Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Computational Geometry

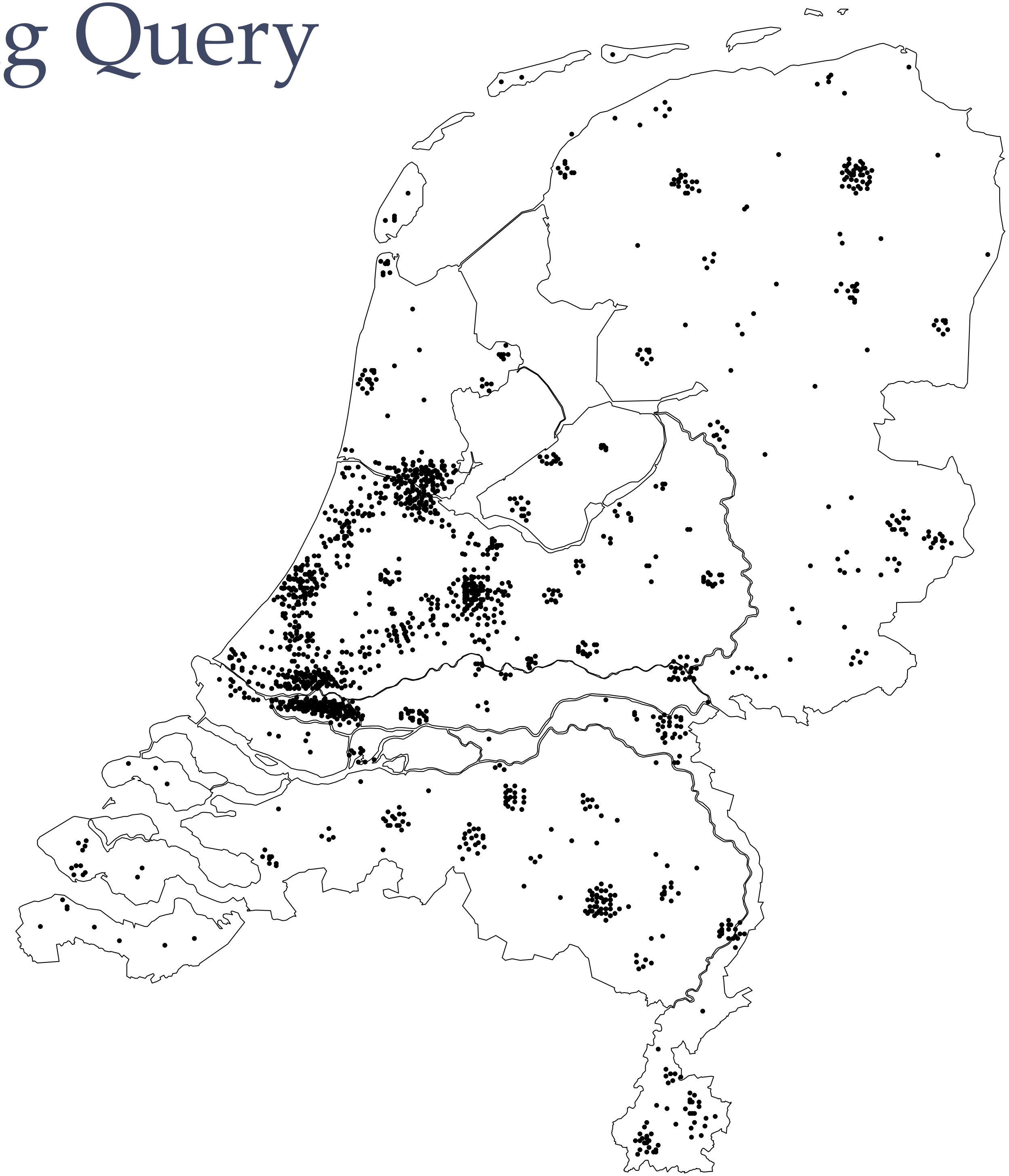
Simple Range Searching

Lecture #11

Dr. Philipp Kindermann

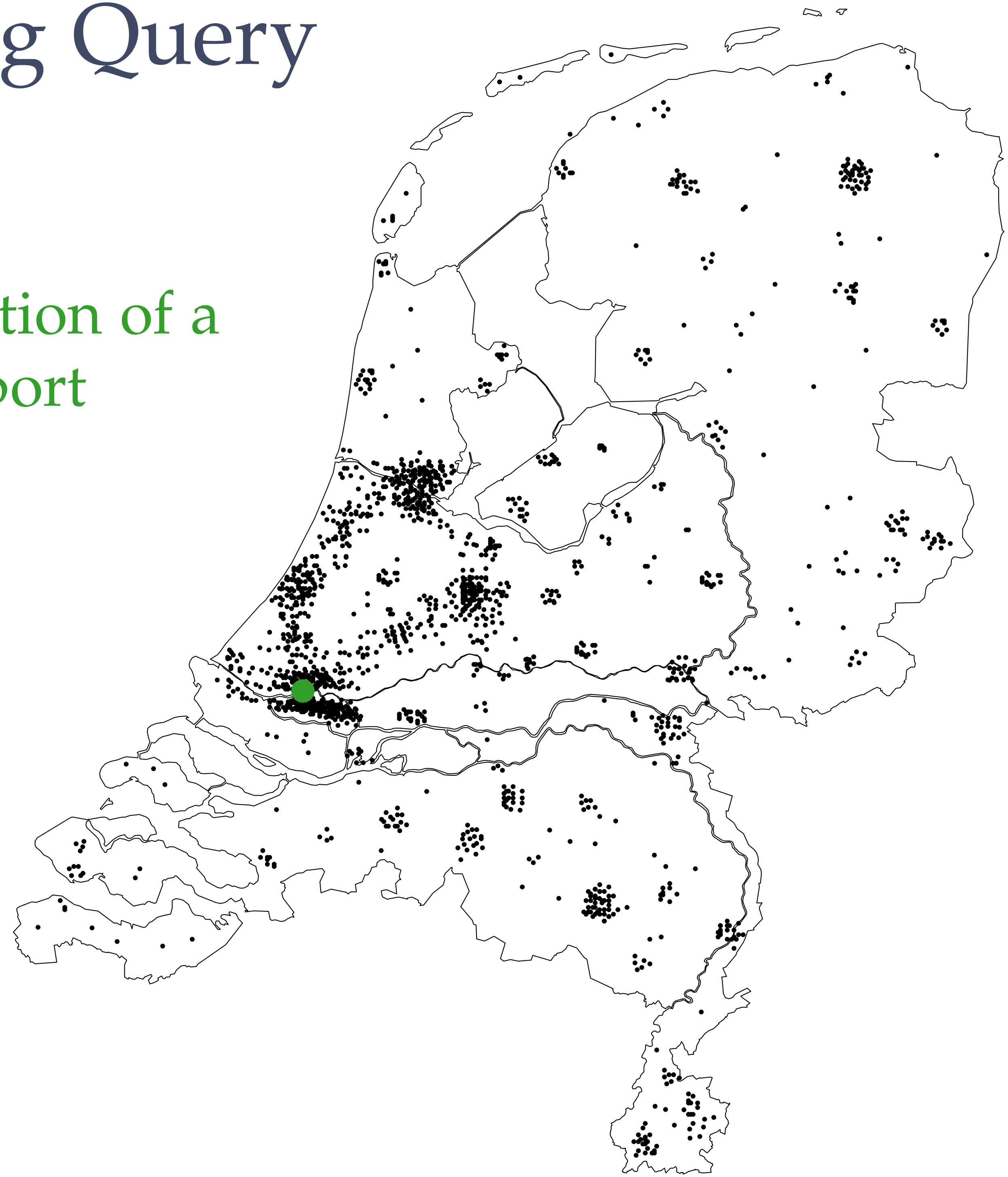
Winter Semester 2018/19

Range-Counting Query



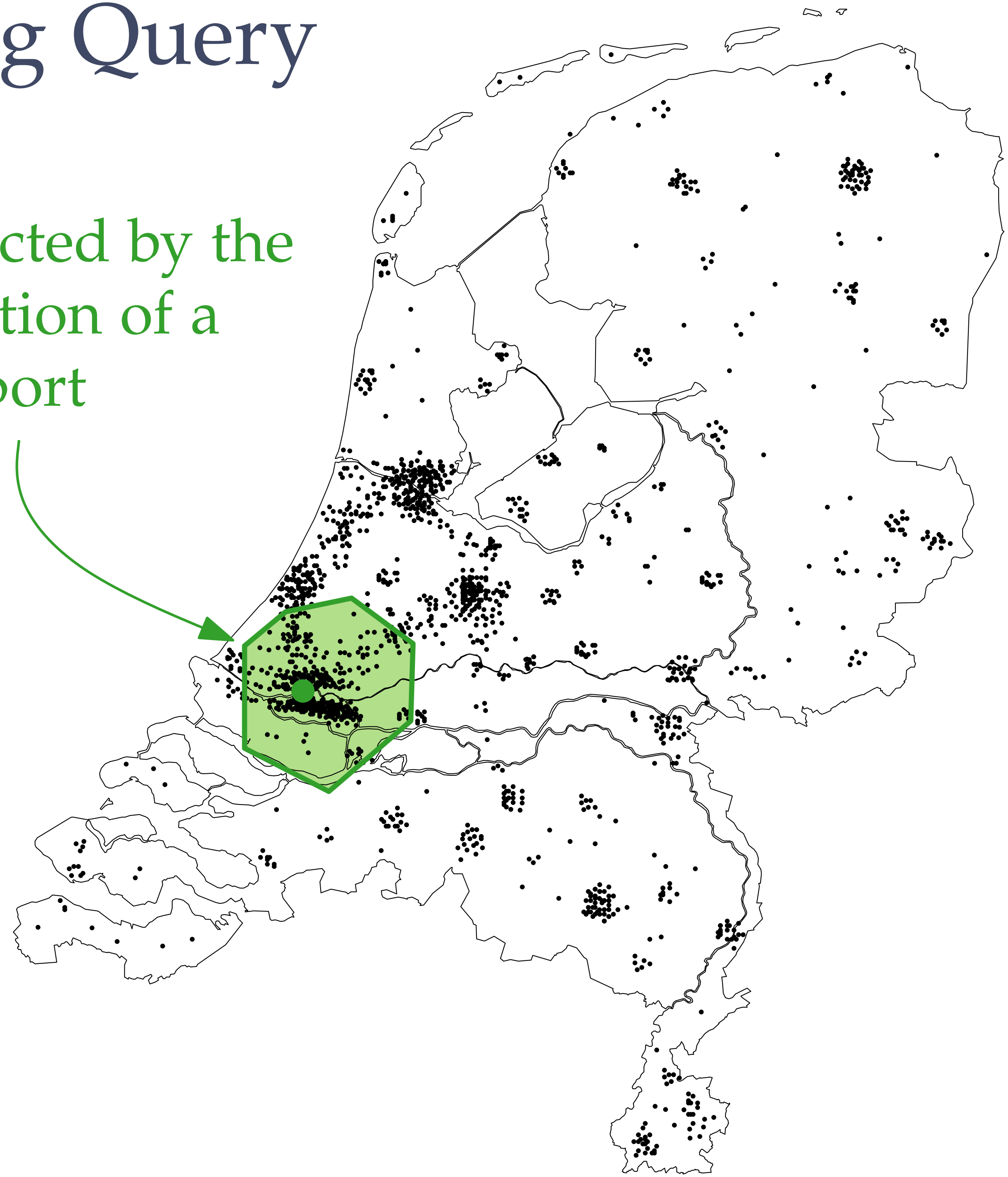
Range-Counting Query

construction of a
new airport



Range-Counting Query

area affected by the construction of a new airport



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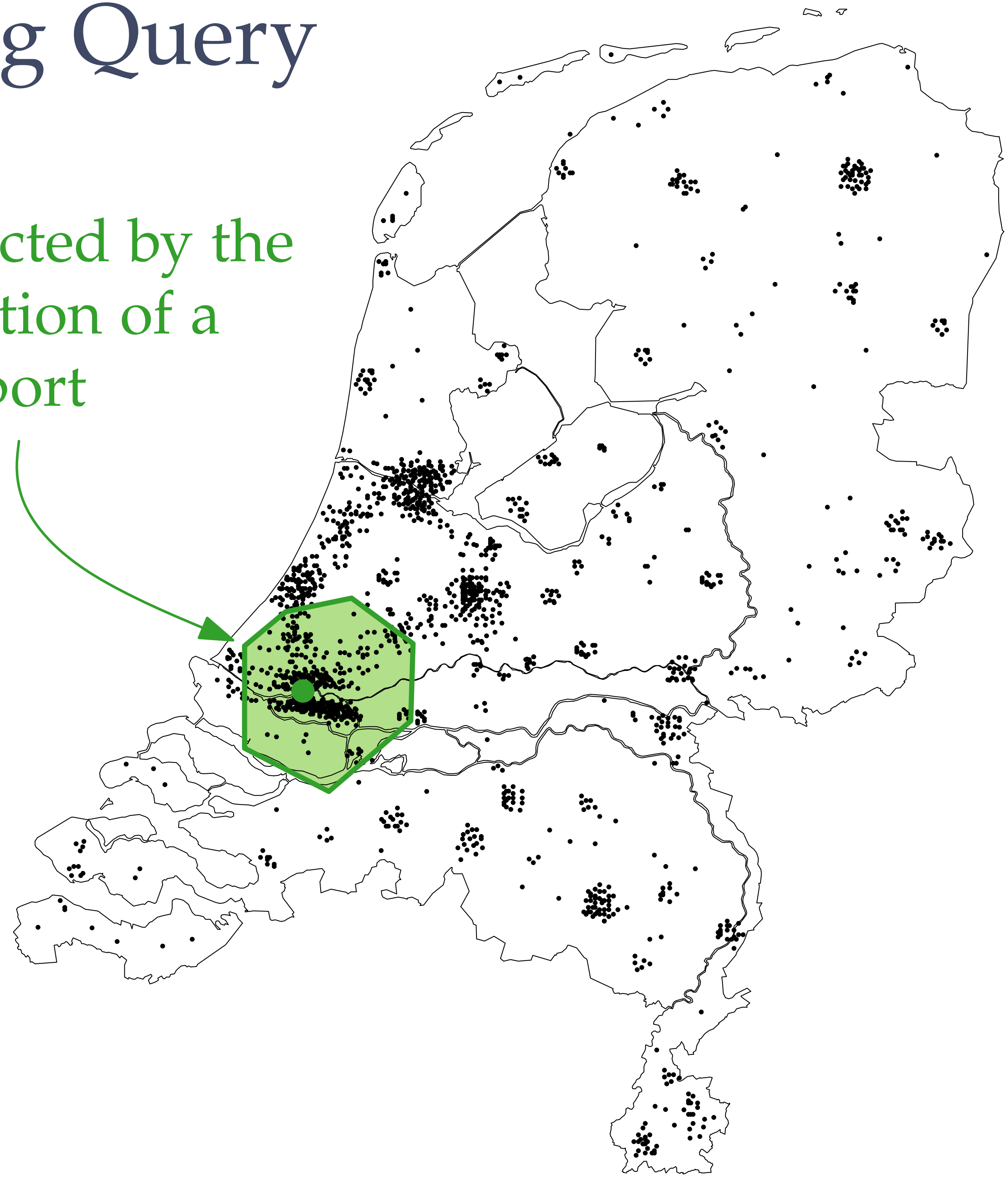

Observation.

Query range depends on, e.g., dominant wind directions

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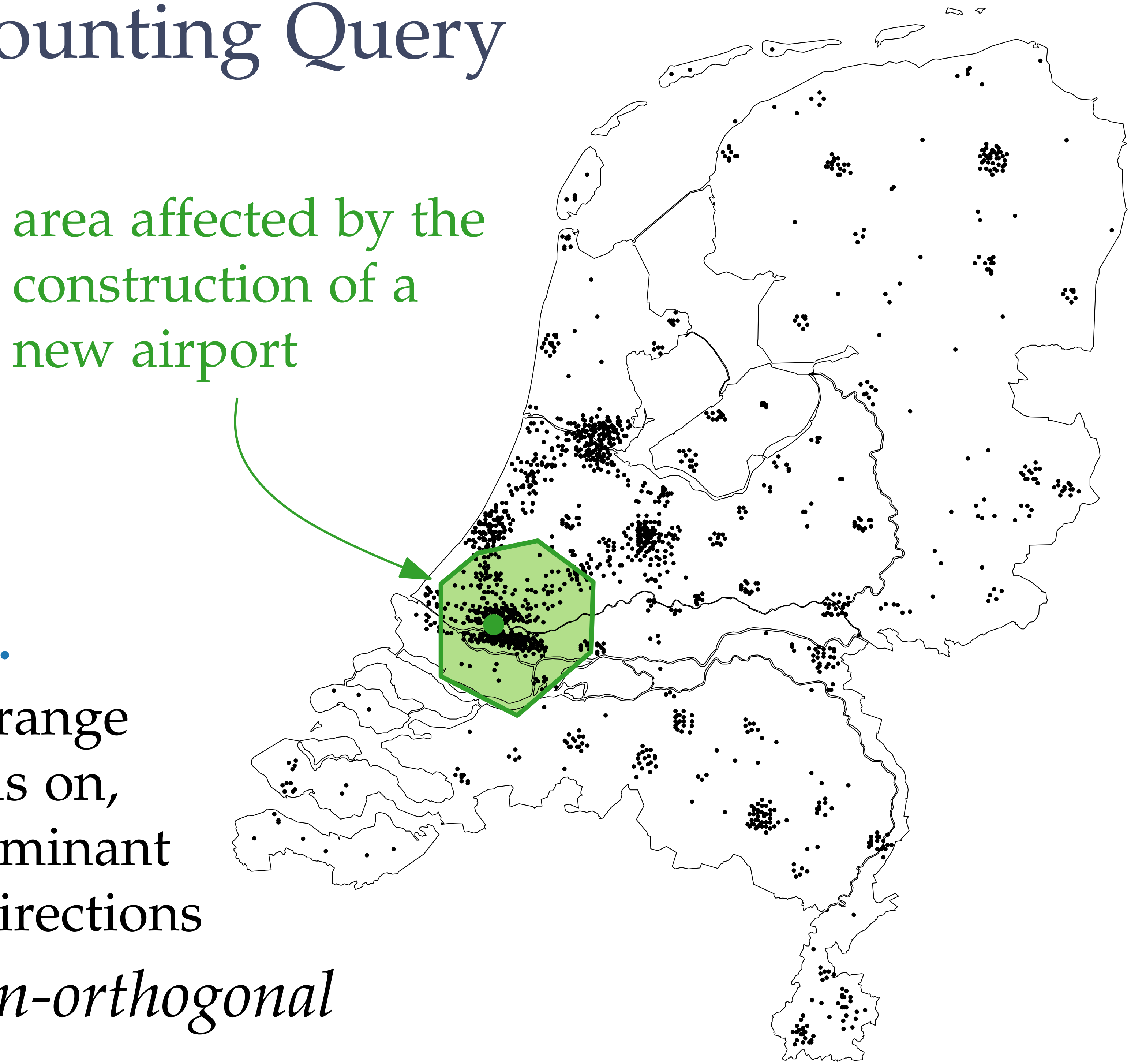

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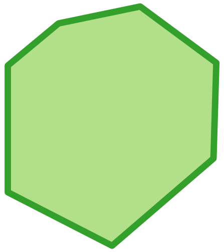
Query range depends on, e.g., dominant wind directions

$\Rightarrow$ *non-orthogonal*

Non-orthogonal range queries

Query range: 

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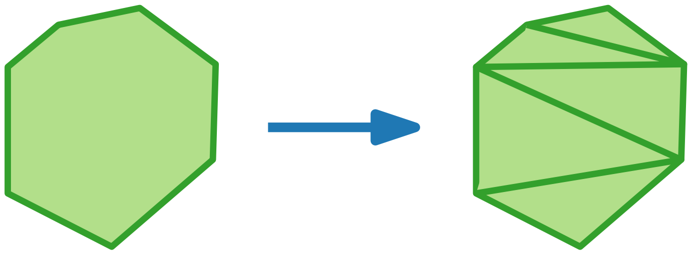

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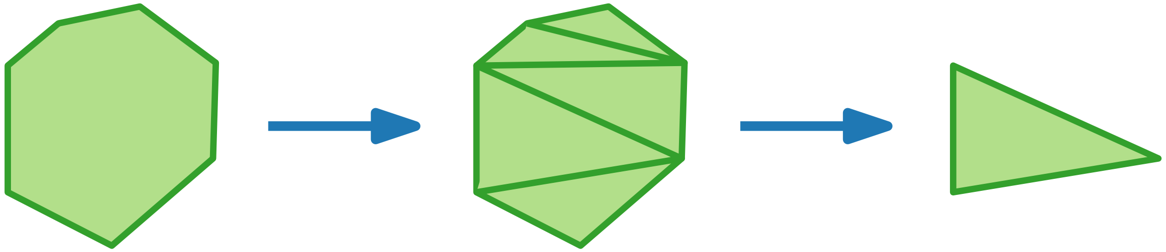

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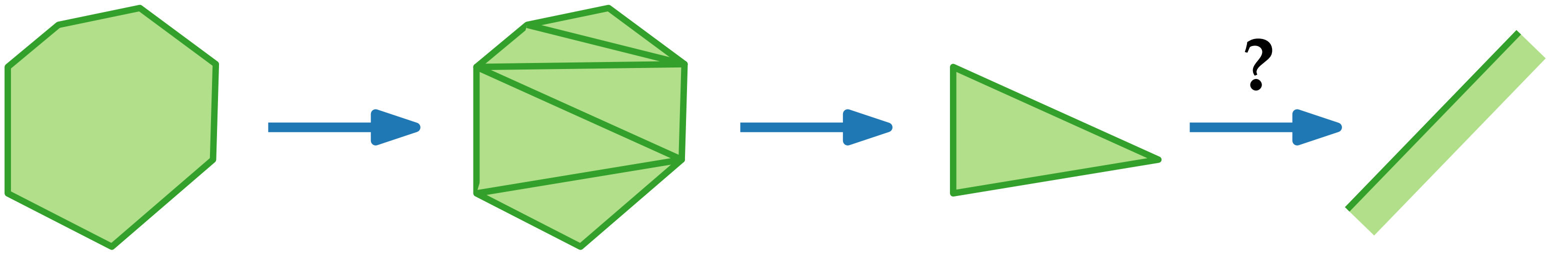

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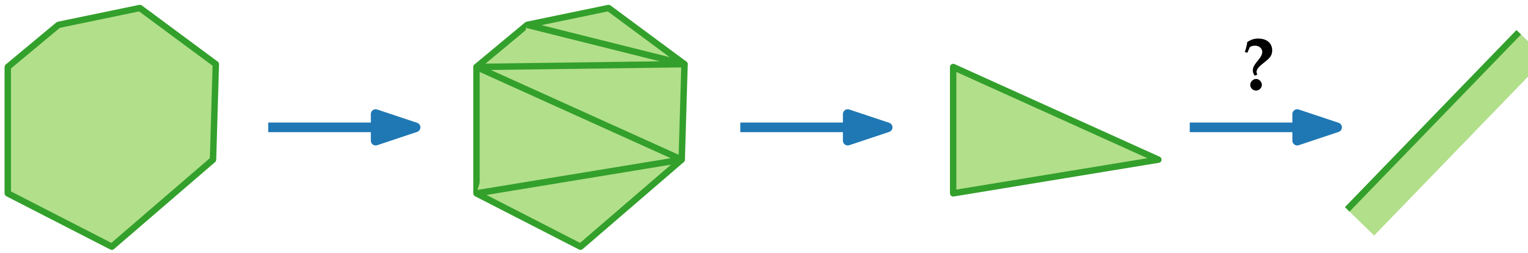

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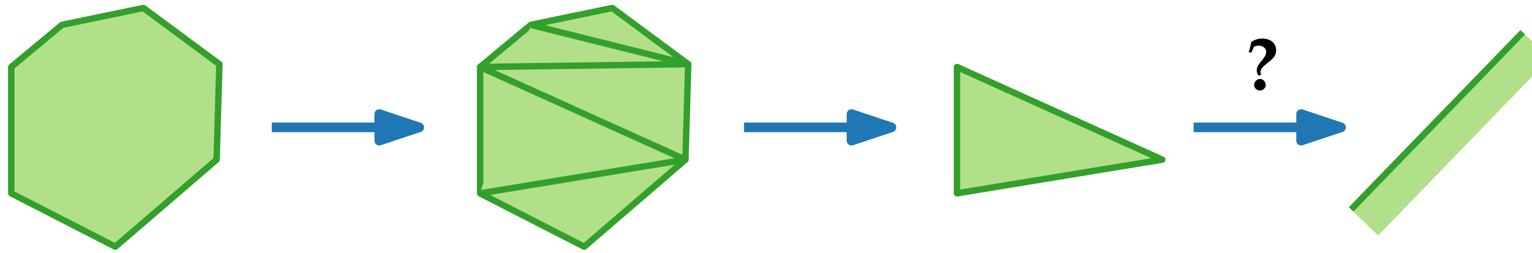
Non-orthogonal range queries

Query range: 

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Task Design a data structure for the 1-dim. case:

Non-orthogonal range queries

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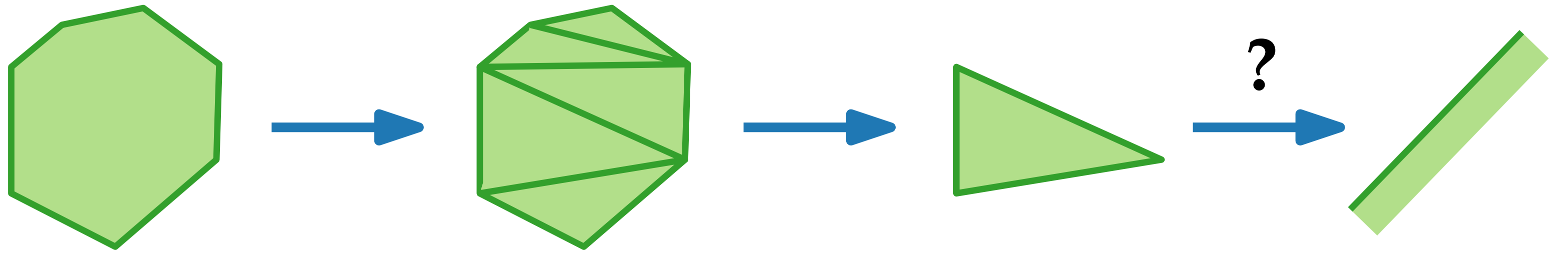

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– Given a number x, return $|P \cap [x, \infty)|$.

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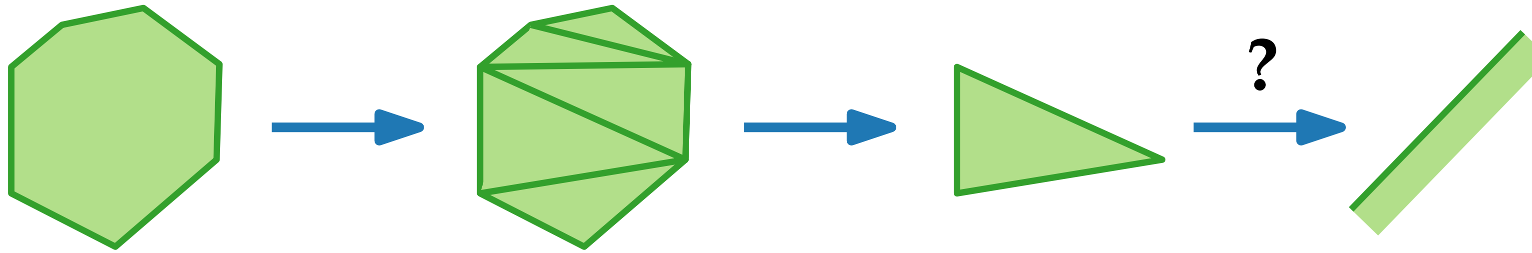
Query range: 

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Task Design a data structure for the 1-dim. case:

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– Consider P static / dynamic!

The 1-Dimensional Case

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Solution.

The 1-Dimensional Case

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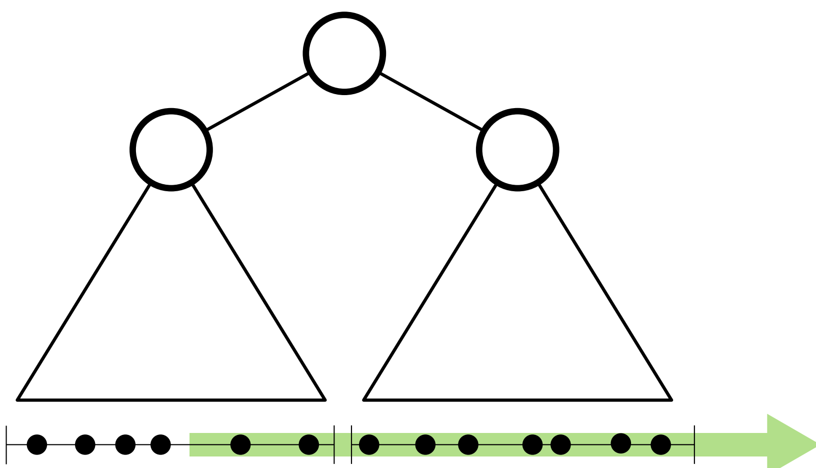

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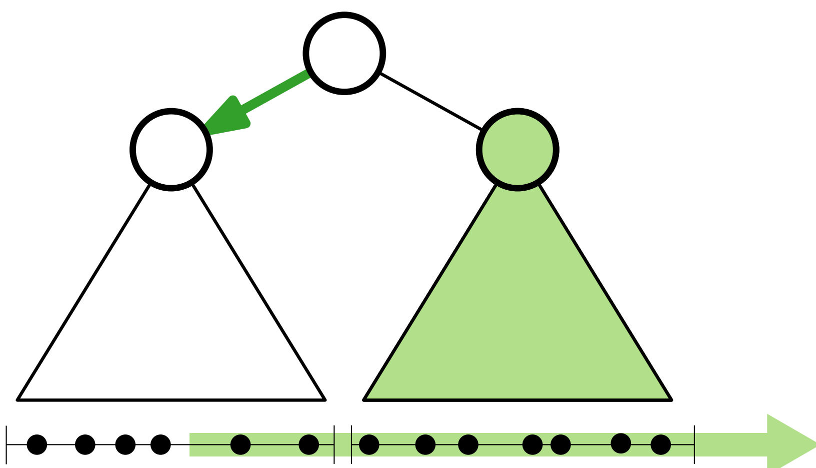

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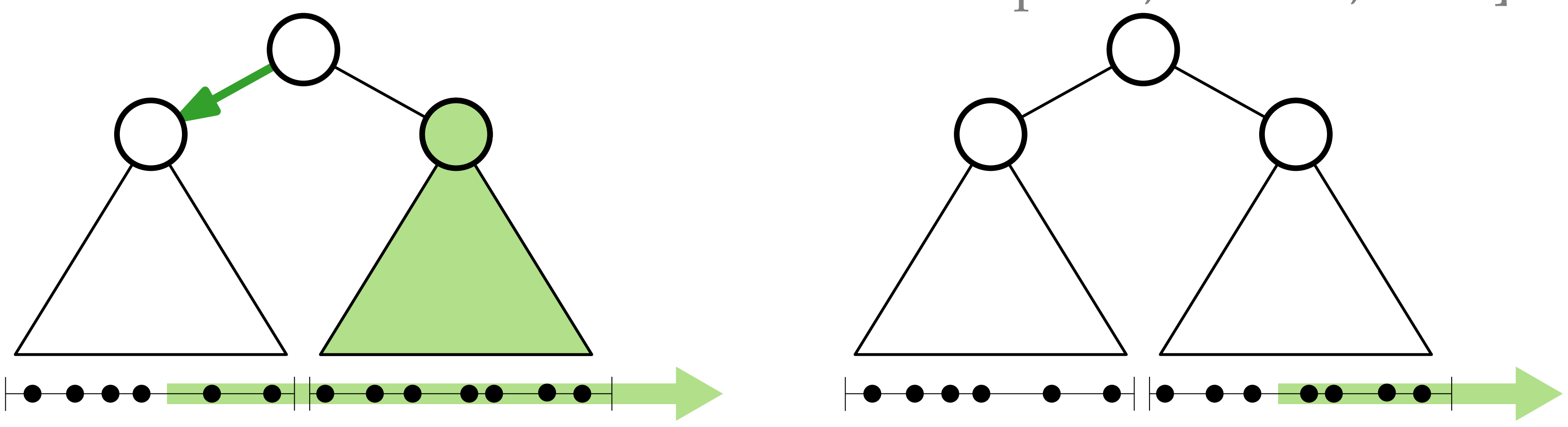

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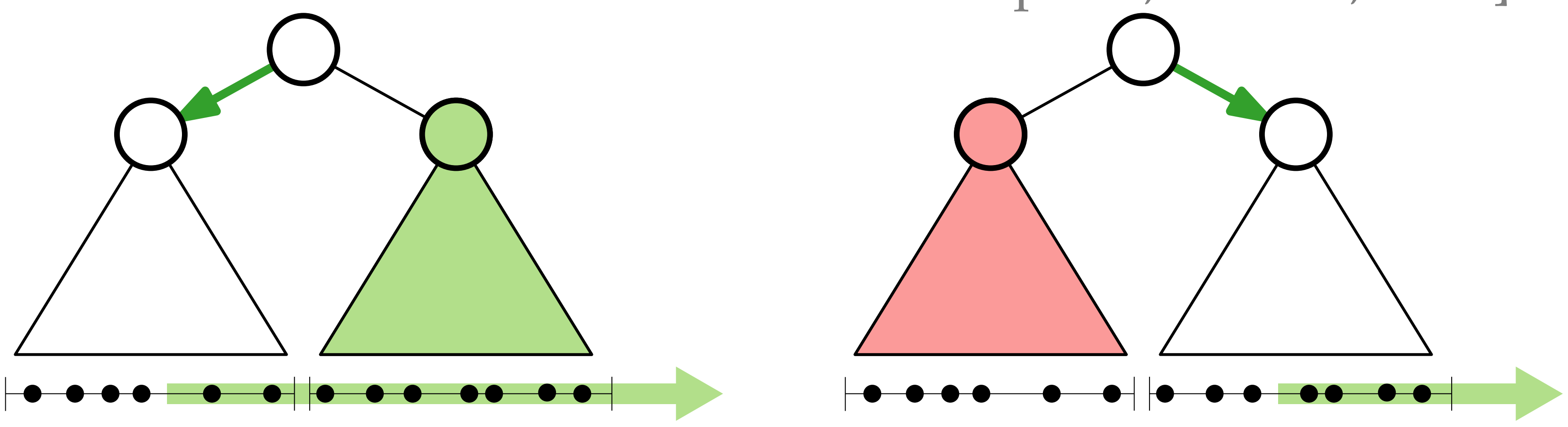

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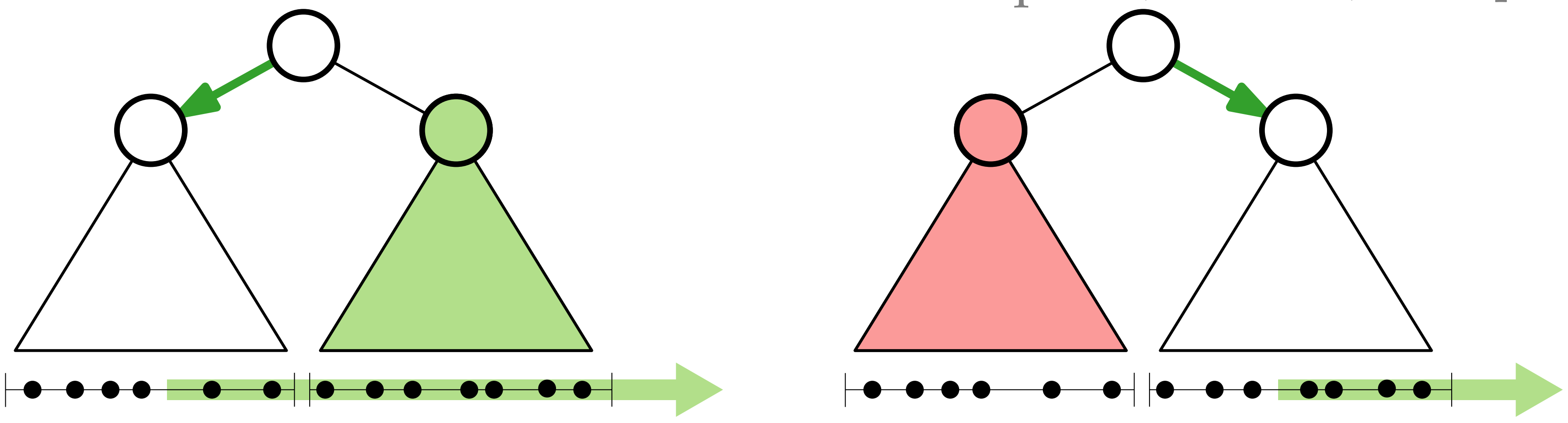

Lesson. On each level, visit ≤ 1 subtree recursively!

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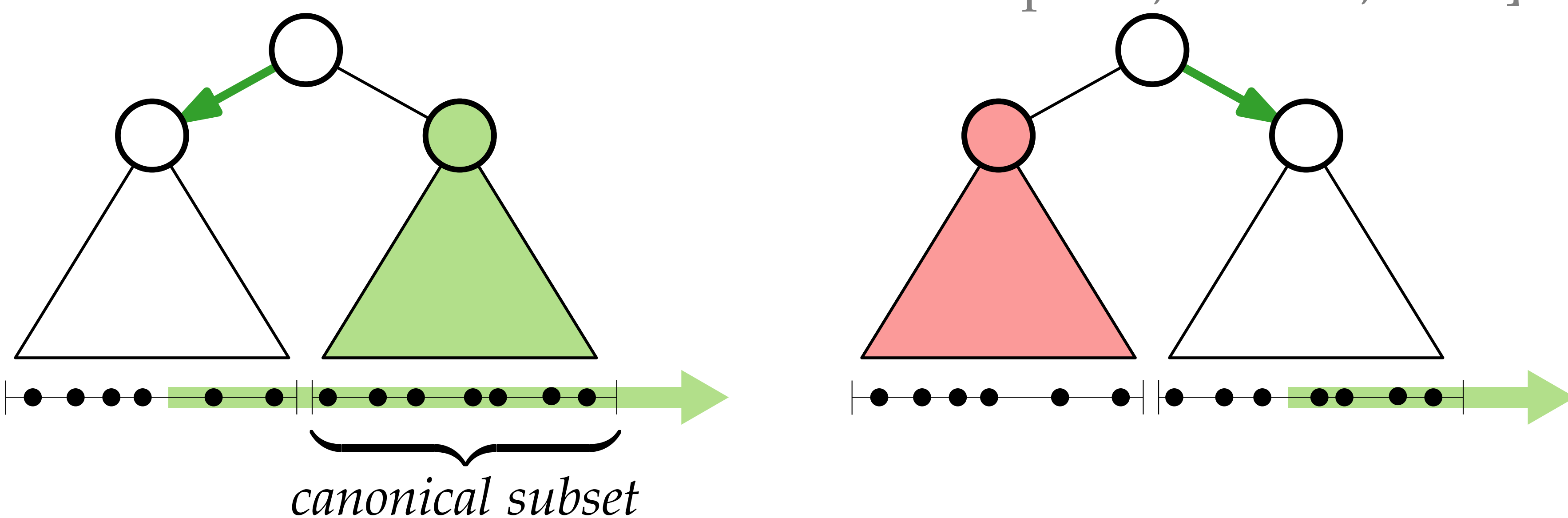

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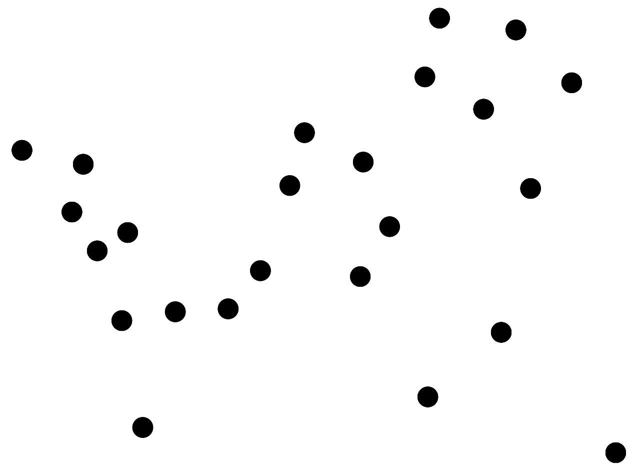
Generalizing to 2 Dimensions

Any ideas?



Generalizing to 2 Dimensions

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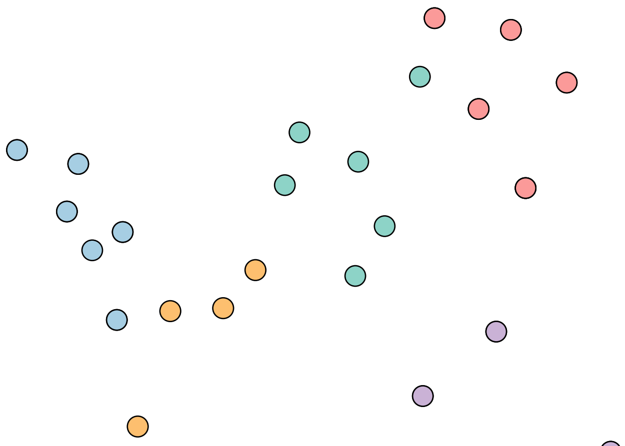
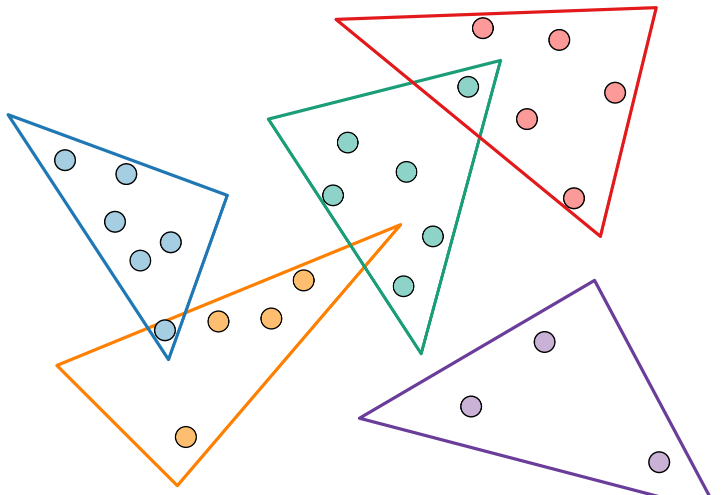


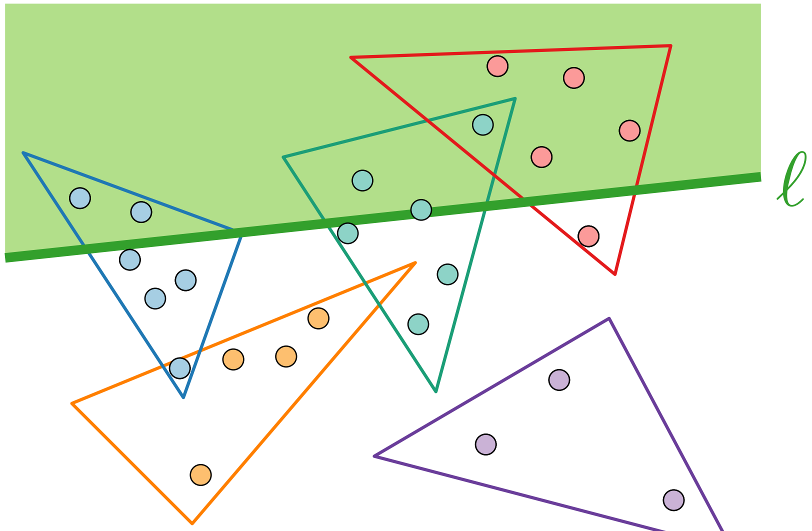
Generalizing to 2 Dimensions

Partition the input!



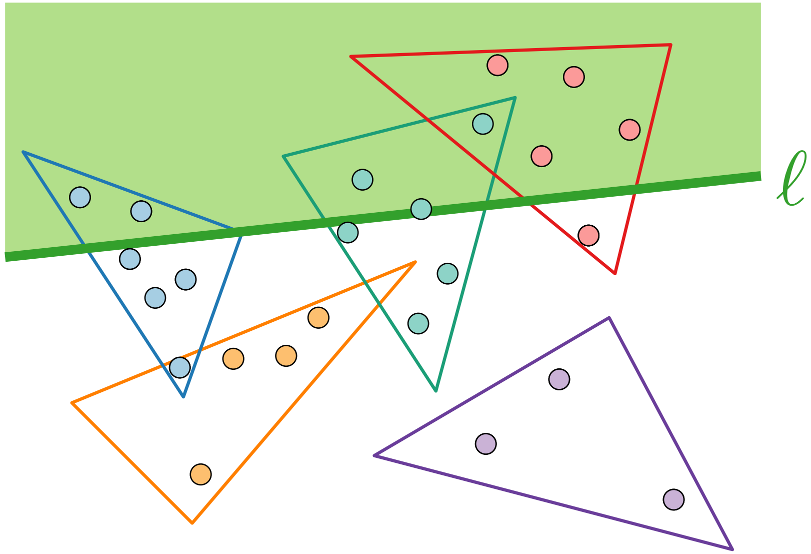
Generalizing to 2 Dimensions

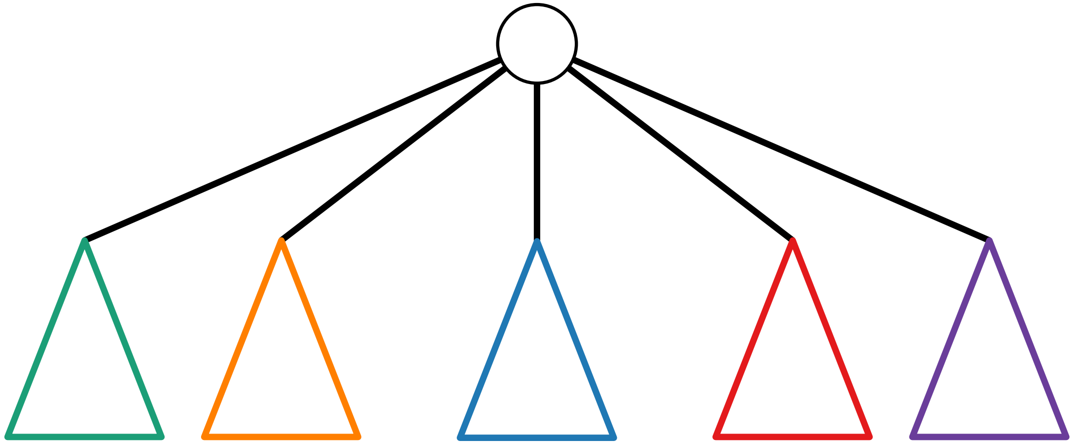
Partition the input! Query…



Generalizing to 2 Dimensions

Partition the input! Query… in a *partition tree*





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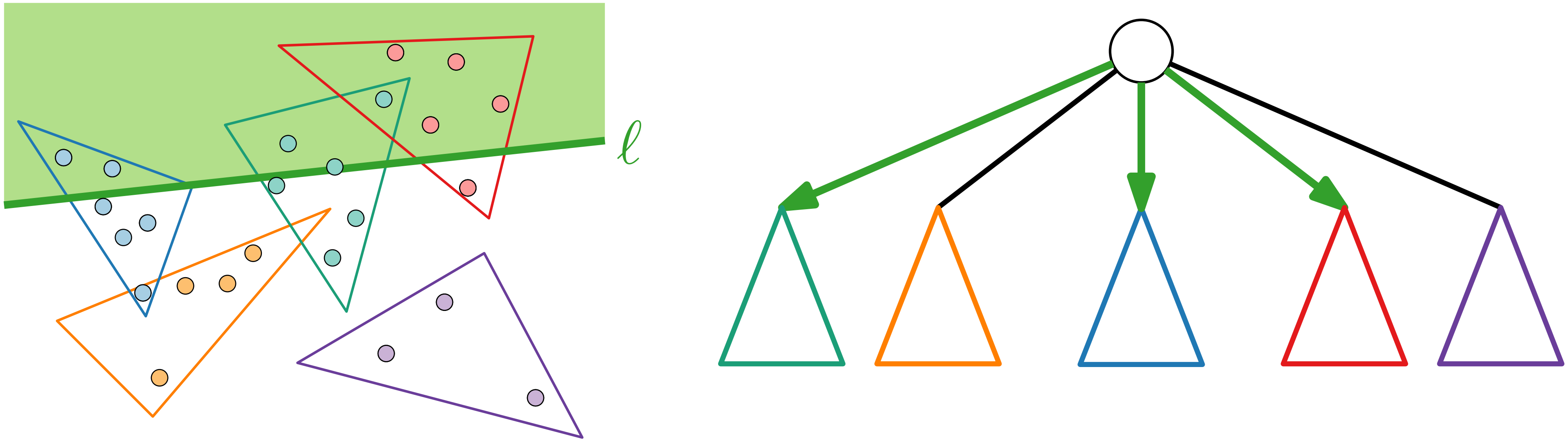

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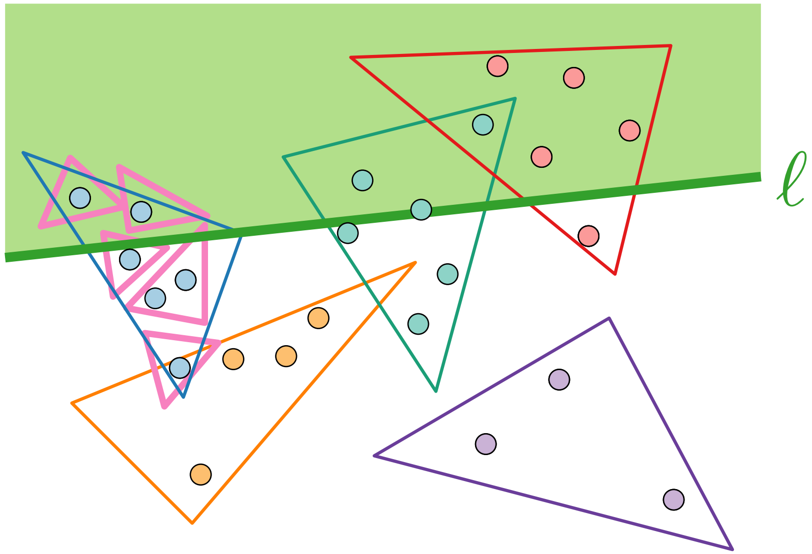

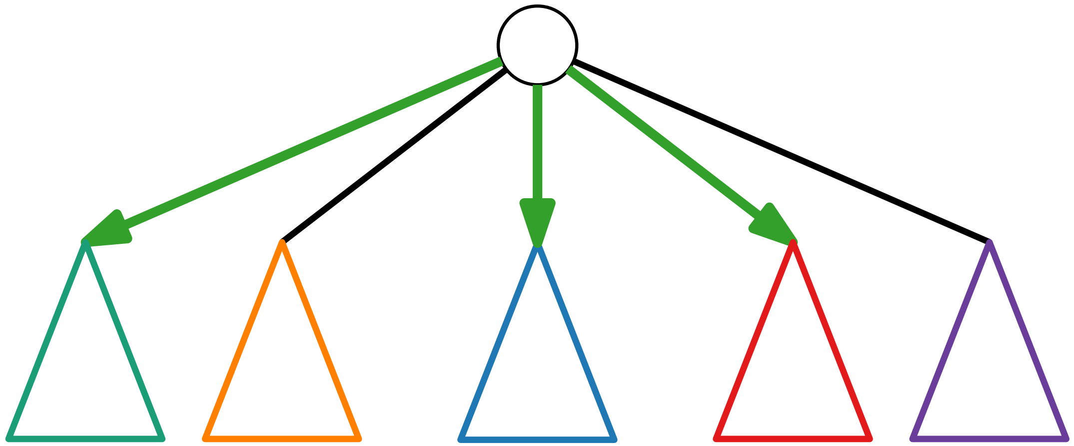

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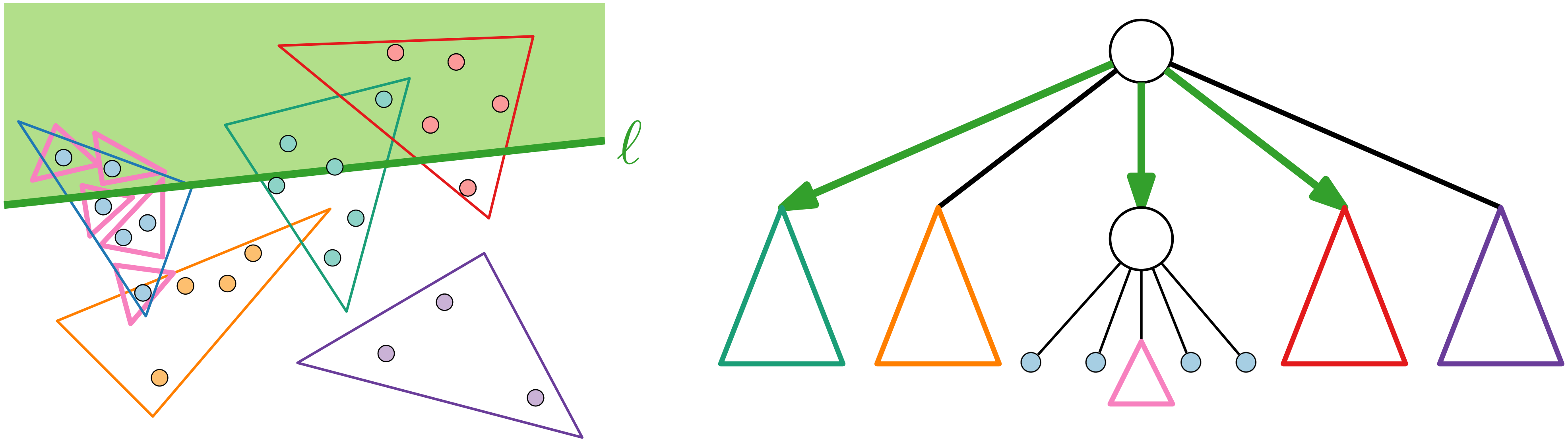

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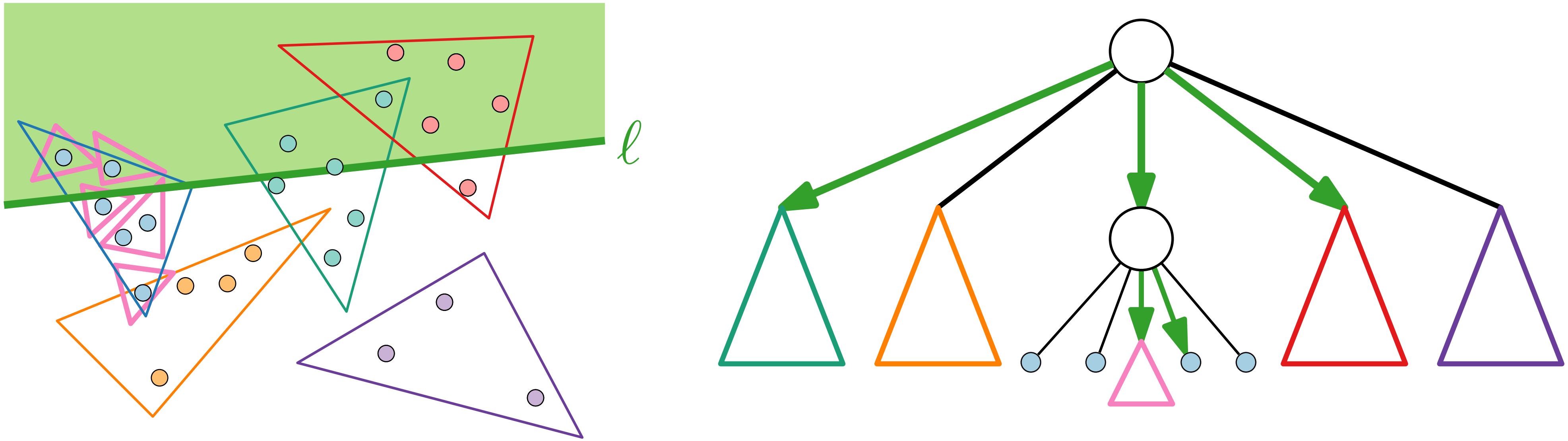

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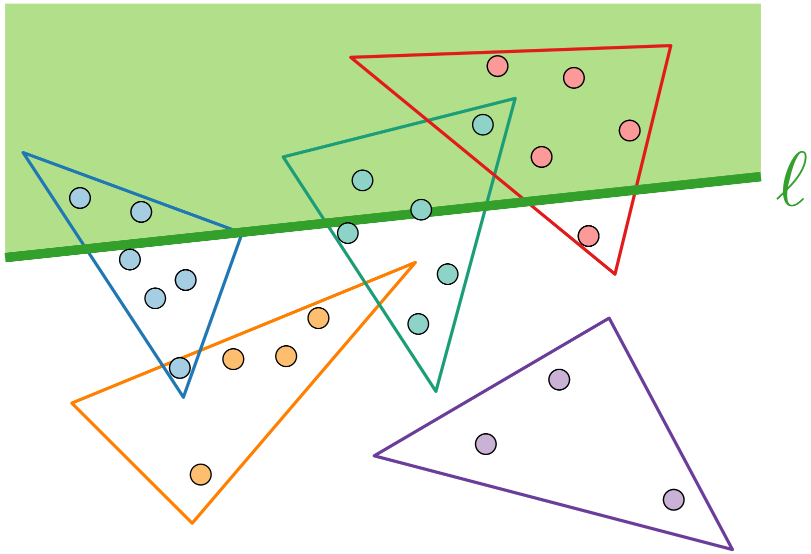

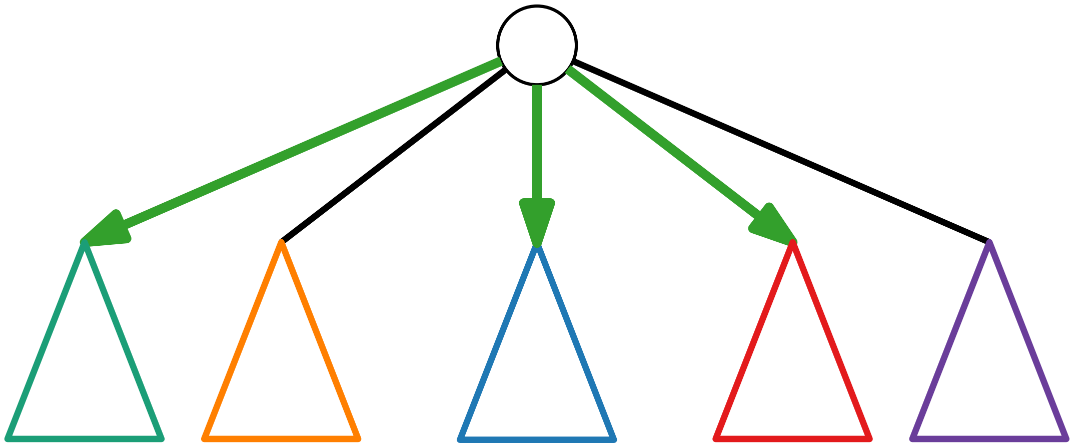

Definition. $\Psi(S) = \{(S_1, t_1), (S_2, t_2), \ldots, (S_r, t_r)\}$ is a *simplicial partition* (of size r) for S if

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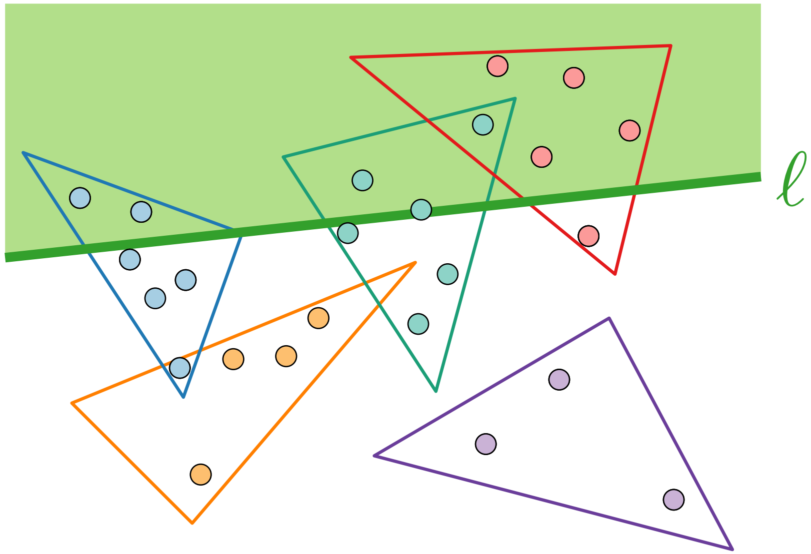

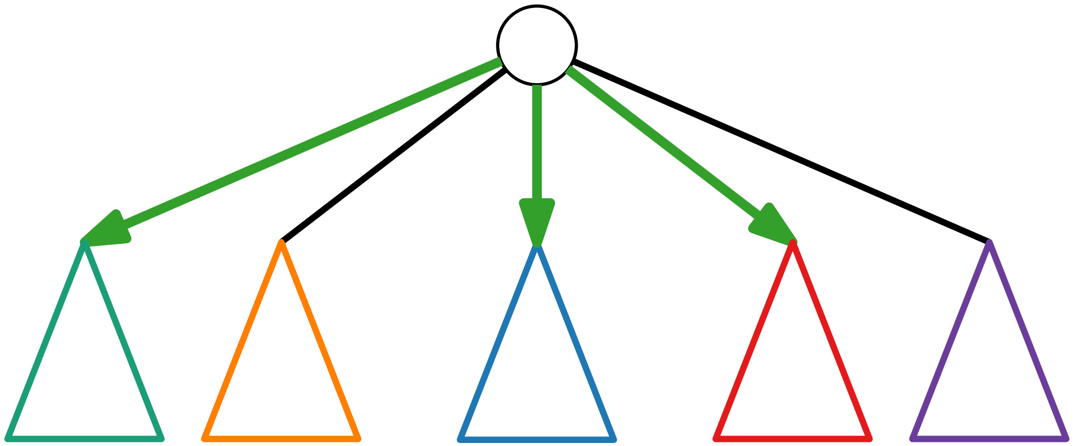

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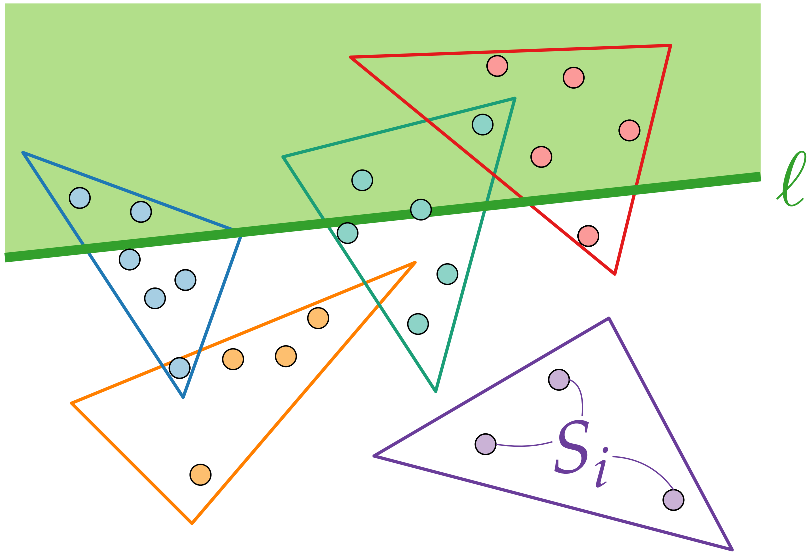

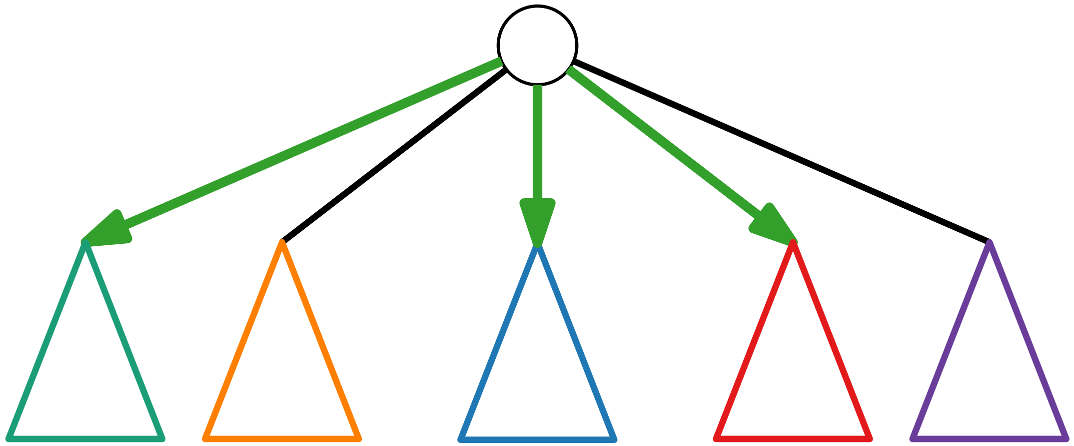

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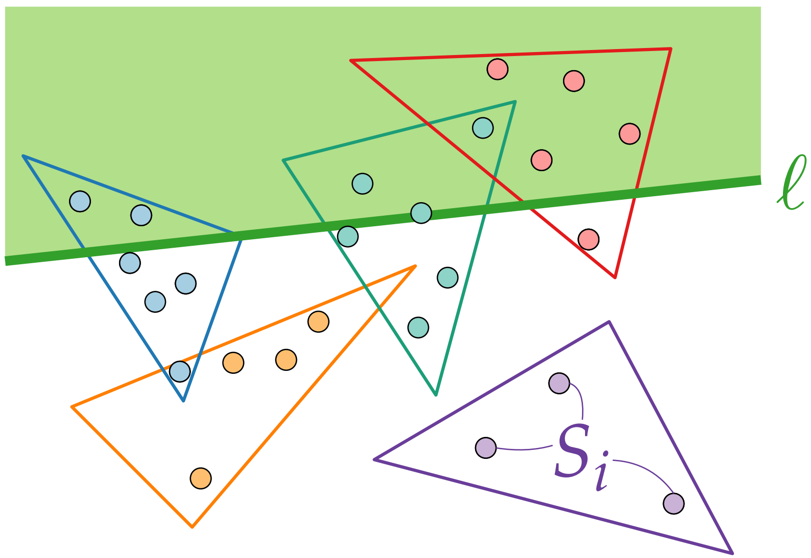

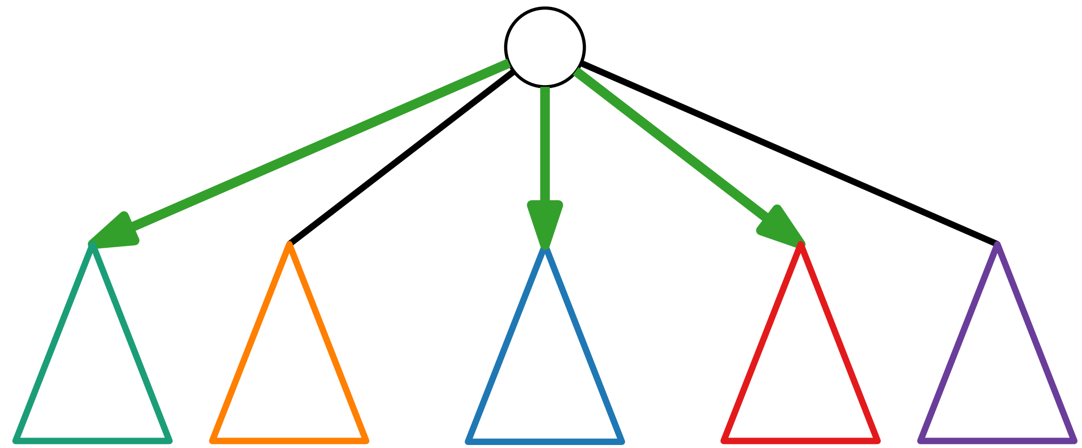

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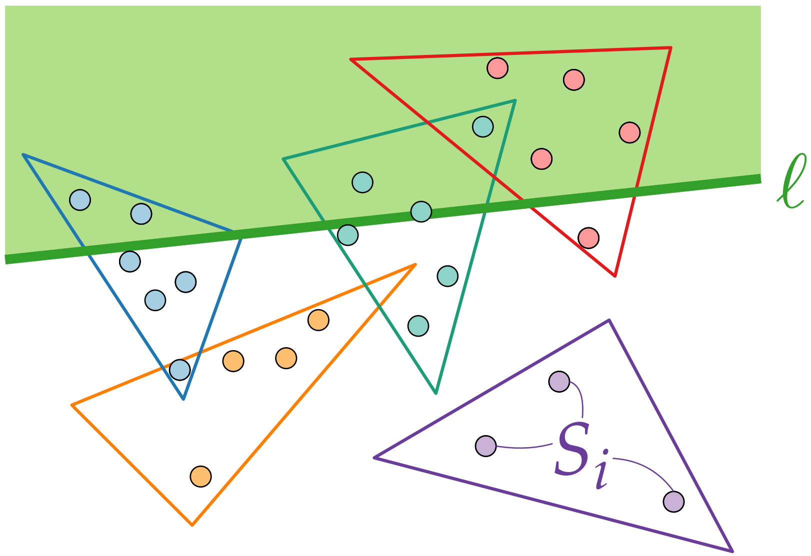

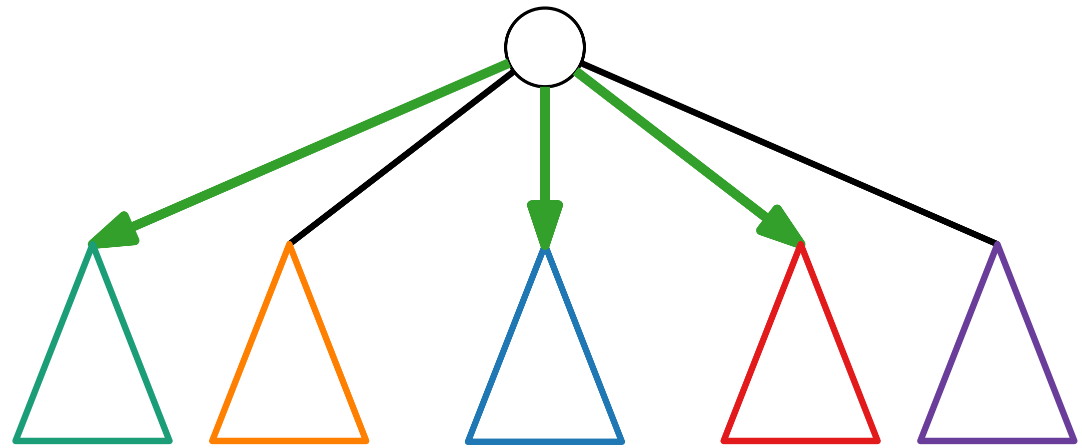

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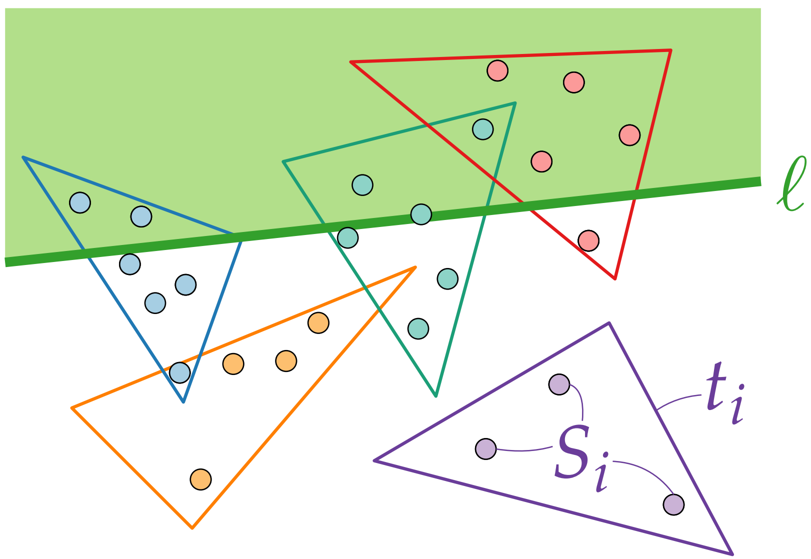

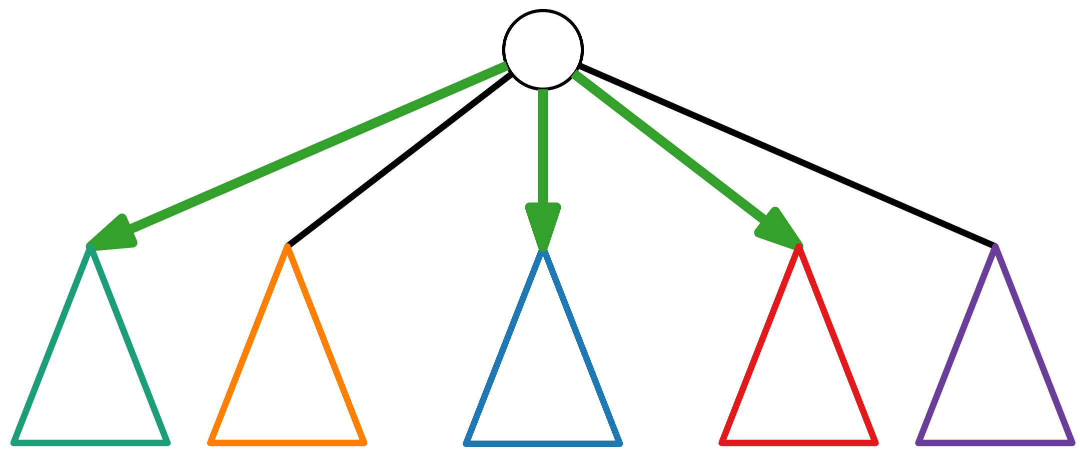

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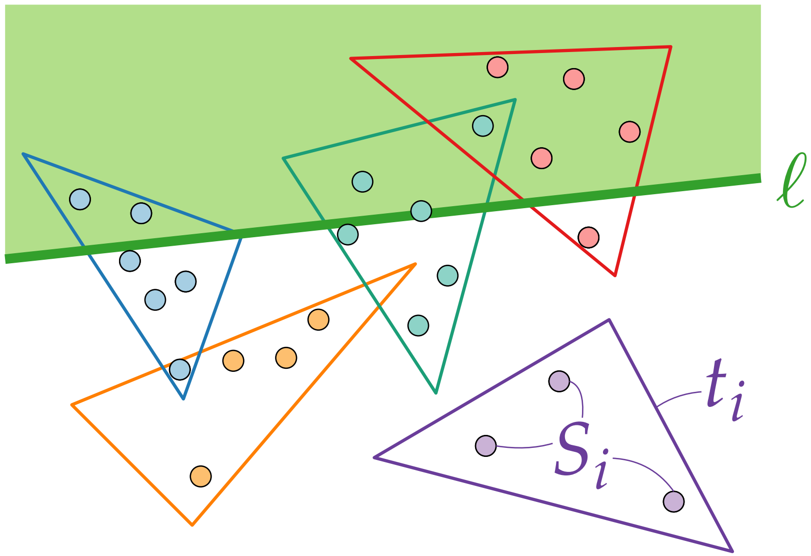

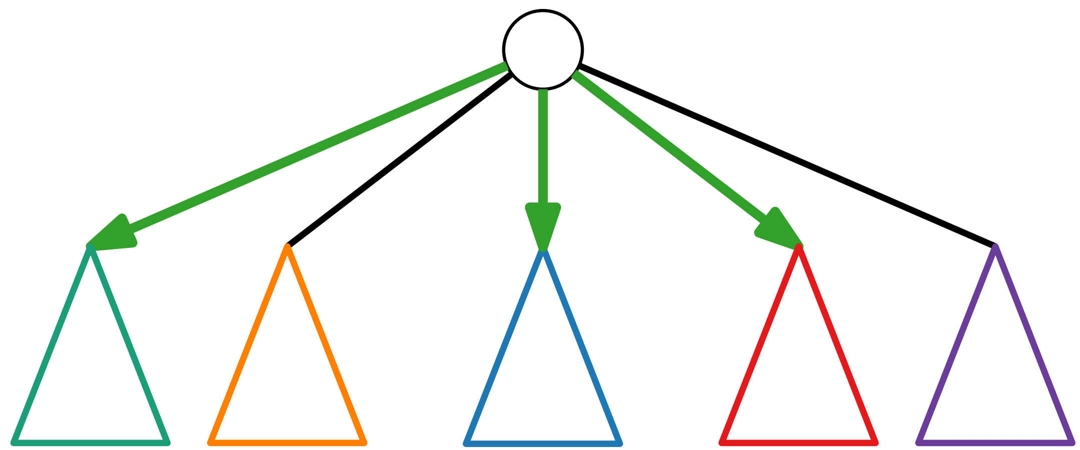

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– for $1 \le i \le r$, t_i is a triangle and $S_i \subset t_i$.

$\Psi(S)$ is *fine* if $|S_i| \le 2\frac{|S|}{r}$ for every $1 \le i \le r$.

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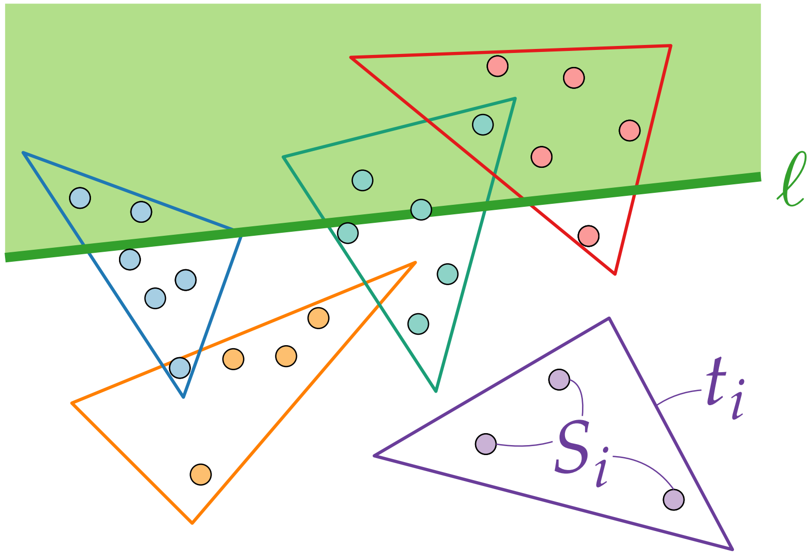

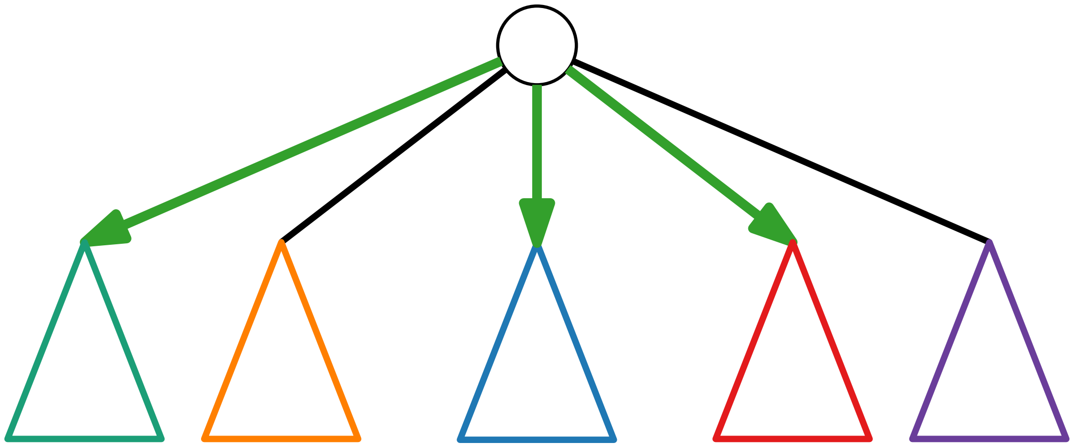

Definition. The *crossing number* of ℓ (w.r.t. $\Psi(S)$) is the number of triangles $t_1, \ldots, t_r$ *crossed* by ℓ.

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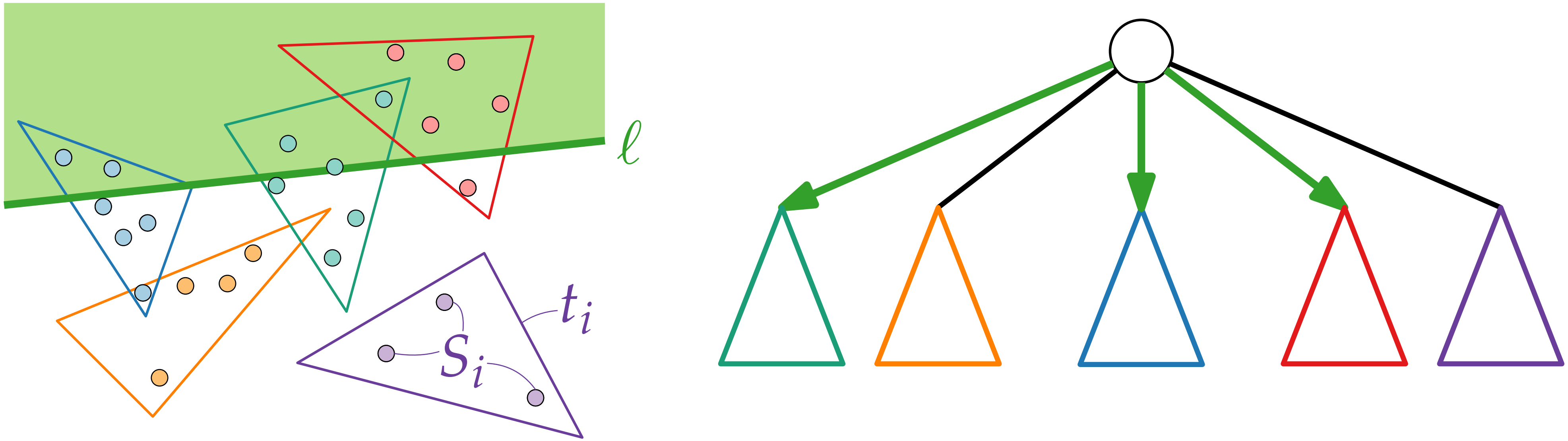

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The *crossing number* of $\Psi(S)$ is the maximum crossing number over all possible lines.

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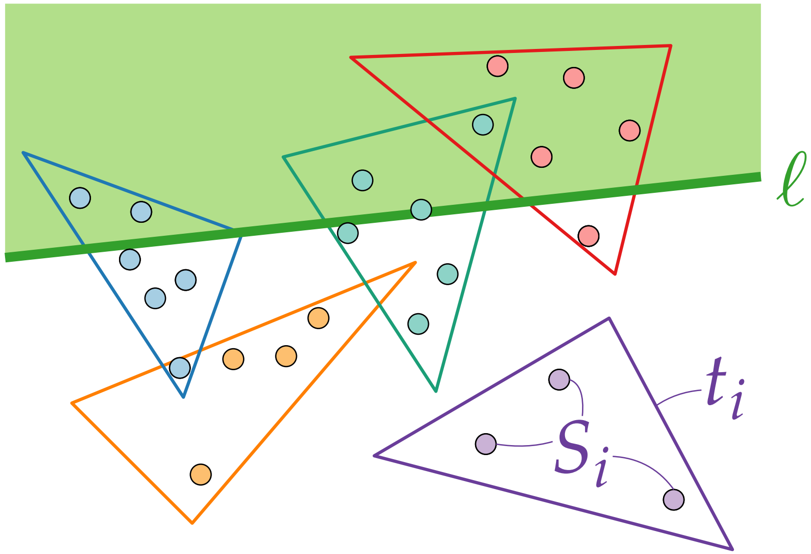

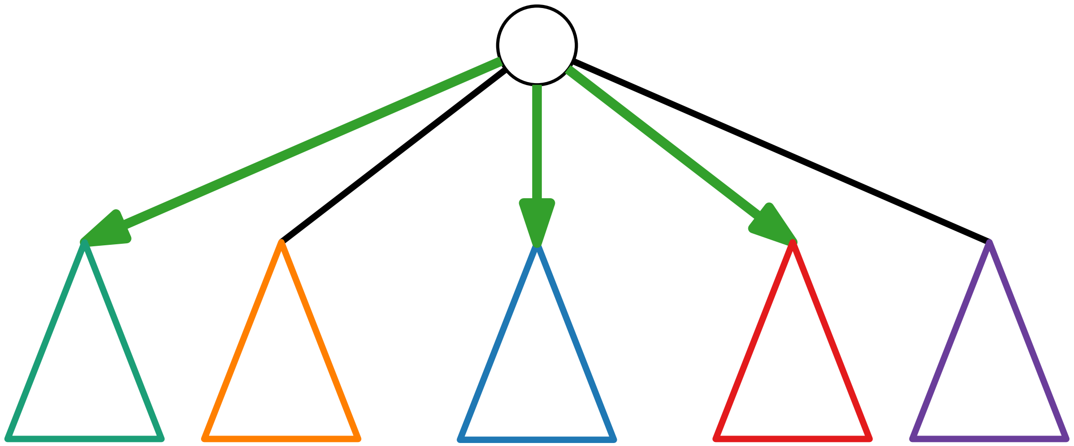

Theorem. [Matoušek, DCG 1992] For any set S of n pts and any $1 \leq r \leq n$, a fine simplicial partition of size r and crossing number $O(\quad)$ exists.

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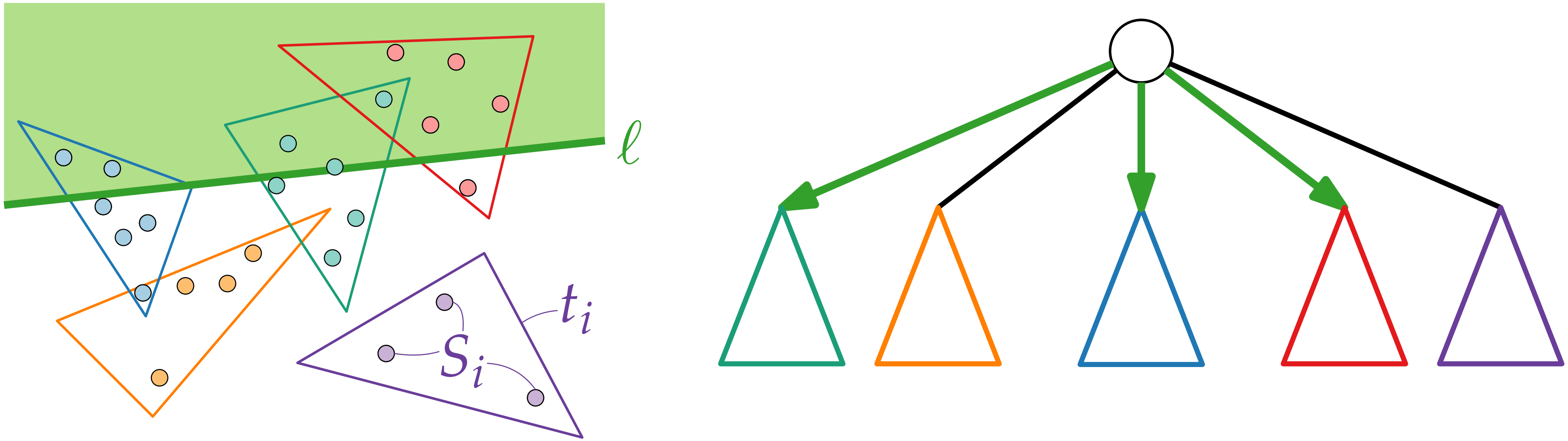

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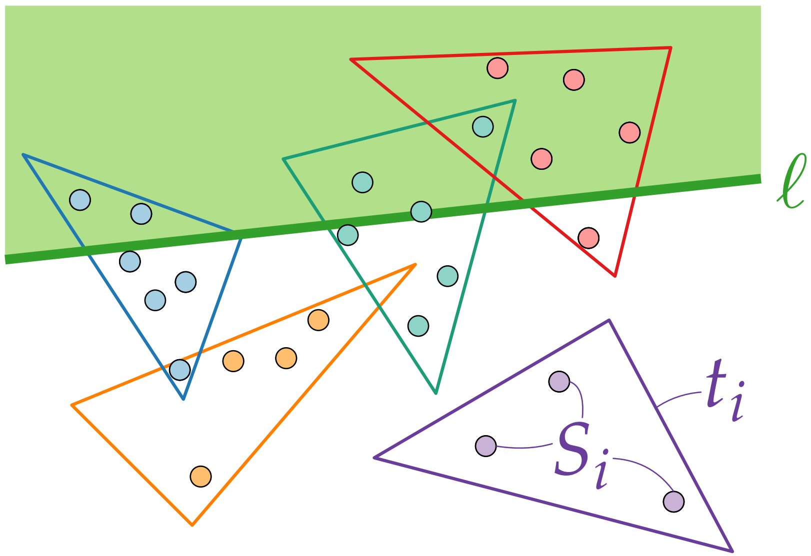

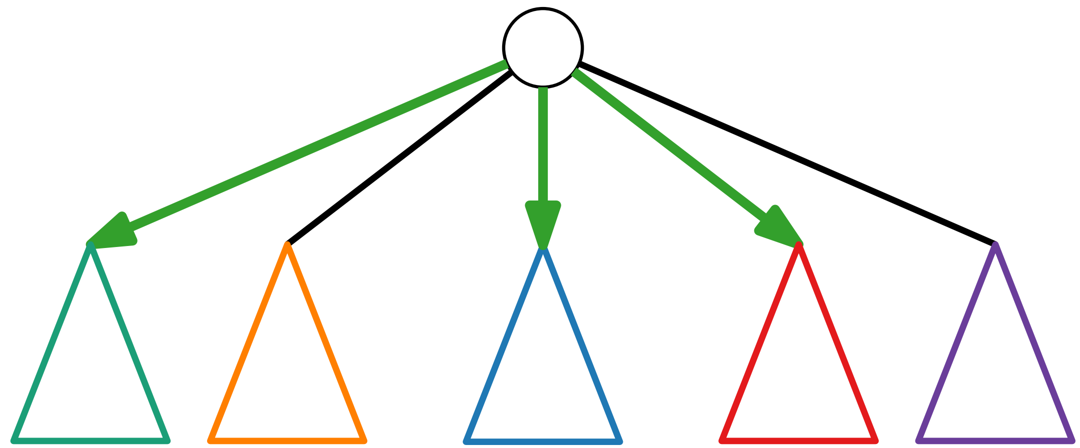

Theorem. [Matoušek, DCG 1992] For any set S of n pts and any $1 \leq r \leq n$, a fine simplicial partition of size r and crossing number $O(\sqrt{r})$ exists. For any $\varepsilon > 0$, such a partition can be built in $O(n^{1+\varepsilon})$ time.

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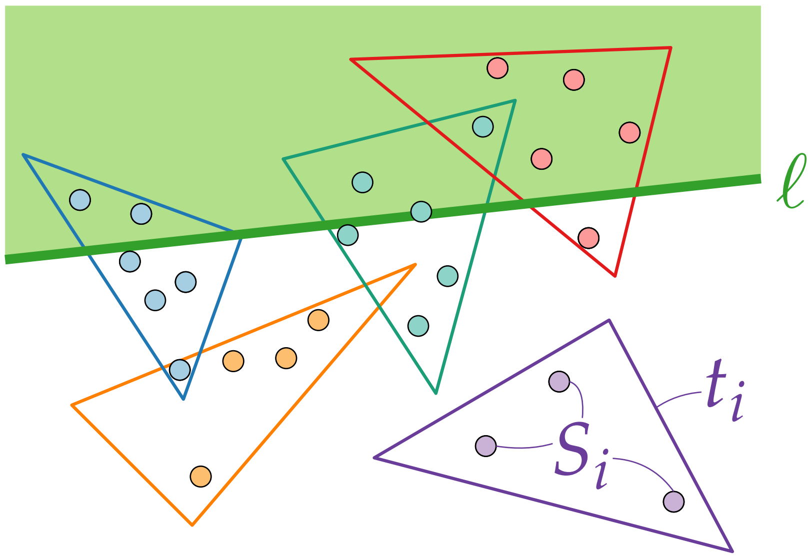

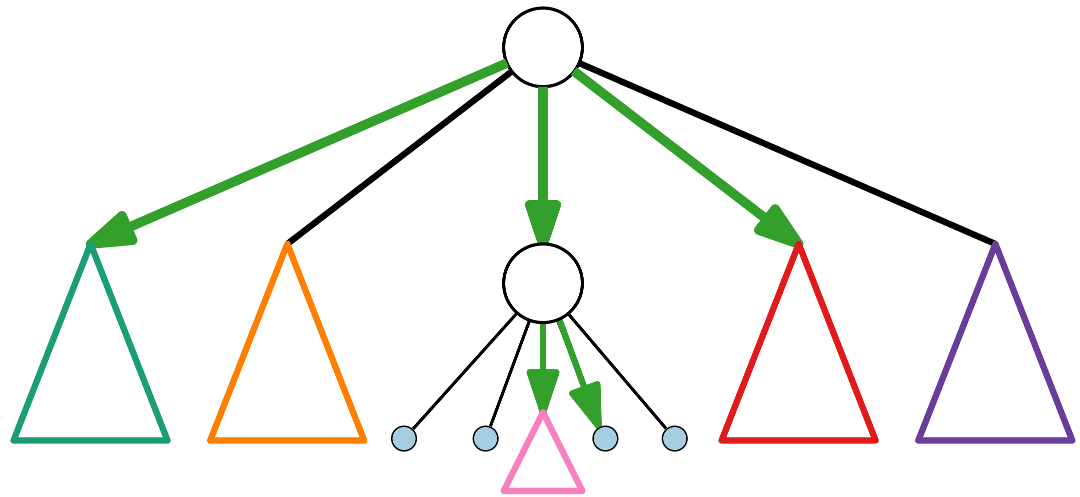

Theorem. [Matoušek, DCG 1992] For any set S of n pts and any $1 \leq r \leq n$, a fine simplicial partition of size r and crossing number $O(\sqrt{r})$ exists. For any $\varepsilon > 0$, such a partition can be built in $O(n^{1+\varepsilon})$ time.

Generalizing to 2 Dimensions

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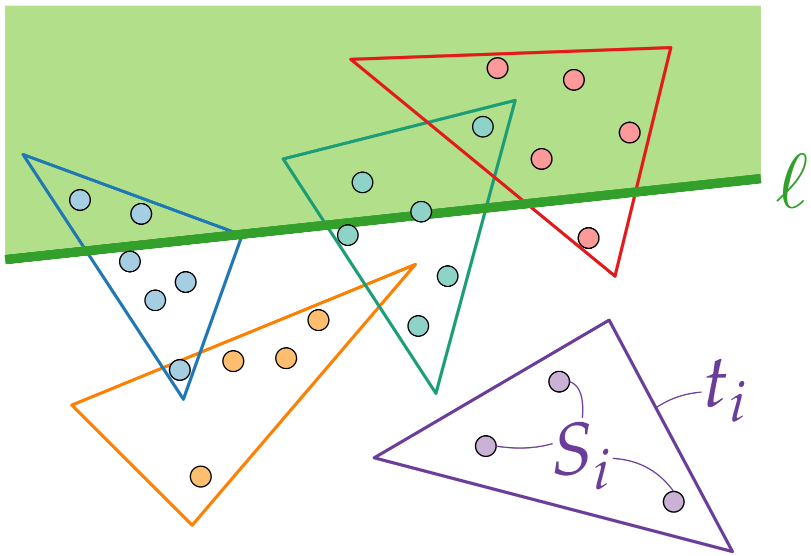

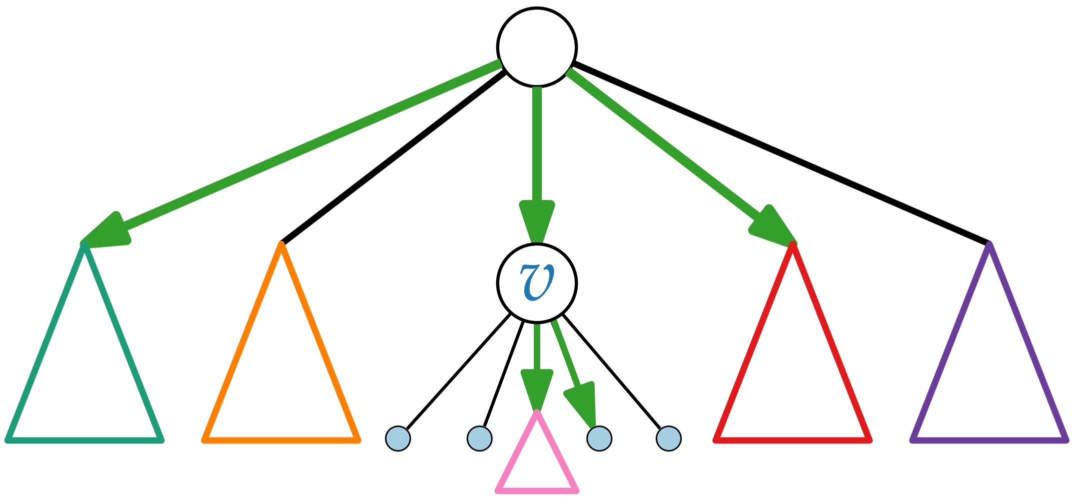

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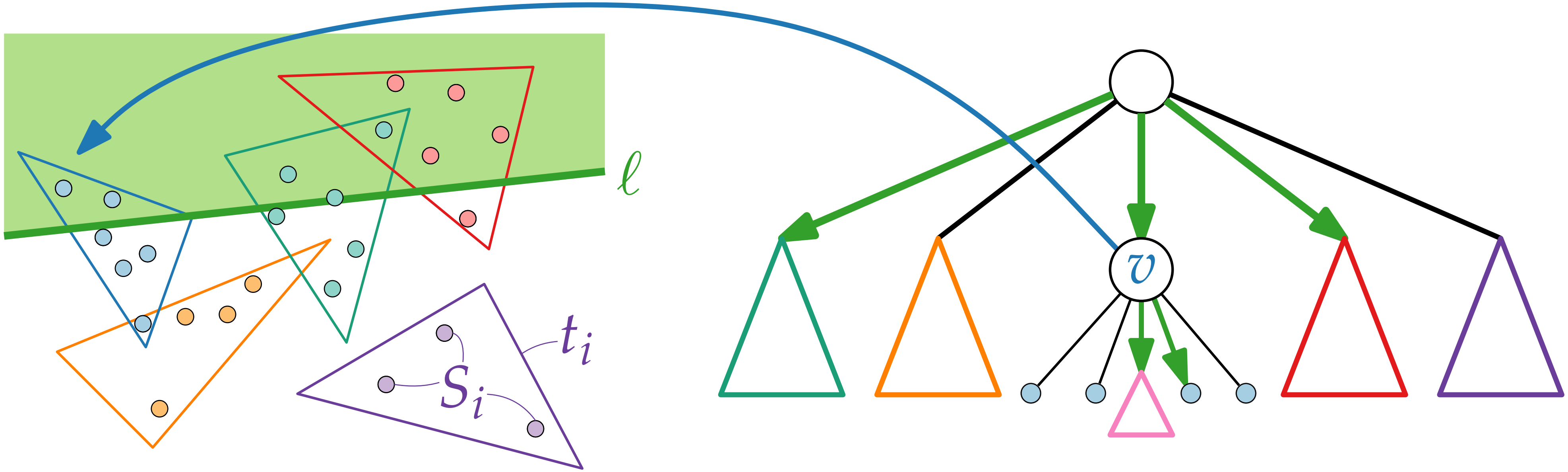

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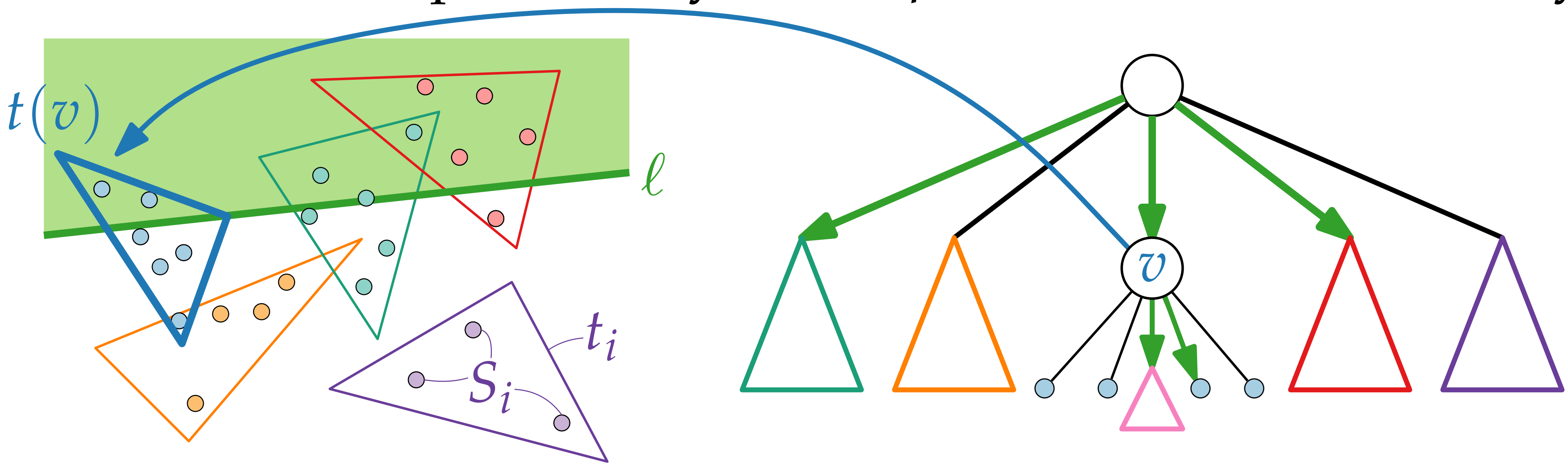

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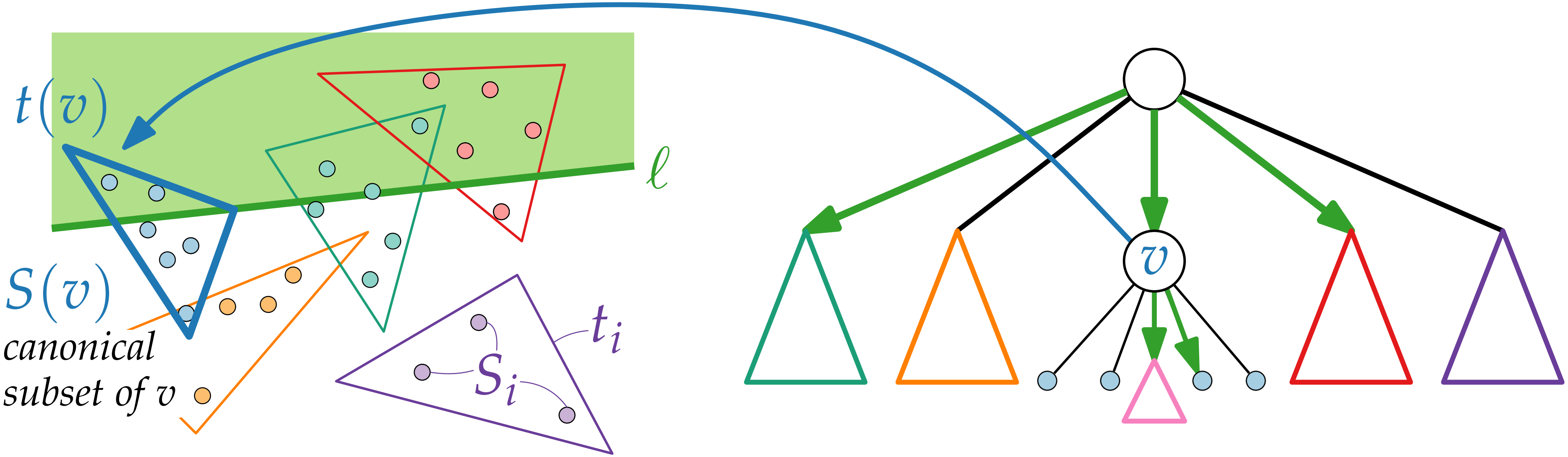

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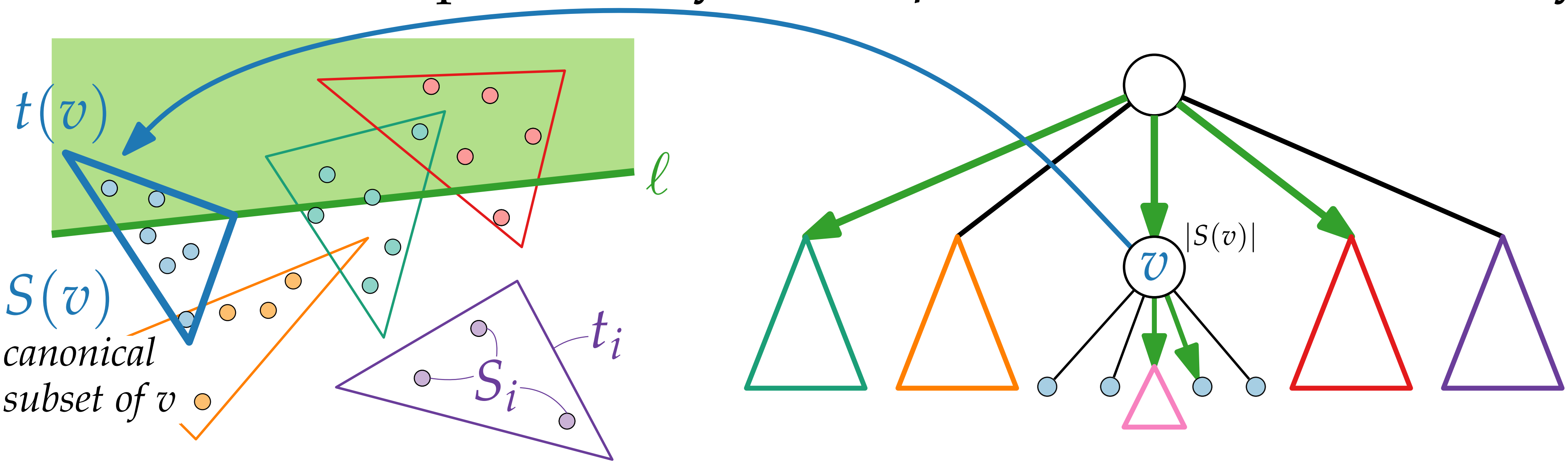

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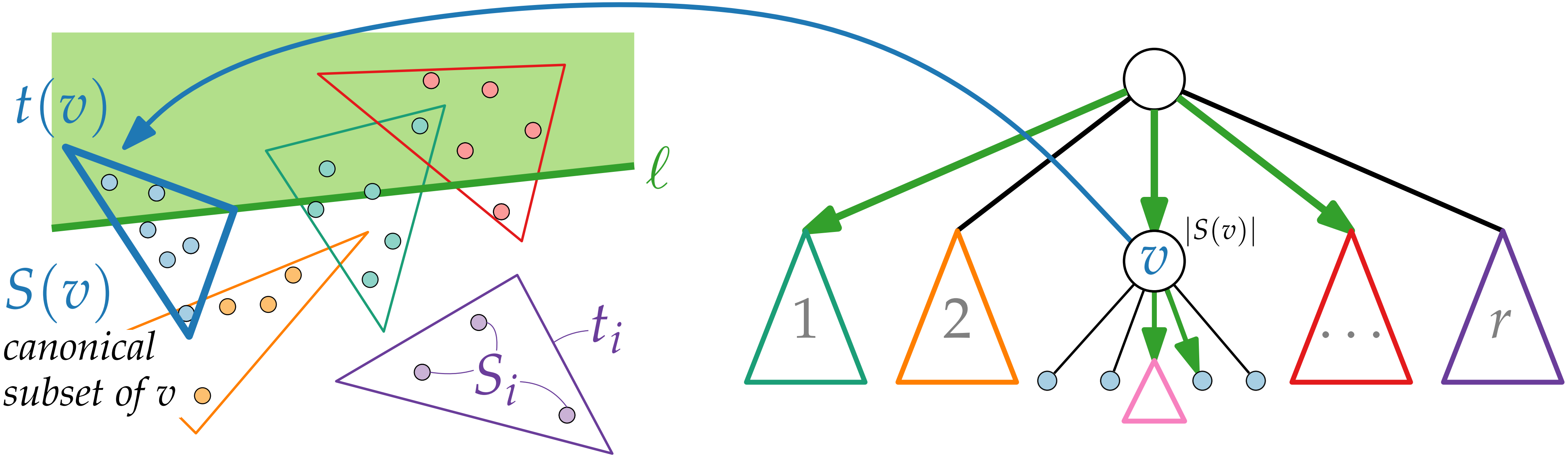

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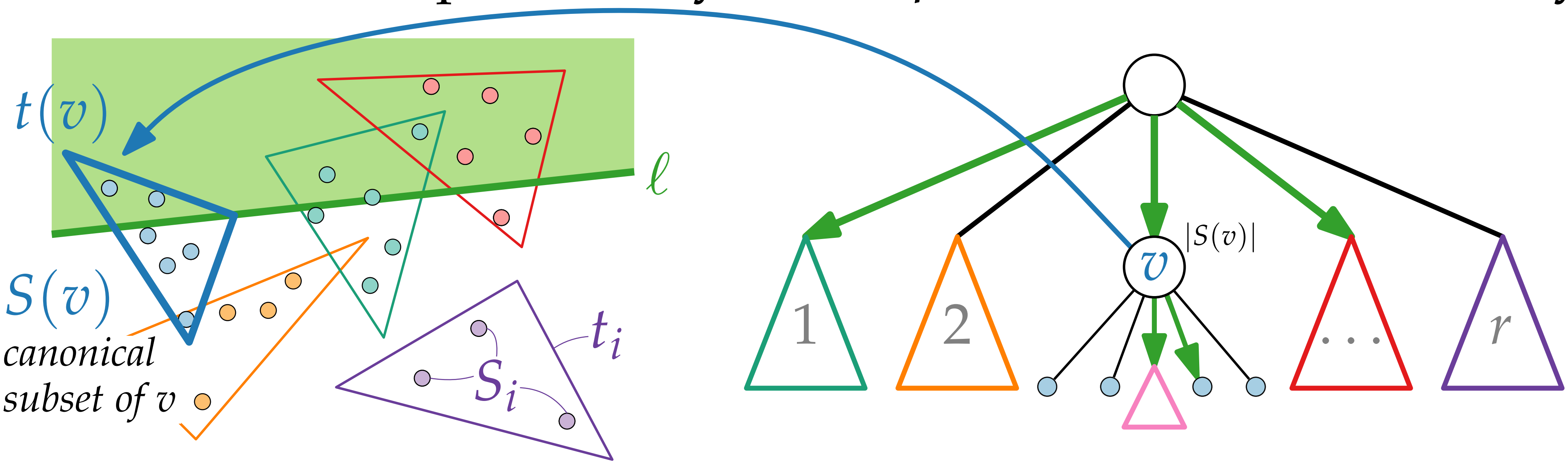

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Lemma. A partition tree for S can be constructed in $O(n^{1+\varepsilon})$ time. The tree uses $O(n)$ storage.

Generalizing to 2 Dimensions

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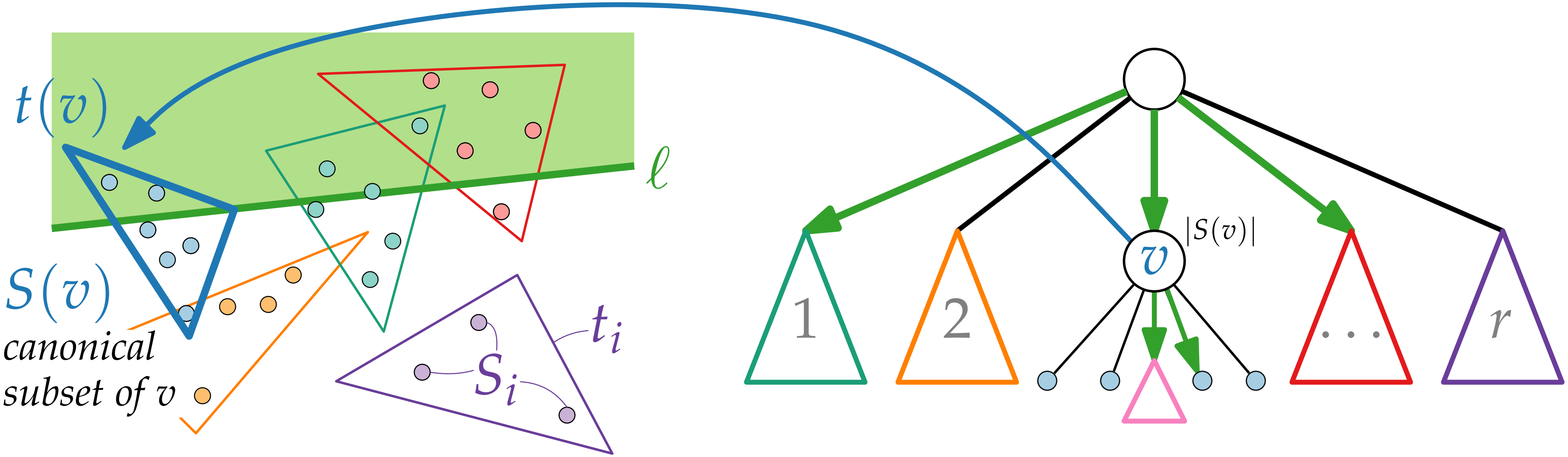

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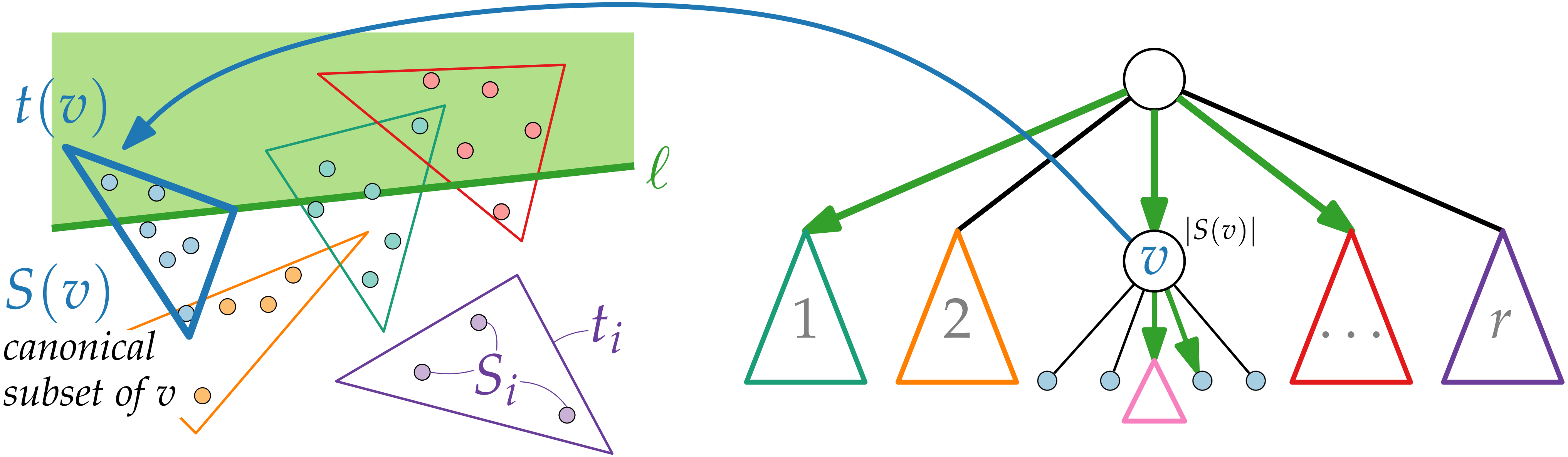

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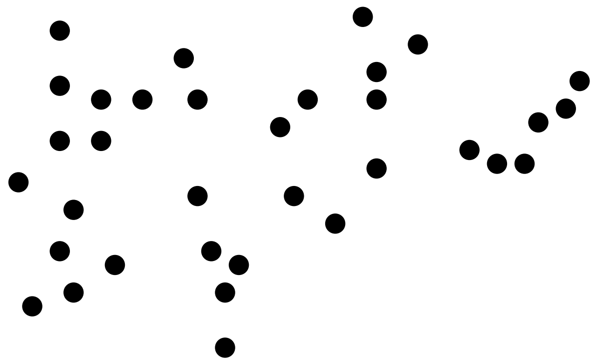
search tree with n leaves

Example for a Query

point set S



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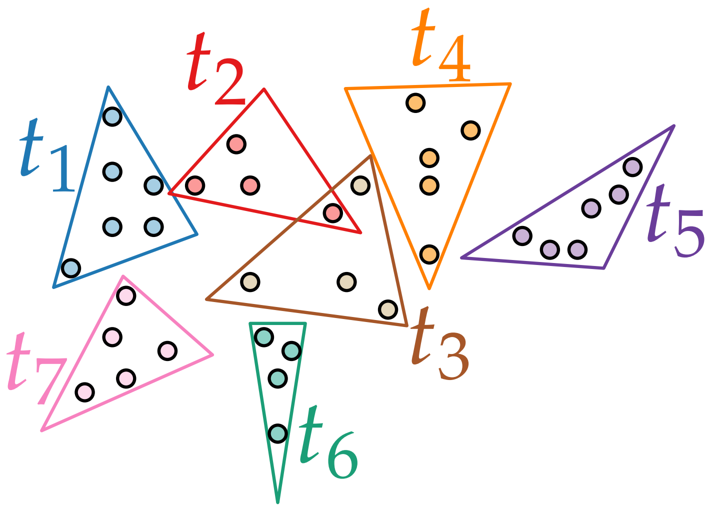
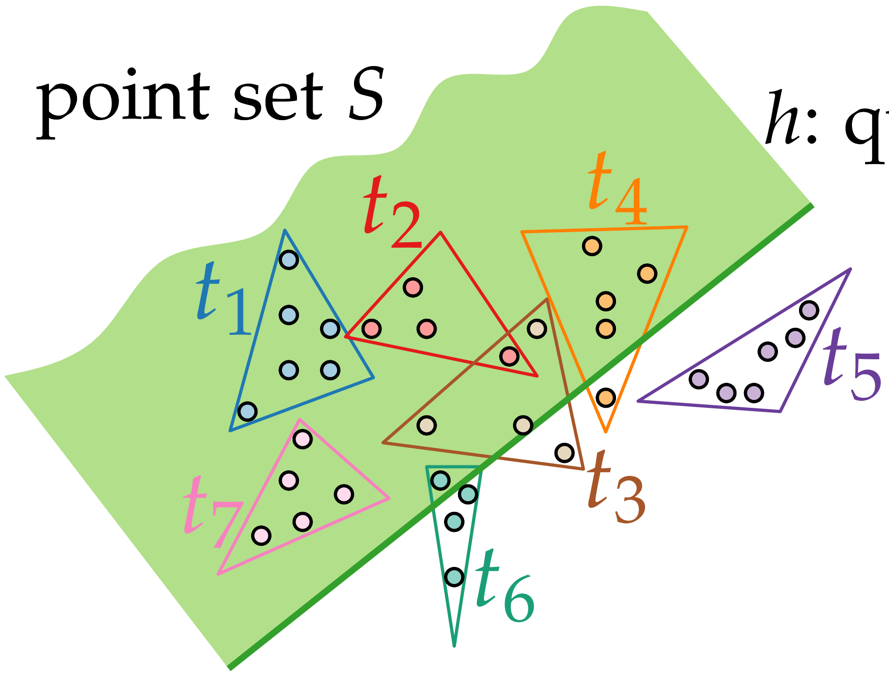


partition by triangles

Example for a Query

point set S

h: query range



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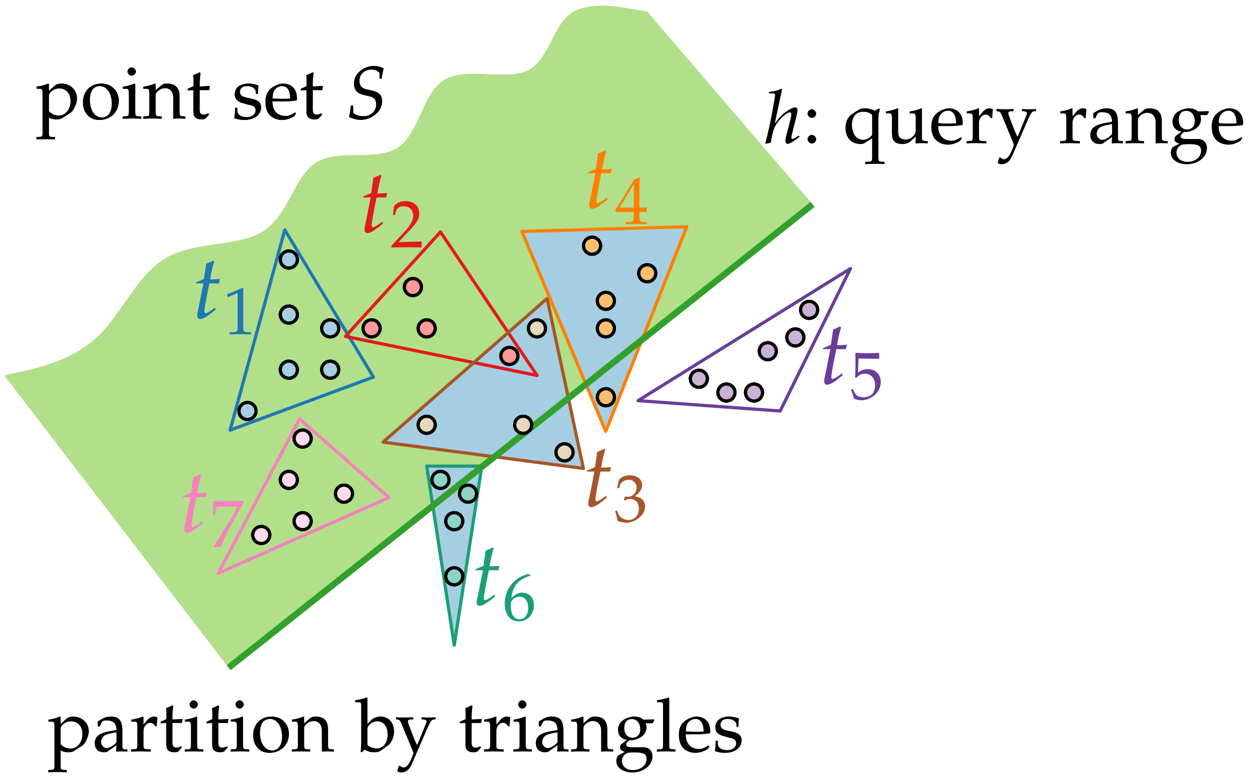

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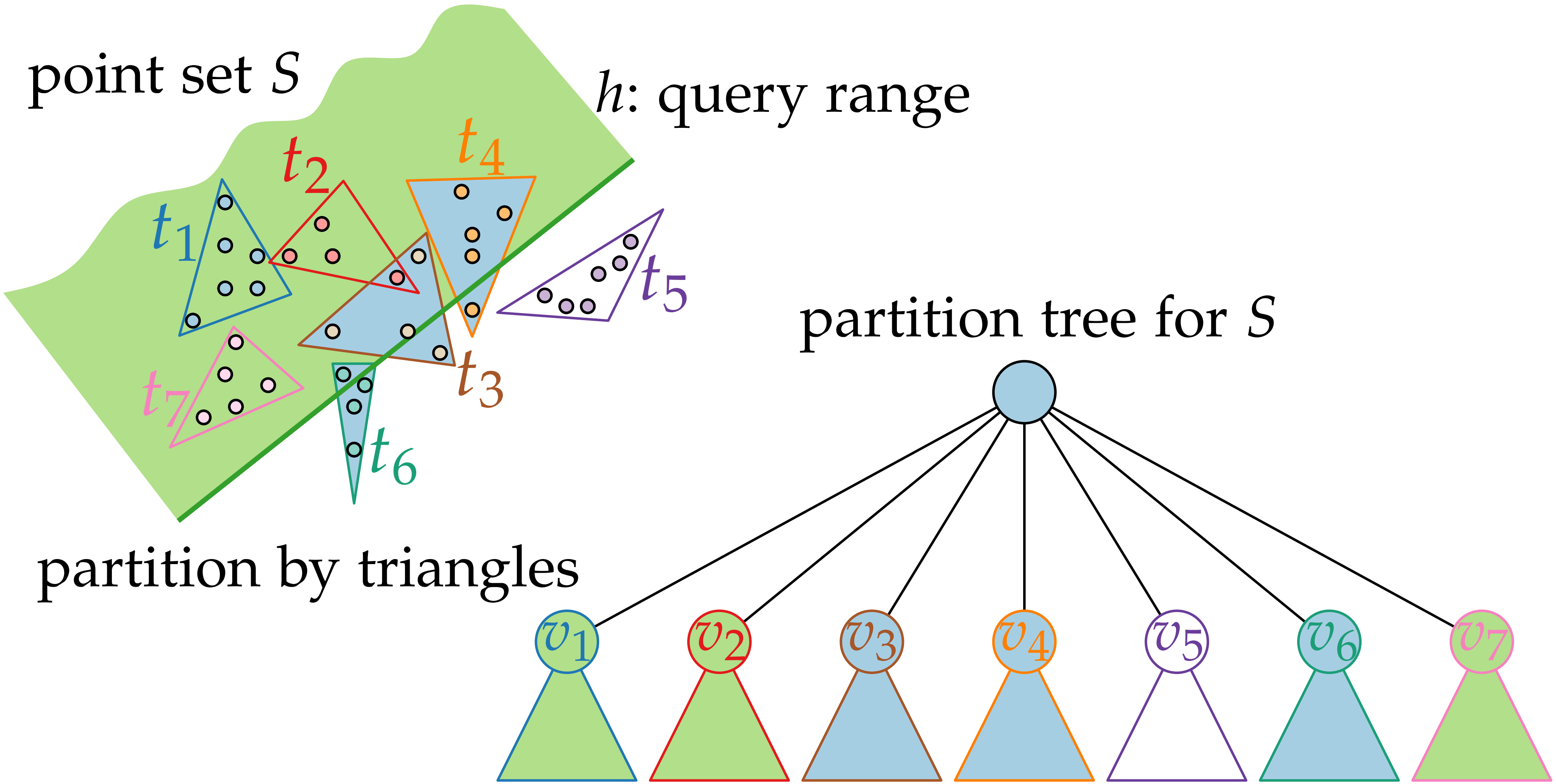

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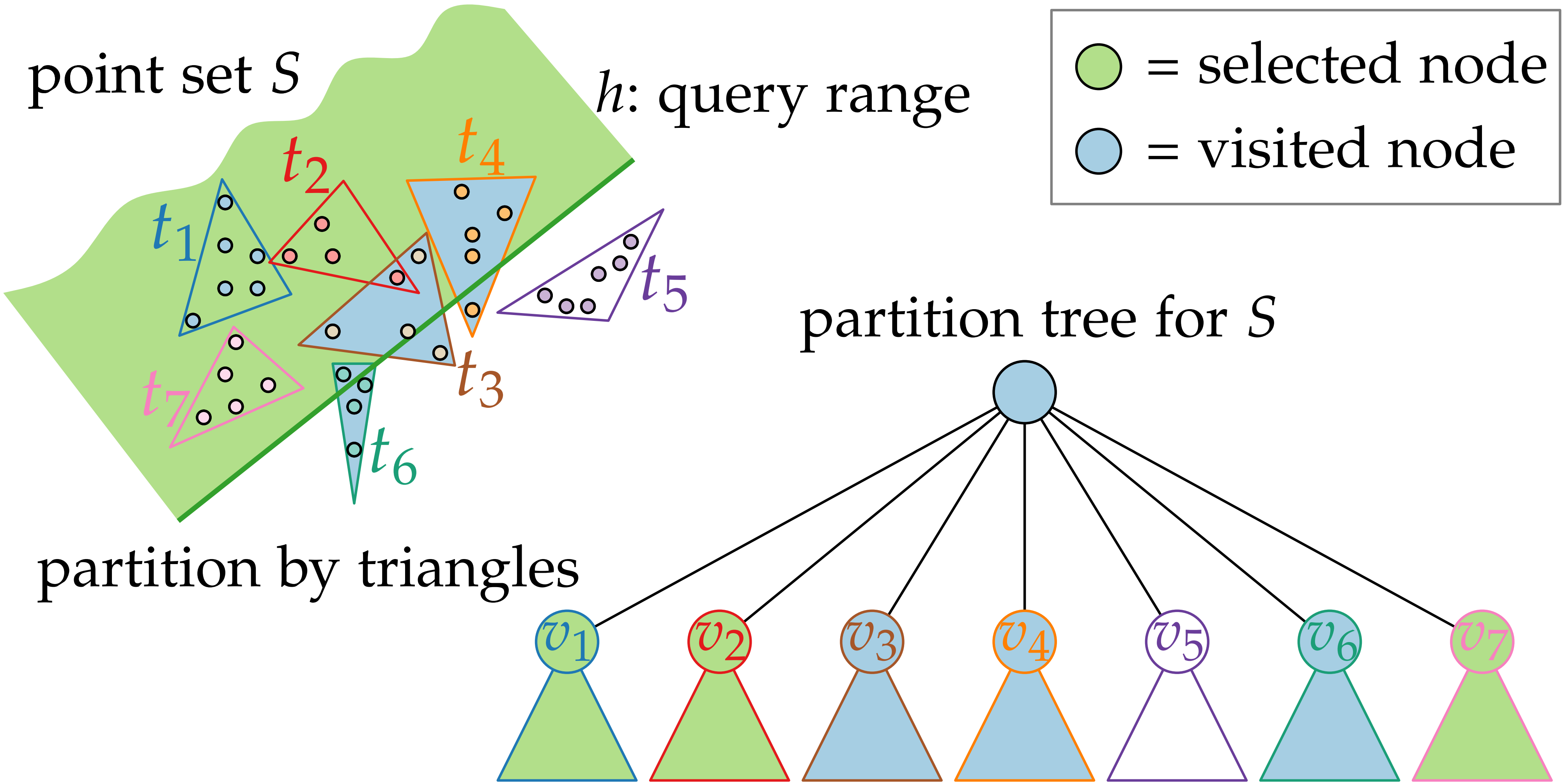

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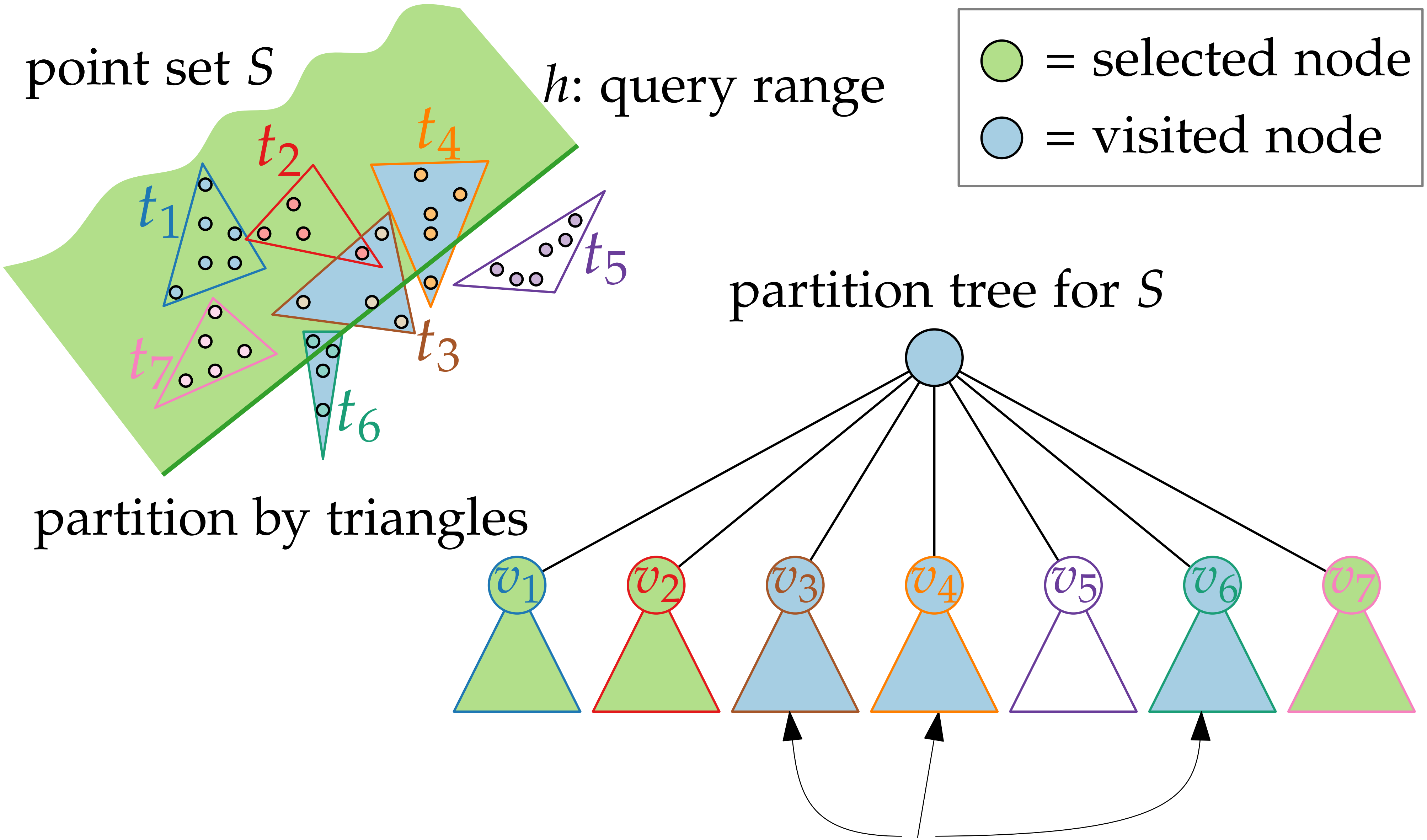

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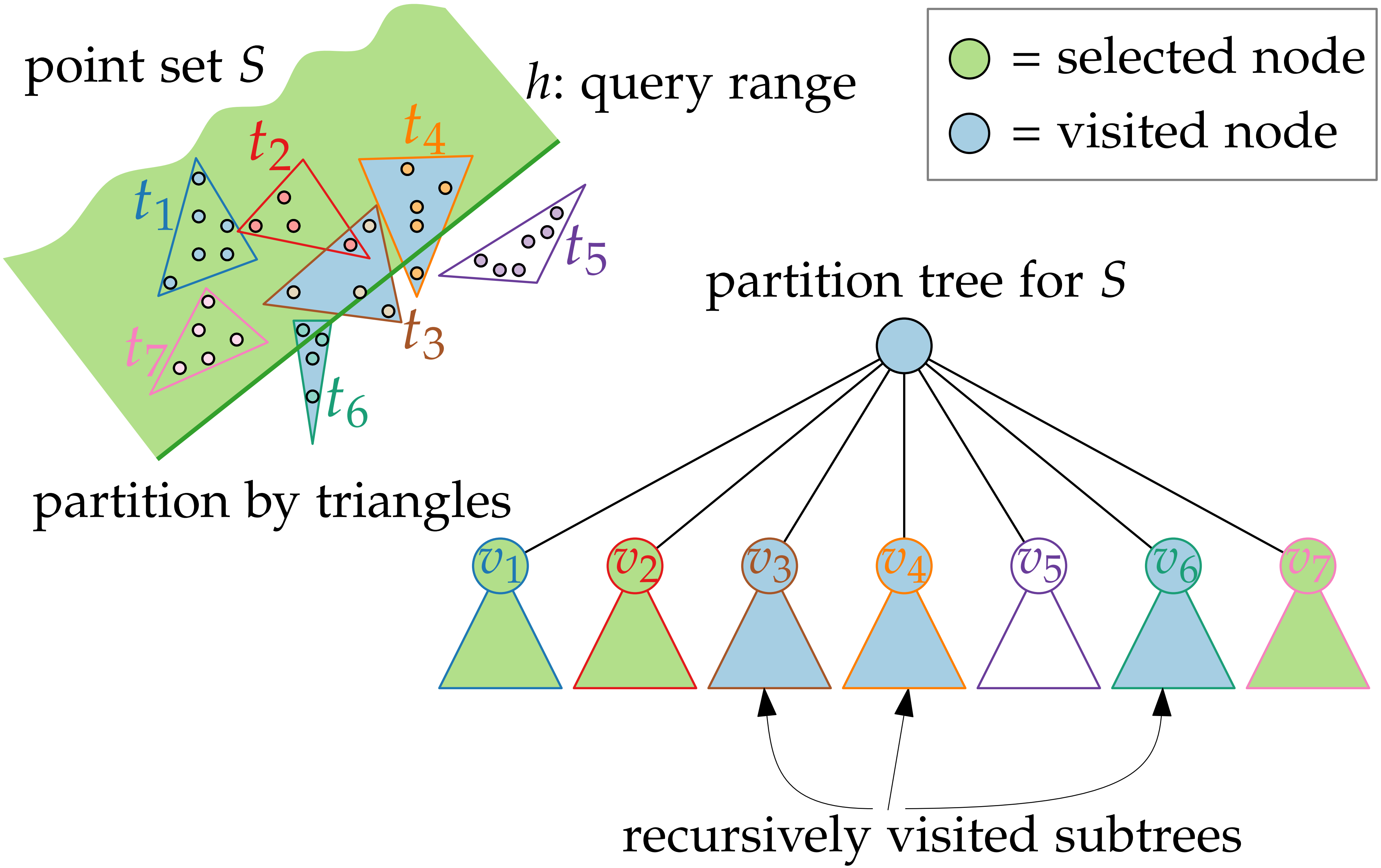
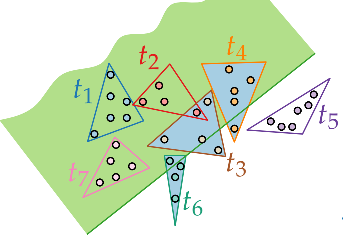
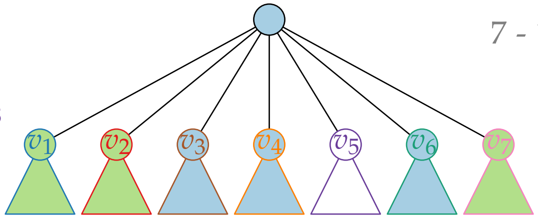


Query Algorithm

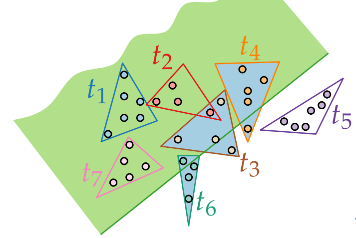


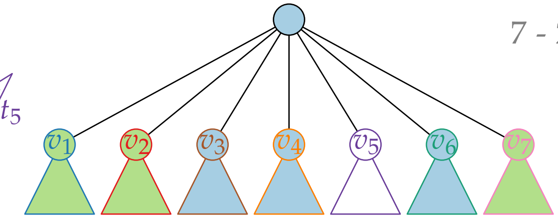


SelectInHalfplane(half-plane h, partit. tree $\mathcal{T}$ for pt set S)

$N \leftarrow \emptyset$ // set of *selected* nodes

Query Algorithm





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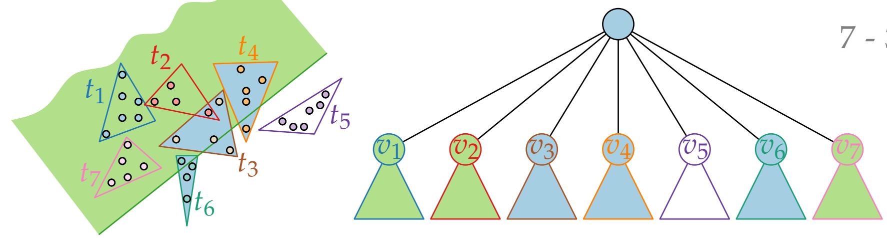

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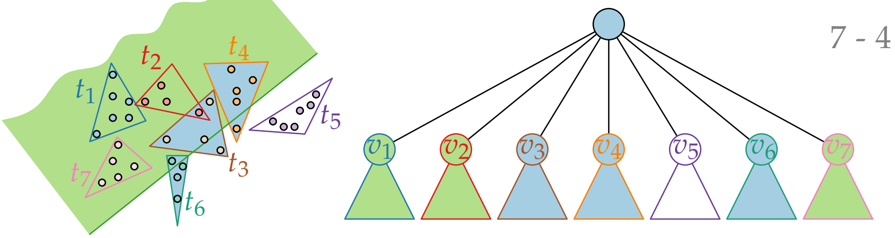

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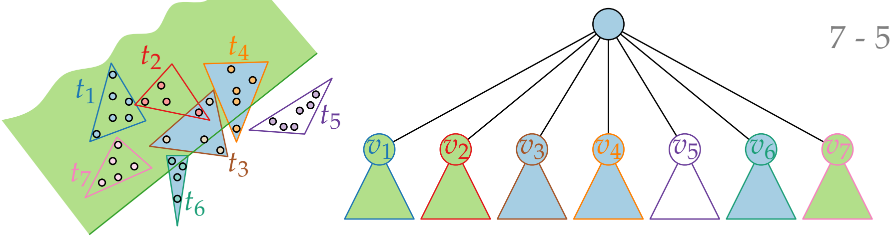

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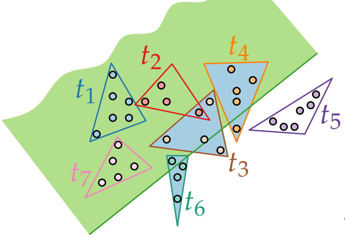

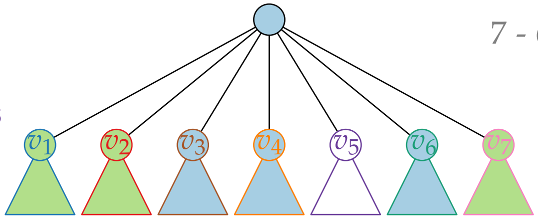

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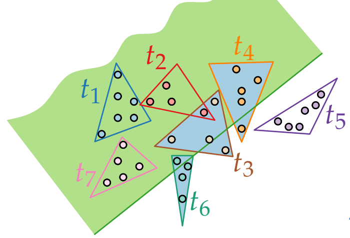

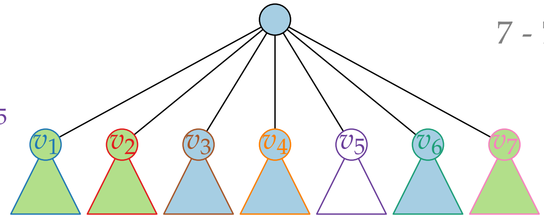

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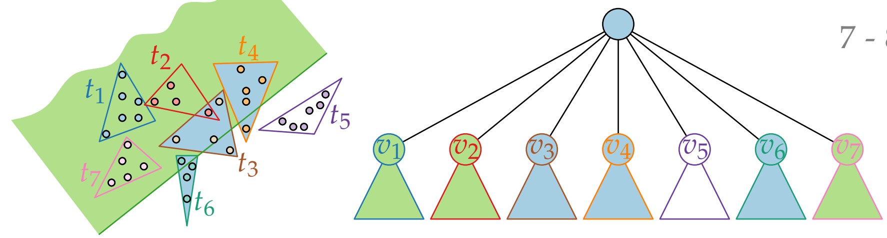

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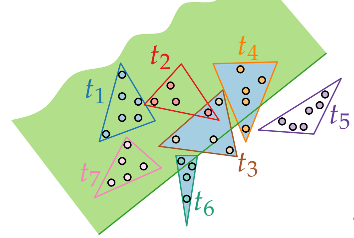

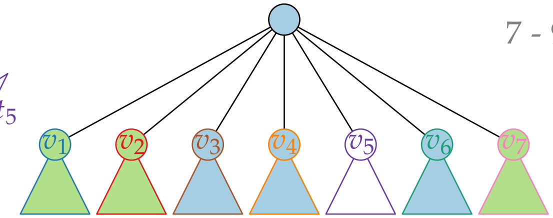

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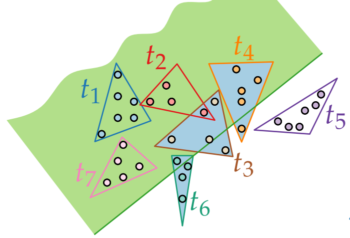
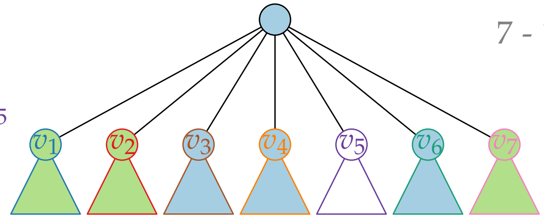
Task:

Turn this into a range *counting* query algorithm!

Query Algorithm





Count
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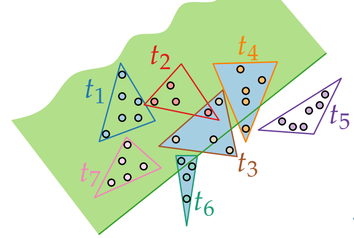

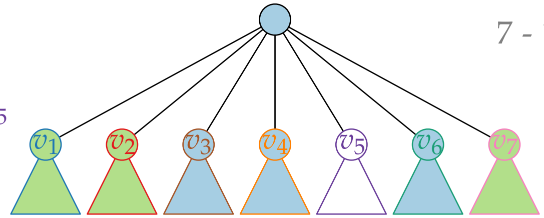

COUNT

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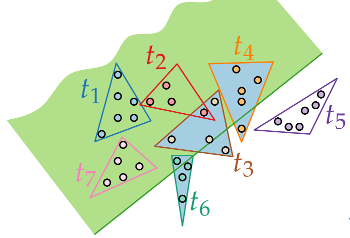

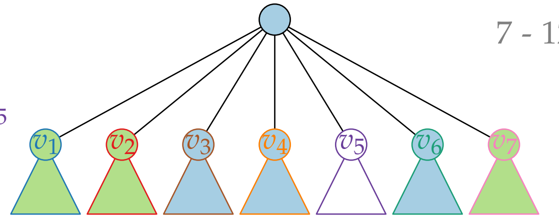

Count

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$N \leftarrow \cancel{\emptyset}$ 0 // ~~set~~ of *selected* nodes

number

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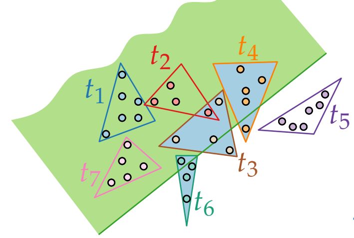

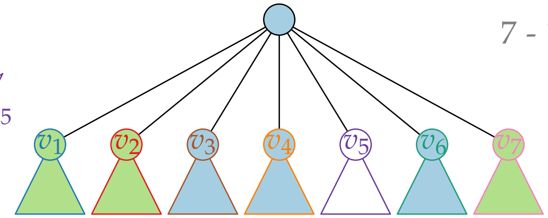

Count ~~Select~~InHalfplane(half-plane h, partit. tree $\mathcal{T}$ for pt set S)

$N \leftarrow \cancel{\emptyset}$ 0 // ~~set~~ number of *selected* nodes

if $\mathcal{T} = \{\mu\}$ **then**
- **if** point stored at μ lies in h **then**
  - $N \leftarrow \cancel{\{\mu\}}$ 1

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- **foreach** child v of the root of $\mathcal{T}$ **do**
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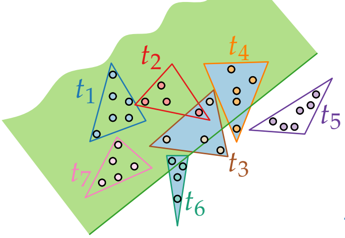

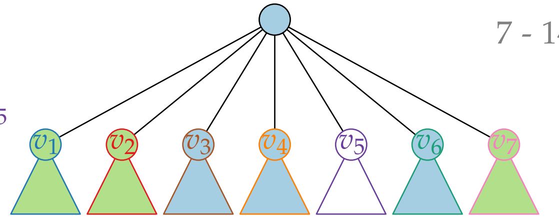

Query Algorithm

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      $N \leftarrow N \cancel{\cup \{v\}} + |S(v)|$

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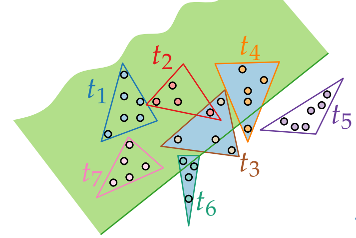
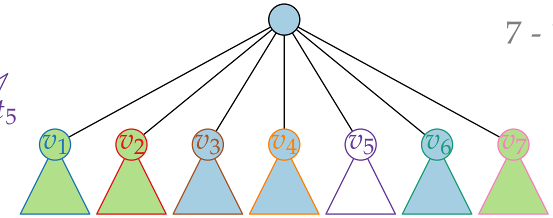
return N // with $S \cap h = \bigcup_{v \in N} S(v)$

Task:

Turn this into a range *counting* query algorithm!

Query Algorithm





COUNT
~~SELECT~~InHalfplane(half-plane h, partit. tree $\mathcal{T}$ for pt set S)

$N \leftarrow \cancel{\emptyset}$ 0   // ~~set~~ of *selected* nodes
number

if $\mathcal{T} = \{\mu\}$ **then**
  if point stored at μ lies in h **then**
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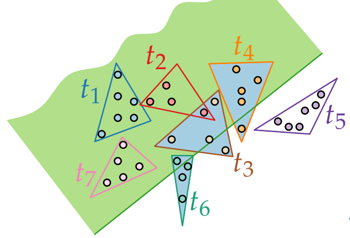

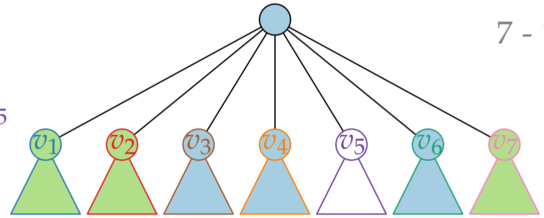

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Lemma. For any $\varepsilon > 0$, there is a partition tree $\mathcal{T}$ for S s.t.:
for a query half-plane h,
SelectInHalfplane selects in $O(n^{1/2+\varepsilon})$ time
a set N of $O(n^{1/2+\varepsilon})$ nodes of $\mathcal{T}$
with the property that $h \cap S = \bigcup_{\nu \in N} S(\nu)$.

Corollary. Half-plane range counting queries can be answered in $O(n^{1/2+\varepsilon})$ time
using $O(n)$ space and $O(n^{1+\varepsilon})$ prep.

Back to *Triangular* Range Queries

Any ideas?

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Any ideas? Just use SelectInHalfplane!

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Theorem. Given a set S of n pts in the plane, for any $\varepsilon > 0$, a triangular range-counting query can be answered in $O(n^{1/2+\varepsilon})$ time using a partition tree.

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Query time $O(\log^3 n)$, prep. & storage $O(n^{2+\varepsilon})$.

Multi-Level Partition Trees

Idea. Store with each internal node not just a number,

Multi-Level Partition Trees

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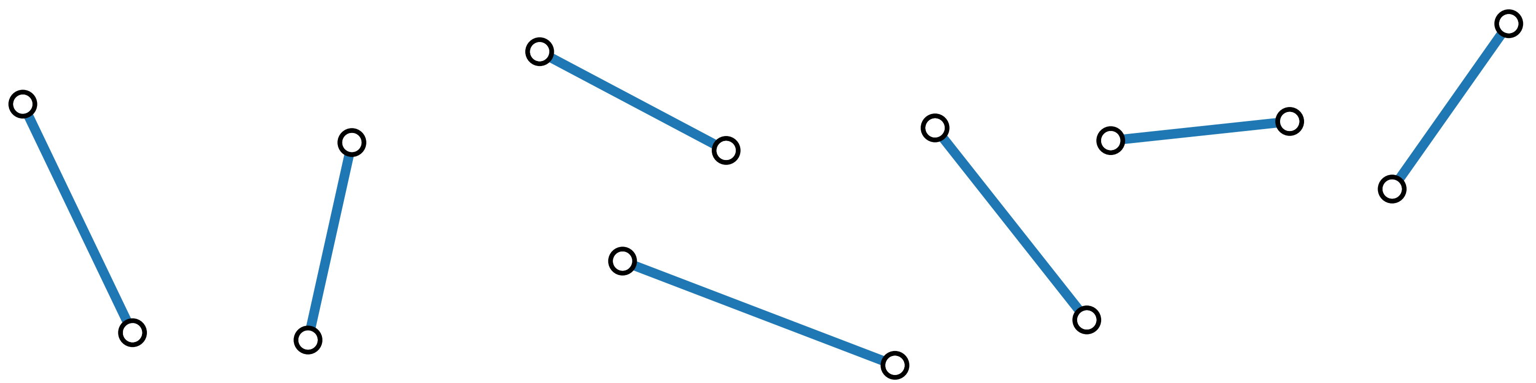

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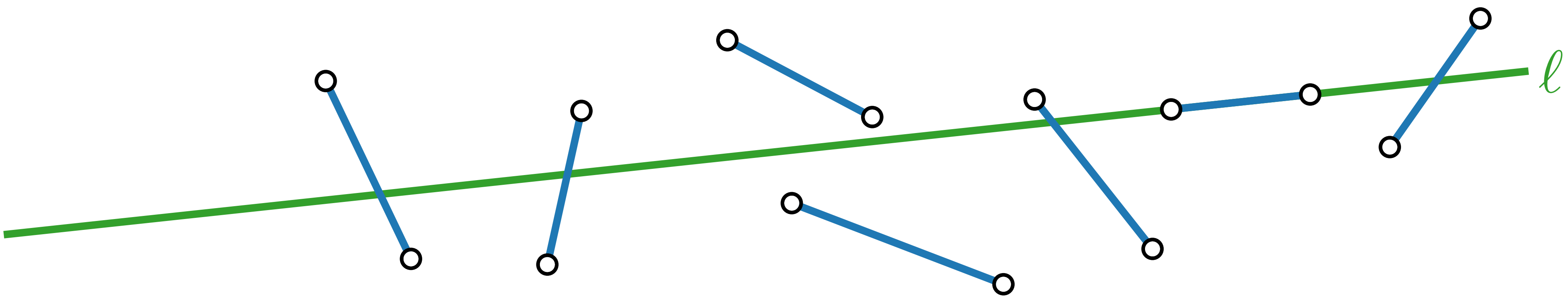

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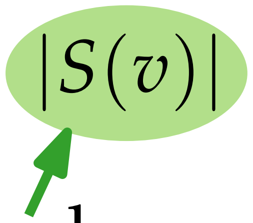

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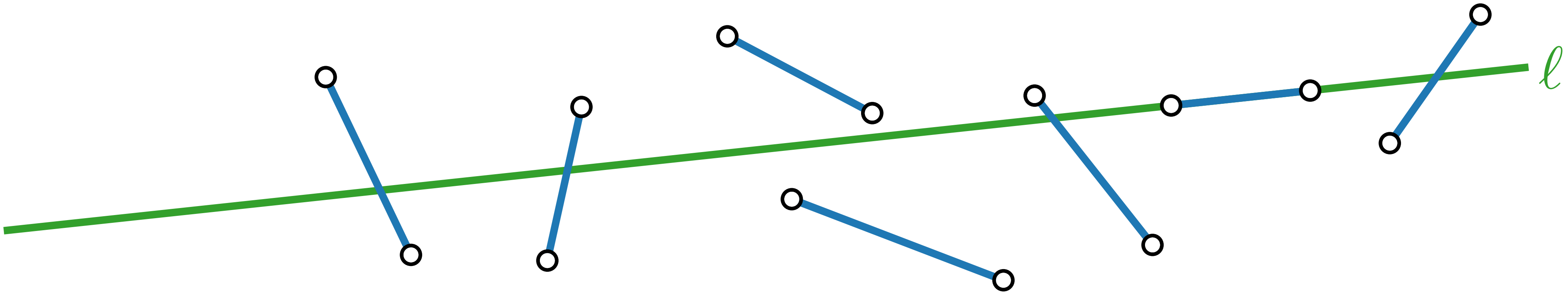

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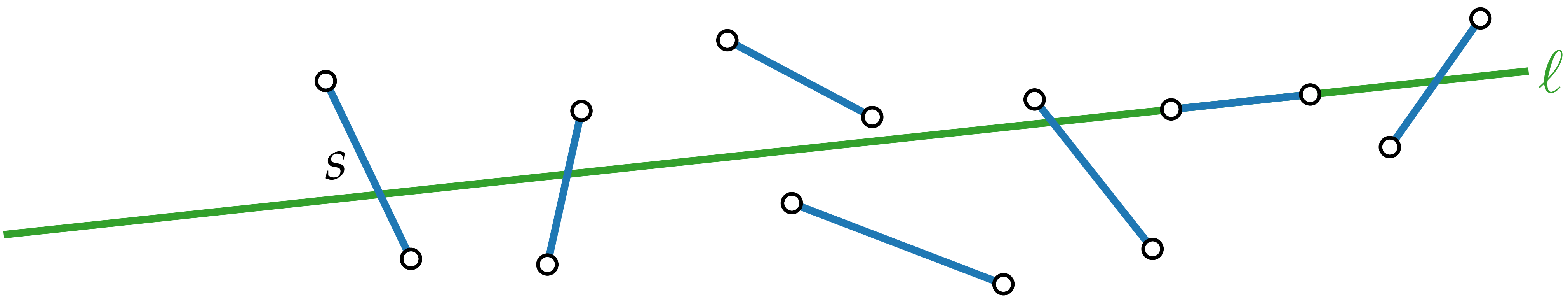

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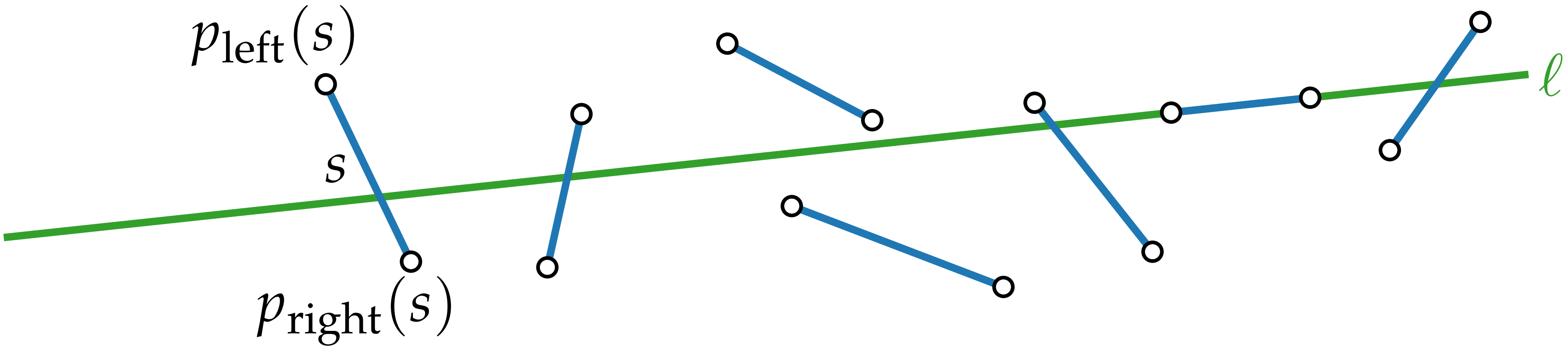

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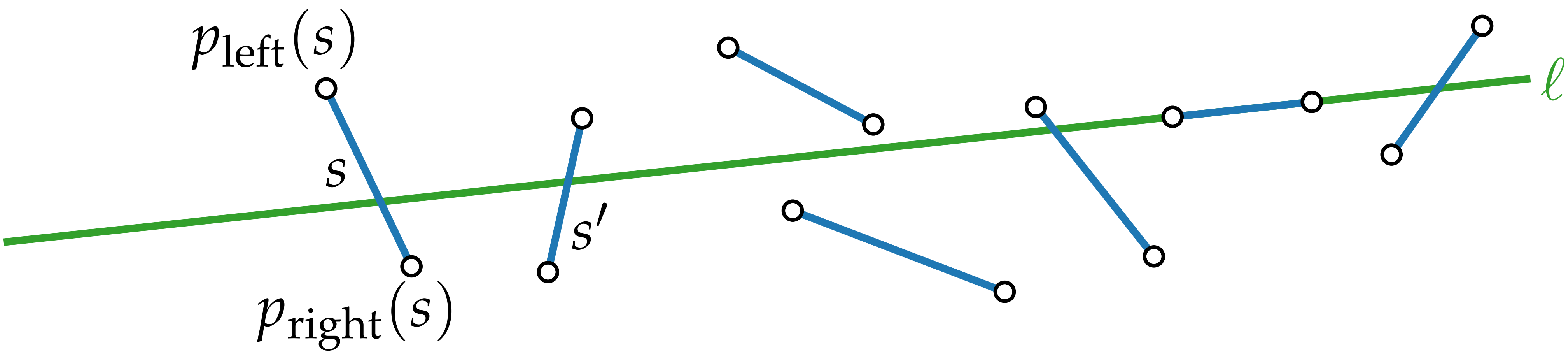

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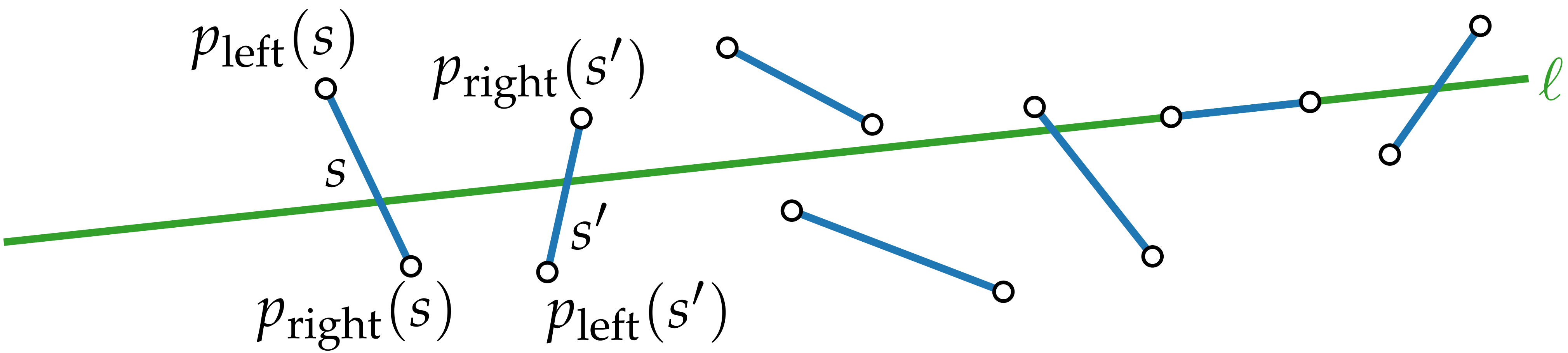

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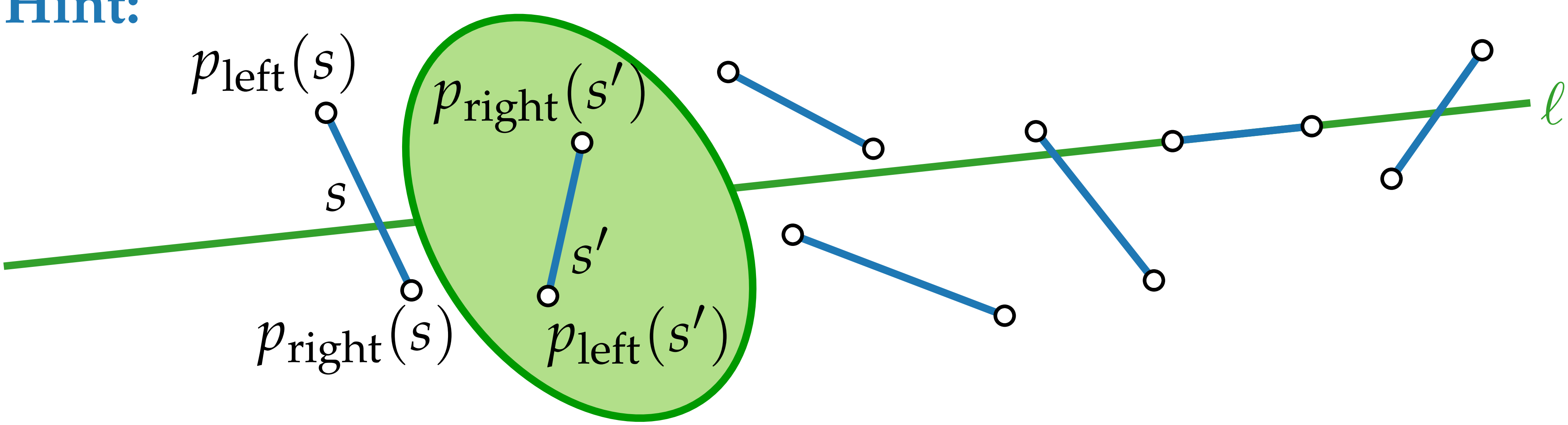

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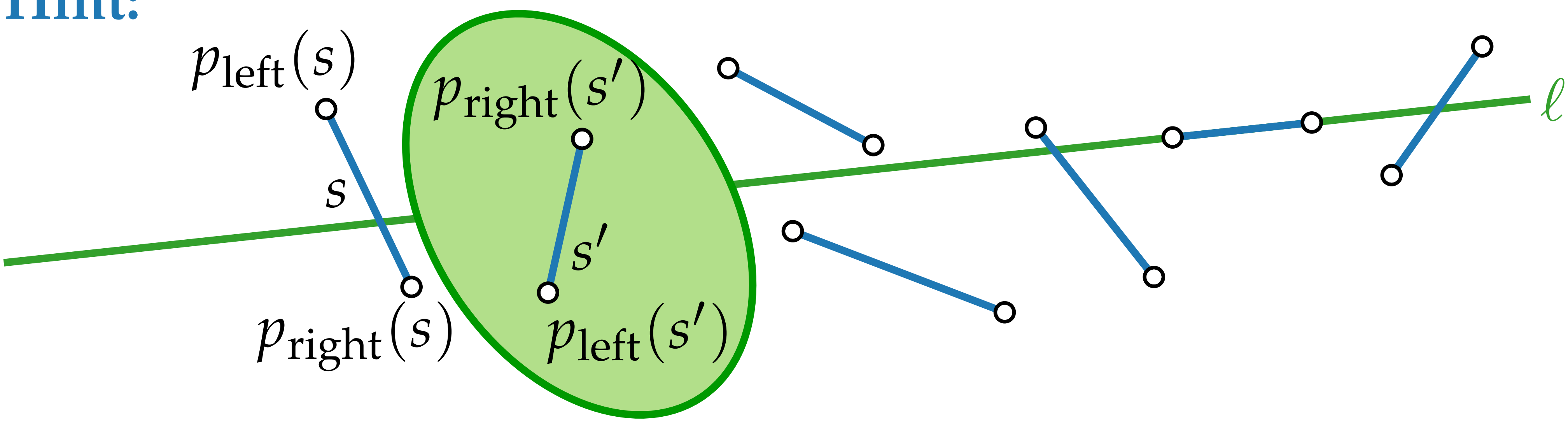

[3 min]

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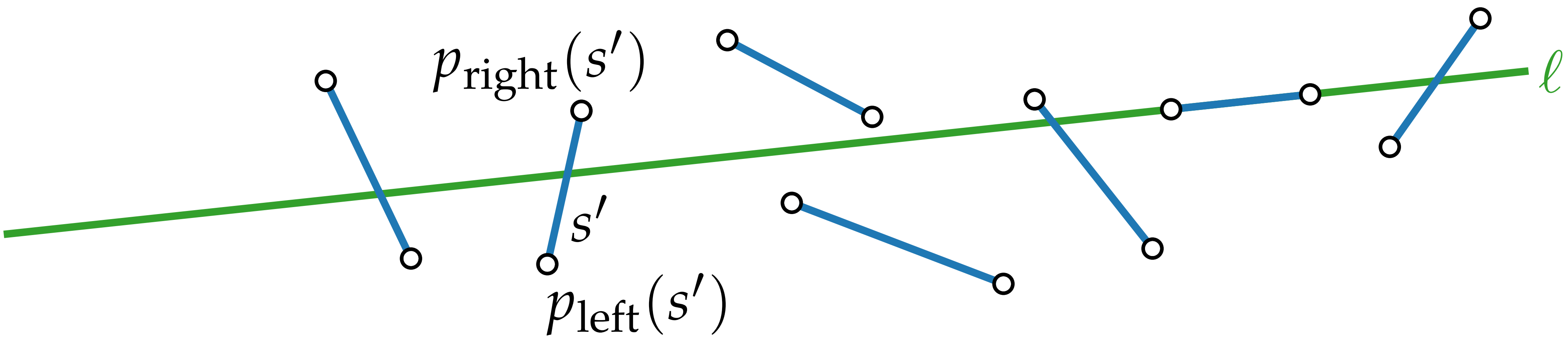

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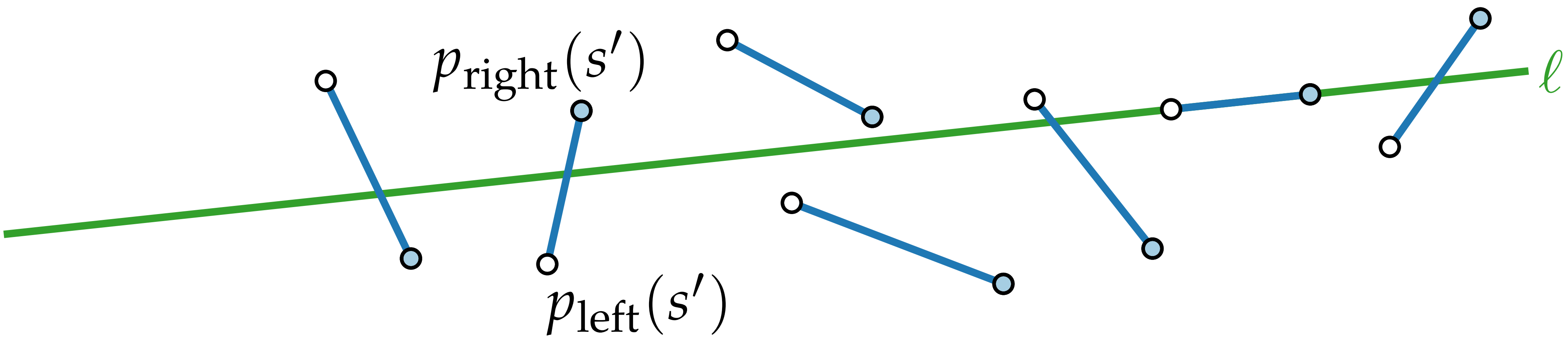

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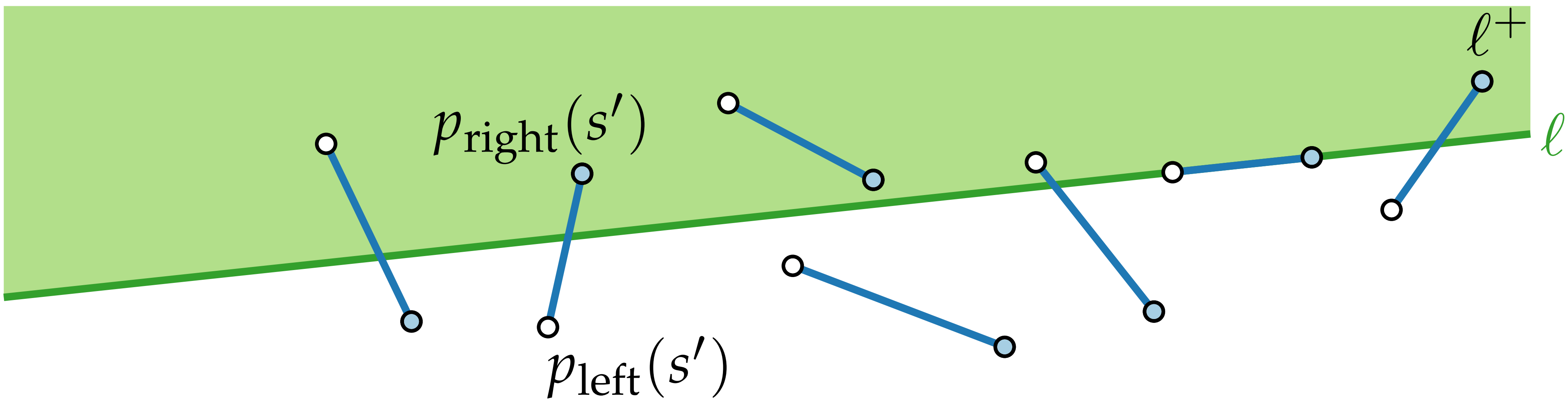

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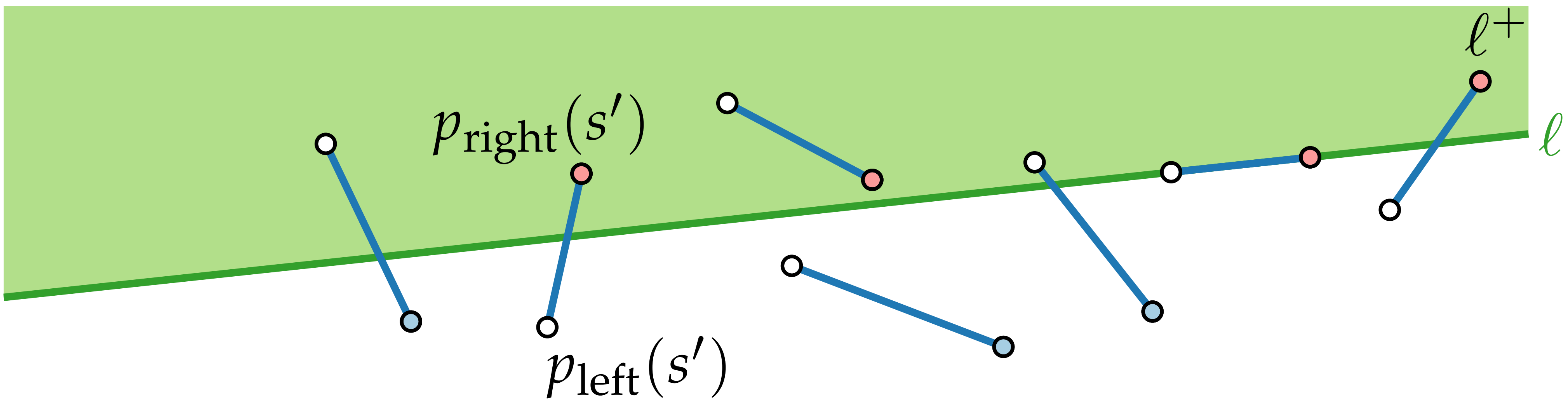

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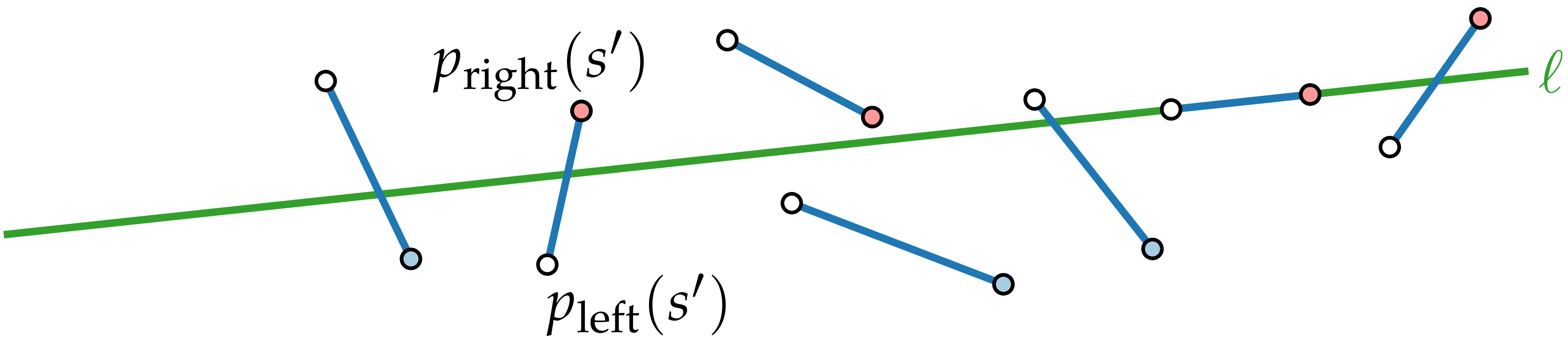

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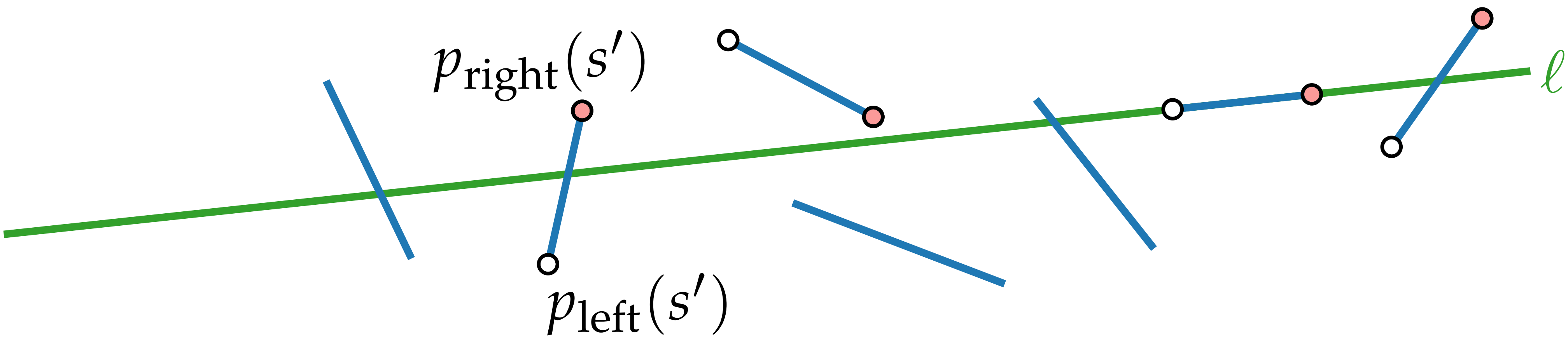

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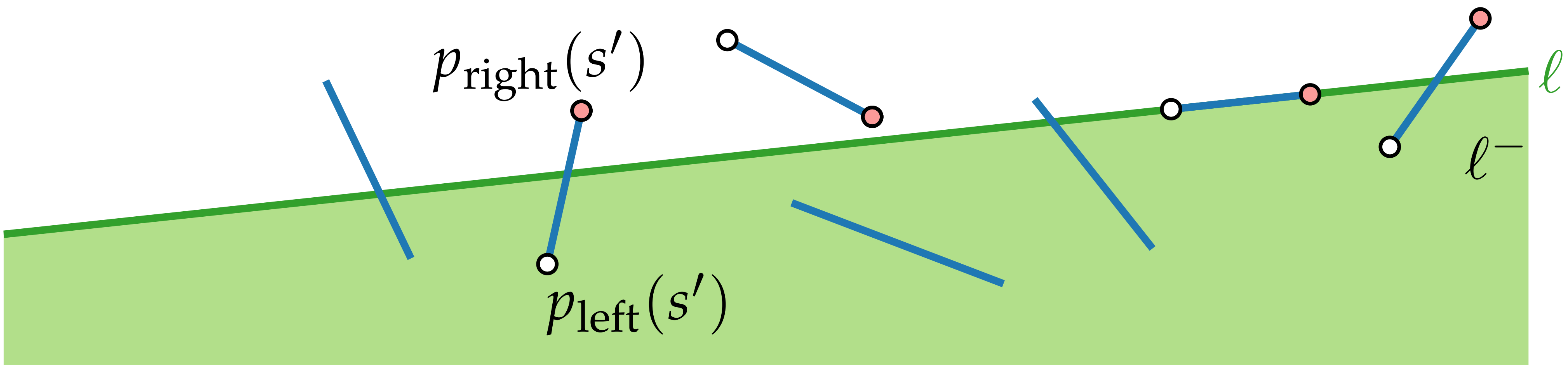

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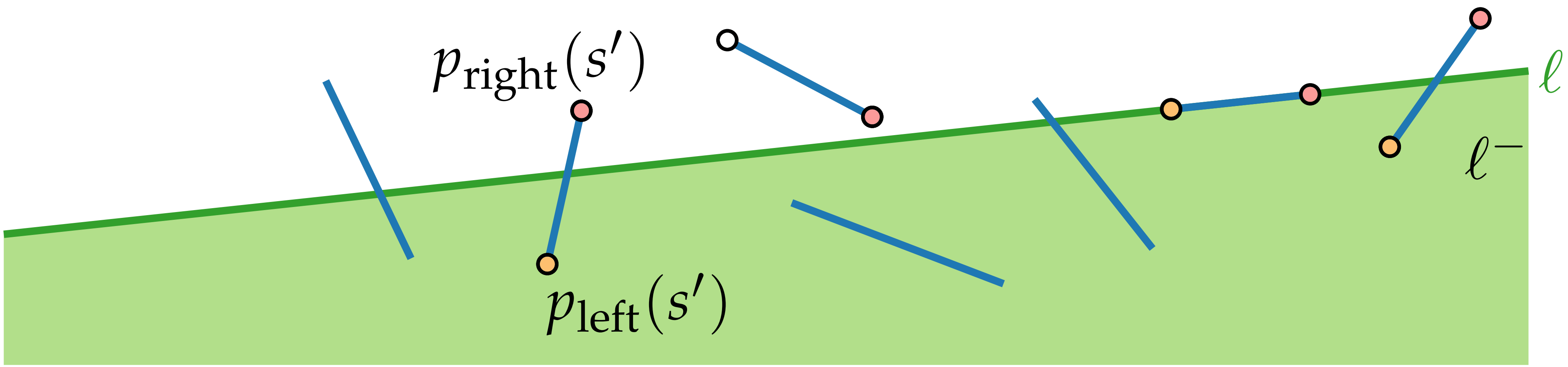

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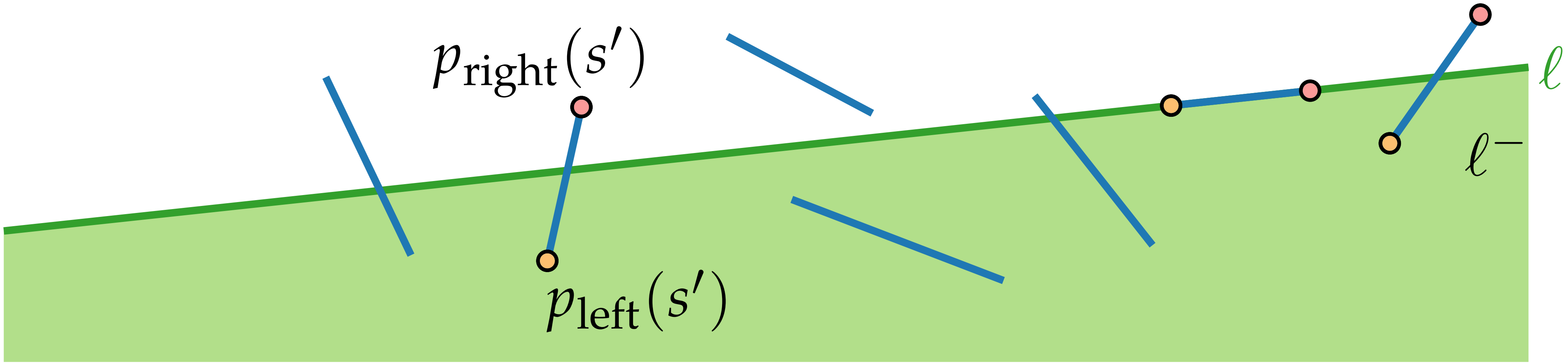

Query Algorithm

SelectIntSegments(line ℓ, two-level partition tree $\mathcal{T}$ for S)

– first-level tree stores $P_{\text{right}}(S)$
– second-level trees store subsets of $P_{\text{left}}(S)$

$N \leftarrow \emptyset$
if $\mathcal{T} = \{\mu\}$ **then**
  if segment stored in μ intersects ℓ **then** $N \leftarrow \{\mu\}$
else
  foreach child ν of $\mathcal{T}$'s root **do**
    if $t(\nu) \subset \ell^+$ **then**
      $N \leftarrow N \cup \text{SelectInHalfplane}(\ell^-, \mathcal{T}_\nu^{\text{assoc}})$
    else
      if $t(\nu) \cap \ell \neq \emptyset$ **then**
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return N

Query Algorithm

For $S' \subseteq S$, let
$P_{\text{right}}^{\text{left}}(S') = \{p_{\text{right}}^{\text{left}}(s) \mid s \in S'\}$

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```
N ← ∅
if 𝒯 = {μ} then
    if segment stored in μ intersects ℓ then N ← {μ}
else
    foreach child v of 𝒯's root do
        if t(v) ⊂ ℓ⁺ then
            N ← N ∪ SelectInHalfplane(ℓ⁻, 𝒯_v^assoc)
        else
            if t(v) ∩ ℓ ≠ ∅ then
                N ← N ∪ SelectIntSegments(ℓ, 𝒯_v)
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```

stores $P_{\text{left}}(S_{\text{seg}}(v))$, where
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!!! $\bigcup_{\nu \in N} S(\nu) = \{s \in S \mid p_{\text{right}}(s) \text{ above } \ell \text{ and } p_{\text{left}}(s) \text{ below } \ell\}$.

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δ-level objects

Corollary. Let S be a set of n ~~segments~~ in the plane. We can count the number of ~~segments~~ in S in a δ-level query ~~intersected by a query line~~ in $O(n^{1/2+\varepsilon})$ time using $O(n \log n)$ space and $O(n^{1+\varepsilon})$ prep.

Results

Lemma. A 2-level partition tree for line-intersection queries among a set of n segments uses $O(n \log n)$ storage.

Lemma. Let S be a set of n segments in the plane. For any $\varepsilon > 0$, there is a 2-level partition tree $\mathcal{T}$ for S s.t.

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δ-level objects

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query
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