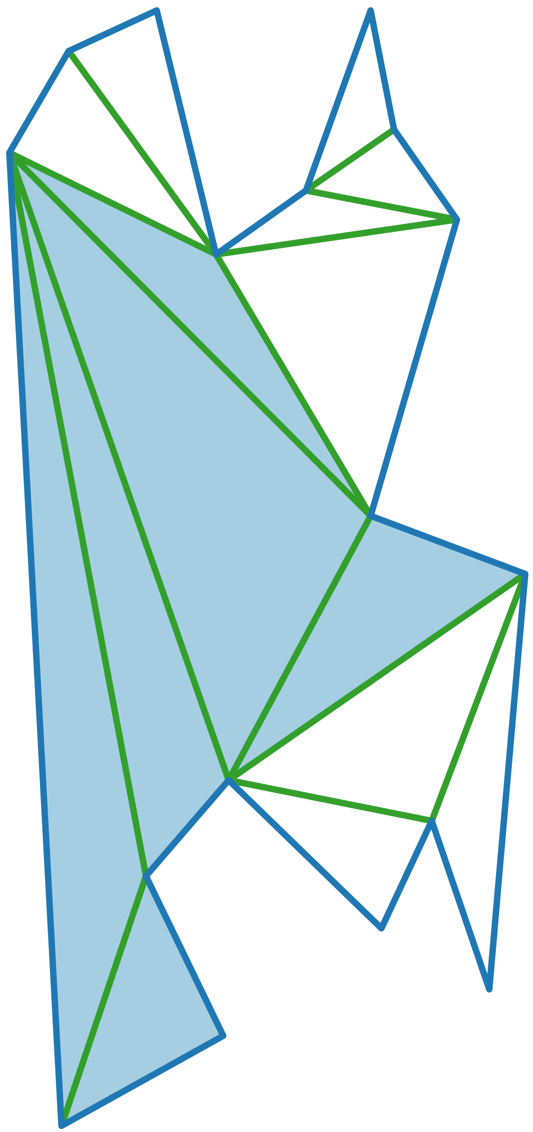

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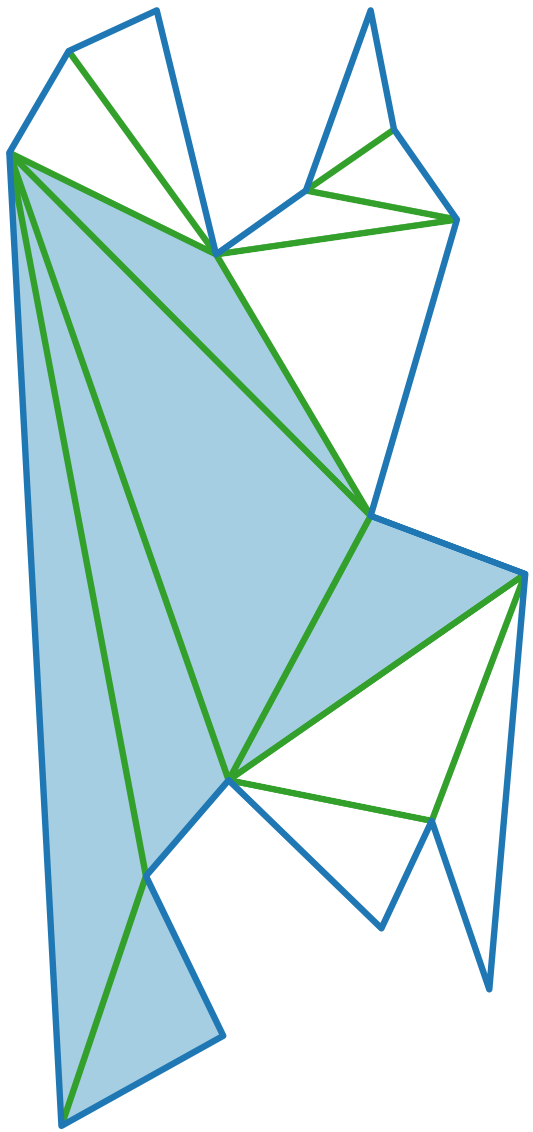

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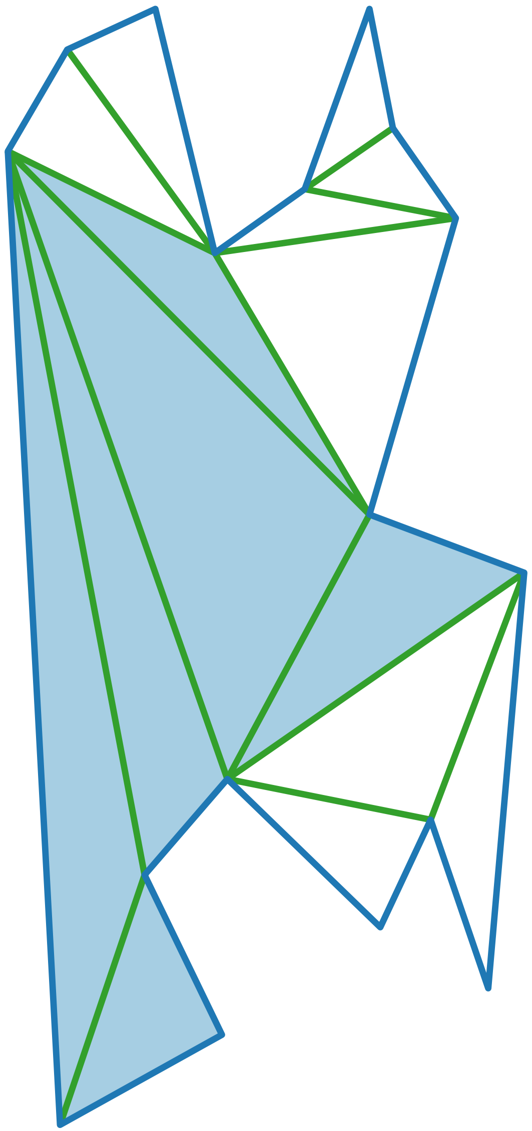

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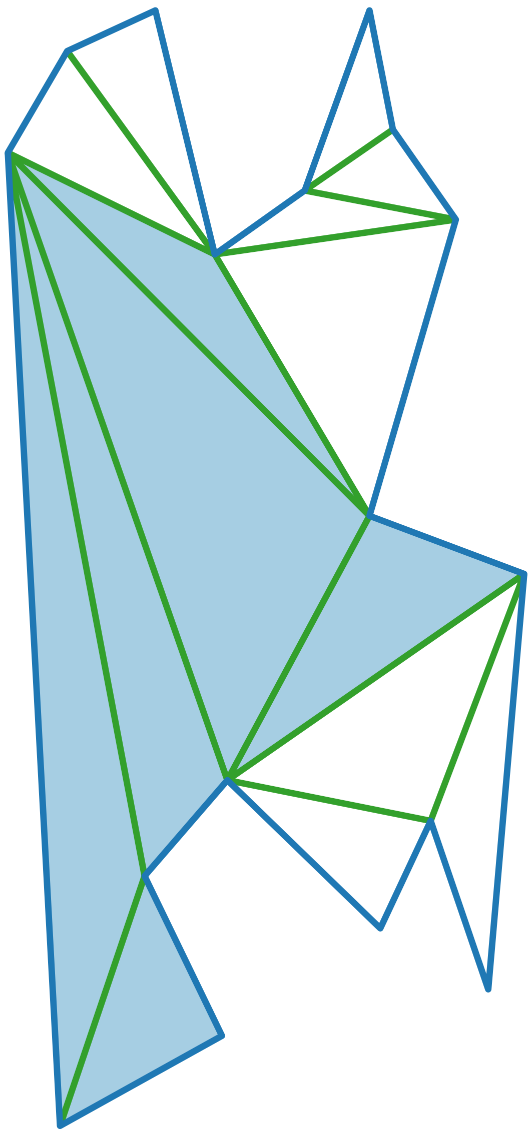

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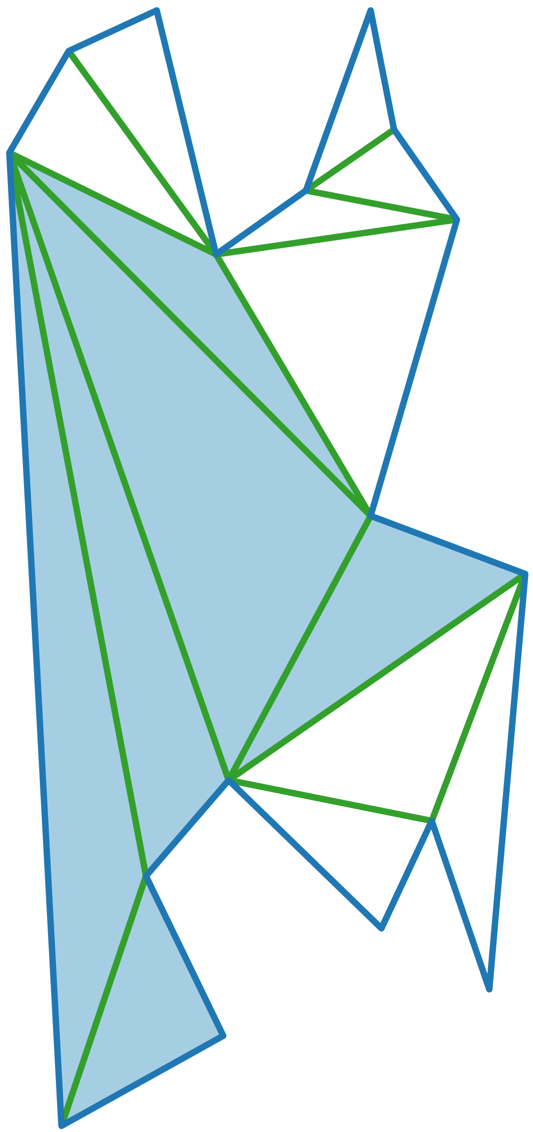

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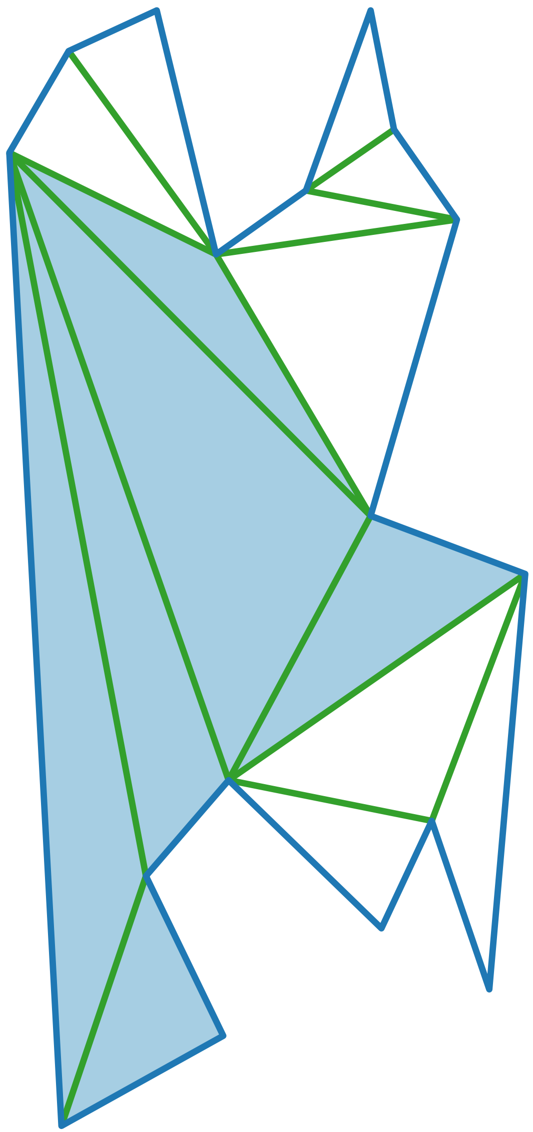

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Lemma 1. Given a trapezoidation, P can be triangulated in linear time.

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Let S be a set of n non-crossing segments

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$\langle s_1, s_2, \ldots, s_n \rangle \leftarrow$ random ordering of S

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Total cost of one step: – location time
– "threading" (updating) time

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(no two points have the same y-coordinate).

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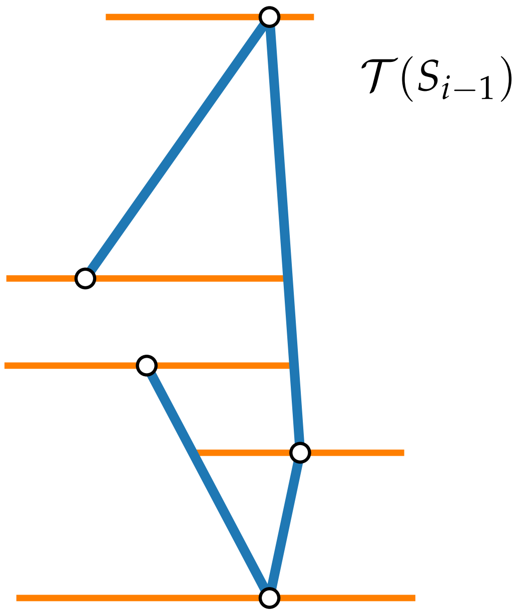

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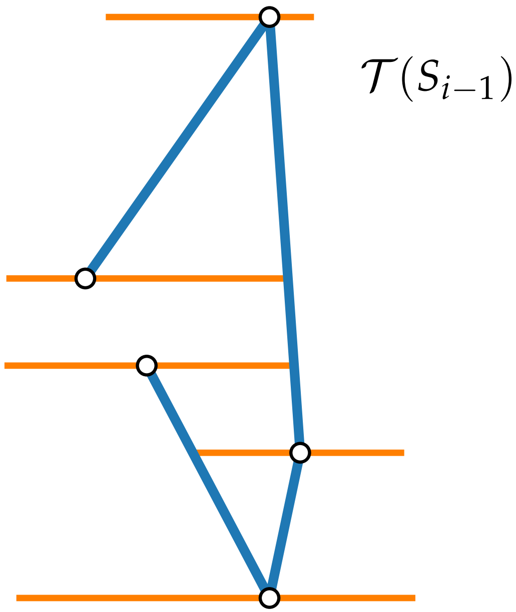

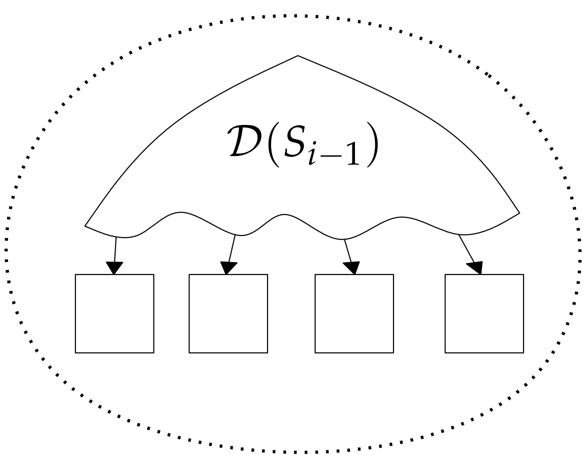

Threading time

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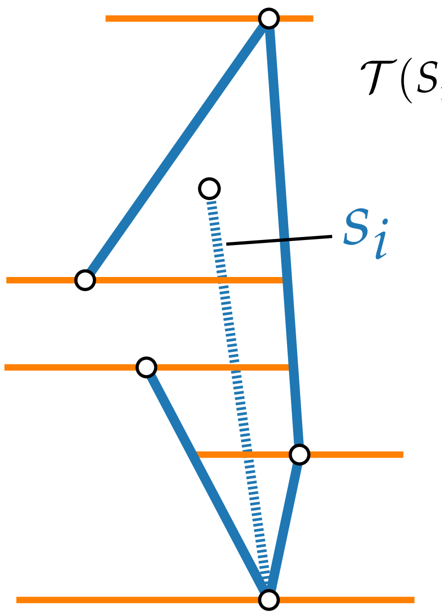

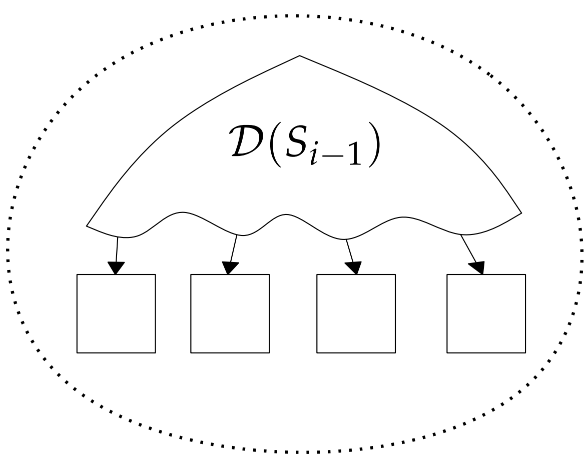

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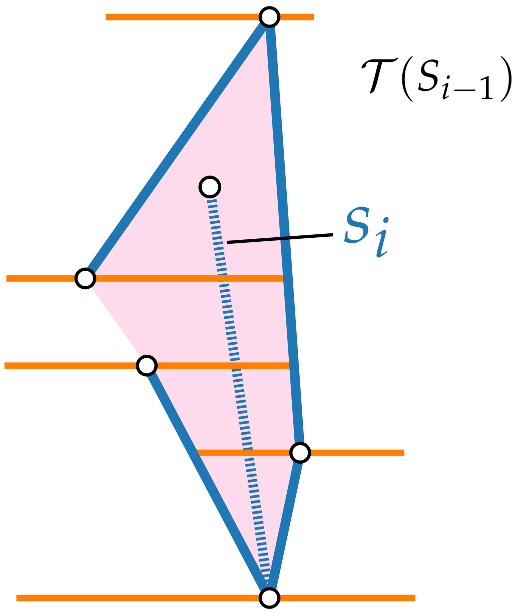

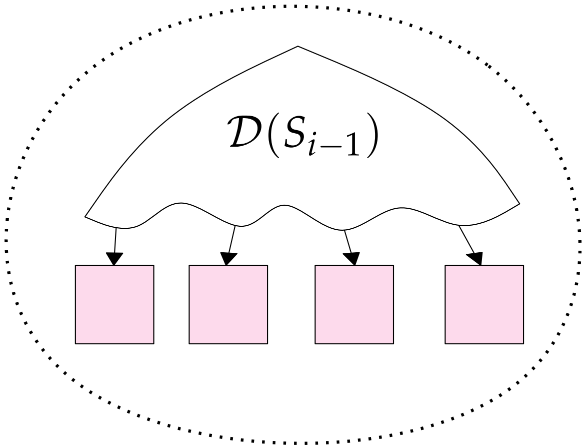

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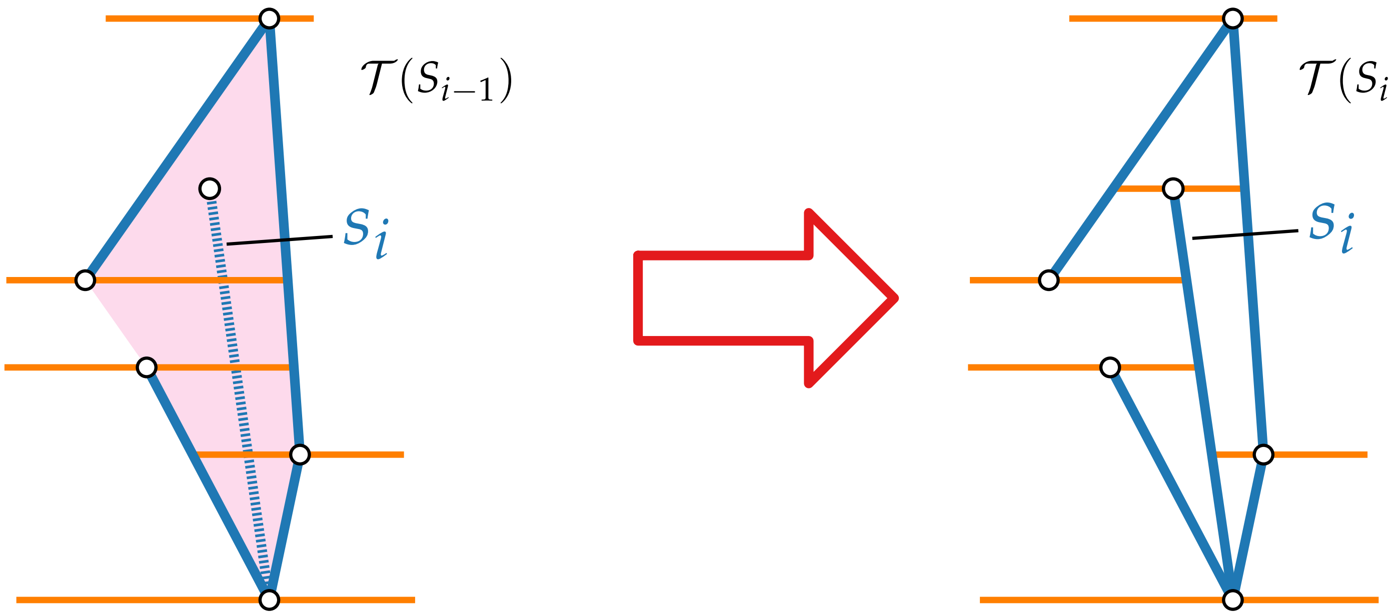

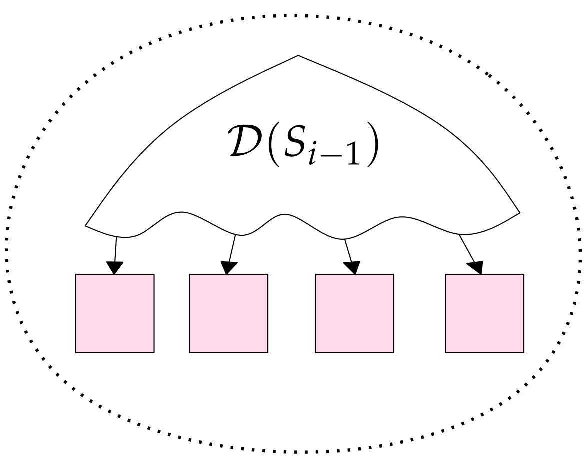

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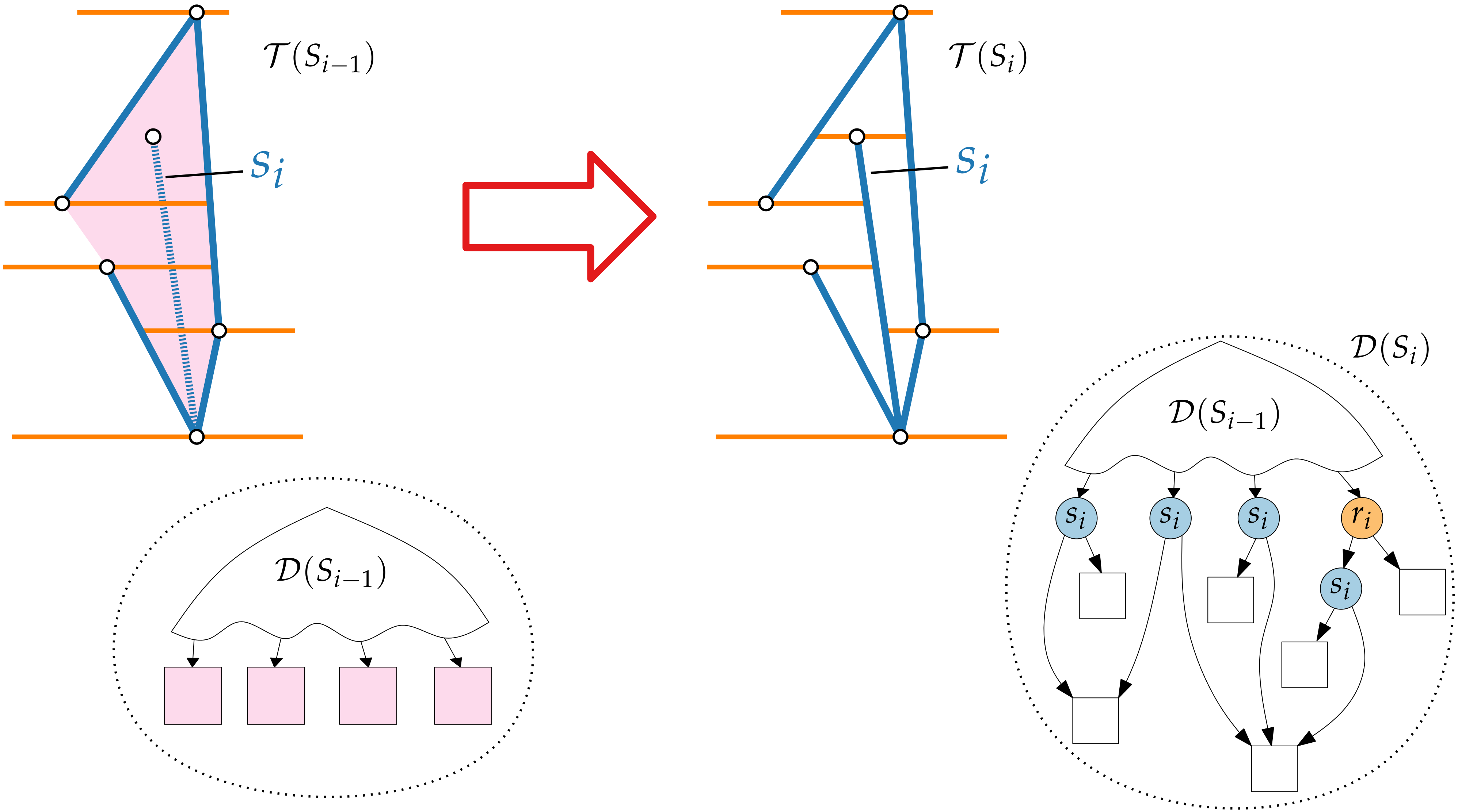

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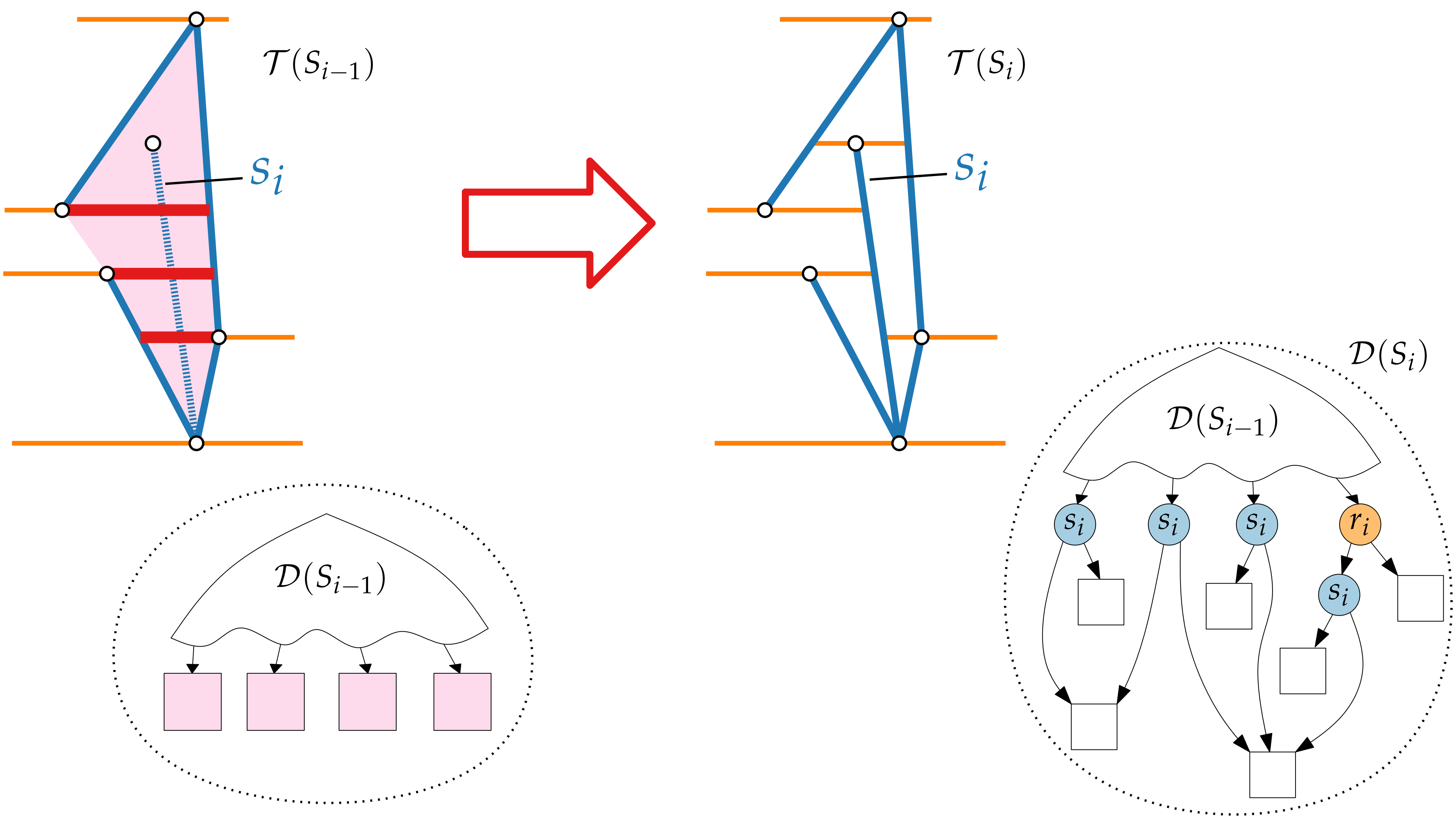

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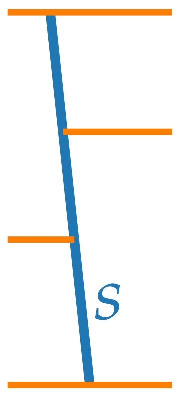

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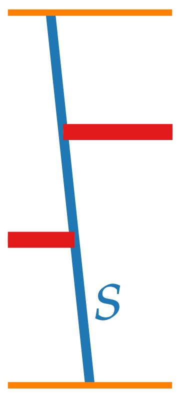

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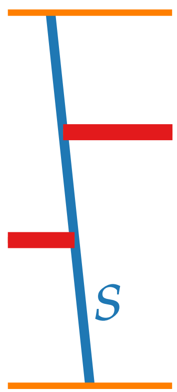

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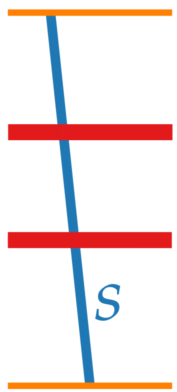

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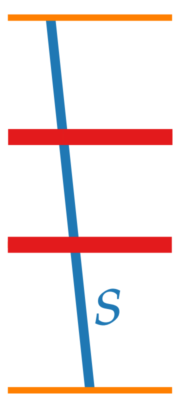

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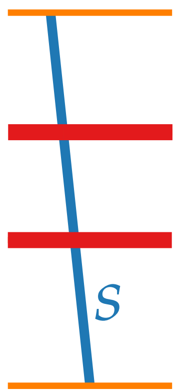

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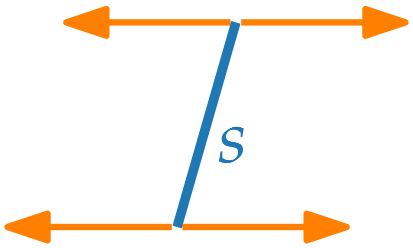

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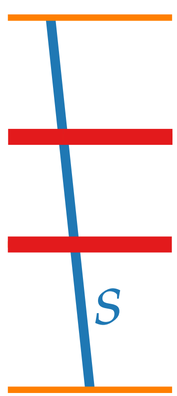

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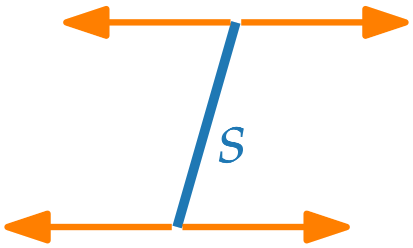

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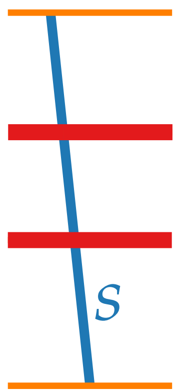

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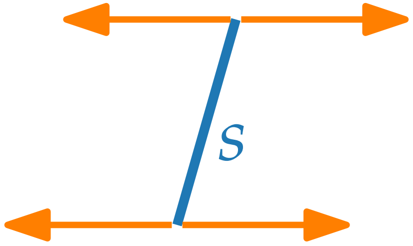

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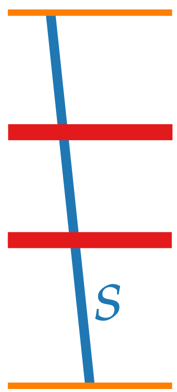

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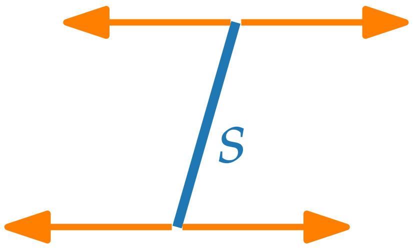

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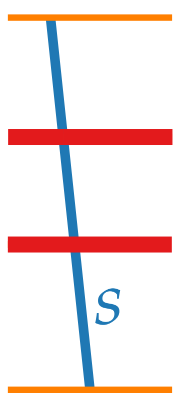

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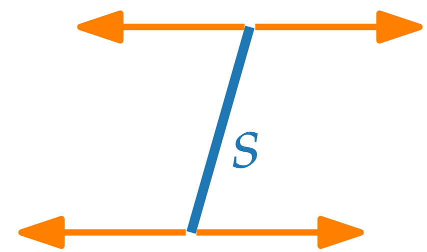

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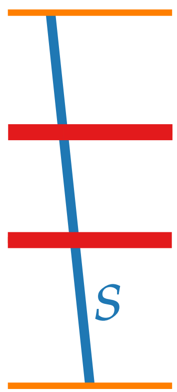

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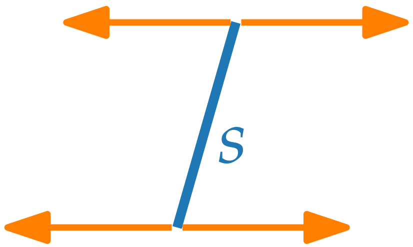

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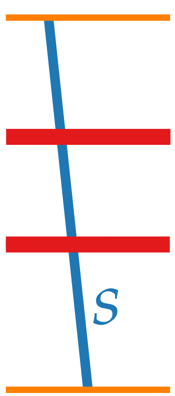

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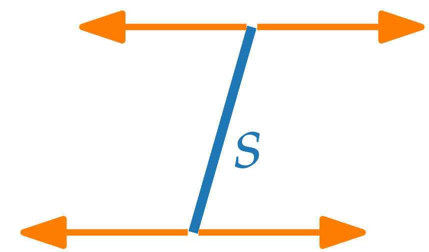

Ordering of S_i random $\Rightarrow E[\deg(s_i, \mathcal{T}(S_i))] \leq 4$ $\square$

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Recall: $$H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} \in \Theta(\qquad)$$

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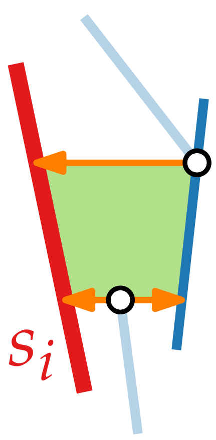

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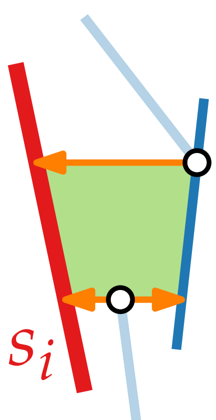

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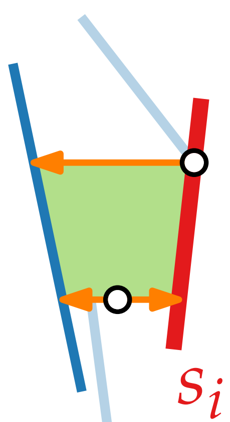

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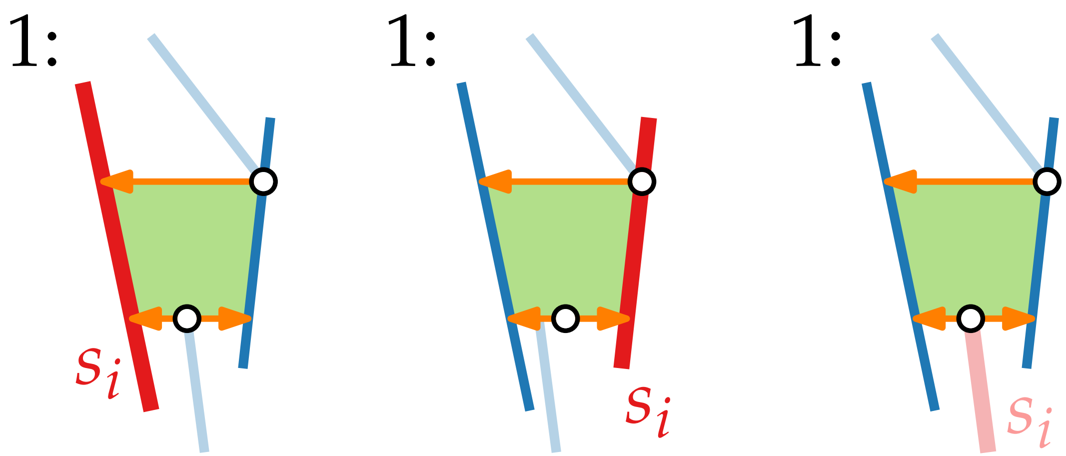

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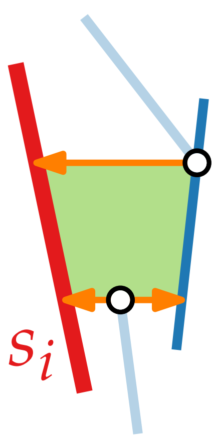
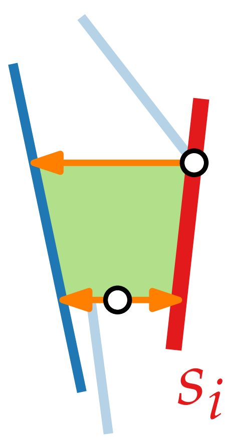
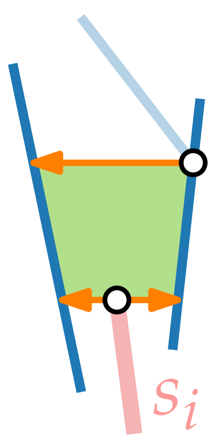
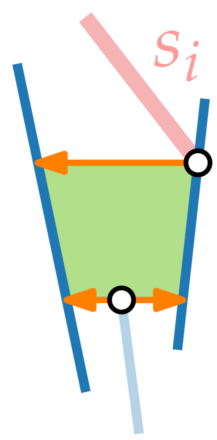
1: 

1: 

1: 

1: 

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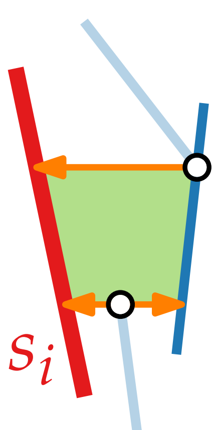

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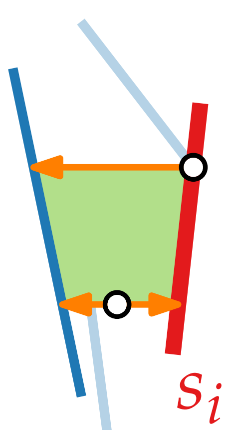

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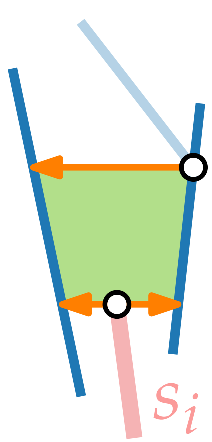

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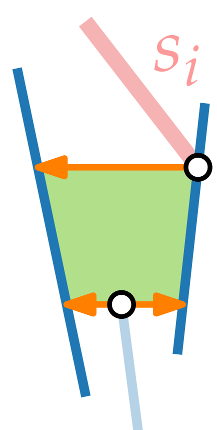

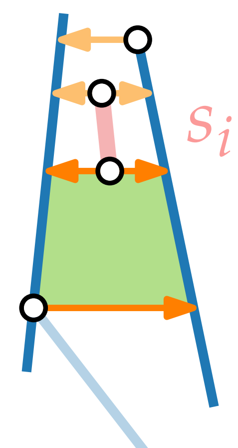

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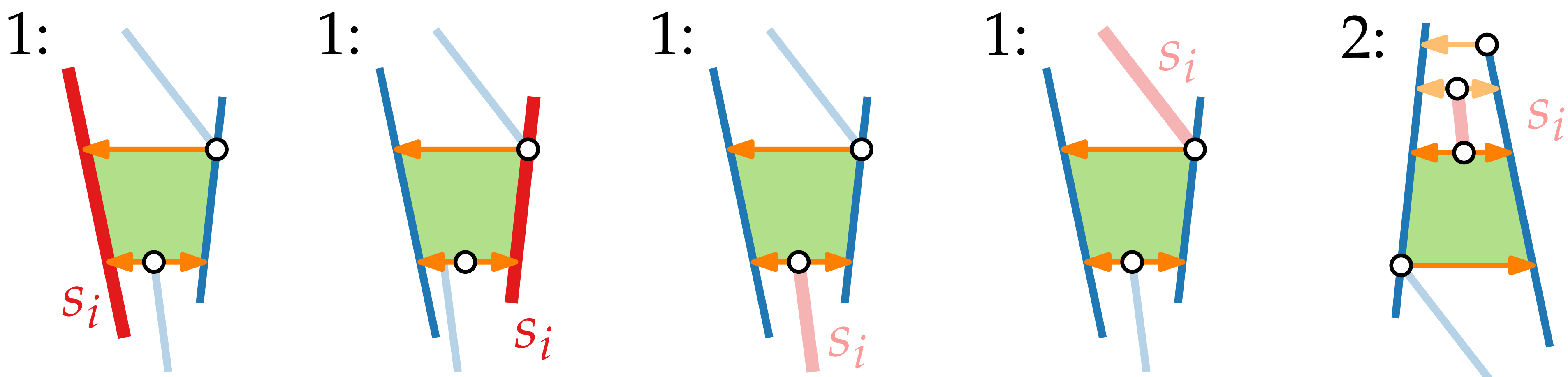

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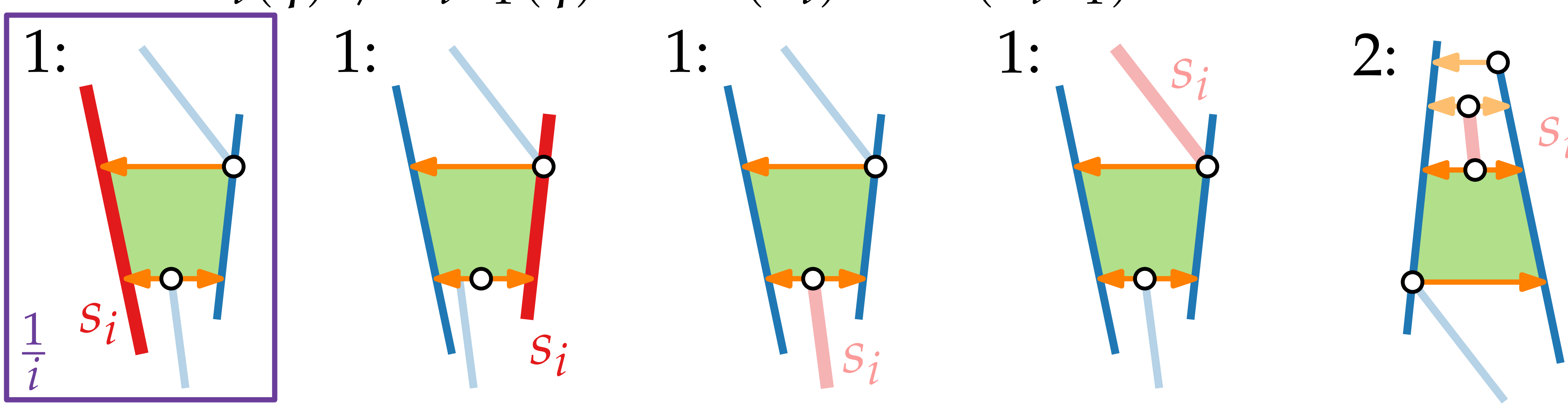

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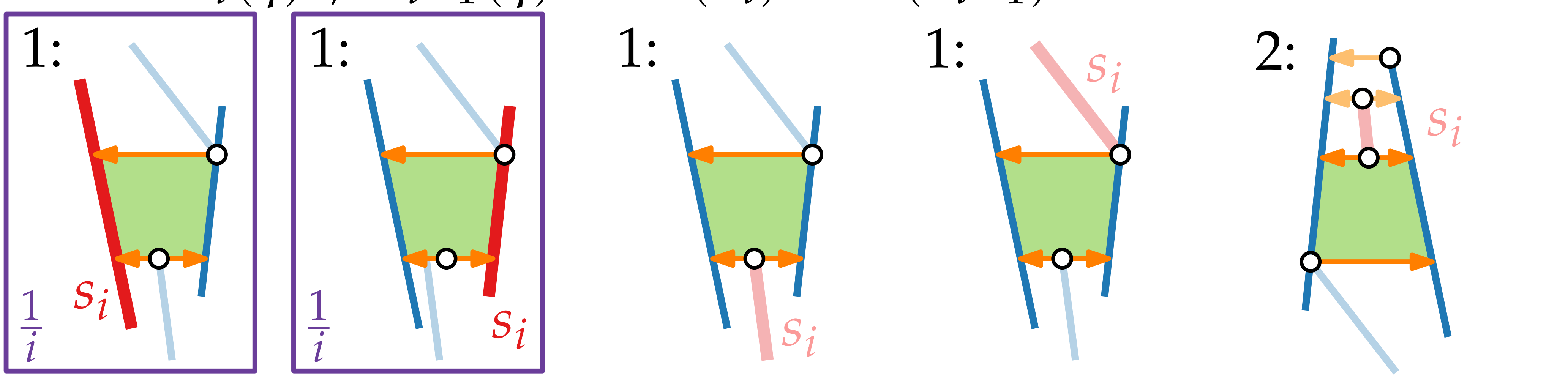

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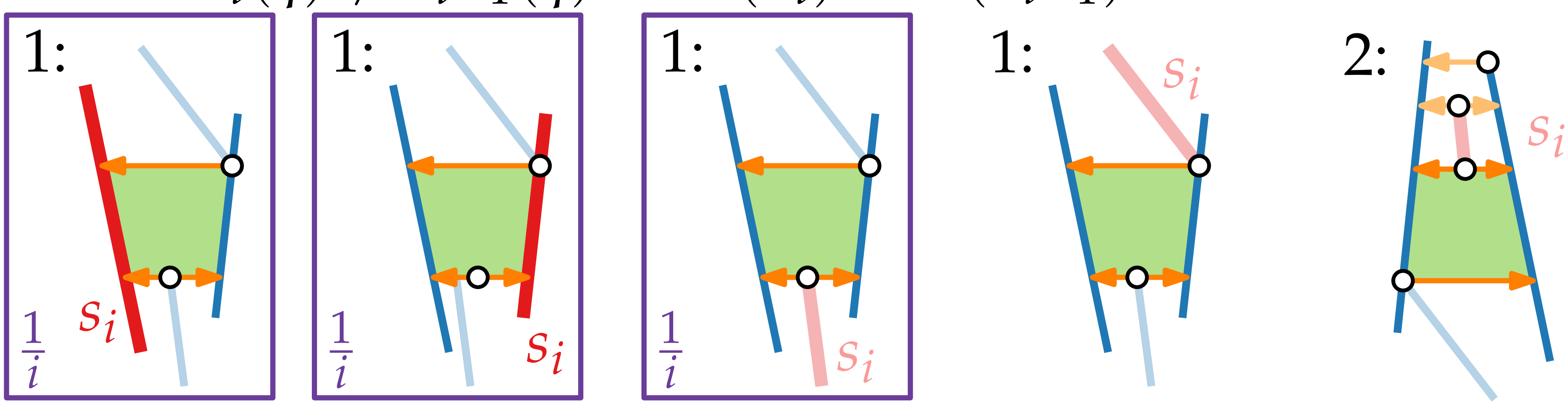

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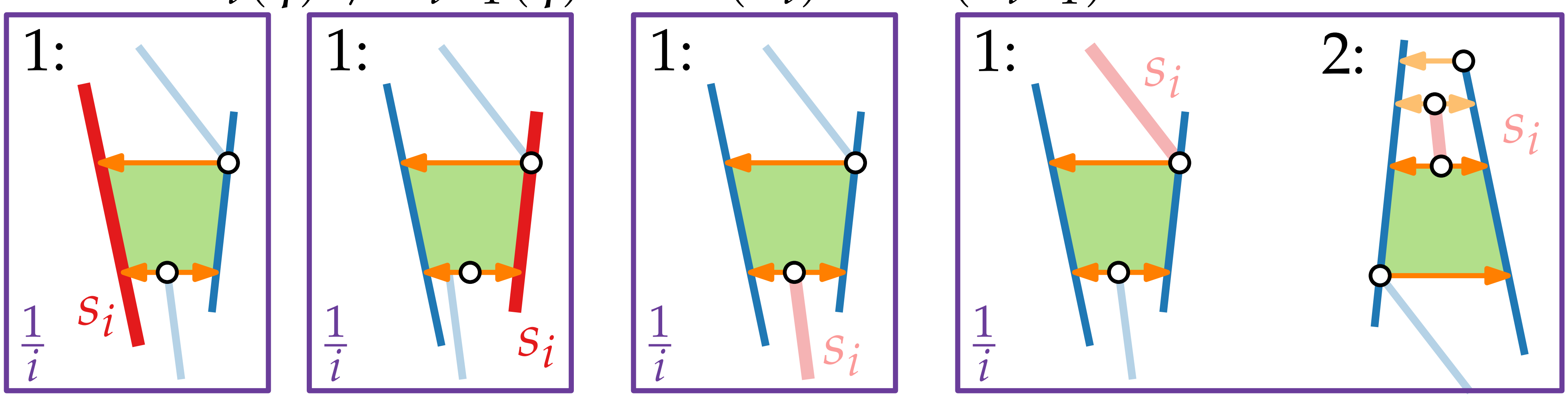

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Aim: Speed-up construction for simple polygons.

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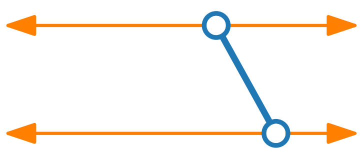

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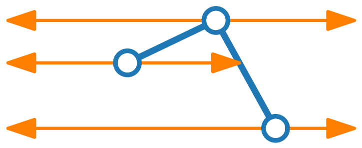

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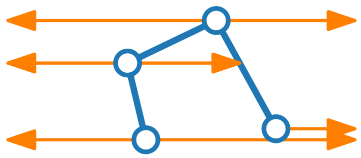

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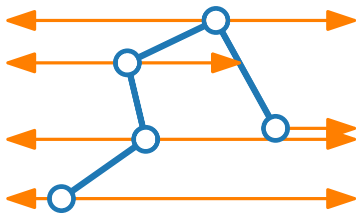

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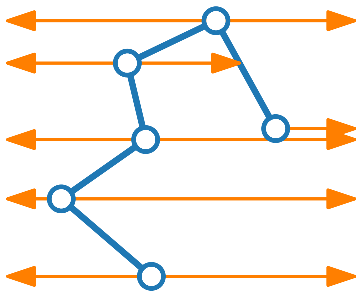

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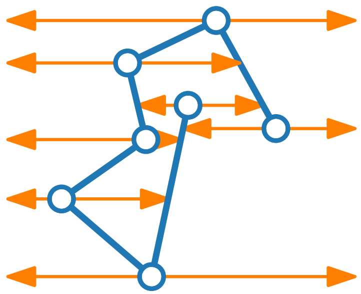

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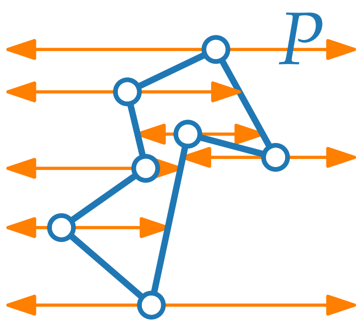

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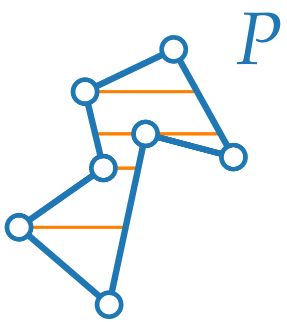

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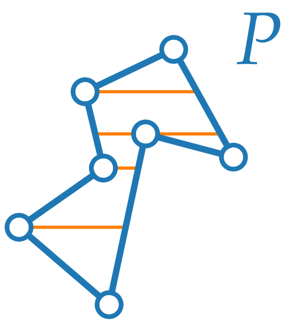

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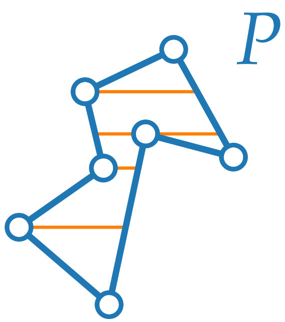

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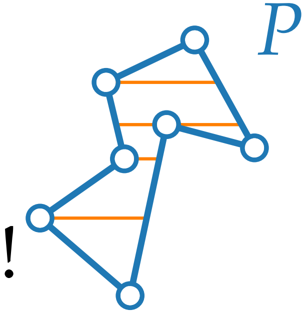

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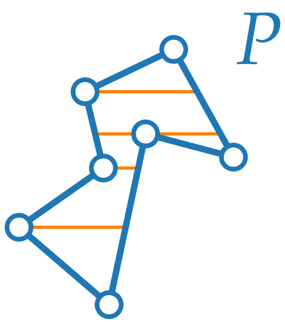

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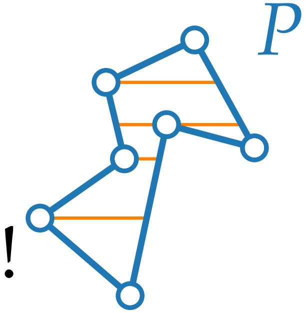

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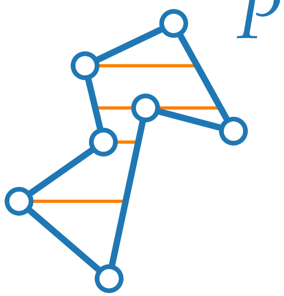

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Logs All Over the Place

Input:

$2^{2^{2^{2}}}$

Open code

Result:

2 003 529 930 406 846 464 979 072 351 560 255 750 447 825 475 569 751 419 265 016 ⋱
973 710 894 059 556 311 453 089 506 130 880 933 348 101 038 234 342 907 263 ⋱
181 822 949 382 118 812 668 869 506 364 761 547 029 165 041 871 916 351 587 ⋱
966 347 219 442 930 927 982 084 309 104 855 990 570 159 318 959 639 524 863 ⋱
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208 500 206 671 563 702 366 126 359 747 144 807 111 774 815 880 914 135 742 ⋱
720 967 190 151 836 282 560 618 091 458 852 699 826 141 425 030 123 391...

More digits

Decimal approximation:

$2.0035299304068464649790723515602557504478254755697 5\ldots \times 10^{19728}$

More digits

Number length:

19 729 decimal digits

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$2^{2^{2^{2^2}}}$

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$2.0035299304068464649790723515602557504478254755697\ldots \times 10^{19728}$

Number length:

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Input interpretation:

estimated number of atoms in the universe

Result:

1×10^{80} atoms

$$\text{x}\{i \mid \log^{(i)} n \geq 1\}.$$

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Logs All Over the Place

Input:

$2^{2^{2^{2^2}}}$

Open code

Result: More digits

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Result:

2×10^{19648} per atom

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The Algorithm

PolygonTrapezoidation ((edges along) simple polygon P)

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    insert $s_i = v_i w_i$ in $\mathcal{T}_{i-1}$ using $\pi(v_i)$ (node in $\mathcal{Q}_{N(h-1)}$)

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  insert $s_i = v_i w_i$ in $\mathcal{T}_{i-1}$ using $\pi(v_i)$ (node in $\mathcal{Q}_{N(h-1)}$)

3.2 walk along P through $\mathcal{T}_{N(h)}$:

The Algorithm

PolygonTrapezoidation ((edges along) simple polygon P)

1. $\langle s_1, s_2, \ldots, s_n \rangle :=$ random ordering of the edges of P
2. Compute $\mathcal{T}_1$ and $\mathcal{Q}_1$ for $\{s_1\}$.

   foreach $v \in P$ **do** $\pi(v) \leftarrow$ pointer to the leaf of $\mathcal{Q}_1$ that contains v.

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4. **for** $i = N(\log^\star n) + 1$ **to** n **do**
  insert $s_i = v_i w_i$ in $\mathcal{T}_{i-1}$ using $\pi(v_i)$ (node in $\mathcal{Q}_{N(\log^\star n)}$)
  return $(\mathcal{T}_n, \mathcal{Q}_n)$

Time Complexity

Step 1: Random permutation

Time Complexity

Step 1: Random permutation $O(n)$

Time Complexity

Step 1: Random permutation $O(n)$

Step 2: Setting up $\mathcal{T}_1$, $\mathcal{Q}_1$, and $\pi(v)$

Time Complexity

Step 1: Random permutation $O(n)$

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Time Complexity

Step 1: Random permutation $O(n)$

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Step 3: Phases 1 to $\log^\star n$

Time Complexity

Step 1: Random permutation $O(n)$

Step 2: Setting up $\mathcal{T}_1$, $\mathcal{Q}_1$, and $\pi(v)$ $O(n)$

Step 3: Phases 1 to $\log^\star n$ $(\log^\star n) \cdot$

Time Complexity

Step 1: Random permutation $O(n)$

Step 2: Setting up $\mathcal{T}_1$, $\mathcal{Q}_1$, and $\pi(v)$ $O(n)$

Step 3: Phases 1 to $\log^\star n$ $(\log^\star n)\,\cdot$

Step 3.2: Walking the polygon

Time Complexity

Step 1: Random permutation $O(n)$

Step 2: Setting up $\mathcal{T}_1$, $\mathcal{Q}_1$, and $\pi(v)$ $O(n)$

Step 3: Phases 1 to $\log^\star n$ $(\log^\star n) \cdot$

Step 3.2: Walking the polygon Lemma 5 $\Rightarrow$

Lemma 5. S as before, $R \subseteq S$ random subset, $r := |R|$. Let I be the number of intersections between rays of $\mathcal{T}(R)$ and segments in $S \setminus R$. Then $E[I] \leq 4(n-r)$, where the expectation is over all size-r subsets of S.

Time Complexity

Step 1: Random permutation $O(n)$

Step 2: Setting up $\mathcal{T}_1$, $\mathcal{Q}_1$, and $\pi(v)$ $O(n)$

Step 3: Phases 1 to $\log^\star n$ $(\log^\star n) \cdot$

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Step 1: Random permutation $O(n)$

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Step 3: Phases 1 to $\log^\star n$ $(\log^\star n) \cdot$

Step 3.2: Walking the polygon Lemma 5 $\Rightarrow$ $O(n)$

Step 3.1: Inserting $s_i = v_i w_i$ using $\mathcal{Q}_{N(h-1)}$

Time Complexity

Step 1: Random permutation $O(n)$

Step 2: Setting up $\mathcal{T}_1$, $\mathcal{Q}_1$, and $\pi(v)$ $O(n)$

Step 3: Phases 1 to $\log^\star n$ $(\log^\star n) \cdot$

Step 3.2: Walking the polygon Lemma 5 $\Rightarrow$ $O(n)$

Step 3.1: Inserting $s_i = v_i w_i$ using $\mathcal{Q}_{N(h-1)}$

– threading cost:

– locating cost:

Time Complexity

Step 1: Random permutation $O(n)$

Step 2: Setting up $\mathcal{T}_1$, $\mathcal{Q}_1$, and $\pi(v)$ $O(n)$

Step 3: Phases 1 to $\log^\star n$ $(\log^\star n) \cdot$

Step 3.2: Walking the polygon Lemma 5 $\Rightarrow$ $O(n)$

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– threading cost:

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Lemma 2. For $i = 1, \ldots, n$, the expected number of rays of $\mathcal{T}(S_{i-1})$ that are intersected by s_i is at most 4.

Time Complexity

Step 1: Random permutation $O(n)$

Step 2: Setting up $\mathcal{T}_1$, $\mathcal{Q}_1$, and $\pi(v)$ $O(n)$

Step 3: Phases 1 to $\log^\star n$ $(\log^\star n) \cdot$

Step 3.2: Walking the polygon Lemma 5 $\Rightarrow$ $O(n)$

Step 3.1: Inserting $s_i = v_i w_i$ using $\mathcal{Q}_{N(h-1)}$

– threading cost: Lem. 2 $\Rightarrow$ expected $O(1)$ per segm.

– locating cost:

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Time Complexity

Step 1: Random permutation $O(n)$

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Step 3: Phases 1 to $\log^\star n$ $(\log^\star n) \cdot$

Step 3.2: Walking the polygon Lemma 5 $\Rightarrow$ $O(n)$

Step 3.1: Inserting $s_i = v_i w_i$ using $\mathcal{Q}_{N(h-1)}$

– threading cost: Lem. 2 $\Rightarrow$ expected $O(1)$ per segm.

– locating cost: Know the location of v_i in $\mathcal{Q}_{N(h-1)}$.

Time Complexity

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Step 3.1: Inserting $s_i = v_i w_i$ using $\mathcal{Q}_{N(h-1)}$

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Lem. 4 $\Rightarrow$ expected location cost

Lemma 4. Let $1 \leq j \leq k \leq n$ and $q \in \mathbb{R}^2$. Suppose location of q in $\mathcal{Q}(S_j)$ is known, then q can be located in $\mathcal{Q}(S_k)$ in expected time $5(H_k - H_j) \in O(\log k/j)$.

Time Complexity

Step 1: Random permutation $O(n)$

Step 2: Setting up $\mathcal{T}_1$, $\mathcal{Q}_1$, and $\pi(v)$ $O(n)$

Step 3: Phases 1 to $\log^\star n$ $(\log^\star n) \cdot$

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$O(\log(i/N(h-1))) \subseteq$

Lemma 4. Let $1 \leq j \leq k \leq n$ and $q \in \mathbb{R}^2$. Suppose location of q in $\mathcal{Q}(S_j)$ is known, then q can be located in $\mathcal{Q}(S_k)$ in expected time $5(H_k - H_j) \in O(\log k/j)$.

Time Complexity

$N(h) := \lceil n / \log^{(h)} n \rceil$

Step 1: Random permutation $O(n)$

Step 2: Setting up $\mathcal{T}_1$, $\mathcal{Q}_1$, and $\pi(v)$ $O(n)$

Step 3: Phases 1 to $\log^\star n$ $(\log^\star n) \cdot$

Step 3.2: Walking the polygon Lemma 5 $\Rightarrow$ $O(n)$

Step 3.1: Inserting $s_i = v_i w_i$ using $\mathcal{Q}_{N(h-1)}$

– threading cost: Lem. 2 $\Rightarrow$ expected $O(1)$ per segm.

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$O(\log(i/N(h-1))) \subseteq O(\log^{(h)} n)$

Time Complexity

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Lem. 4 $\Rightarrow$ expected location cost

$O(\log(i / N(h-1))) \subseteq O(\log^{(h)} n)$

$\cdot N(h) =$

Time Complexity

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Step 2: Setting up $\mathcal{T}_1$, $\mathcal{Q}_1$, and $\pi(v)$ $O(n)$

Step 3: Phases 1 to $\log^\star n$ $(\log^\star n) \cdot$

Step 3.2: Walking the polygon Lemma 5 $\Rightarrow$ $O(n)$

Step 3.1: Inserting $s_i = v_i w_i$ using $\mathcal{Q}_{N(h-1)}$

– threading cost: Lem. 2 $\Rightarrow$ expected $O(1)$ per segm.

– locating cost: Know the location of v_i in $\mathcal{Q}_{N(h-1)}$.

Lem. 4 $\Rightarrow$ expected location cost $O(\log(i/N(h-1))) \subseteq O(\log^{(h)} n)$

$\cdot N(h) = O(n)$

Time Complexity

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Step 3.2: Walking the polygon Lemma 5 $\Rightarrow$ $O(n)$

Step 3.1: Inserting $s_i = v_i w_i$ using $\mathcal{Q}_{N(h-1)}$

$\left.\begin{array}{ll} \text{– threading cost:} & \text{Lem. 2} \Rightarrow \text{expected } O(1) \text{ per segm.} \\ \text{– locating cost:} & \text{Know the location of } v_i \text{ in } \mathcal{Q}_{N(h-1)}. \\ & \text{Lem. 4} \Rightarrow \text{expected location cost} \\ & O(\log(i/N(h-1))) \subseteq O(\log^{(h)} n) \end{array}\right\} \cdot N(h) = O(n)$

Step 4: Inserting s_i (for $N(\log^\star n) < i \le n$) using $\mathcal{Q}_{N(\log^\star n)}$

Time Complexity

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Lemma 2. For $i = 1, \ldots, n$, the expected number of rays of $\mathcal{T}(S_{i-1})$ that are intersected by s_i is at most 4.

Step 4: Inserting s_i (for $N(\log^\star n) < i \leq n$) using $\mathcal{Q}_{N(\log^\star n)}$

– threading cost:

– locating cost:

Time Complexity

$N(h) := \lceil n / \log^{(h)} n \rceil$

Step 1: Random permutation $O(n)$

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Step 3: Phases 1 to $\log^\star n$ $(\log^\star n) \cdot$

Step 3.2: Walking the polygon Lemma 5 $\Rightarrow$ $O(n)$

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– threading cost: Lem. 2 $\Rightarrow$ $O(1)$

– locating cost:

Time Complexity

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Step 3.2: Walking the polygon Lemma 5 $\Rightarrow$ $O(n)$

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Step 4: Inserting s_i (for $N(\log^\star n) < i \le n$) using $\mathcal{Q}_{N(\log^\star n)}$

– threading cost: Lem. 2 $\Rightarrow O(1)$

– locating cost: Lem. 4 $\Rightarrow O(\log n / N(\log^\star n)) =$

Time Complexity

$N(h) := \lceil n / \log^{(h)} n \rceil$

Step 1: Random permutation $O(n)$

Step 2: Setting up $\mathcal{T}_1$, $\mathcal{Q}_1$, and $\pi(v)$ $O(n)$

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Step 3.2: Walking the polygon Lemma 5 $\Rightarrow$ $O(n)$

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Step 4: Inserting s_i (for $N(\log^\star n) < i \leq n$) using $\mathcal{Q}_{N(\log^\star n)}$

– threading cost: Lem. 2 $\Rightarrow O(1)$

– locating cost: Lem. 4 $\Rightarrow O(\log n / \underbrace{N(\log^\star n)}) =$

Time Complexity

$N(h) := \lceil n / \log^{(h)} n \rceil$

Step 1: Random permutation $O(n)$

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Step 4: Inserting s_i (for $N(\log^\star n) < i \leq n$) using $\mathcal{Q}_{N(\log^\star n)}$

– threading cost: Lem. 2 $\Rightarrow O(1)$

– locating cost: Lem. 4 $\Rightarrow O(\log n / \underbrace{N(\log^\star n)}_{> n/2}) =$

Time Complexity

$N(h) := \lceil n / \log^{(h)} n \rceil$

Step 1: Random permutation $O(n)$

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Lemma 4. Let $1 \leq j \leq k \leq n$ and $q \in \mathbb{R}^2$. Suppose location of q in $\mathcal{Q}(S_j)$ is known, then q can be located in $\mathcal{Q}(S_k)$ in expected time $5(H_k - H_j) \in O(\log k/j)$.

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Time Complexity

$N(h) := \lceil n / \log^{(h)} n \rceil$

Step 1: Random permutation $O(n)$

Step 2: Setting up $\mathcal{T}_1$, $\mathcal{Q}_1$, and $\pi(v)$ $O(n)$

Step 3: Phases 1 to $\log^\star n$ $(\log^\star n) \cdot$

Step 3.2: Walking the polygon Lemma 5 $\Rightarrow$ $O(n)$

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Lem. 4 $\Rightarrow$ expected location cost

$O(\log(i/N(h-1))) \subseteq O(\log^{(h)} n)$

$\cdot N(h) = O(n)$

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The Results

Theorem. Let S be the edge set of a polygon, $|S| = n$.

- We can build $\mathcal{T}(S)$ and $\mathcal{Q}(S)$ in $O(n \log^{\star} n)$ expected time.
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Theorem. Let S be the edge set of a plane straight-line graph with k connected components, $|S| = n$.

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- The expected time for locating a point in $\mathcal{T}(S)$ via $\mathcal{Q}(S)$ is $O(\log n)$.