Julius-Maximilians-
UNIVERSITÄT WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and Knowledge-Based Systems

Institute for Informatics

Computational Geometry

Voronoi Diagrams
or
The Post-Office Problem

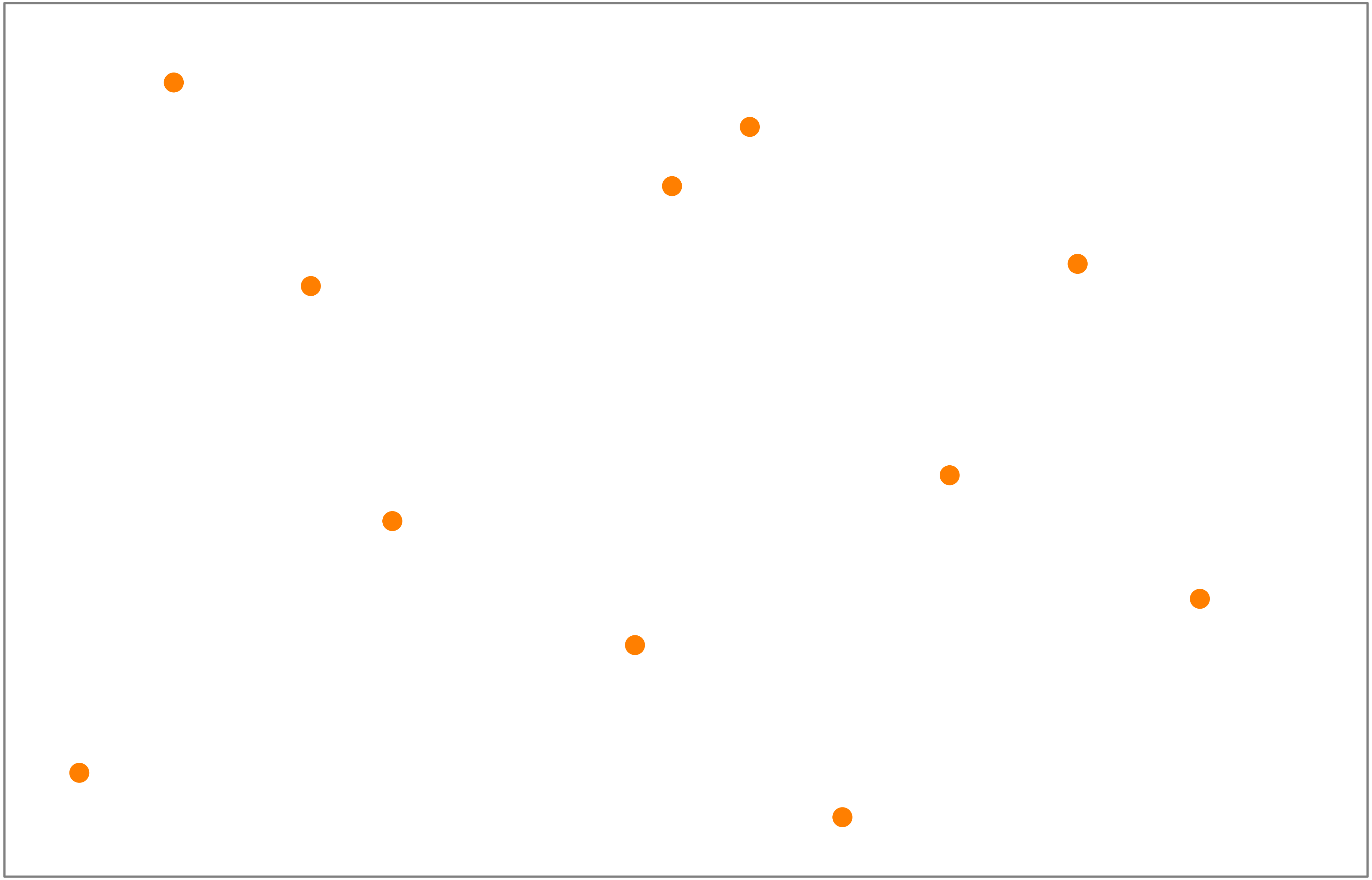
Lecture #7

[Comp. Geom A&A : Chapter 7]

Steven Chaplick

Winter Semester 2019/20

The Post-Office Problem



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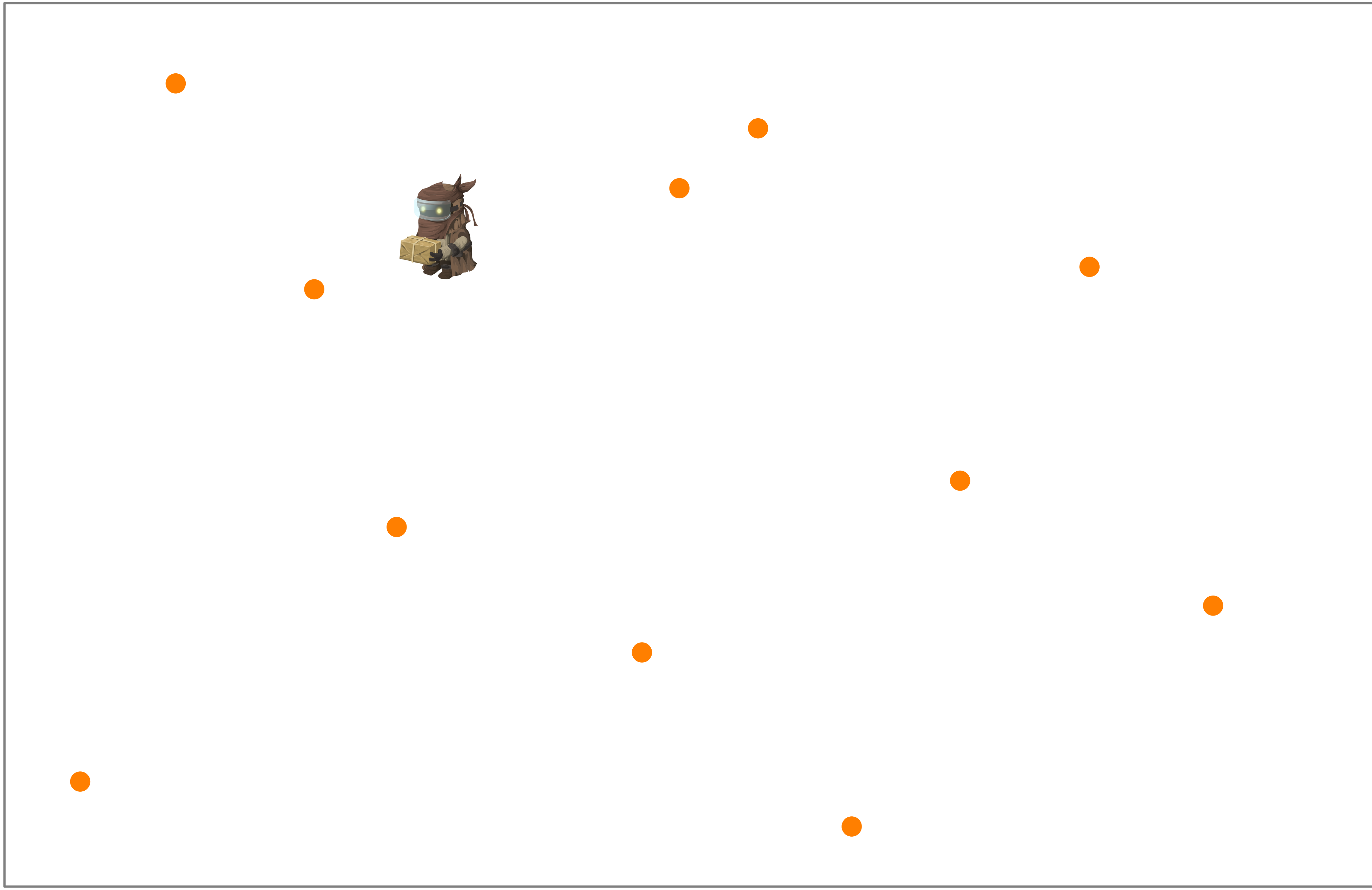

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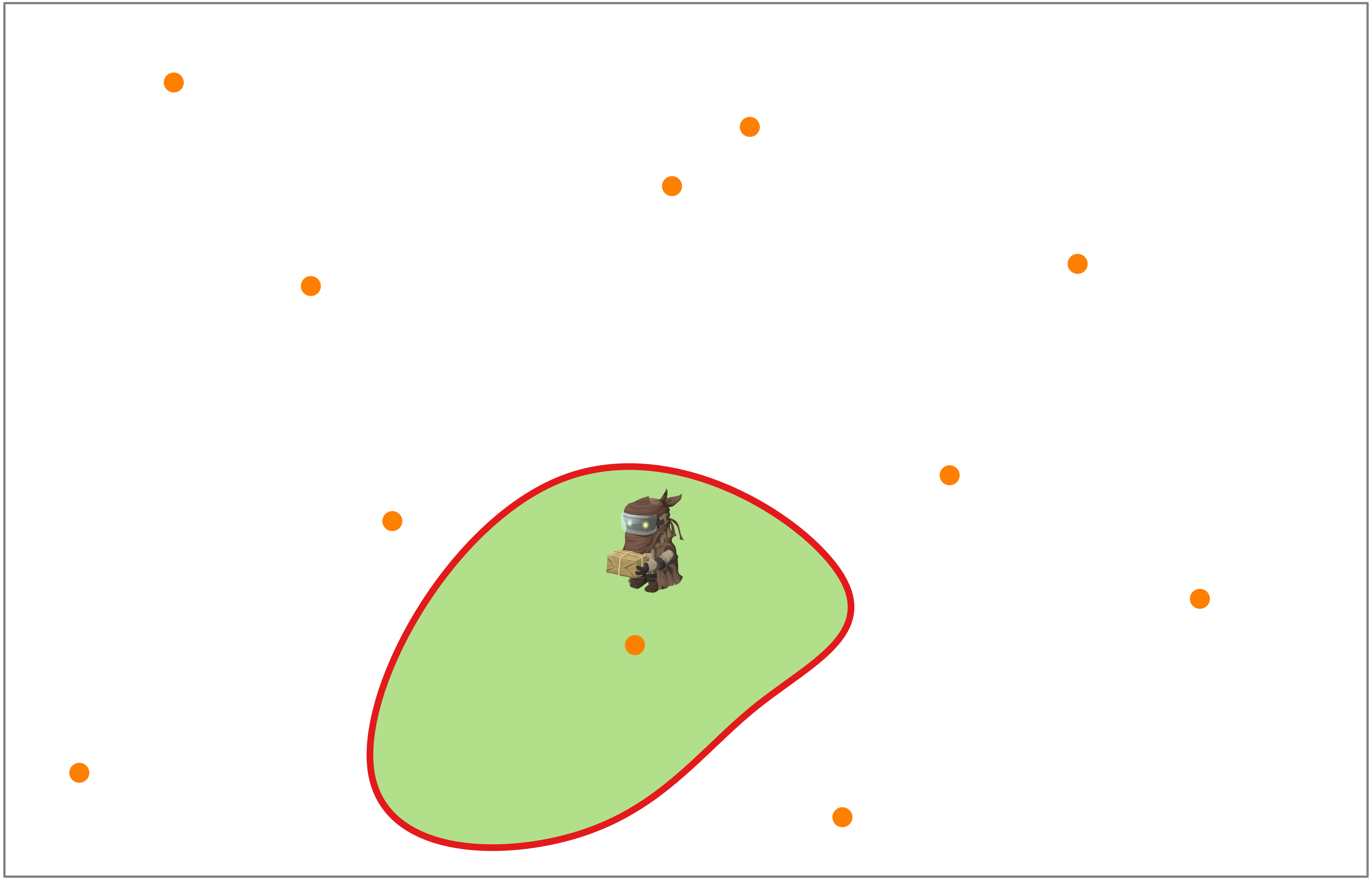

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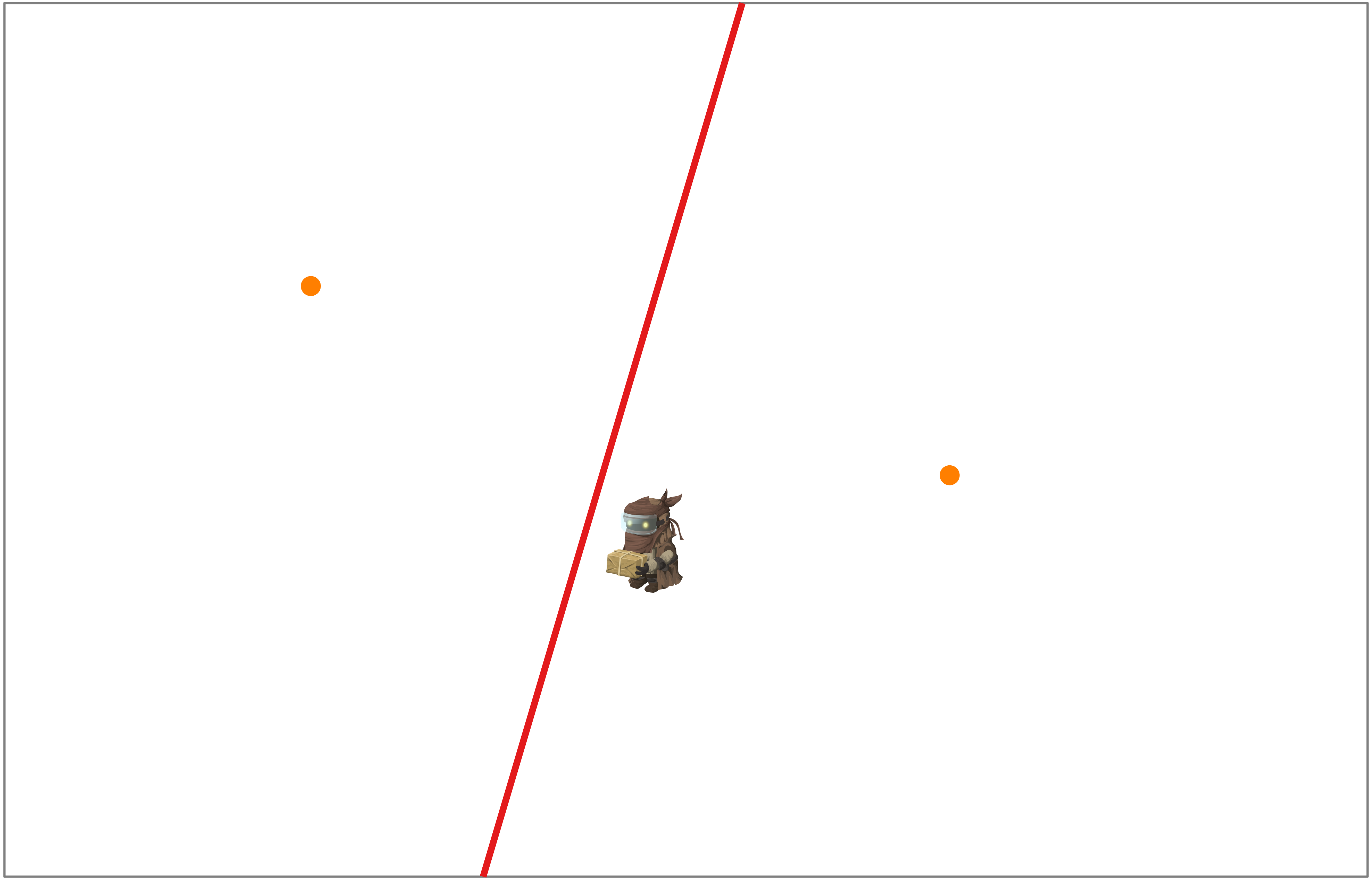

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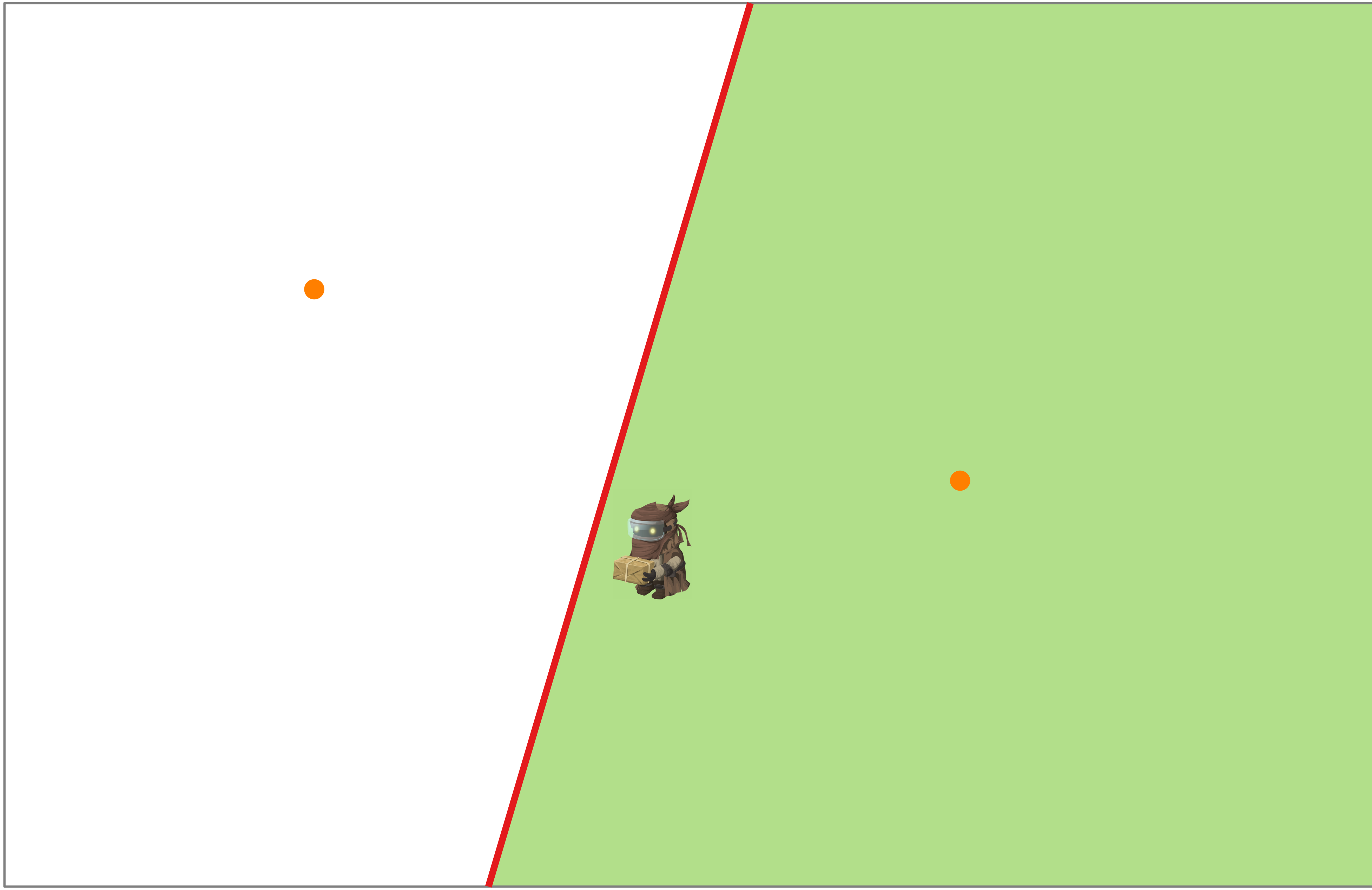

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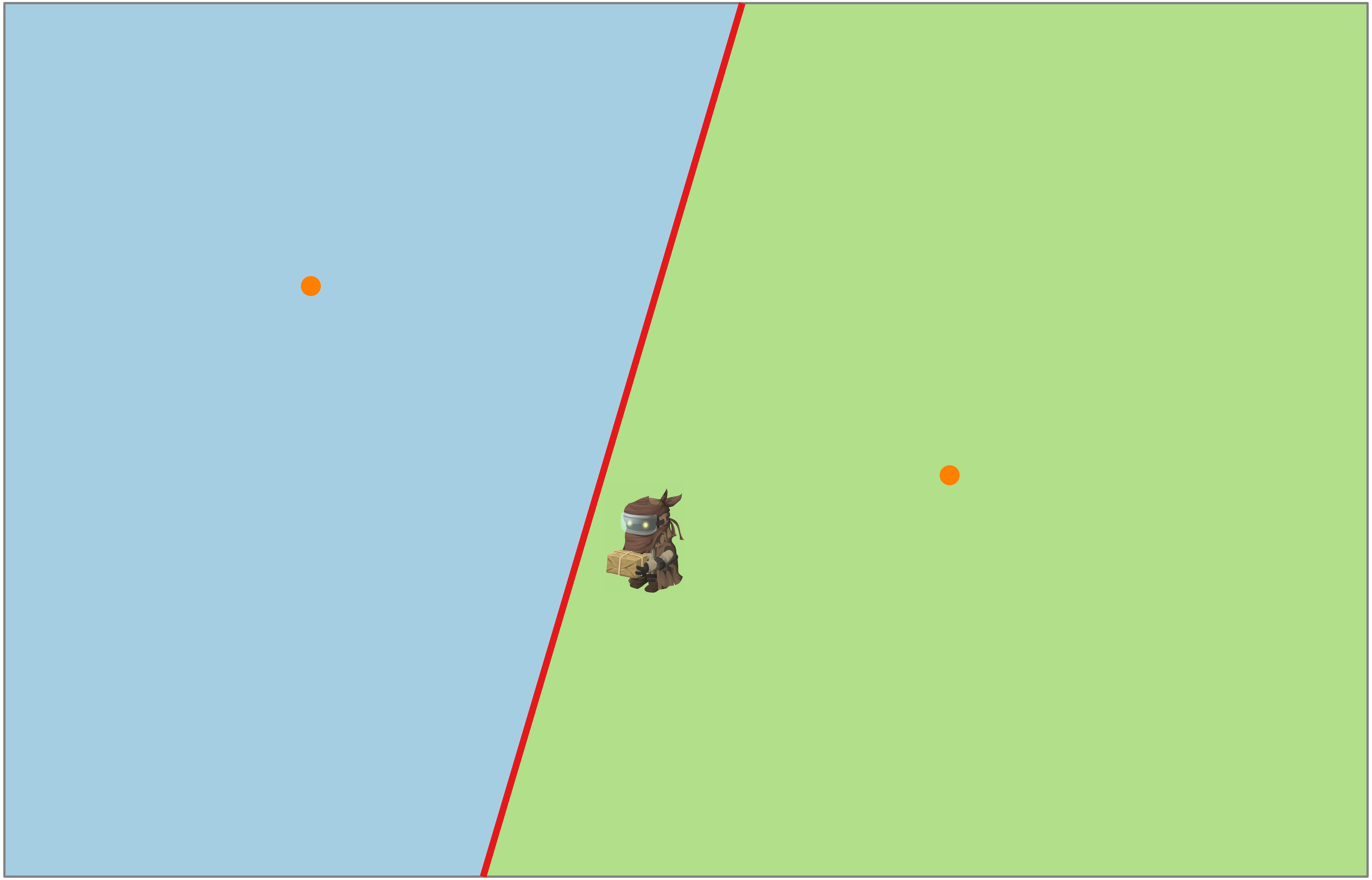

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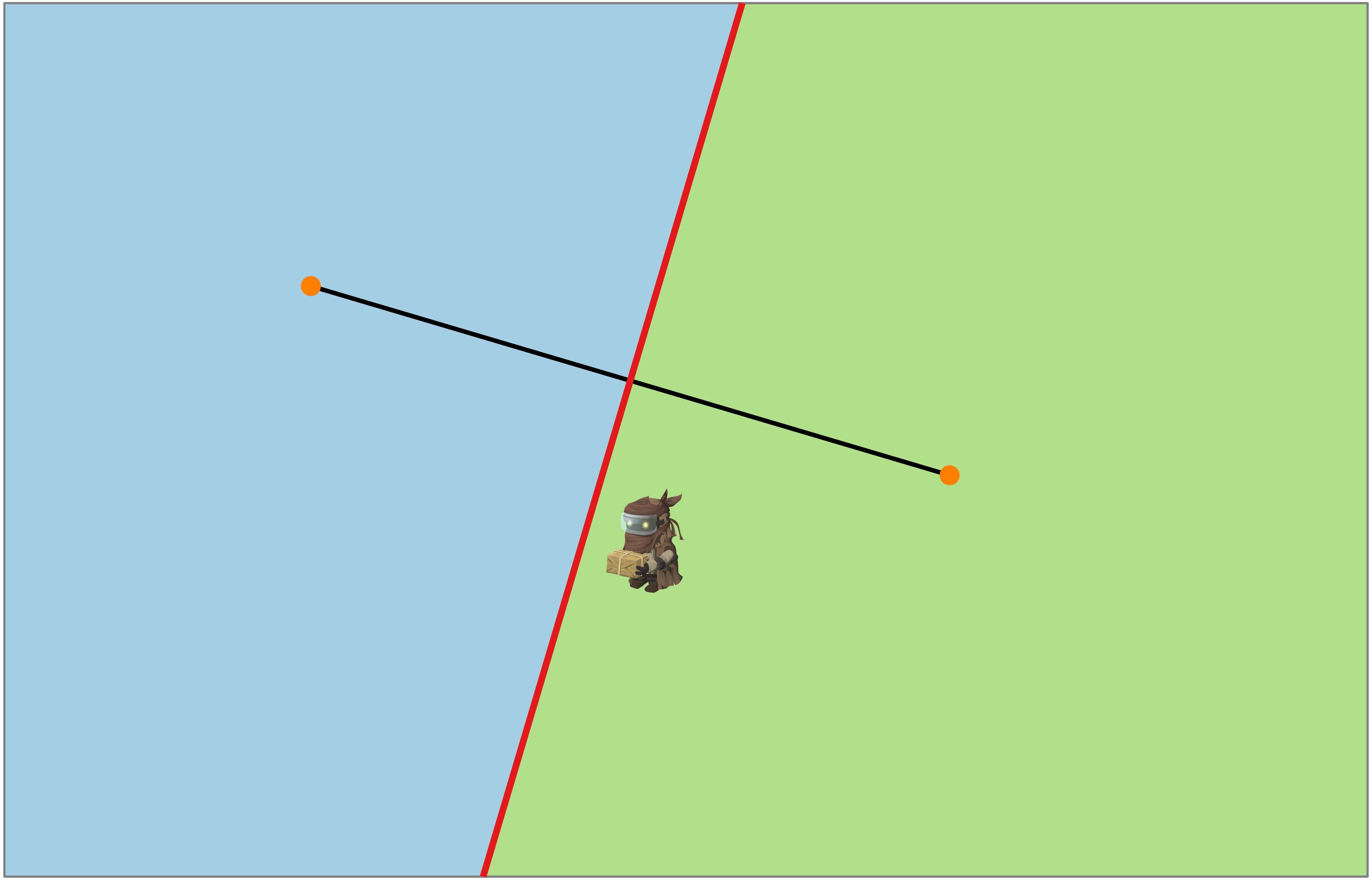

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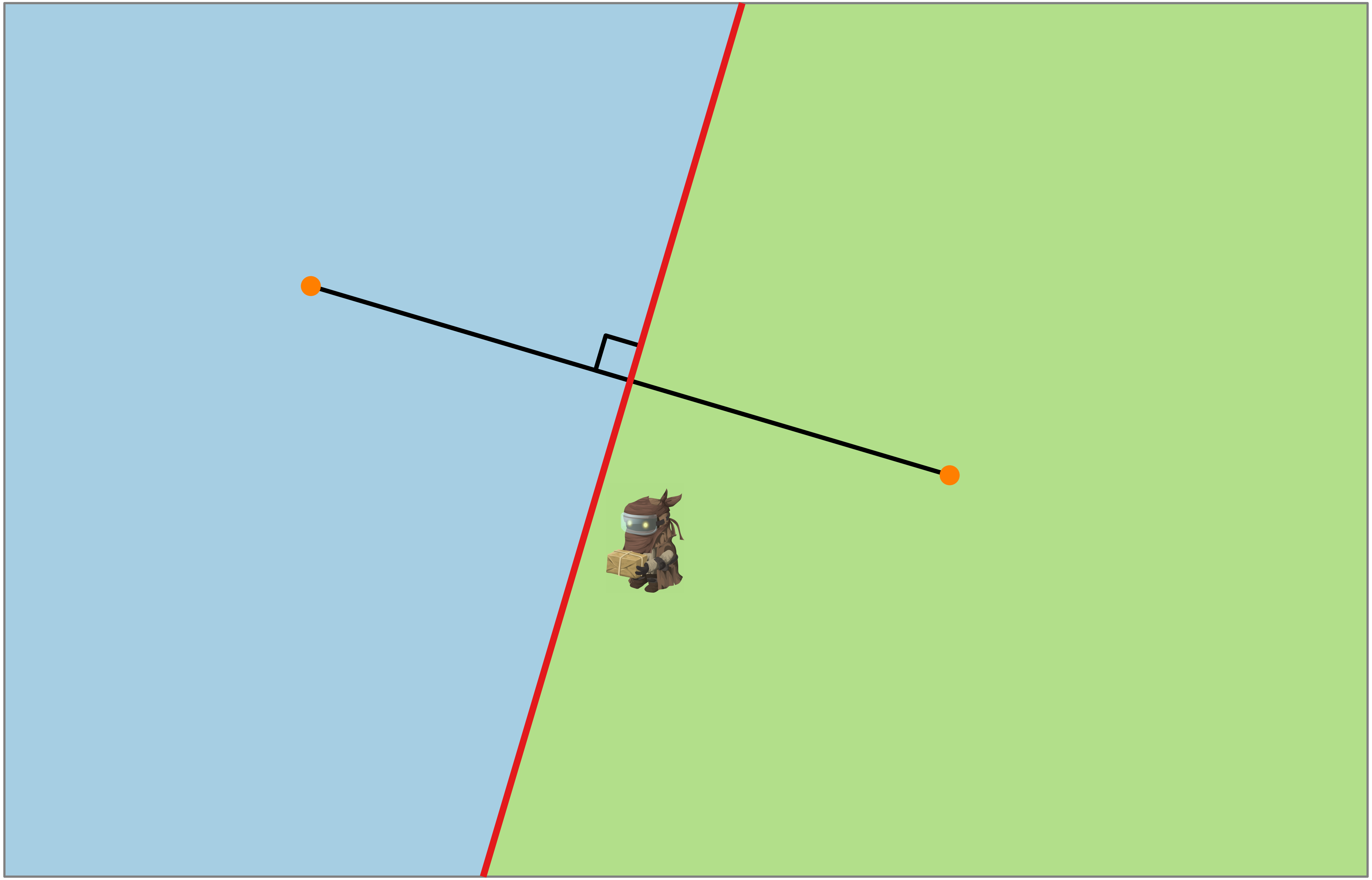

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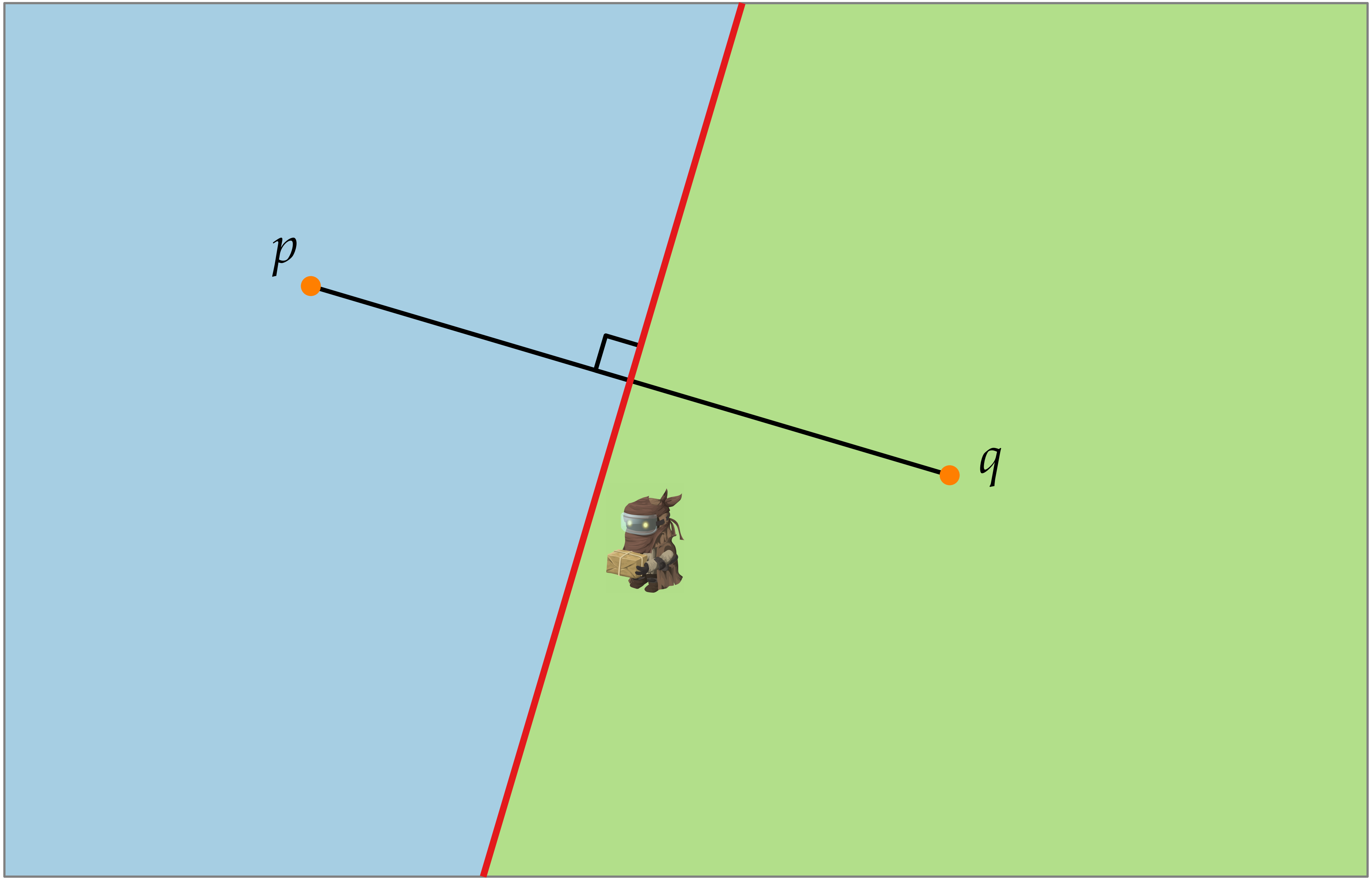

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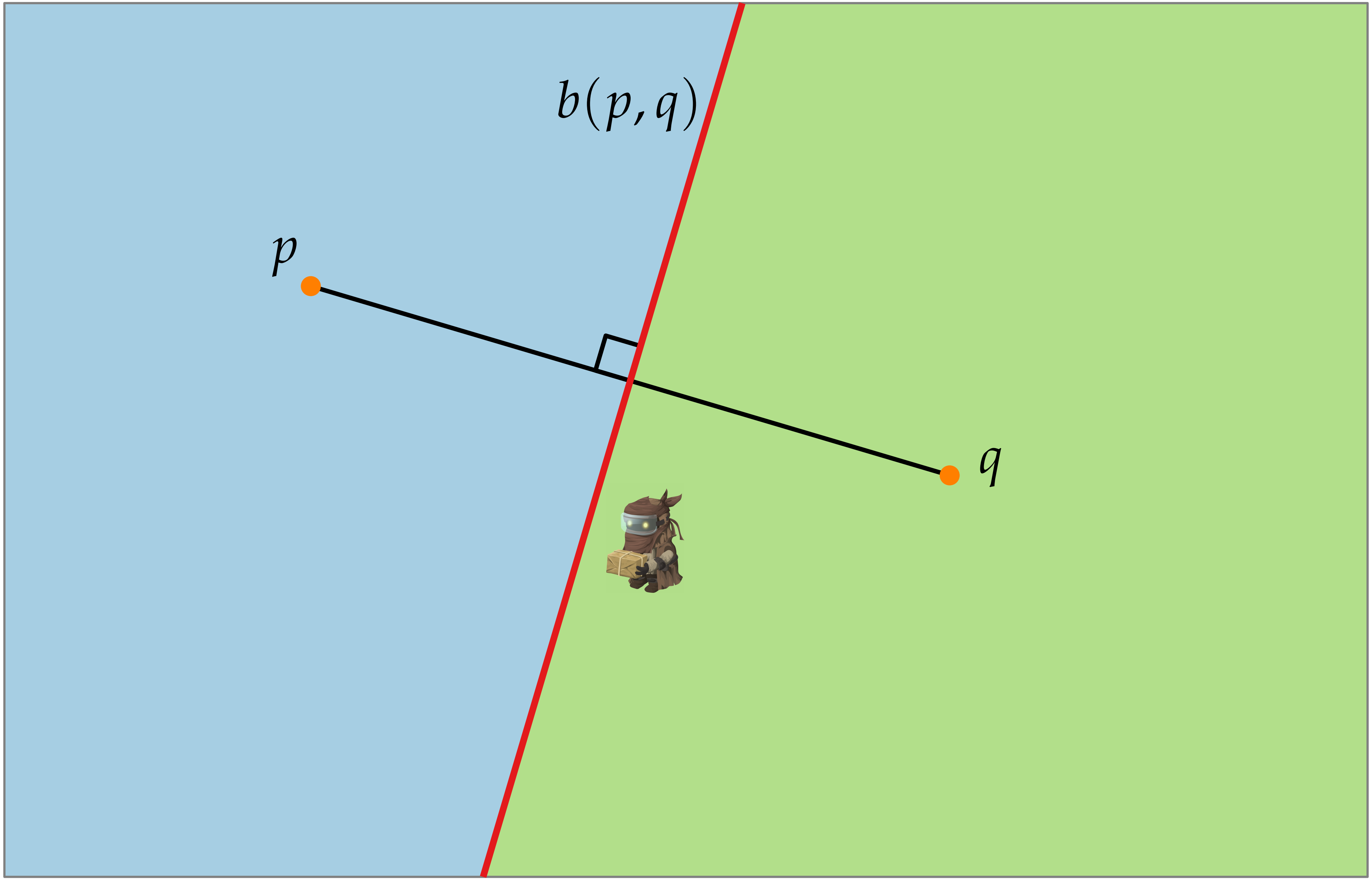

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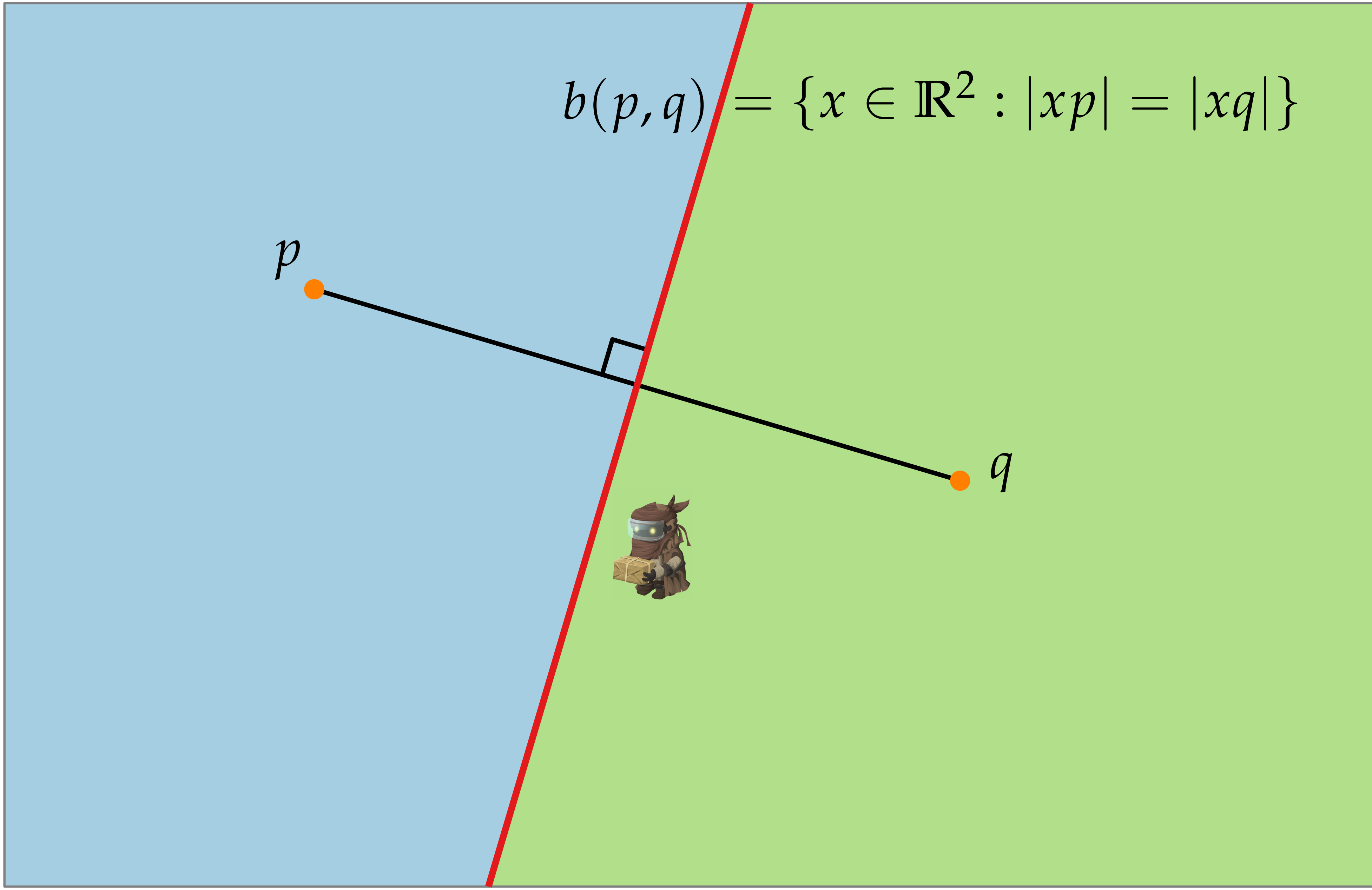

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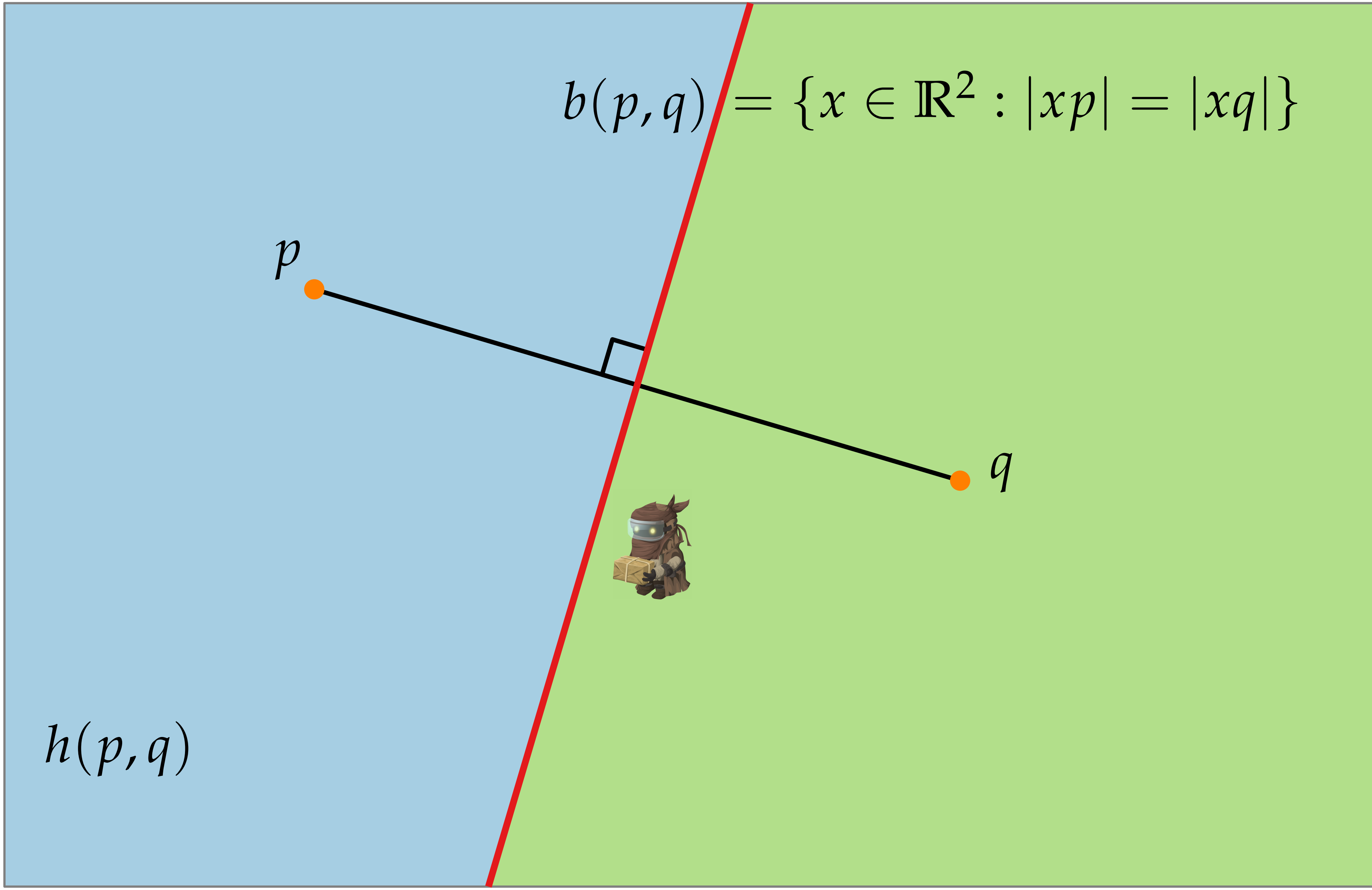

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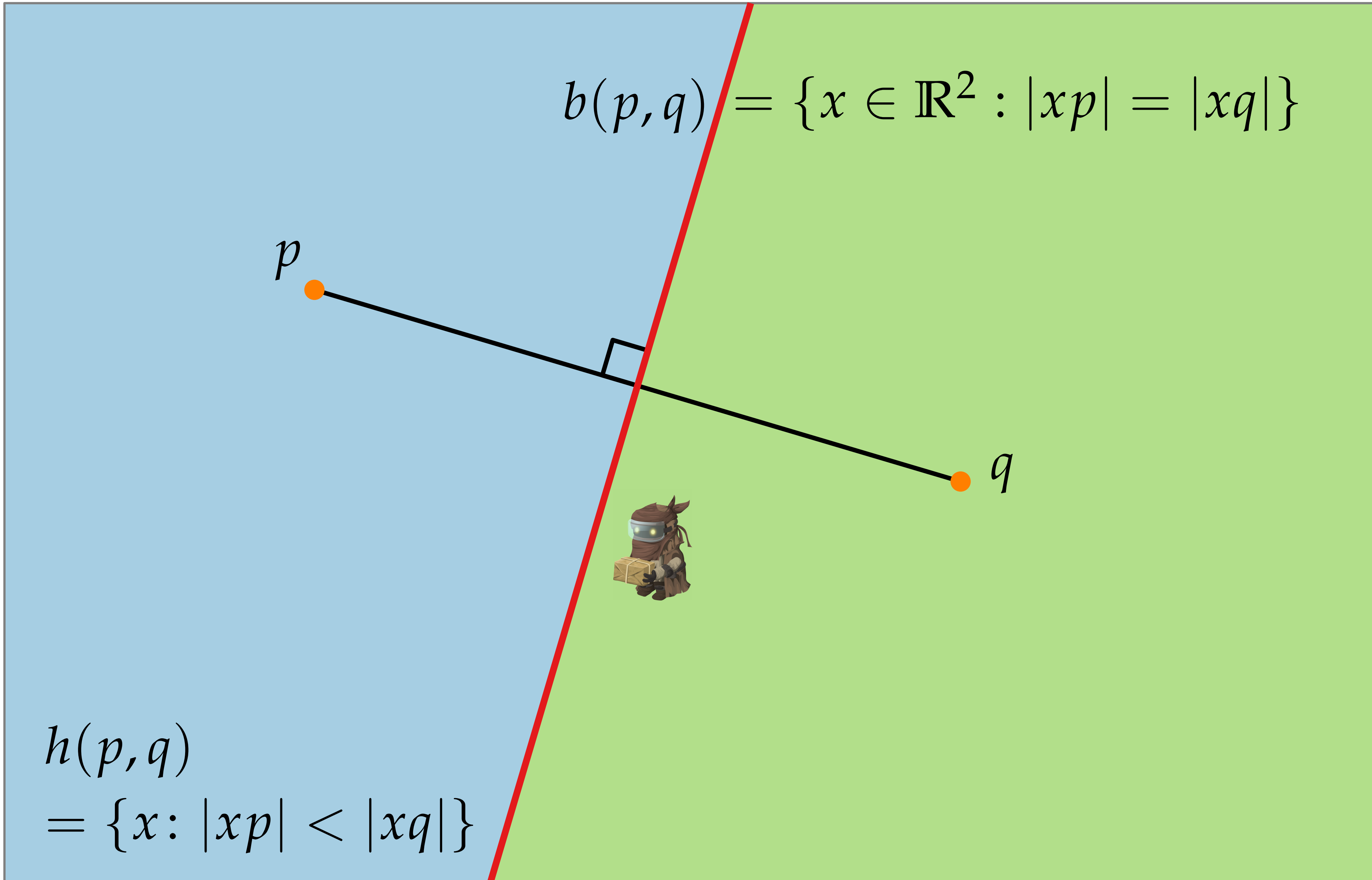

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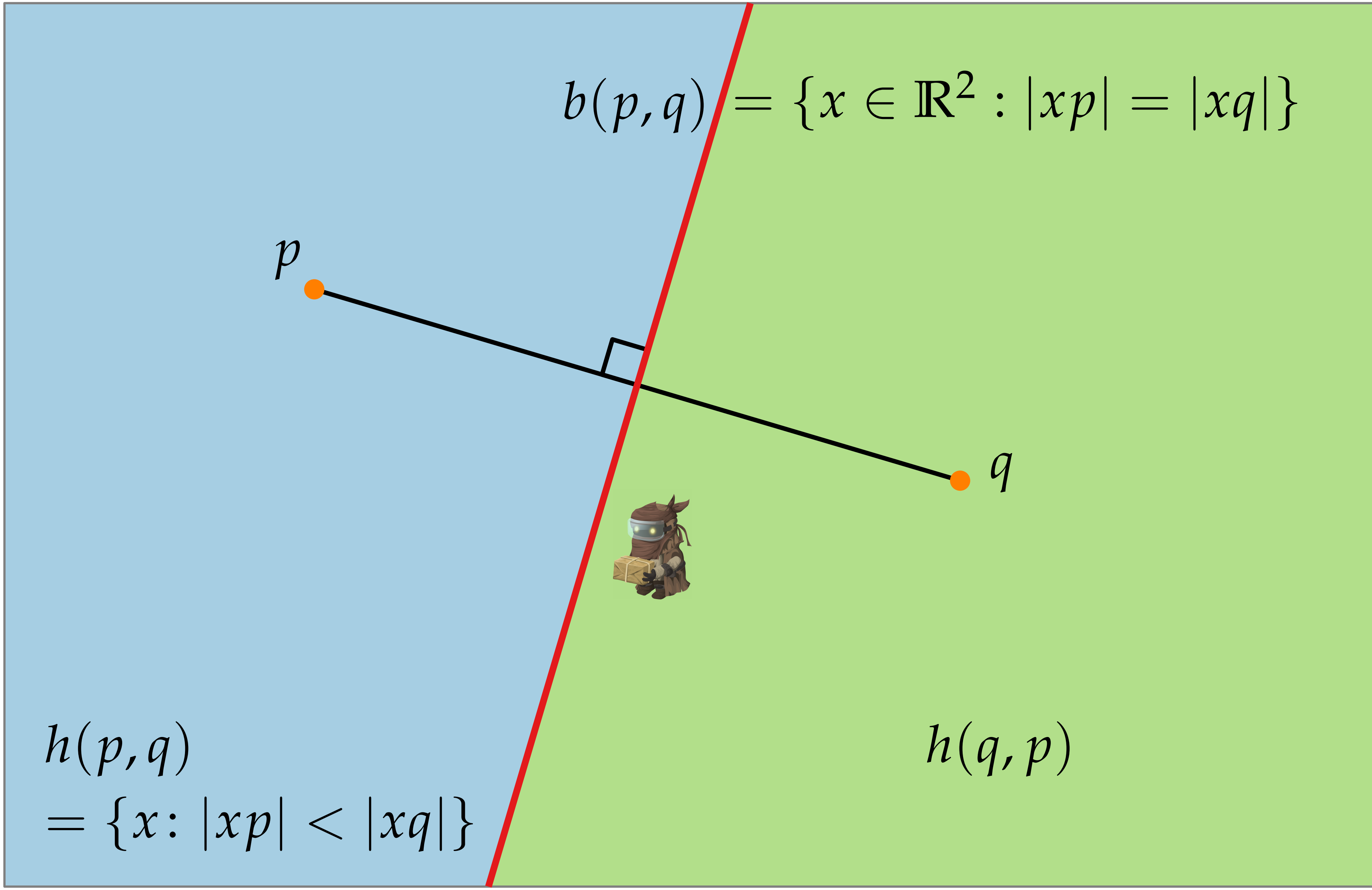

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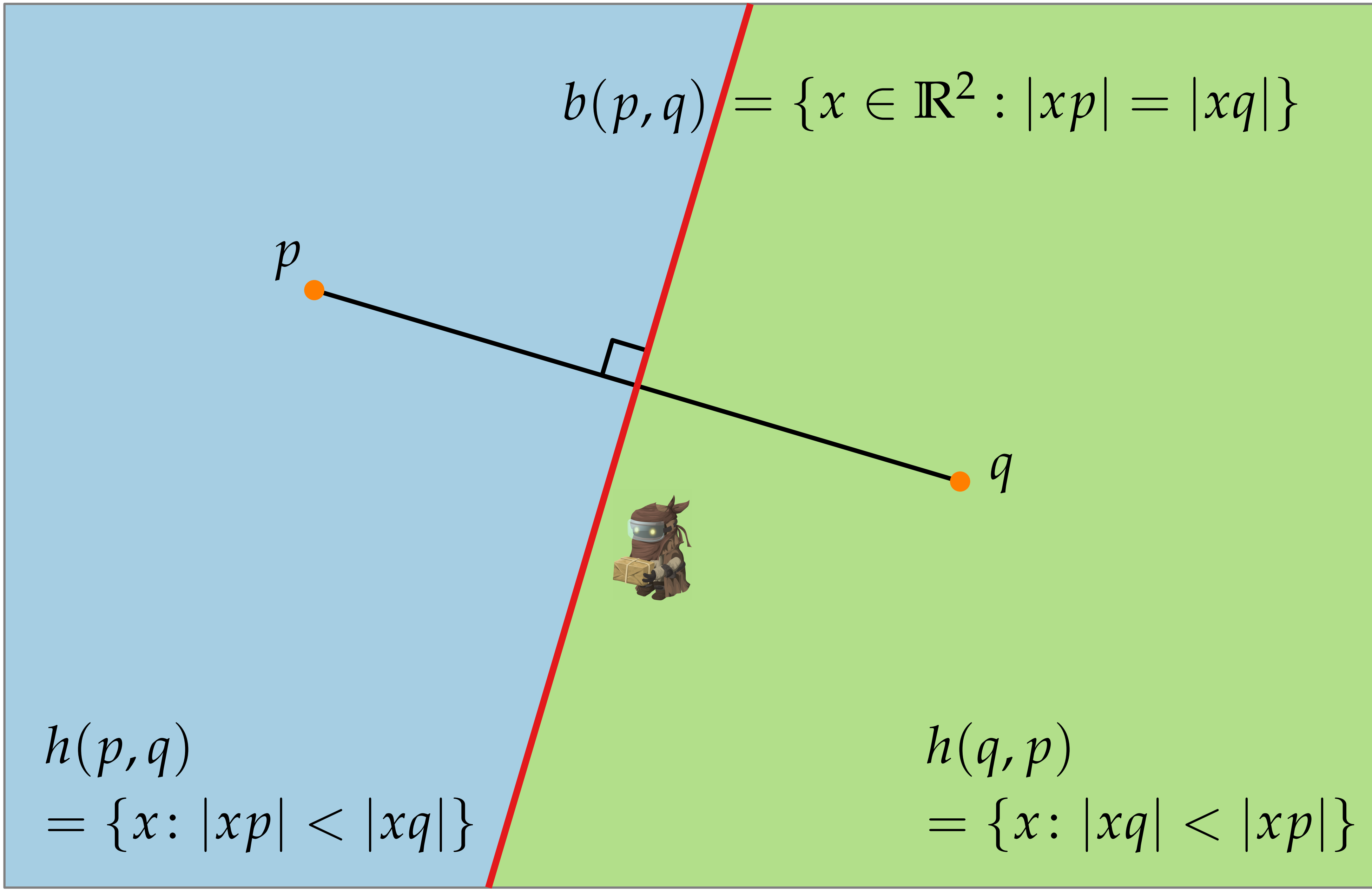

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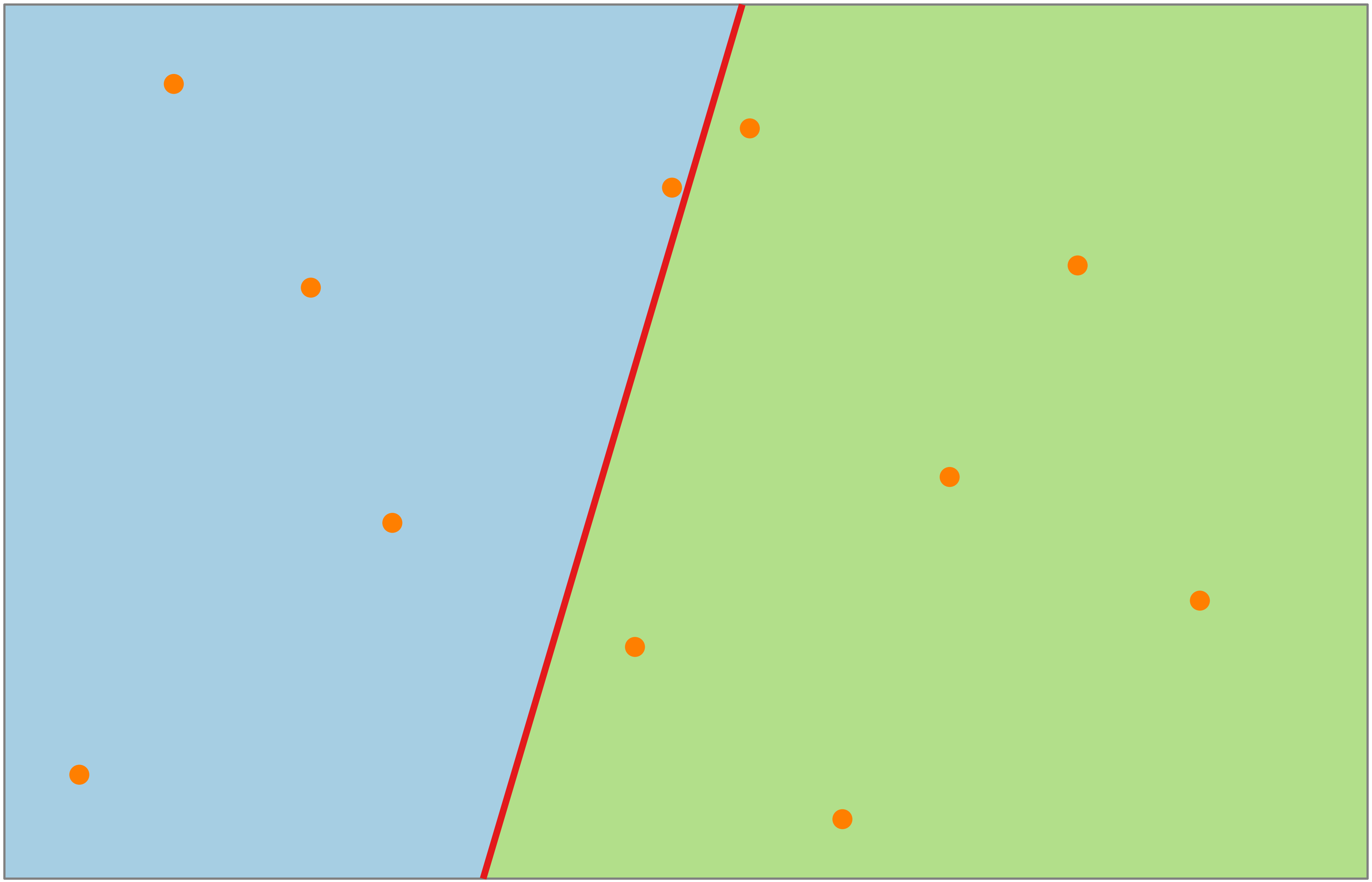

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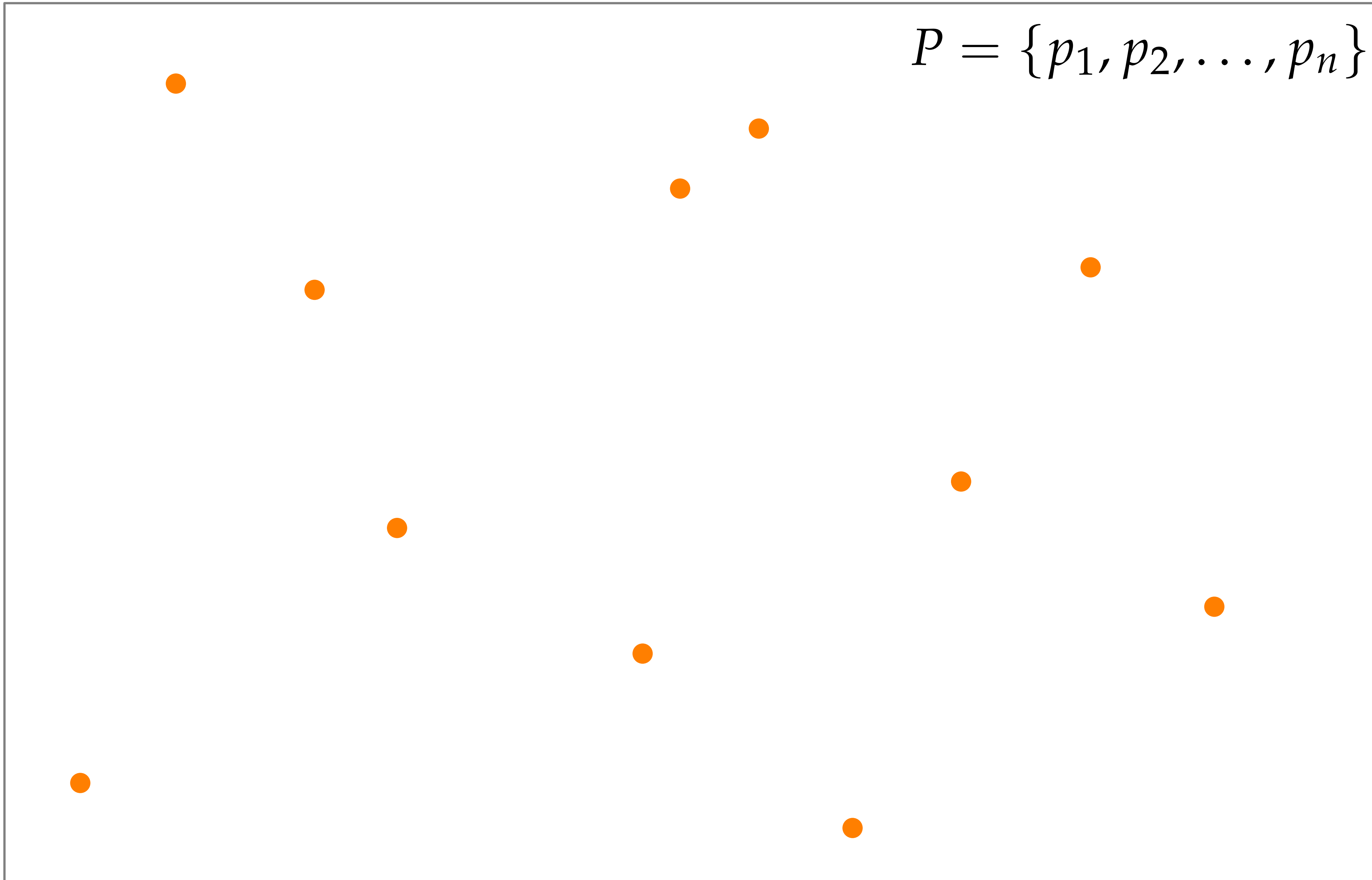

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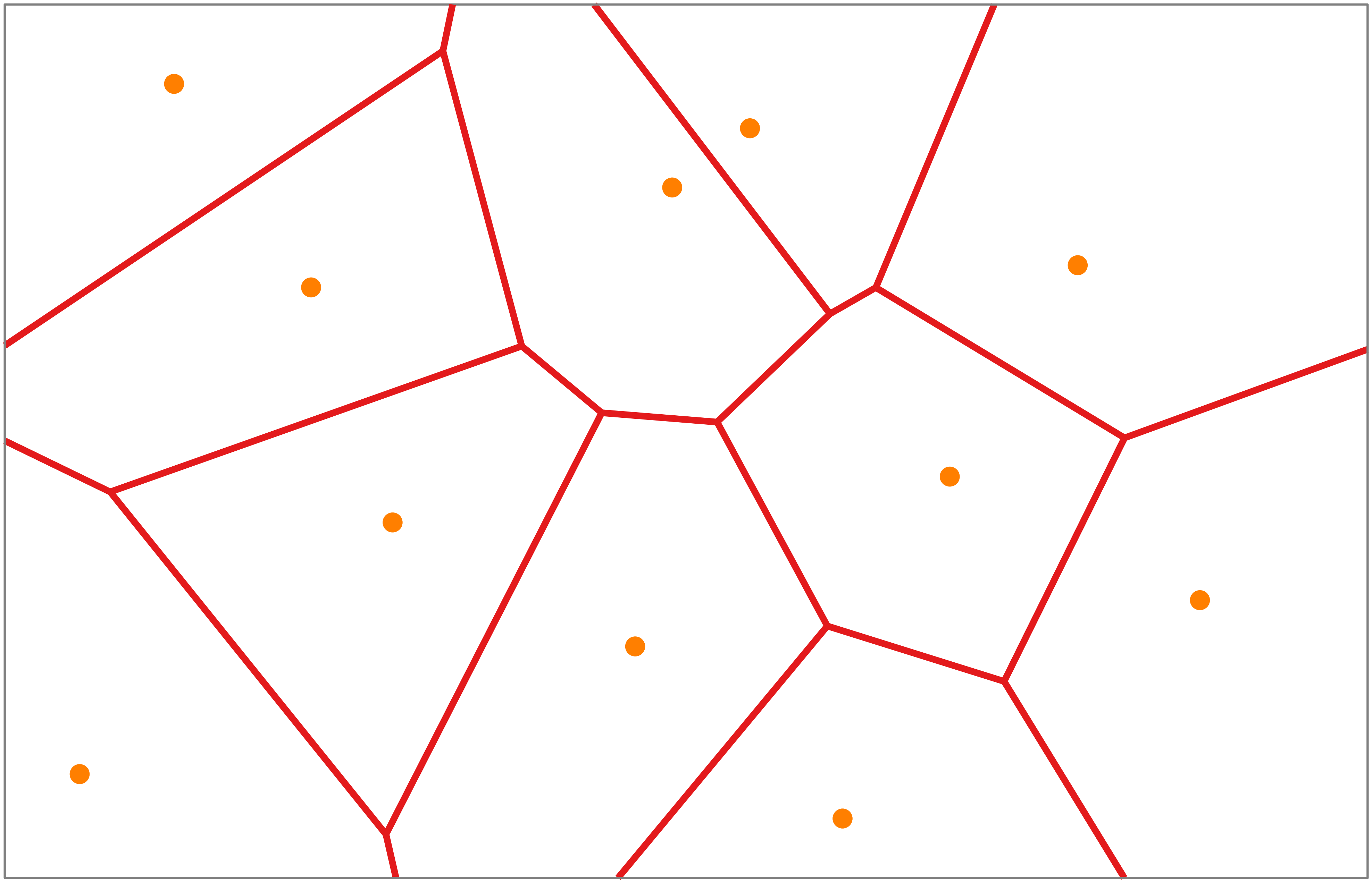

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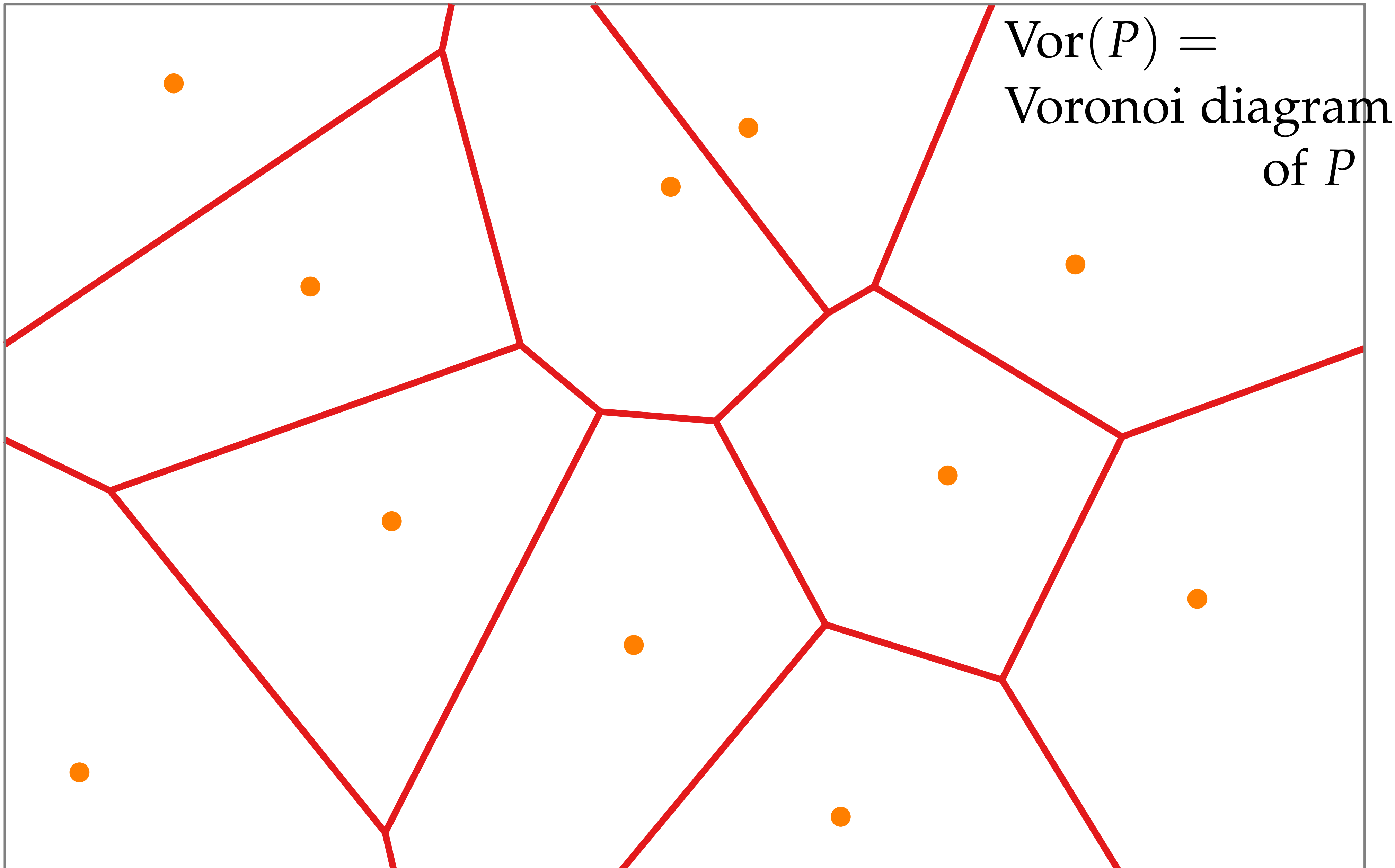

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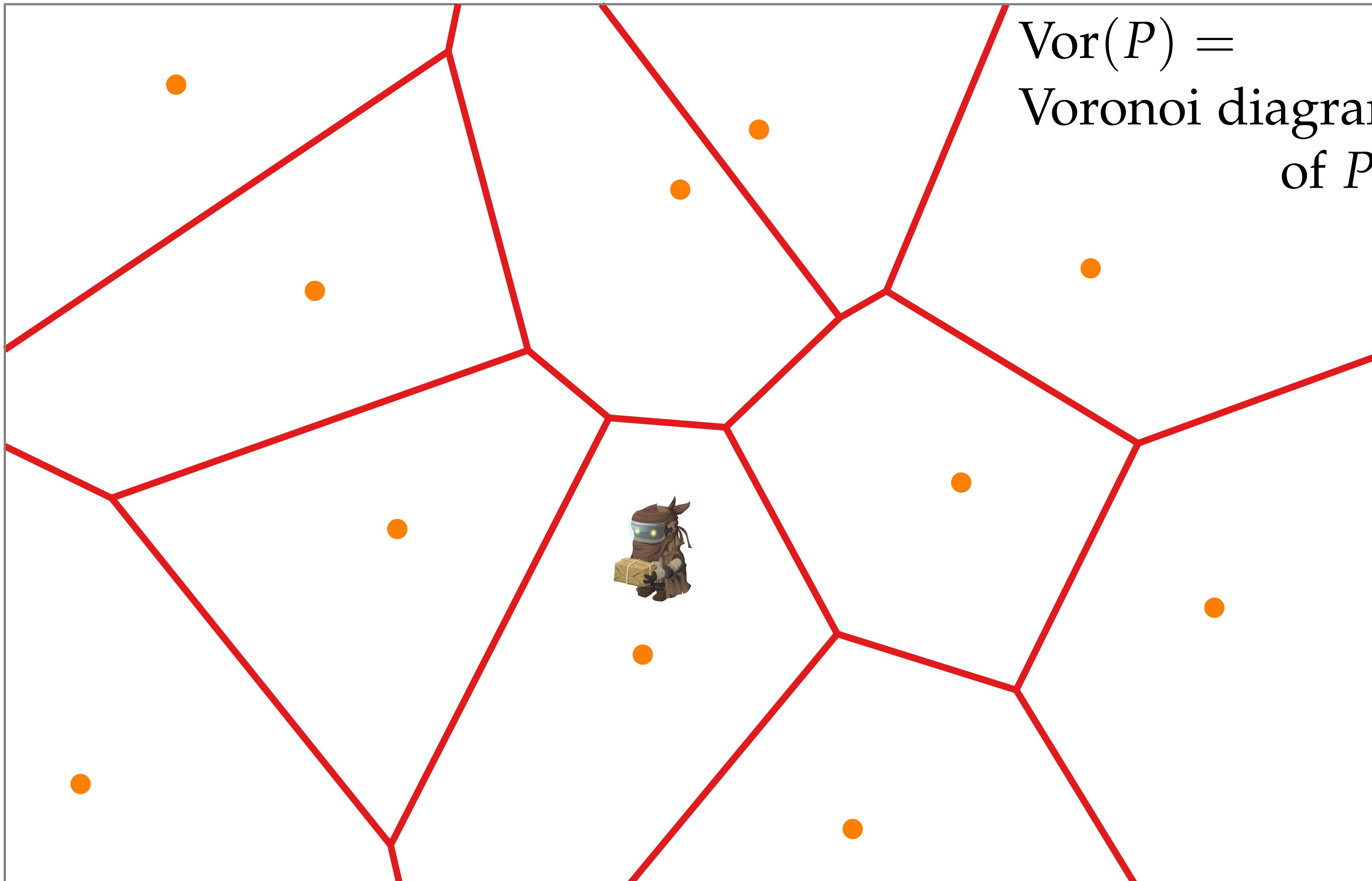

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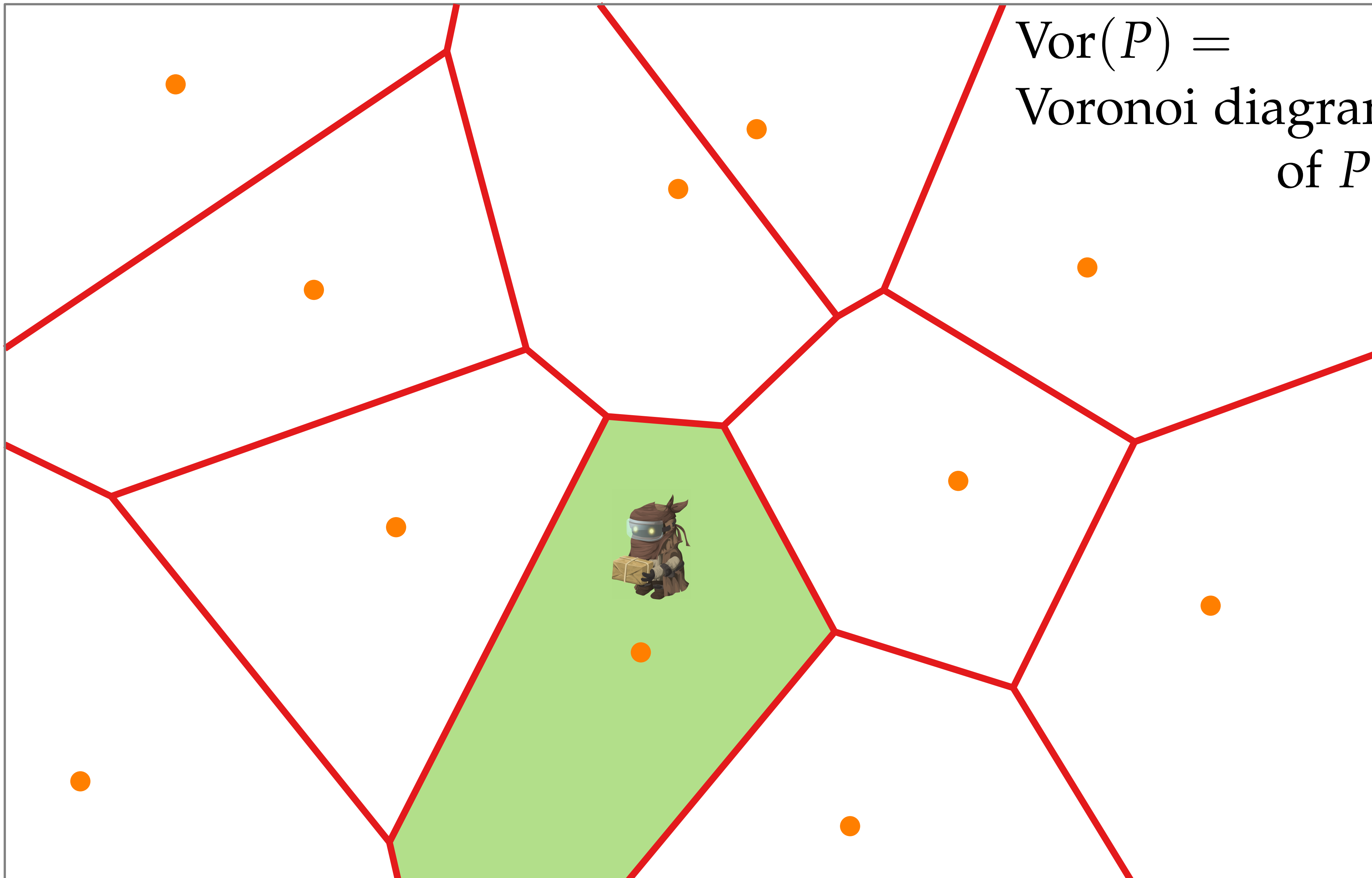
Voronoi diagram
of P

The Post-Office Problem



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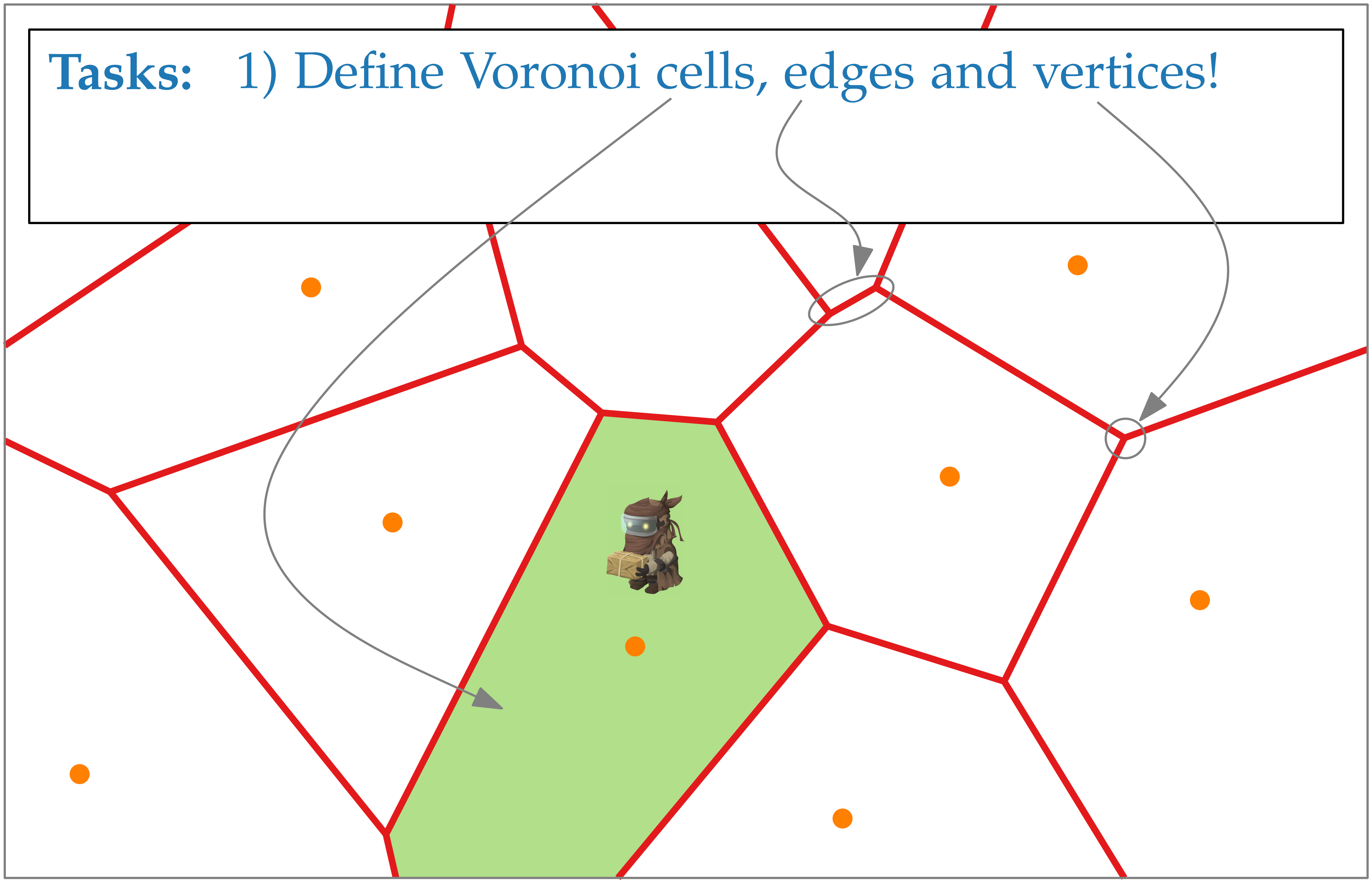
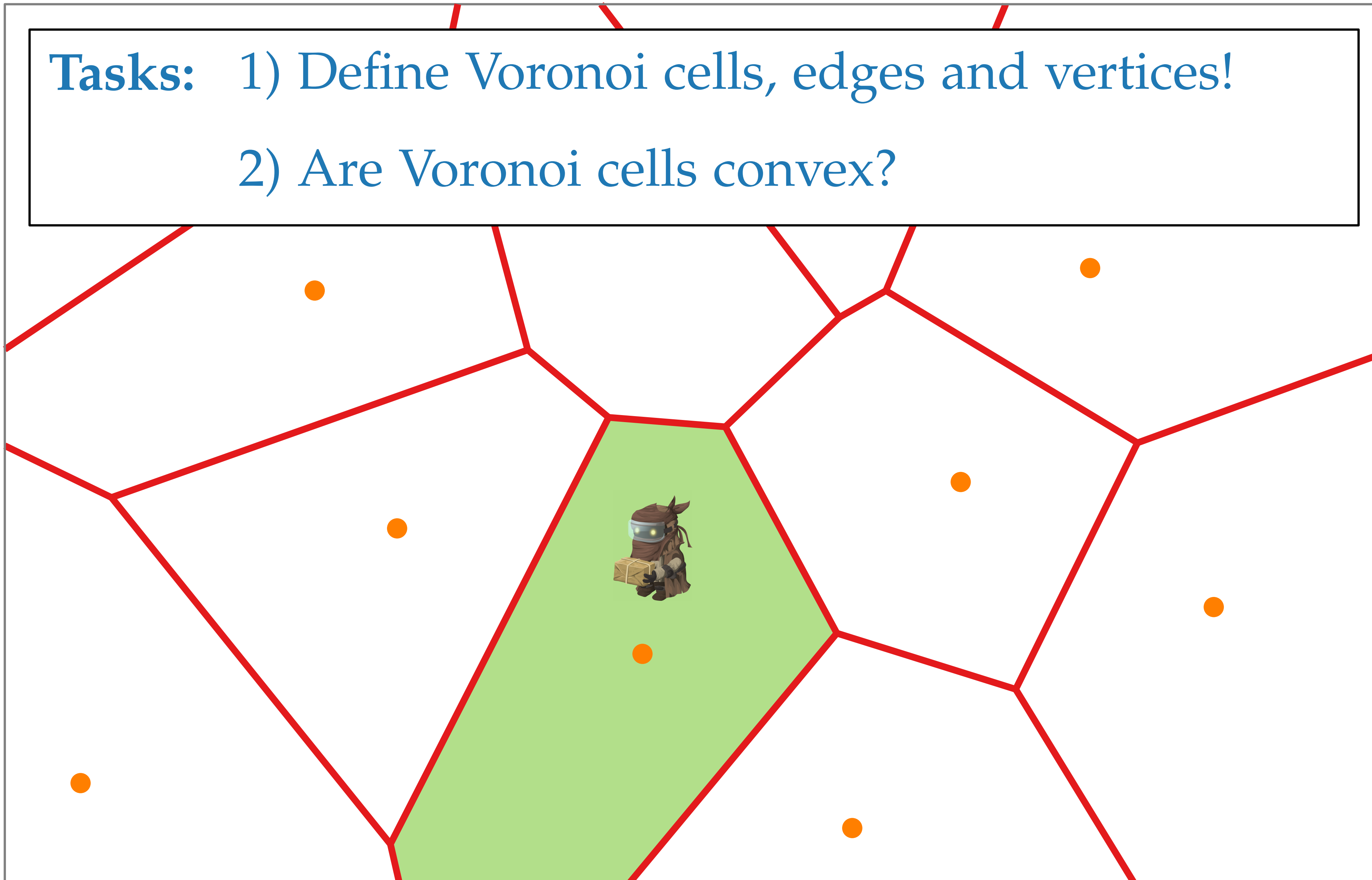
Tasks: 1) Define Voronoi cells, edges and vertices!



The Post-Office Problem

Tasks: 1) Define Voronoi cells, edges and vertices!

2) Are Voronoi cells convex?



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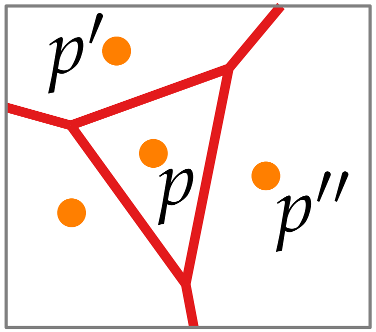

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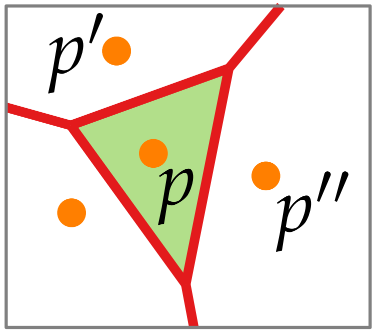

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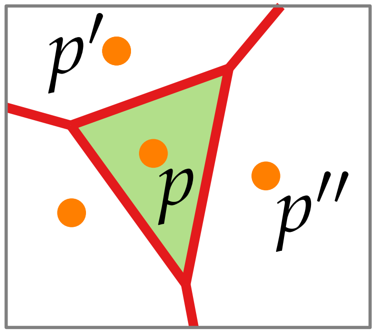

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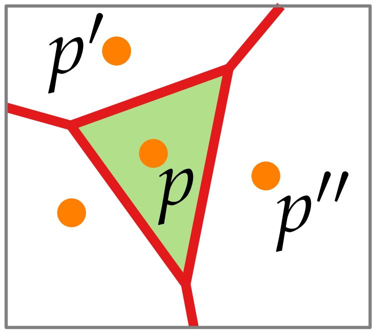

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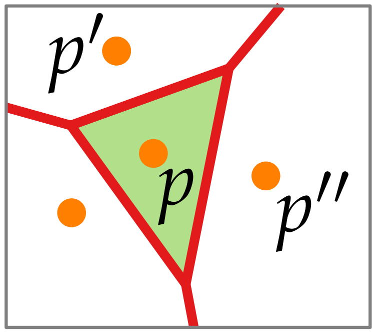

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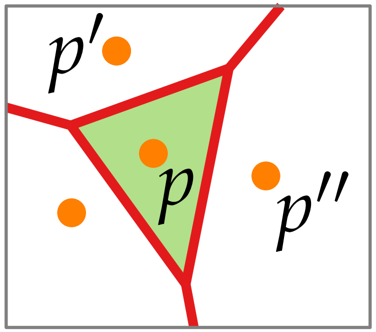

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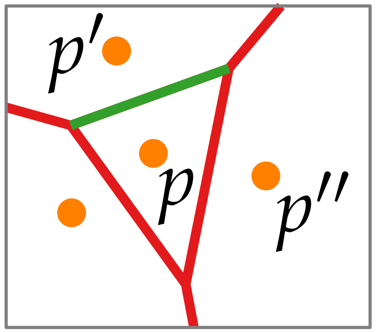

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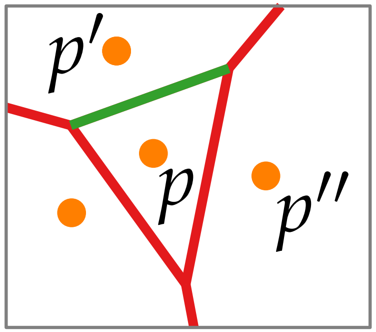

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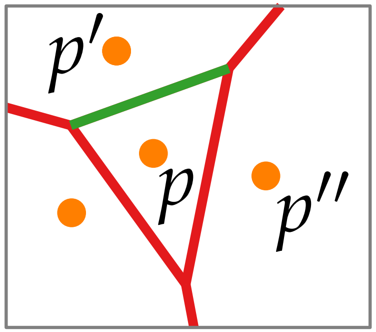

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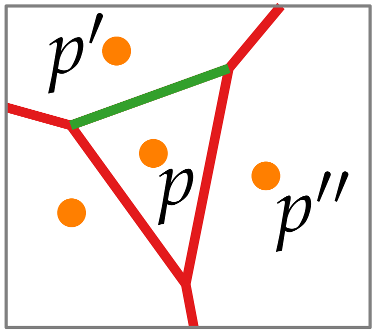

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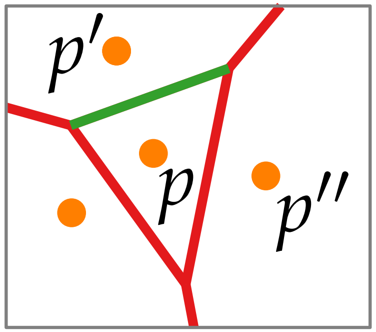

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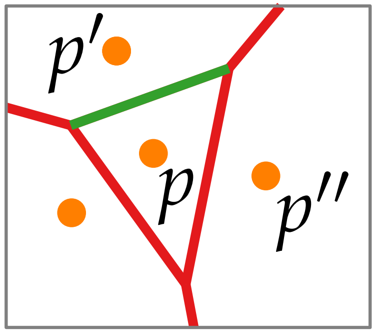

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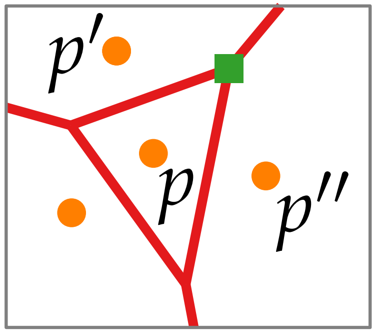

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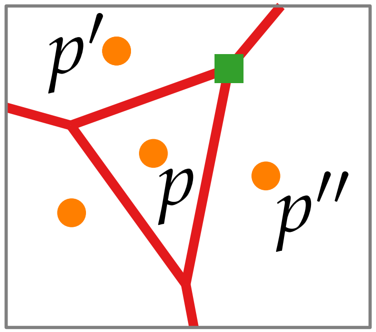

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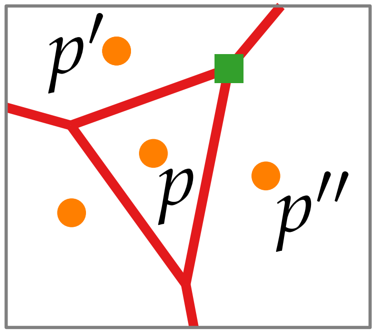

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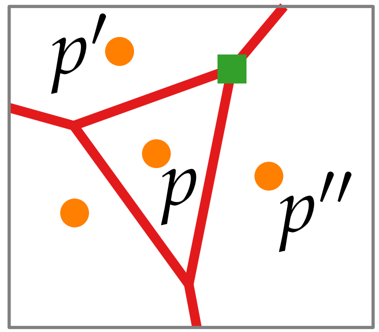

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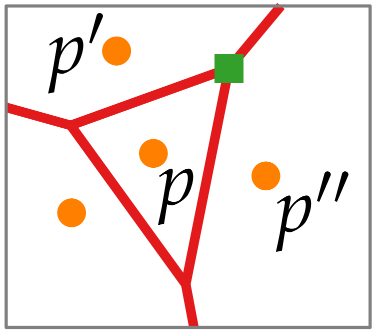

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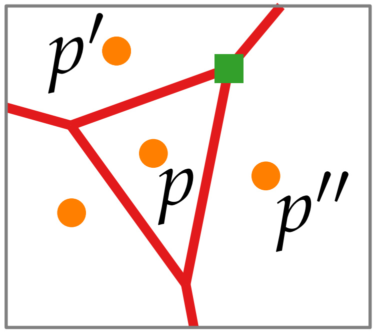

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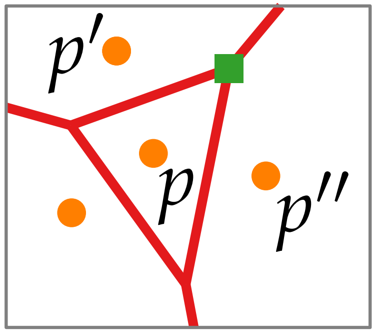
Let P be a set of points in the plane and let $p, p', p'' \in P$.

[Voronoi diagram]



$\text{Vor}(P)$ → subdivision of $\mathbb{R}^2$

[Voronoi cell]

$$\mathcal{V}(\{p\}) = \mathcal{V}(p) = \{x \in \mathbb{R}^2 : |xp| < |xq| \text{ for all } q \in P \setminus \{p\}\}$$
$$= \bigcap_{q \neq p} h(p,q)$$

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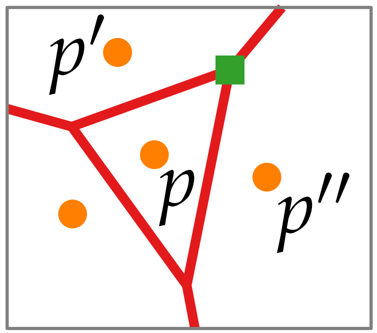

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Theorem. Let $P \subset \mathbb{R}^2$ be a set of n pts (called *sites*). If all sites are collinear, Vor(P) consists of $n-1$ parallel lines. Otherwise, Vor(P) is connected and its edges are line segments or half-lines.

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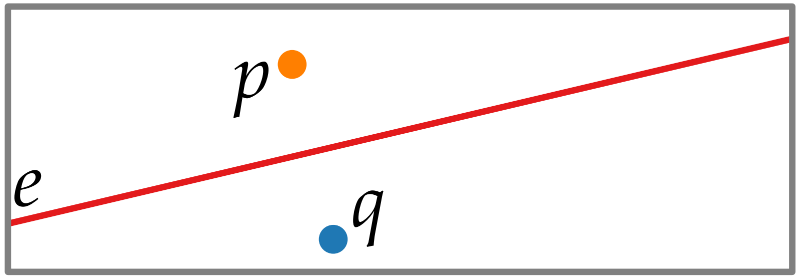

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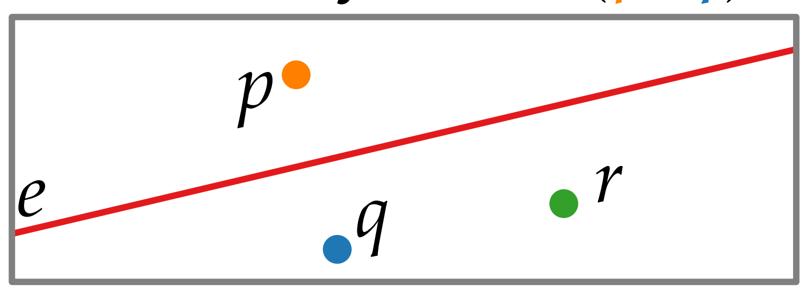

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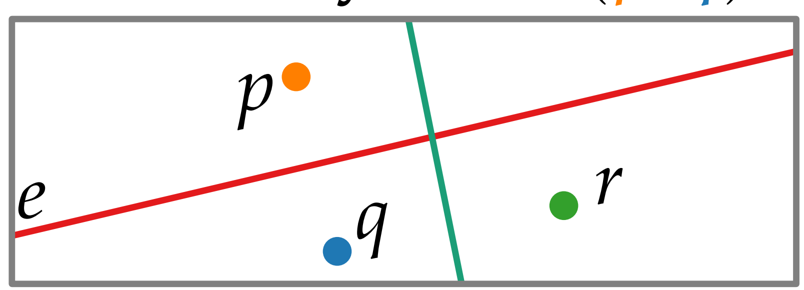

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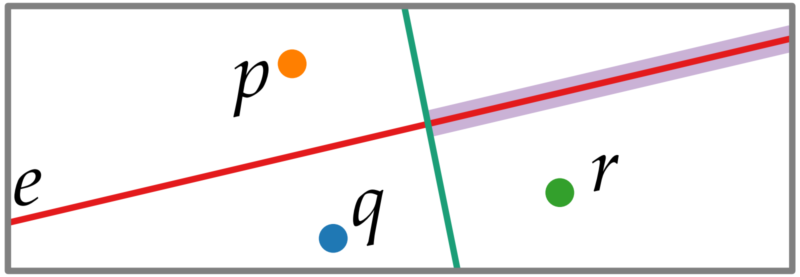

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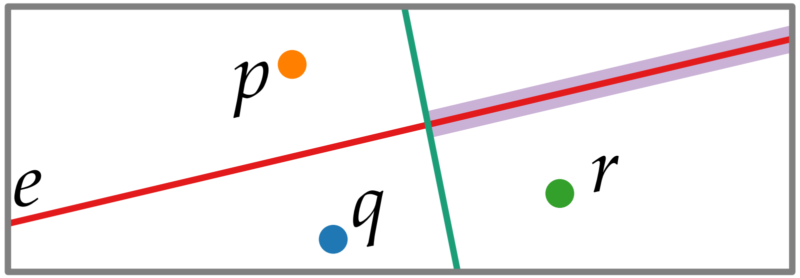

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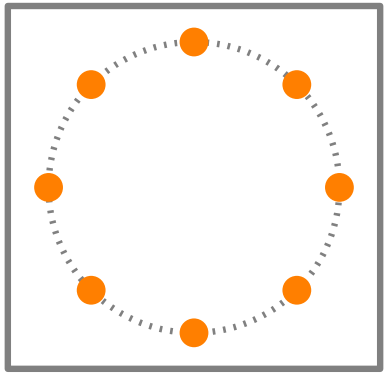

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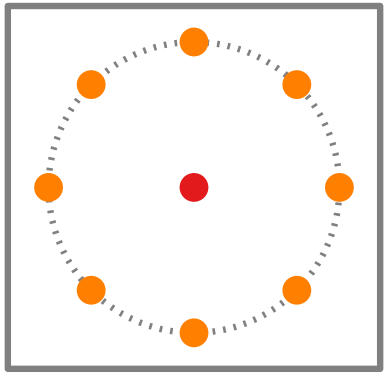

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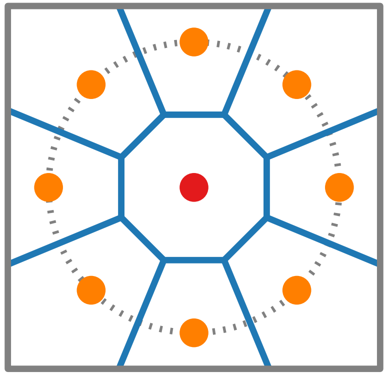

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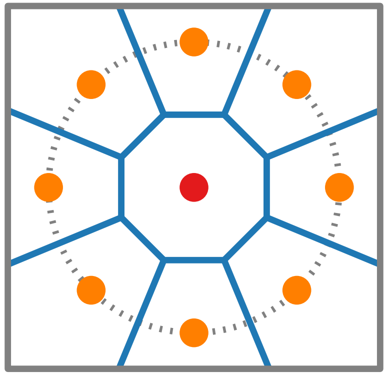

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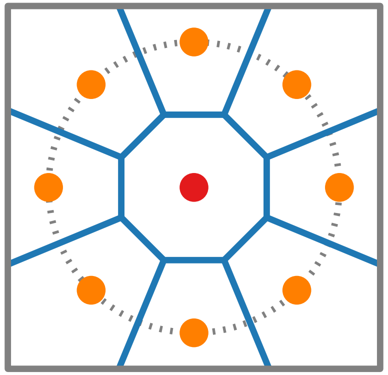

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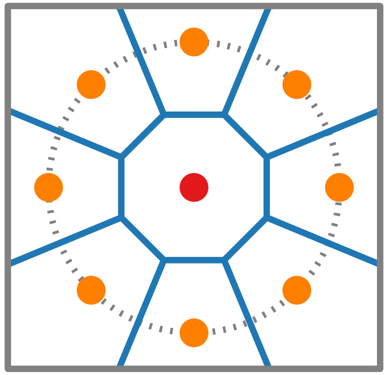

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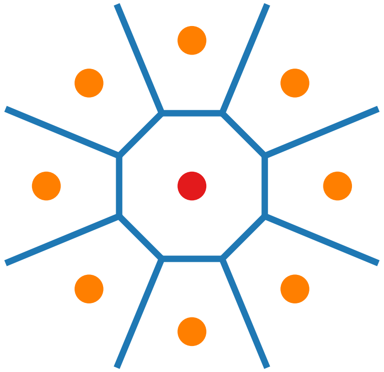

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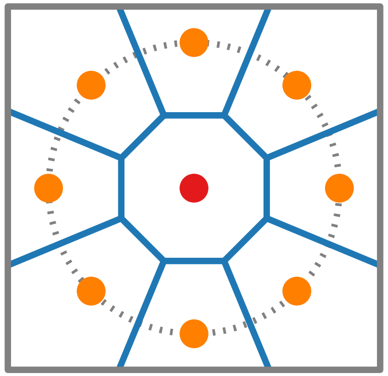

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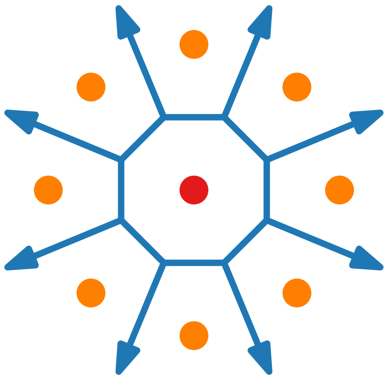

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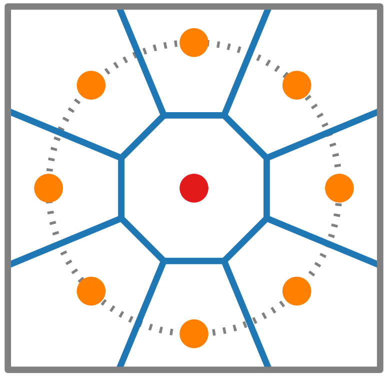

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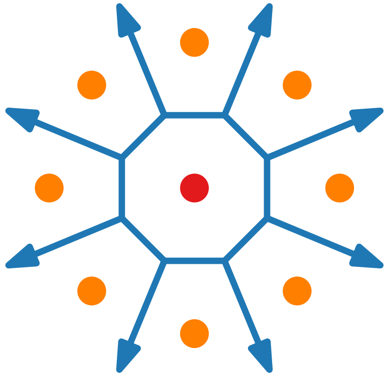

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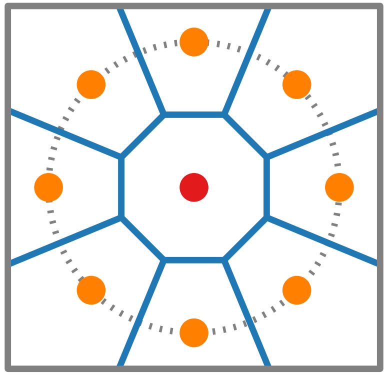

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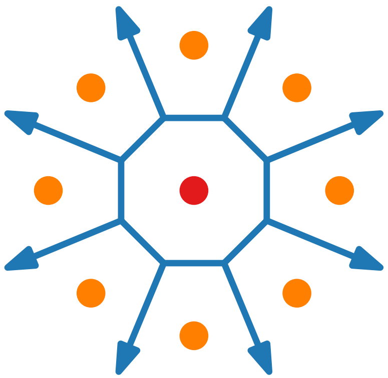

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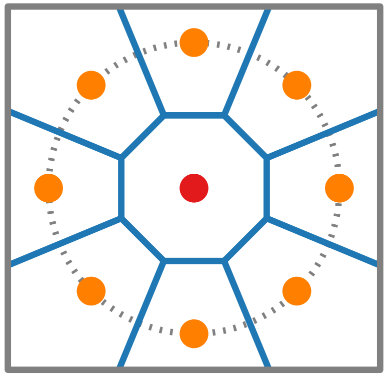

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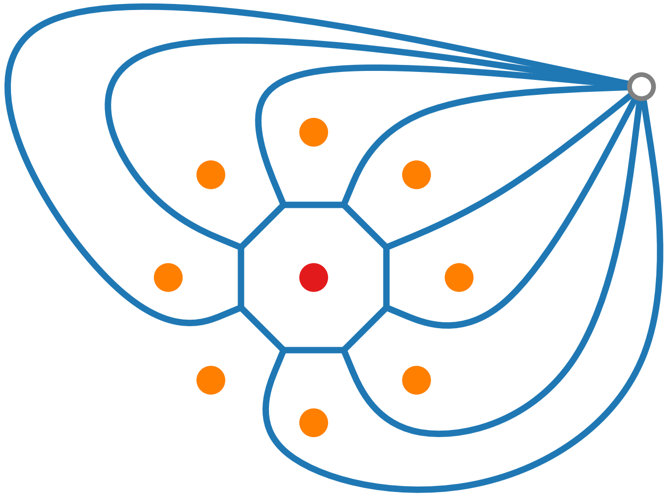

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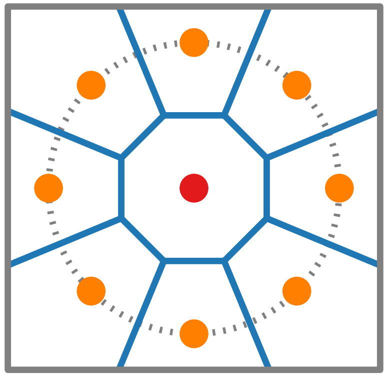

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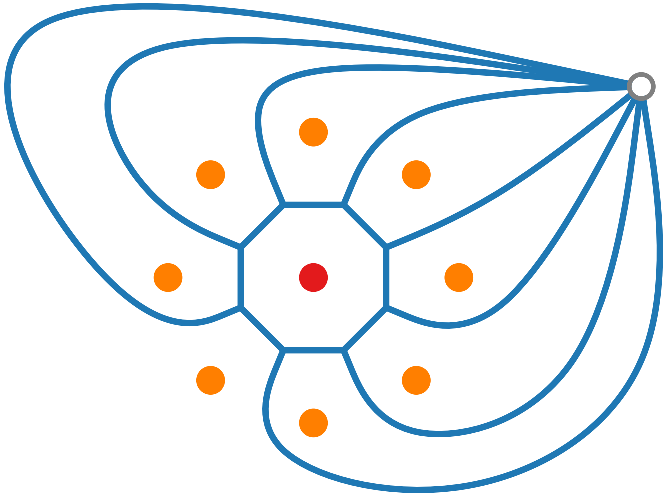

$|F| = n$

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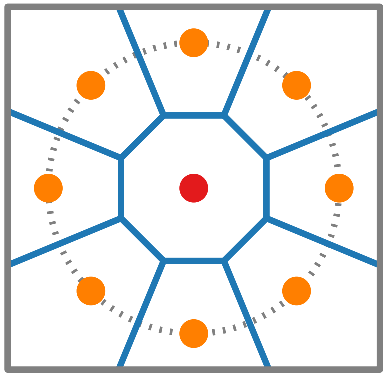

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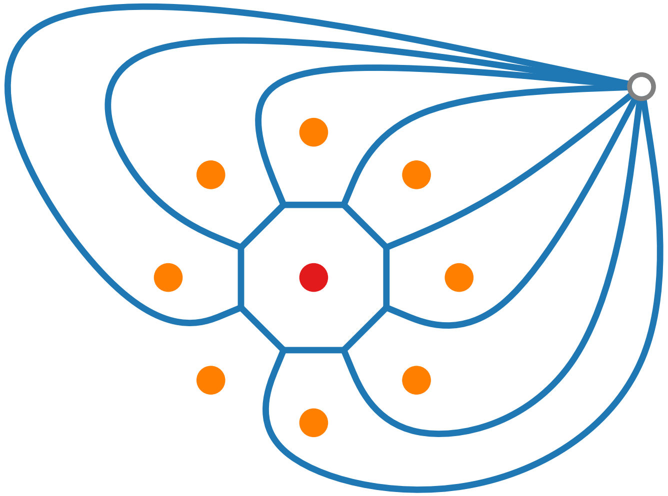

$$|F| = n \Rightarrow (|V| + 1) - |E| + n = 2$$

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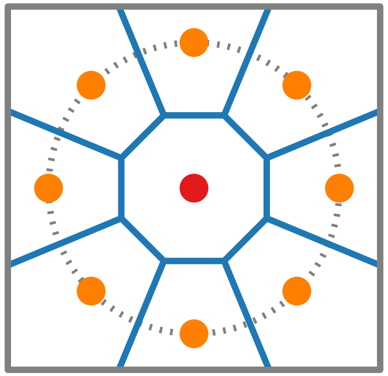

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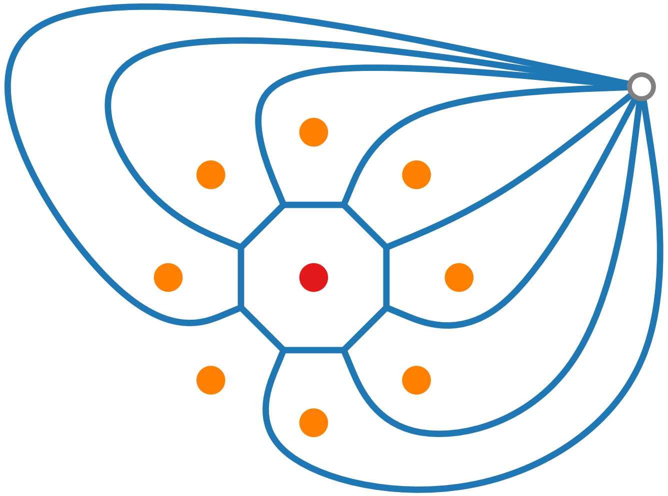

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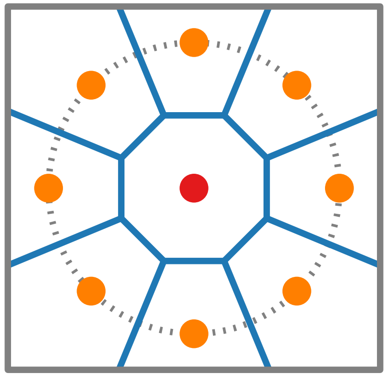

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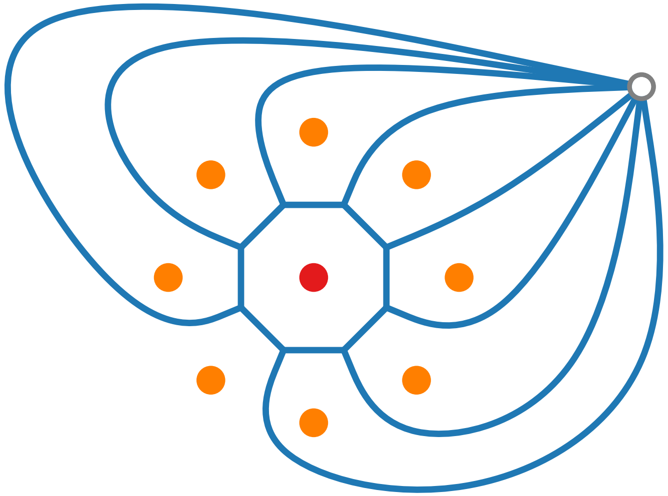

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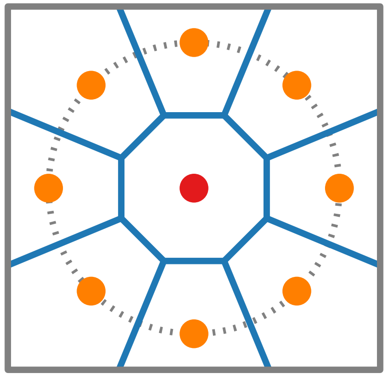

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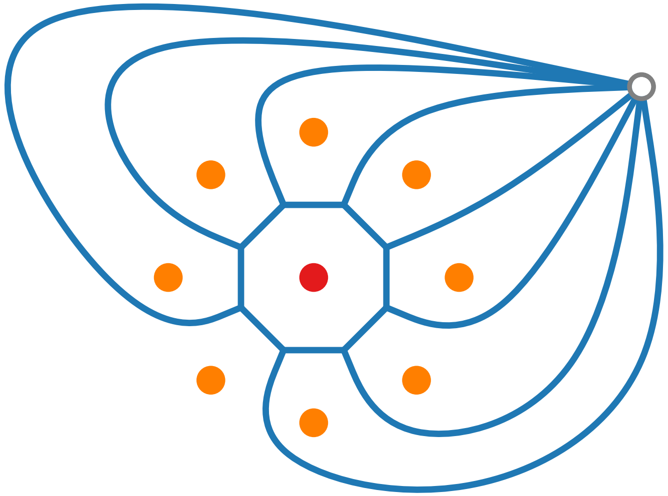

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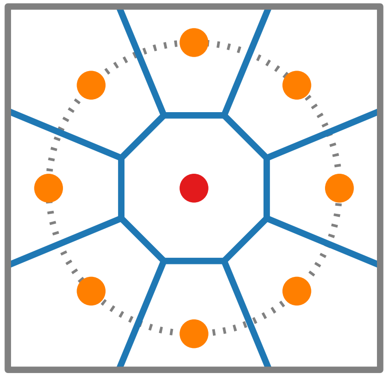

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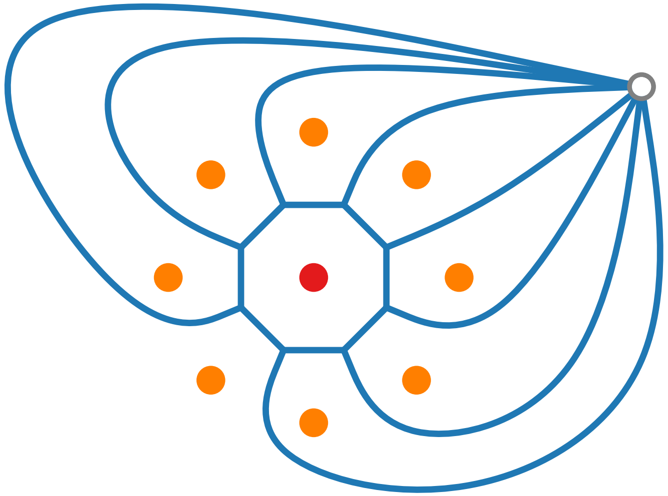

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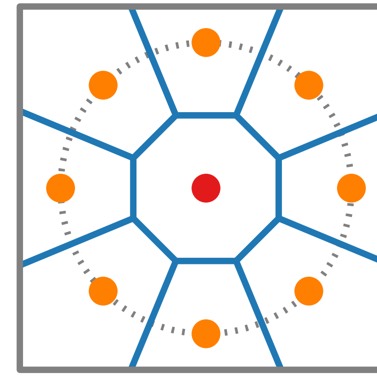

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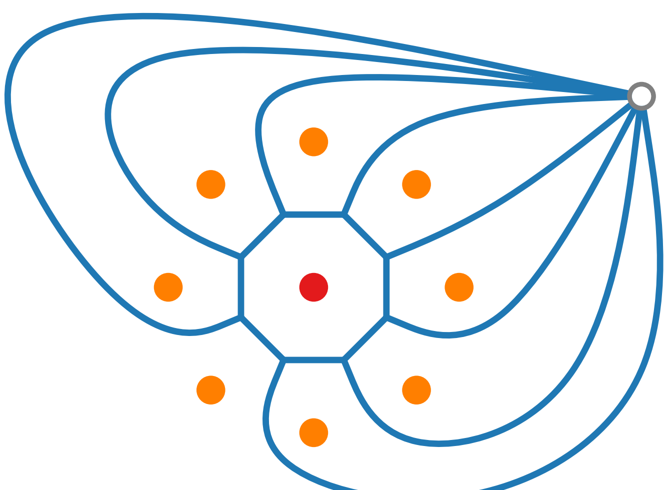

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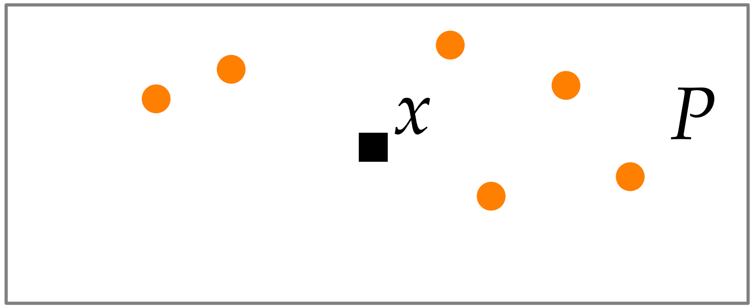
Characterization of Voronoi vtc and edges

$C_P(x) :=$ largest circle centered at x w/o sites in its interior



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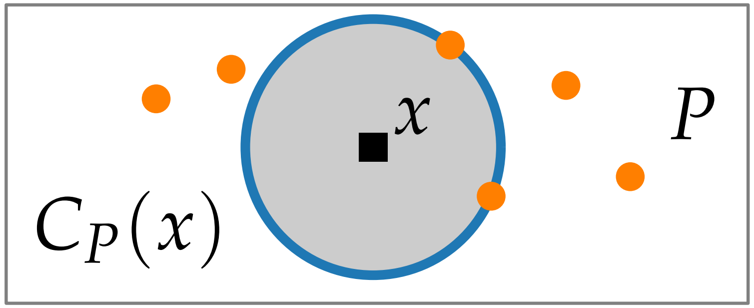

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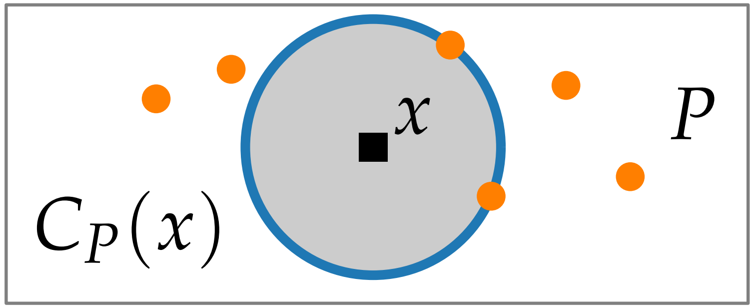

Theorem: (i) x Voronoi vtx $\Leftrightarrow$

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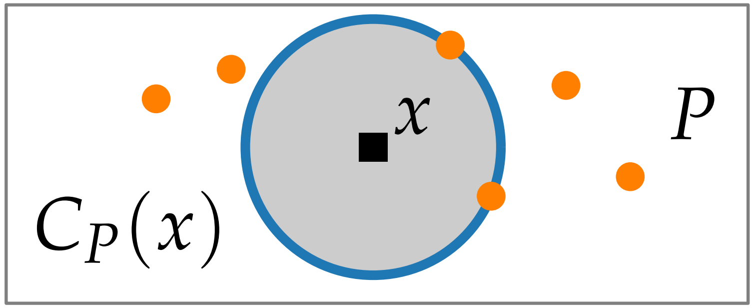

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Characterization of Voronoi vtc and edges

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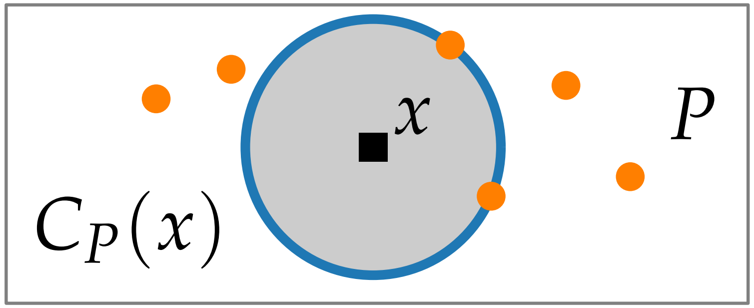

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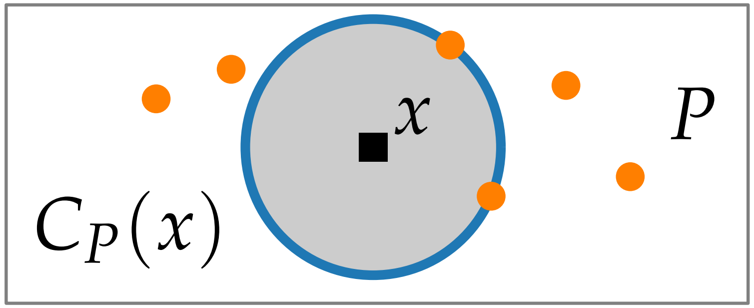

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(ii) $b(p, p')$ contains a Voronoi edge $\Leftrightarrow$
$\exists x \in b(p, p')\colon C_P(x) \cap P = \{p, p'\}$

Computation

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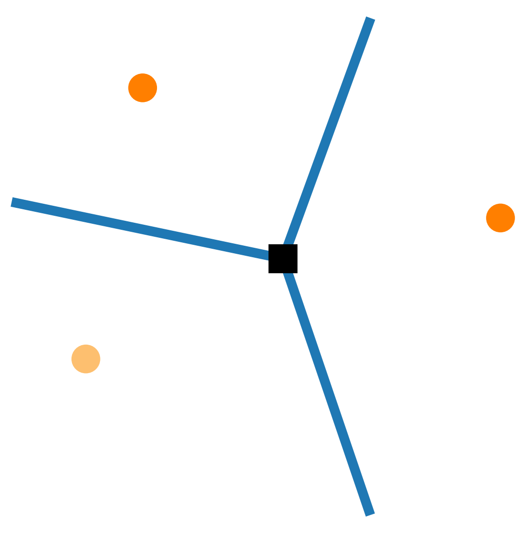

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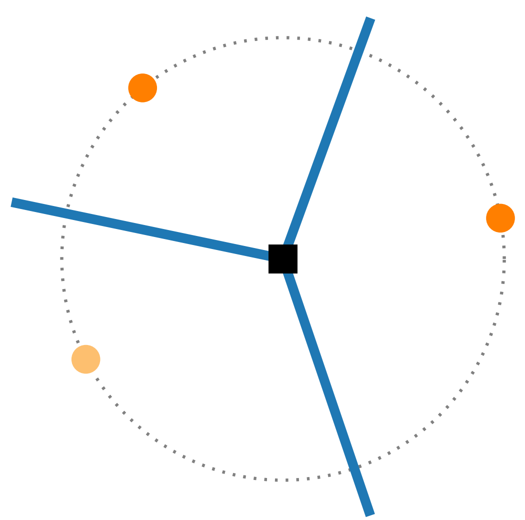

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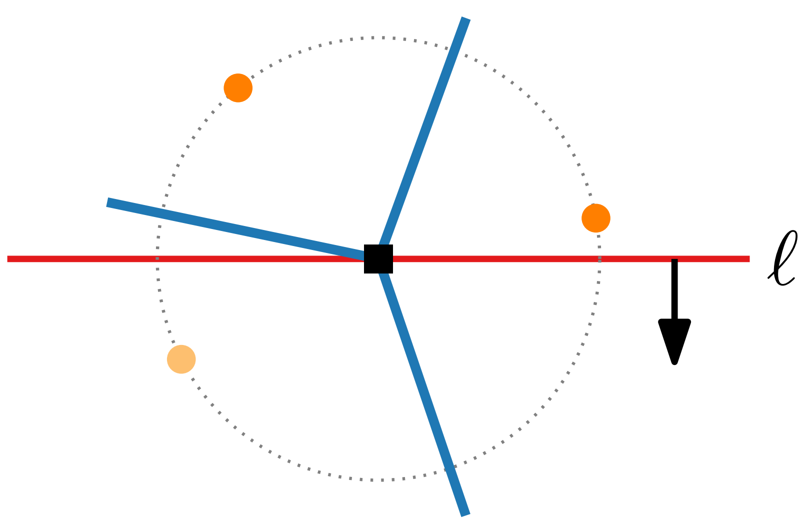

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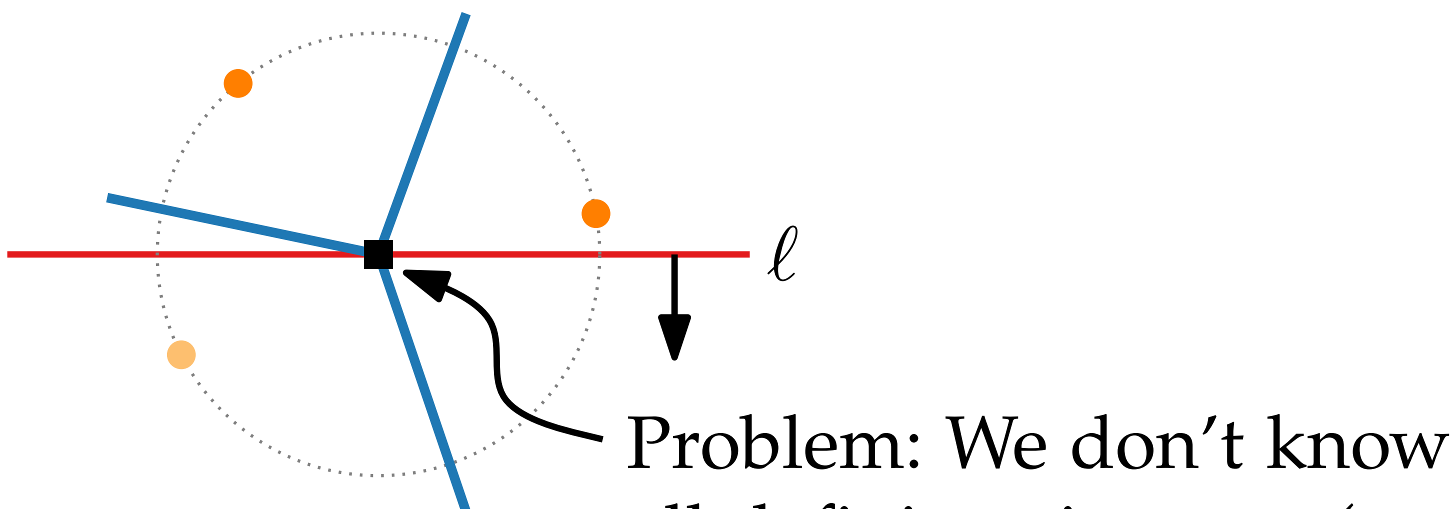

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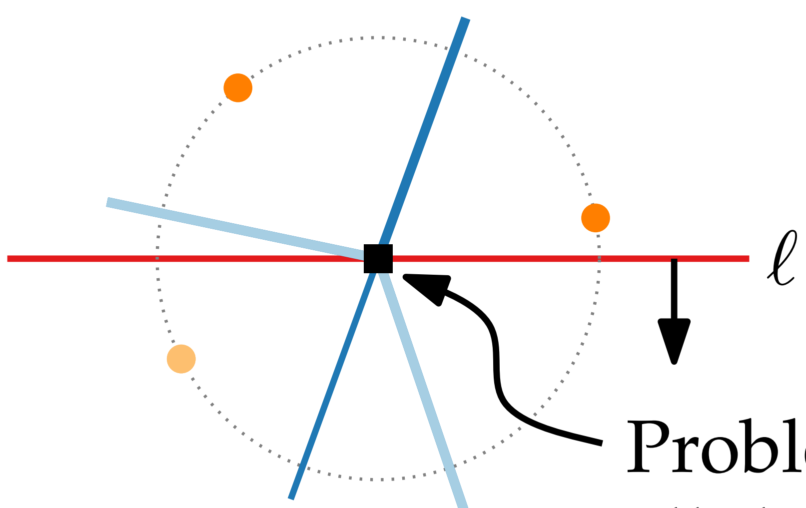

Problem: We don't know all defining sites yet :(

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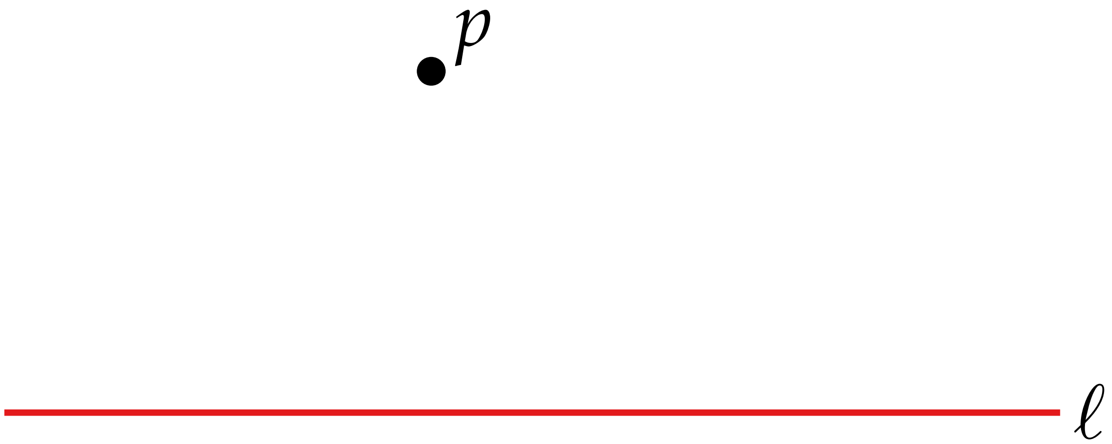

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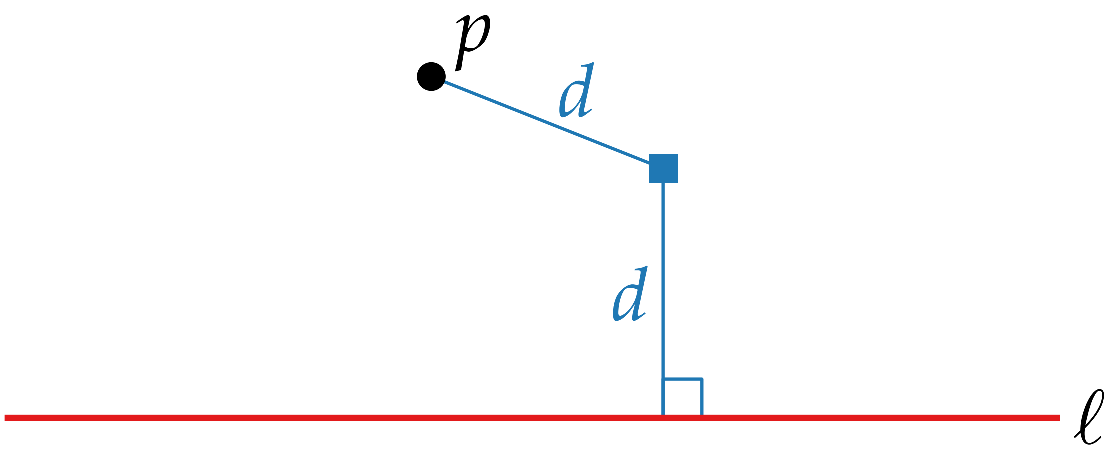

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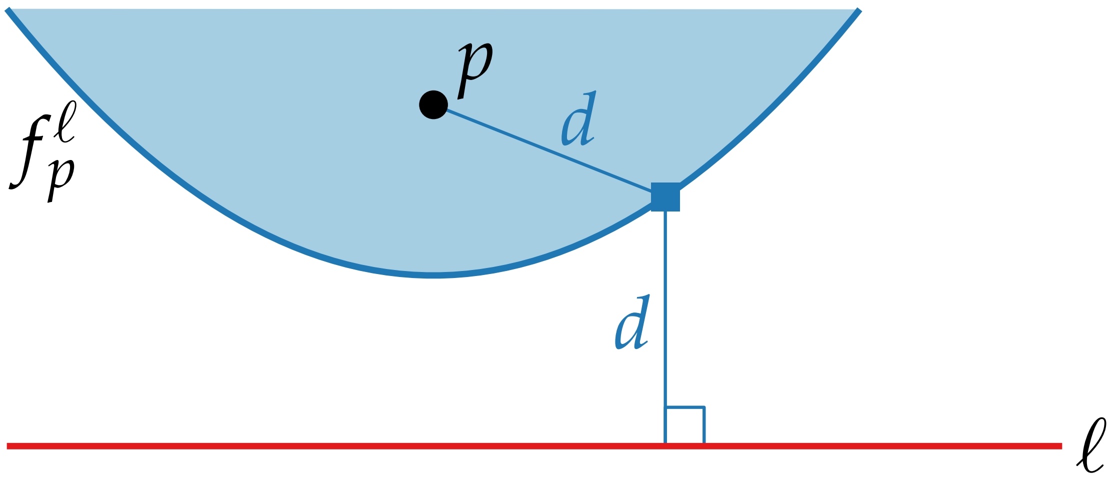

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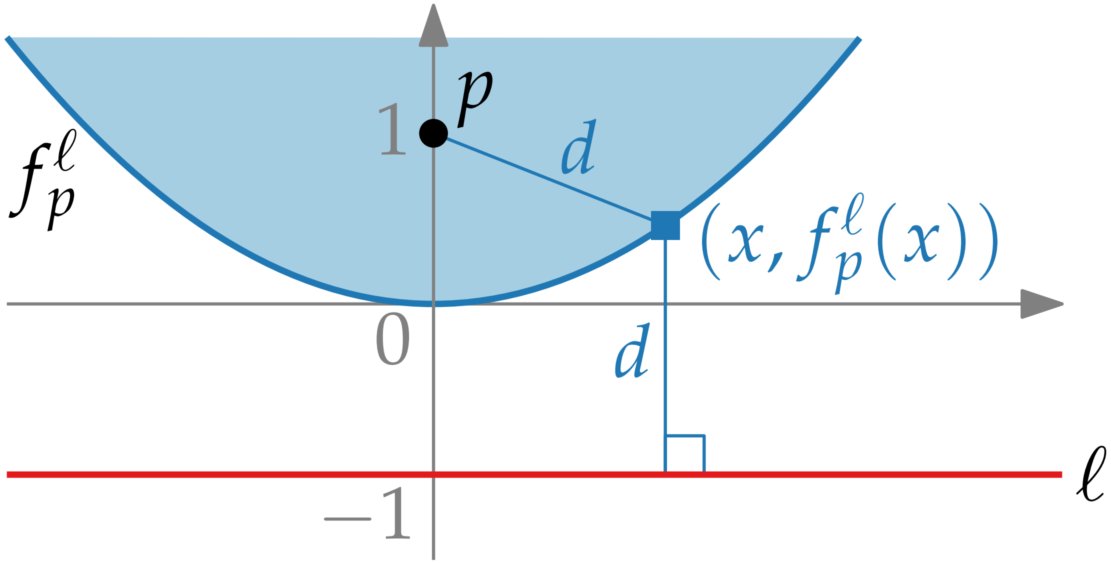

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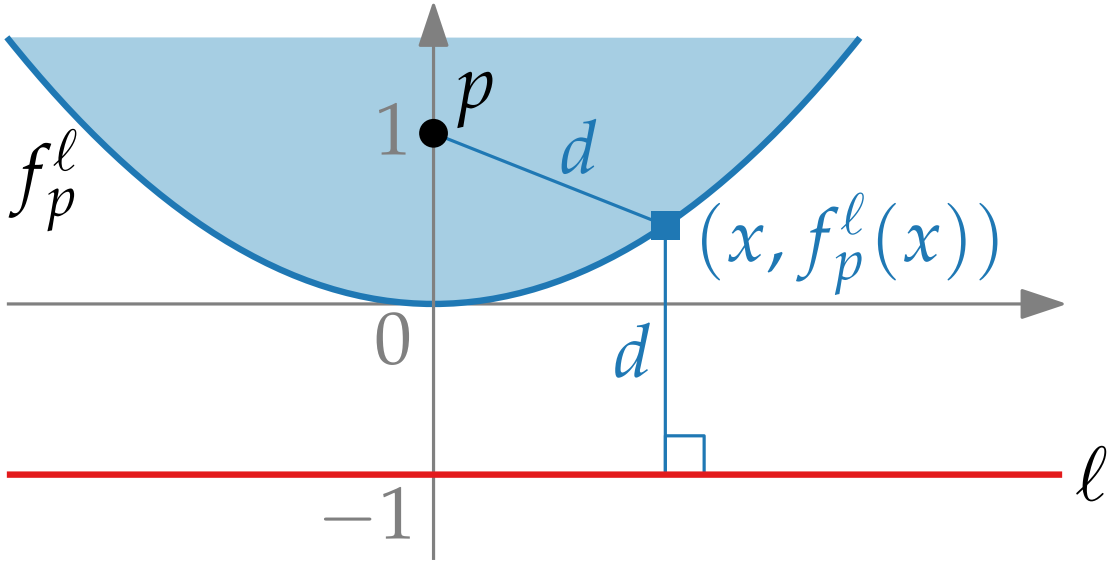

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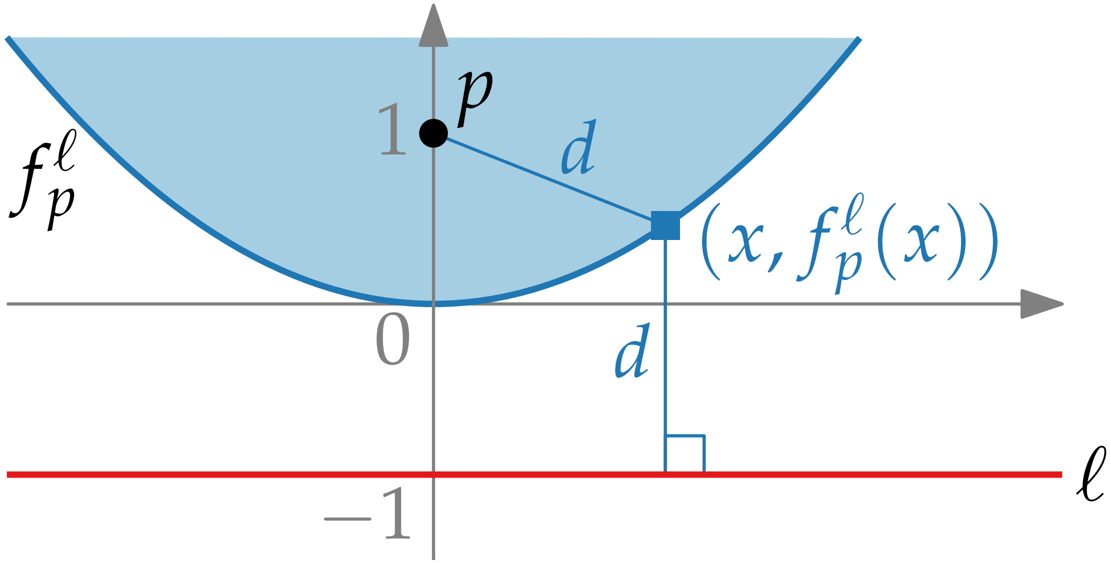

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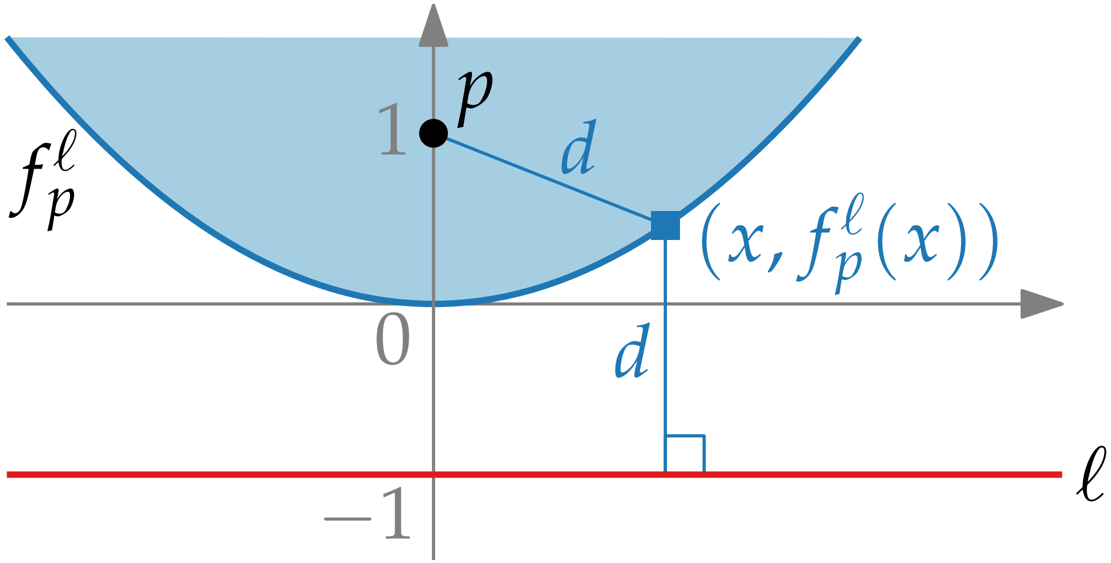

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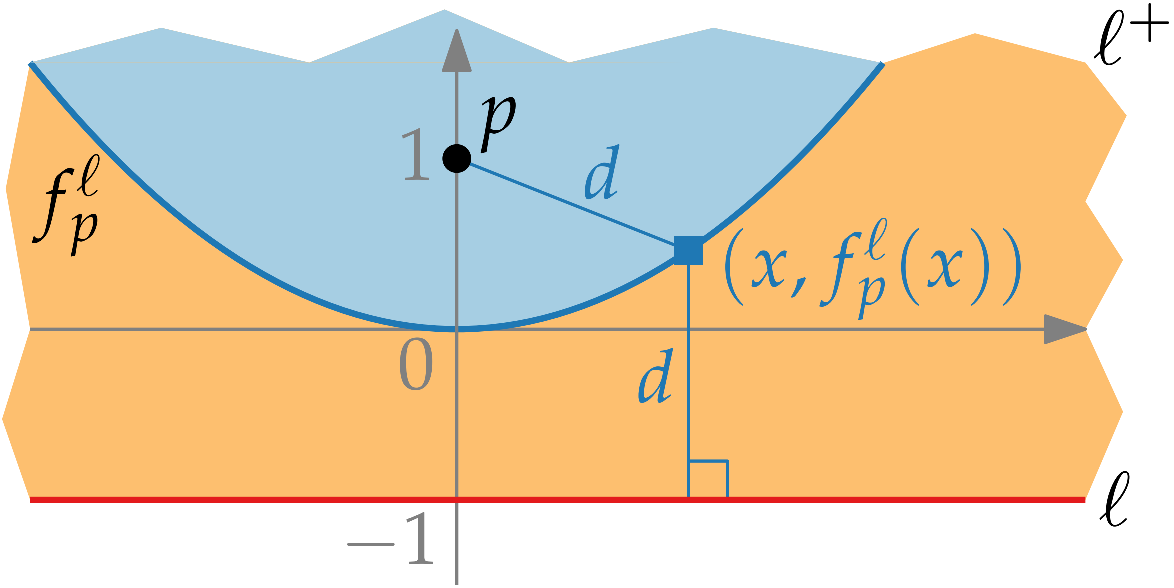

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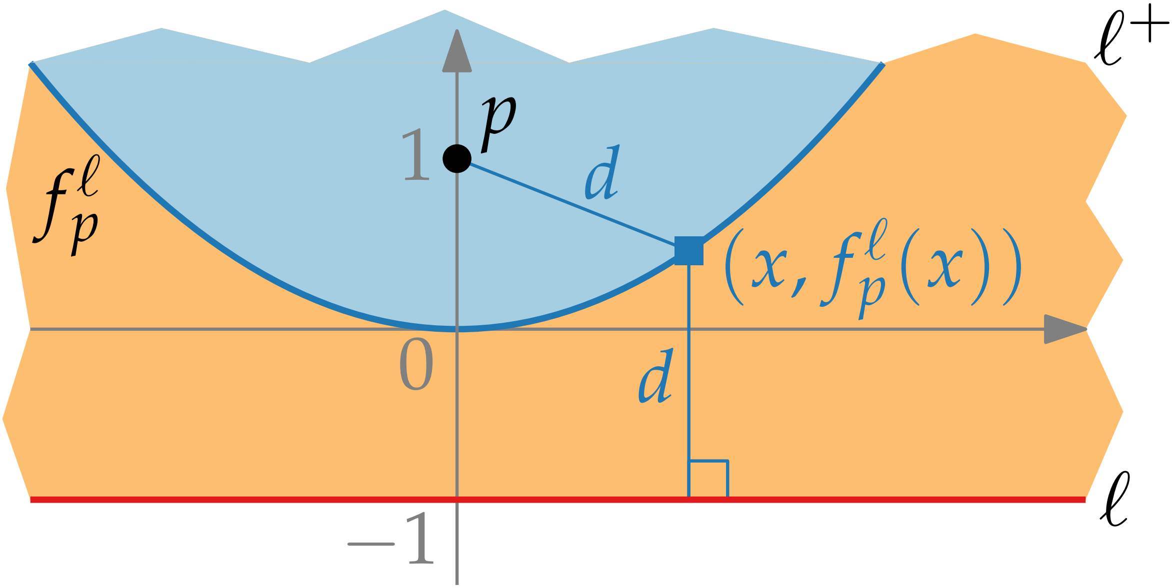

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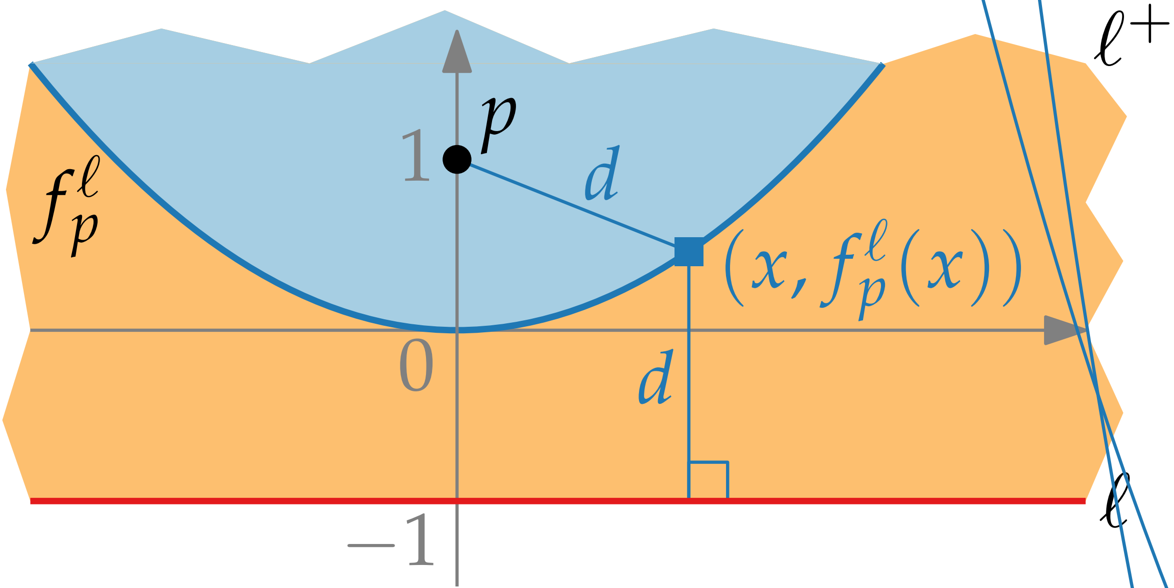

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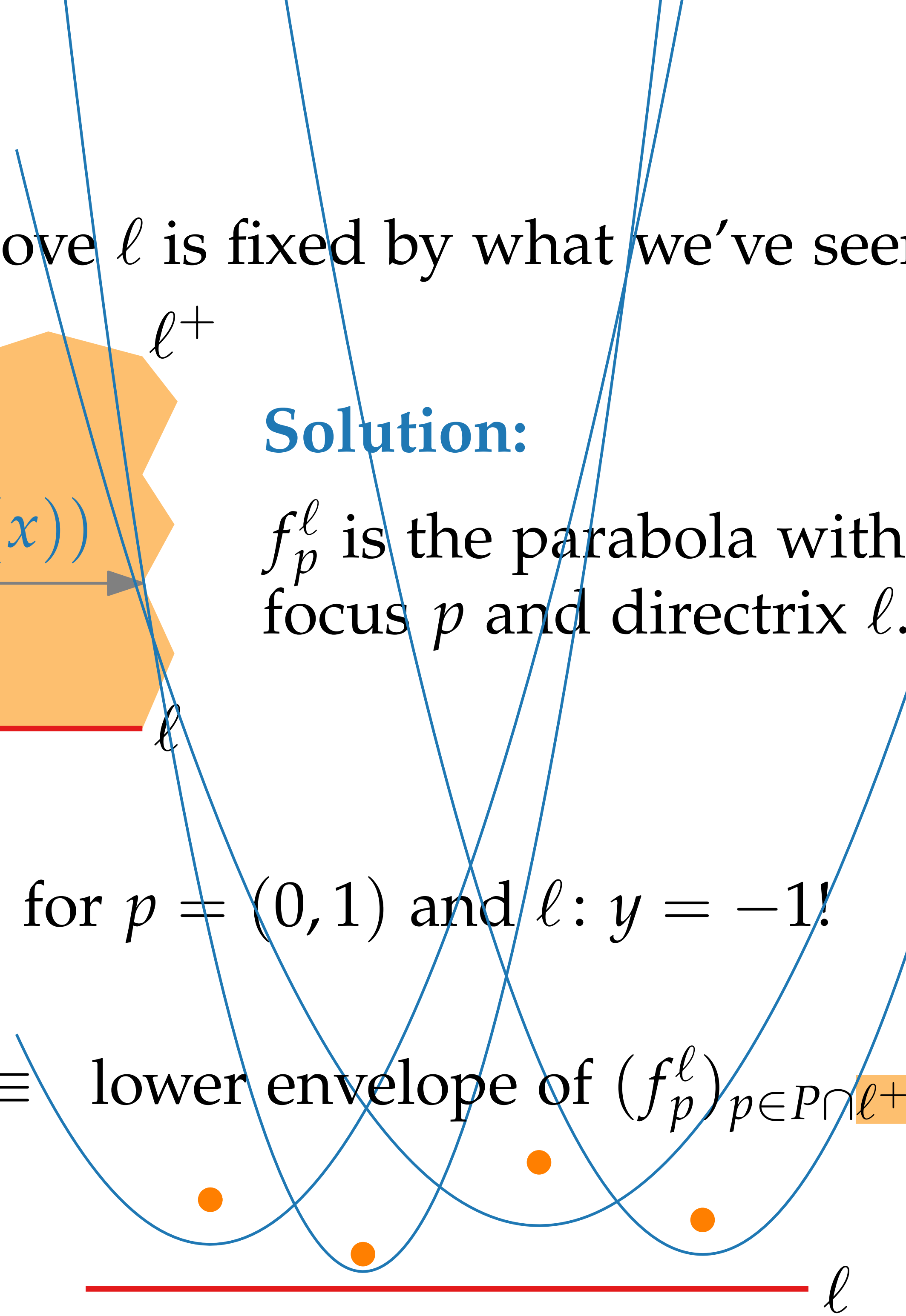

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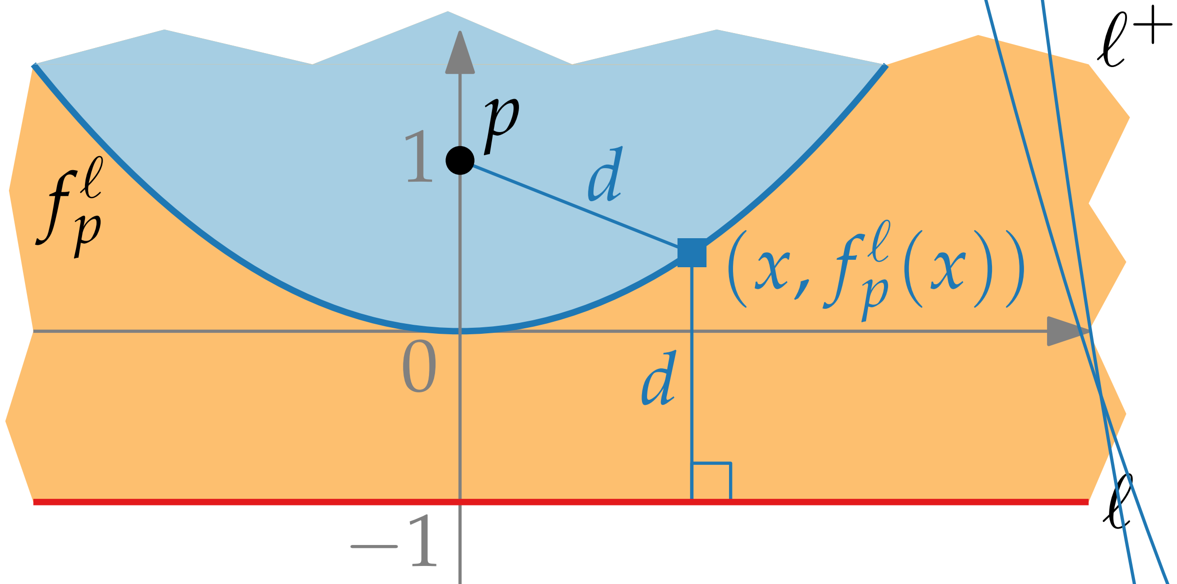

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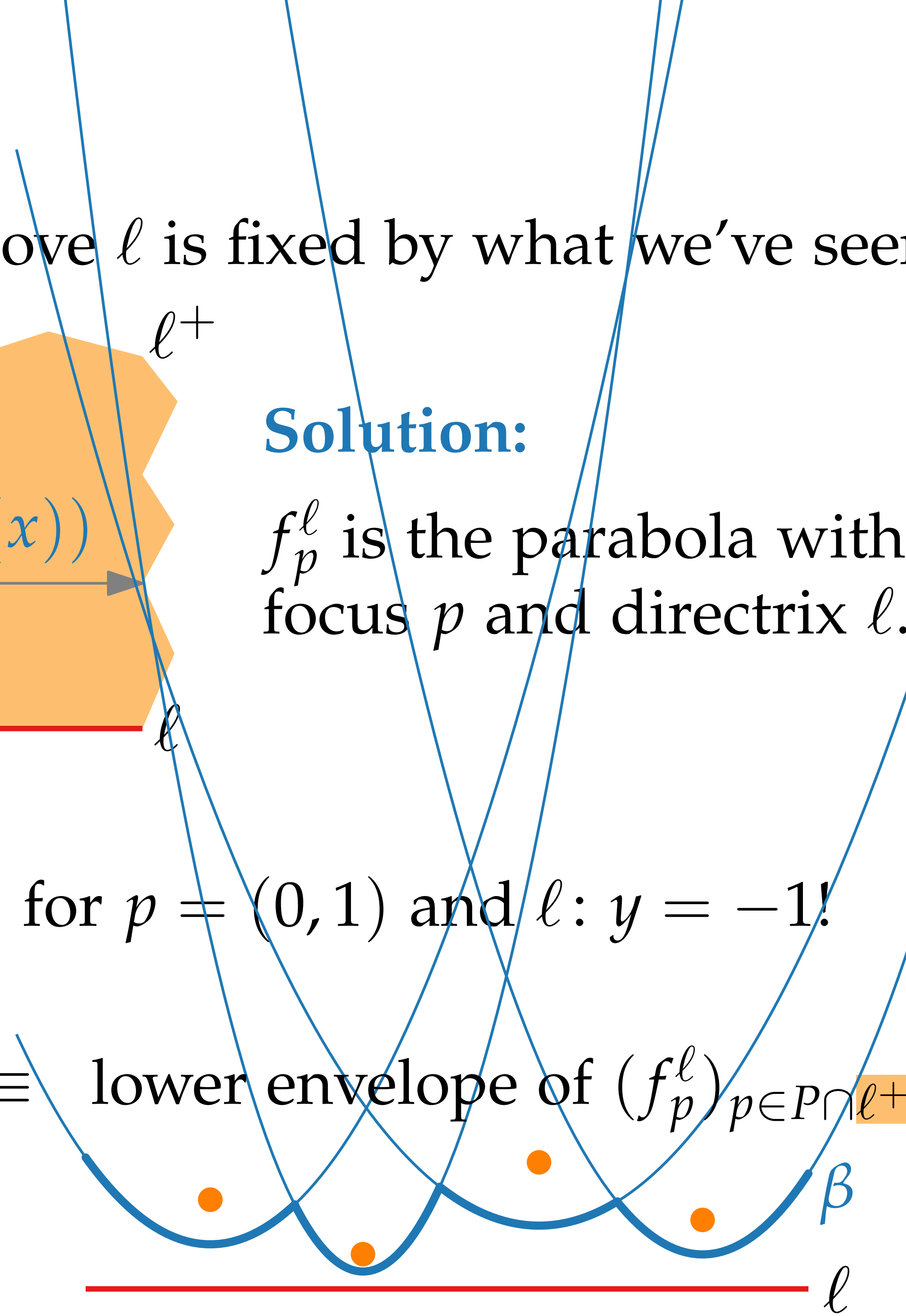

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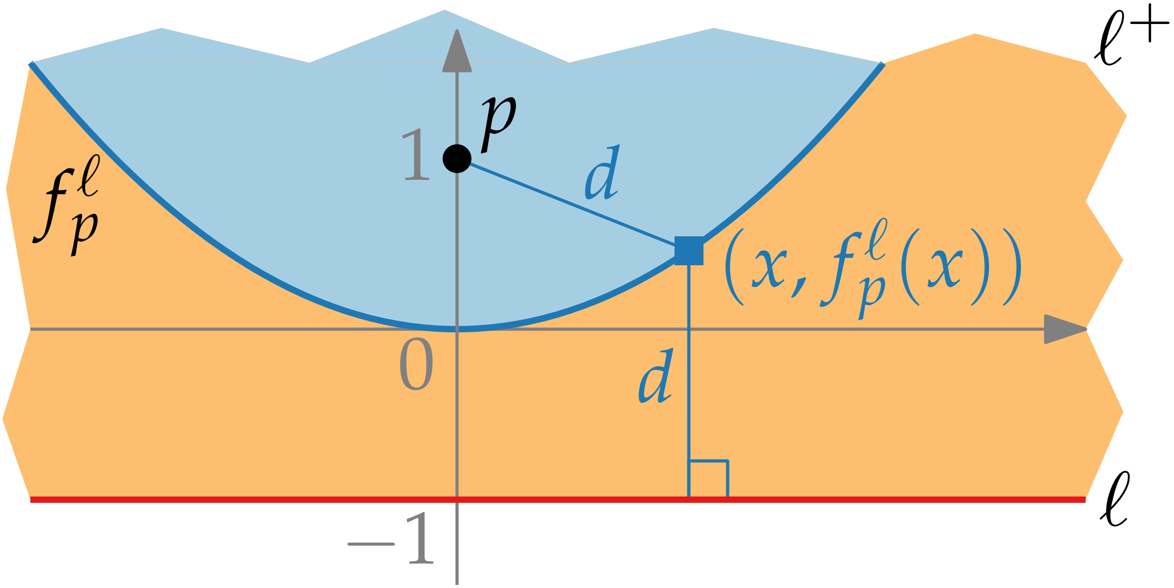

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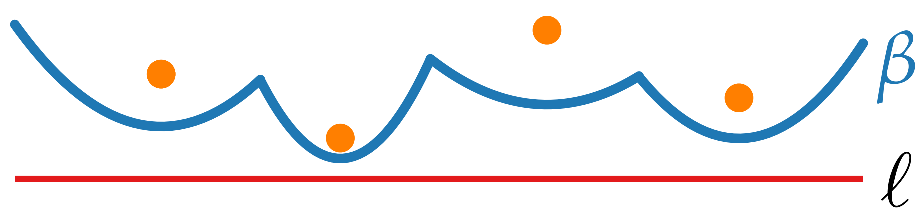

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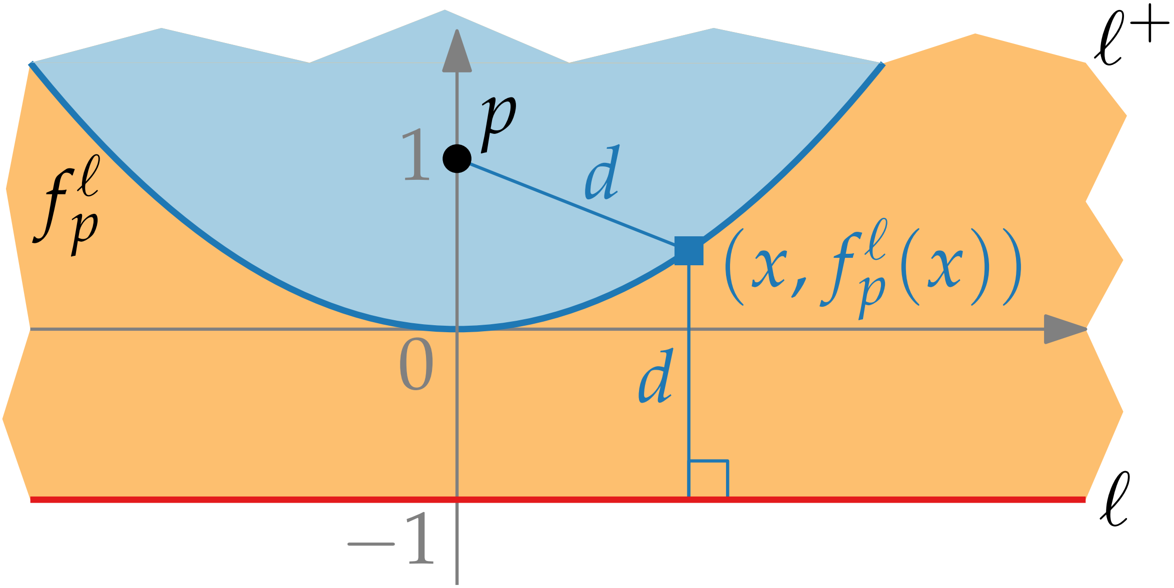

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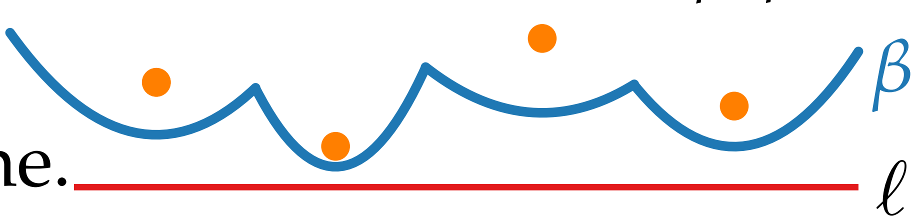
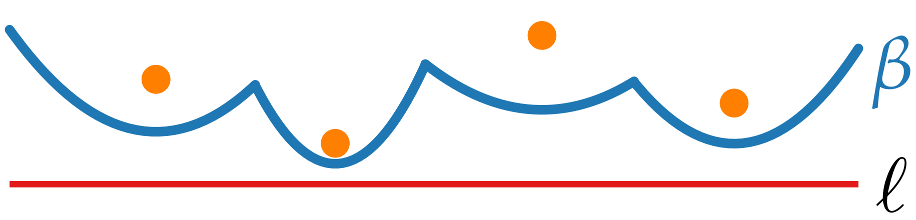
Observation. β is x-monotone.



The beachline β

Question: What does β have to do with $\text{Vor}(P)$?



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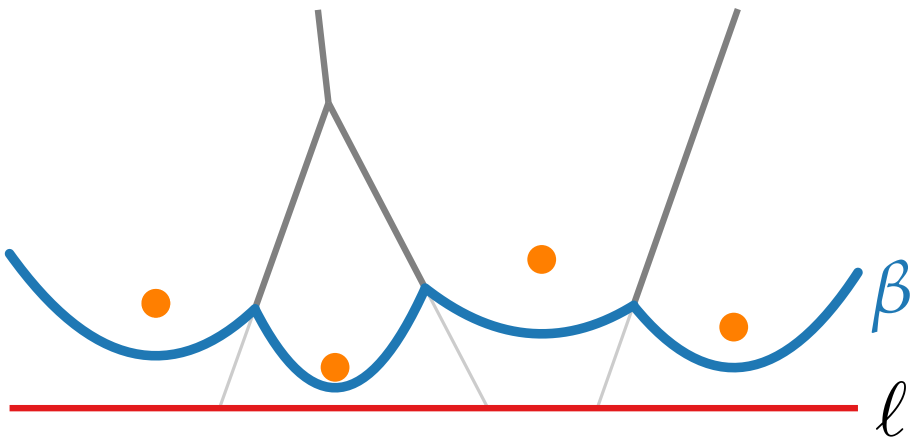

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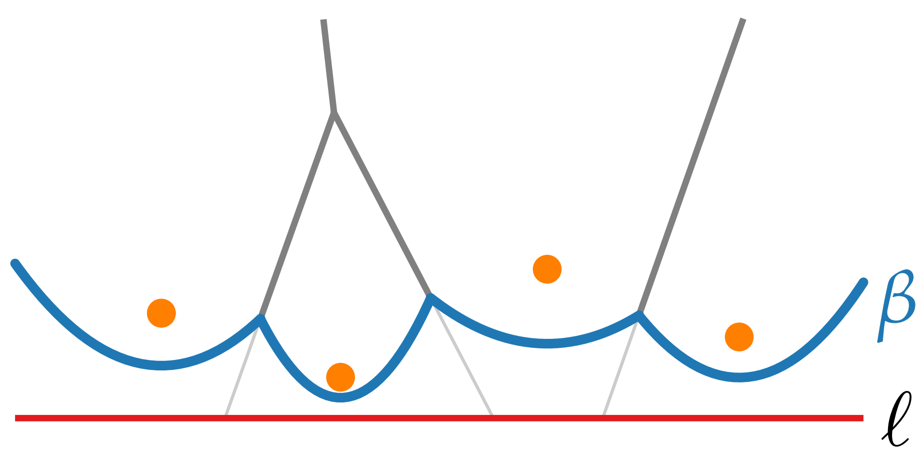

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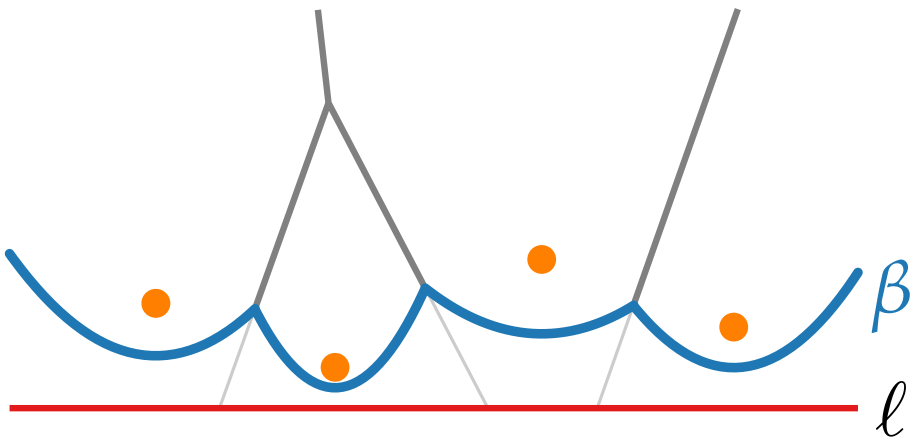

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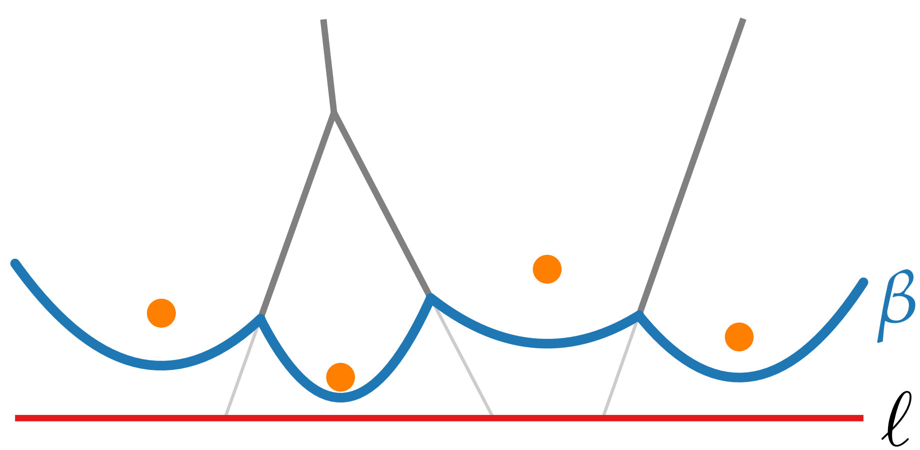

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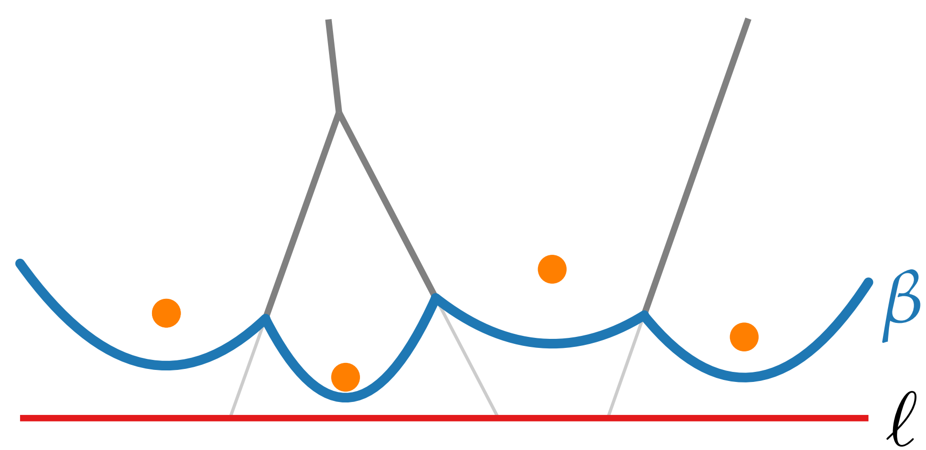
Corollary. β consists of at most $2n - 1$ arcs.



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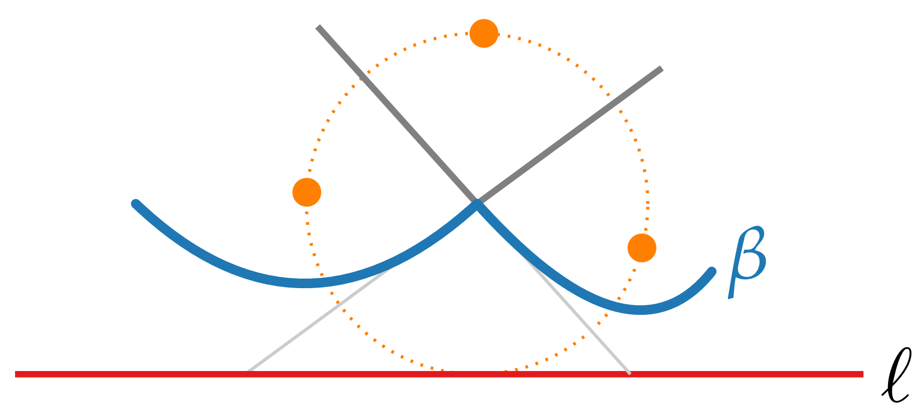

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Lemma. The Voronoi vtc correspond 1:1 to circle events.

Fortune's Sweep

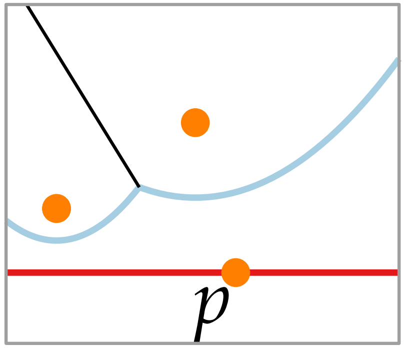
VoronoiDiagram($P \subset \mathbb{R}^2$)
$\mathcal{Q} \leftarrow$ new PriorityQueue(P) // site events sorted by y-coord.
$\mathcal{T} \leftarrow$ new BalancedBinarySearchTree() // sweep status (β)
$\mathcal{D} \leftarrow$ new DCEL() // to-be Vor(P)
while not $\mathcal{Q}$.empty() **do**

treat remaining int. nodes of $\mathcal{T}$ ($\equiv$ unbnd. edges of Vor(P))
return $\mathcal{D}$

Fortune's Sweep

```
VoronoiDiagram(P ⊂ ℝ²)
 Q ← new PriorityQueue(P)    // site events sorted by y-coord.
 T ← new BalancedBinarySearchTree()    // sweep status (β)
 D ← new DCEL()    // to-be Vor(P)
 while not Q.empty() do
     p ← Q.ExtractMax()
     if p site event then
         HandleSiteEvent(p)
     else
         α ← arc on β that will disappear
         HandleCircleEvent(α)
 treat remaining int. nodes of T (≡ unbnd. edges of Vor(P))
 return D
```

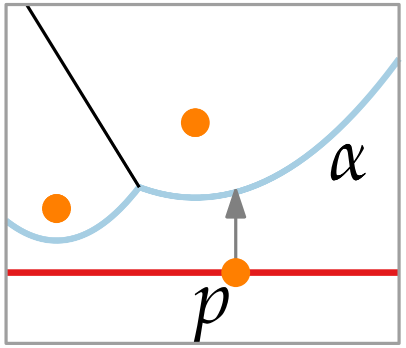
Handling Events



HandleSiteEvent(point p)

HandleCircleEvent(arc α)

Handling Events

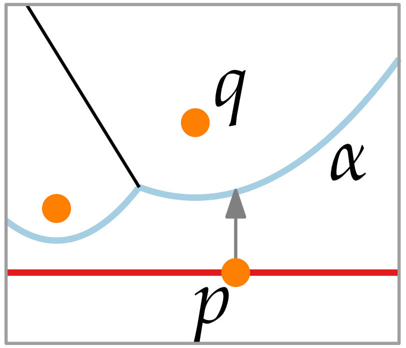


HandleSiteEvent(point p)

- Search in $\mathcal{T}$ for the arc α vertically above p.
  If α has pointer to circle event in $\mathcal{Q}$, delete this event.

HandleCircleEvent(arc α)

Handling Events



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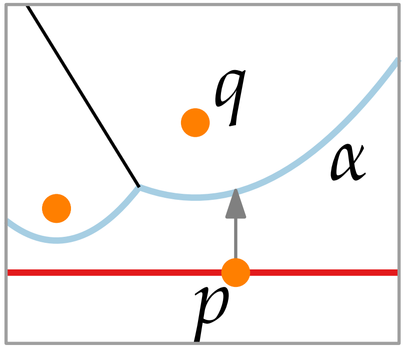

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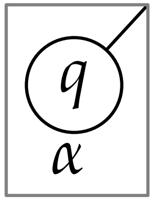

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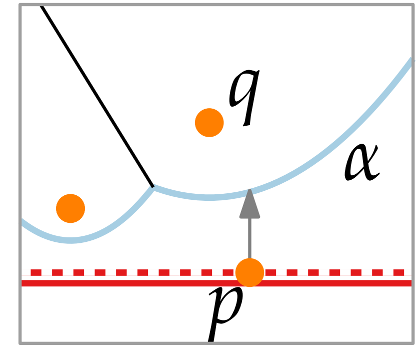

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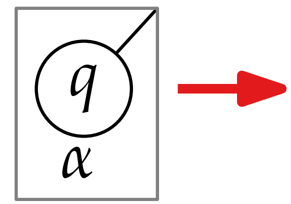
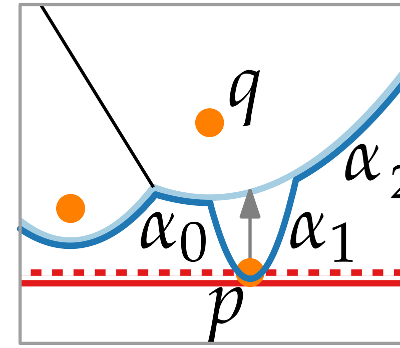


HandleCircleEvent(arc α)

Handling Events



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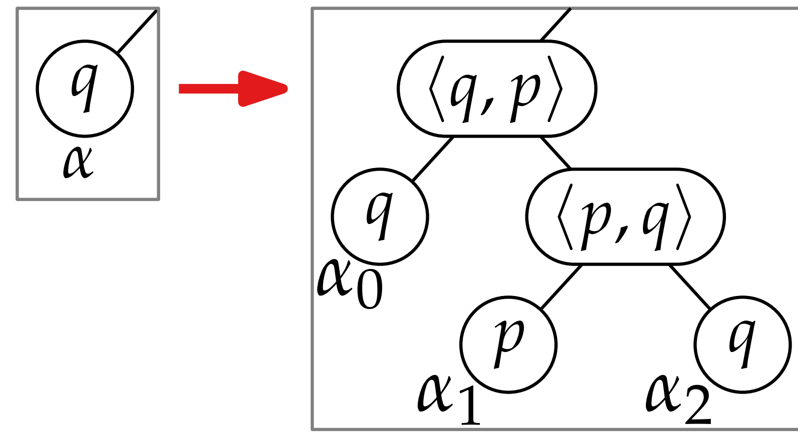

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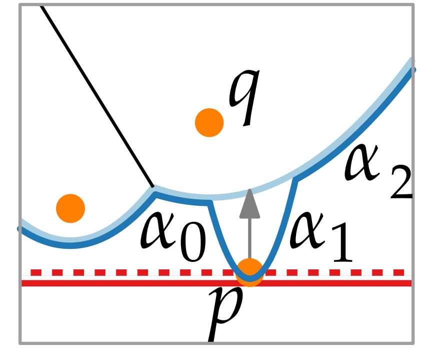

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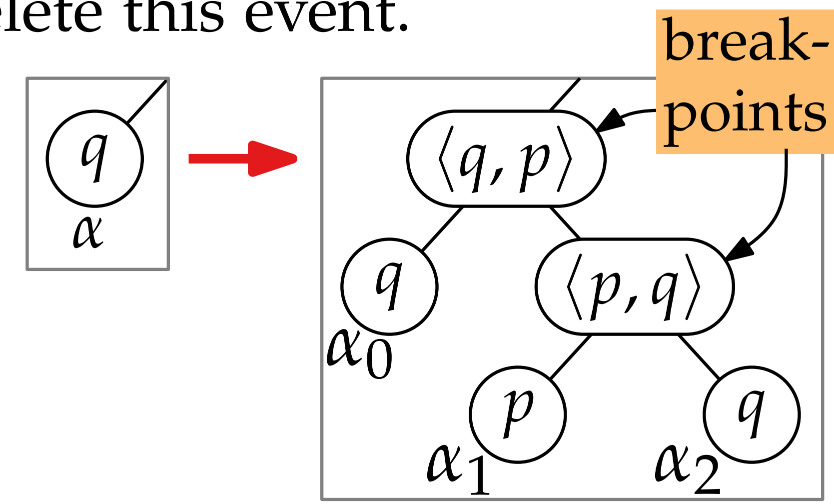

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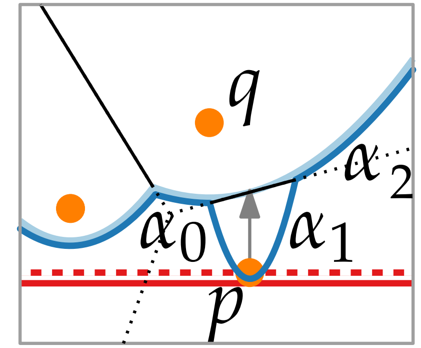

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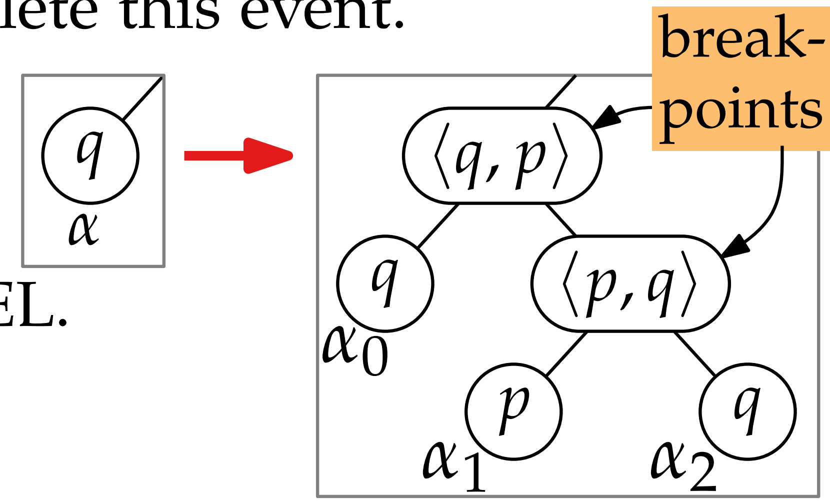

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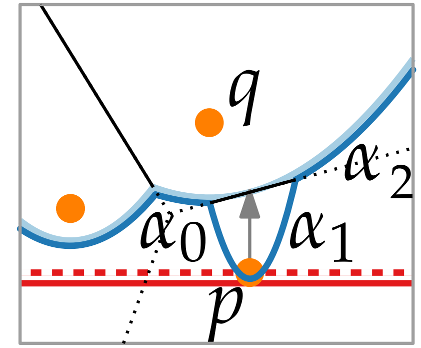
Handling Events



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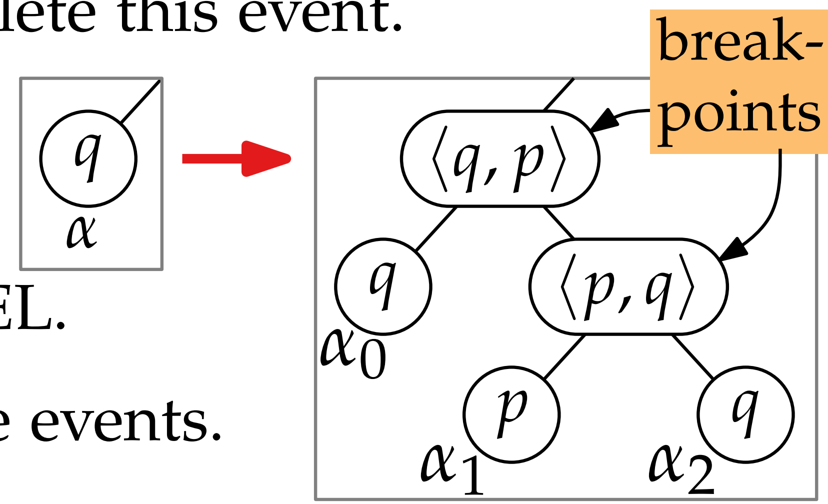

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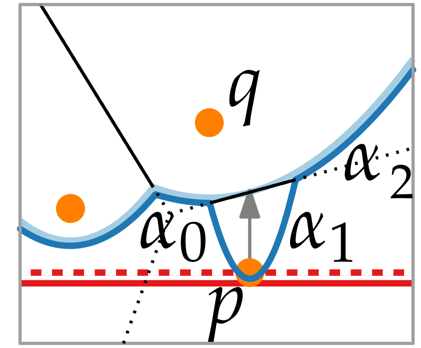

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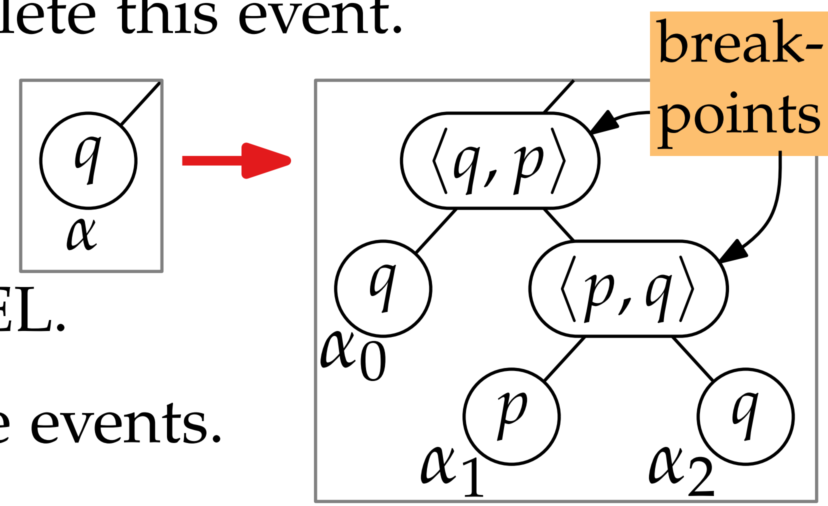

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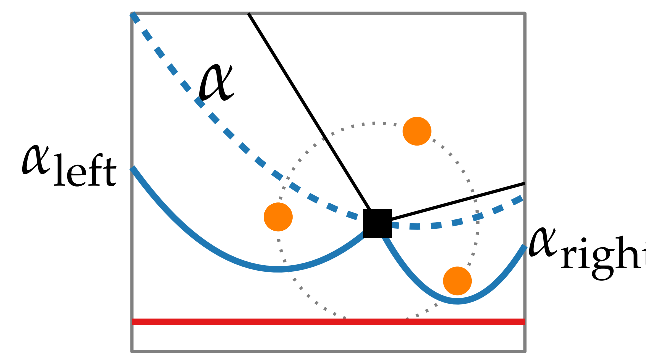

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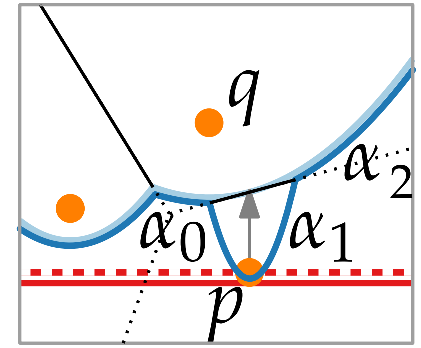

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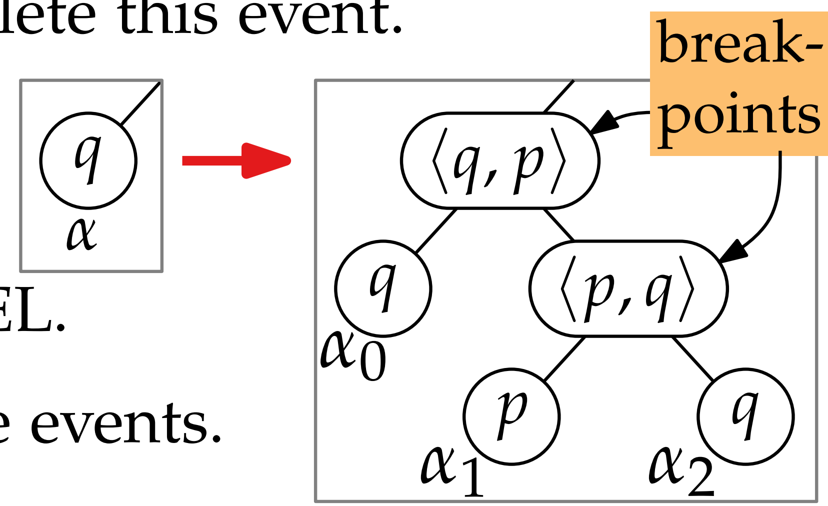

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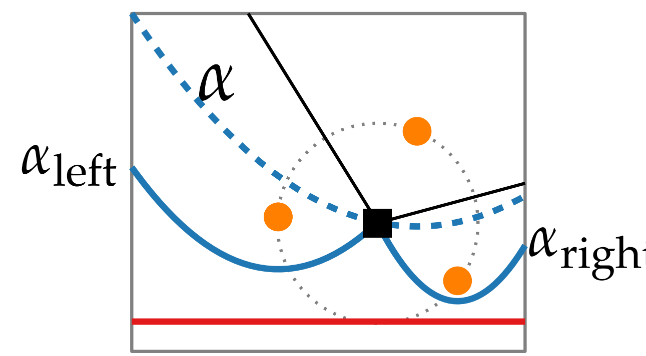

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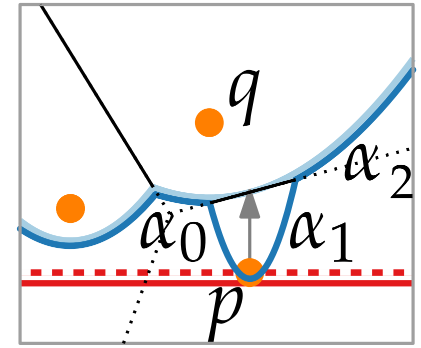

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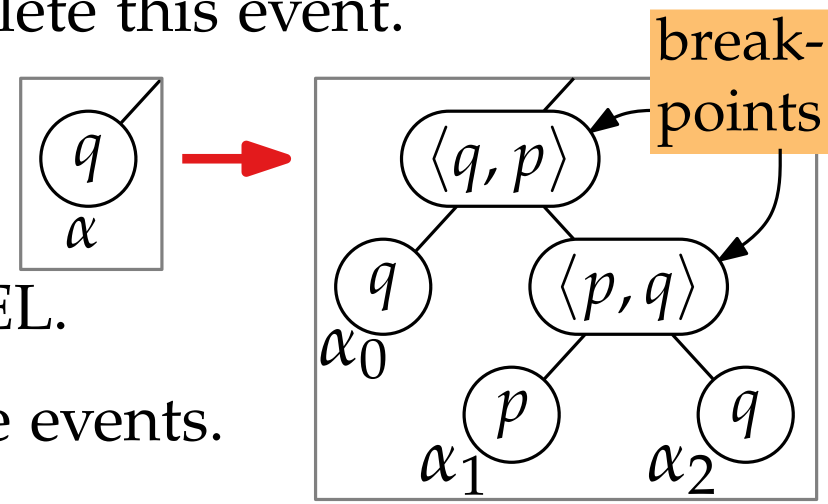

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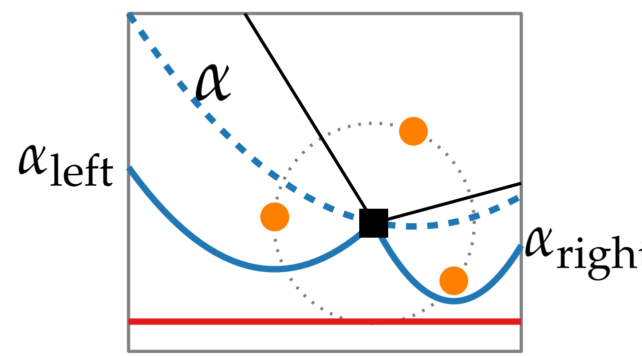

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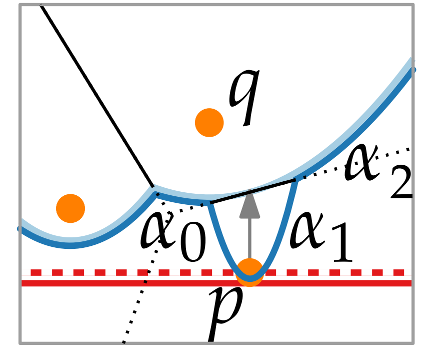

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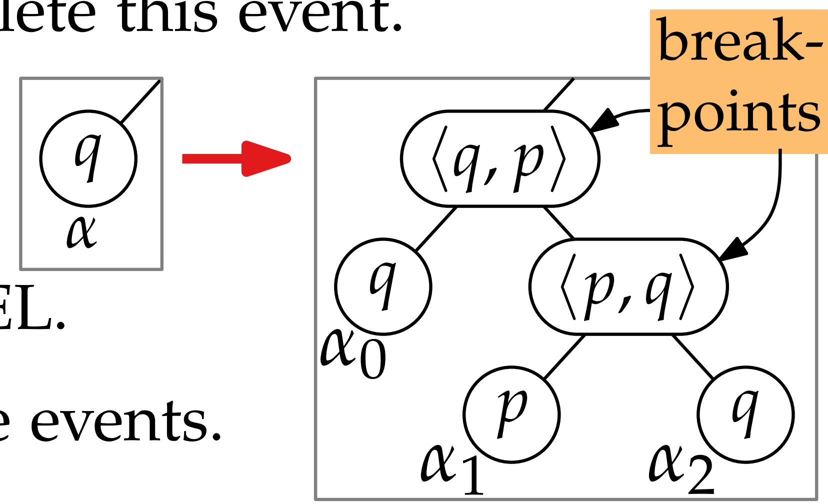

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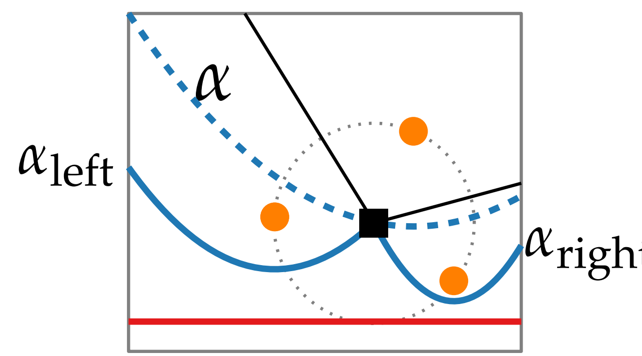

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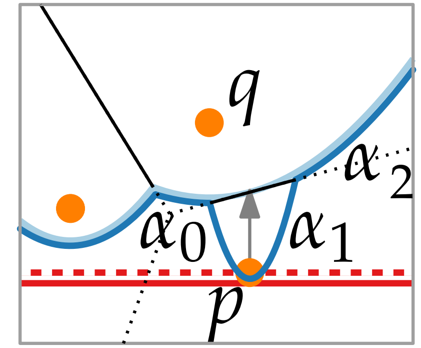

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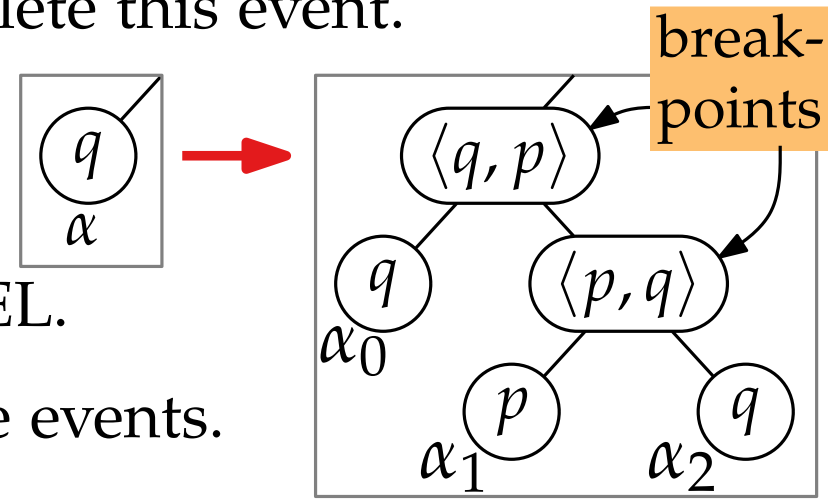

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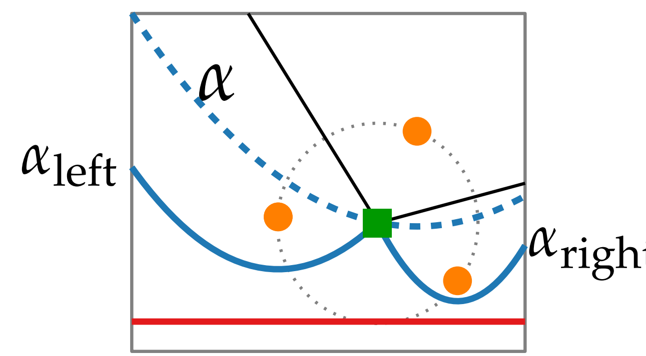

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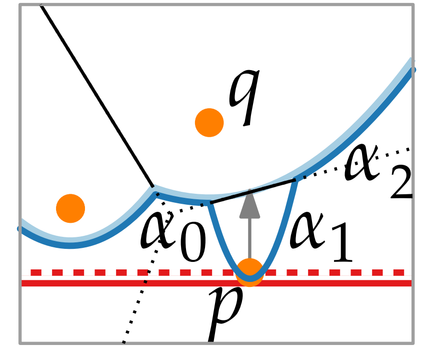

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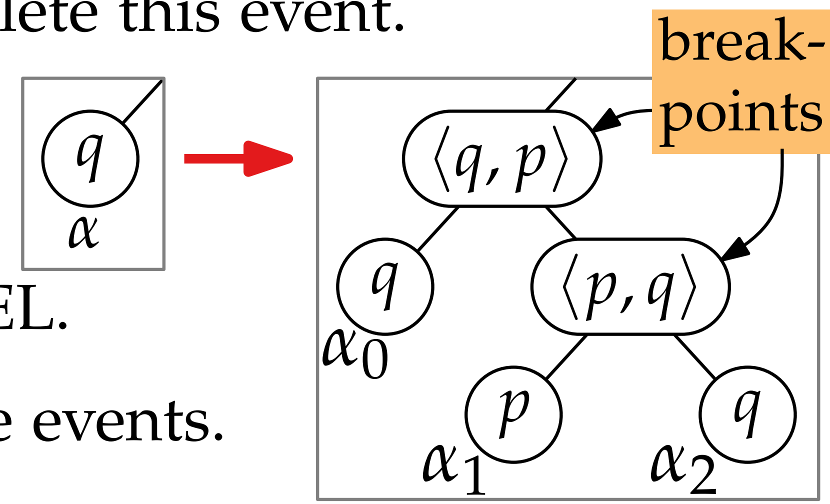

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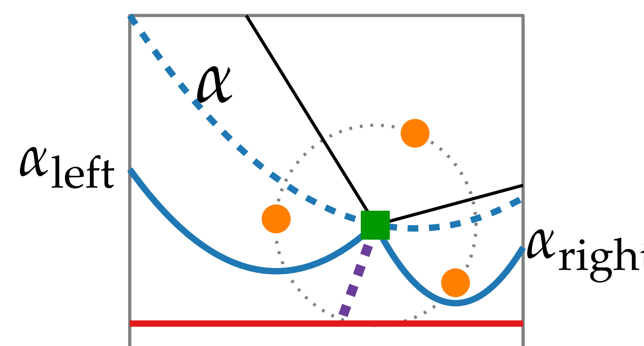

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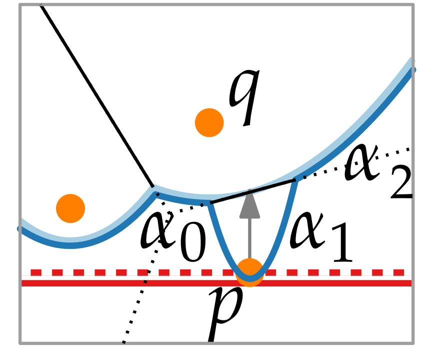

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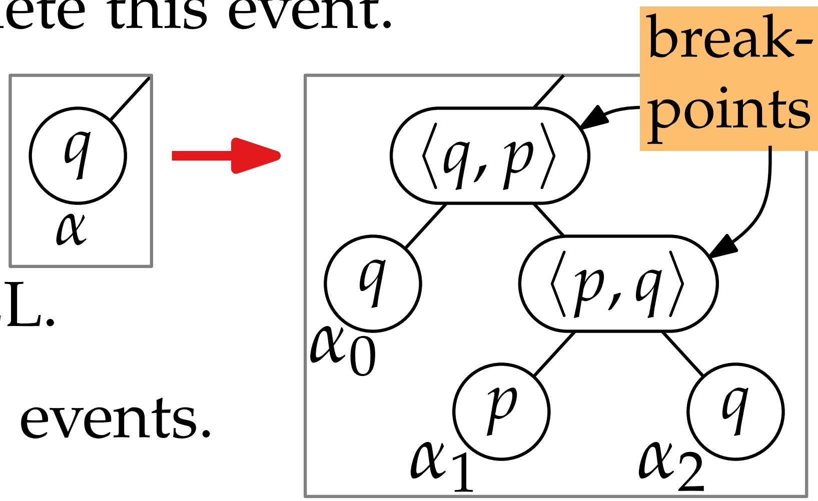

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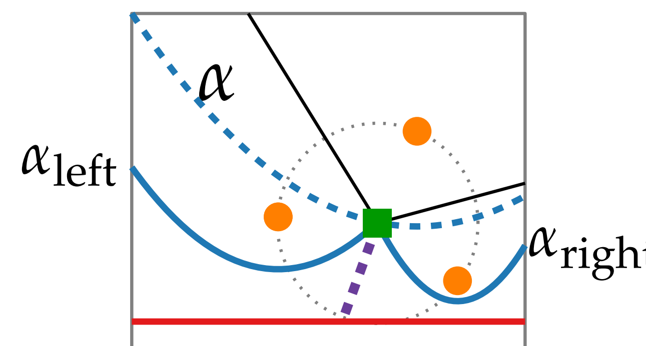

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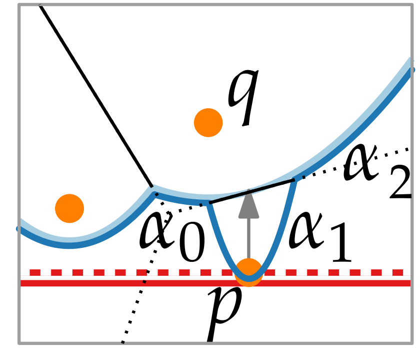

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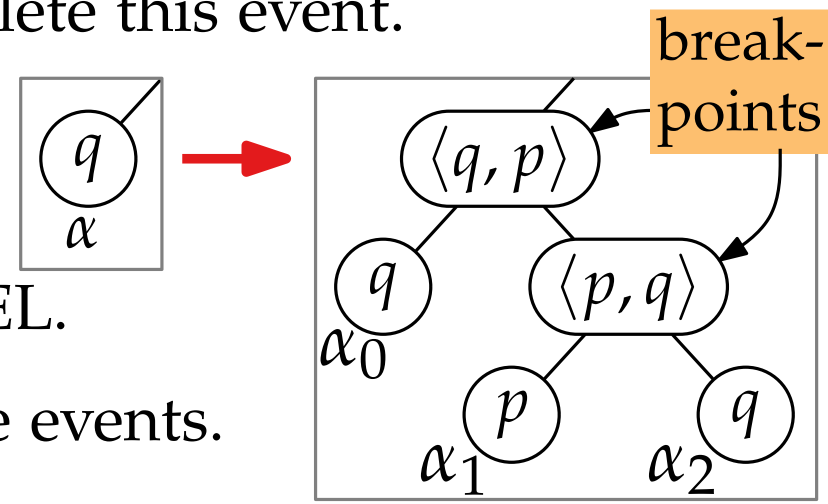

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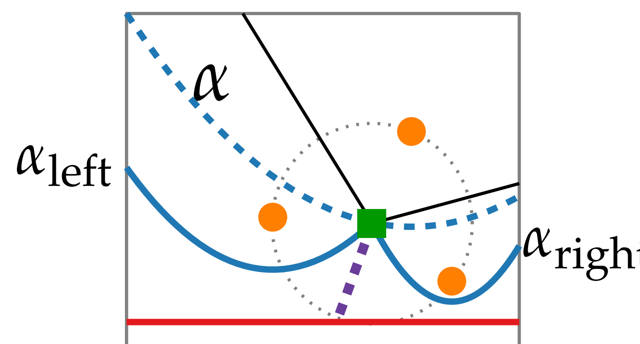

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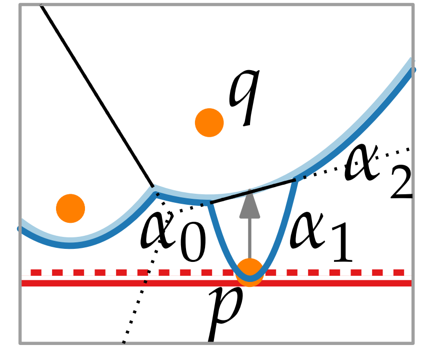
Running time?

Handling Events



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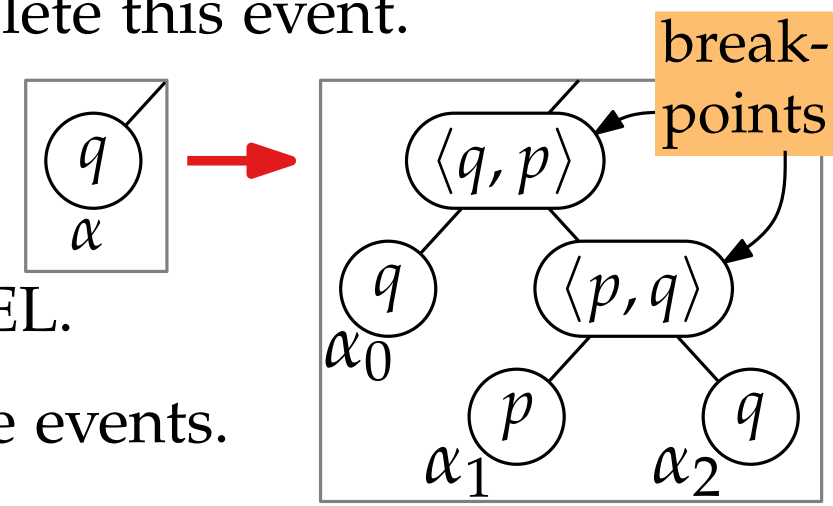

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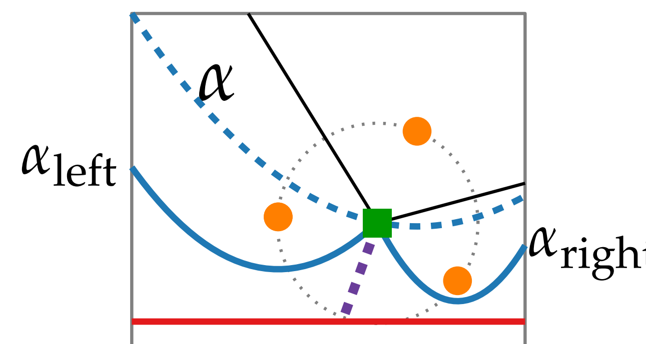

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Running time? $O(\log n)$ per event…

Running Time?

```
VoronoiDiagram(P ⊂ ℝ²)
  Q ← new PriorityQueue(P)      // site events sorted by y-coord.
  T ← new BalancedBinarySearchTree()     // sweep status (β)
  D ← new DCEL()     // to-be Vor(P)
  while not Q.empty() do
      p ← Q.ExtractMax()
      if p site event then
          HandleSiteEvent(p)
      else
          α ← arc on β that will disappear
          HandleCircleEvent(α)
  treat remaining int. nodes of T (≡ unbnd. edges of Vor(P))
  return D
```

Running Time?

VoronoiDiagram($P \subset \mathbb{R}^2$)

$\mathcal{Q} \leftarrow$ new PriorityQueue(P) // site events sorted by y-coord.

$\mathcal{T} \leftarrow$ new BalancedBinarySearchTree() // sweep status (β)

$\mathcal{D} \leftarrow$ new DCEL() // to-be Vor(P)

while not $\mathcal{Q}$.empty() **do**

  $p \leftarrow \mathcal{Q}$.ExtractMax()

  if p site event **then**

    HandleSiteEvent(p) exactly n such events

  else

    $\alpha \leftarrow$ arc on β that will disappear

    HandleCircleEvent(α)

treat remaining int. nodes of $\mathcal{T}$ ($\equiv$ unbnd. edges of Vor(P))

return $\mathcal{D}$

Running Time?

```
VoronoiDiagram(P ⊂ ℝ²)
 𝒬 ← new PriorityQueue(P)        // site events sorted by y-coord.
 𝒯 ← new BalancedBinarySearchTree()    // sweep status (β)
 𝒟 ← new DCEL()    // to-be Vor(P)
 while not 𝒬.empty() do
     p ← 𝒬.ExtractMax()
     if p site event then
         HandleSiteEvent(p)        exactly n such events
     else
         α ← arc on β that will disappear
         HandleCircleEvent(α)      at most 2n − 5 such events
 treat remaining int. nodes of 𝒯 (≡ unbnd. edges of Vor(P))
 return 𝒟
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Summary

Theorem. Given a set P of n pts in the plane, Fortune's sweep computes $\text{Vor}(P)$ in $O(n \log n)$ time and $O(n)$ space.

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Bell Labs

Steven Fortune. A sweepline algorithm for Voronoi diagrams. *Proc. 2nd Annual ACM Symposium on Computational Geometry.* Yorktown Heights, NY, pp. 313–322. 1986.