Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Computational Geometry

Orthogonal Range Queries
or
Fast Access to Data Bases

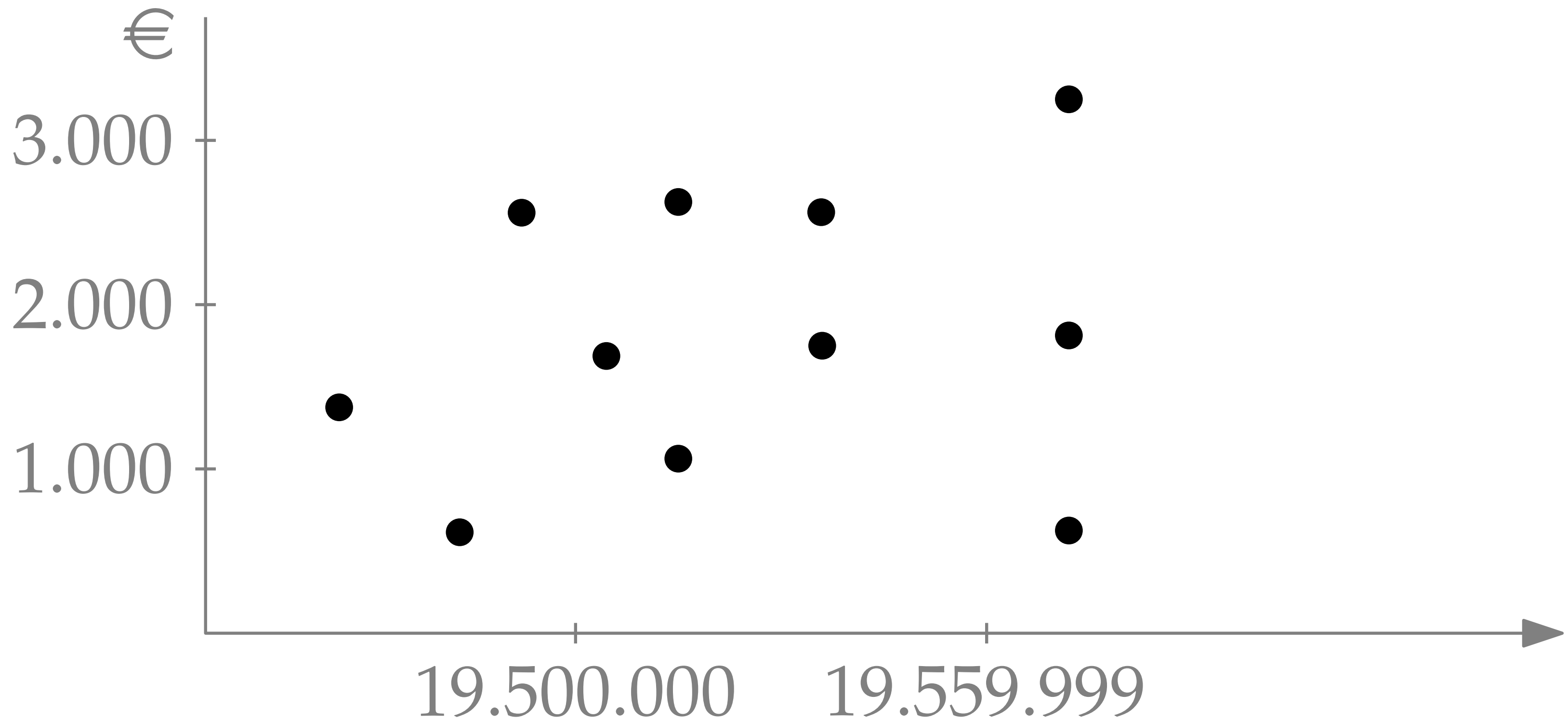
Lecture #4

Dr. Philipp Kindermann

Winter Semester 2019/20

Orthogonal Range Queries

Example: Personnel management in a company



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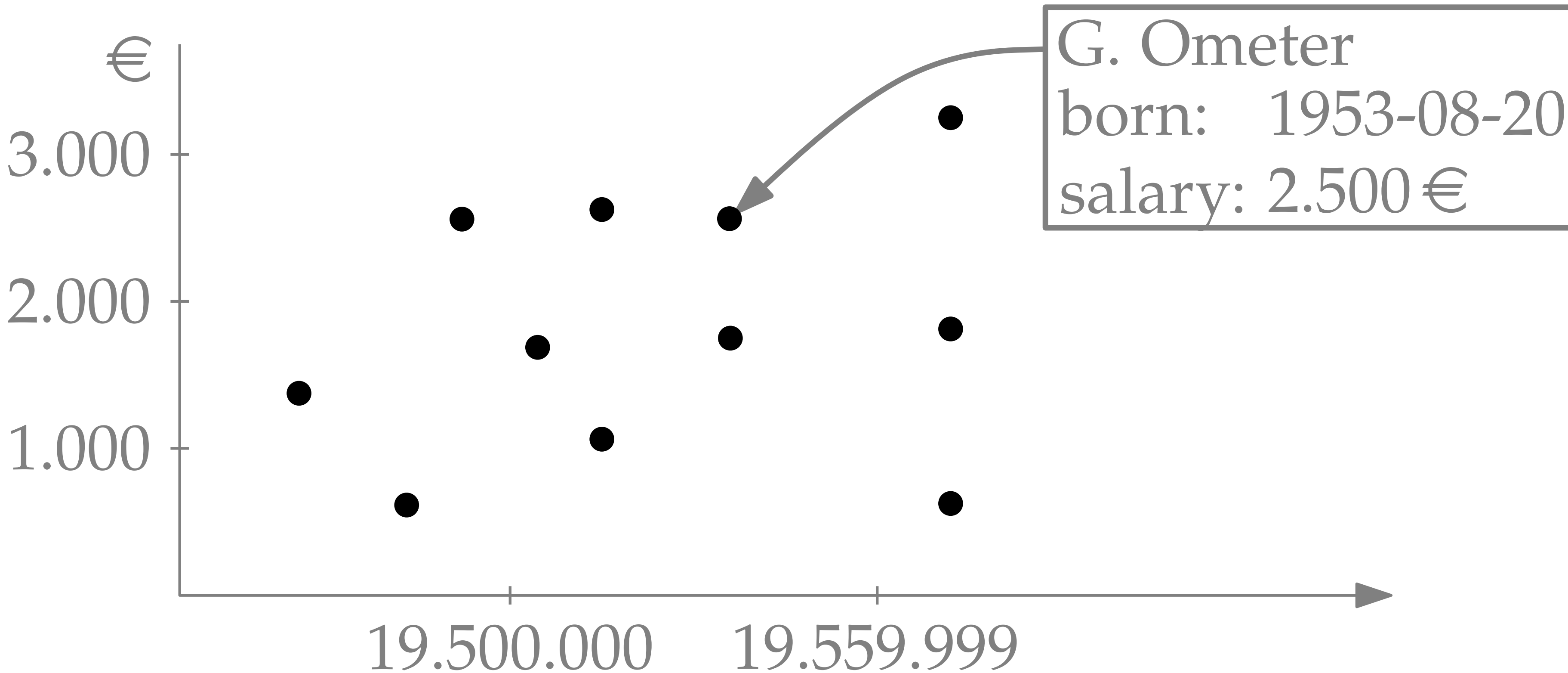

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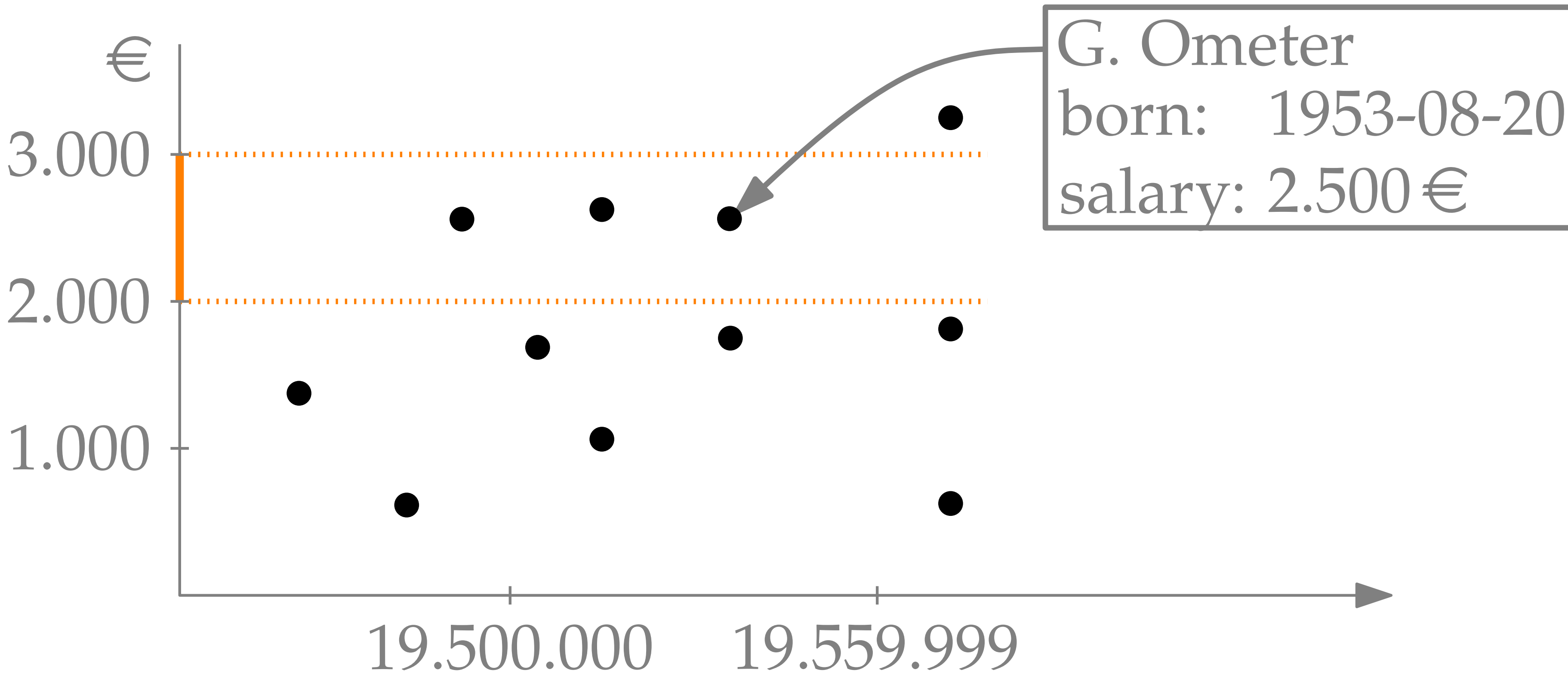

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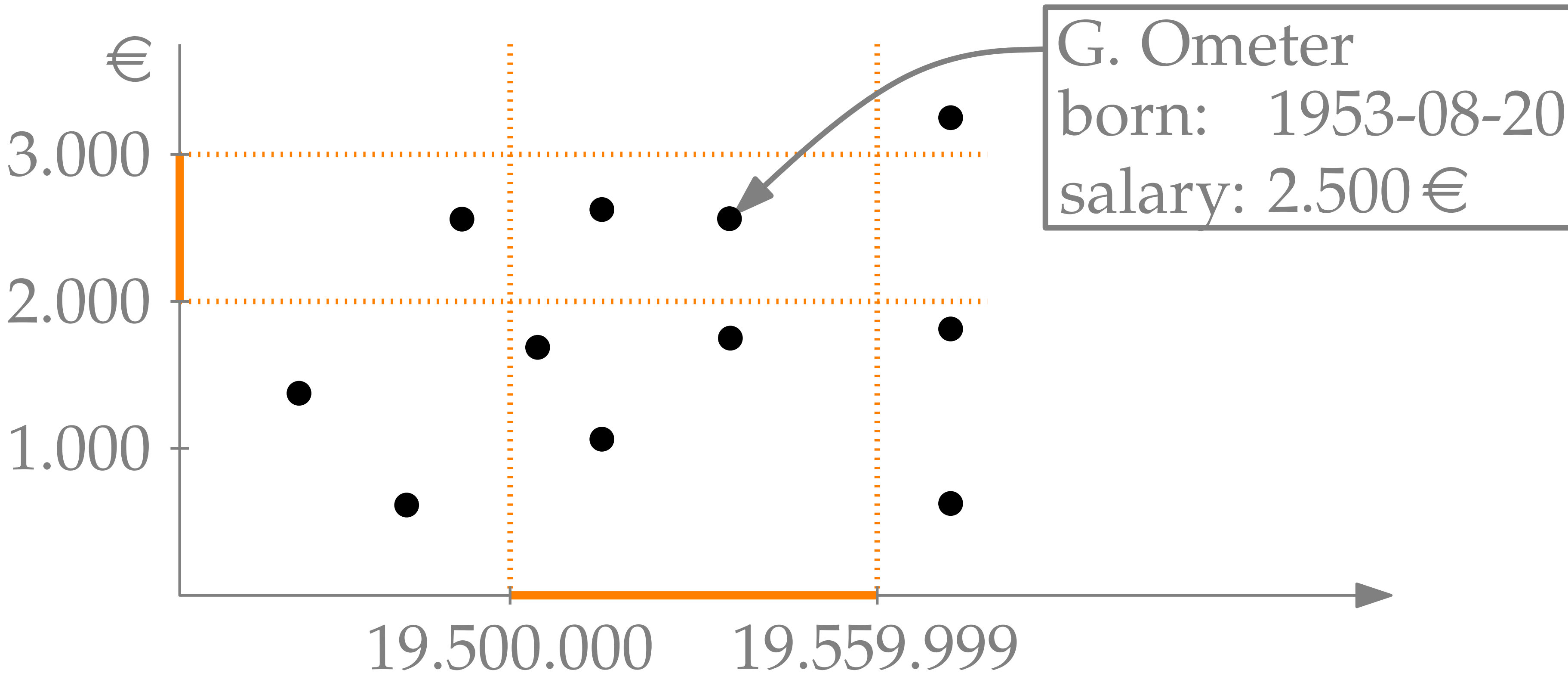

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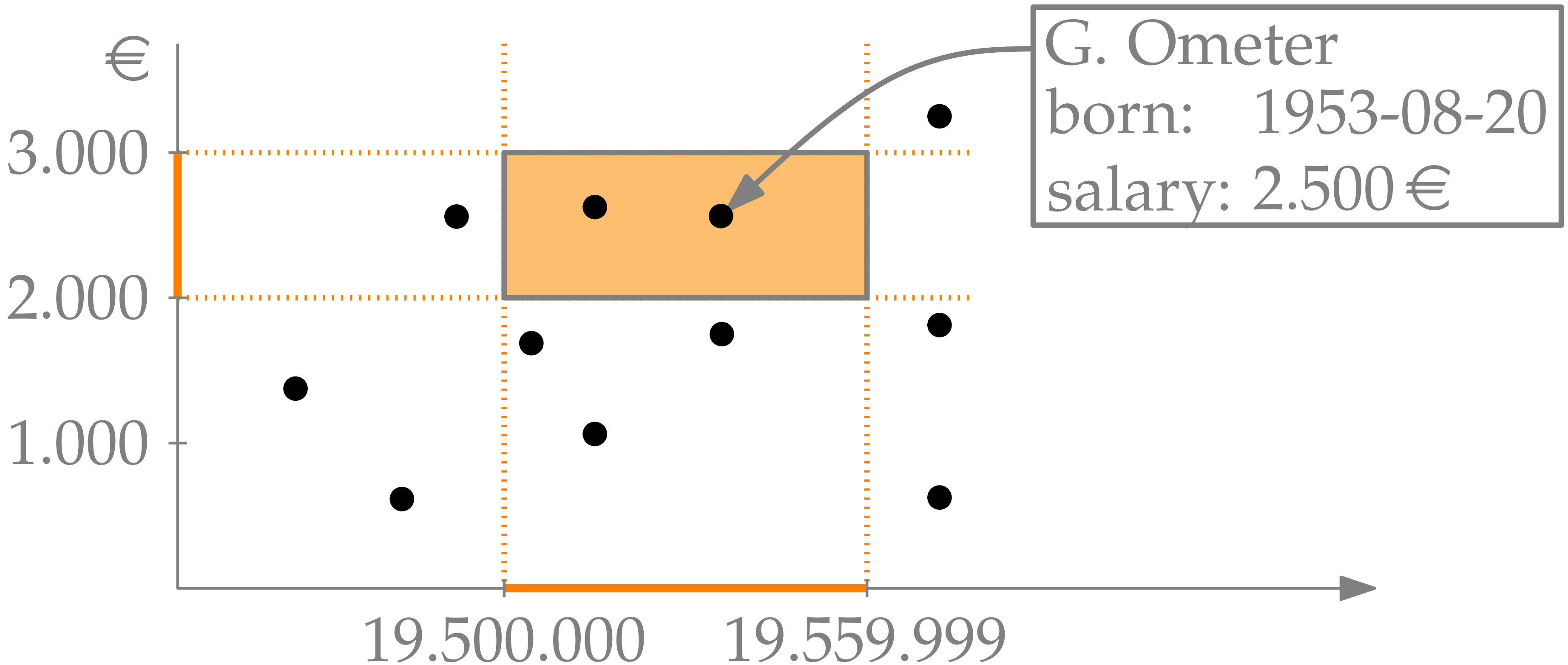

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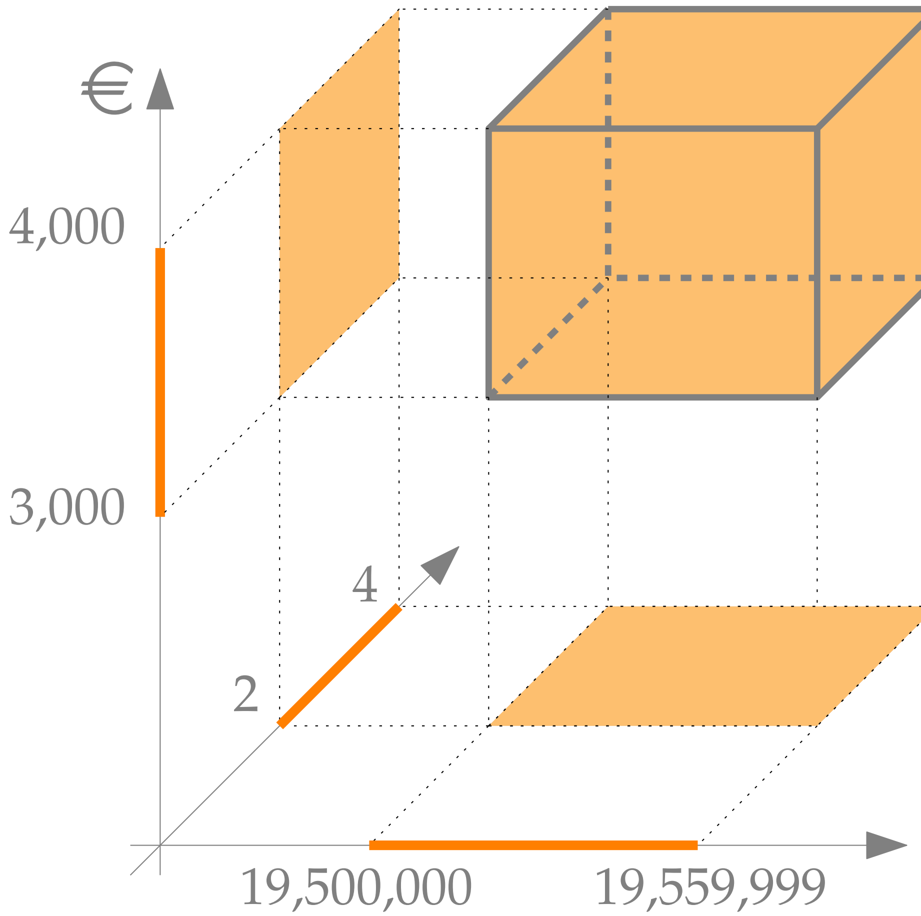

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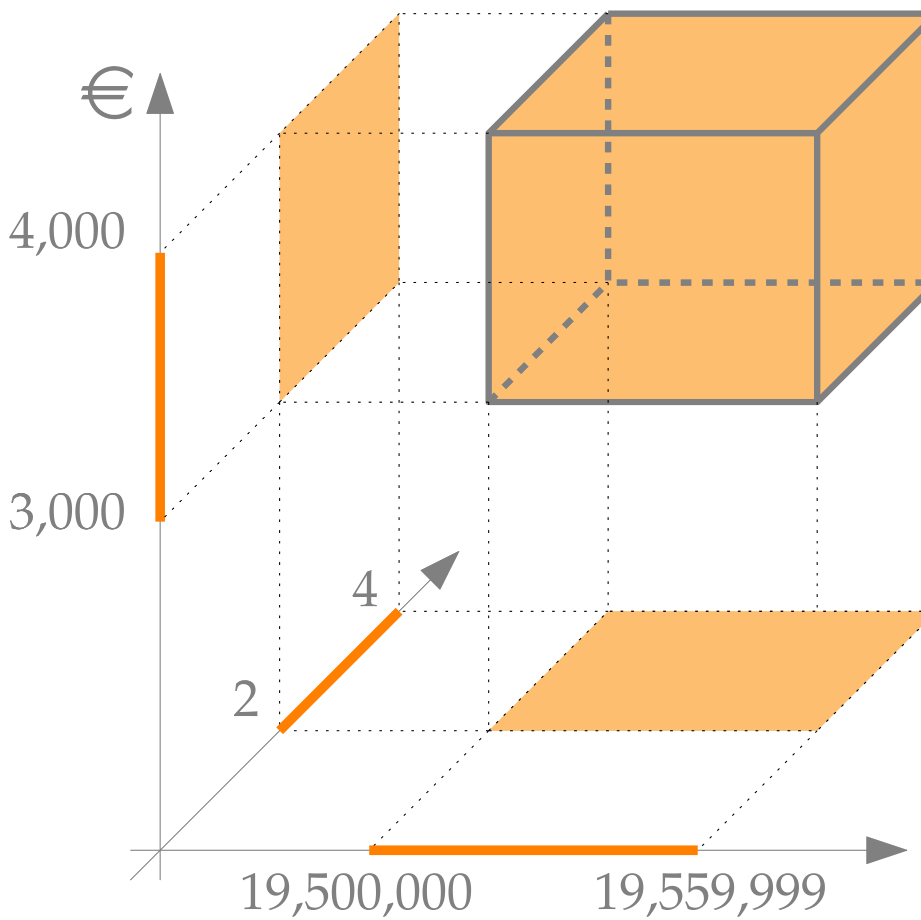

Typical queries for data bases!

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Task: Preprocess a finite set $P \subset \mathbb{R}$ such that for any interval $[x, x']$ the set $P \cap [x, x']$ can be reported quickly. [2 min]

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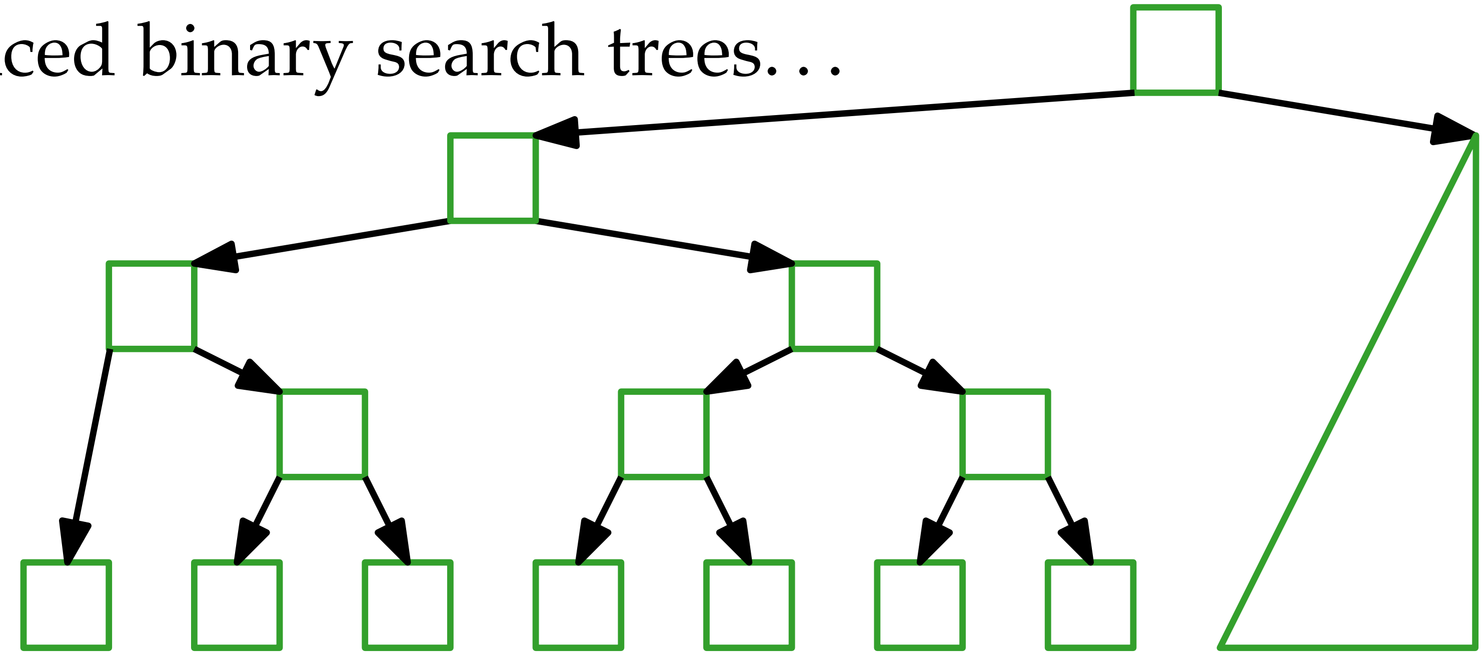

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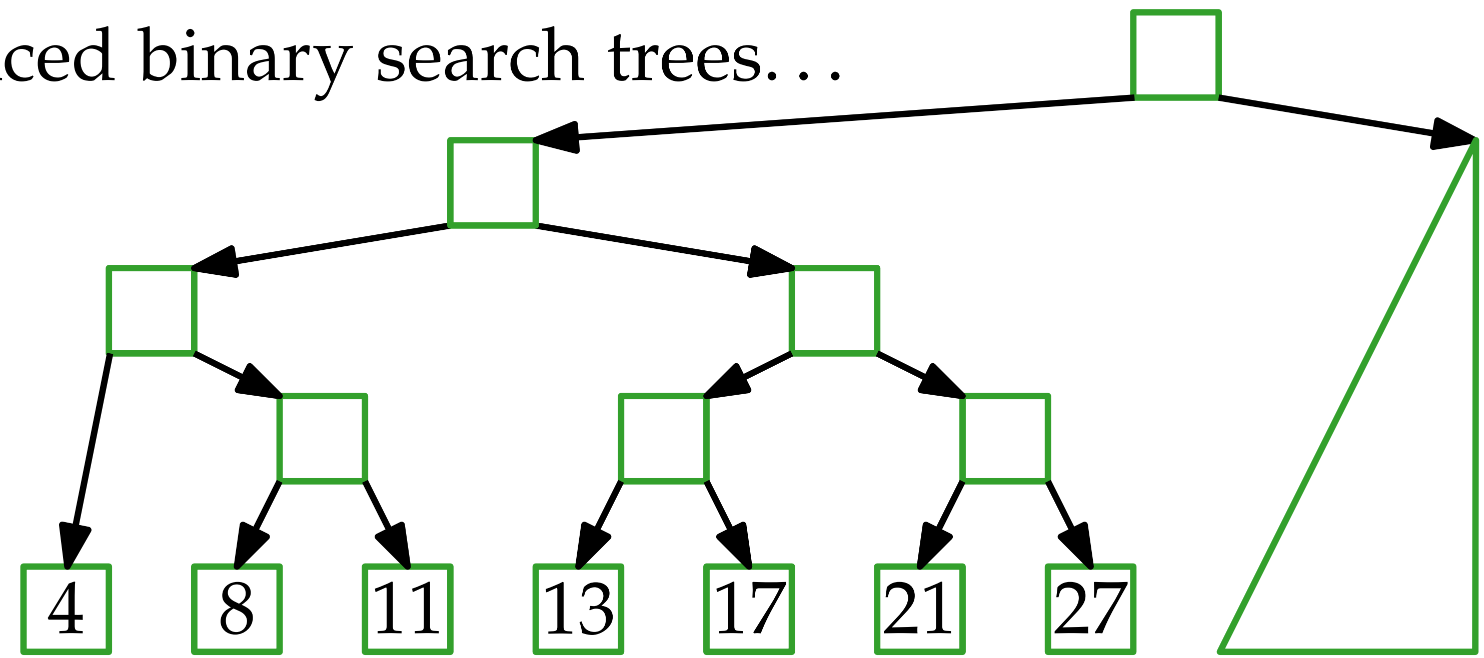

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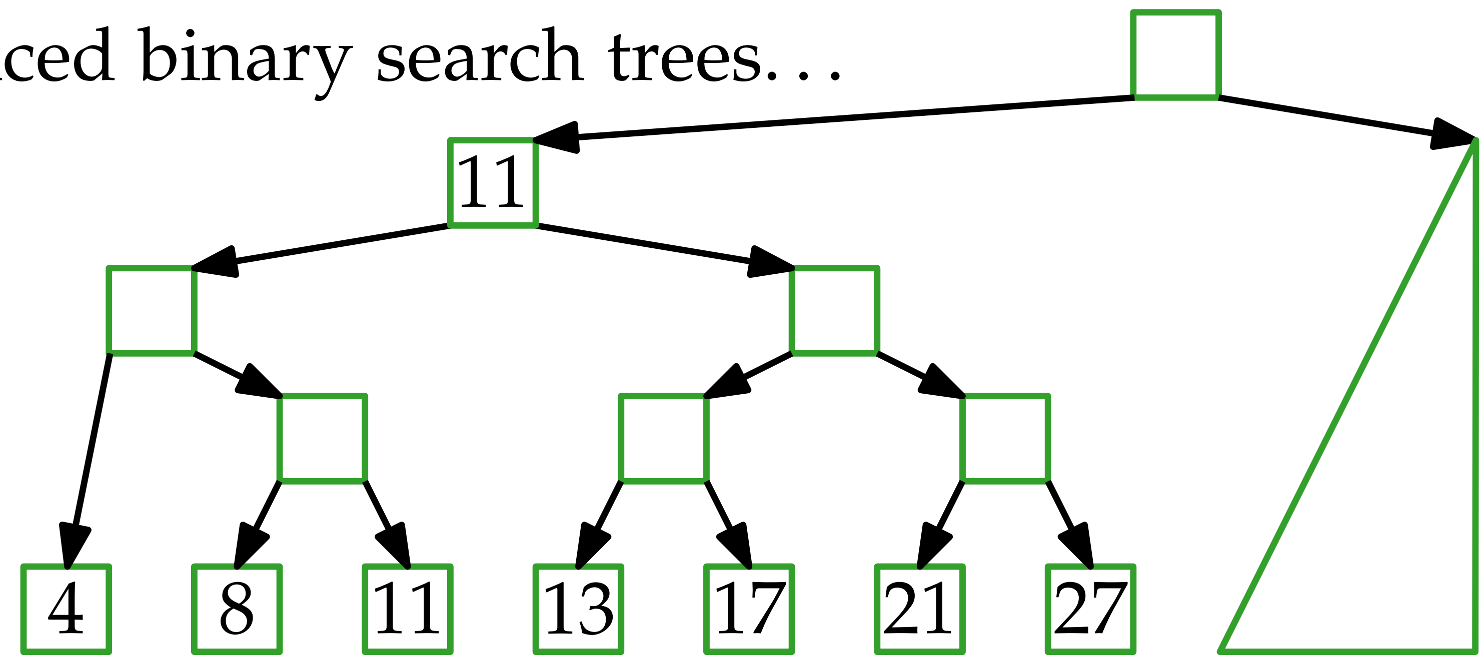

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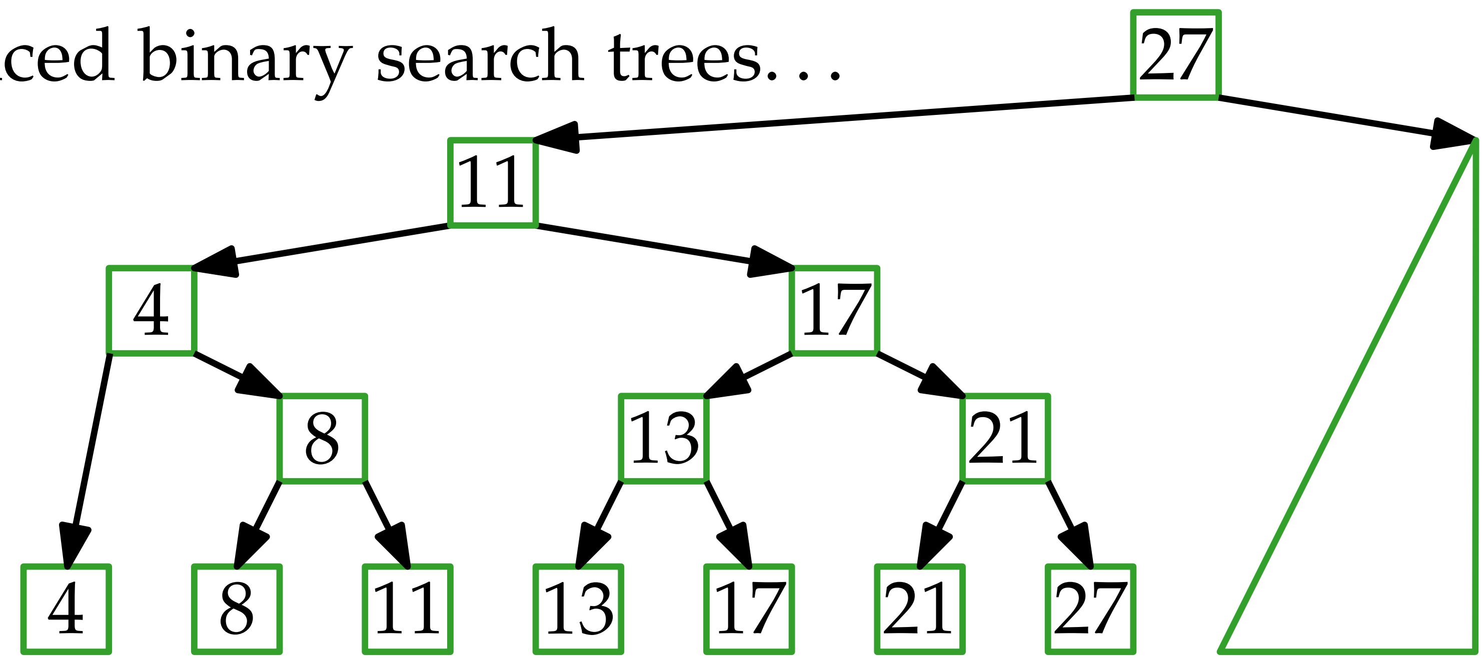

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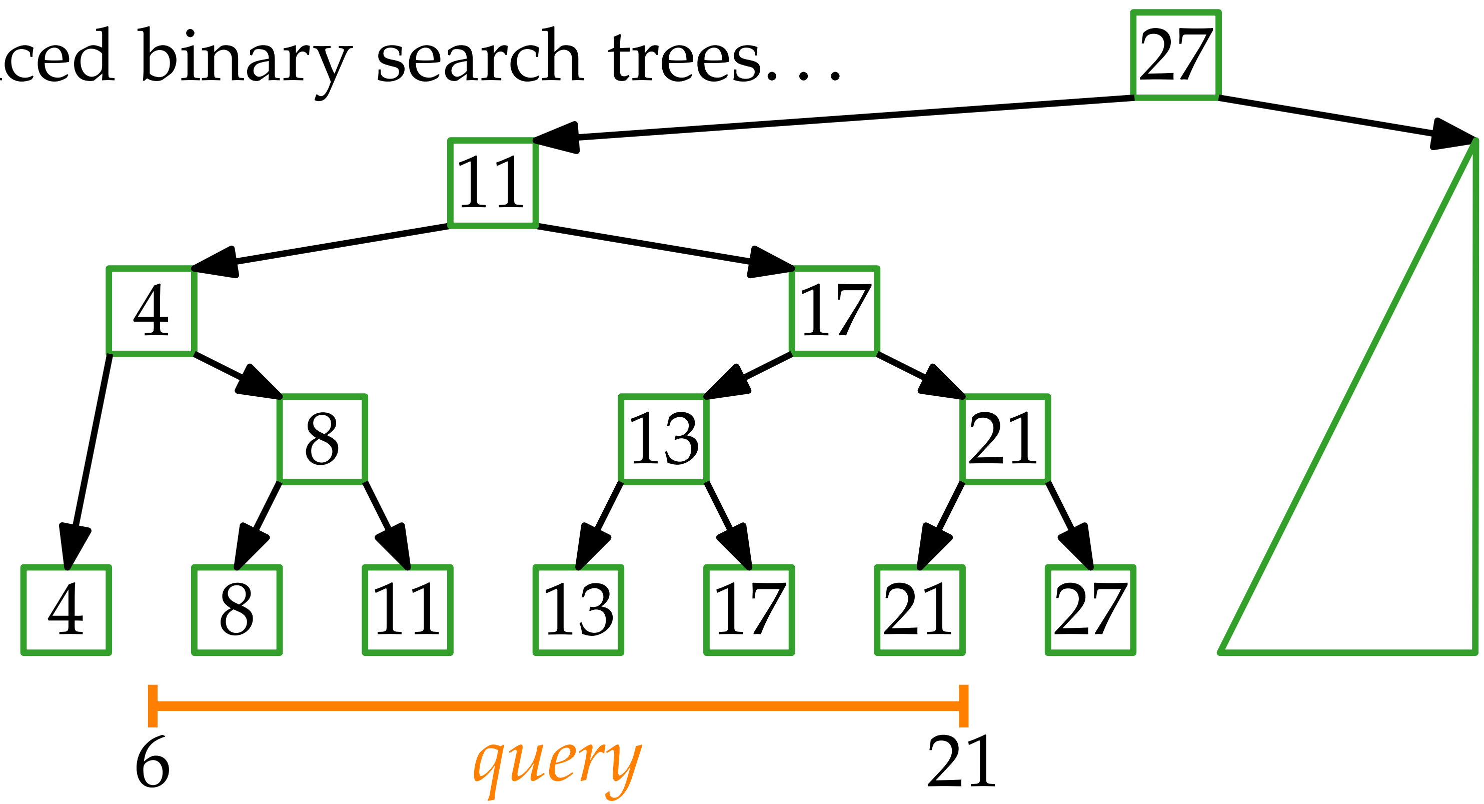

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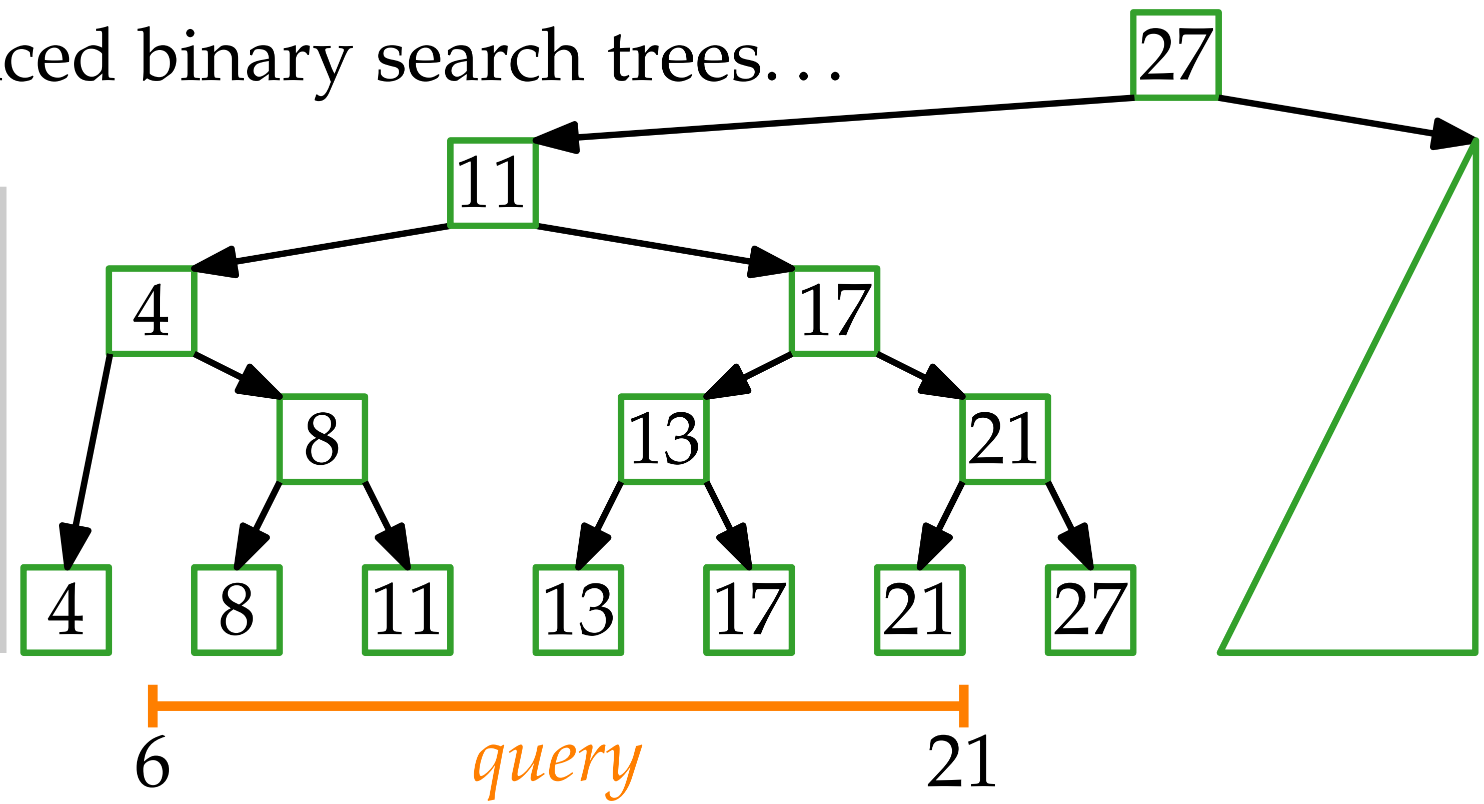

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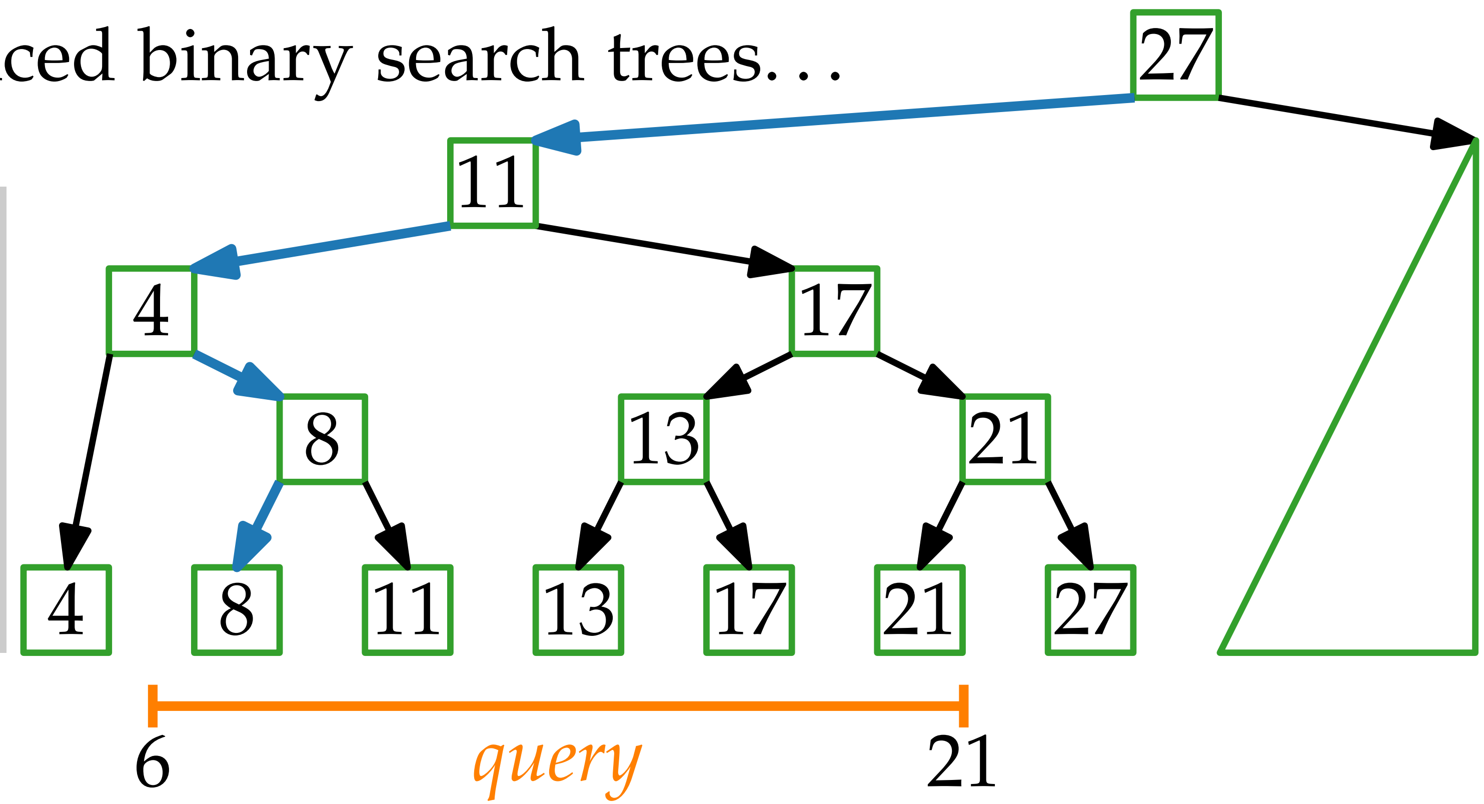

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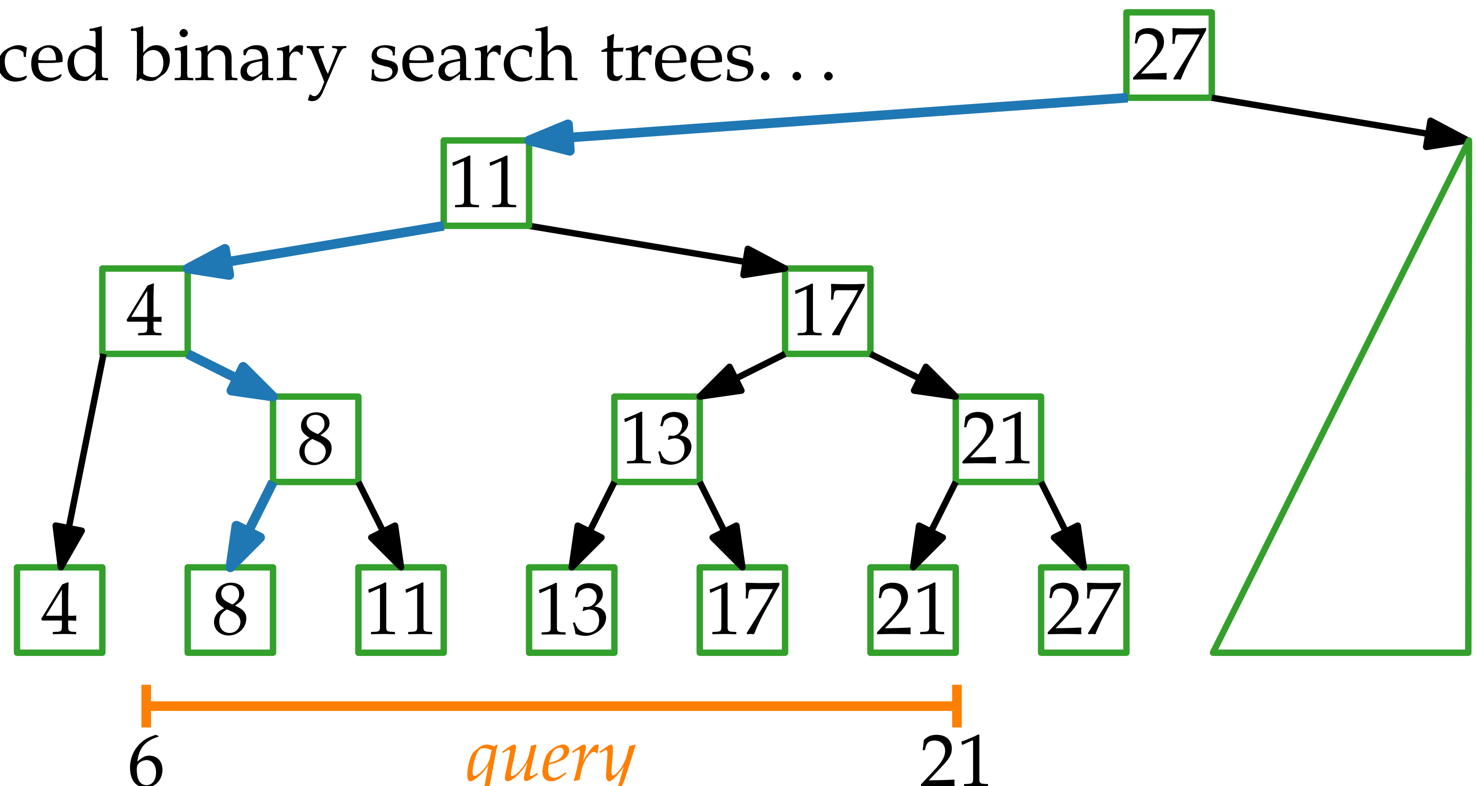

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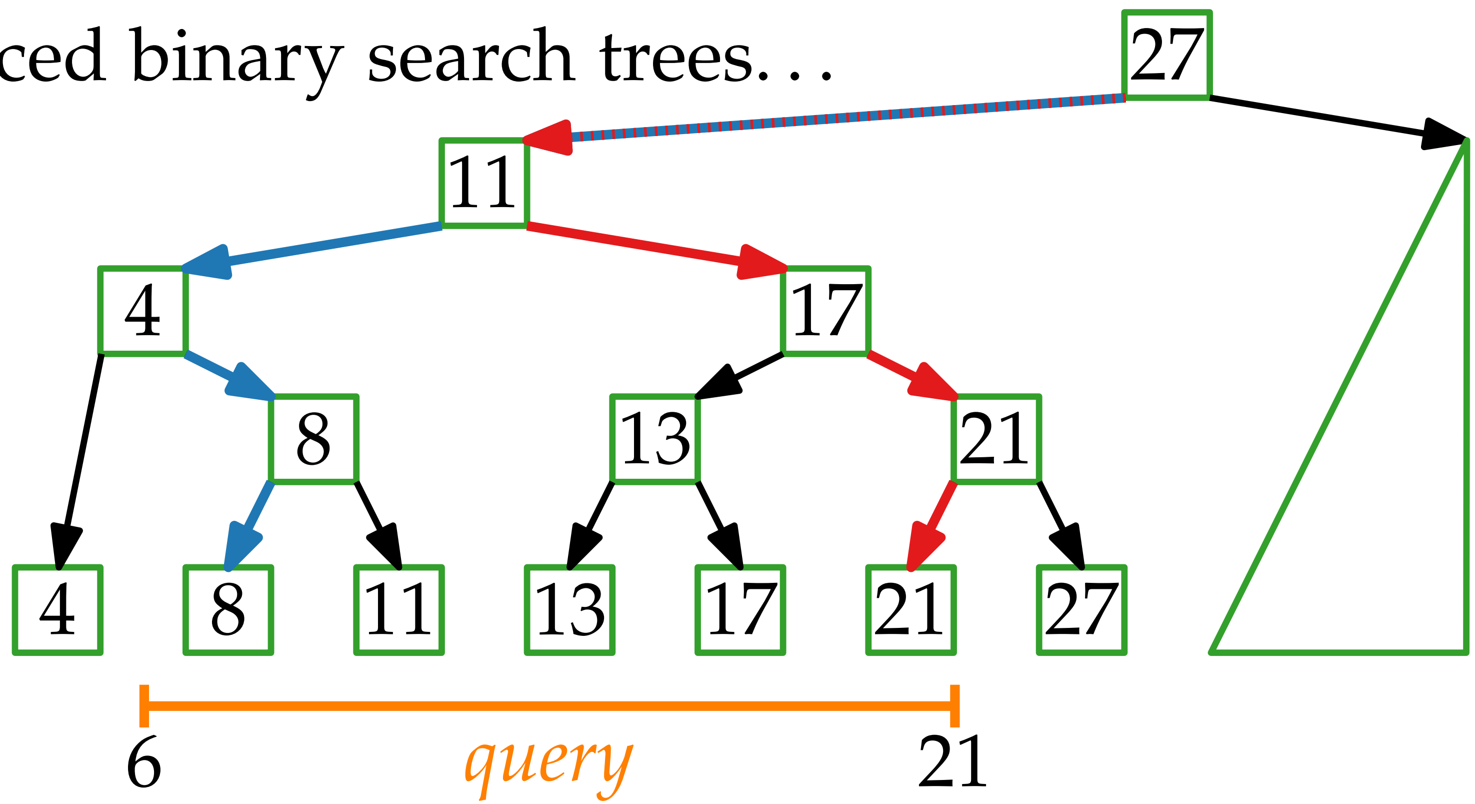

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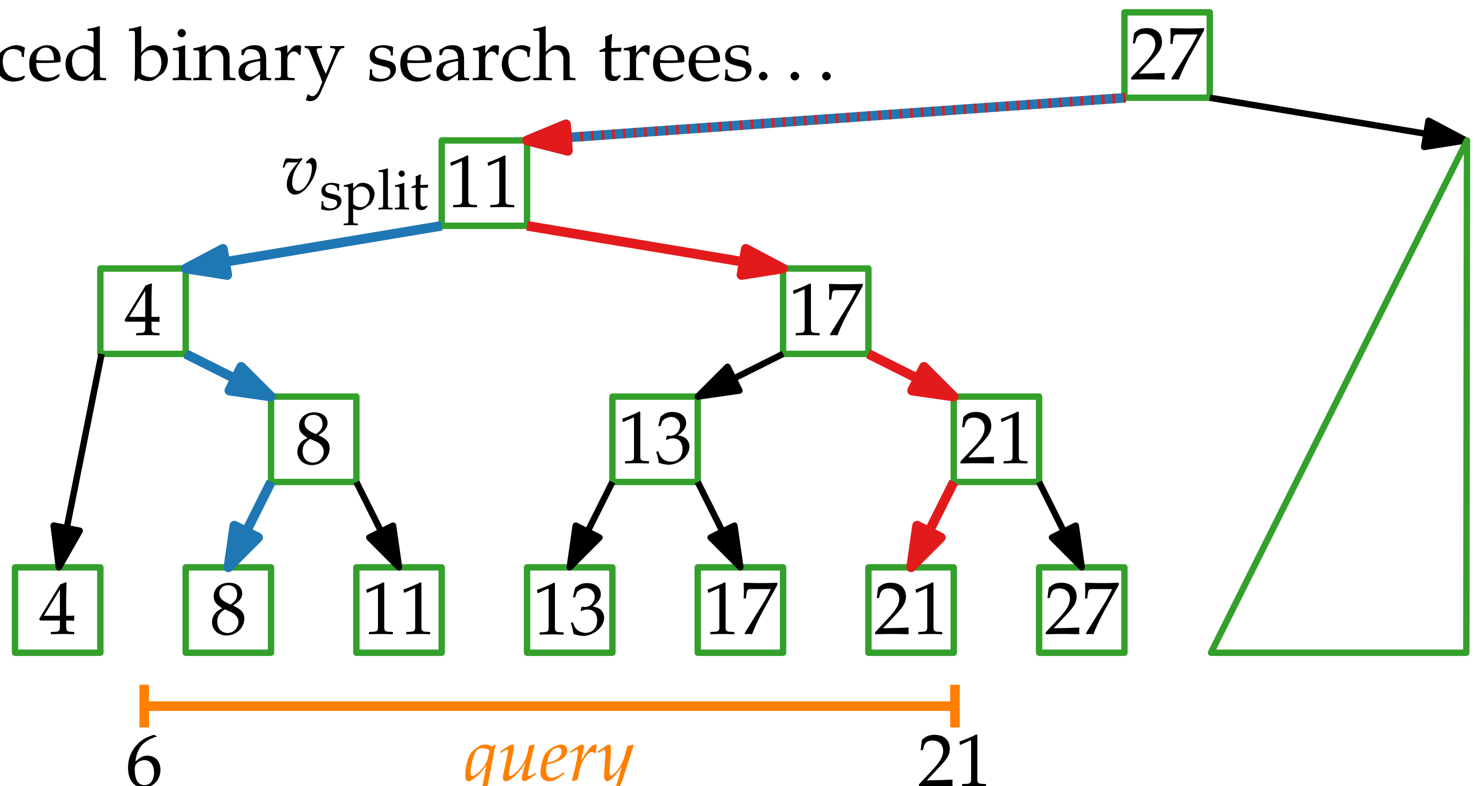

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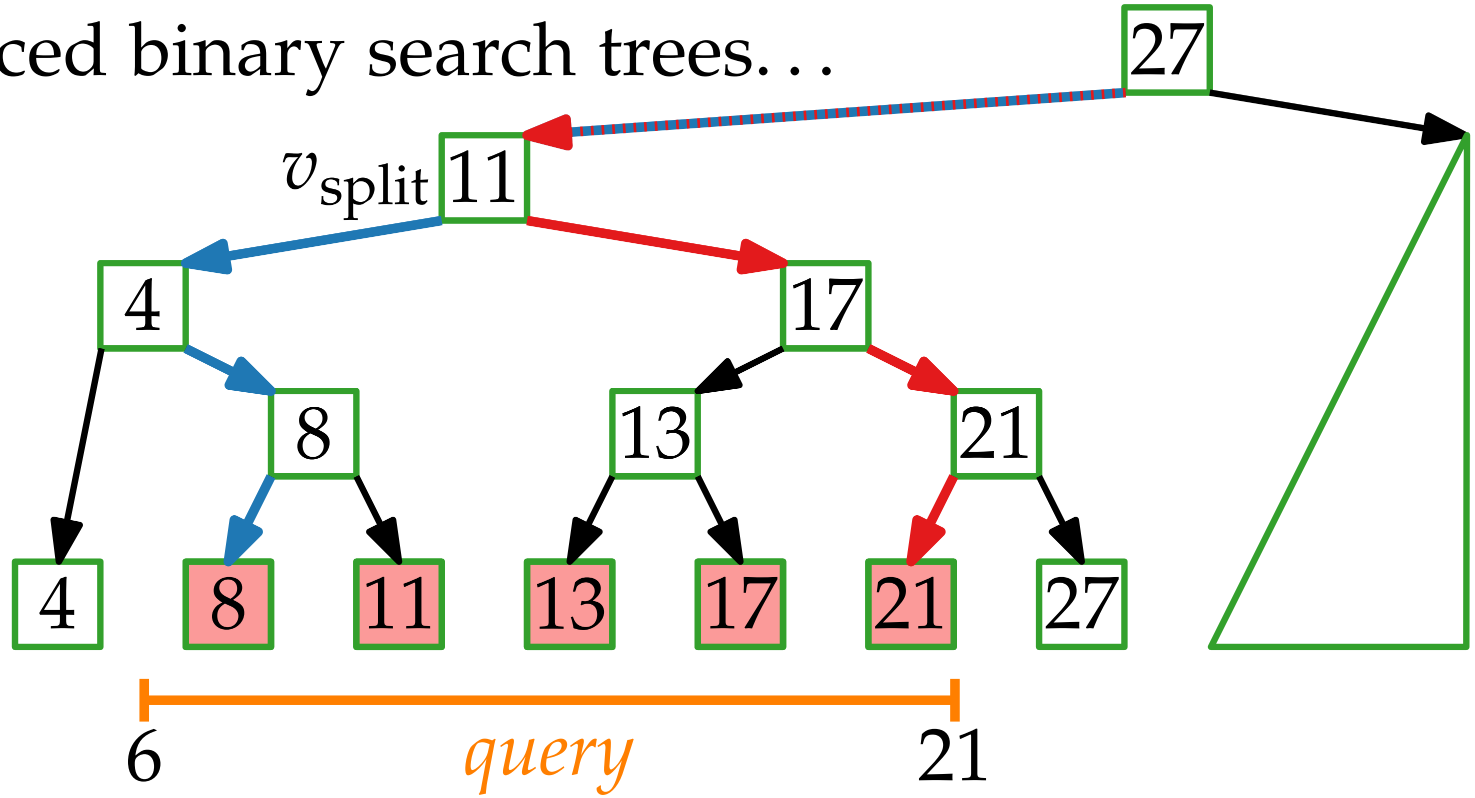

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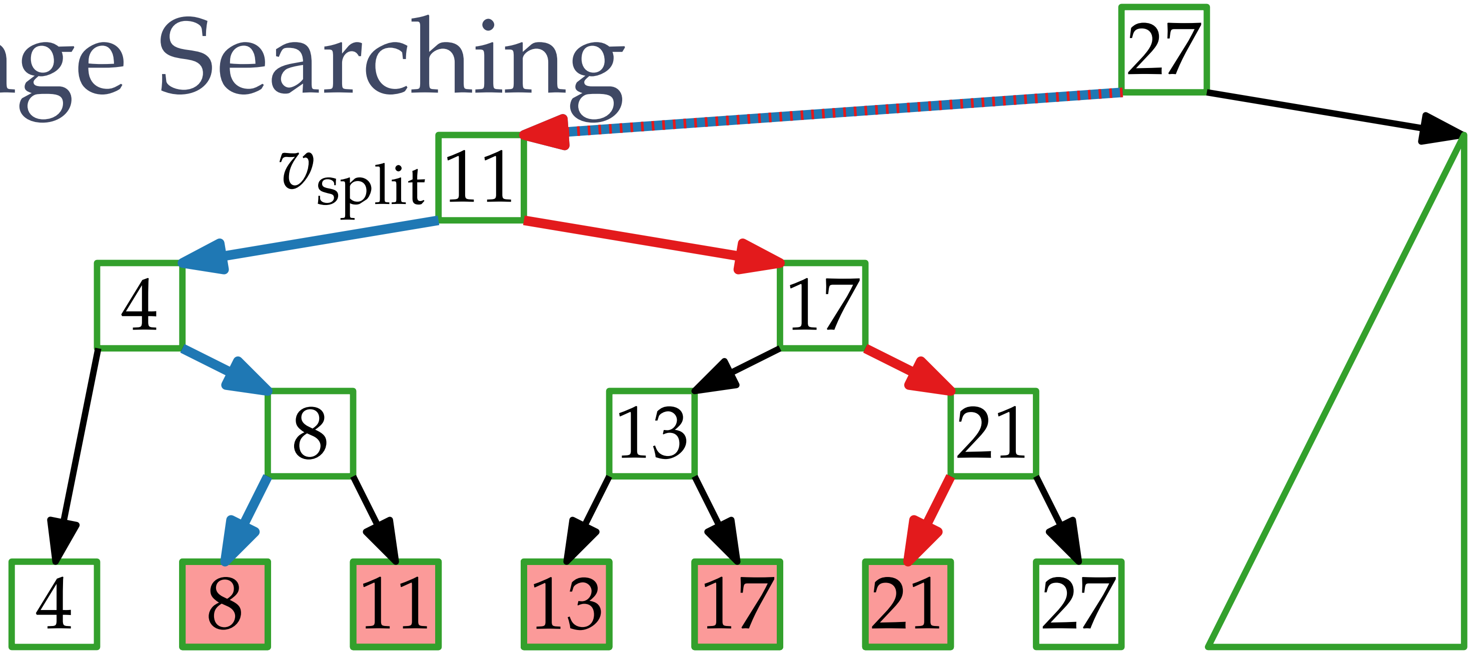

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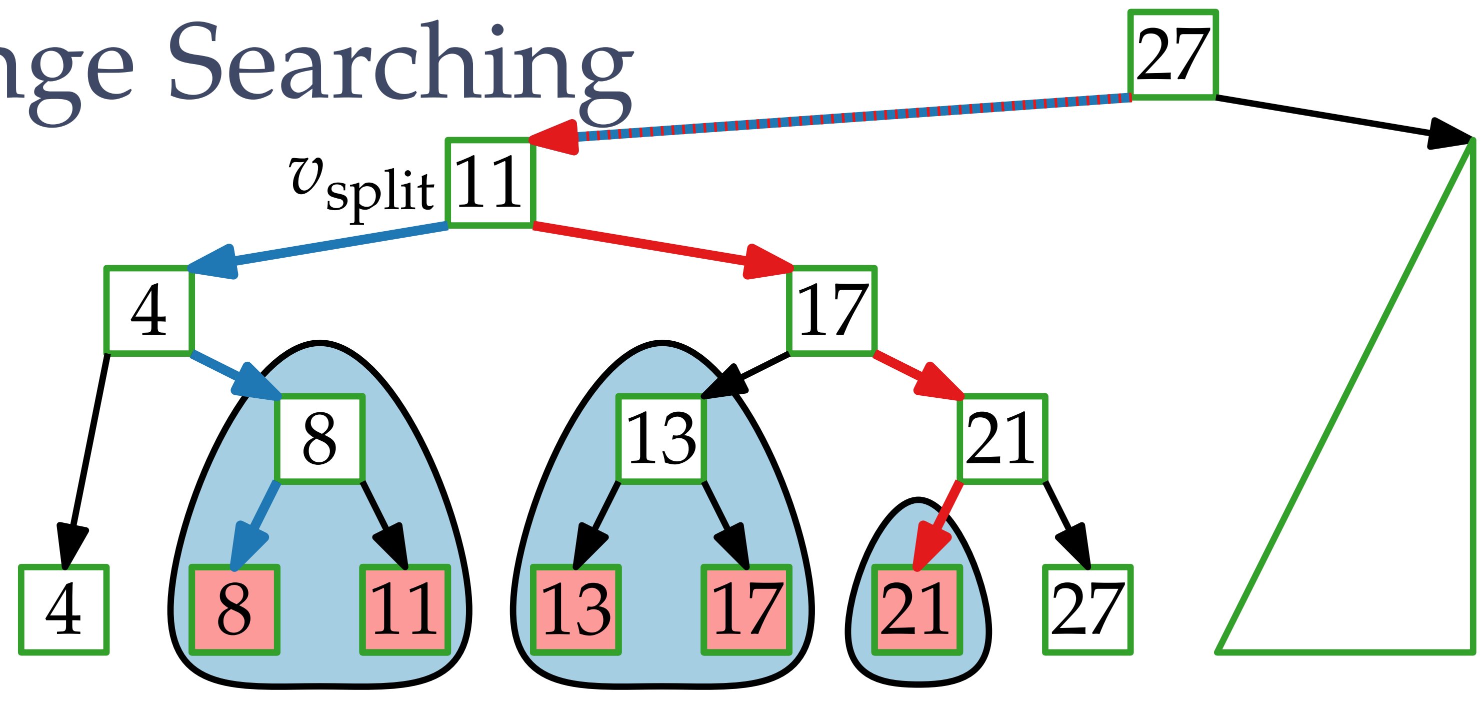

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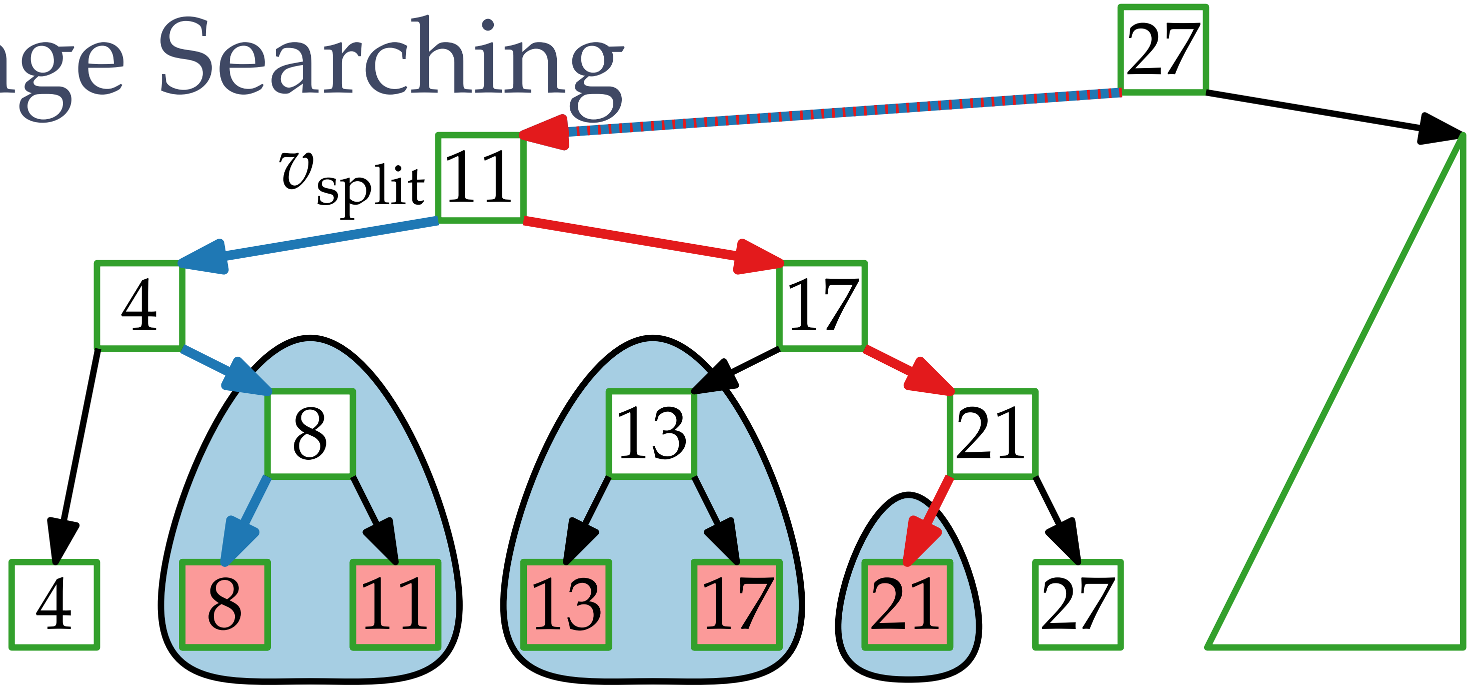

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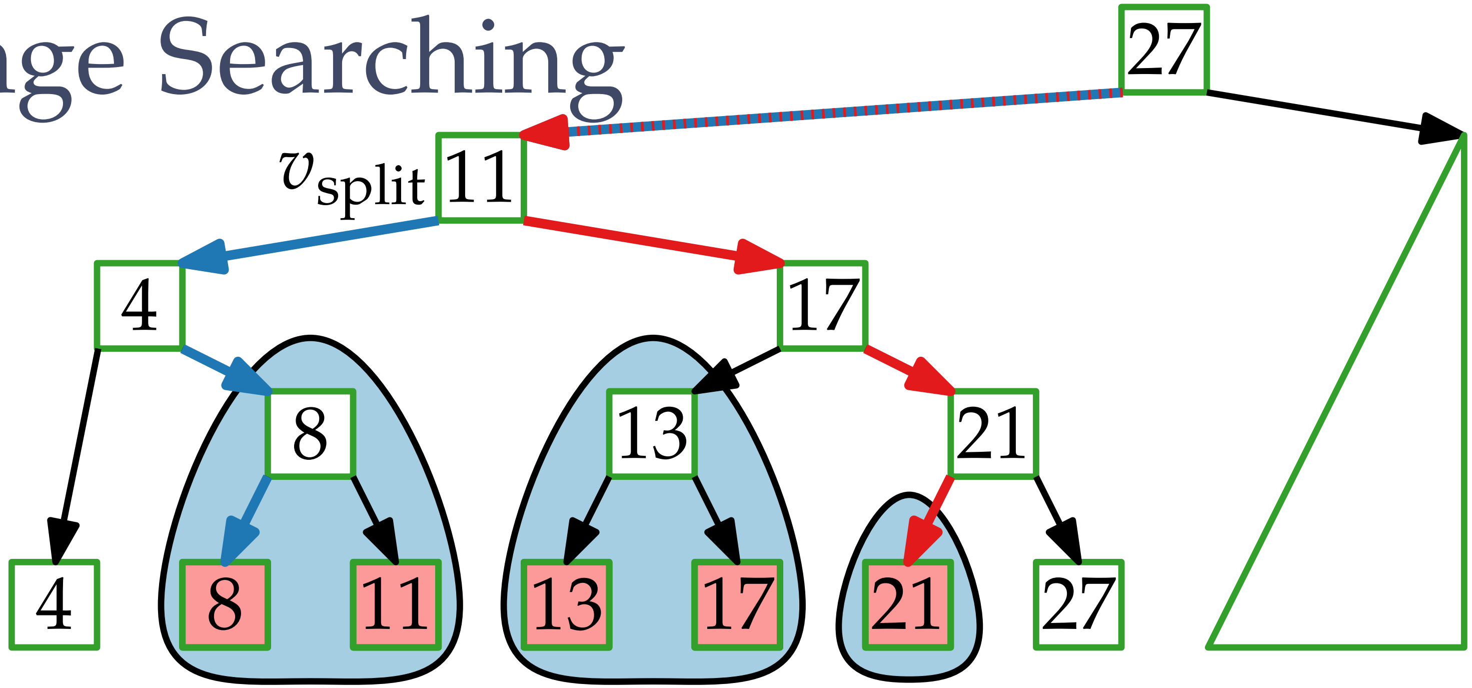

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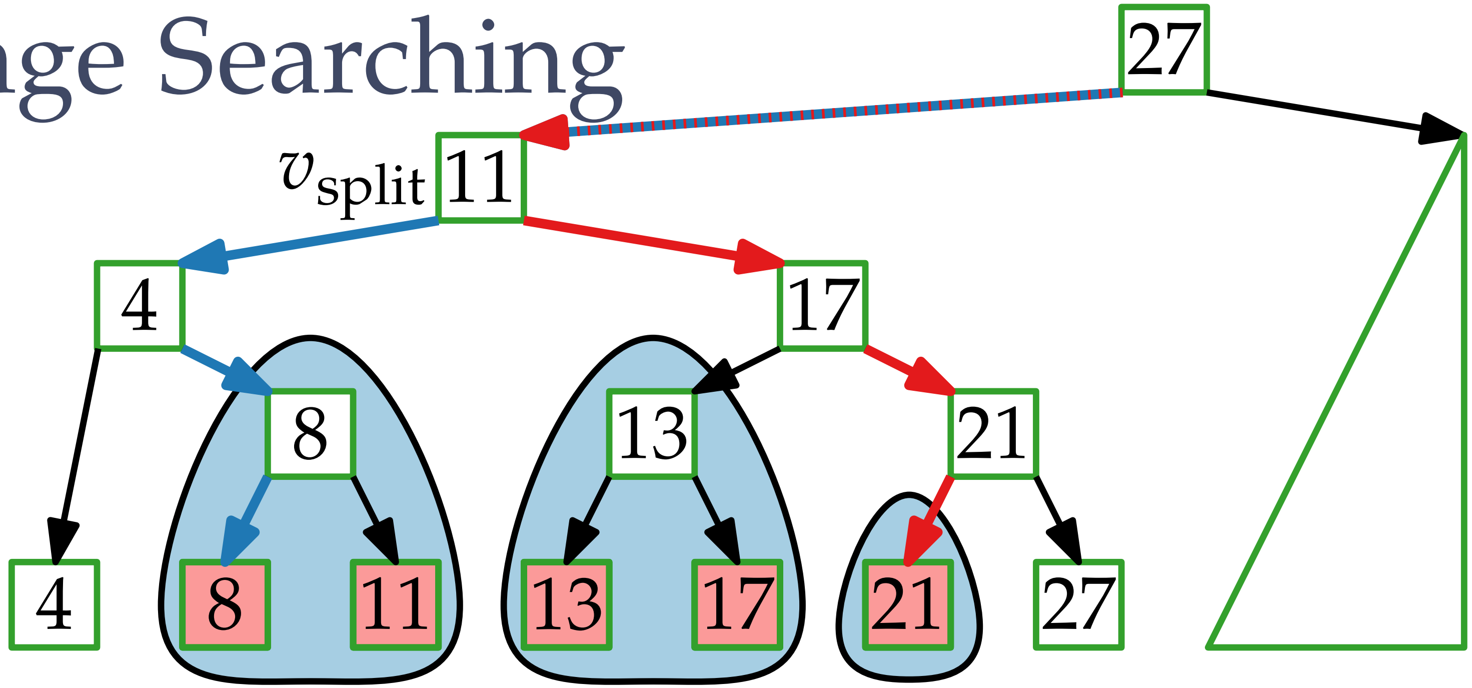

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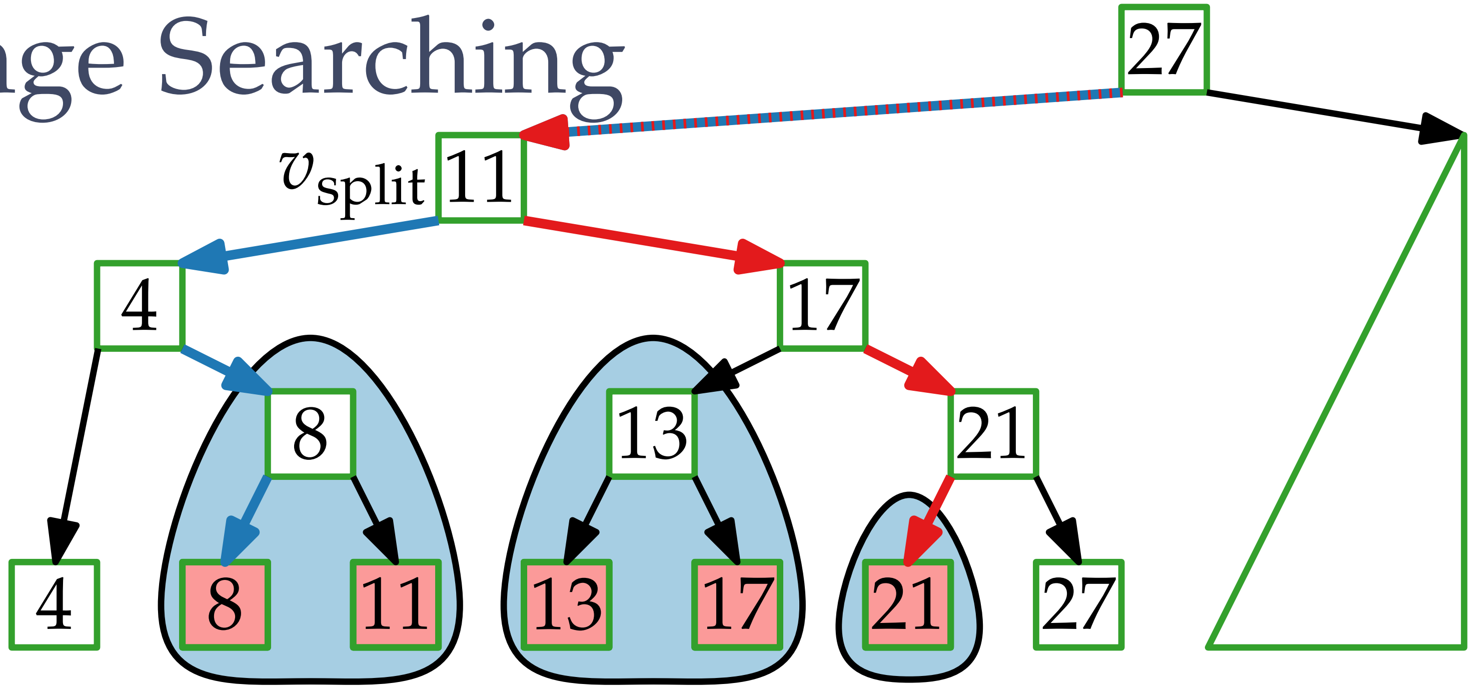

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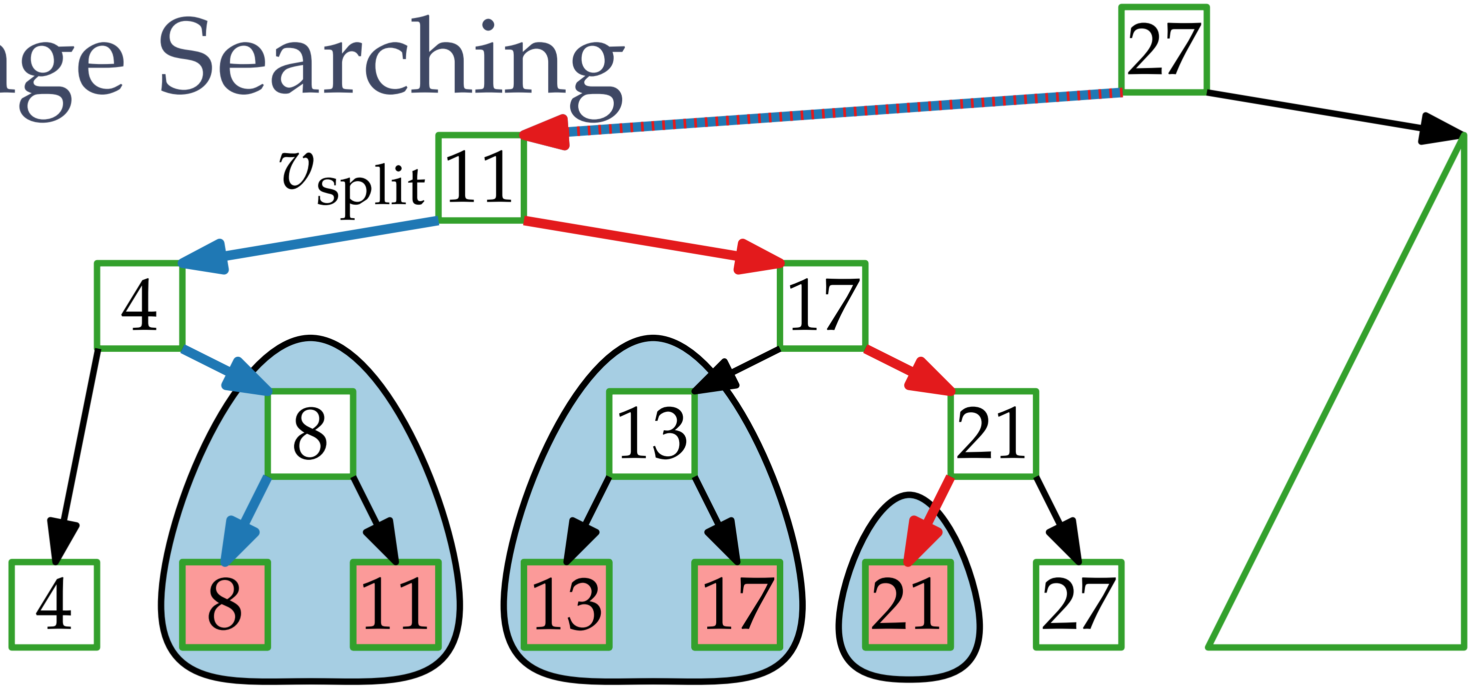

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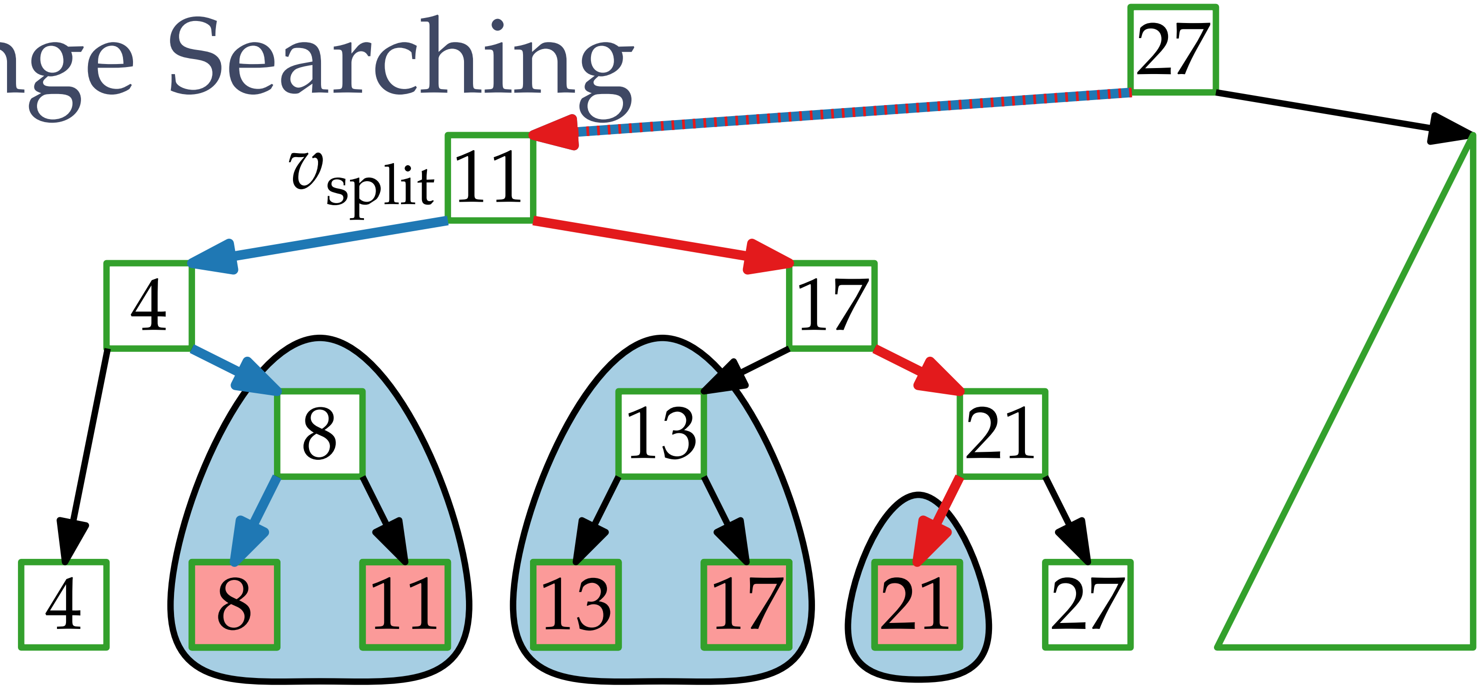

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output sensitive!

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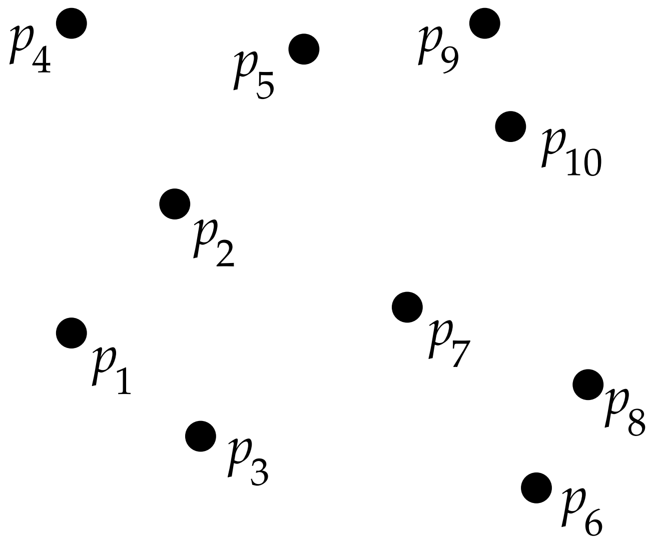
[dBCvKO'08]



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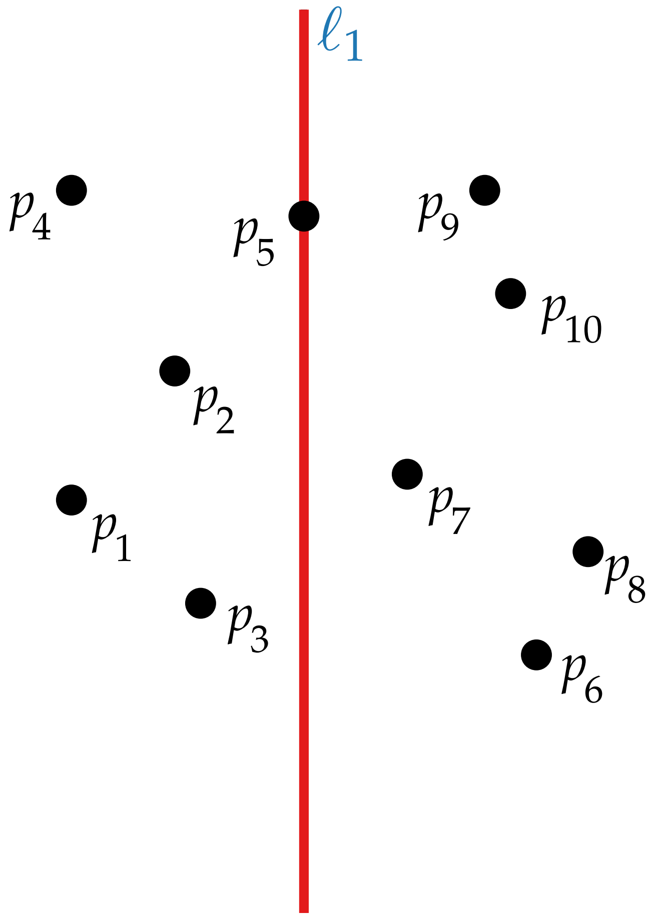

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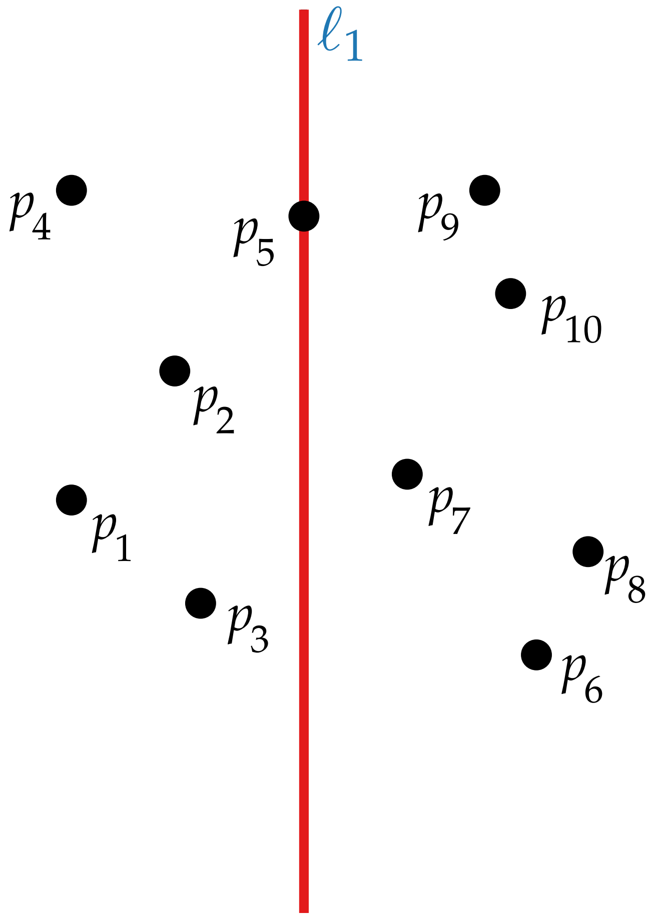

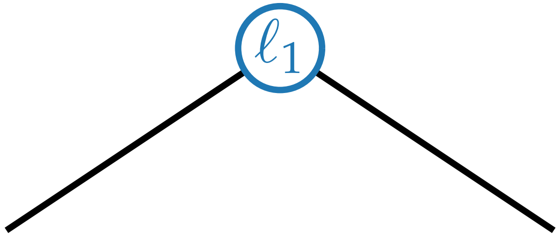

- Split any region that contains more than one point.

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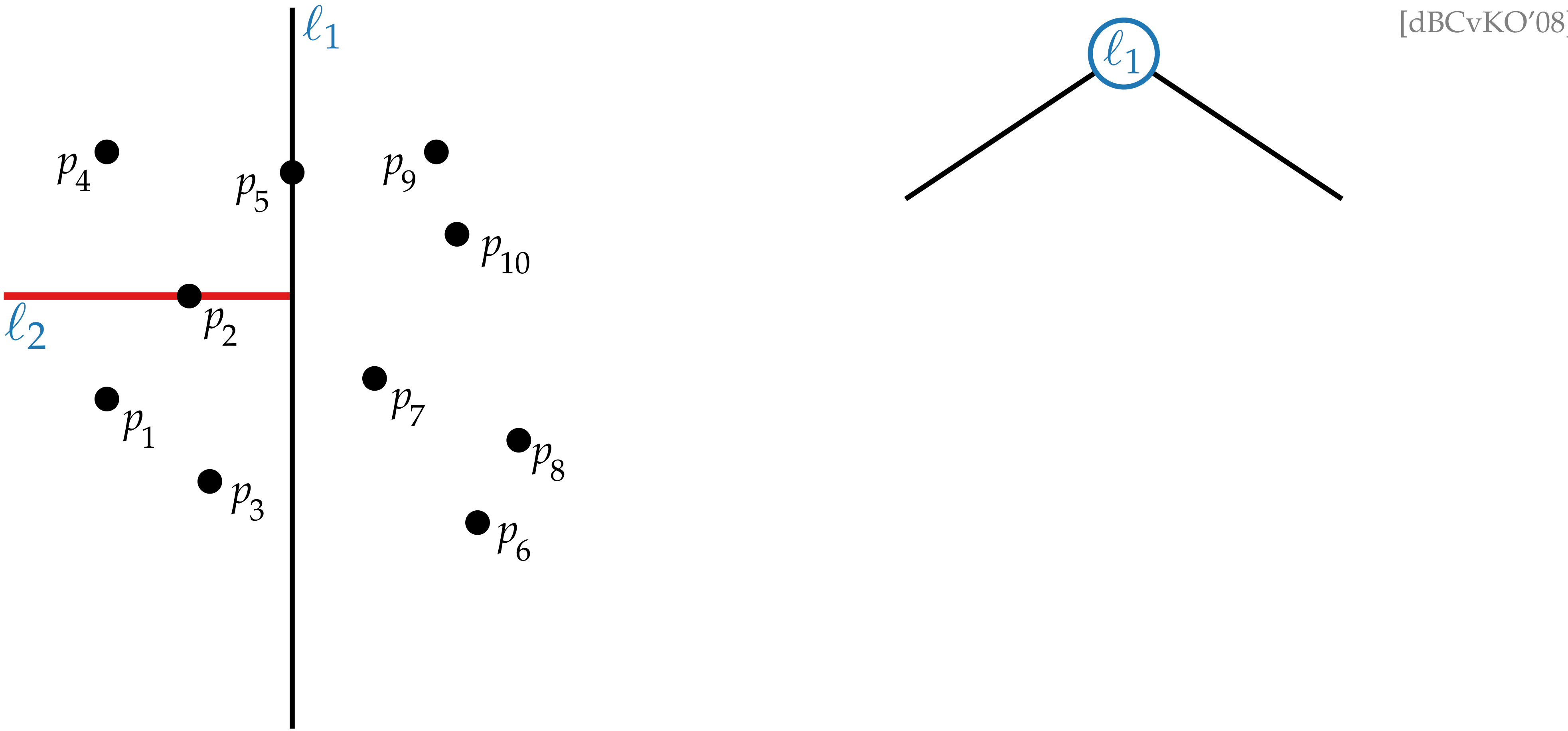

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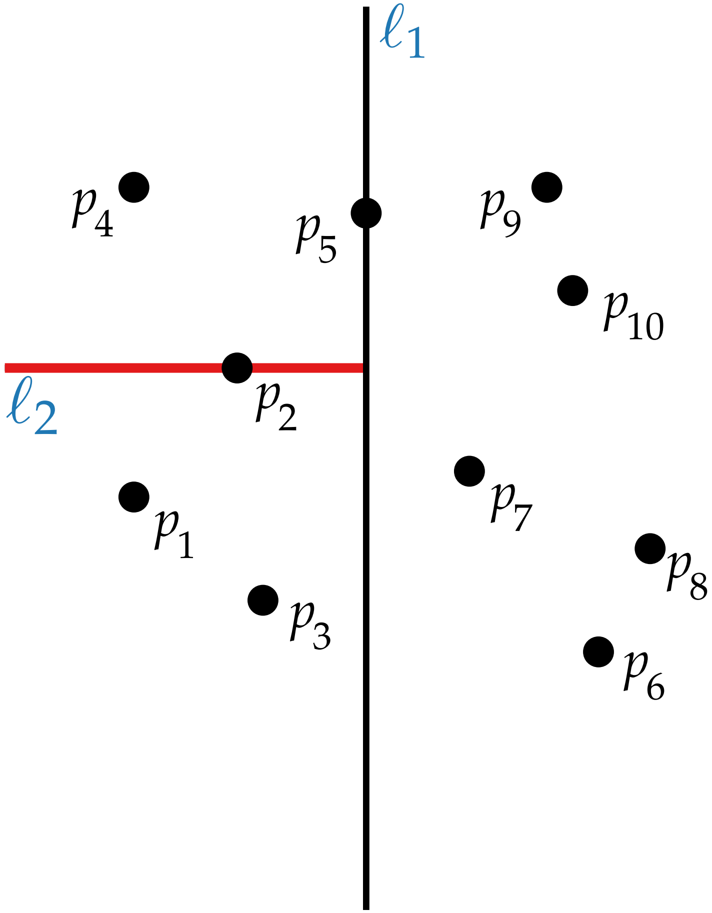

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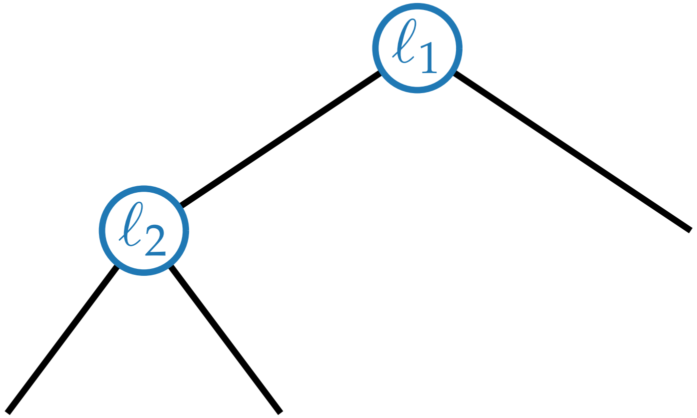

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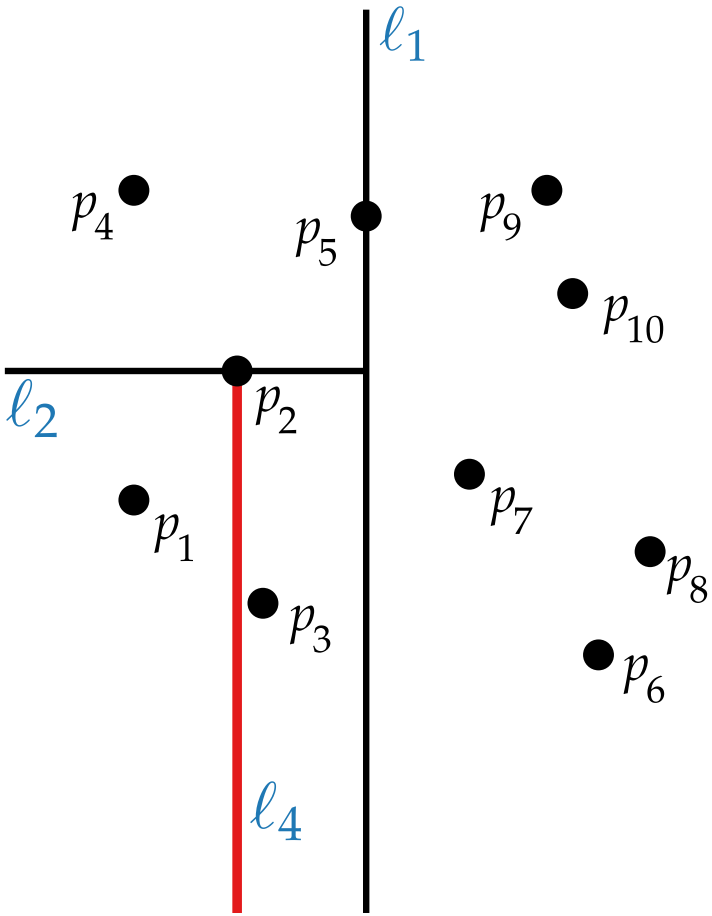

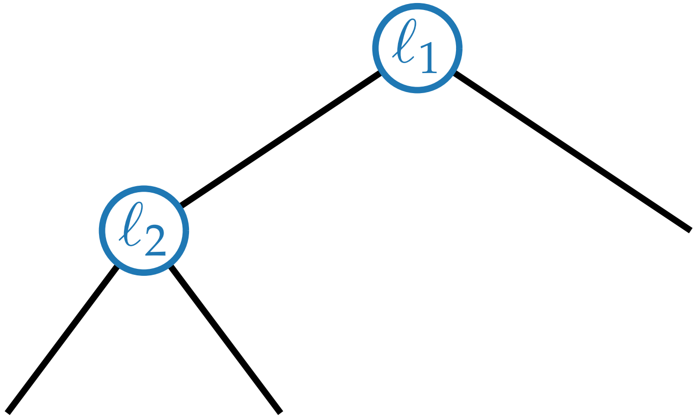

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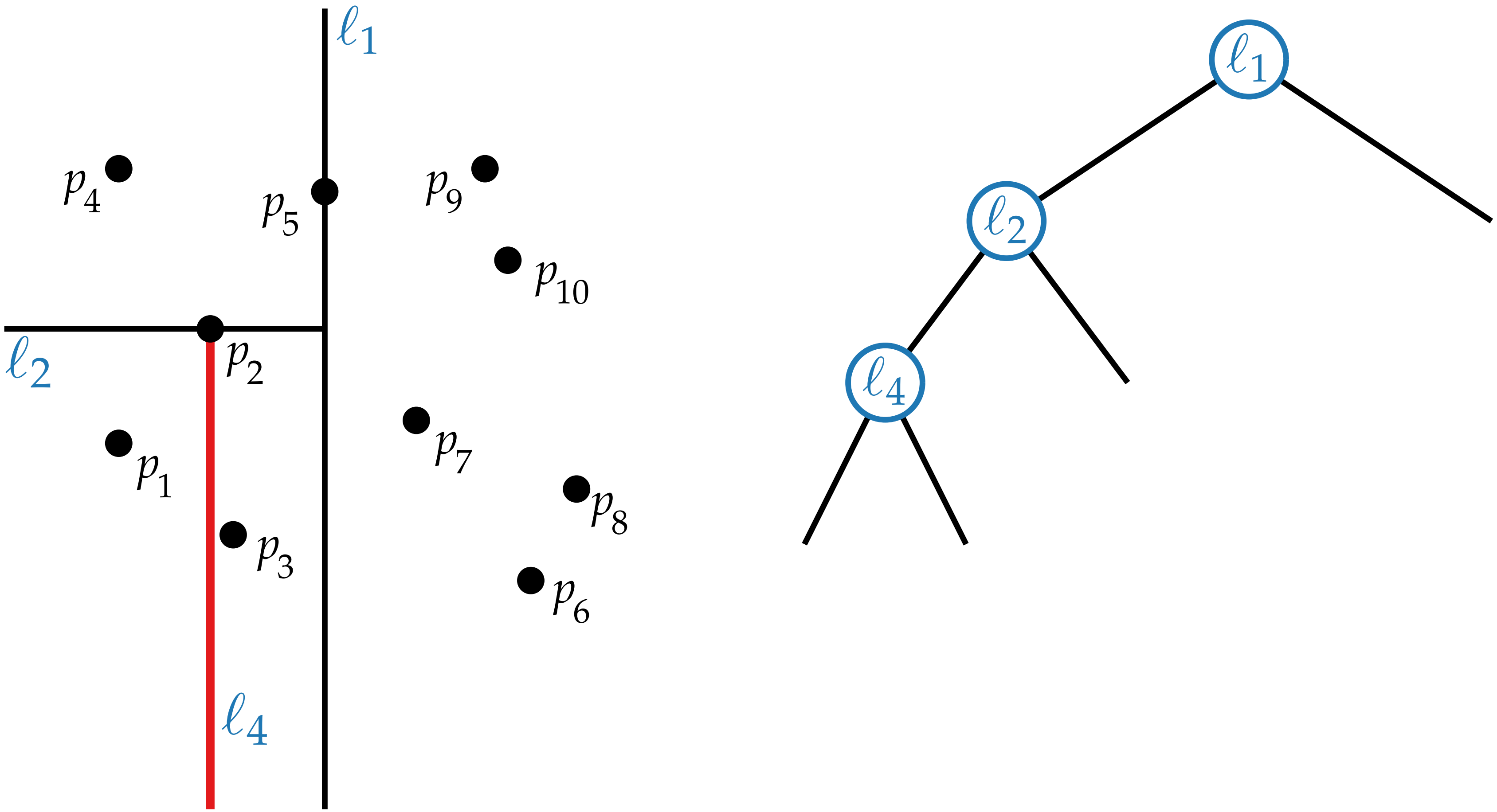

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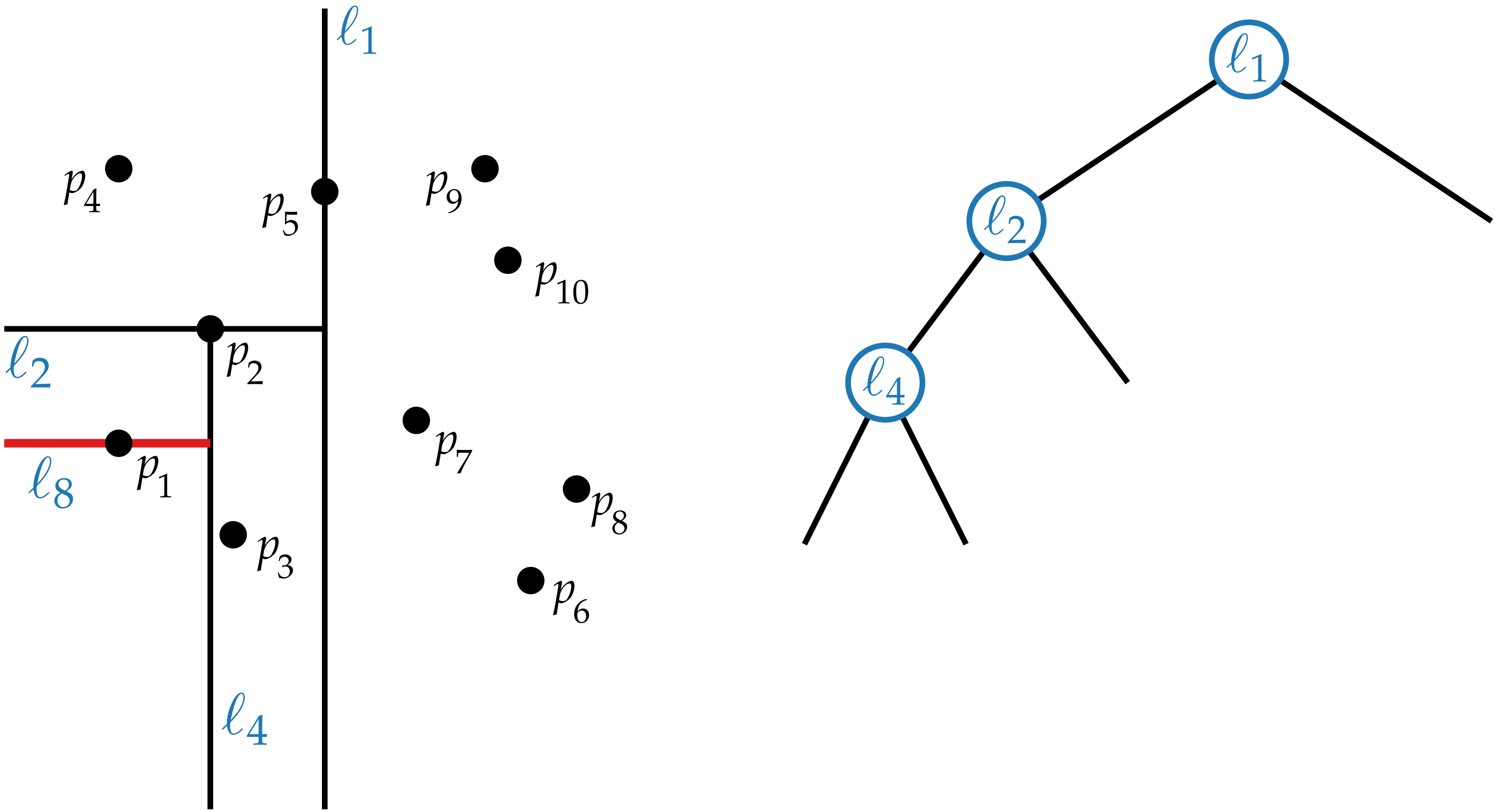

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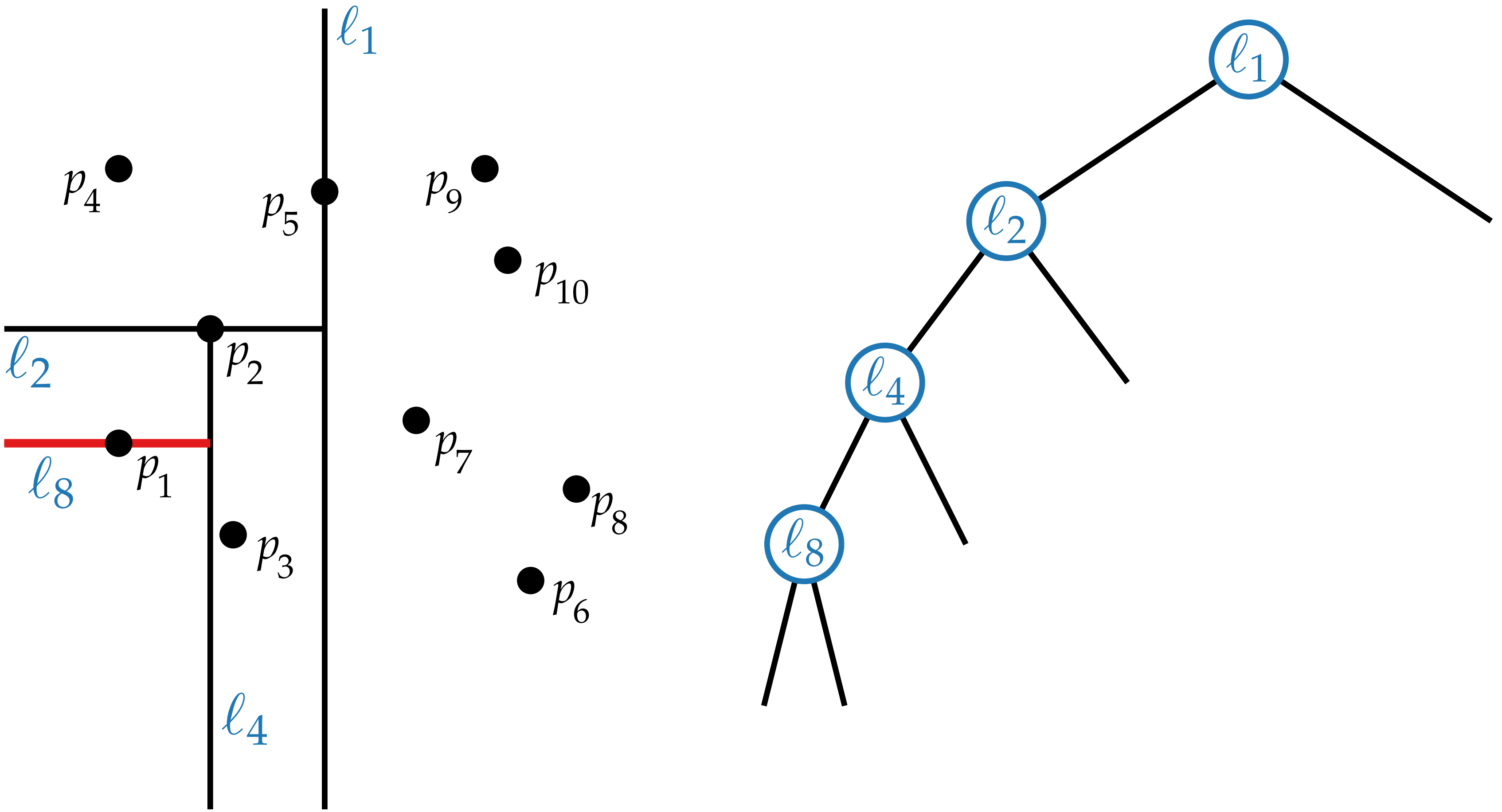

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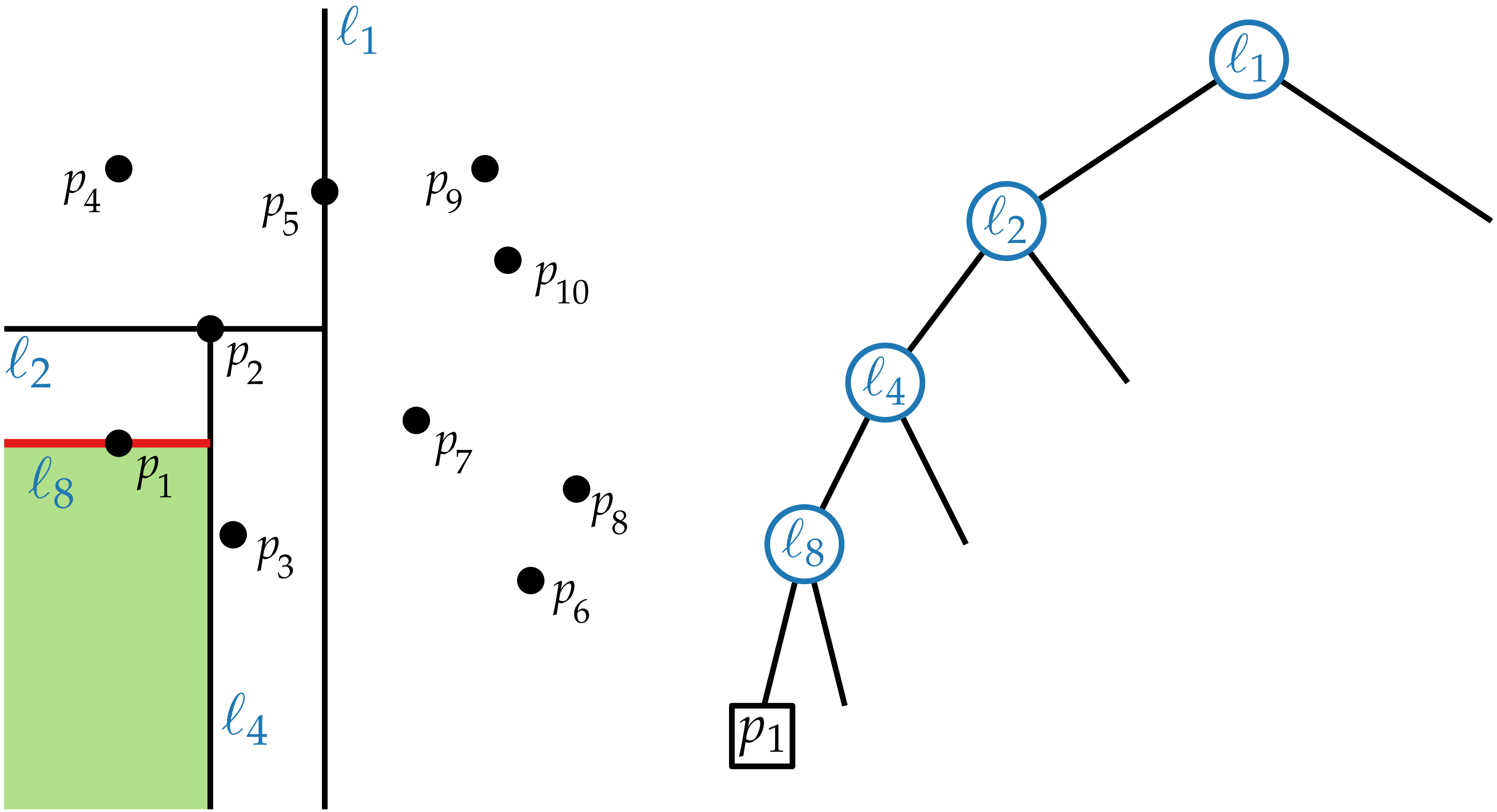

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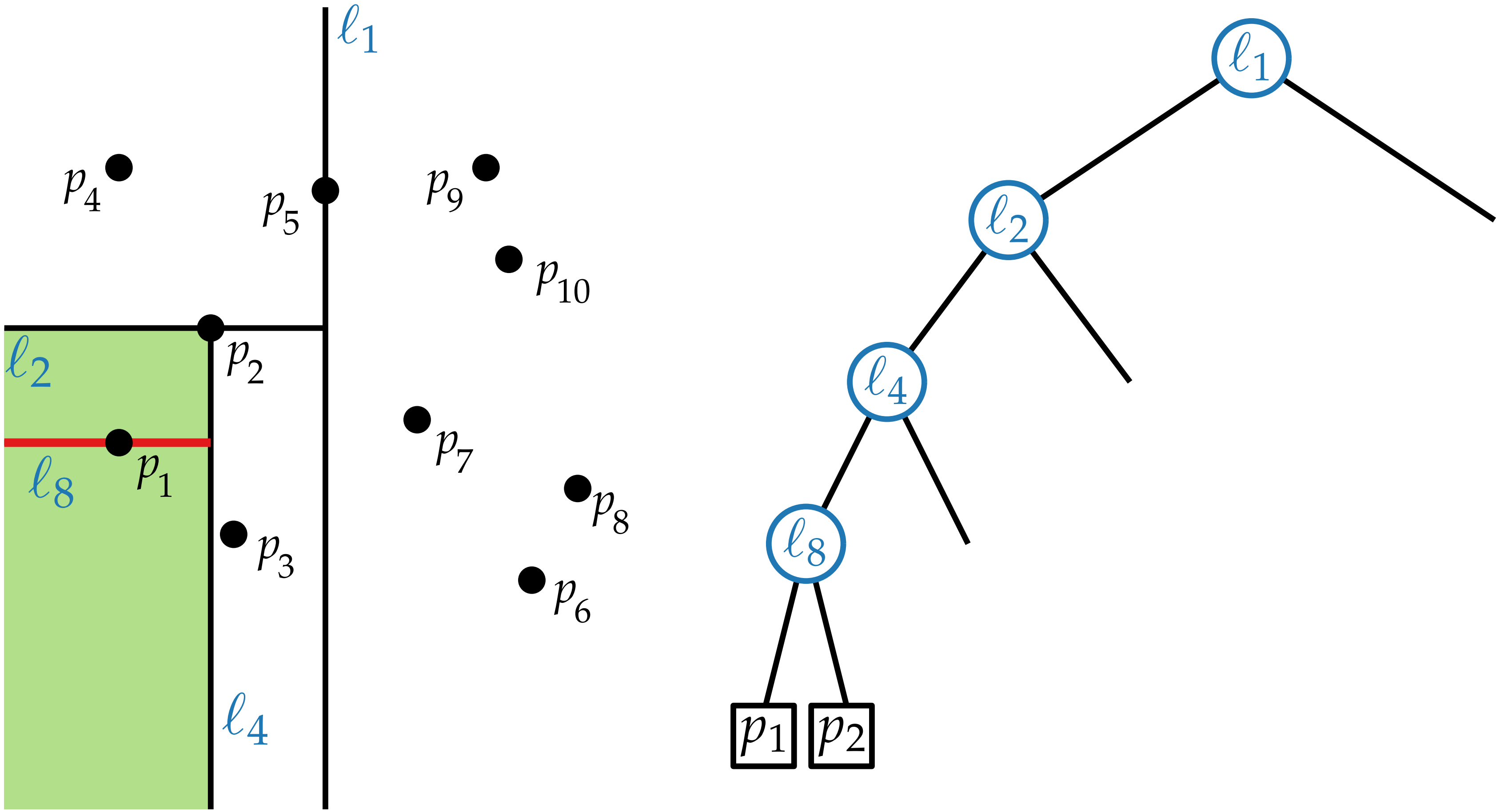

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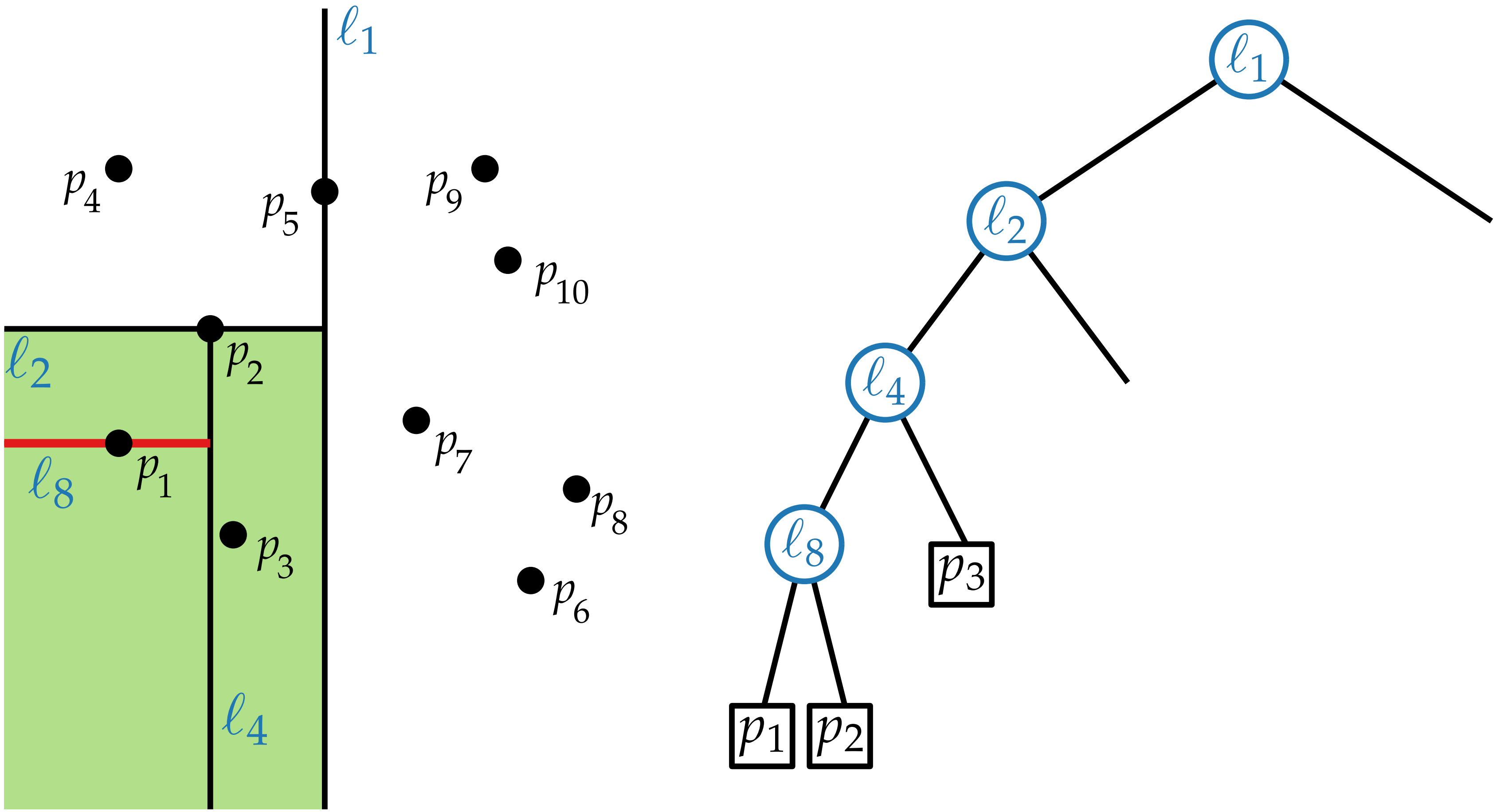

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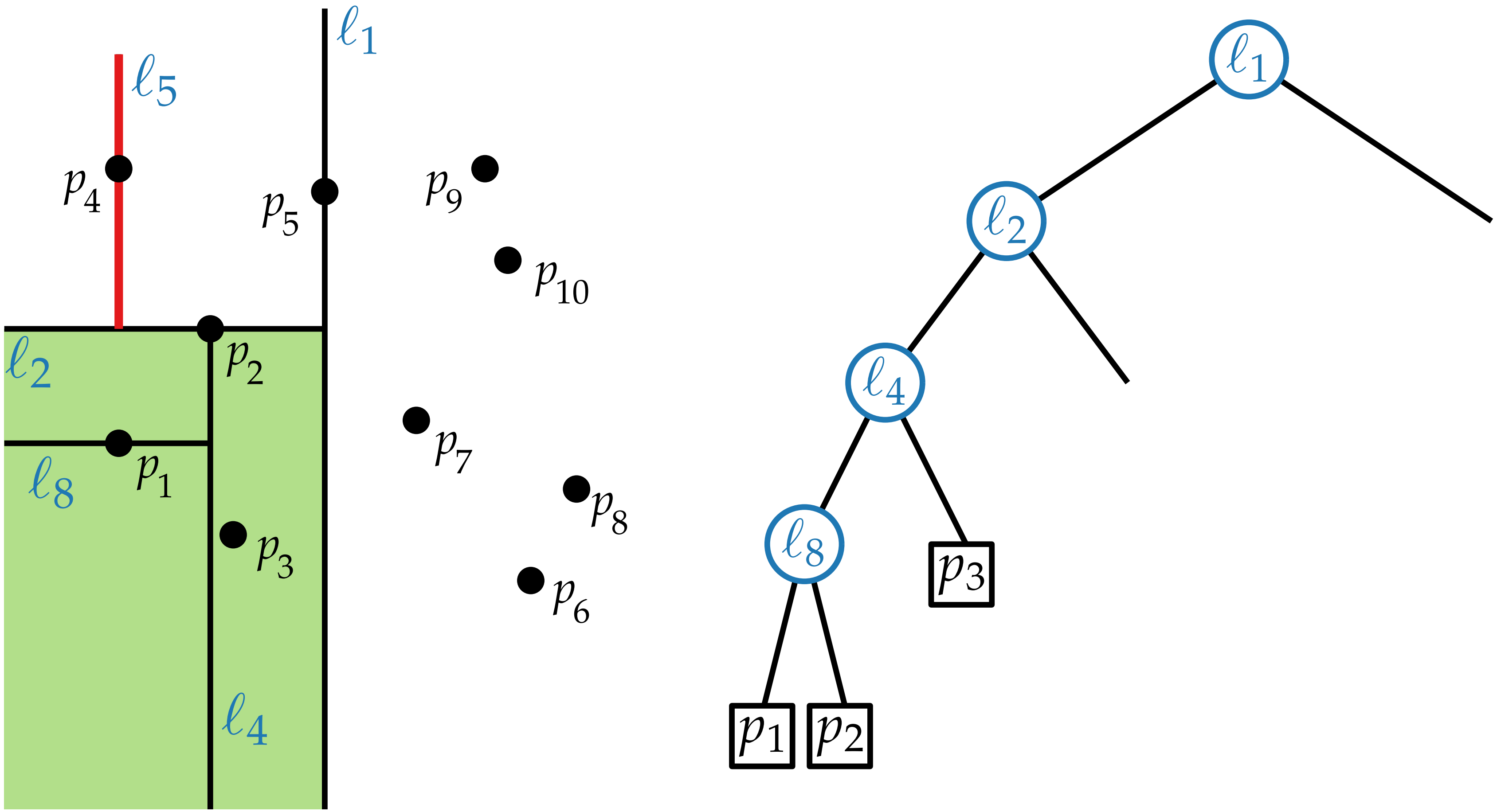

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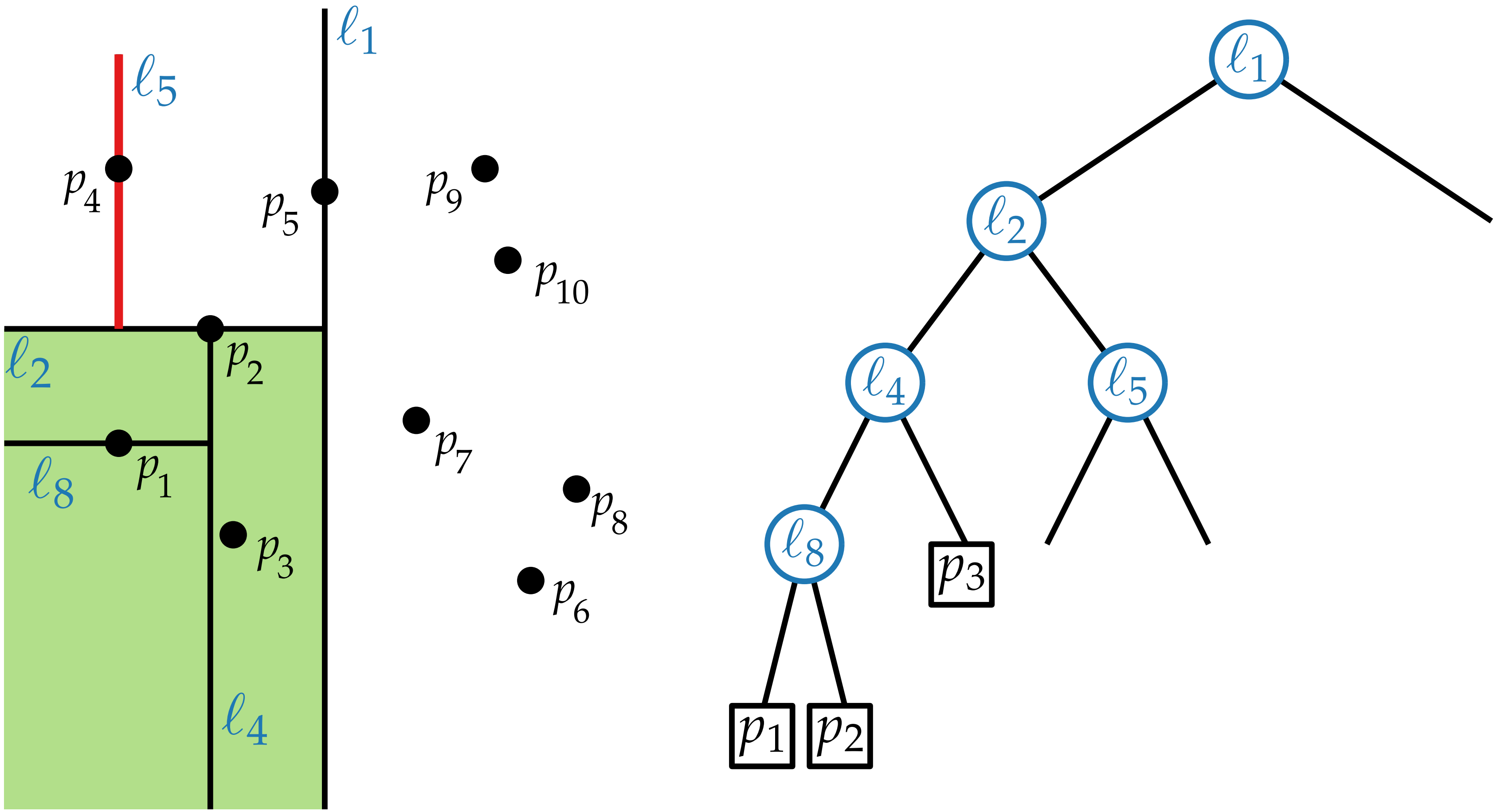

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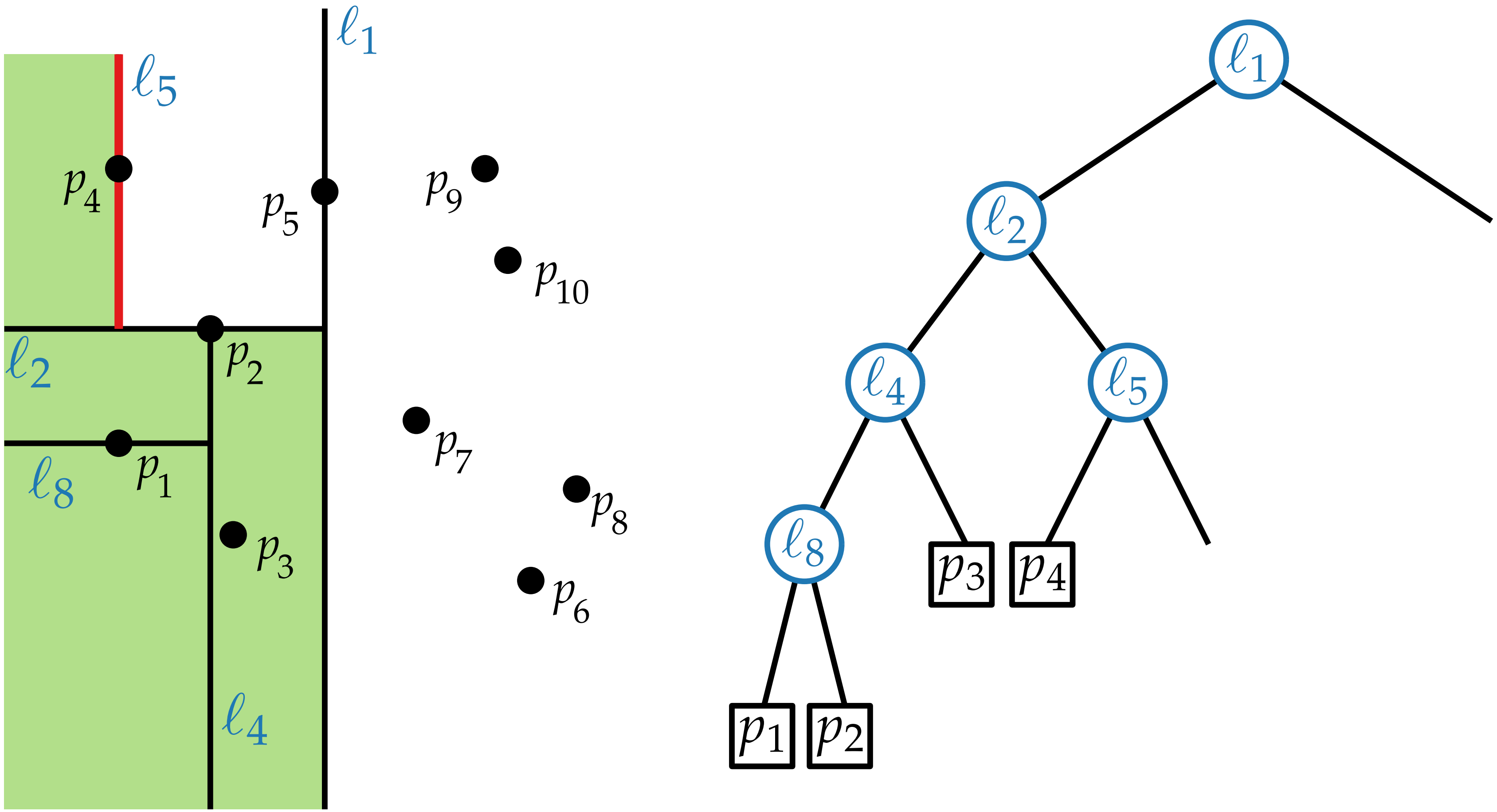

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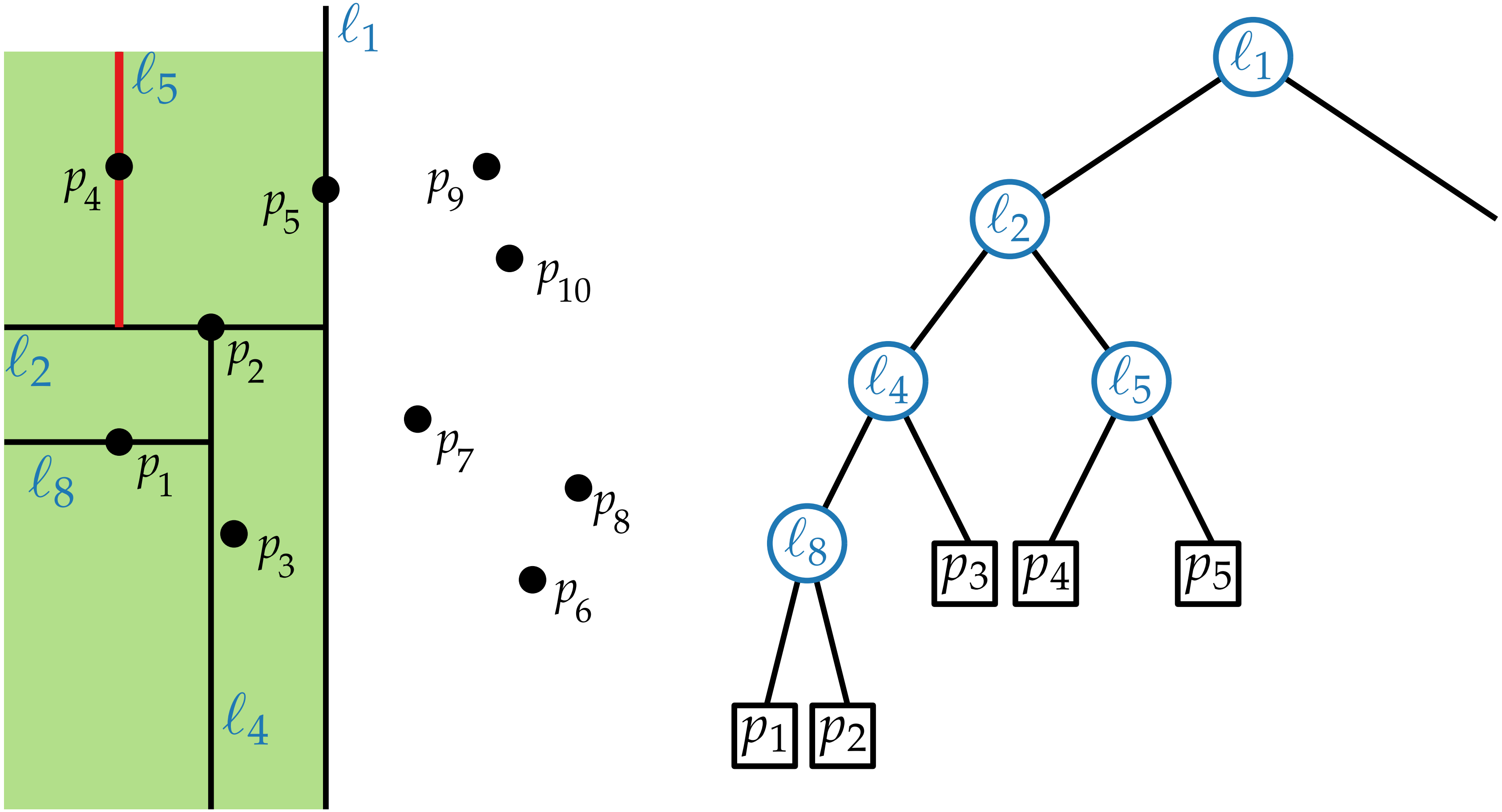

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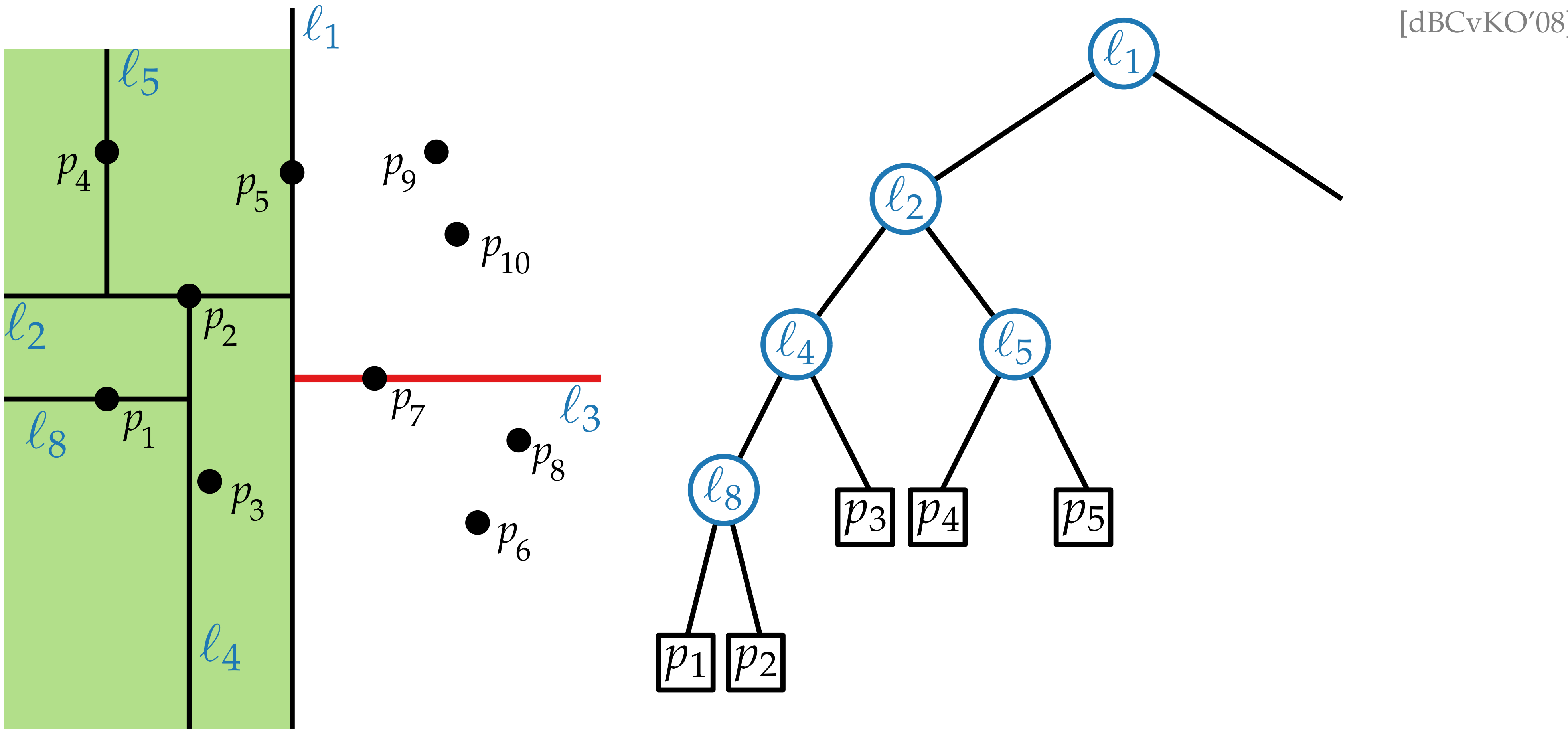

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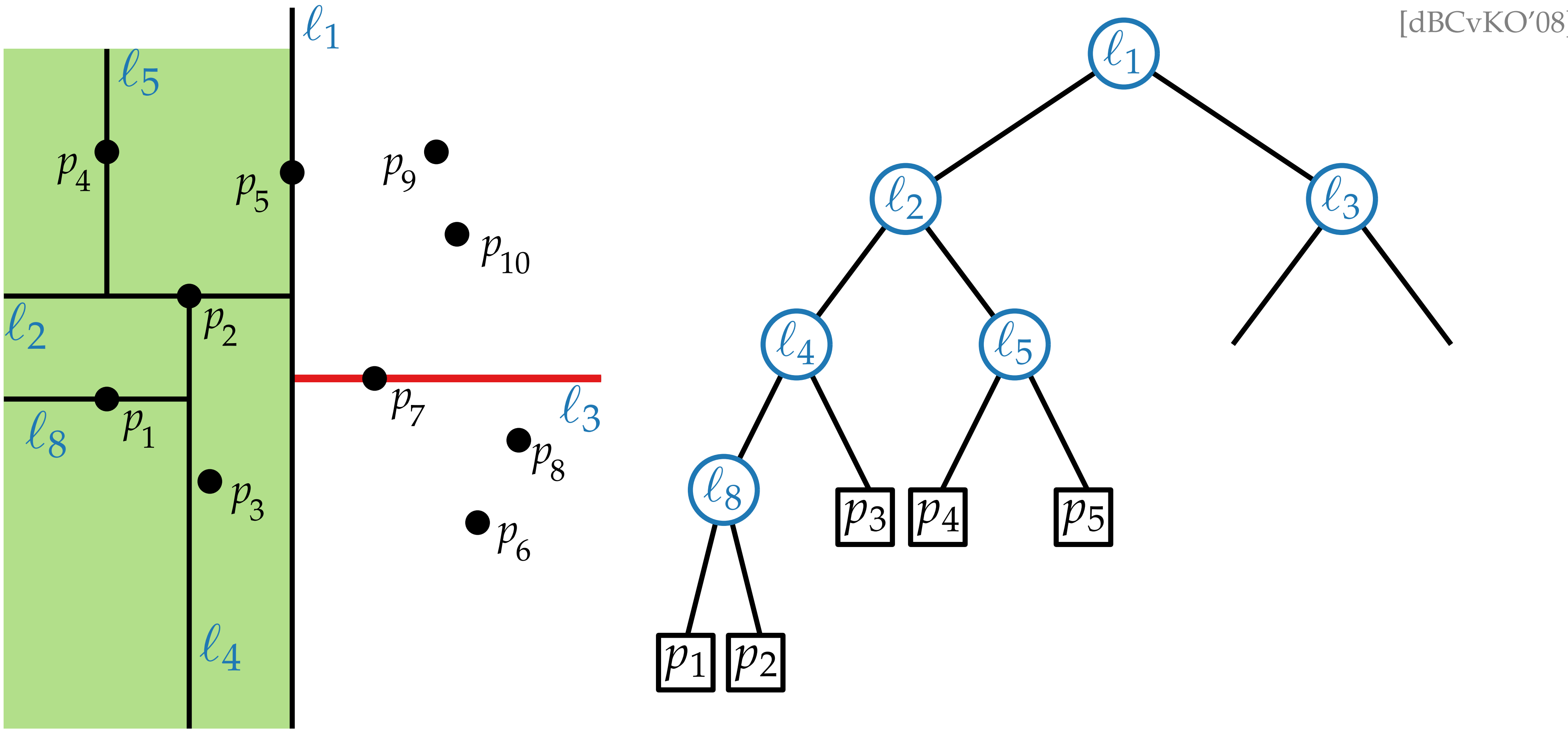

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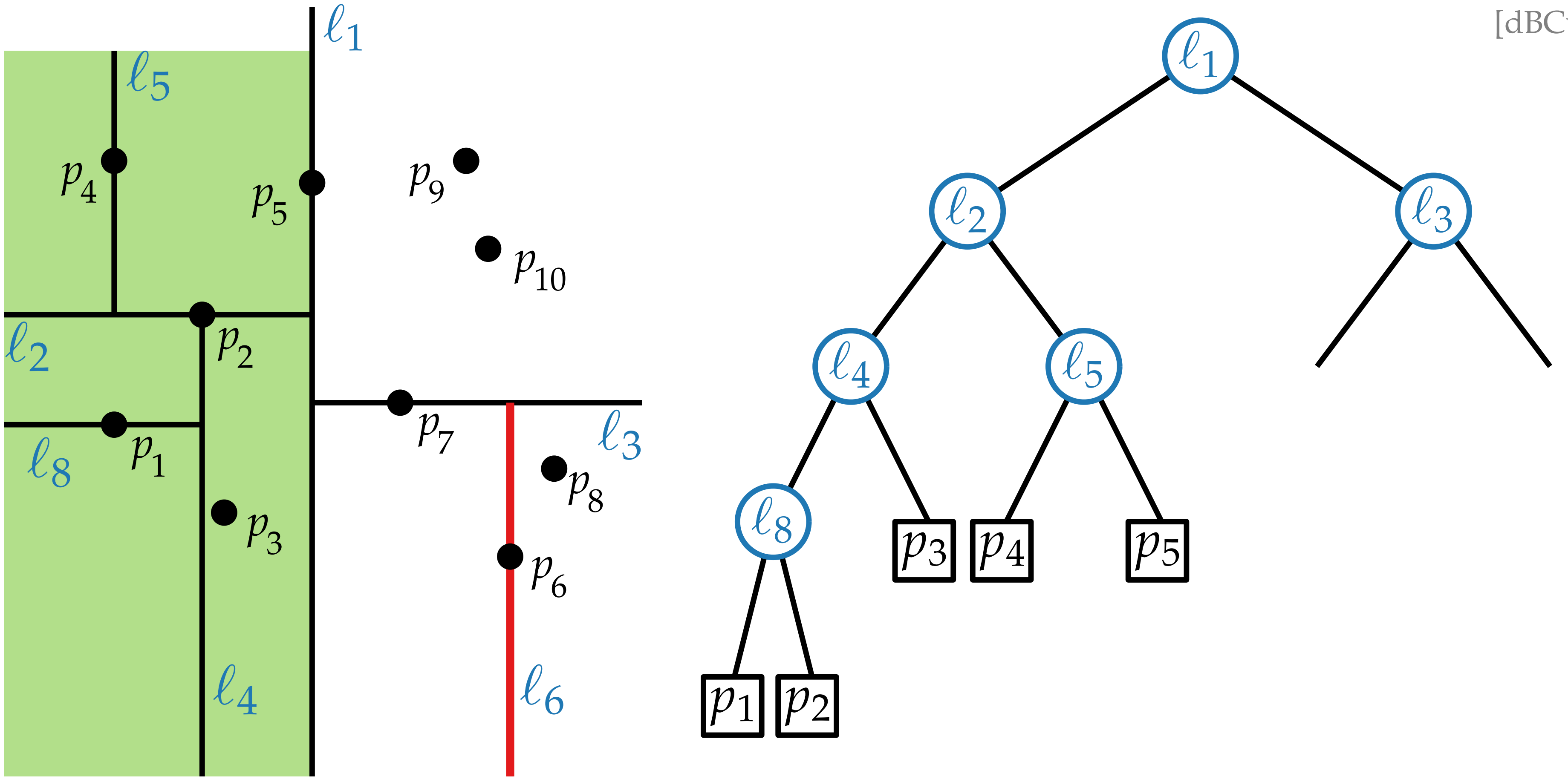

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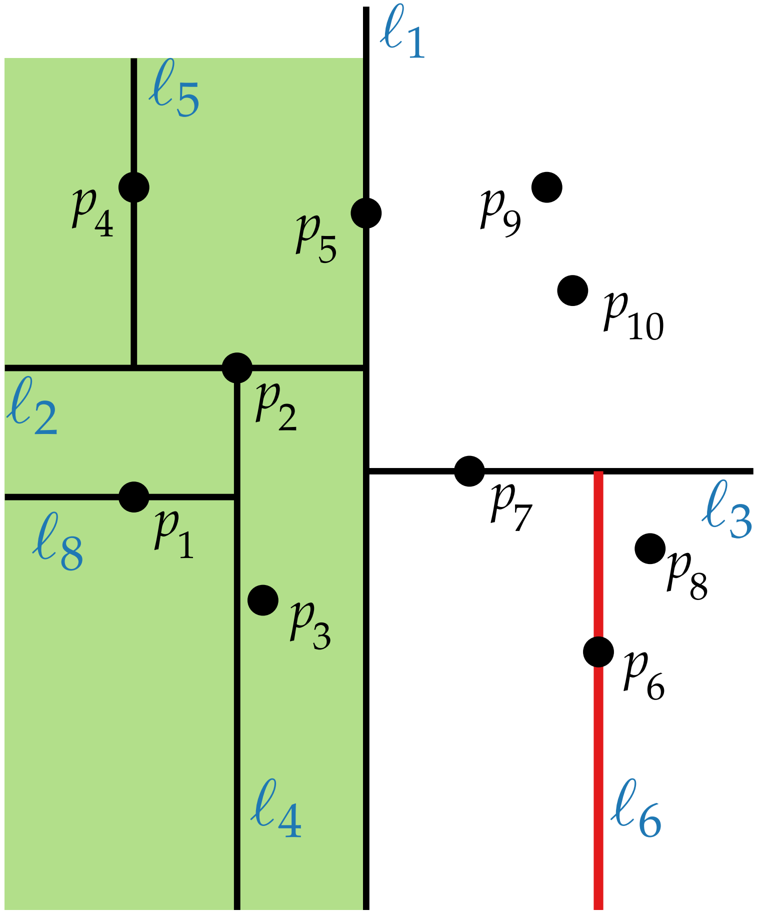

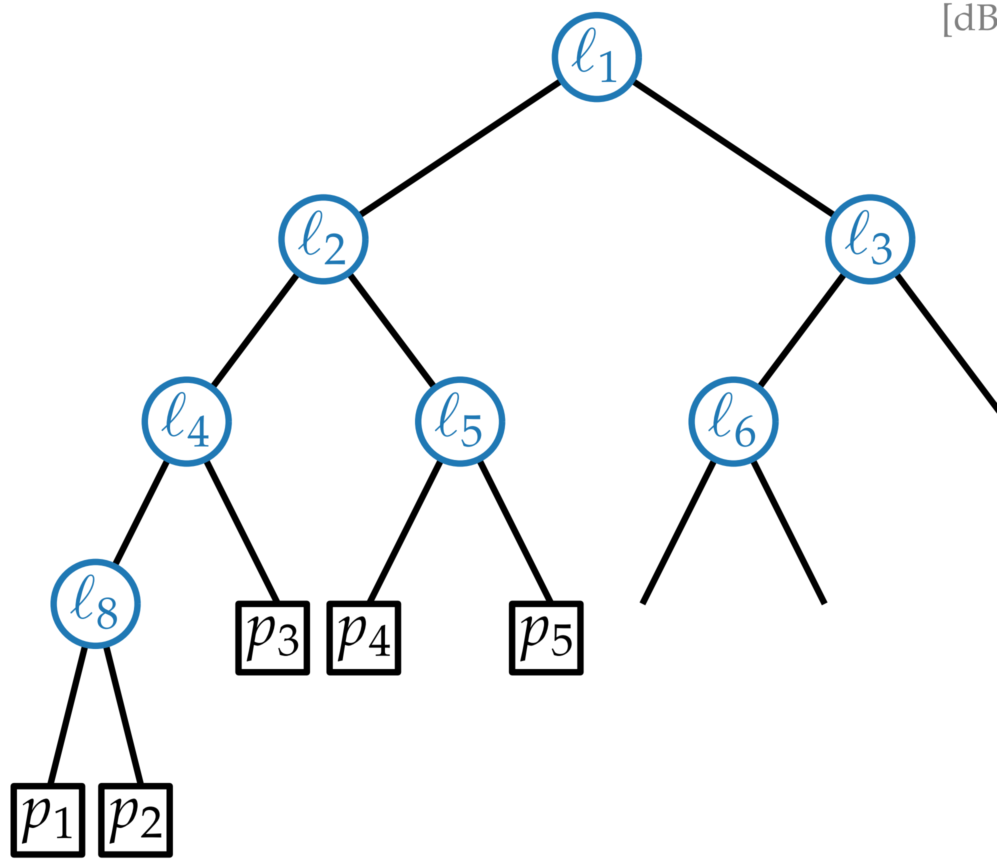

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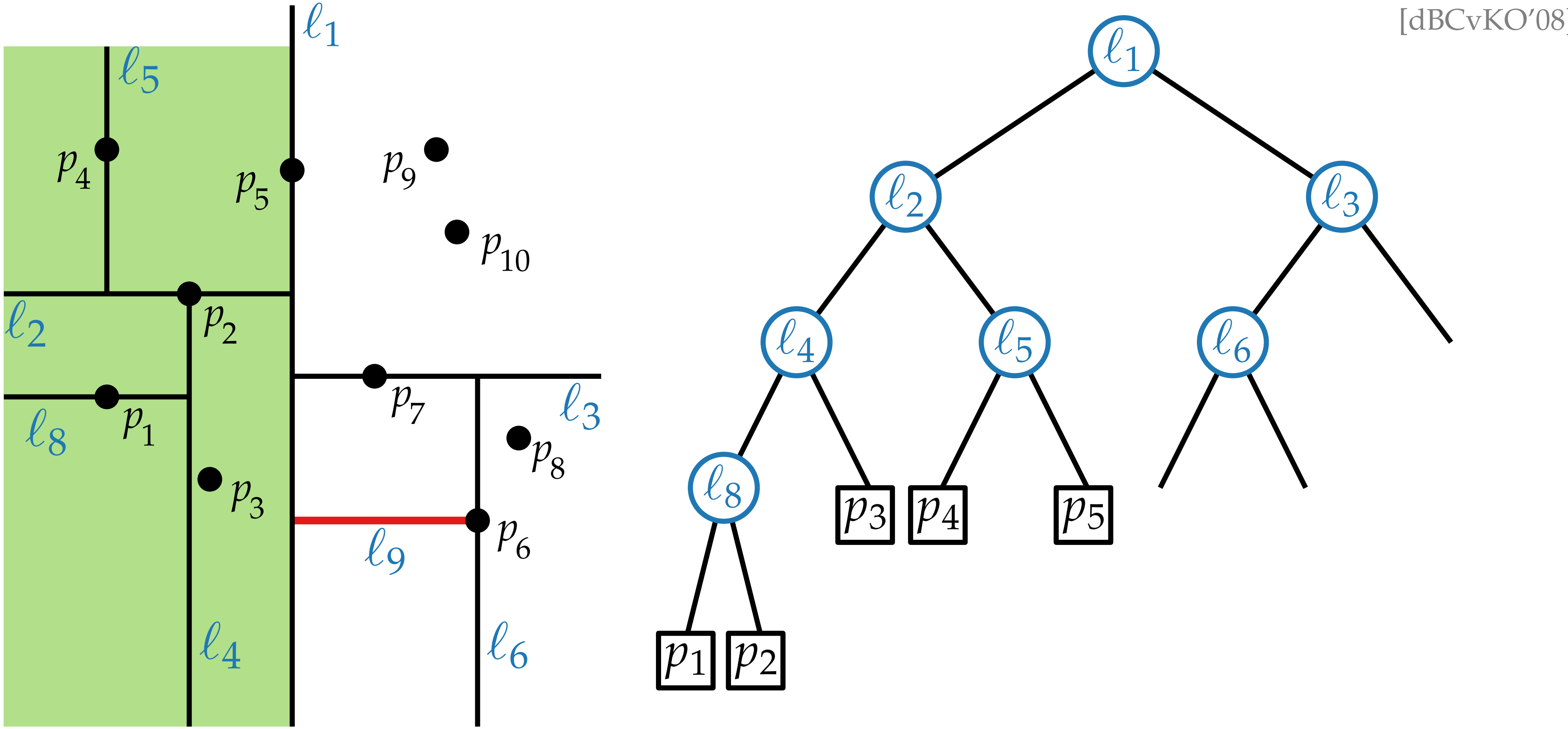

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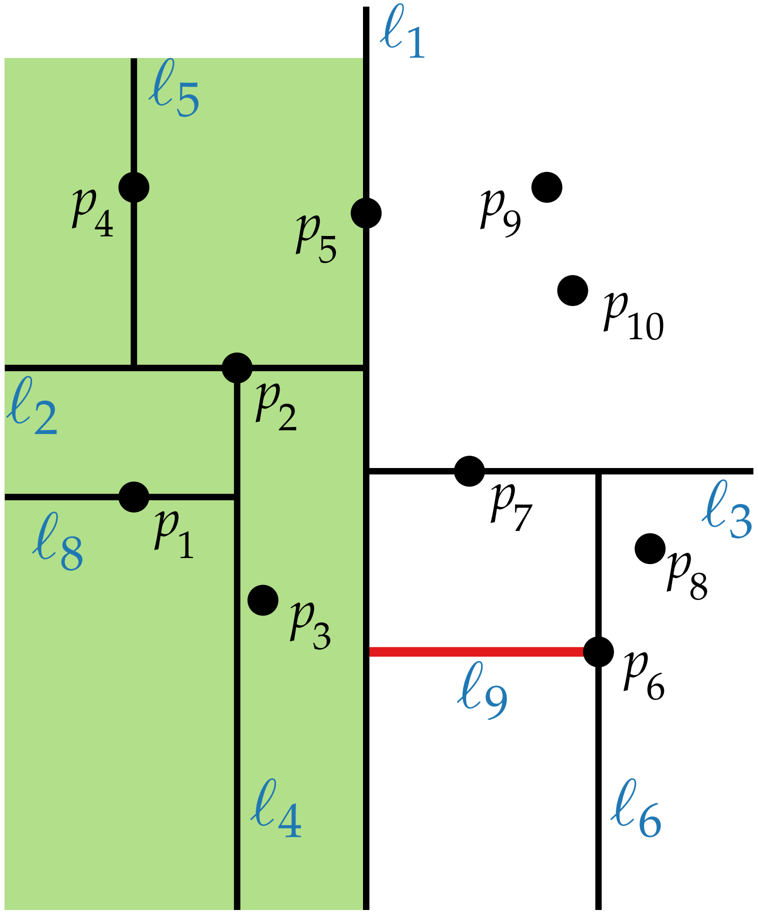

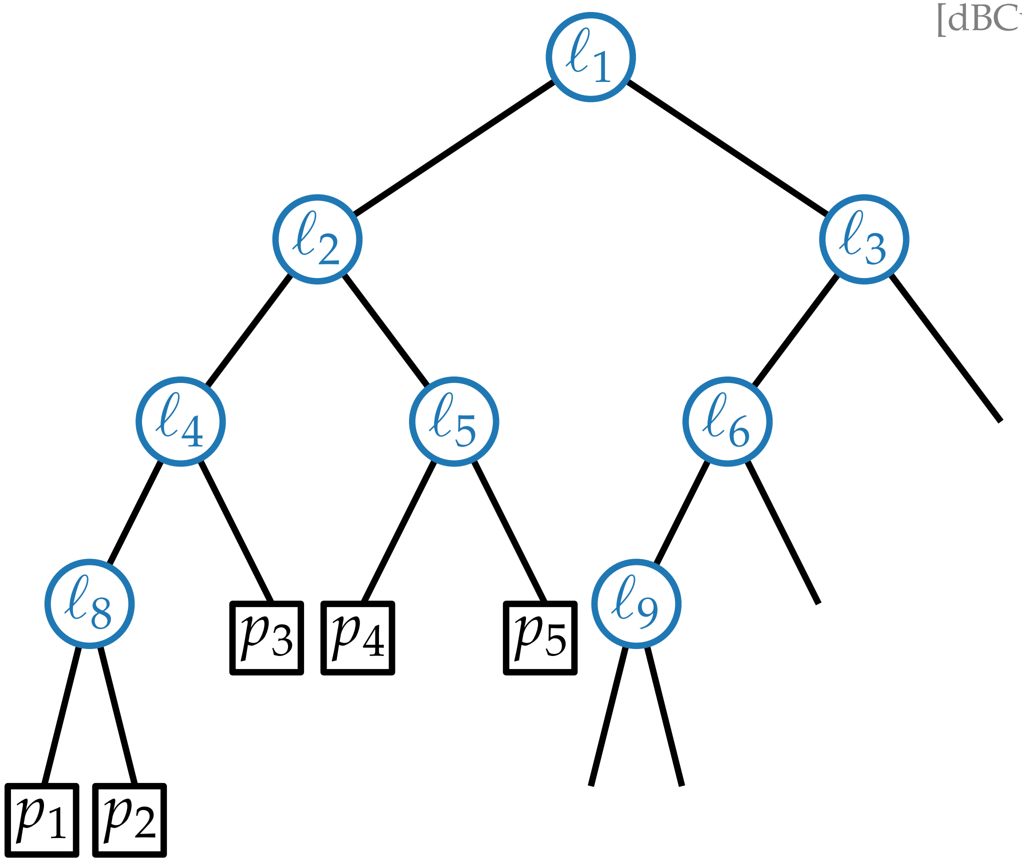

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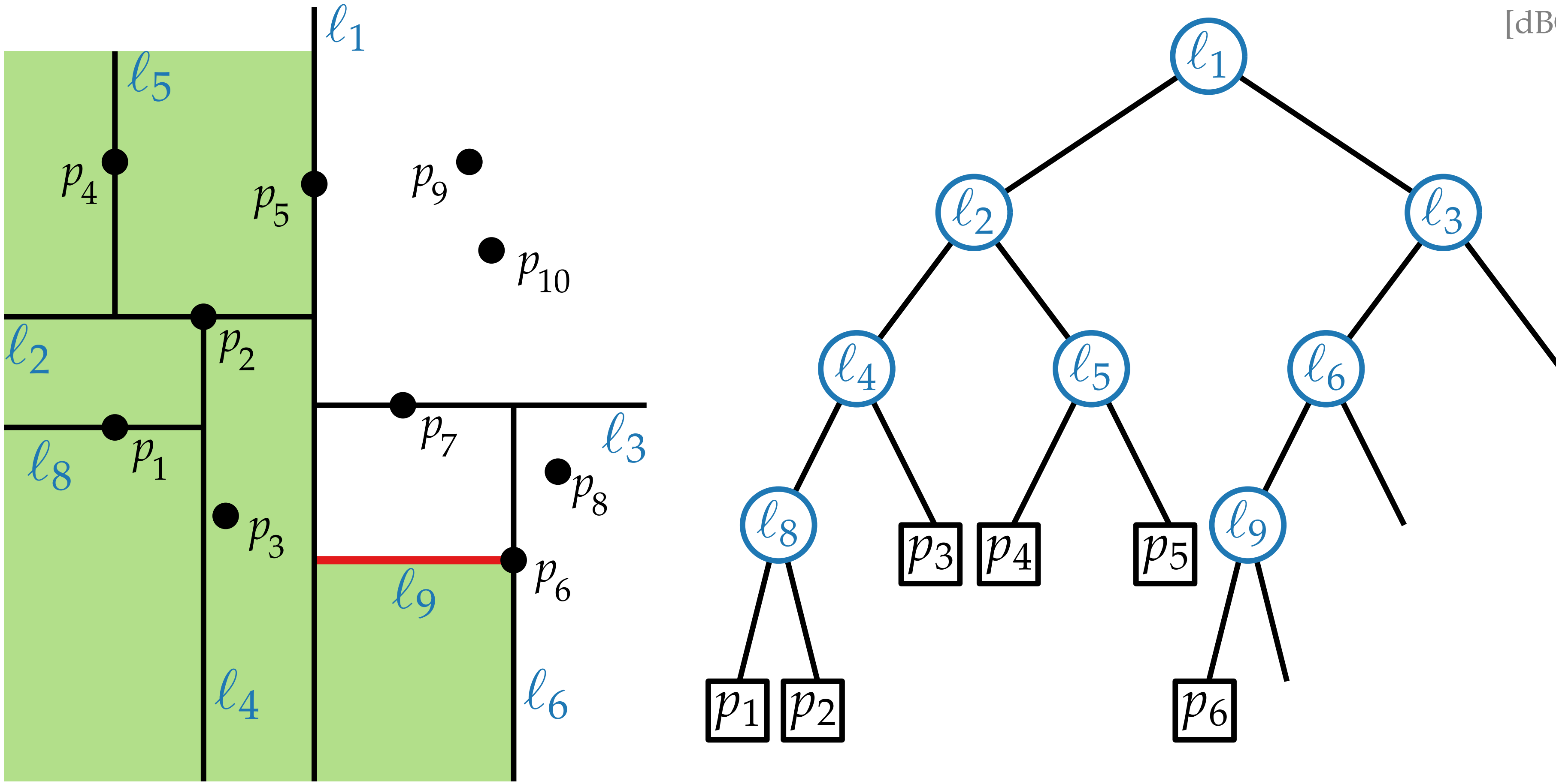

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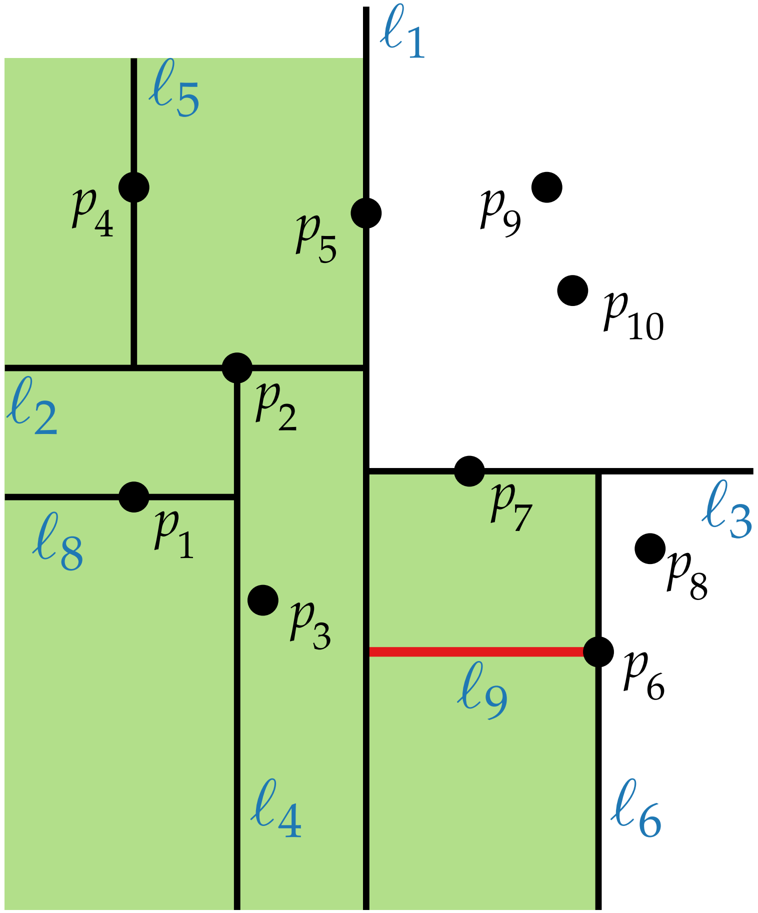

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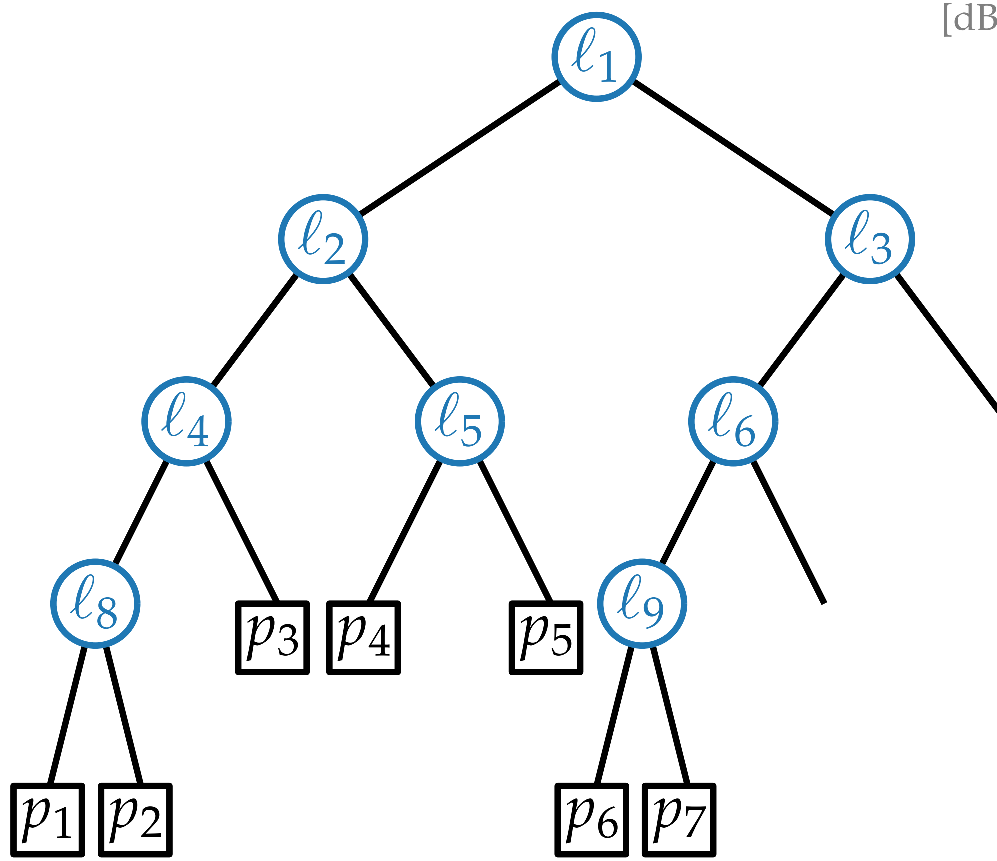

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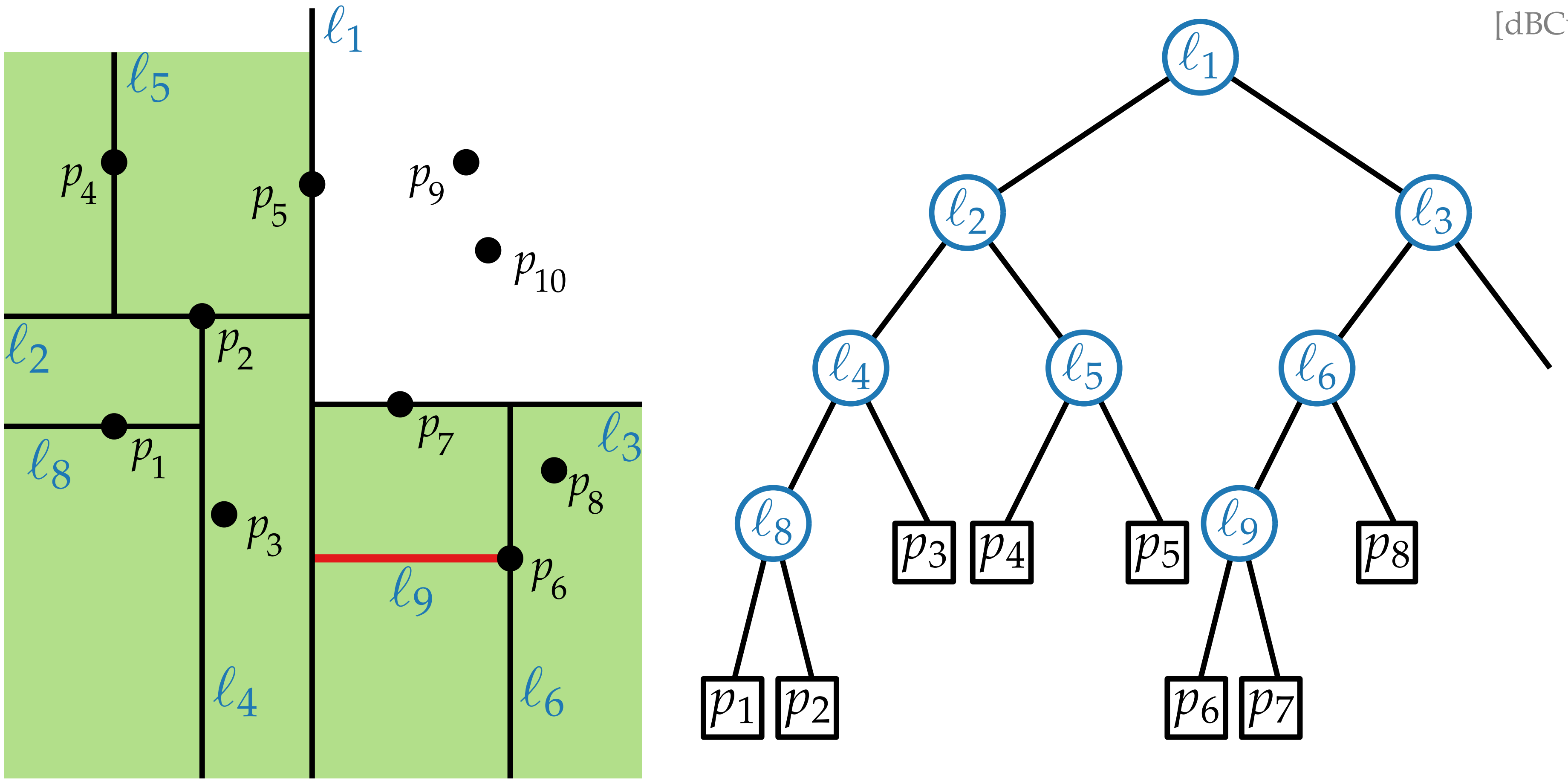

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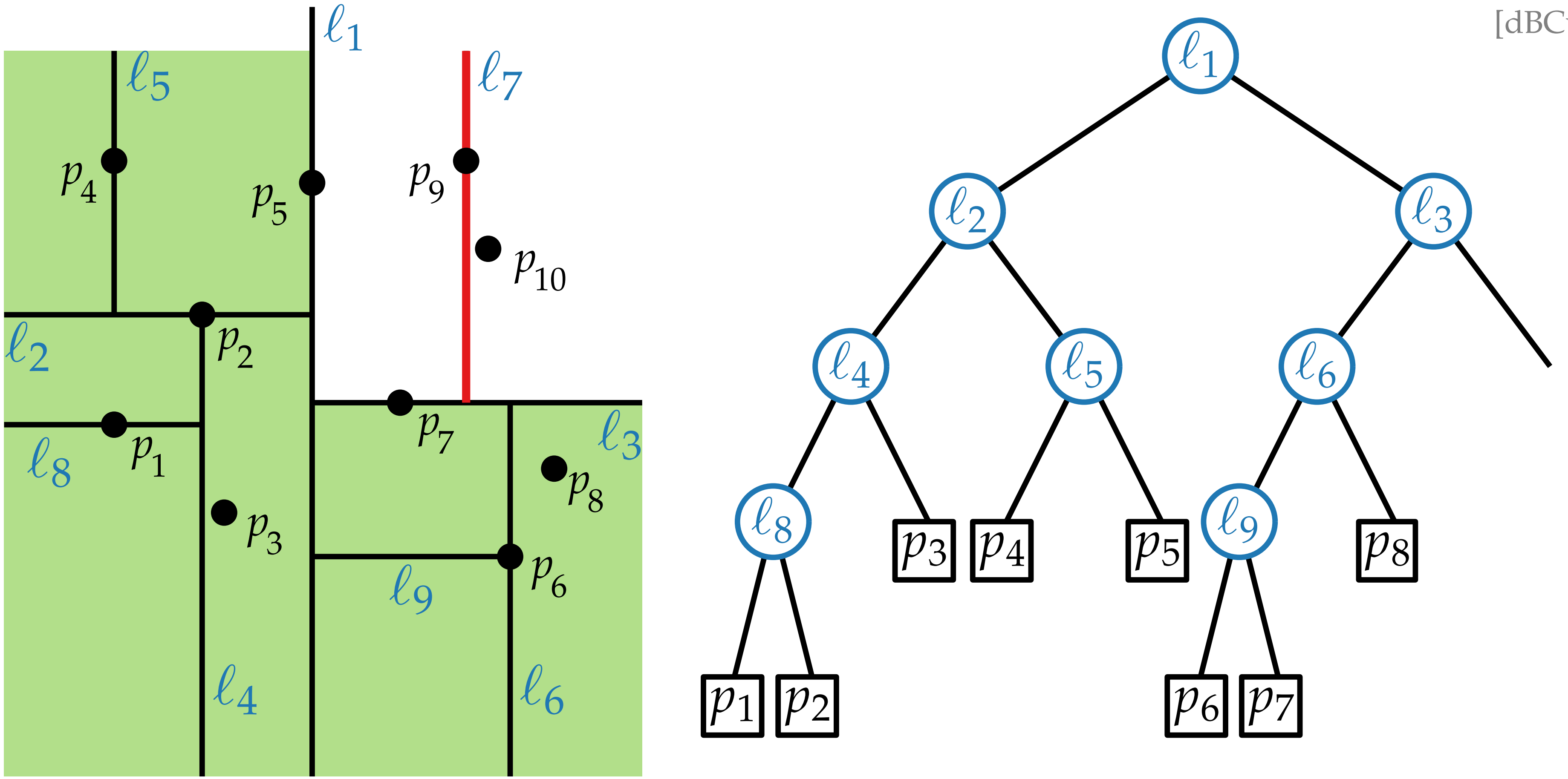

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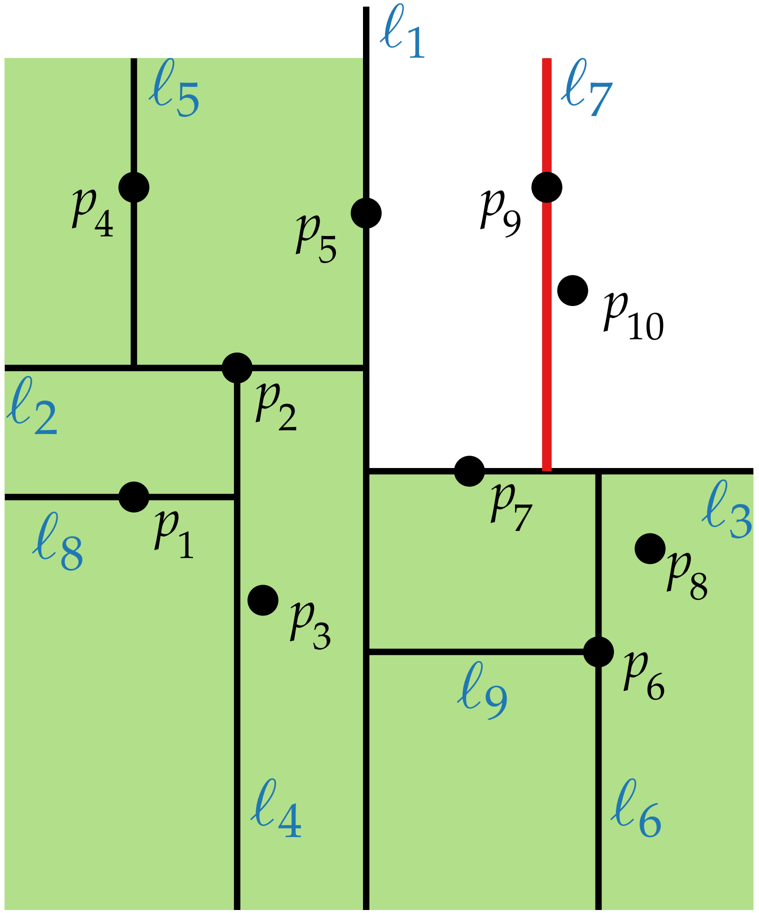

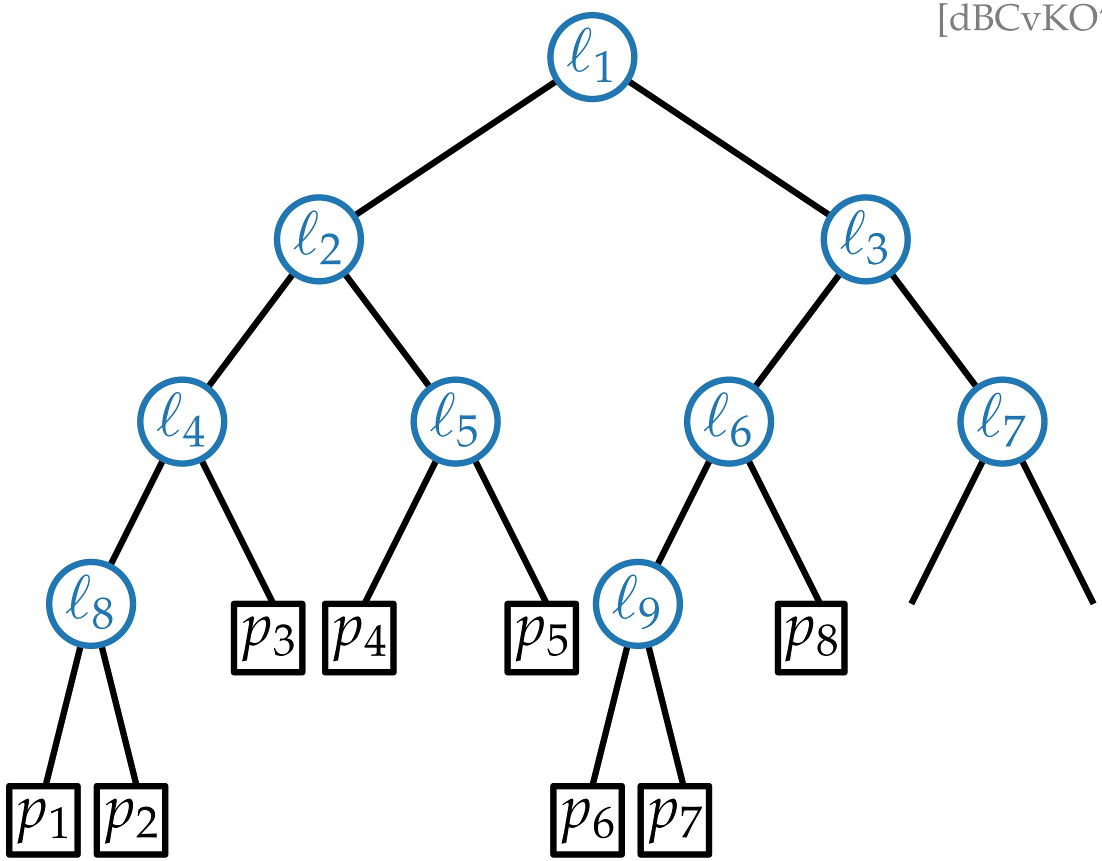

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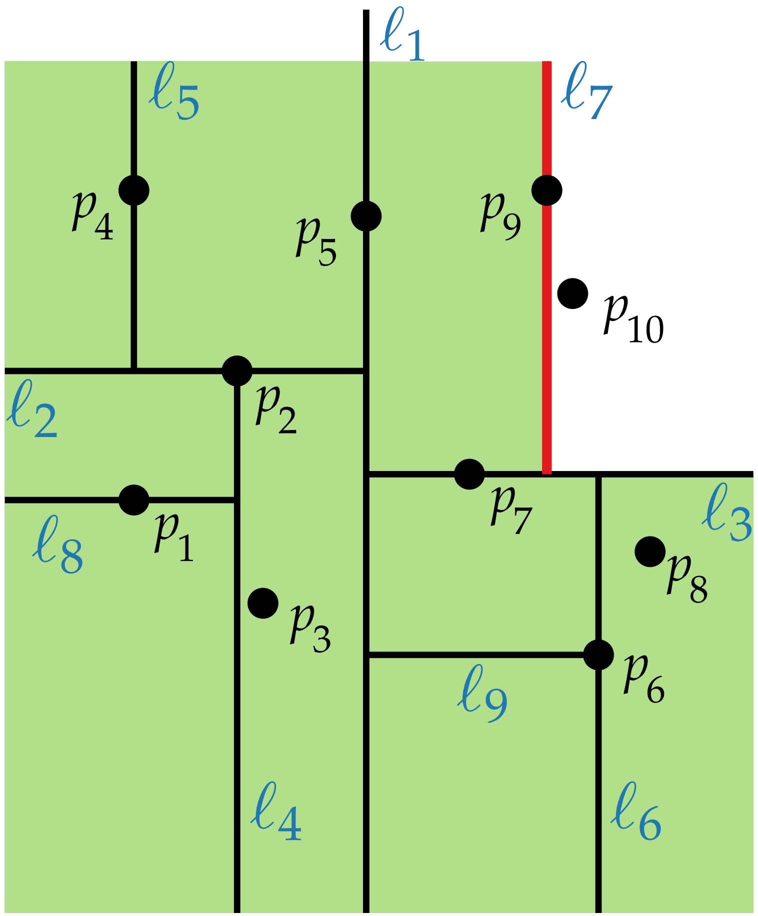

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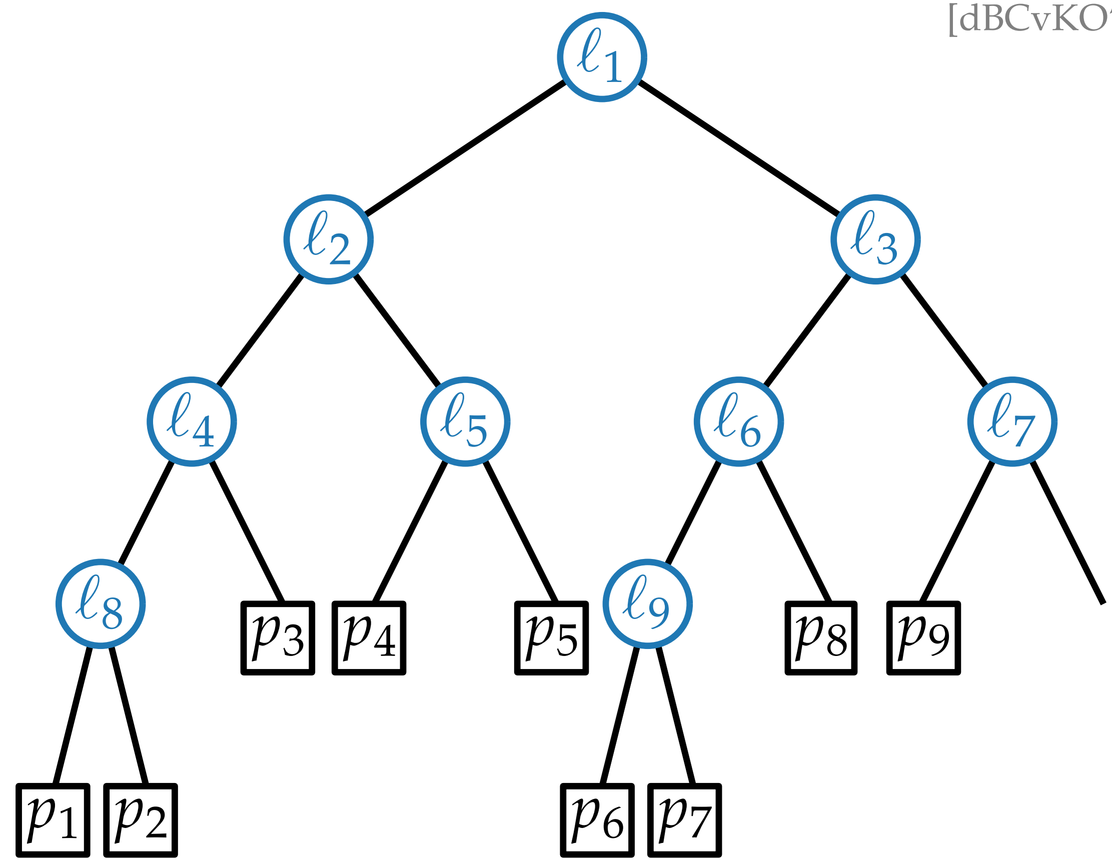

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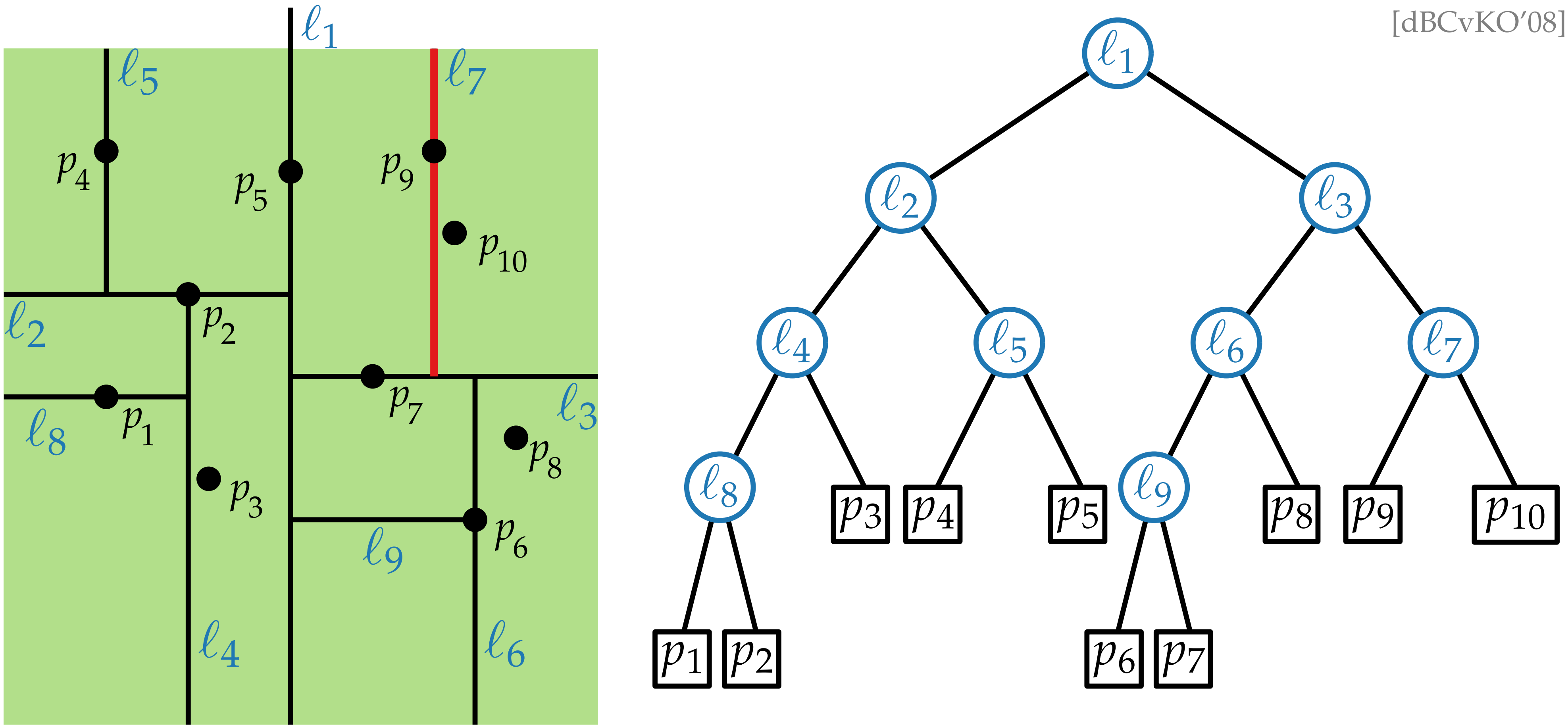

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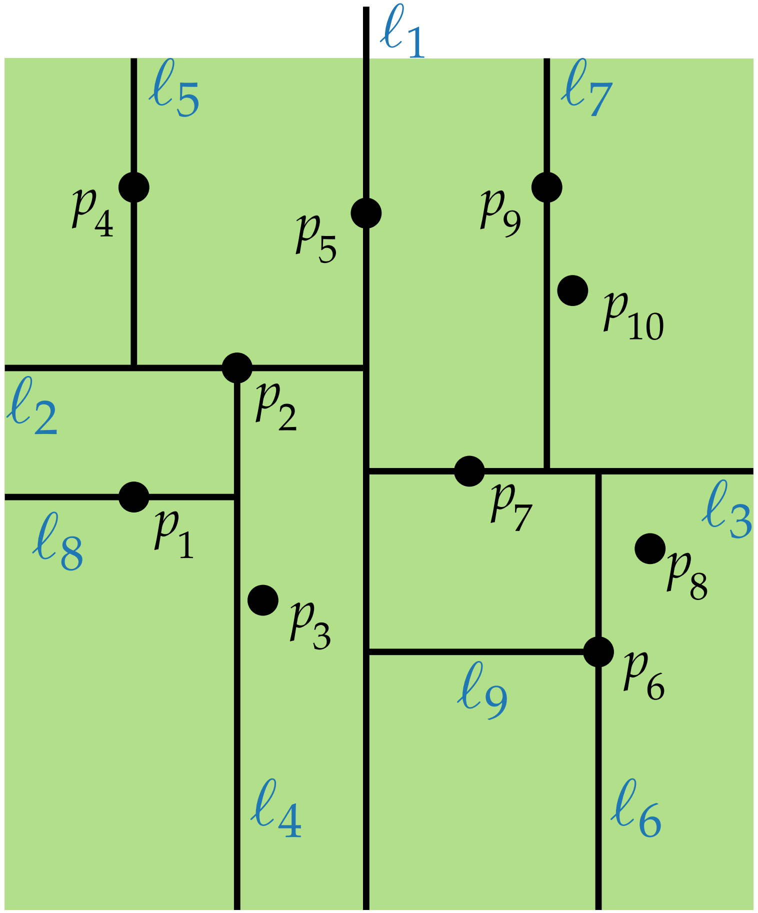

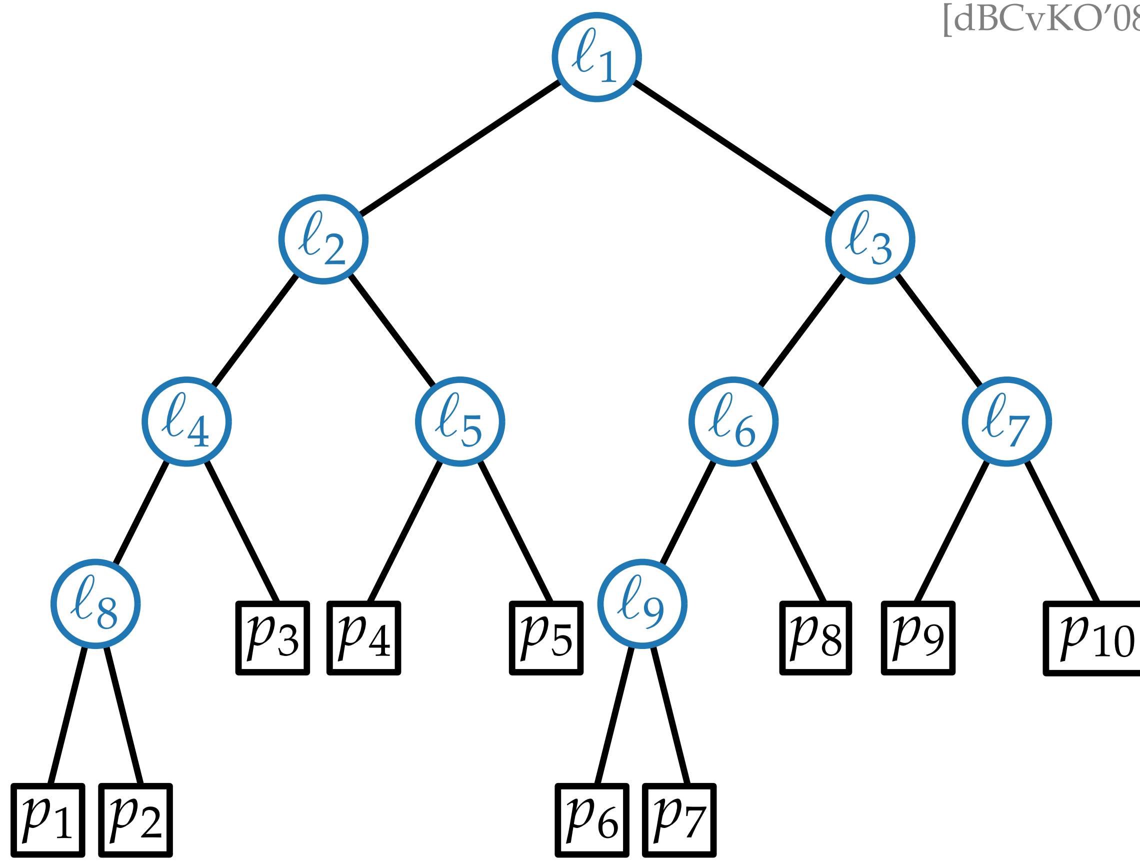

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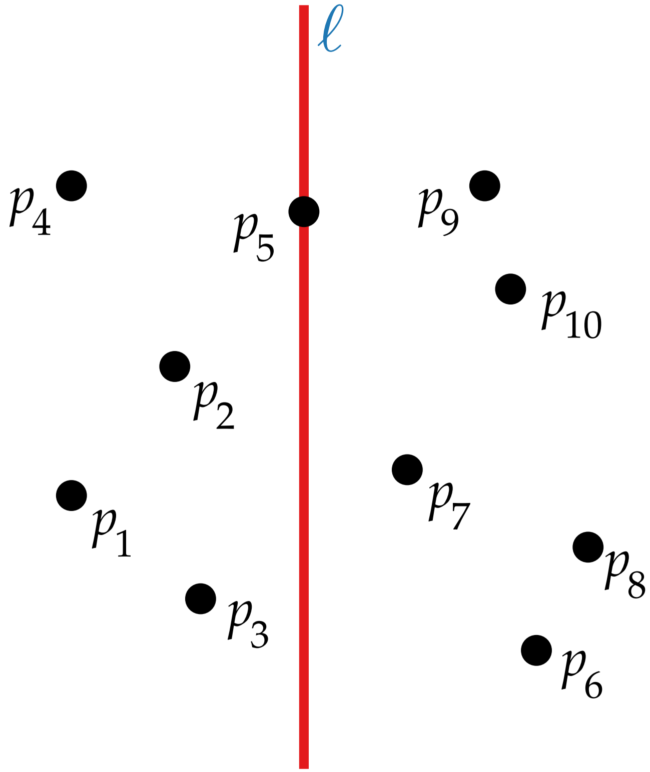

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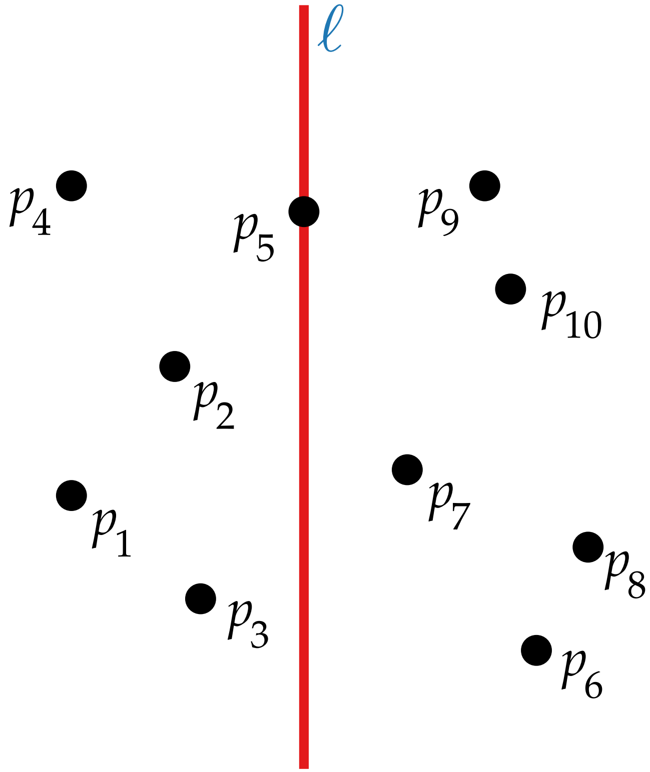

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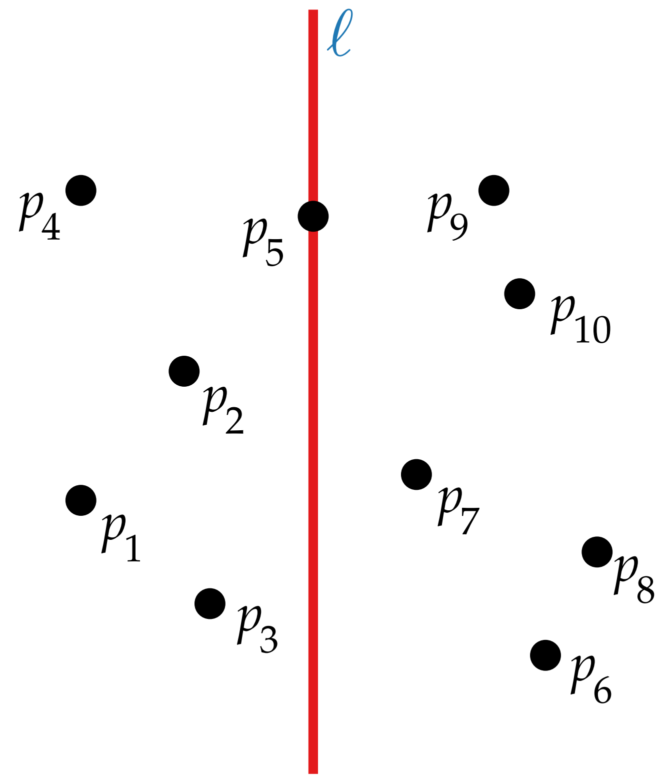

Pseudo-code:

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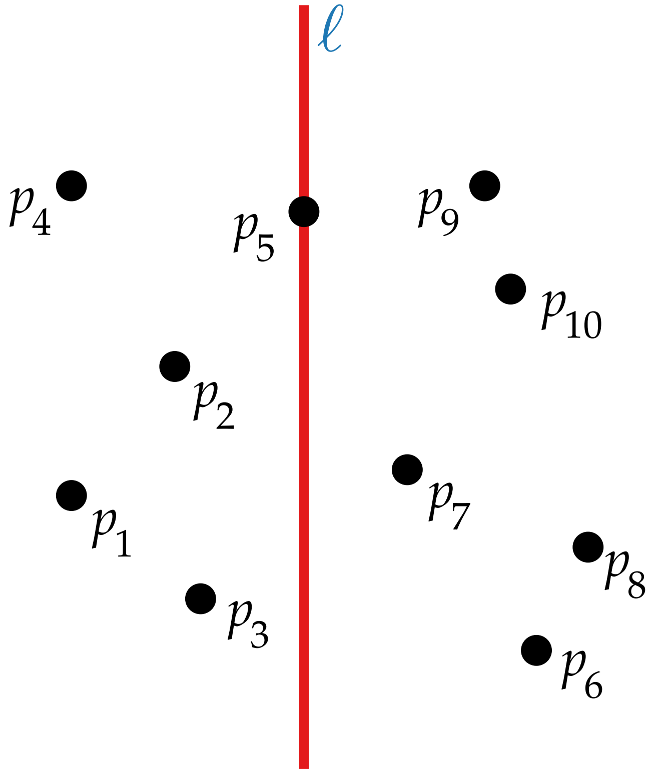

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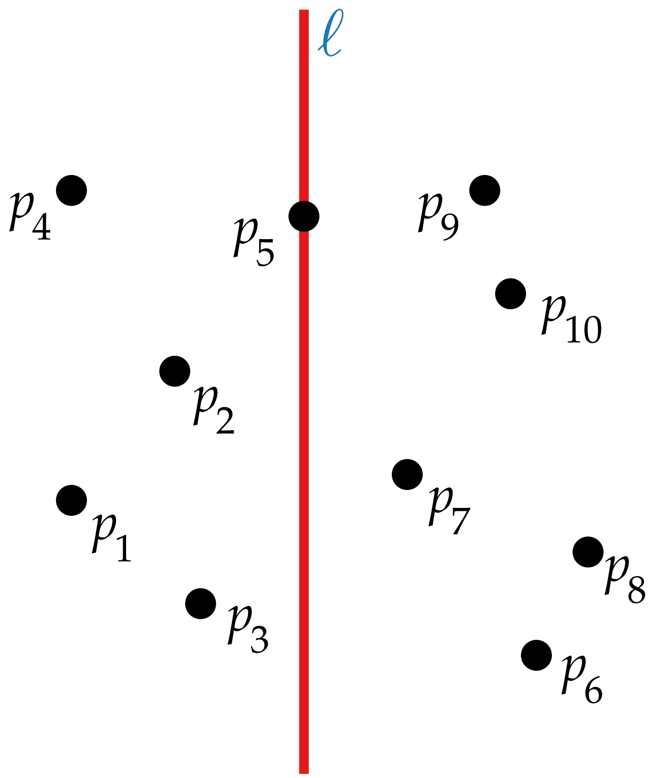

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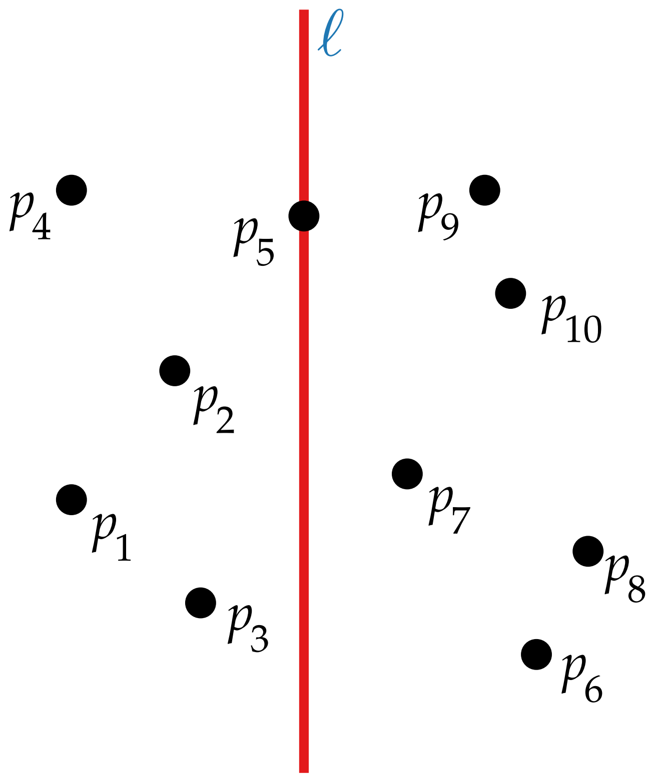

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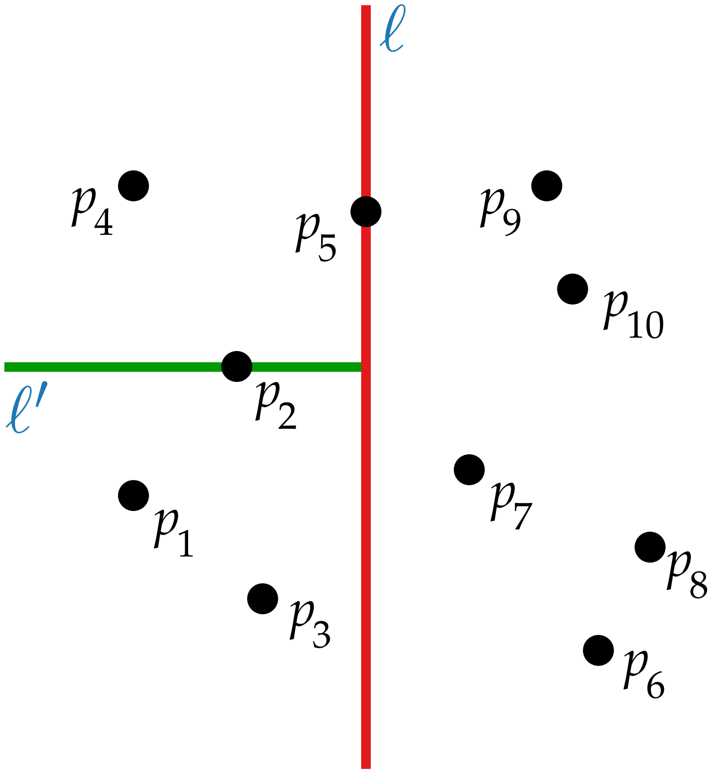

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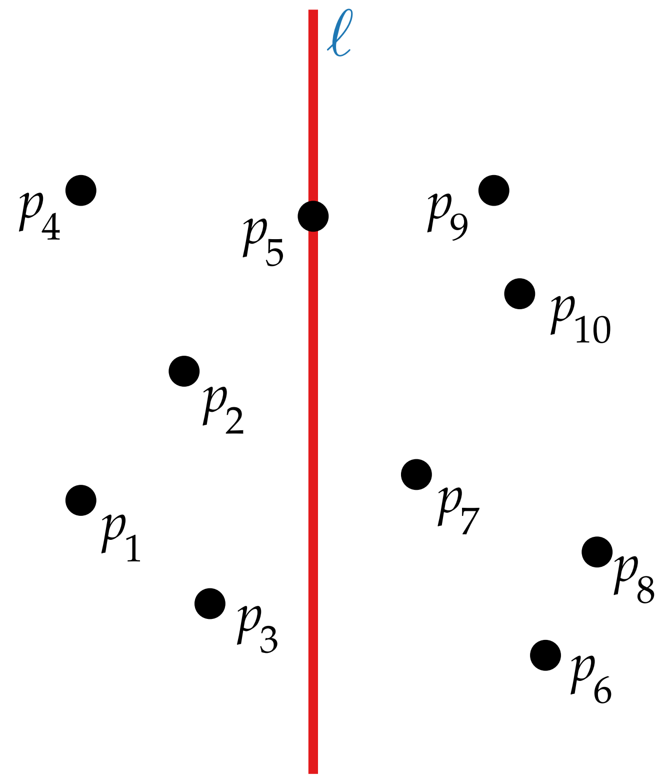

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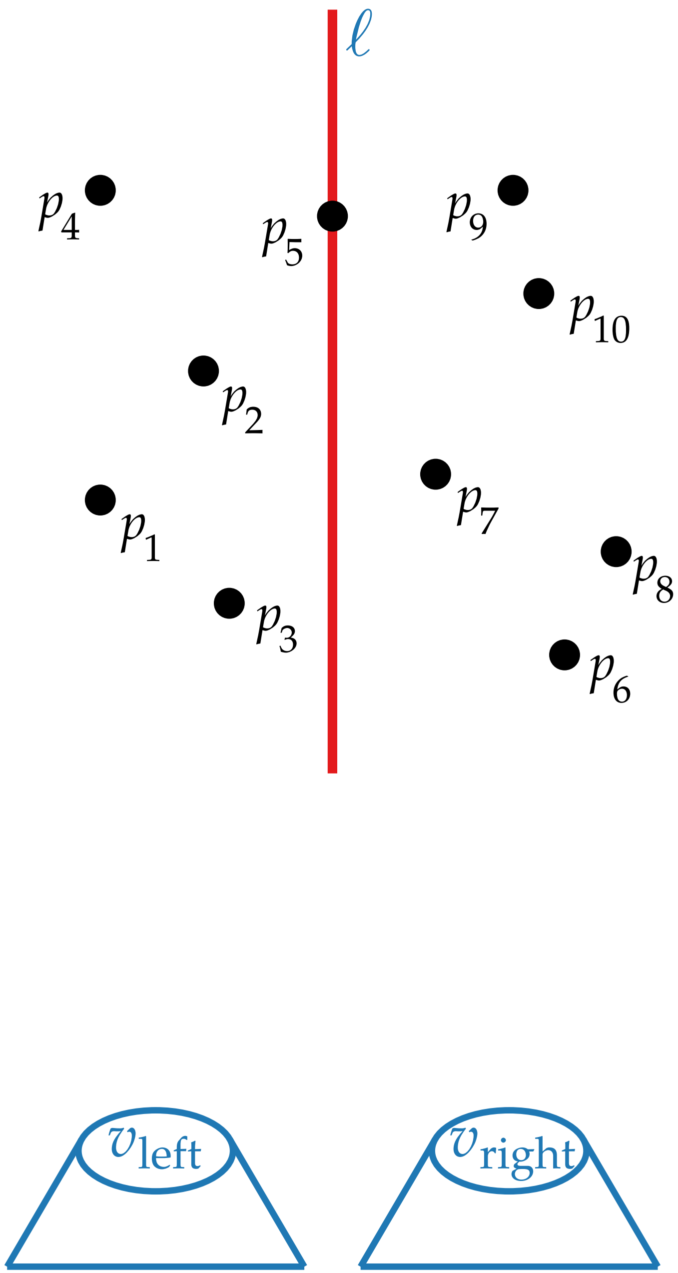

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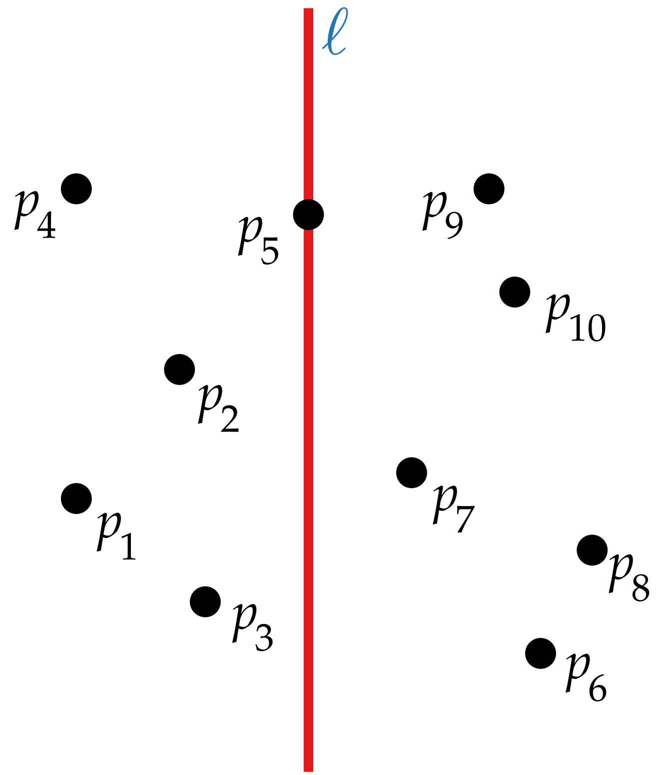

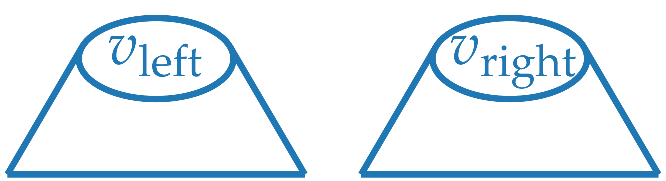

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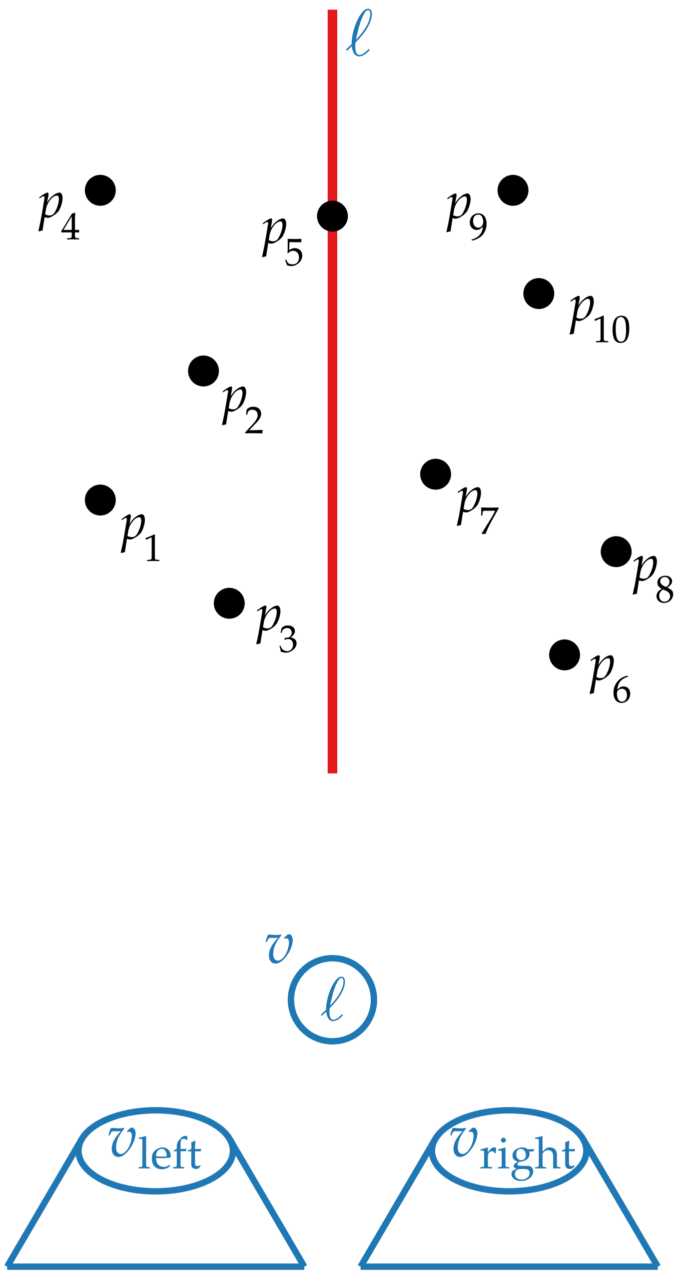

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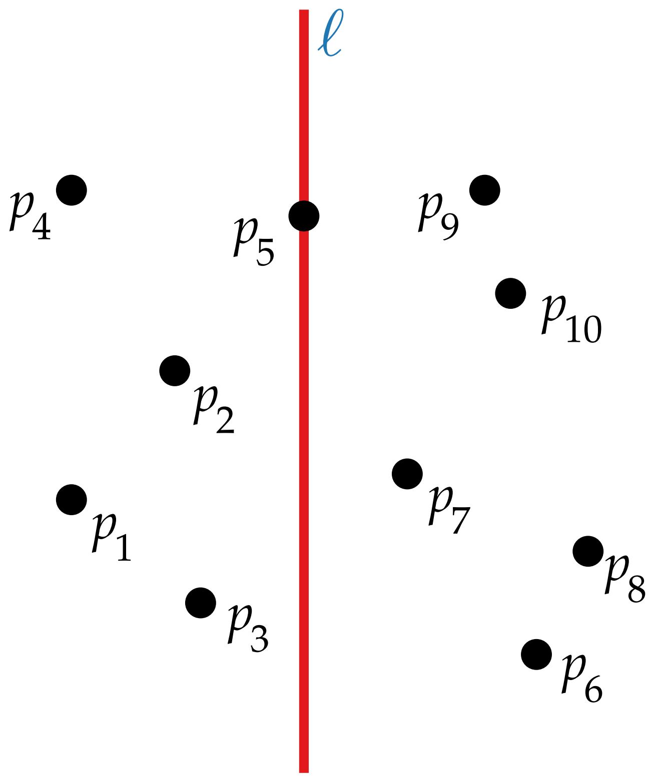

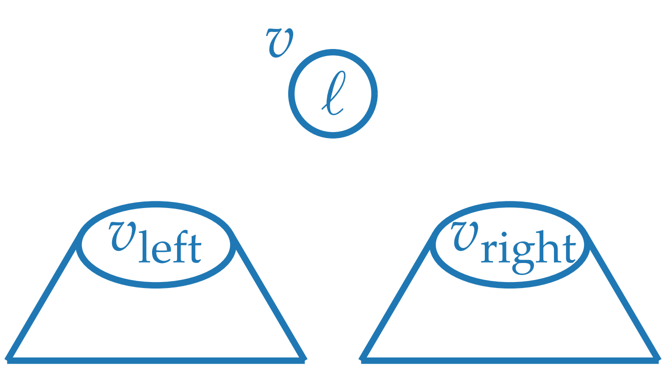

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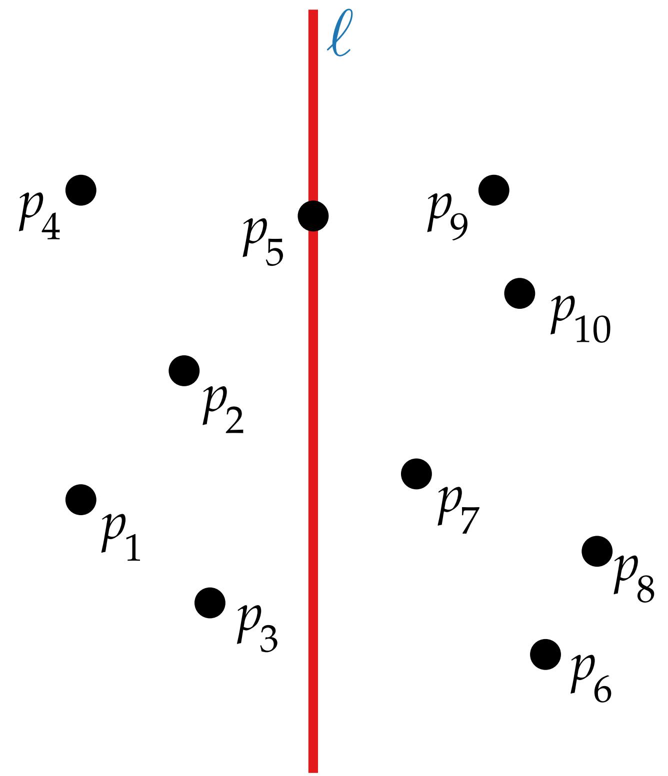

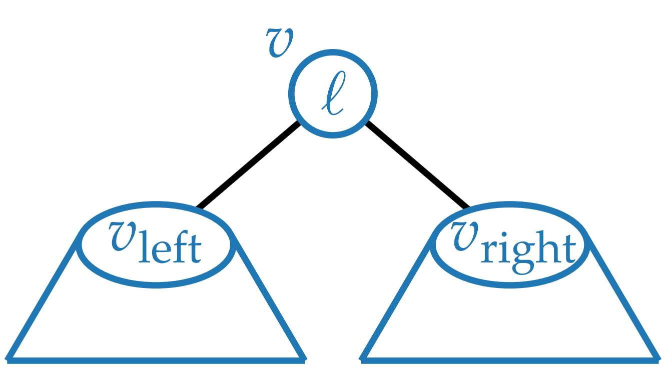

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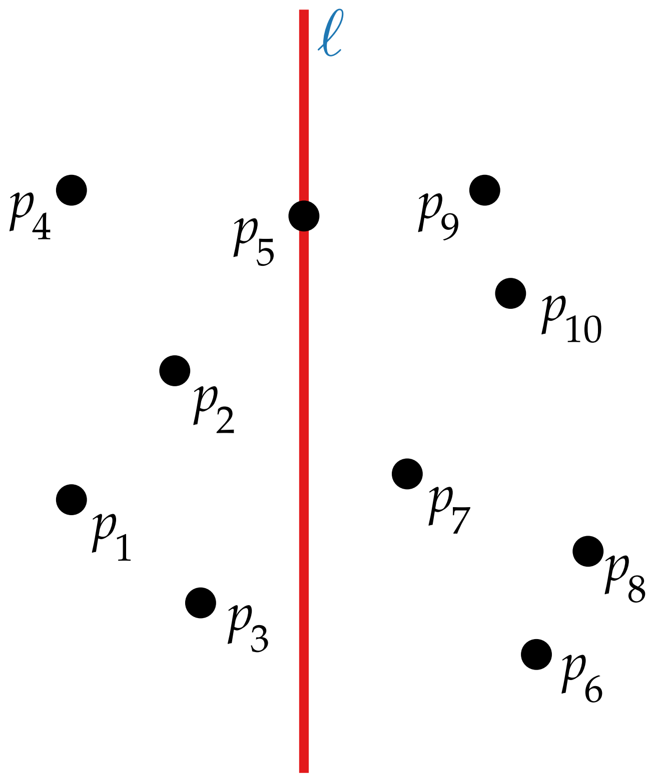

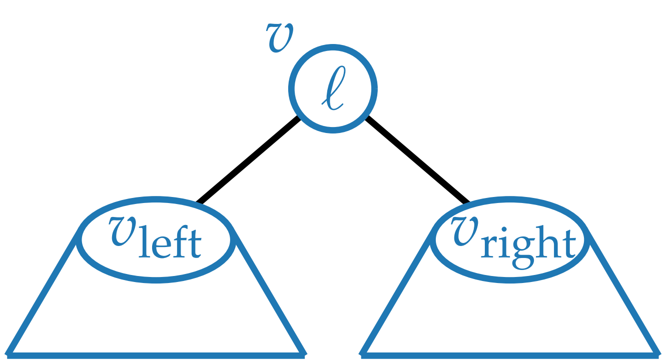

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  return (v)

Kd-Trees: Analysis

Construction time?

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$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ O(n) + 2T(\lceil n/2 \rceil) & \text{else.} \end{cases}$$

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see Mergesort!

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Lemma: A kd-tree for a set of n pts in the plane takes $O(n \log n)$ time to construct and uses $O(n)$ storage.

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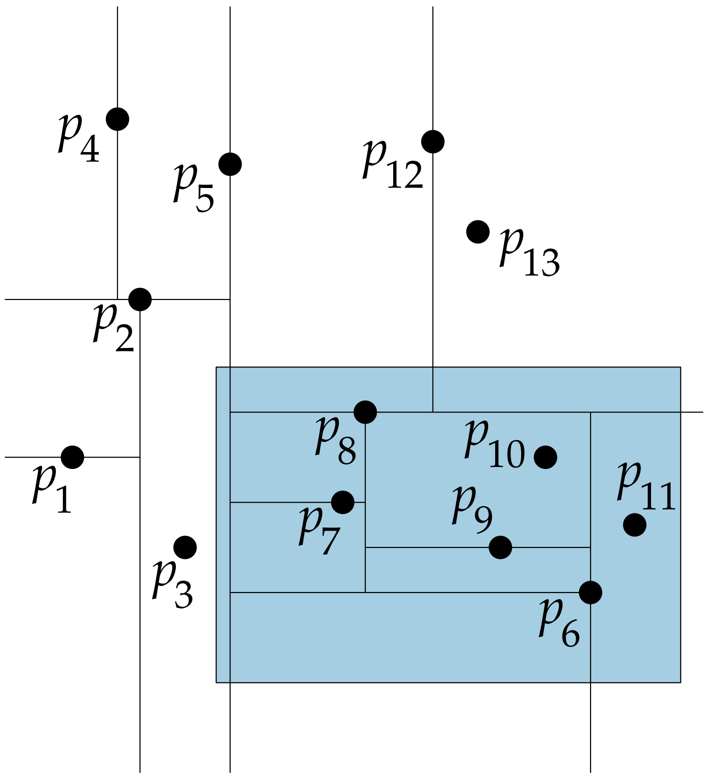

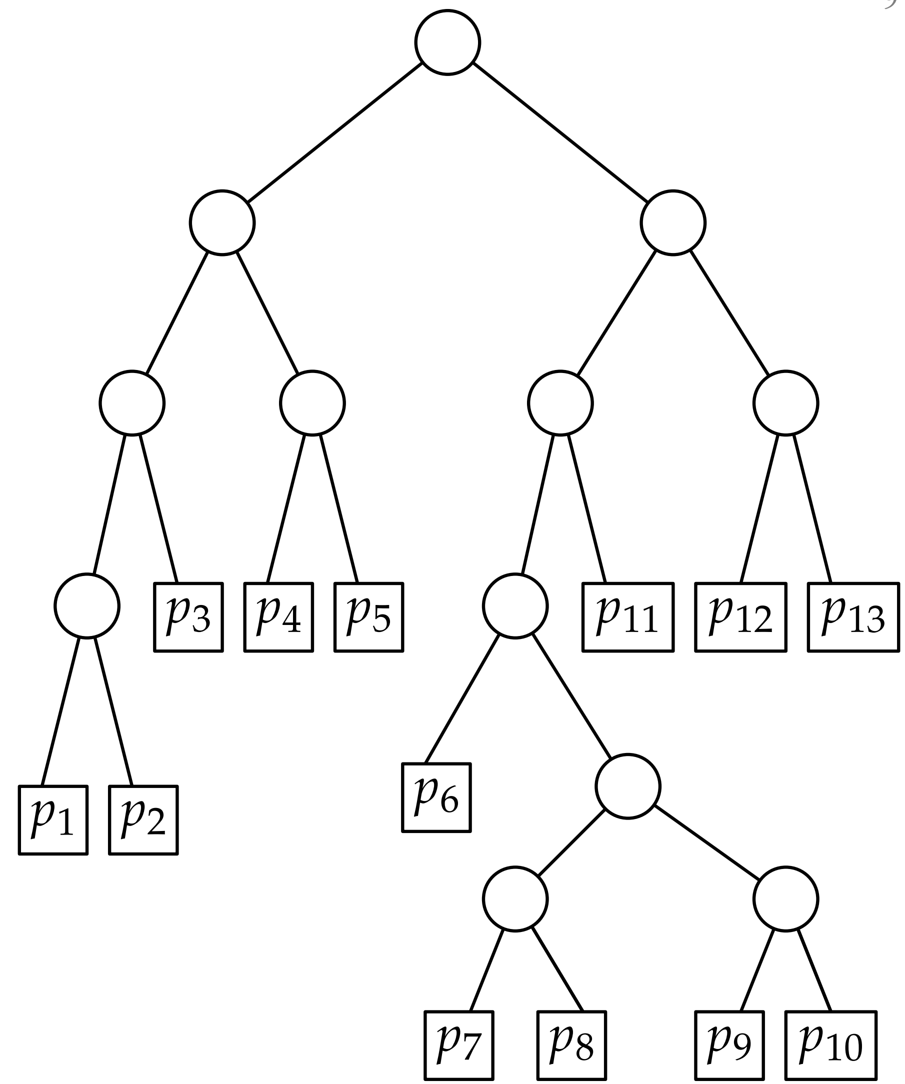

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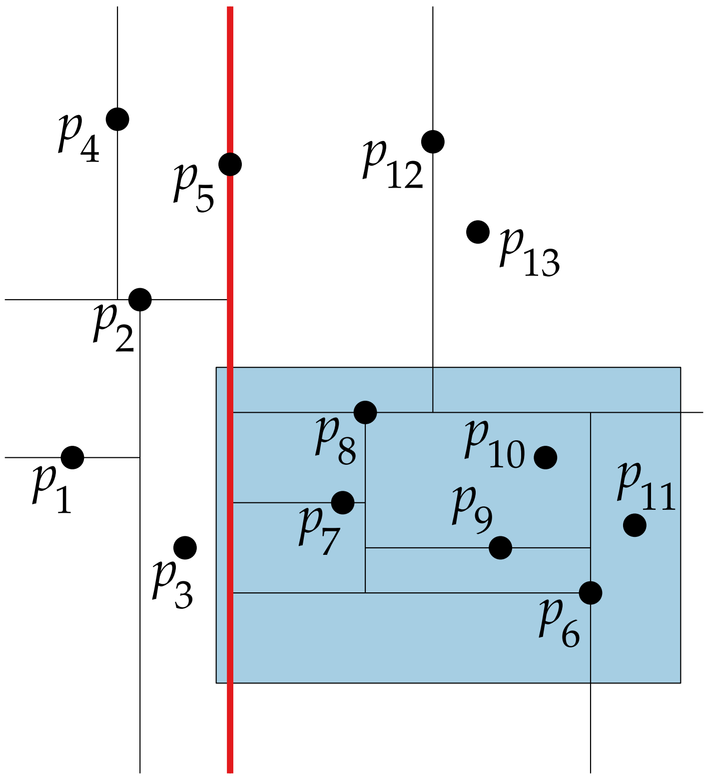

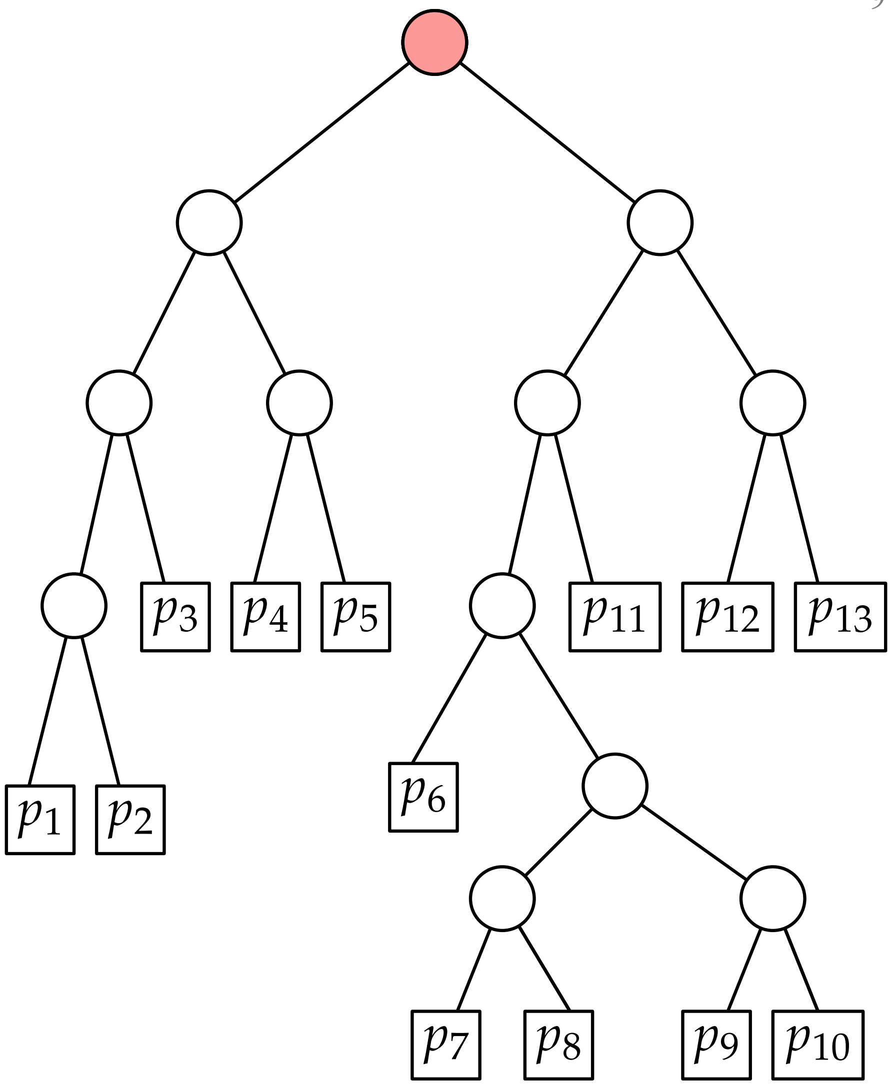

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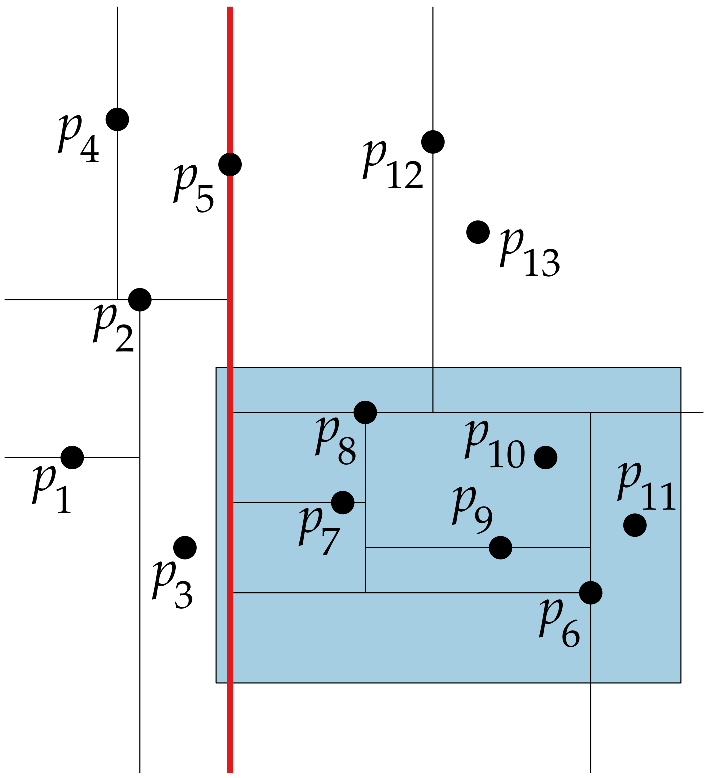

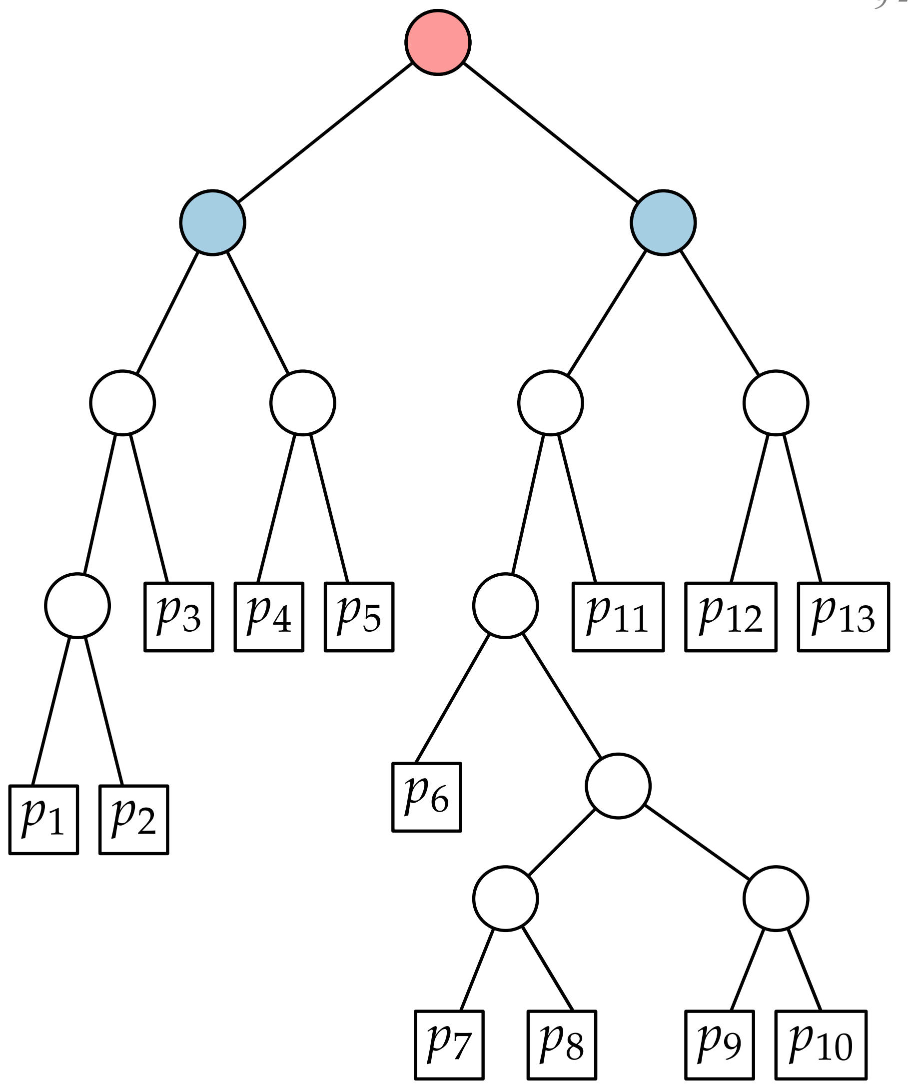

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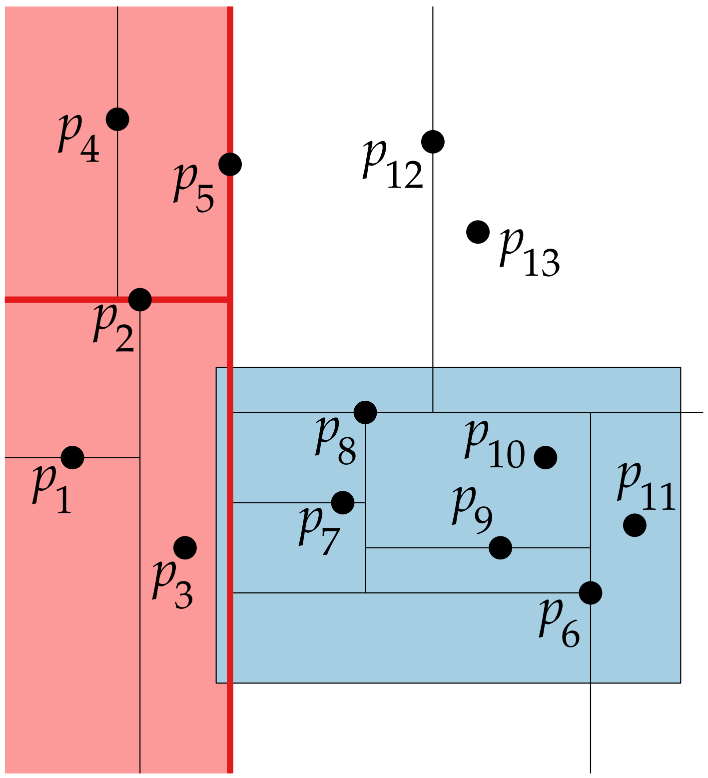
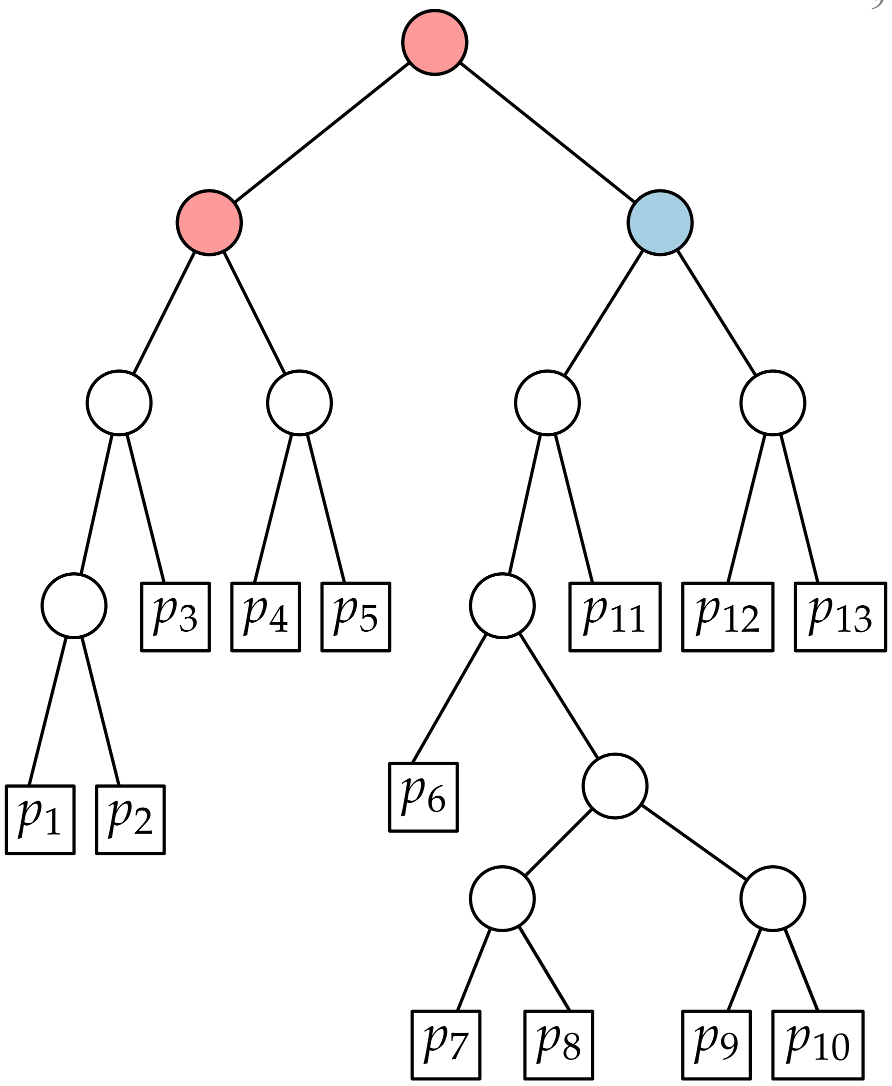
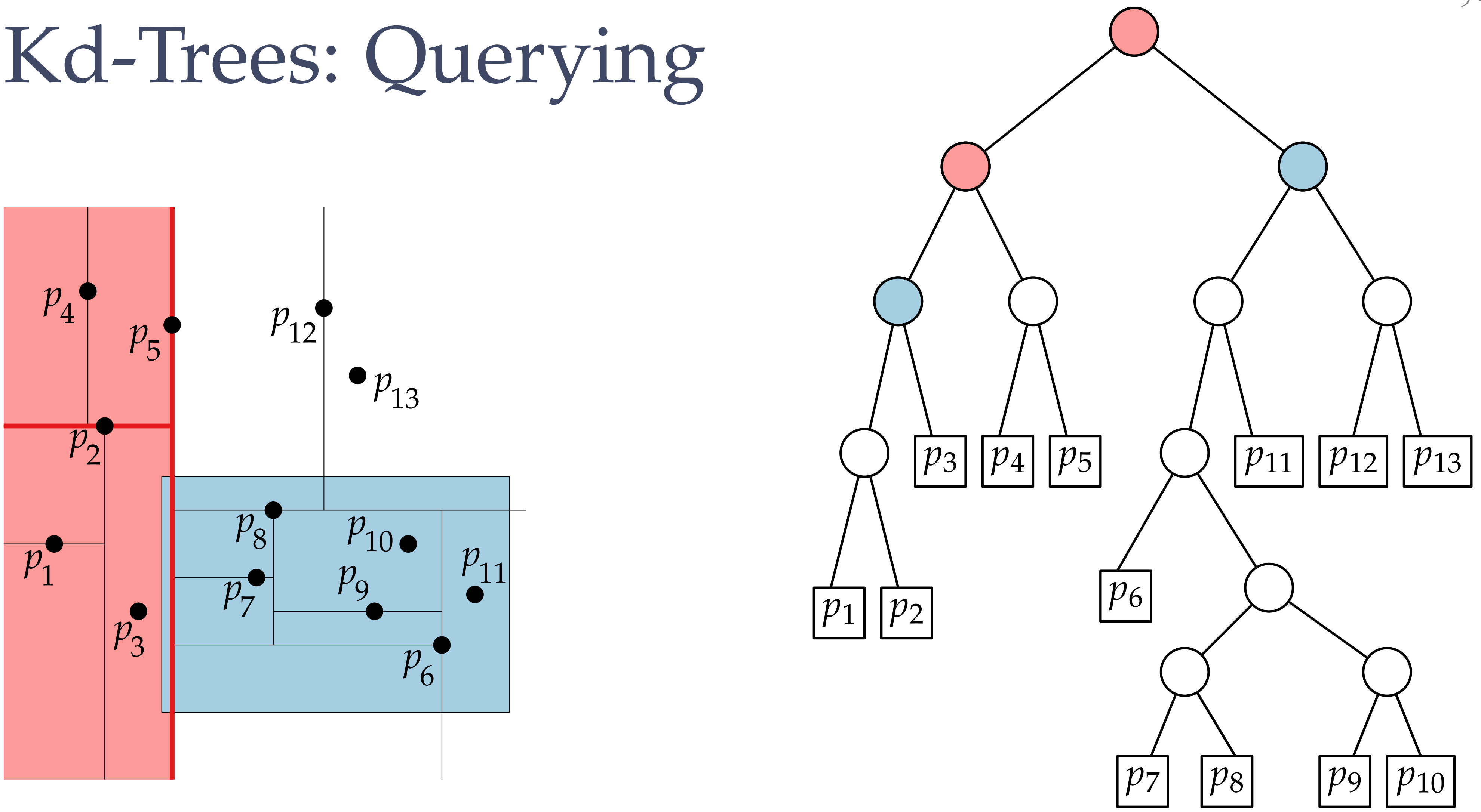




Kd-Trees: Querying



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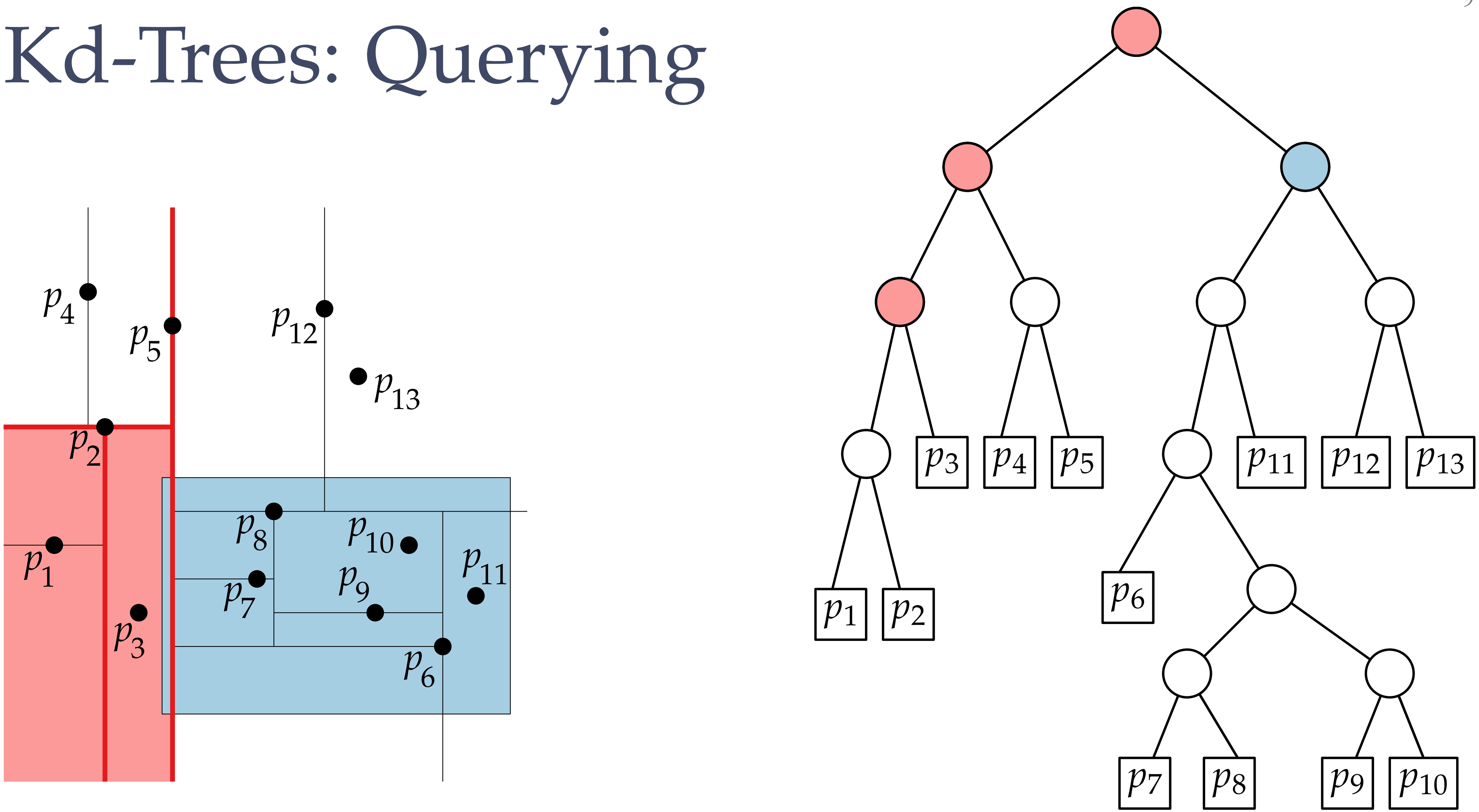

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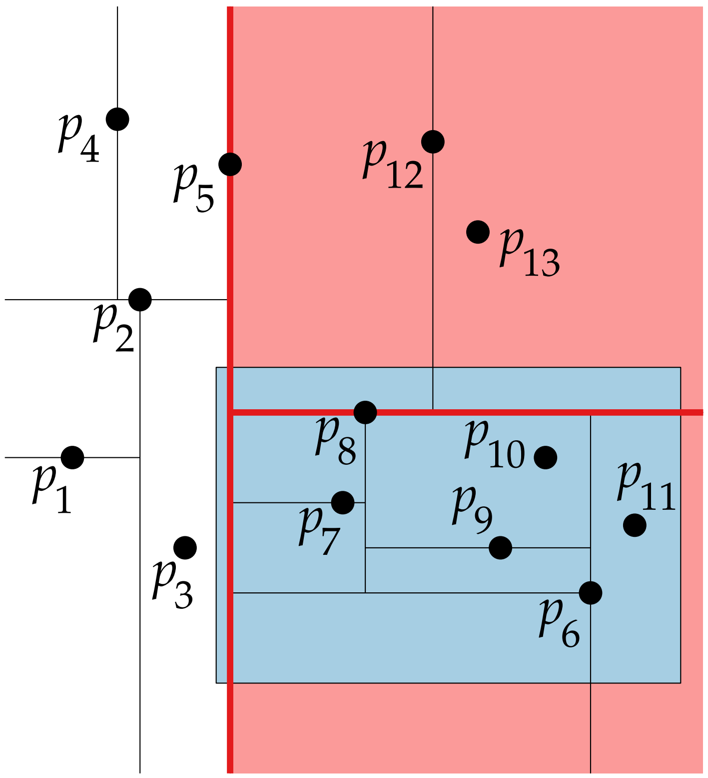

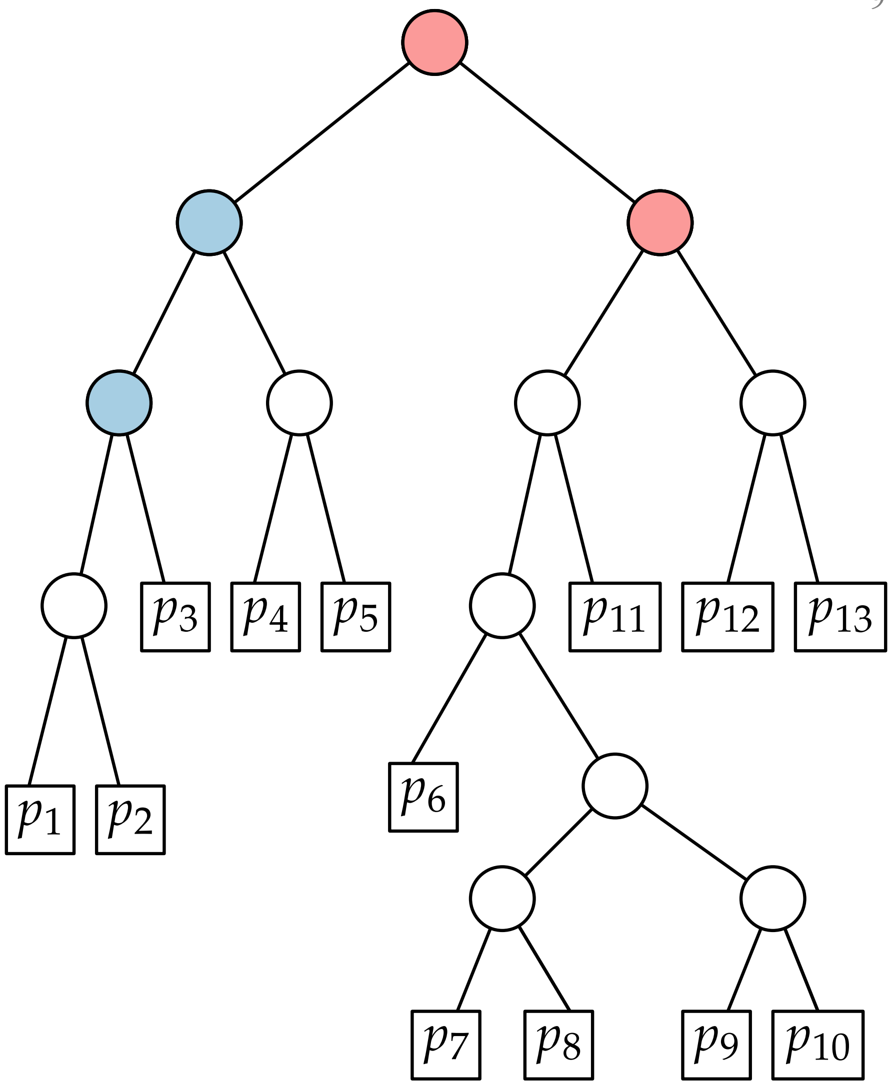

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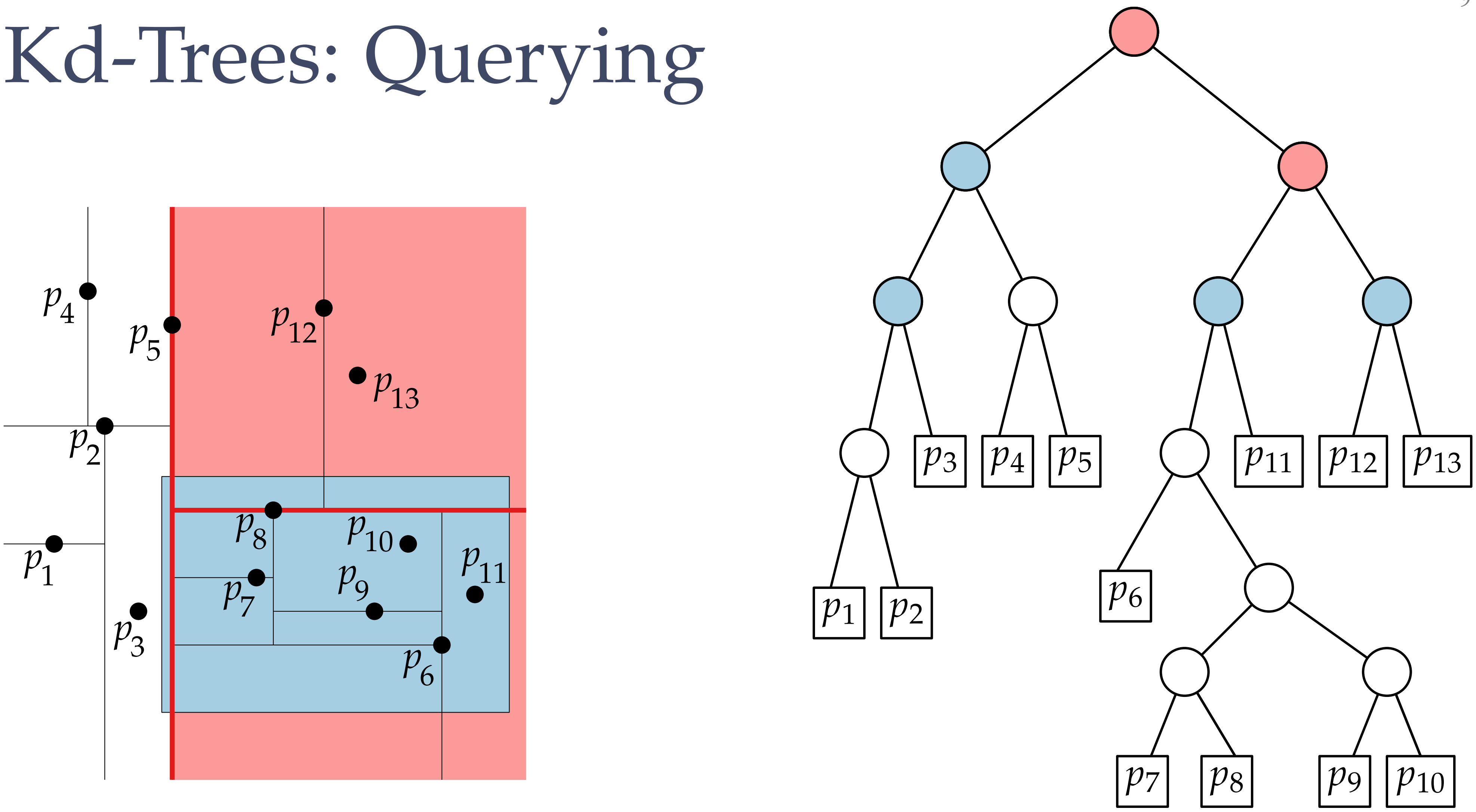

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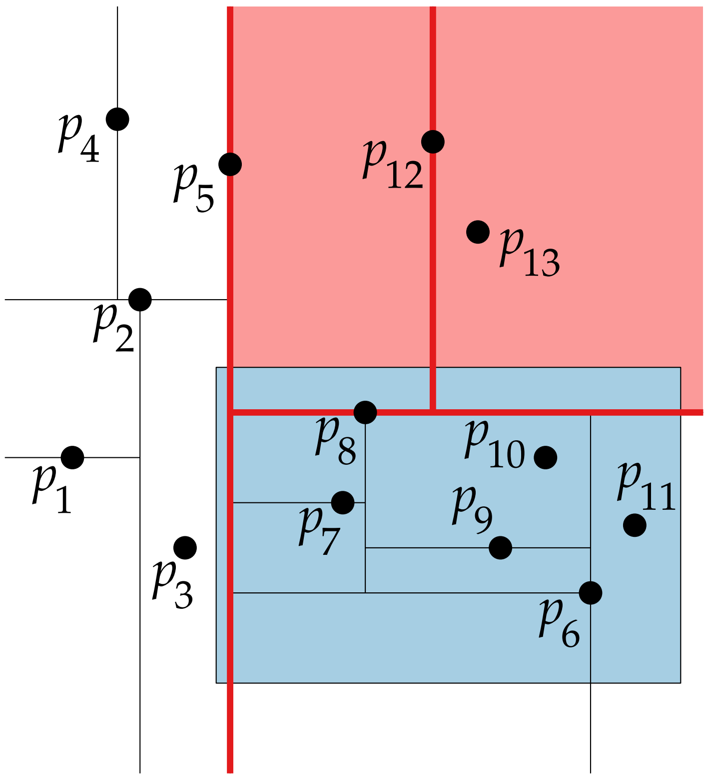

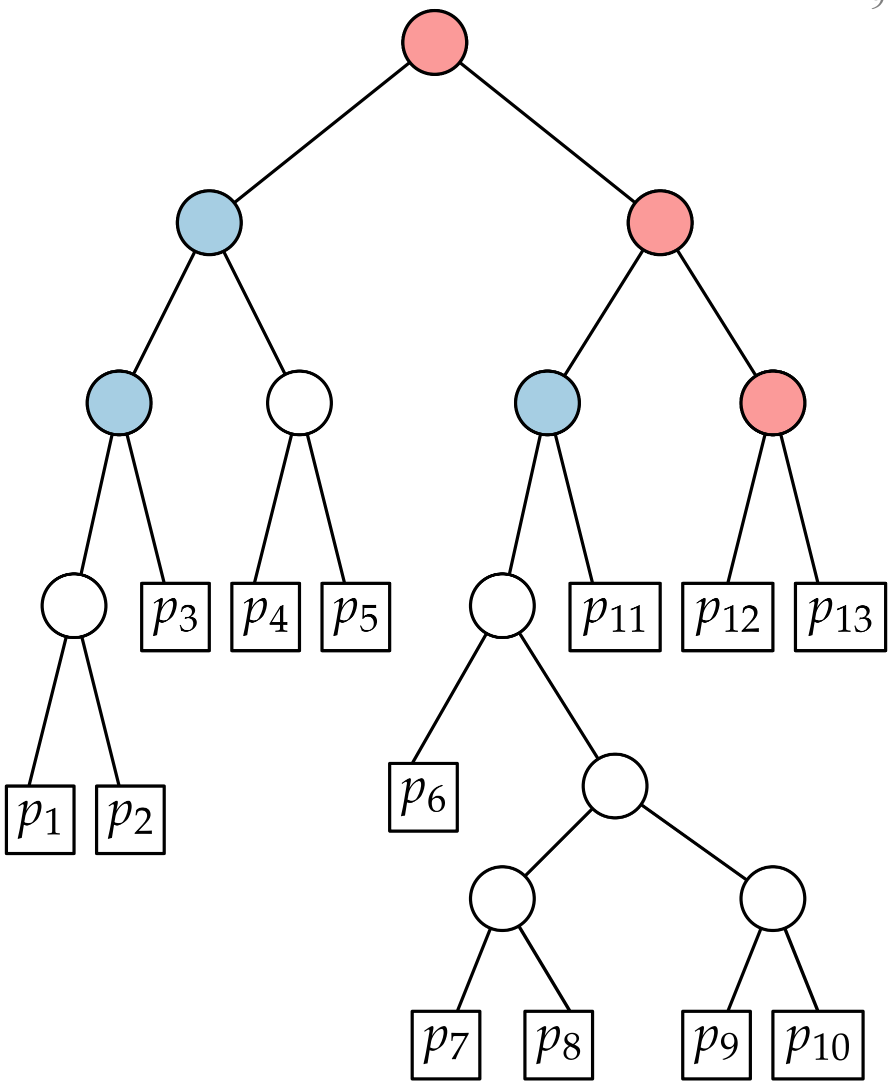

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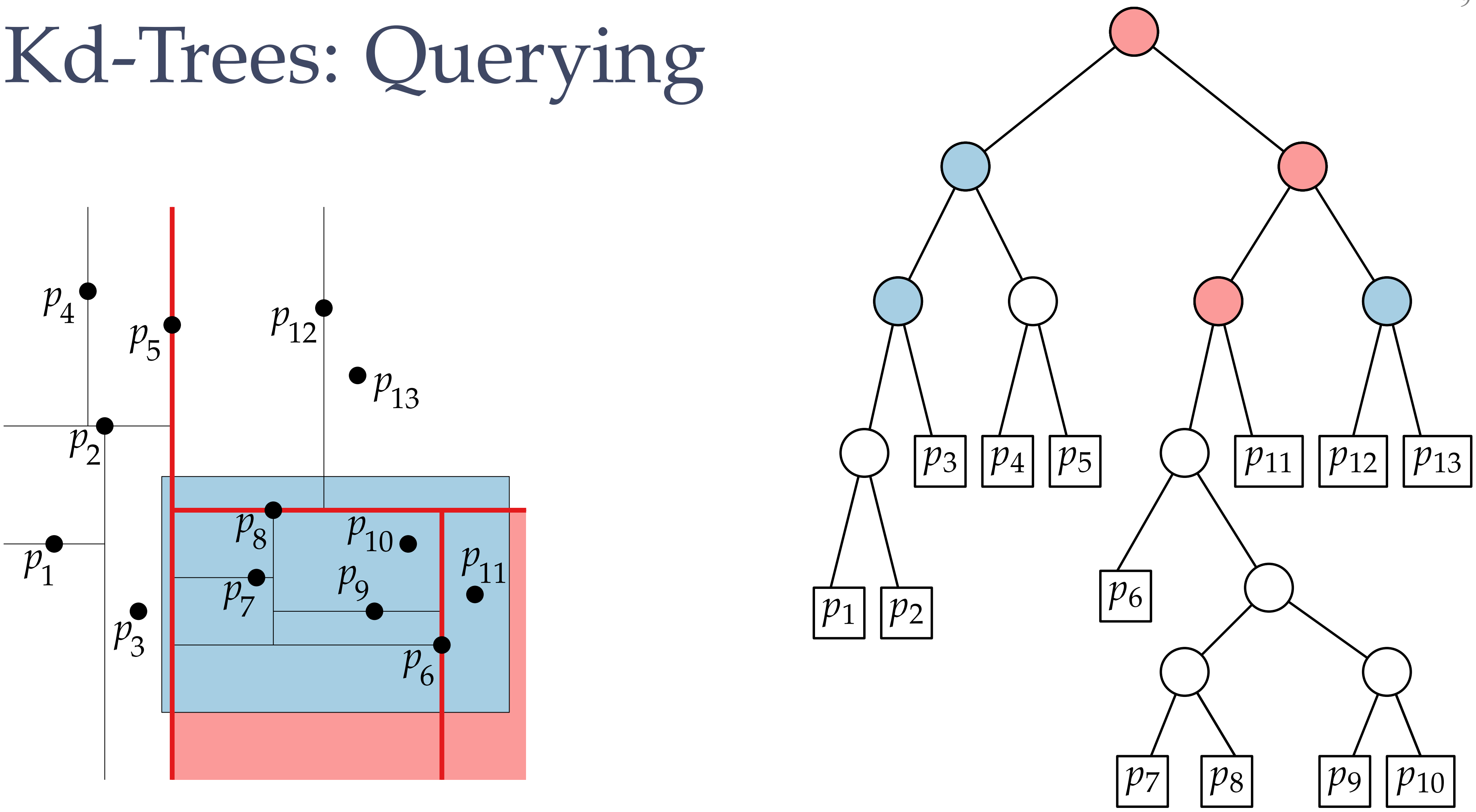

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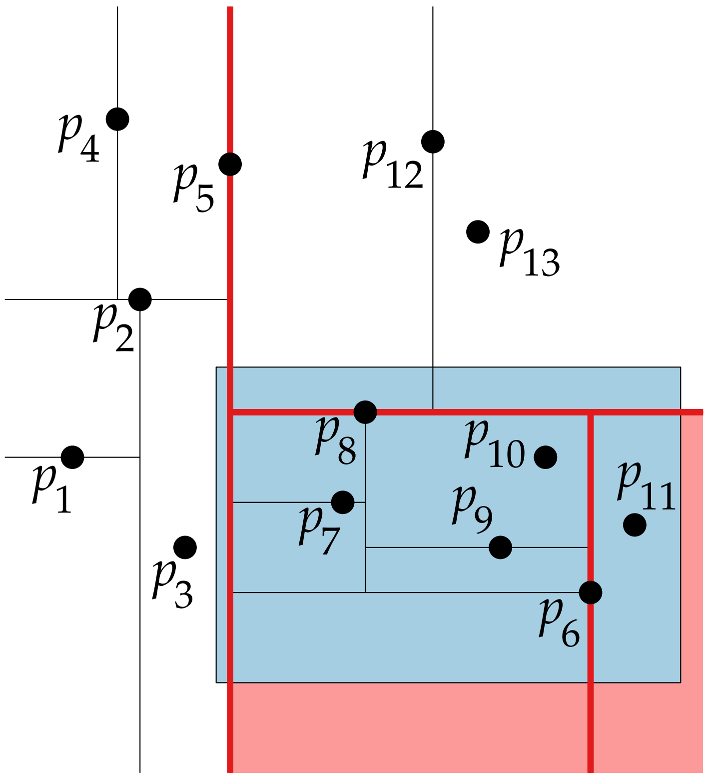

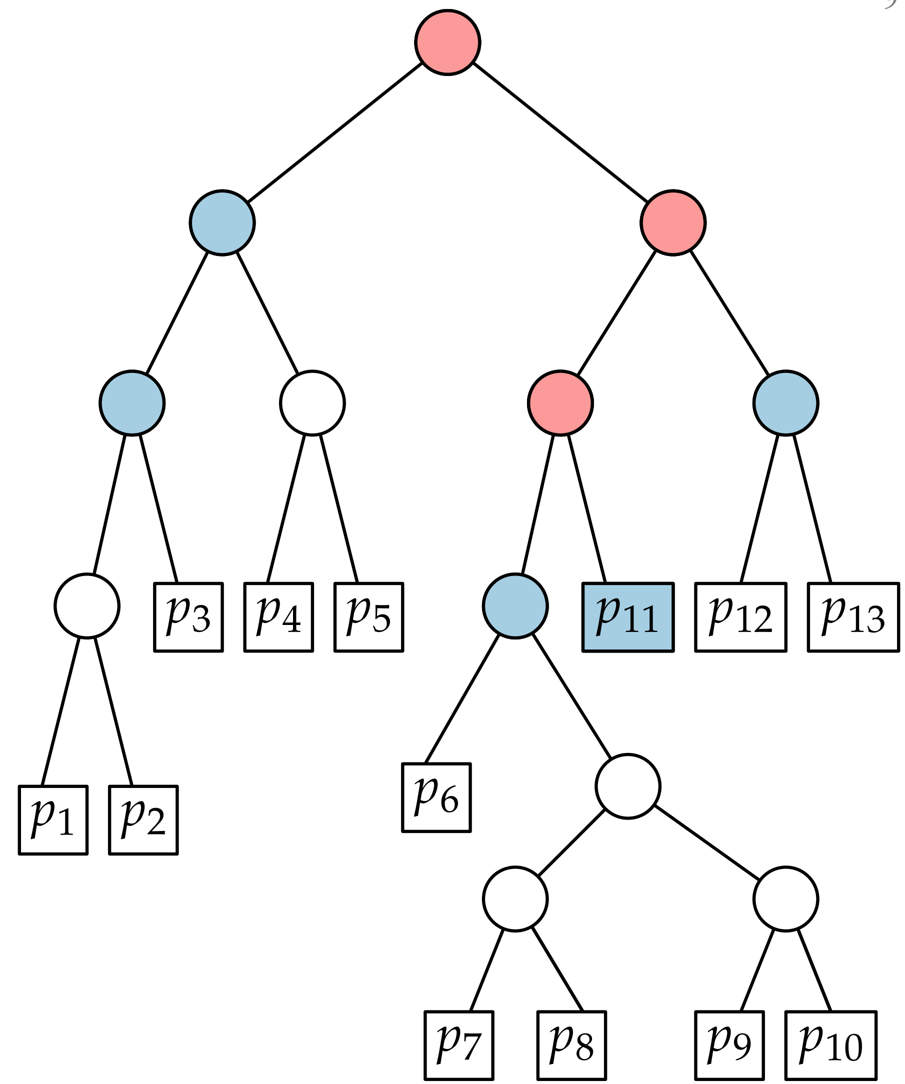

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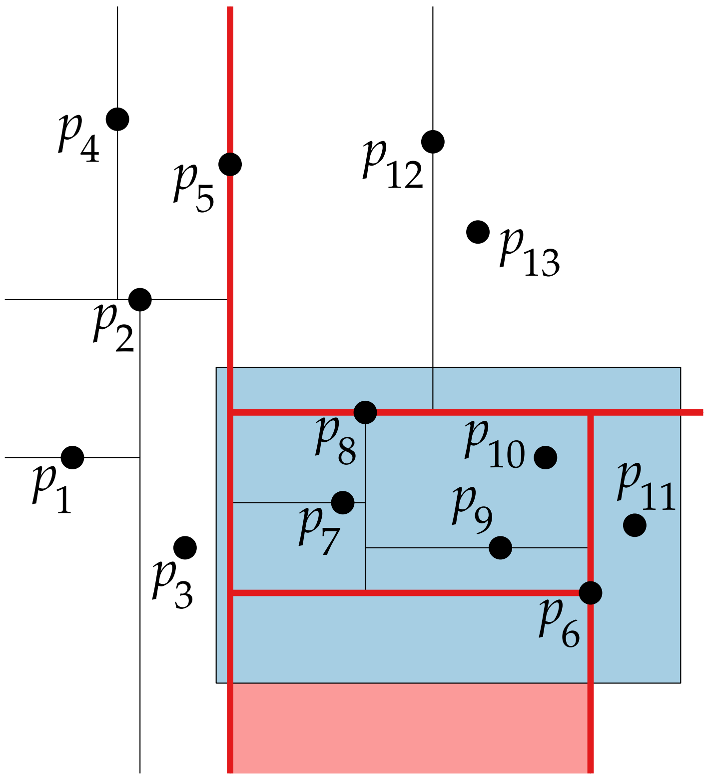

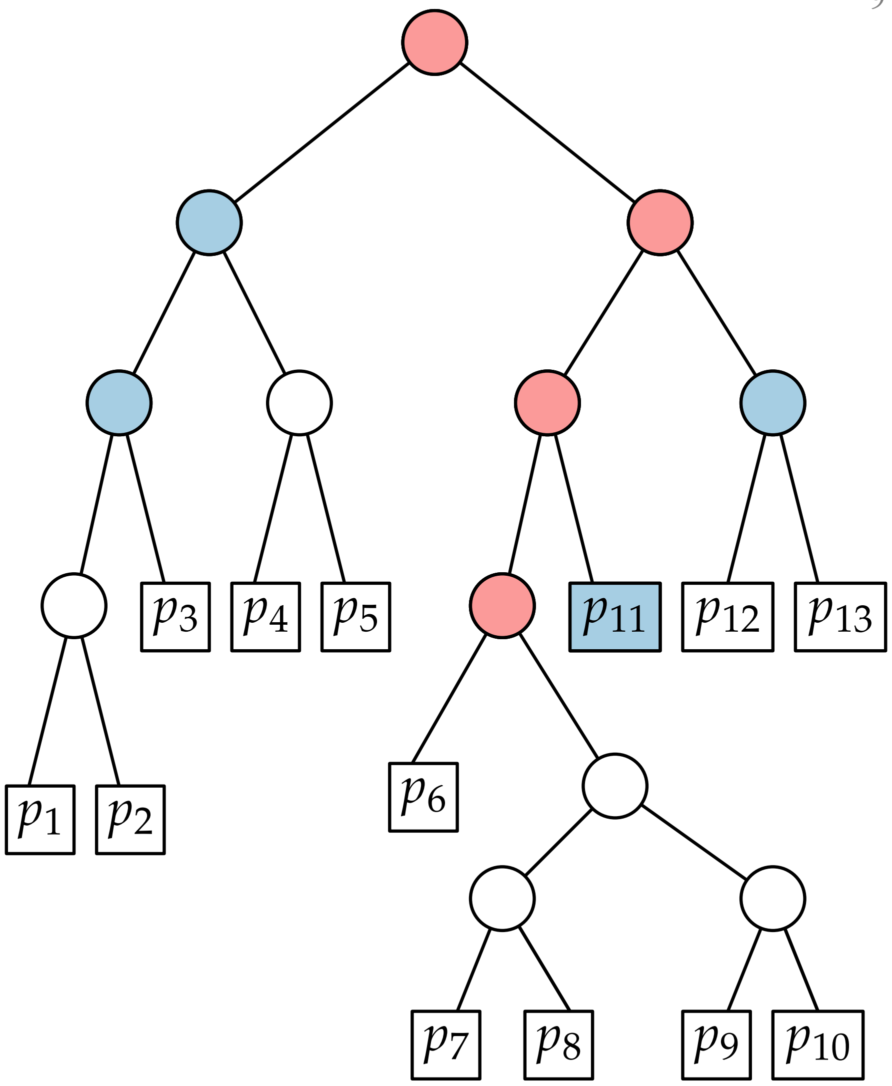

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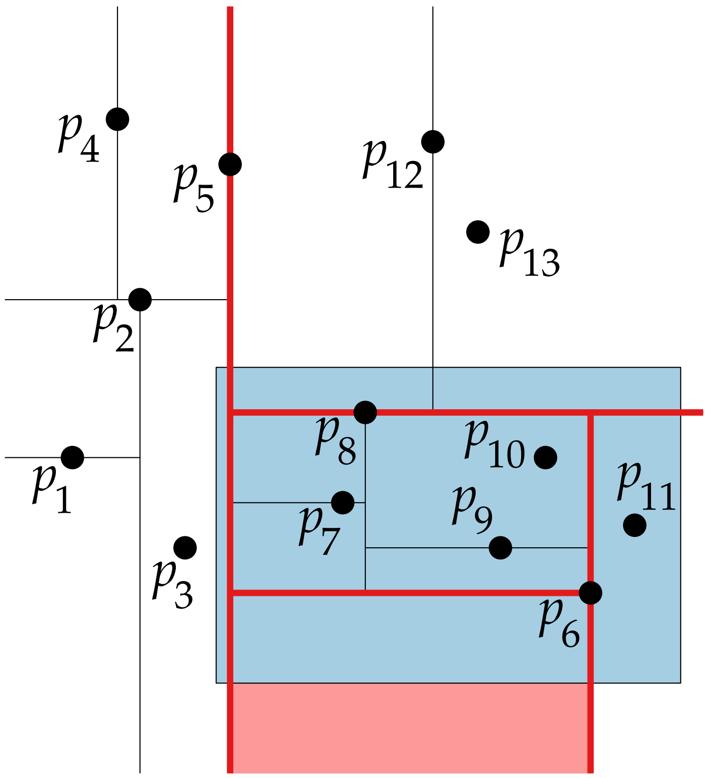

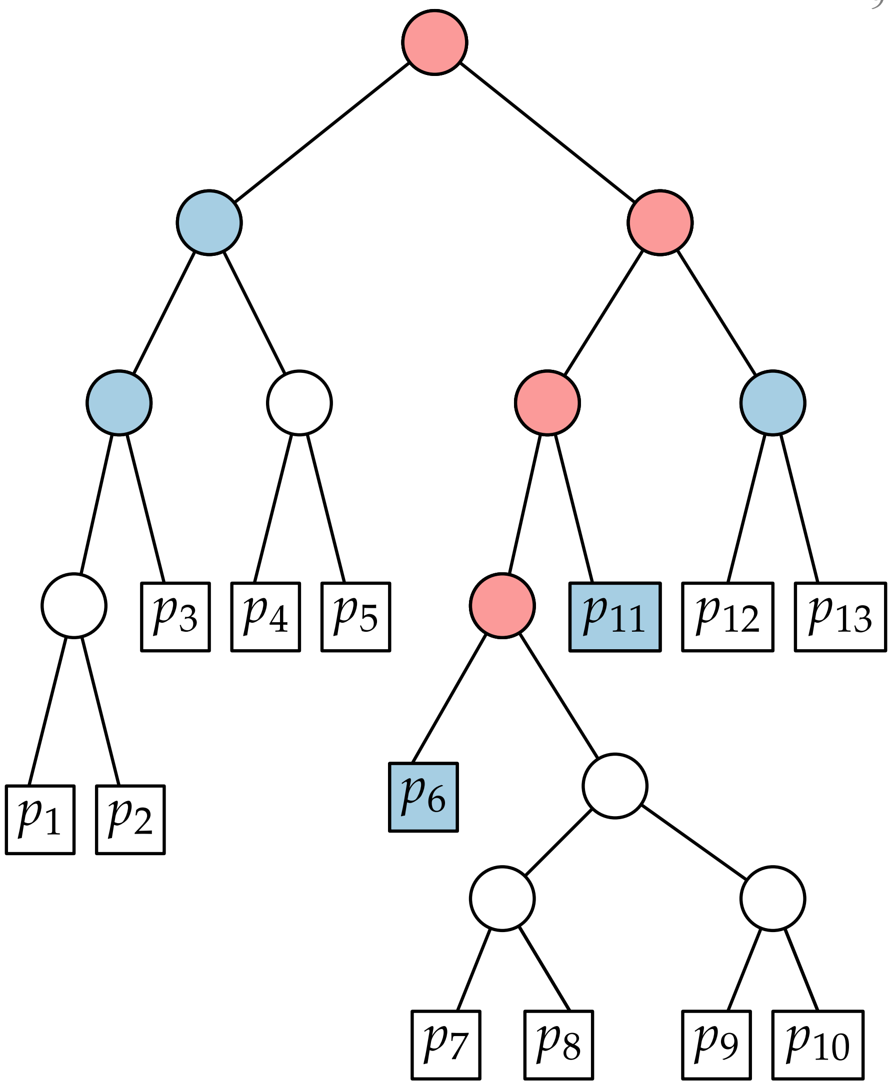

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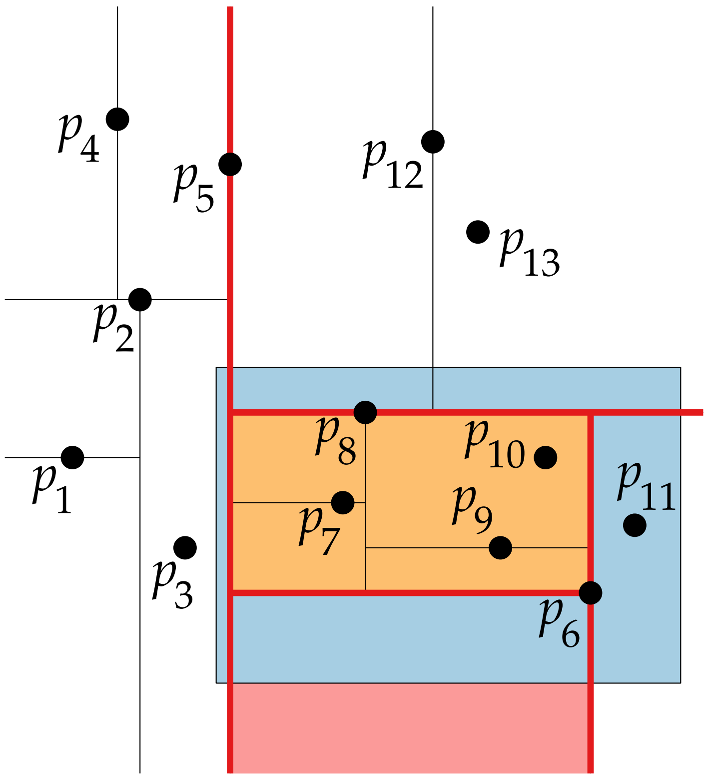

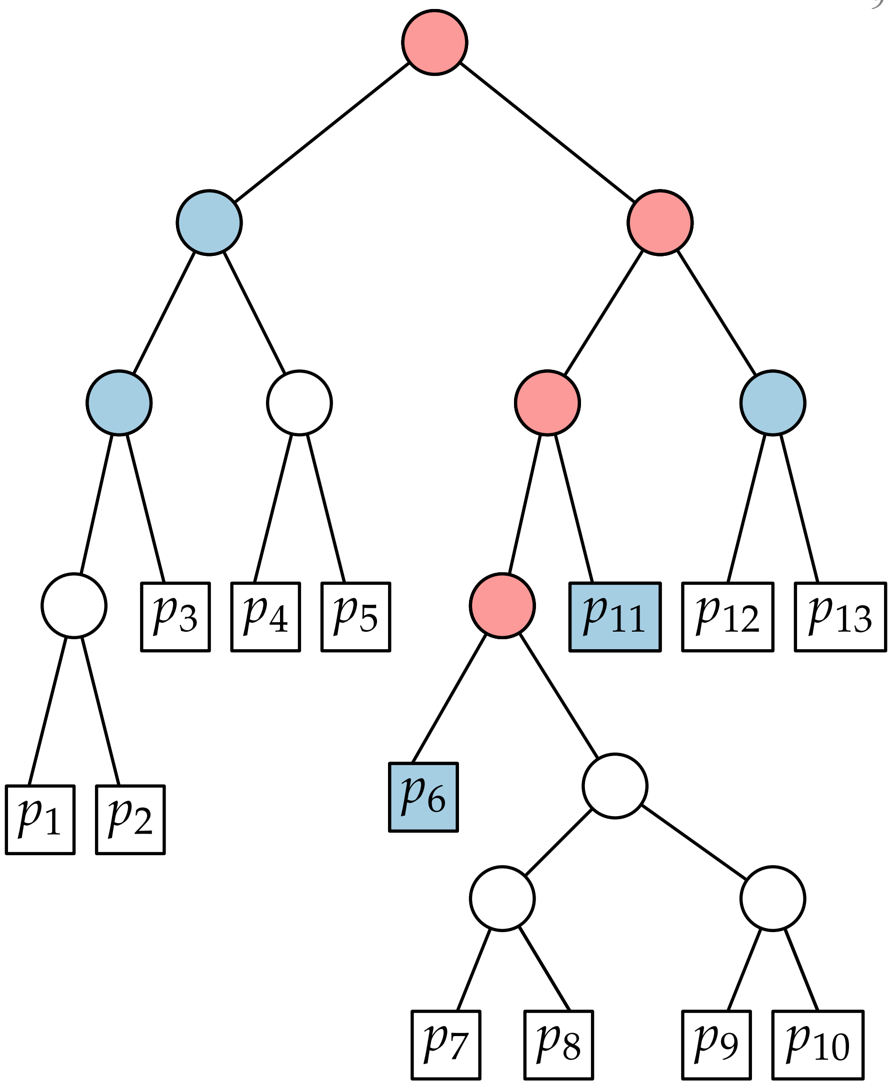

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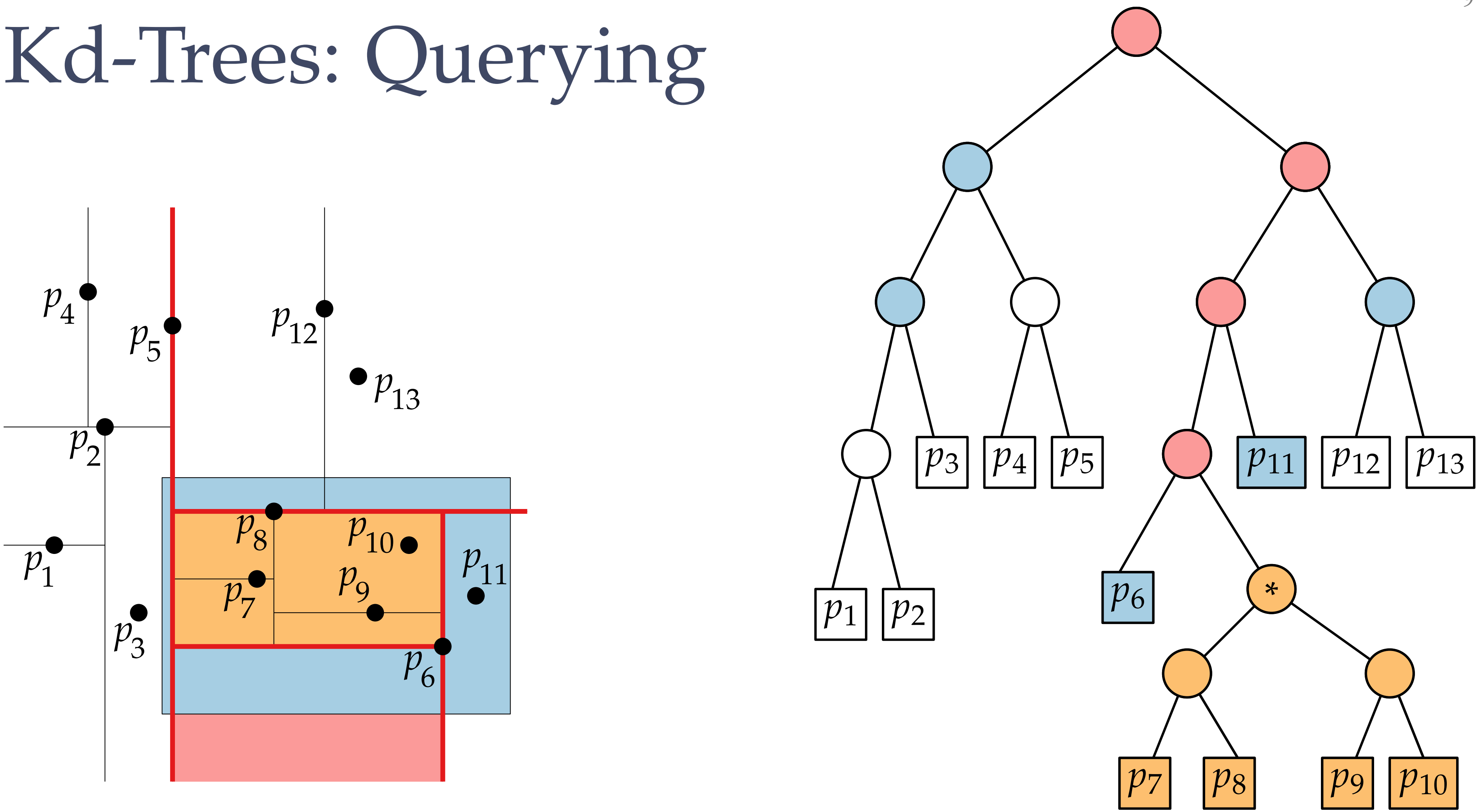

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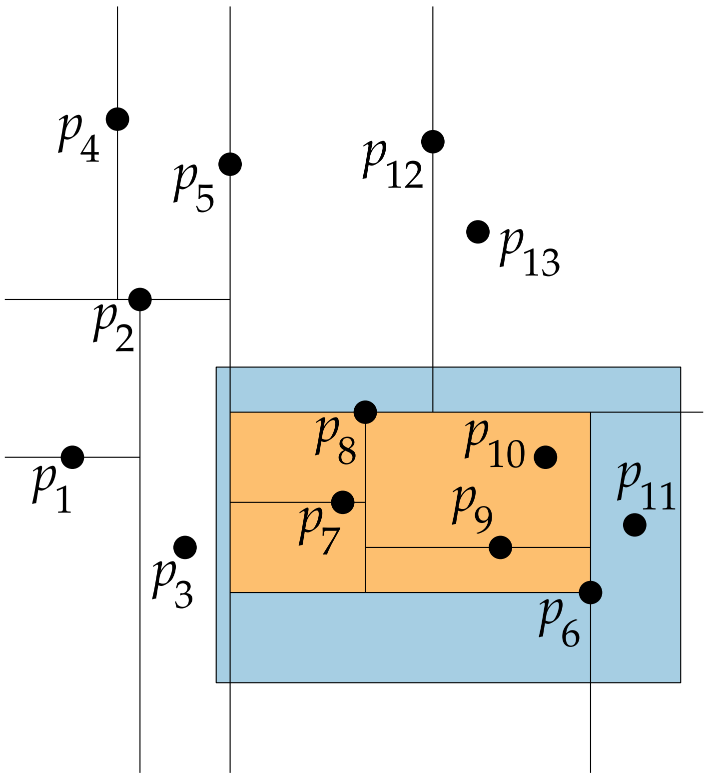
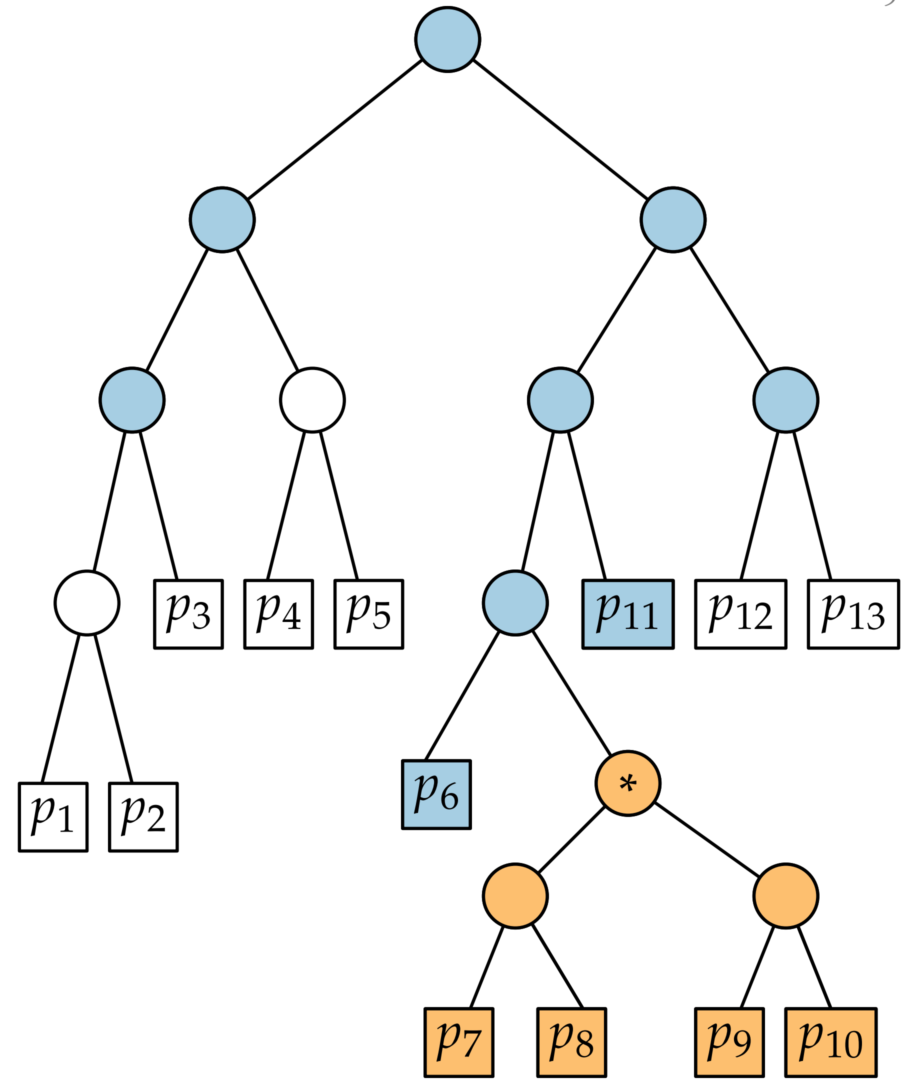
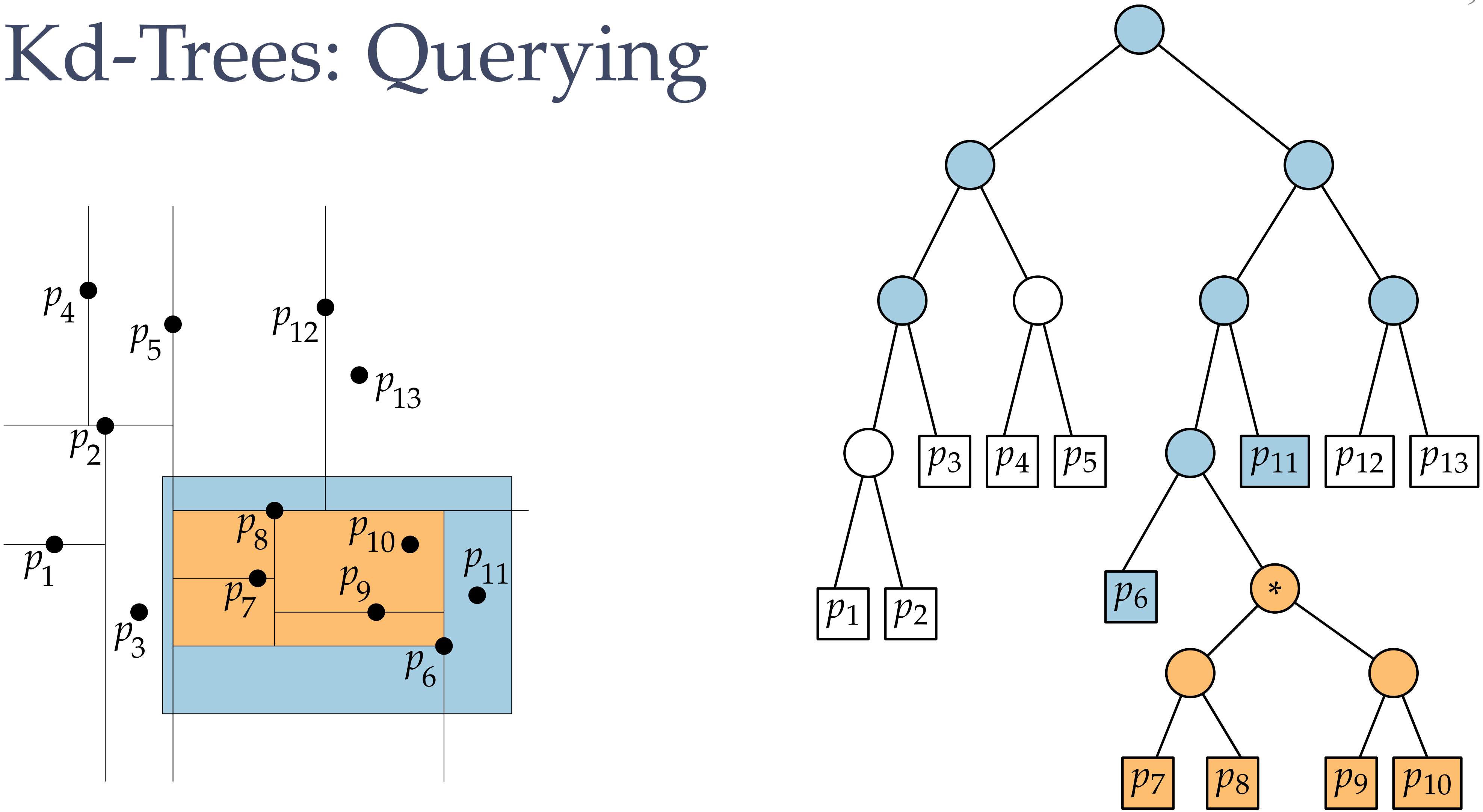




Lemma. Querying a kd-tree for n pts in the plane with an axis-parallel rectangle R takes $O(k + \sqrt{n}\,)$ time, where $k = |\text{output}|$.

Kd-Trees: Querying



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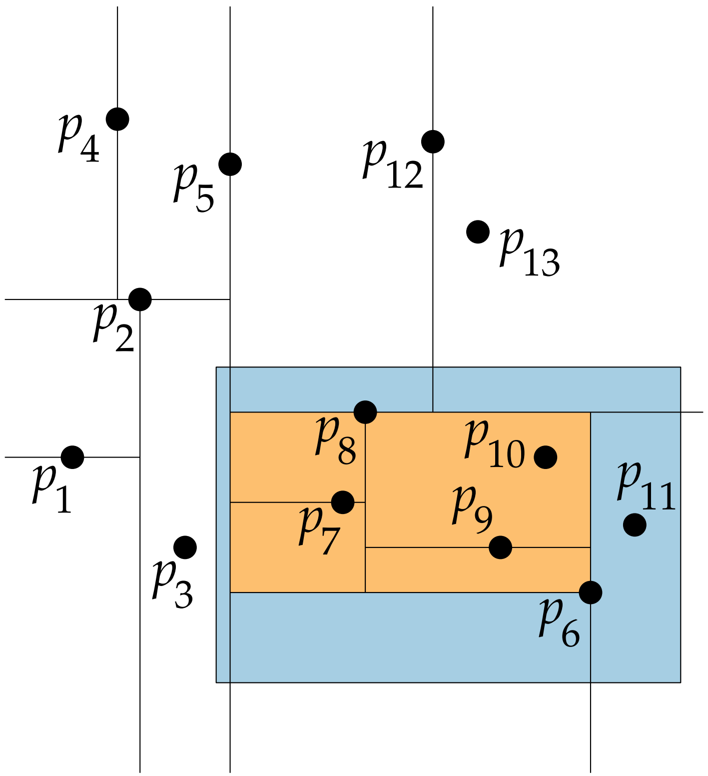
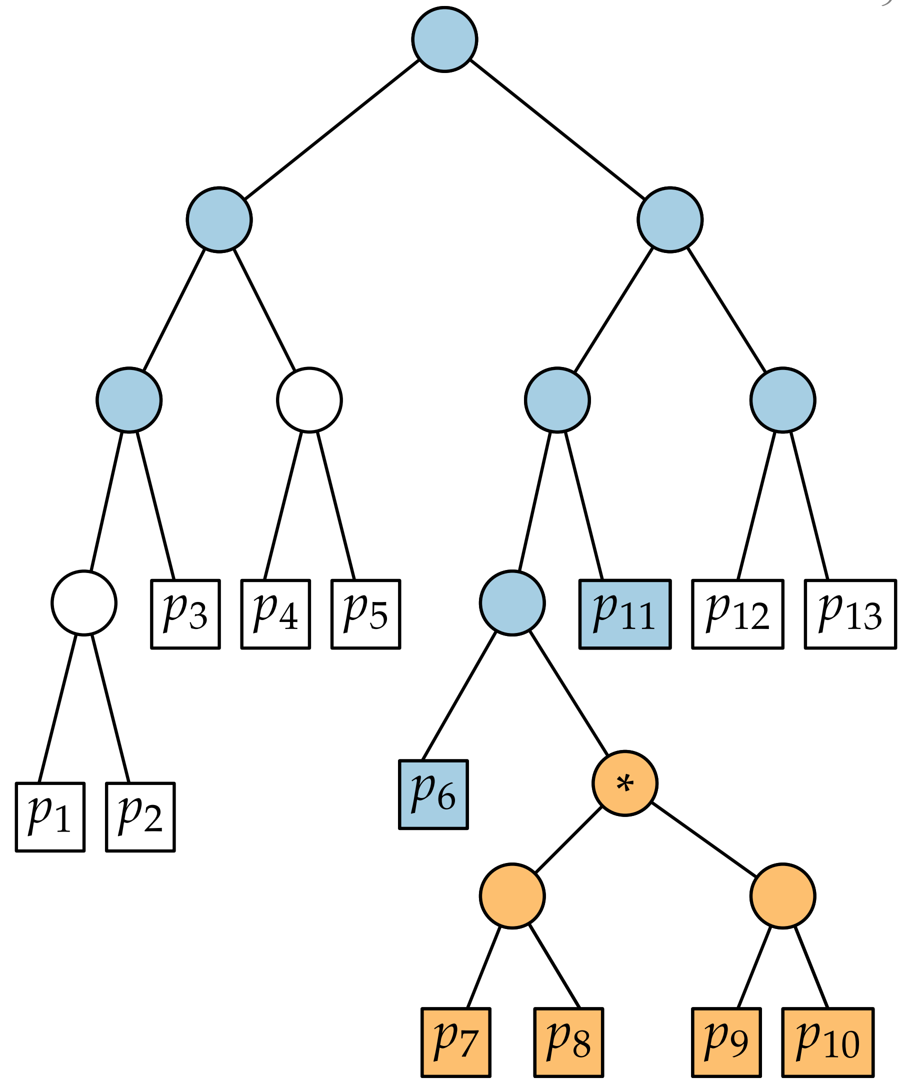
Idea: $O(\sqrt{n})$ regions of the kd-tree intersect a vertical/horizontal line.

Kd-Trees: Querying





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↓ in $\mathbb{R}^d$

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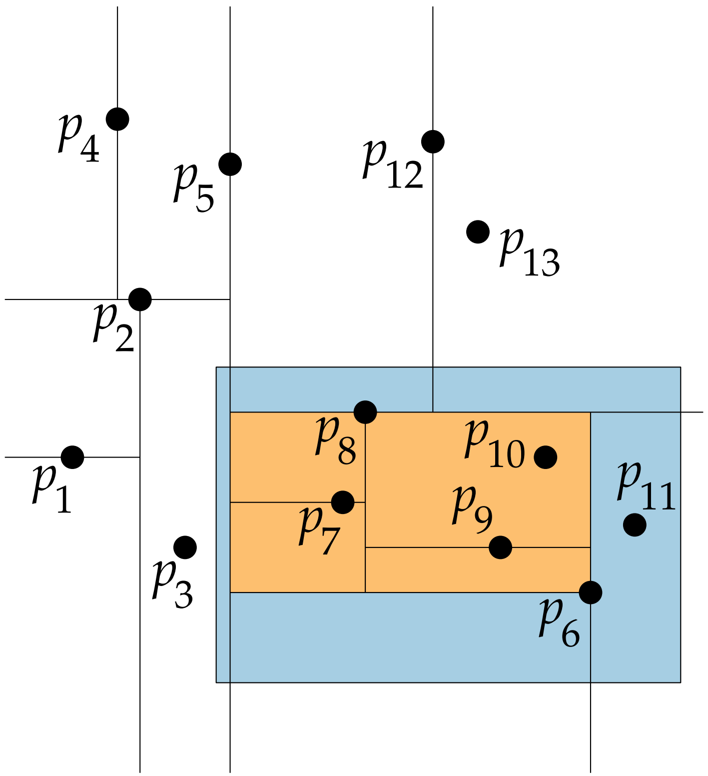

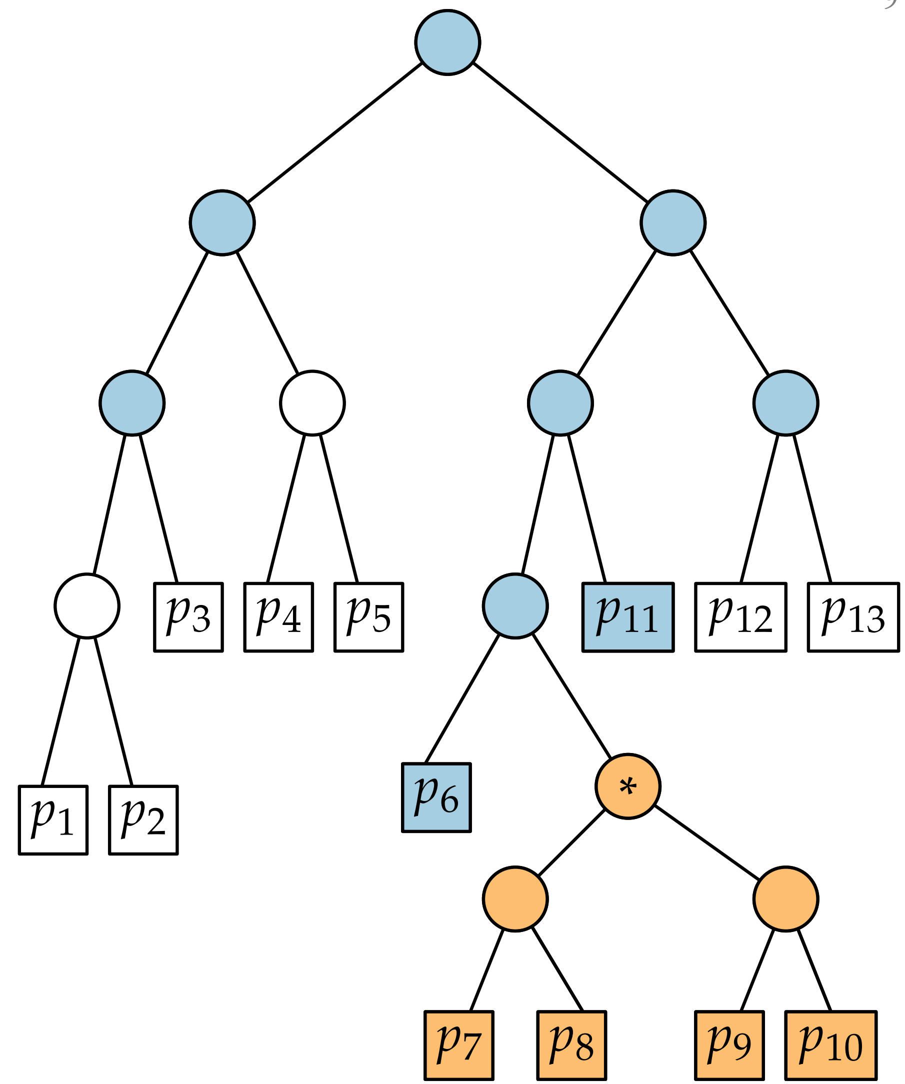

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in $\mathbb{R}^d$ → $O(k + n^{1-1/d})$

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Extensions to 2D

Task: Preprocess a finite set $P \subset \mathbb{R}^2$ such that for any range query $R = [x, x'] \times [y, y']$ the set $P \cap R$ can be reported quickly.

Solutions:

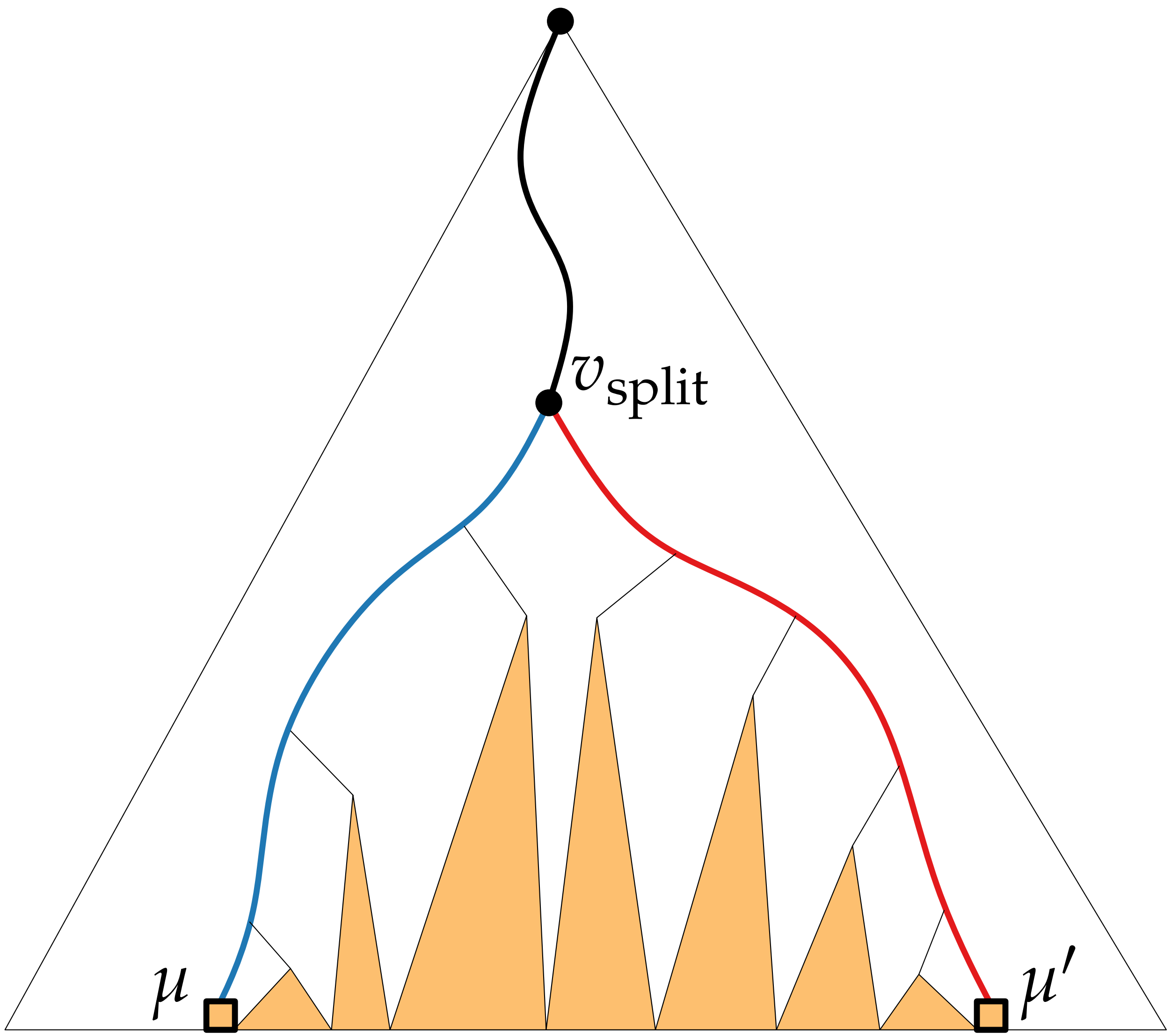
- *one* tree; query path alternates between x- and y-coord. } *kd-tree*
- first-level tree for x-coordinates; many second-level trees for y-coord. } *range tree*

Assume: *General position!*

Here: no two points have the same x- or y-coordinate.

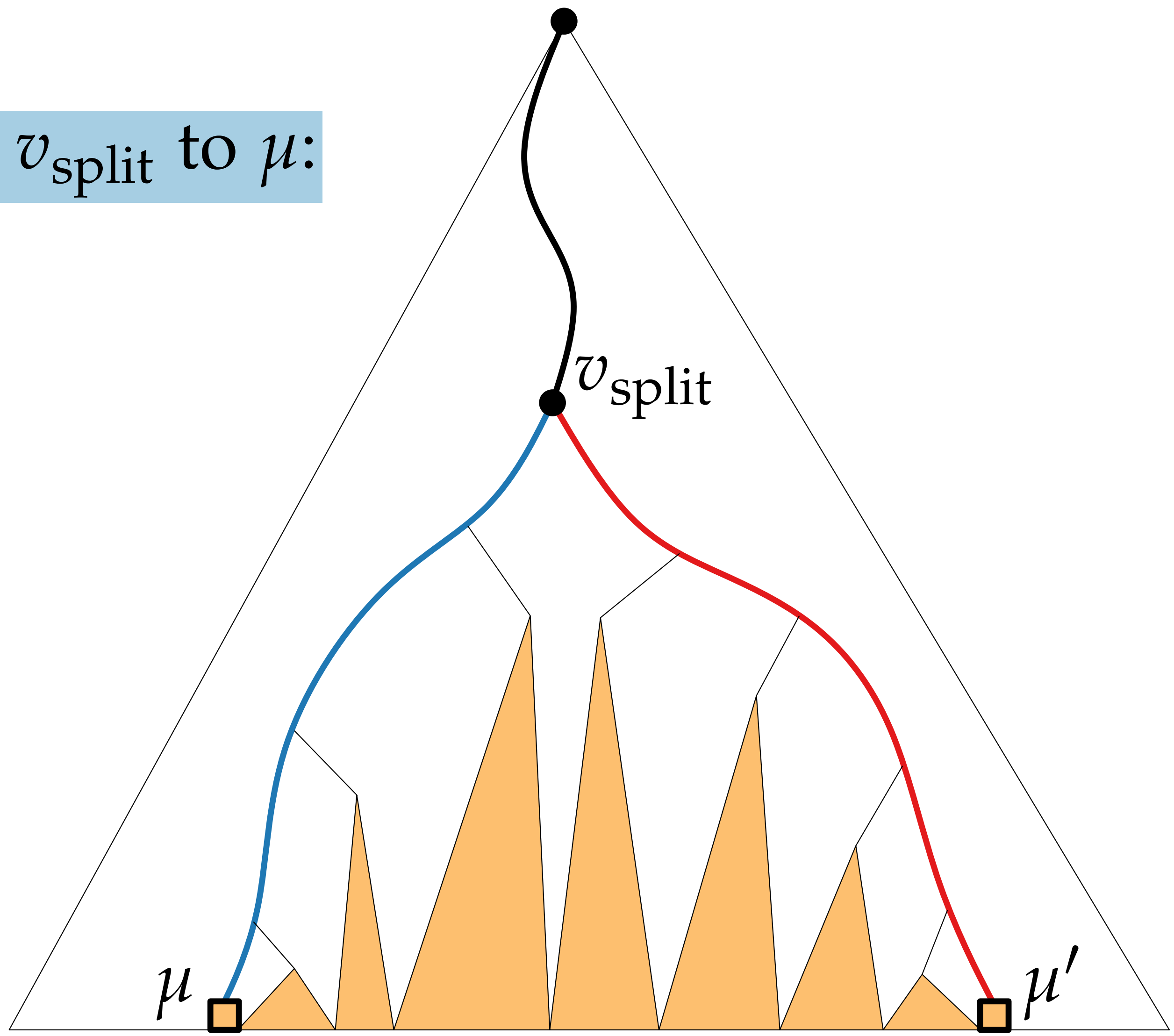
Range Trees: Query Algorithm

1. Search in main tree for x-coordinate



Range Trees: Query Algorithm

1. Search in main tree for x-coordinate
2. For each node u
   on the path from v_{split} to μ:



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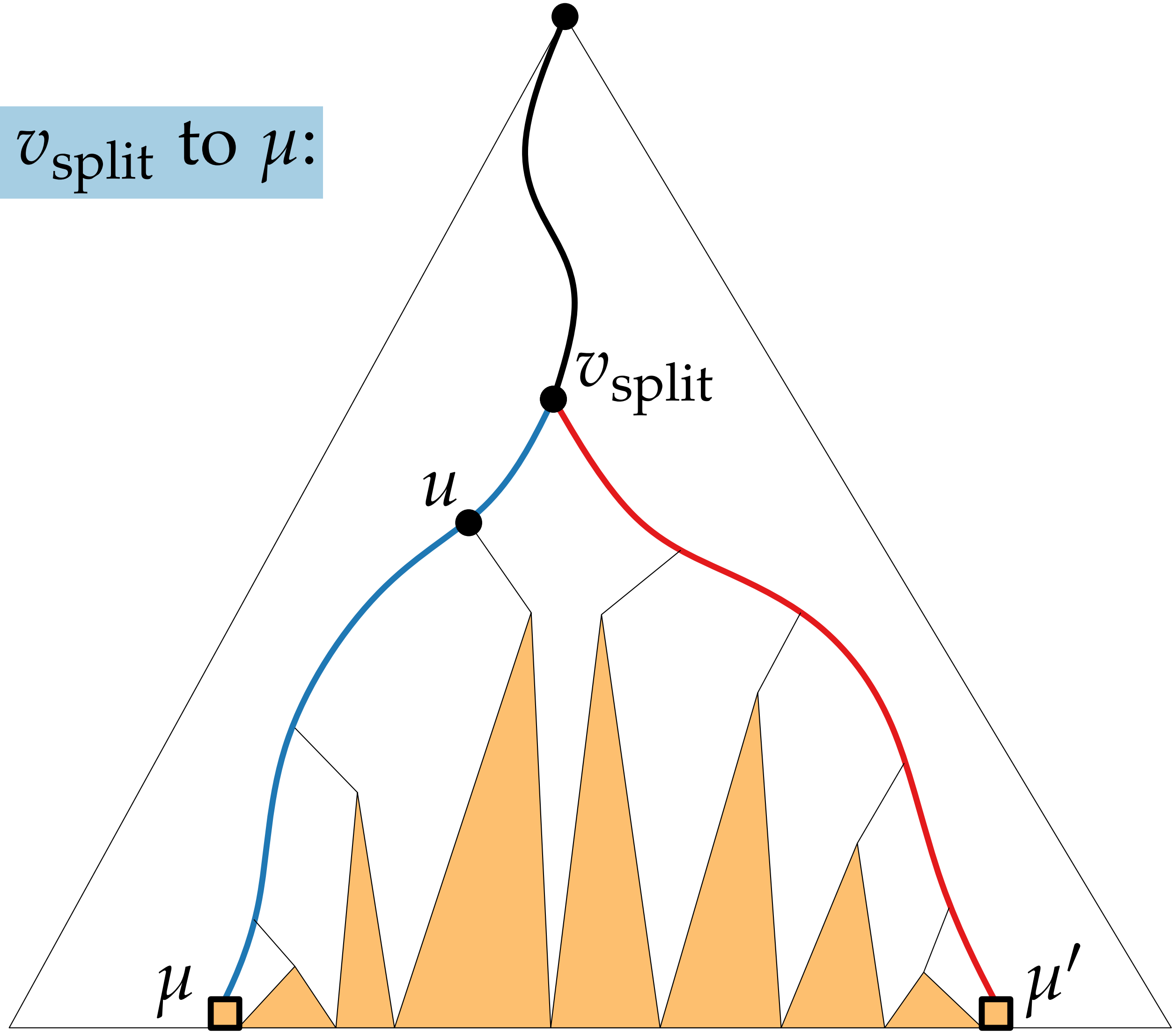

Range Trees: Query Algorithm

1. Search in main tree for x-coordinate
2. For each node u
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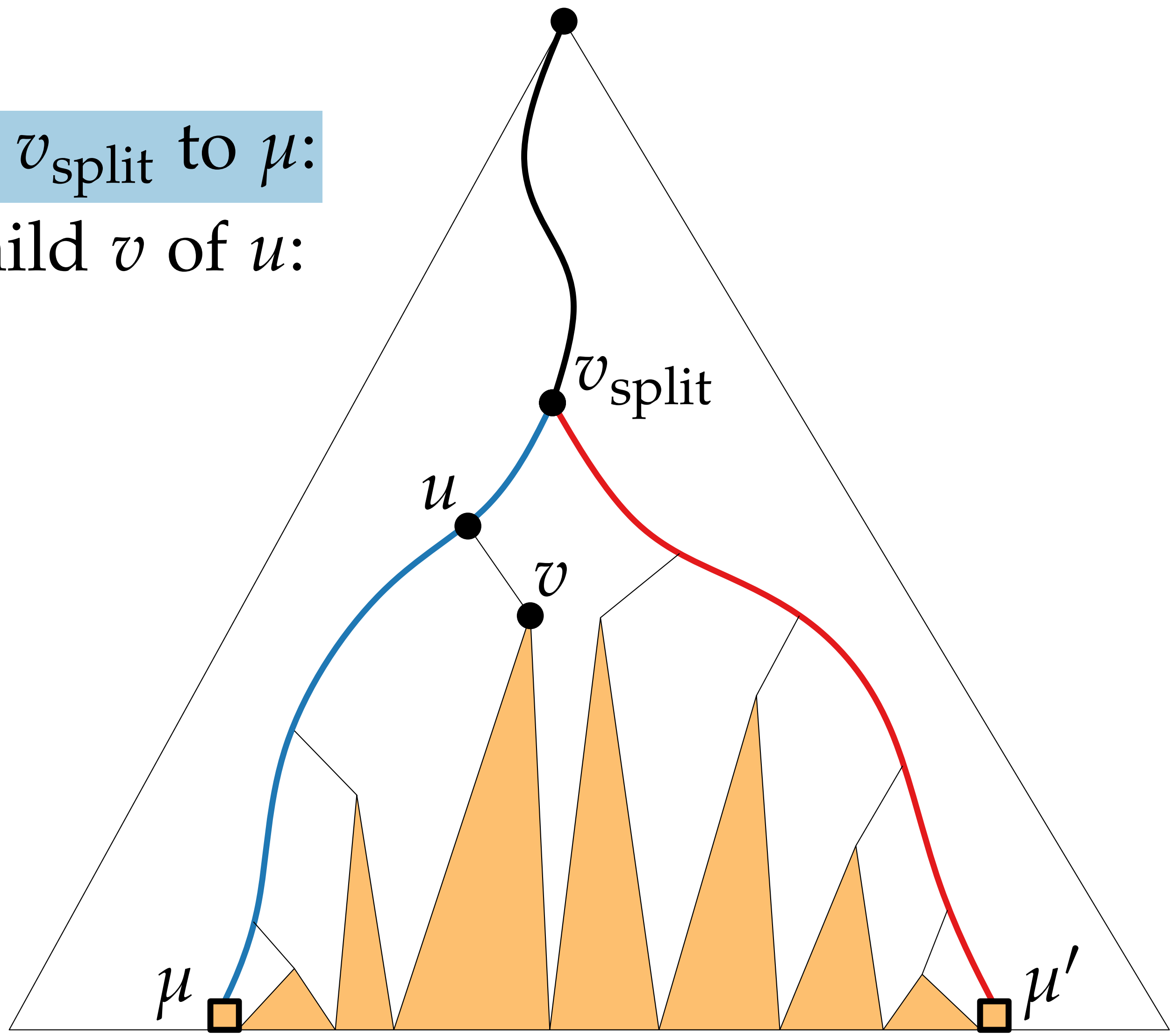
      For the right child v of u:



Range Trees: Query Algorithm

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2. For each node u
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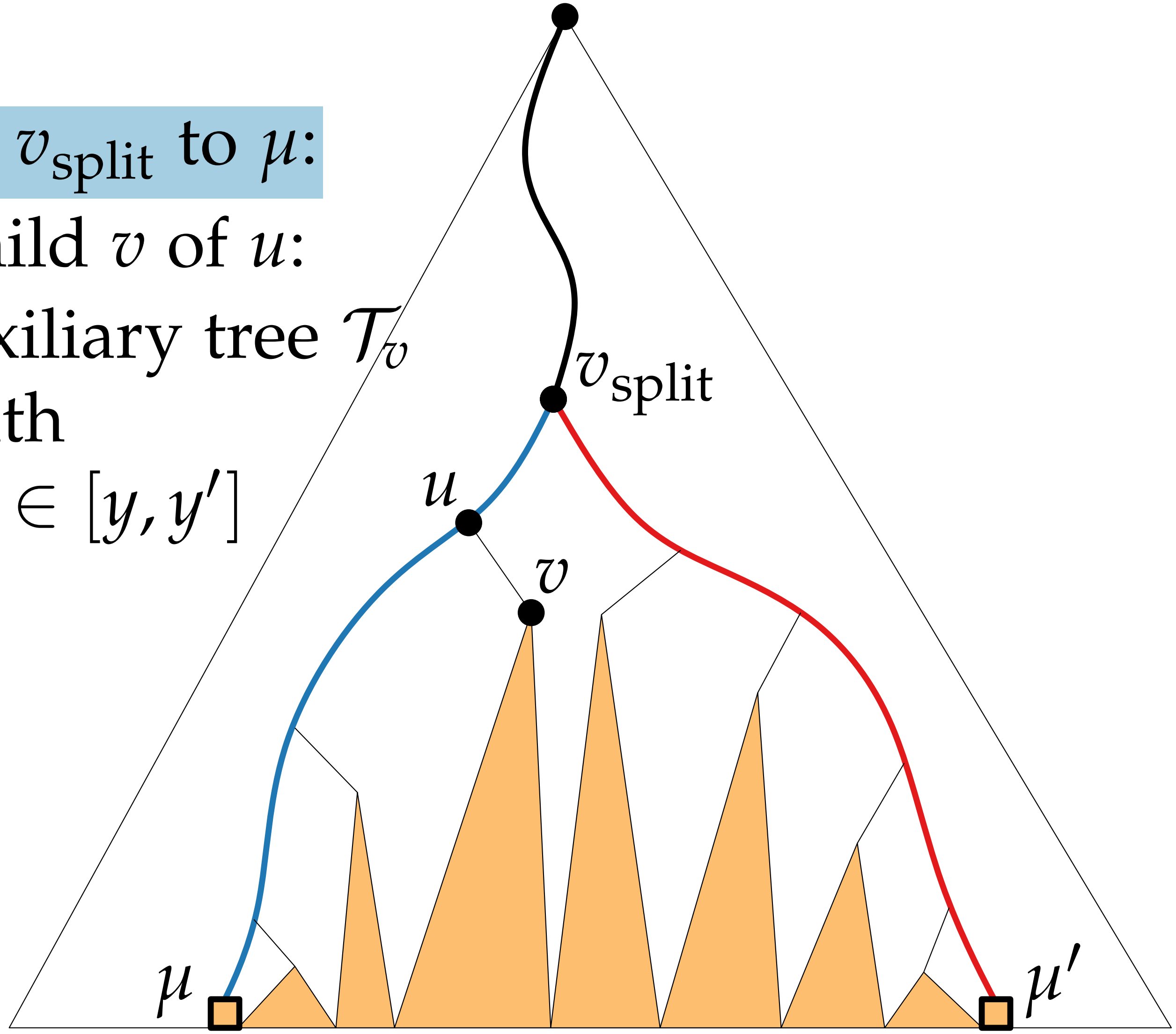
         Search in auxiliary tree $\mathcal{T}_v$
         for points with
         y-coordinate $\in [y, y']$



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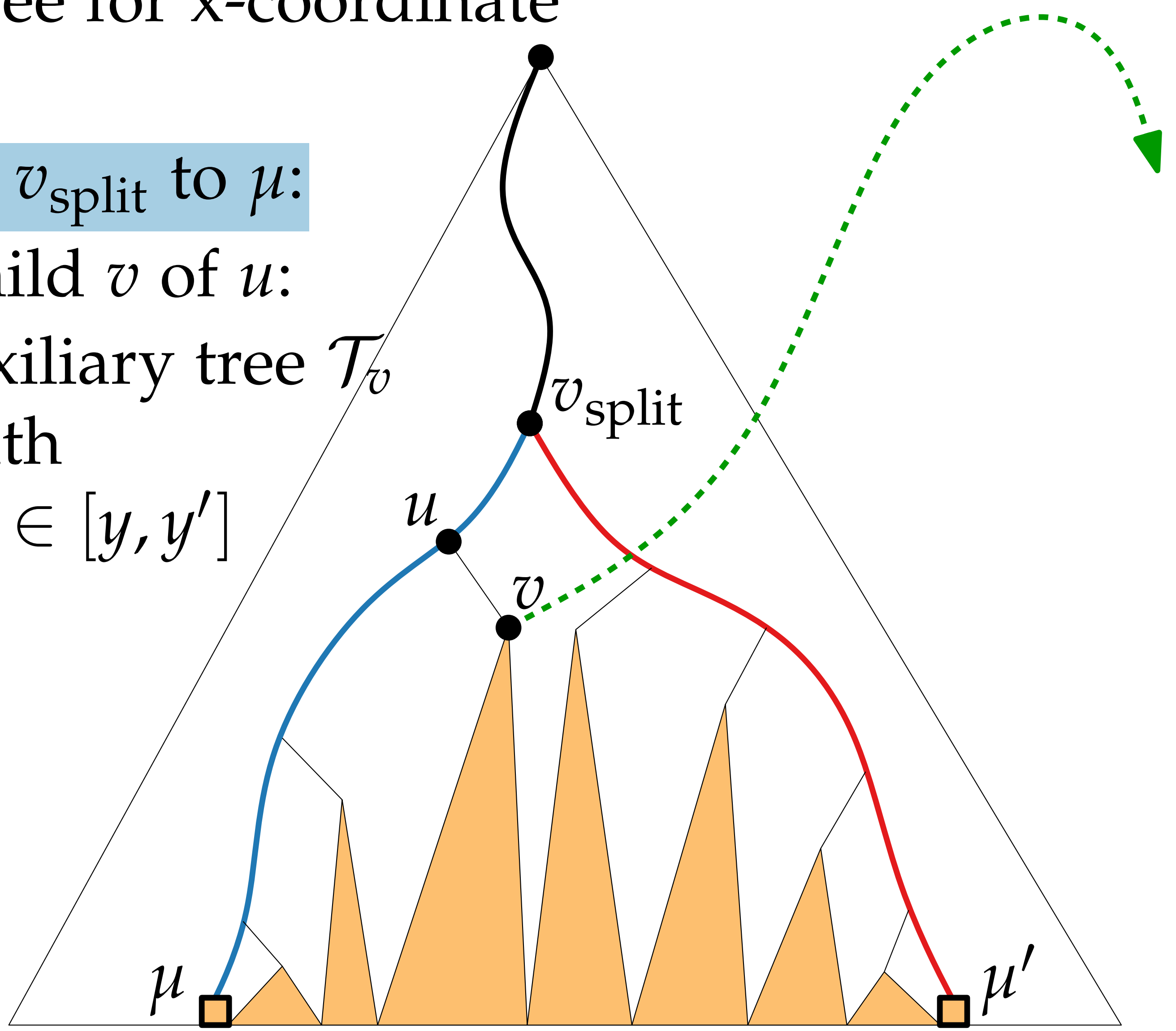

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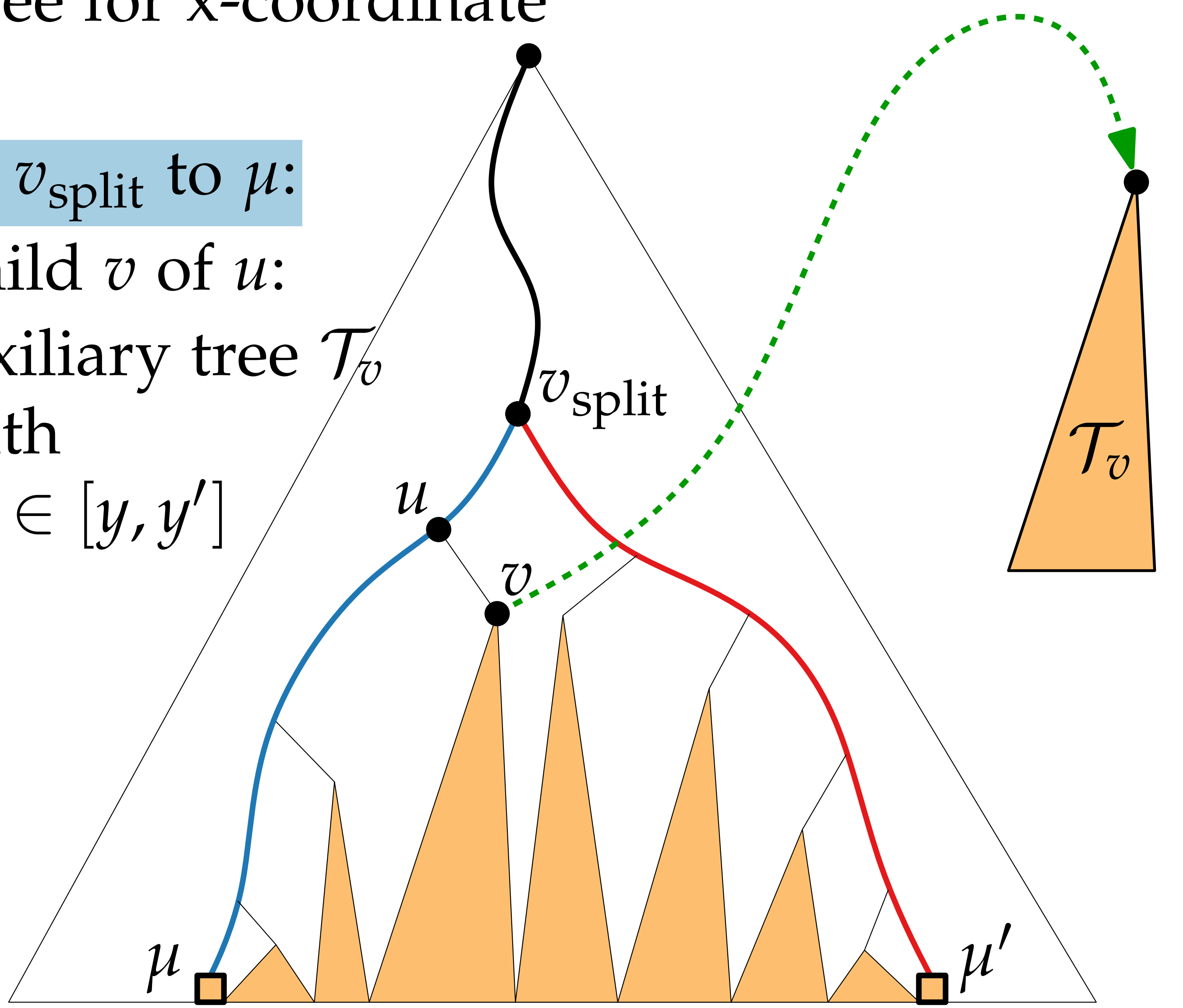

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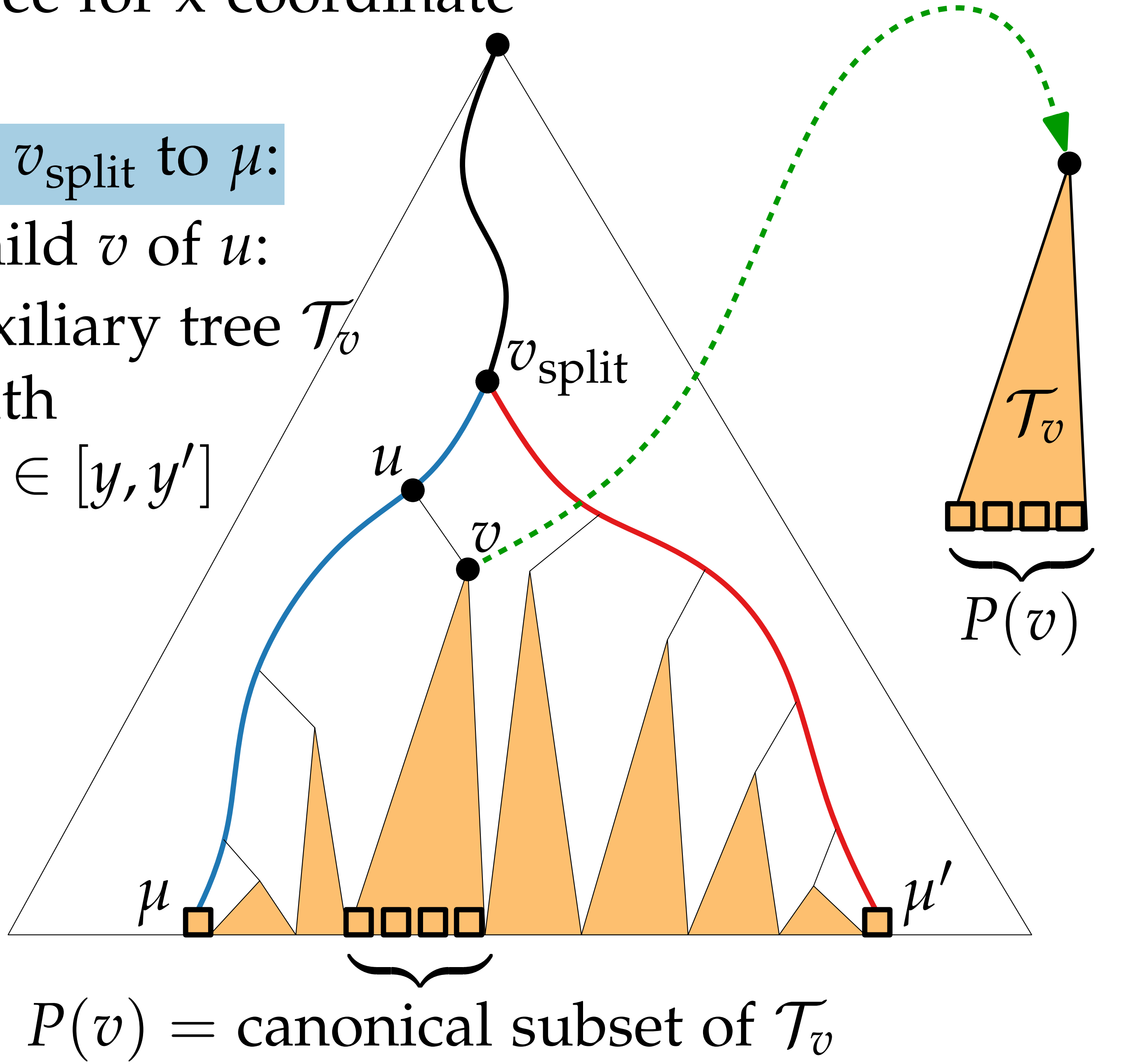

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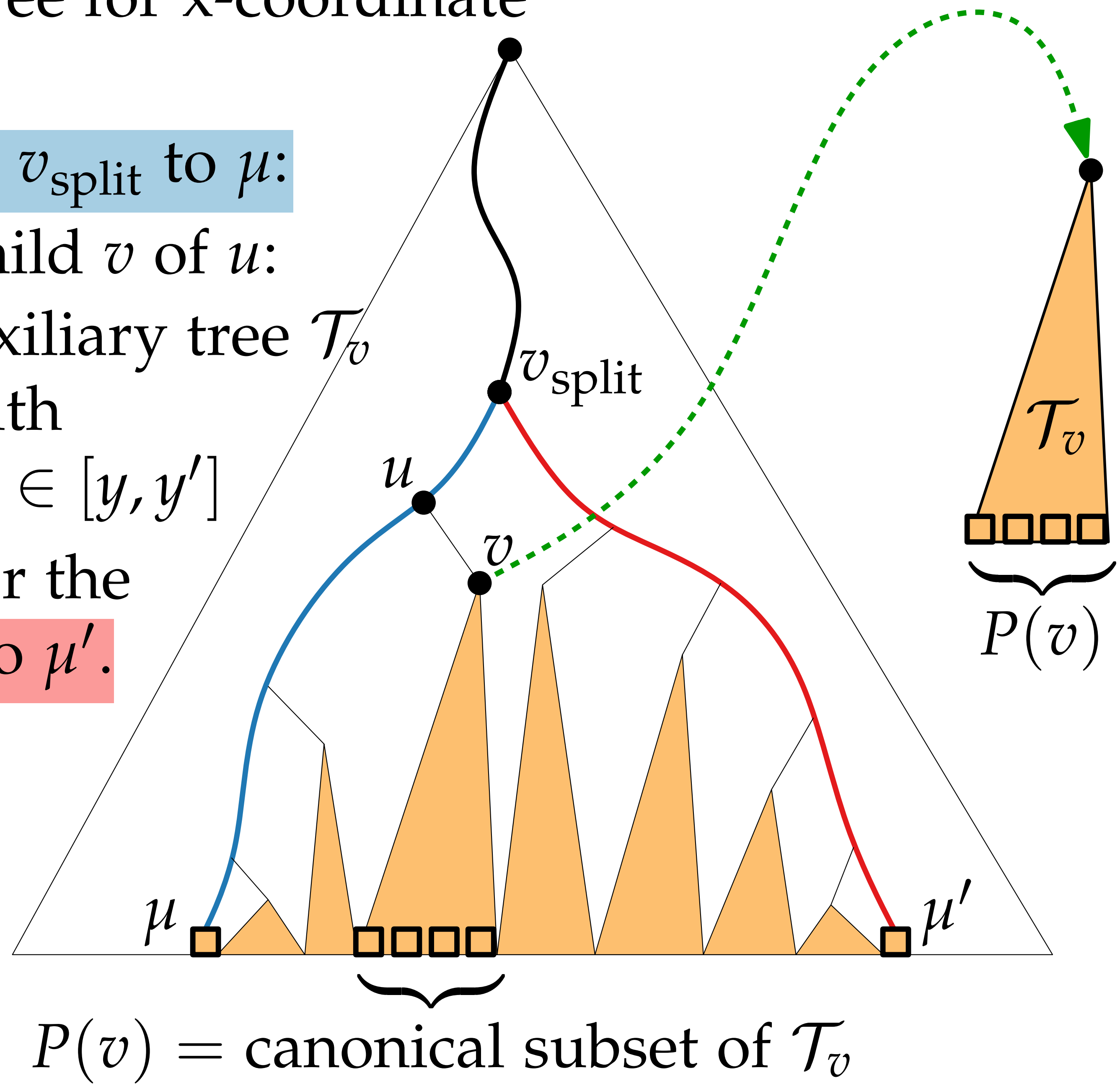

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Build2DRangeTree(point[ ] P)
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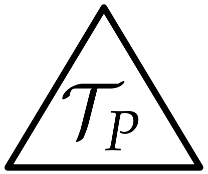

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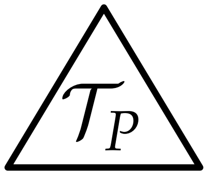

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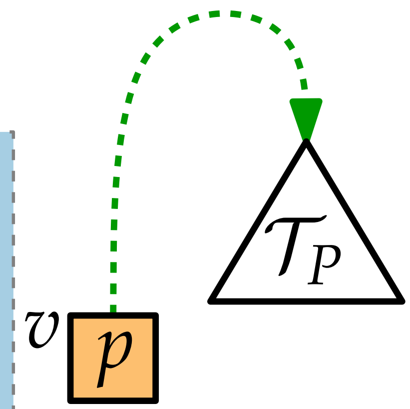

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  |   P_right =                             >
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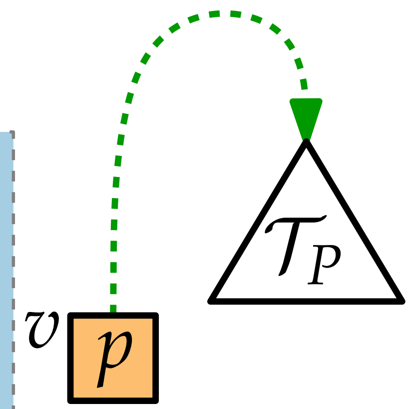

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  |   v_left  = Build2DRangeTree(P_left)
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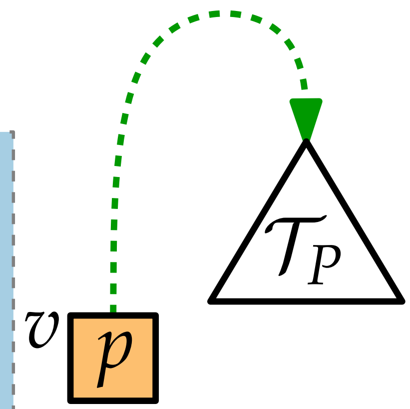

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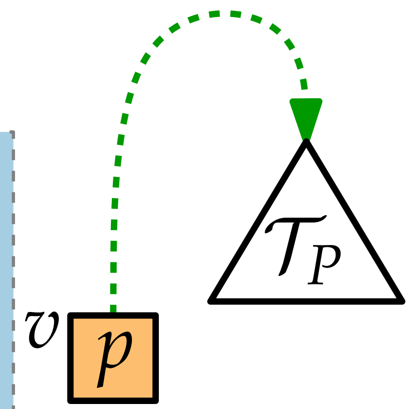

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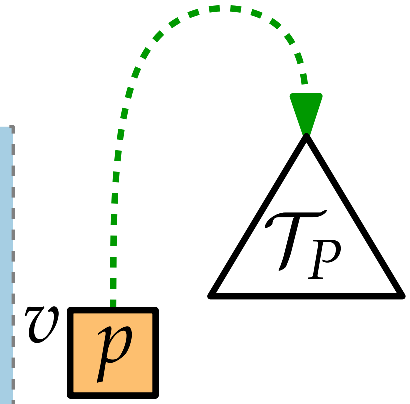

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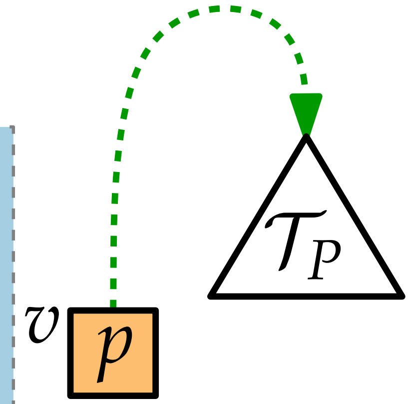

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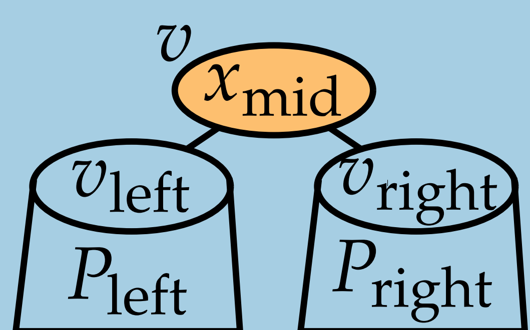

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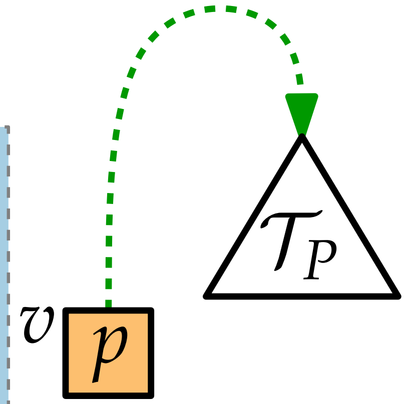

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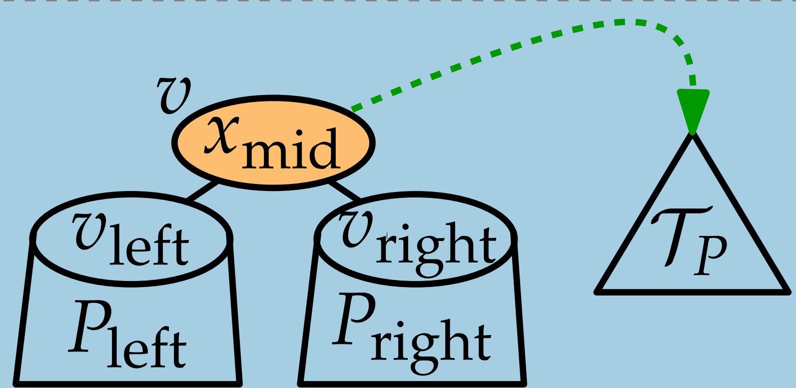

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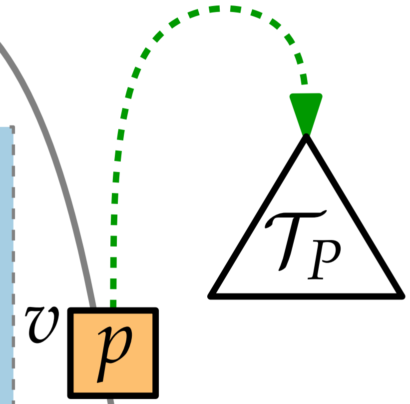

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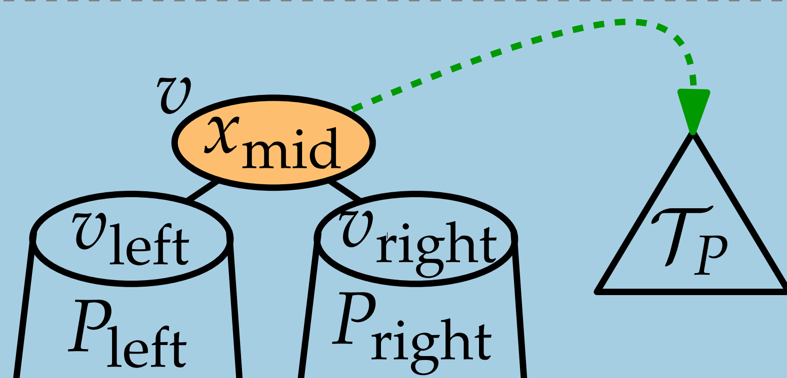

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Running time?

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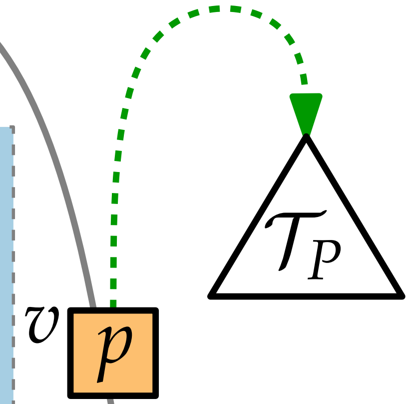

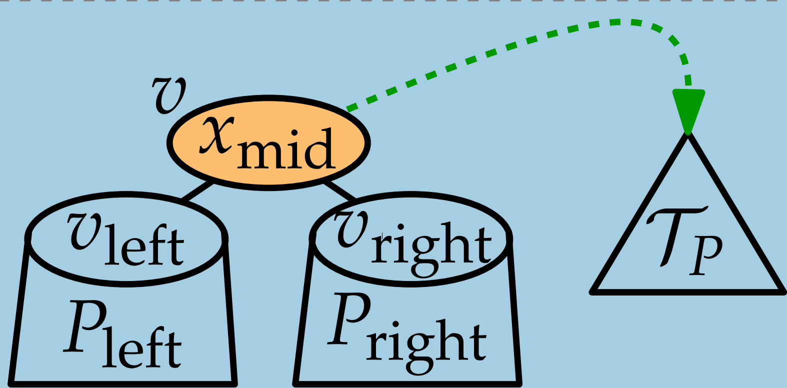

Running time?

$O(n \log n)$:-(

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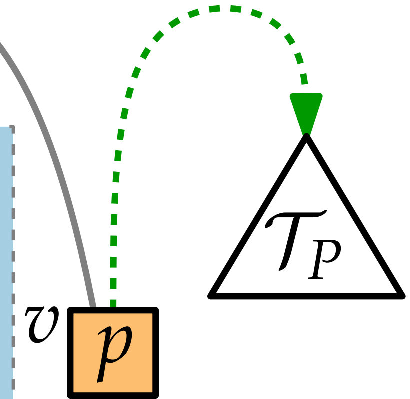

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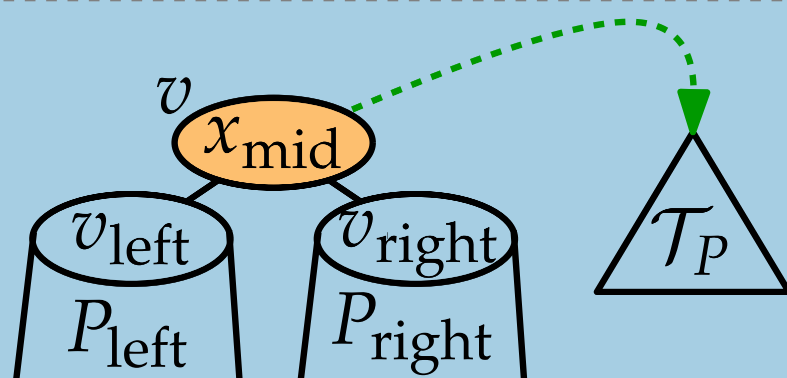


return v

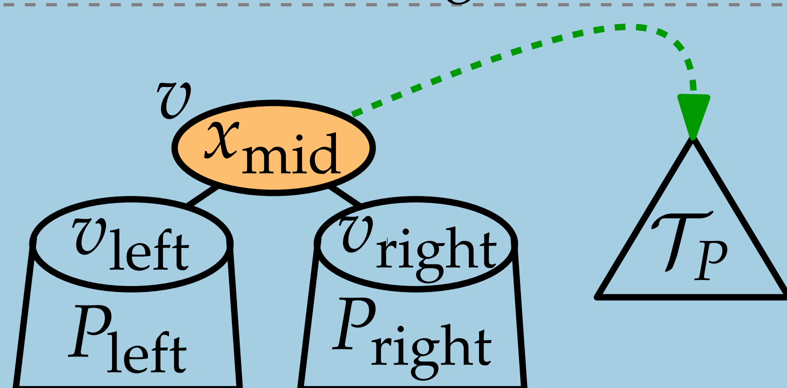
Running time?

$O(n \log n)$:-(

Better:
Pre-sort once,
then build tree
bottom-up
in linear time.

Range Trees: Construction

Build2DRangeTree(point[ ] P)
construct 2nd-level tree $\mathcal{T}_P$ on P (y-order)
if $P = \{p\}$ **then**
  create leaf v:
else
  x_{mid} = median x-coordinate of P
  P_{left} = pts in P with x-coordinate $\leq x_{\mathrm{mid}}$
  P_{right} = $>$
  v_{left} = Build2DRangeTree(P_{left})
  v_{right} = Build2DRangeTree(P_{right})
  create node v:



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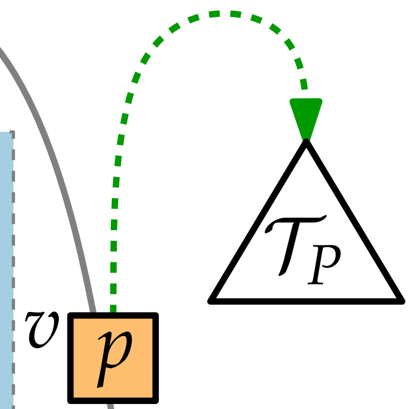

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Pre-sort once, then build tree bottom-up in linear time.
$\Downarrow$
Total construction time $O(n \log n)$

Range Trees: Space Consumption

Each node v of the 1st-level tree has a pointer to a 2nd-level tree $\mathcal{T}_v$ with $|\mathcal{T}_v| = \Theta(|P(v)|)$.

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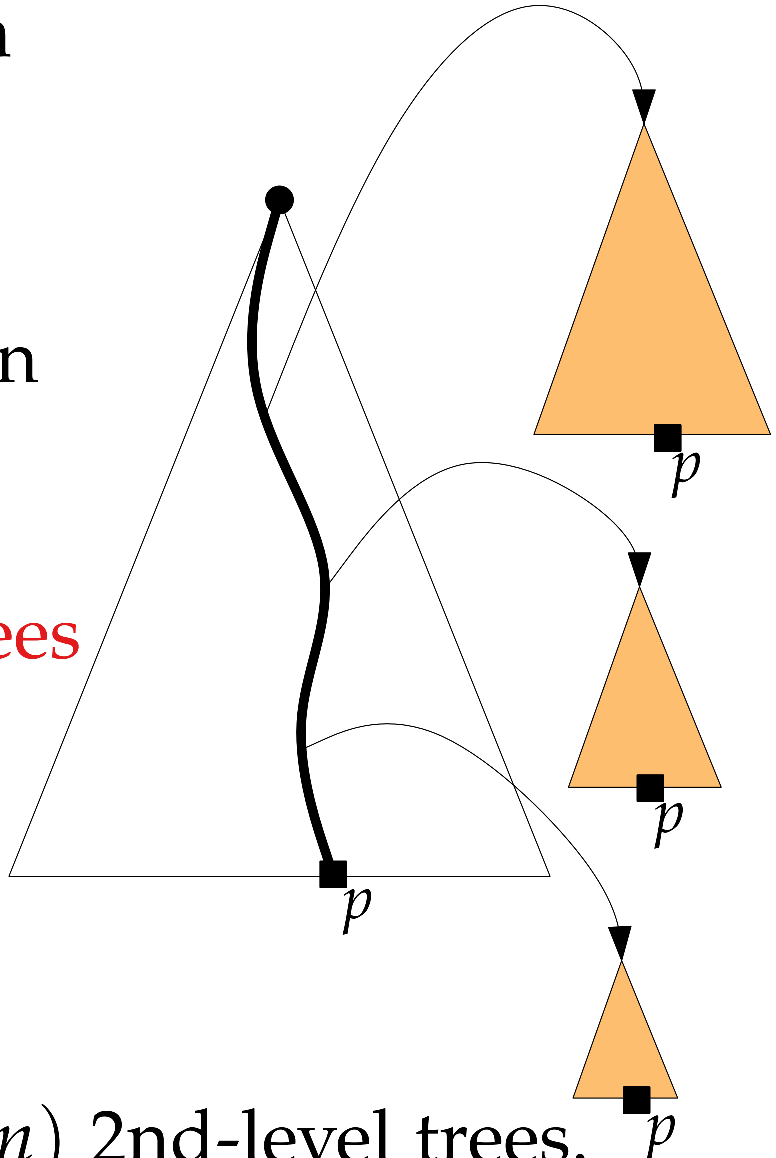

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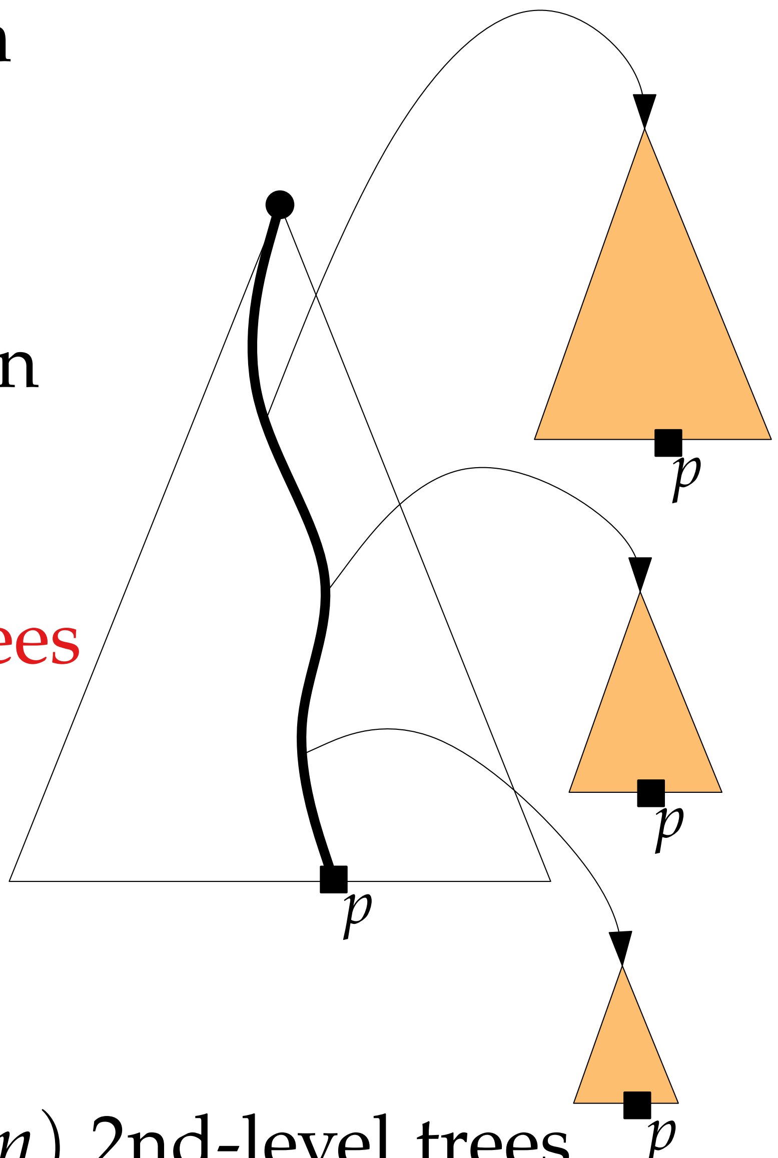

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$\Rightarrow$ $\Theta(n \log n)$ space

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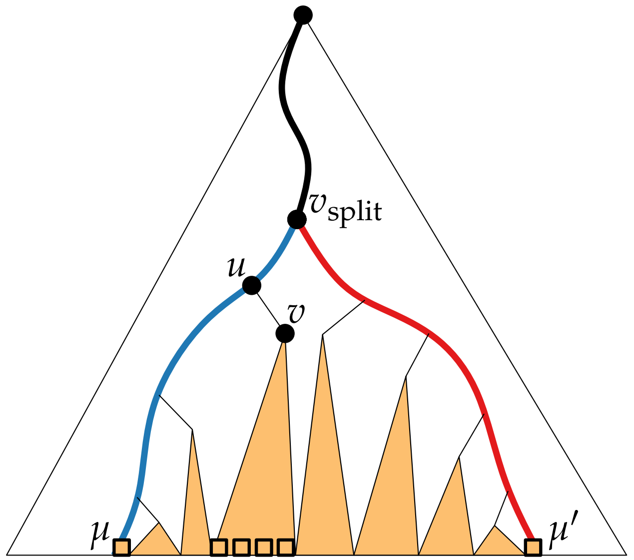

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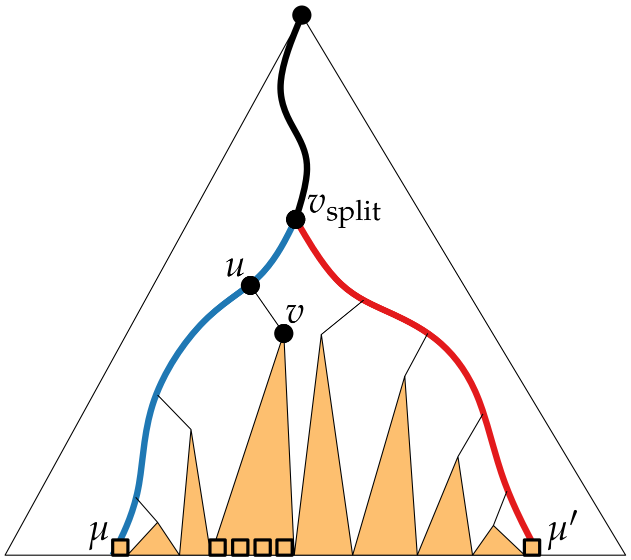

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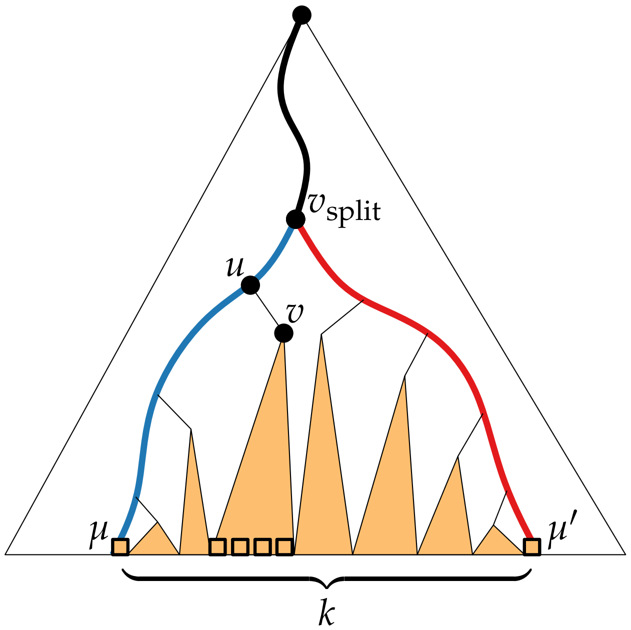

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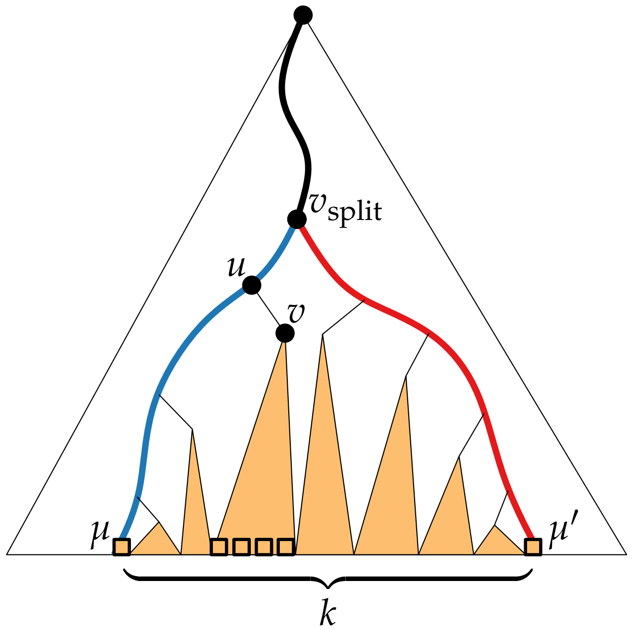

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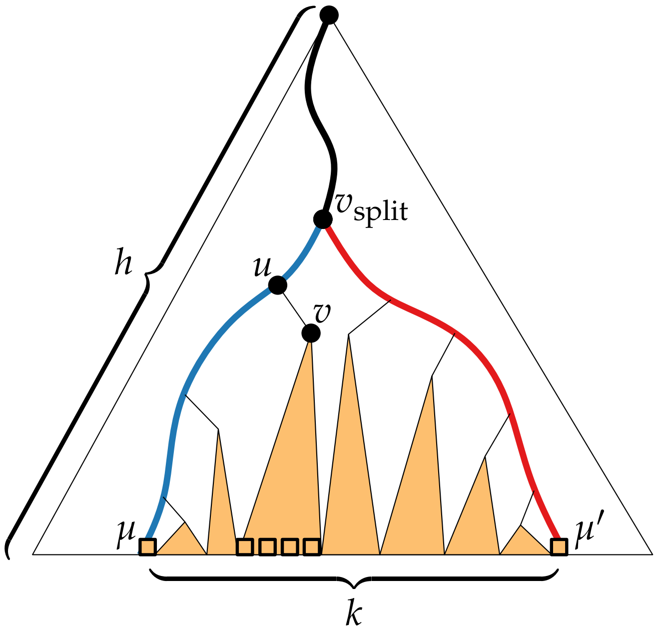

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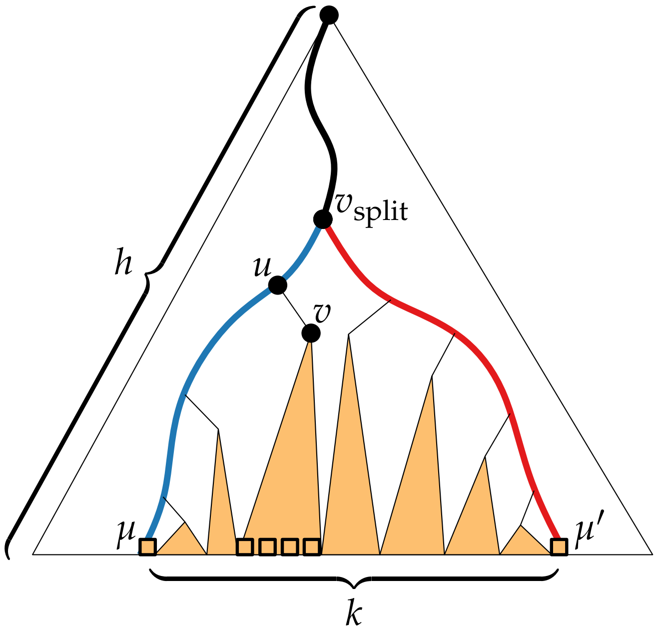

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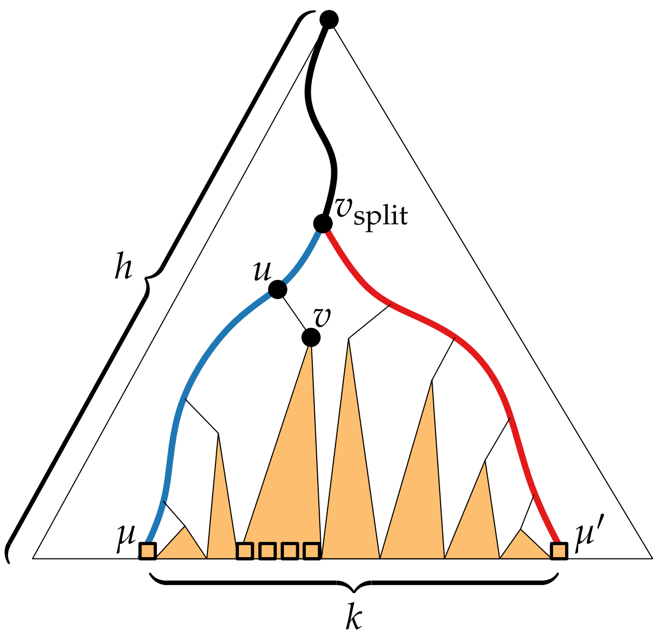

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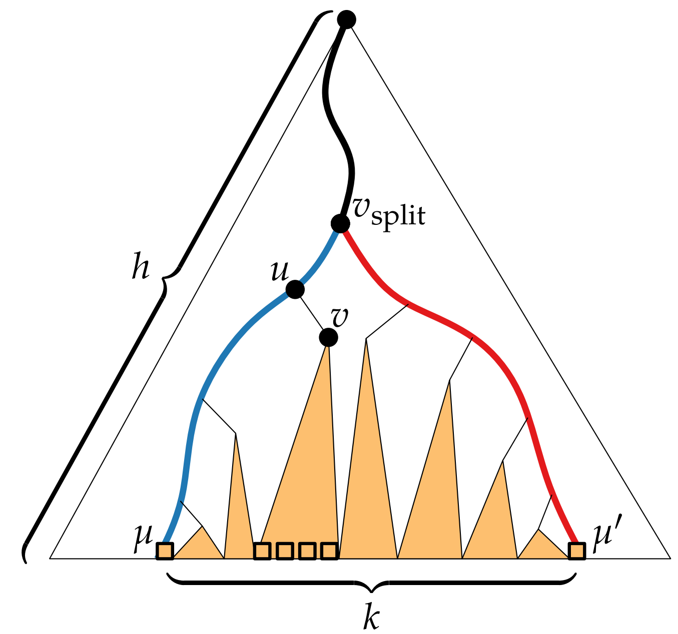

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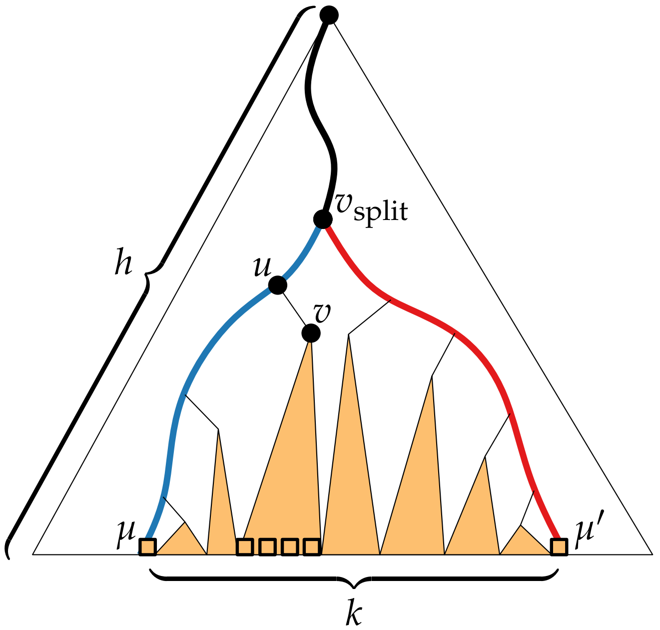

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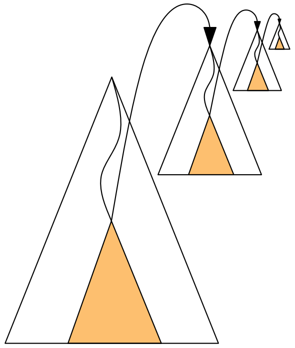

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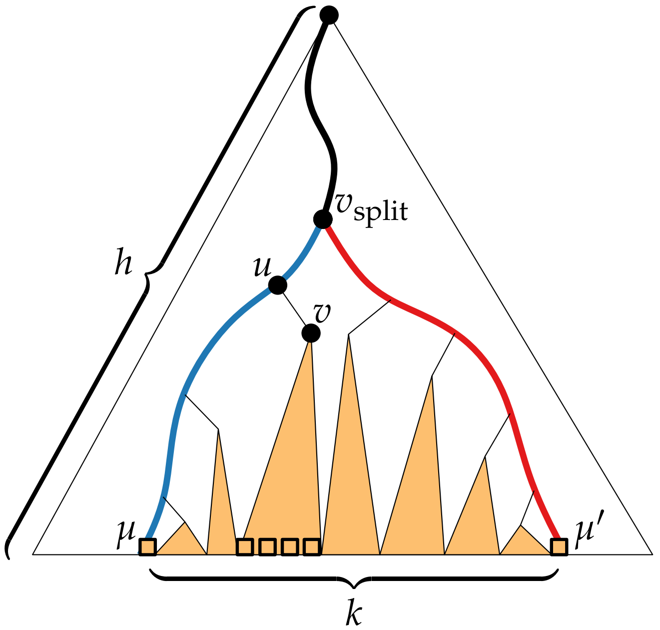
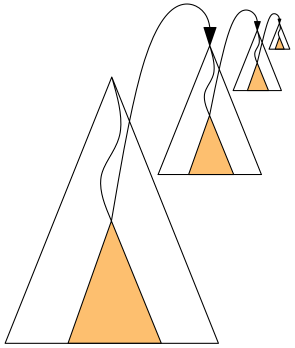


$\mathbb{R}^d$? $O(n \log^{d-1} n)$ storage and construction time

$O(k + \log^d n)$ query time



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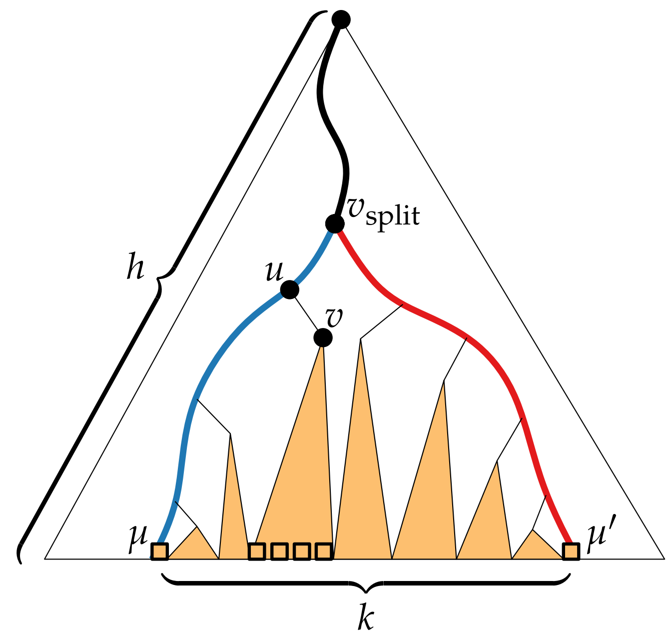

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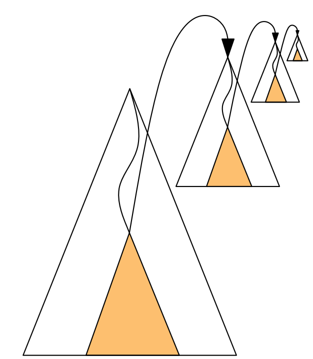

See Chapter 5.4 in Comp. Geom A&A

Comparison

	kd-tree	range tree
construction time	$O(n \log n)$	$O(n \log n)$
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Note: *trade-off* between space and query time

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We can use kd-trees and range trees for *any* set of points; no matter how many points have the same x- or y-coord.

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Task 1: Given sets $B \subset A \subset \mathbb{N}$ stored in sorted order in arrays $A[1..n]$ and $B[1..m]$, support 1d range queries in the multiset $A \cup B$ in $k + 1 \cdot \log n$ time!

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[dBCvKO'08]

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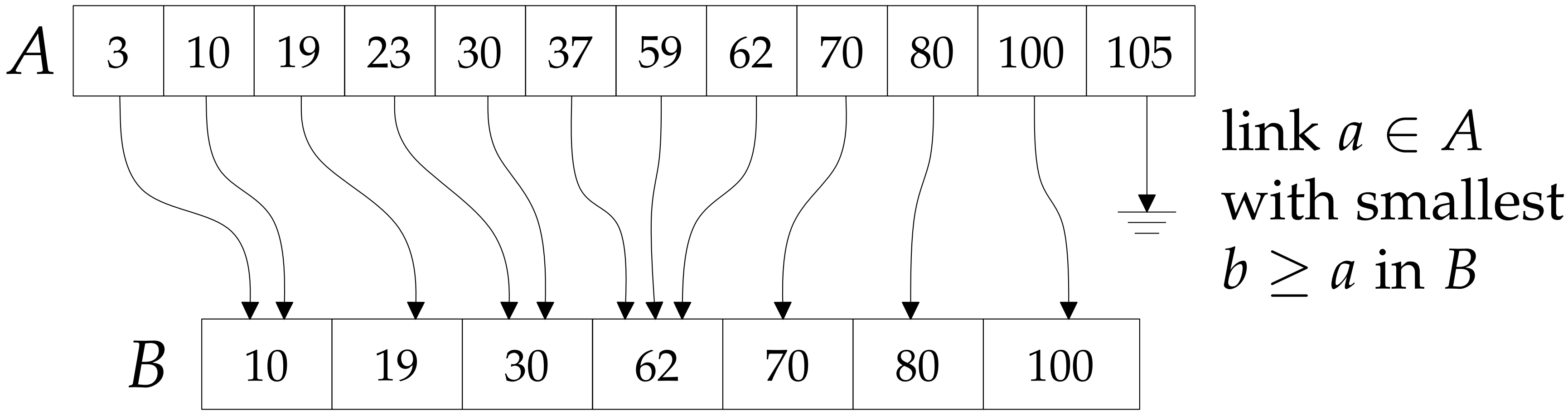

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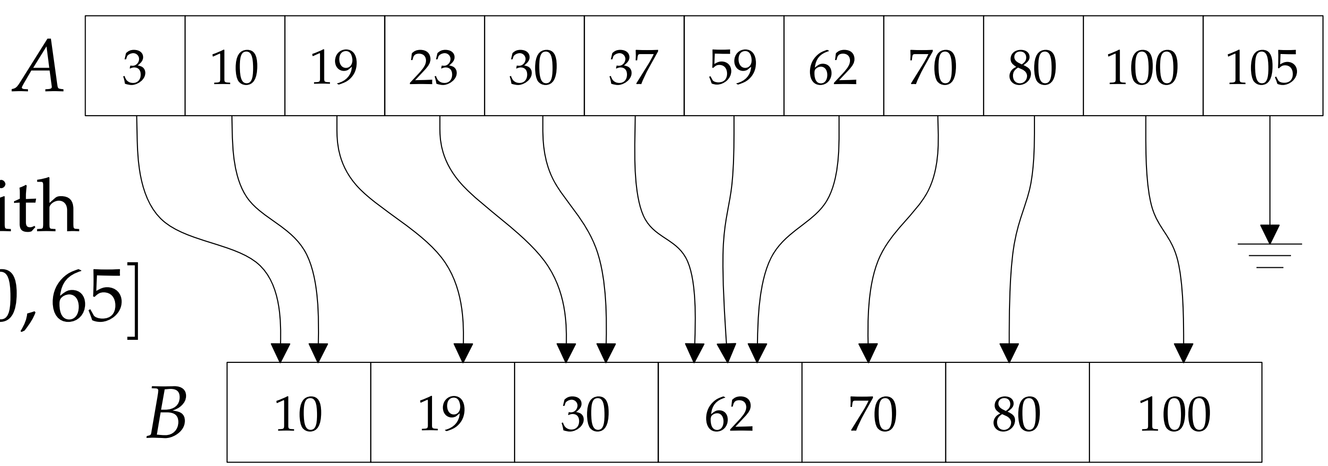

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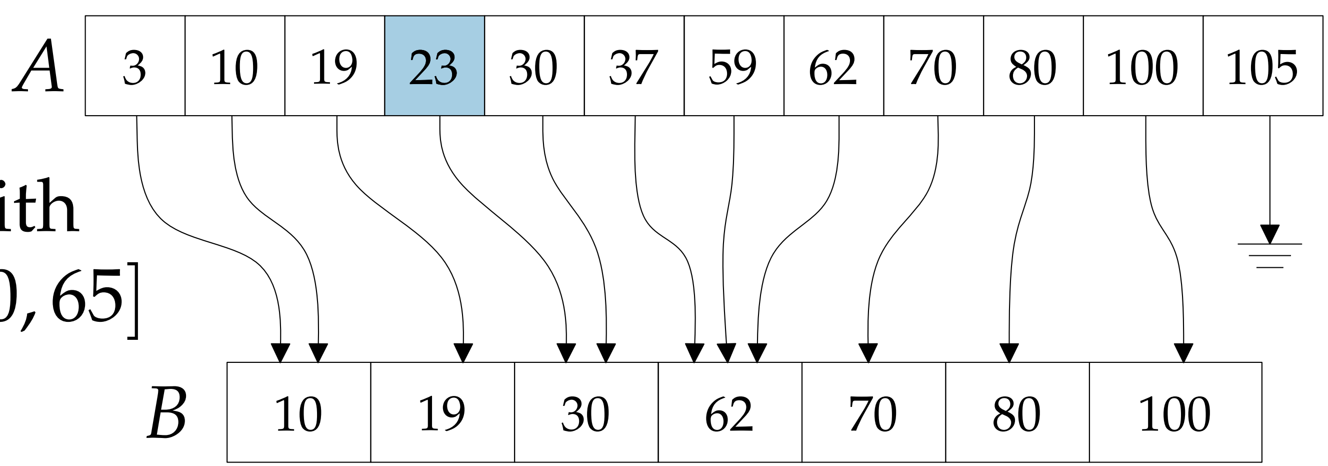

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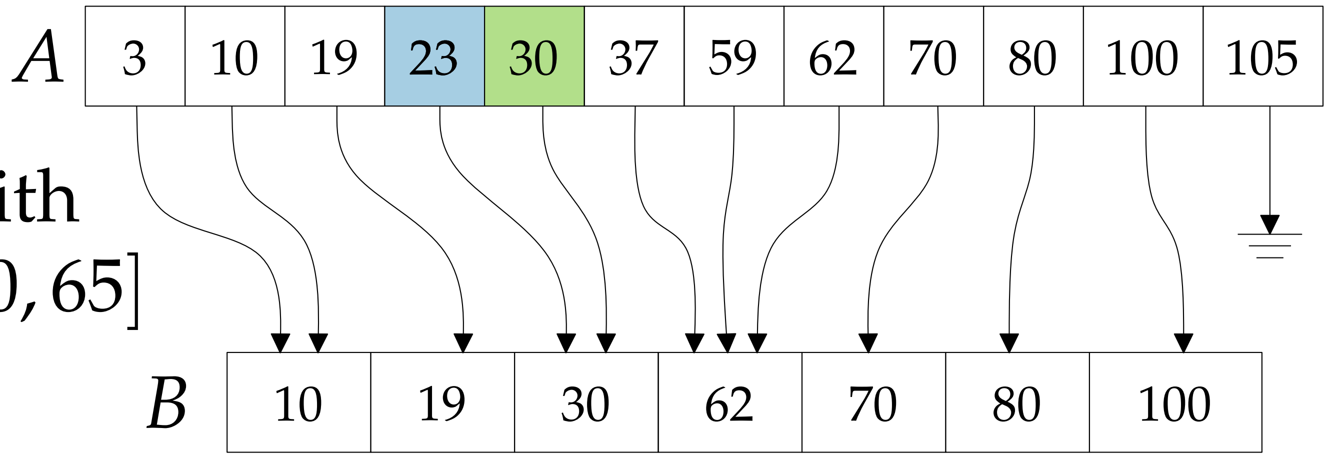

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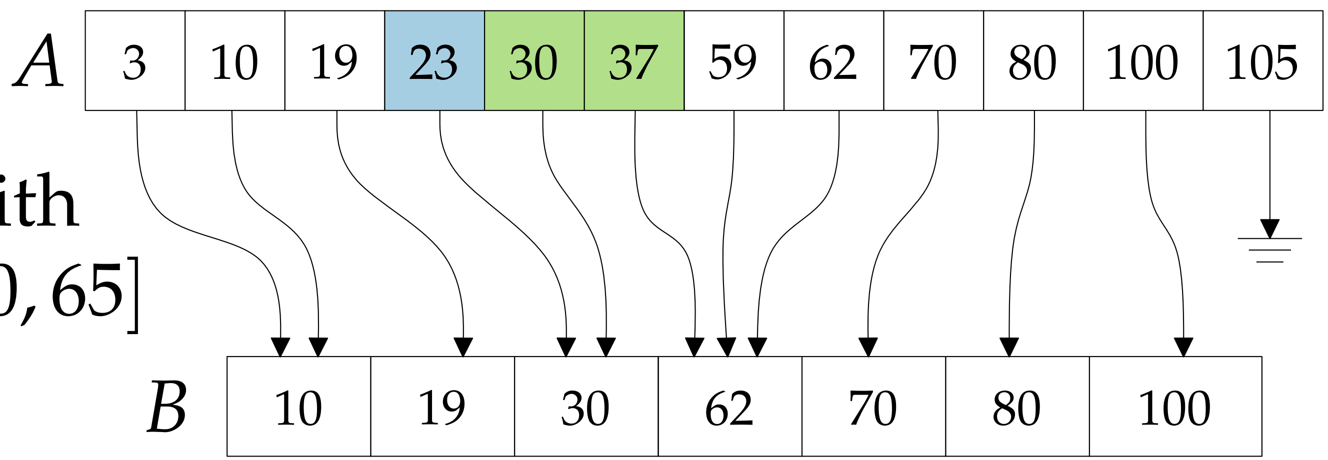

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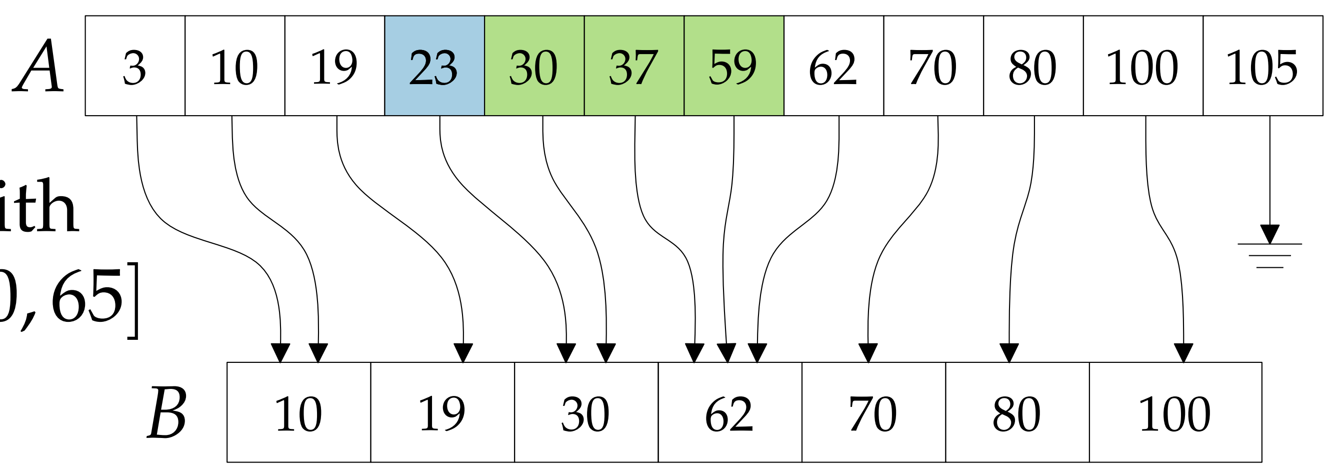

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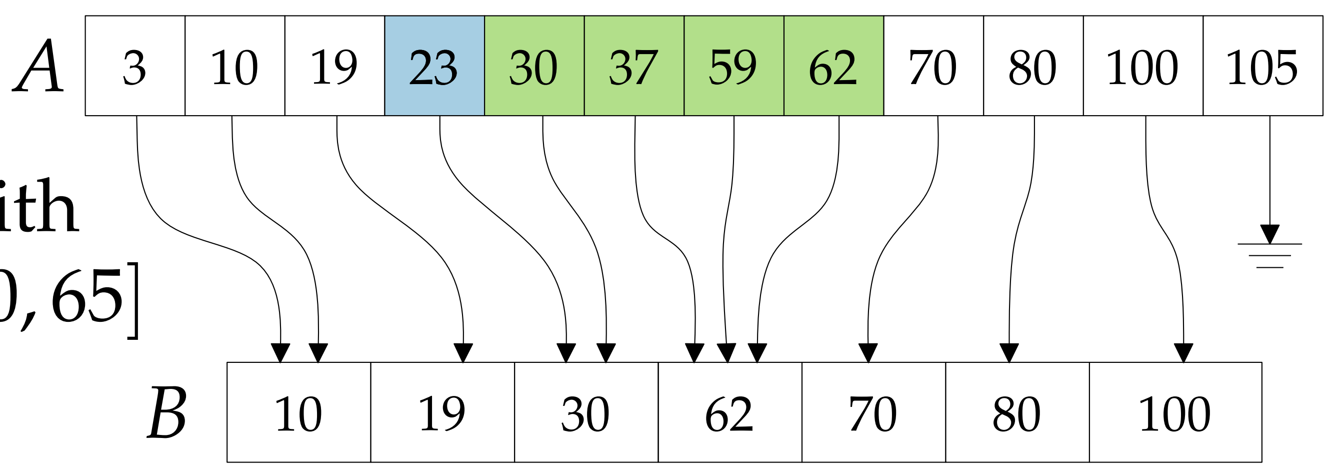

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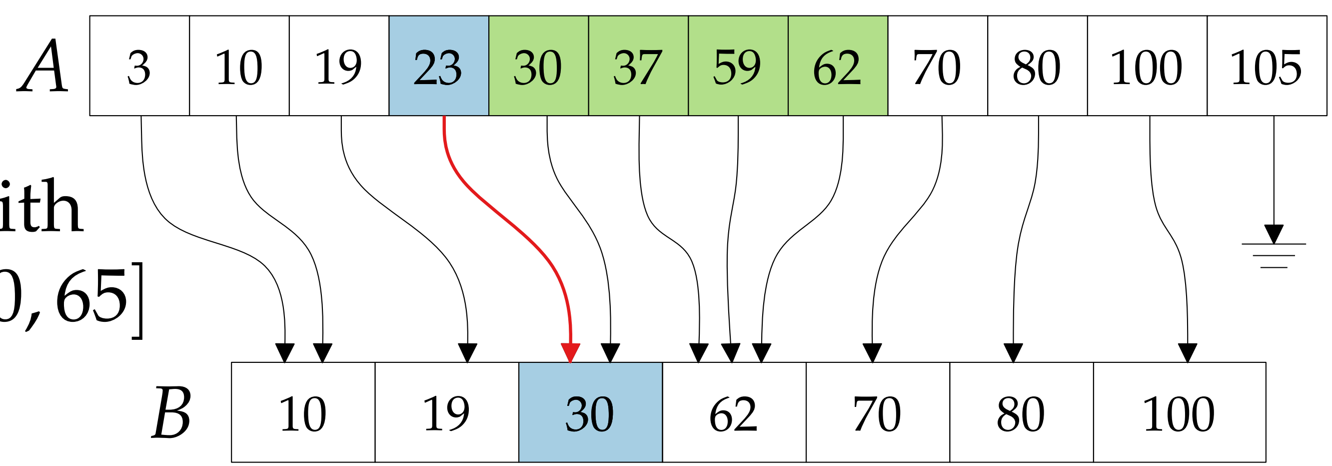
Task 2: Assuming that task 1 can be solved, speed up 2d range queries: $O(k + \log^2 n) \rightarrow O(k + \log n)$ time!

query with range $[20, 65]$



link $a \in A$ with smallest $b \geq a$ in B

[dBCvKO'08]

Fractional Cascading

Task 1: Given sets $B \subset A \subset \mathbb{N}$ stored in sorted order in arrays $A[1..n]$ and $B[1..m]$, support 1d range queries in the multiset $A \cup B$ in $k + 1 \cdot \log n$ time!

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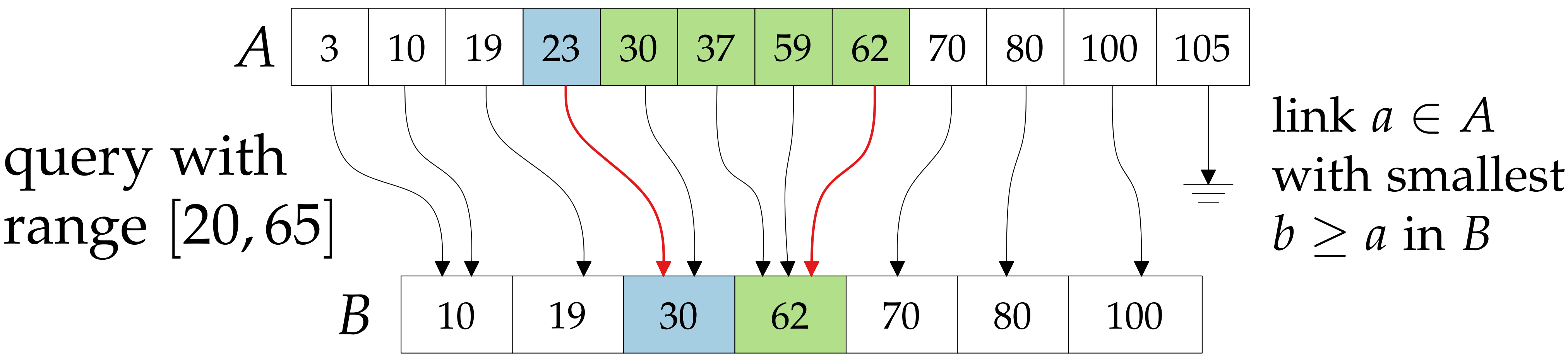
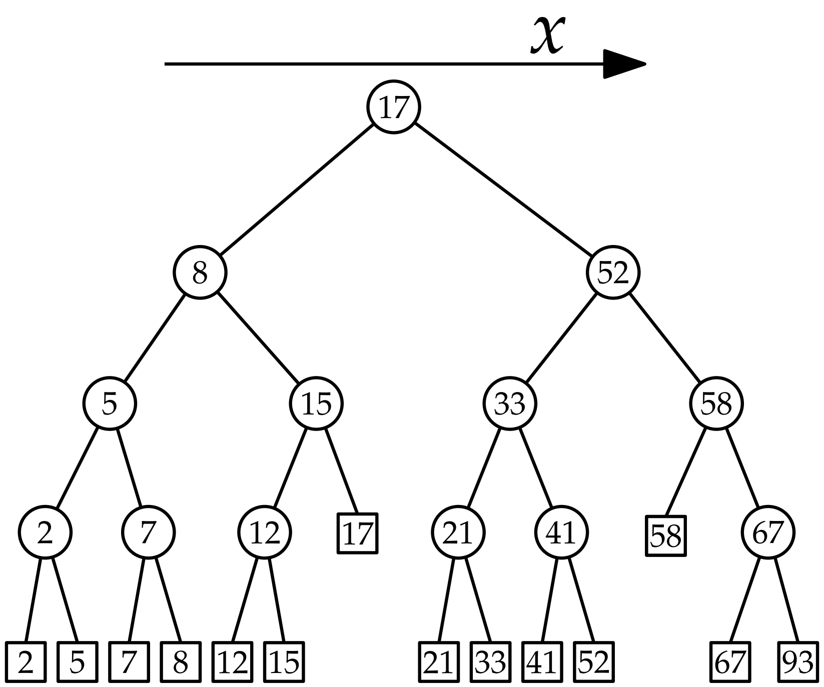


[dBCvKO'08]

Layered Range Trees



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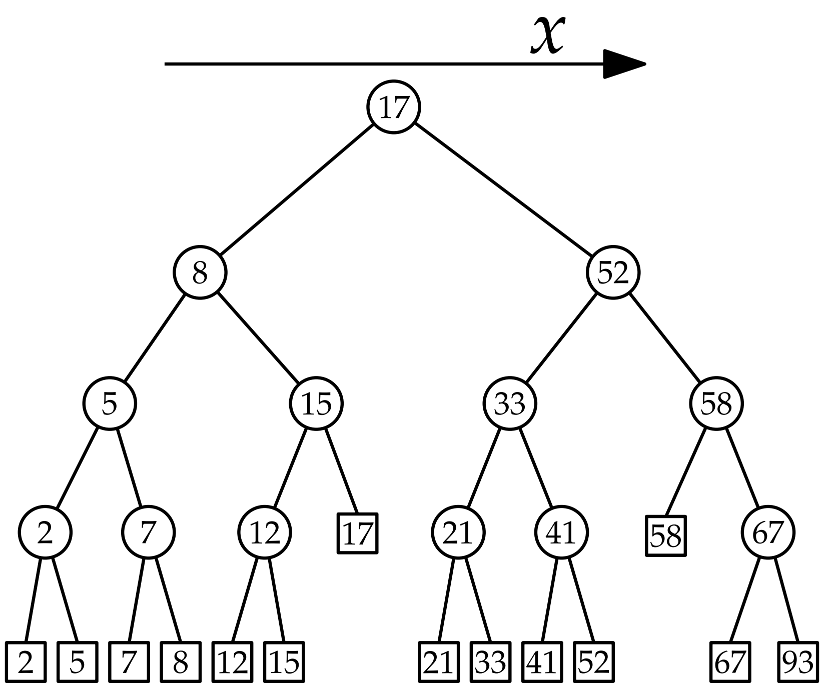

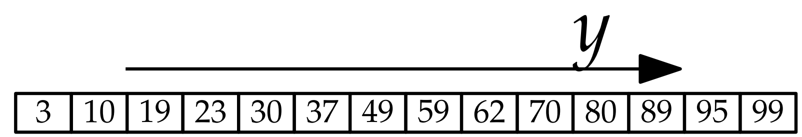

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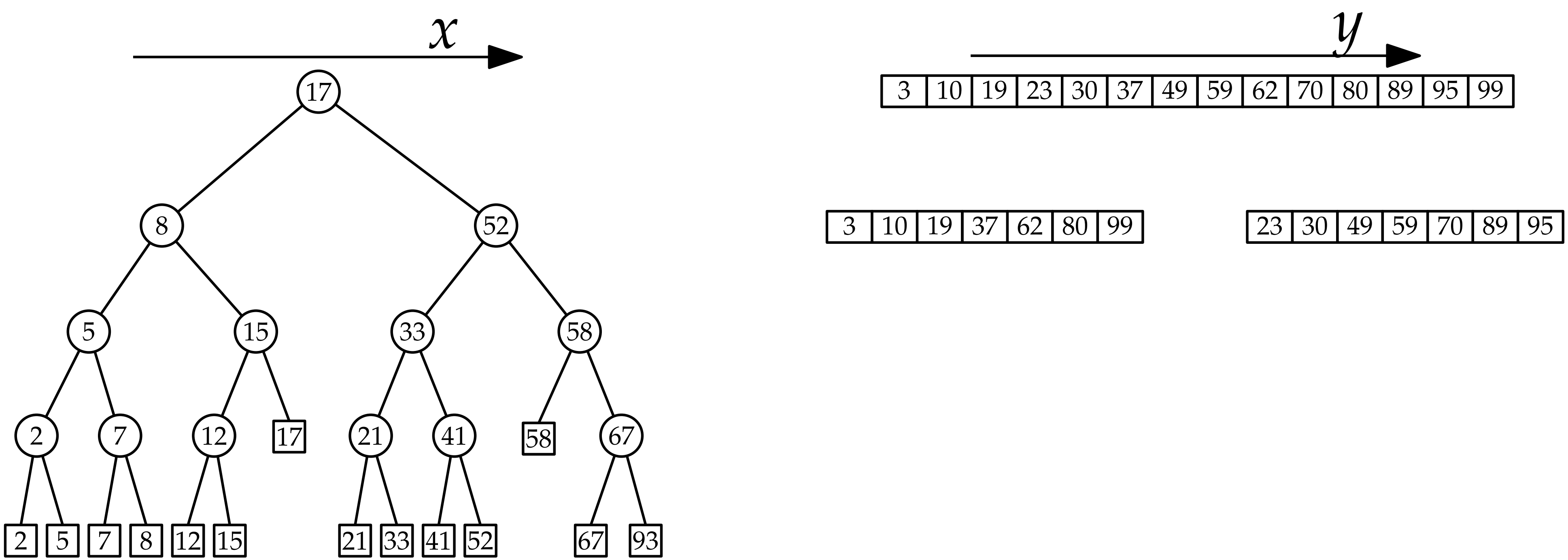

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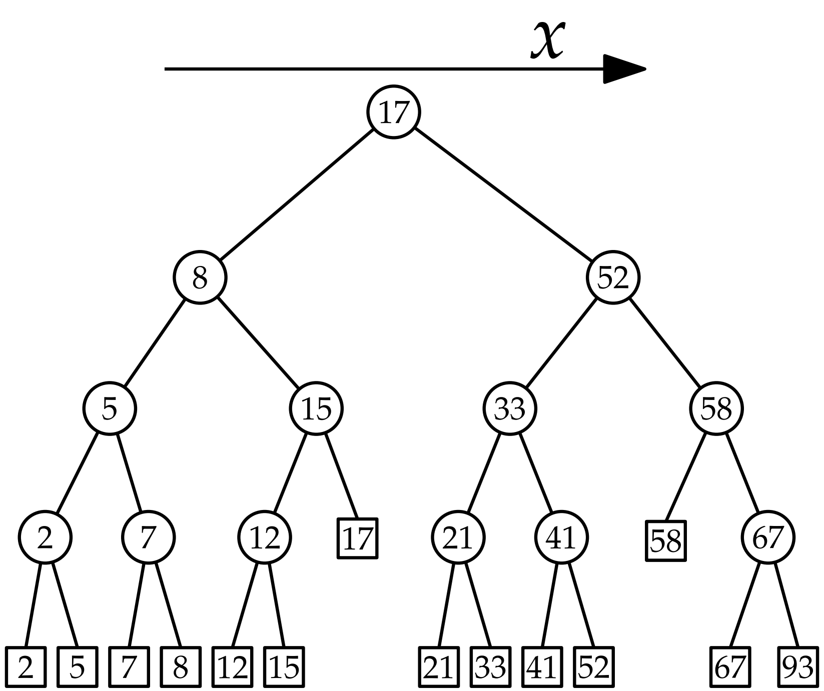

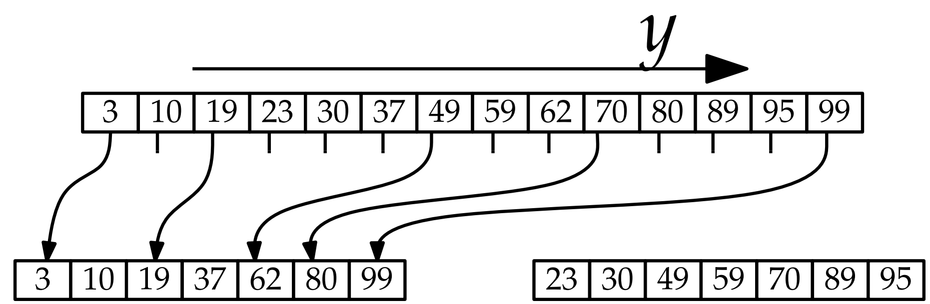

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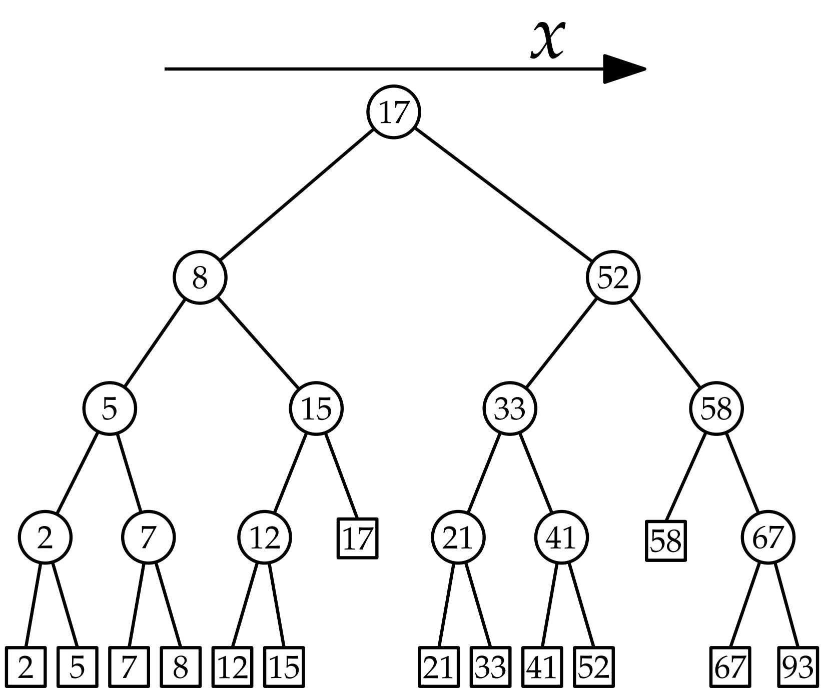

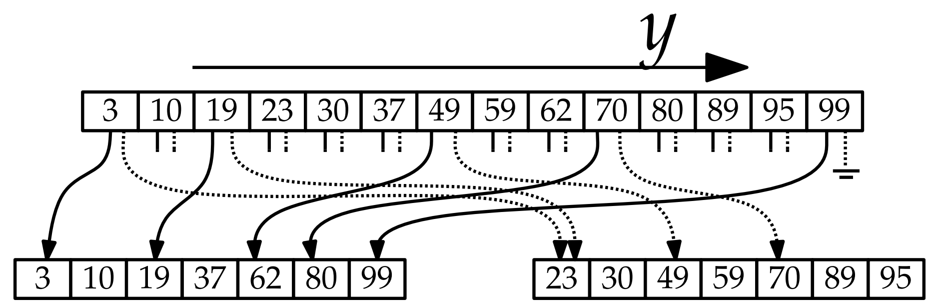

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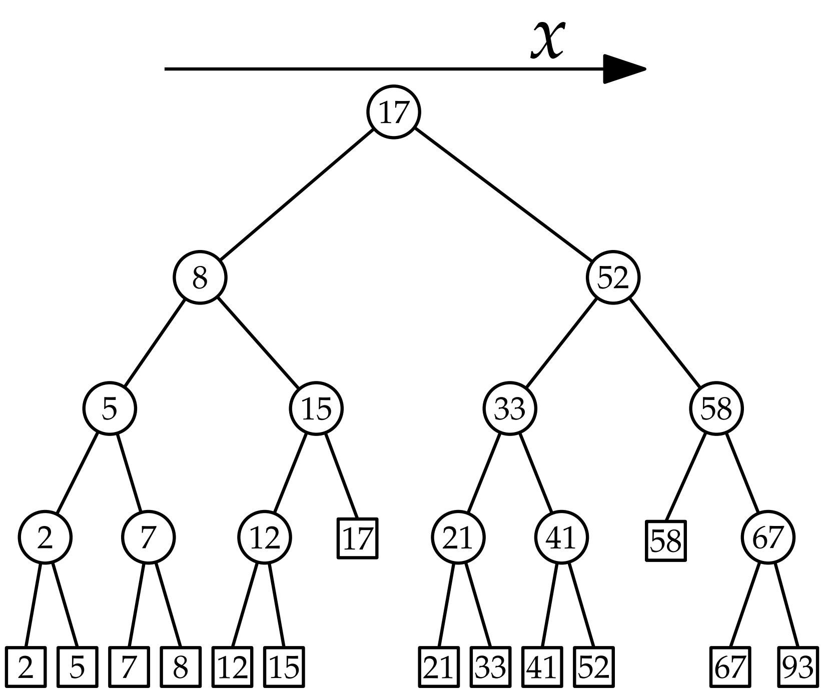

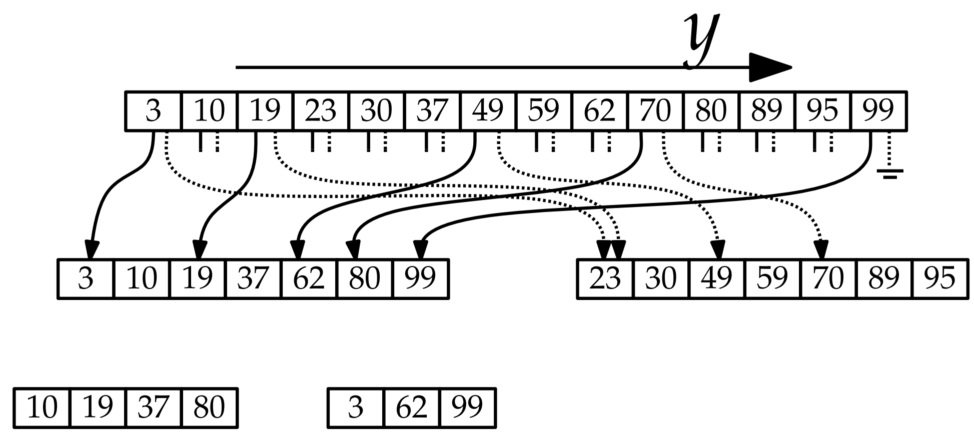

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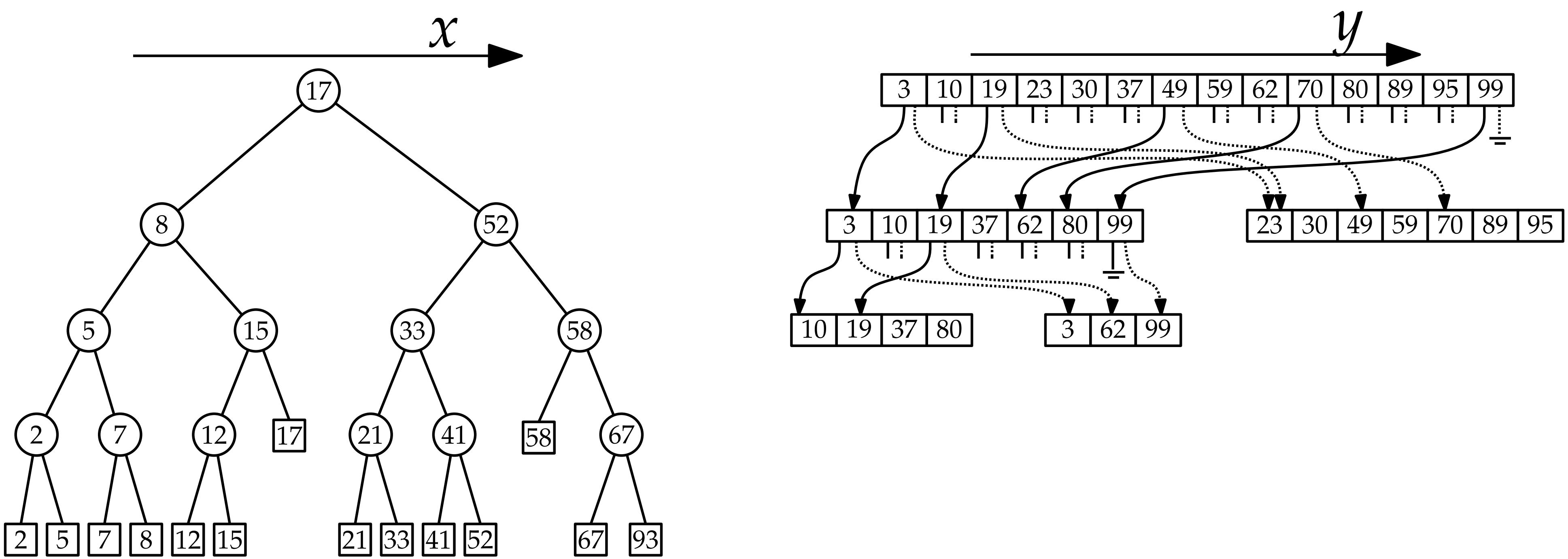

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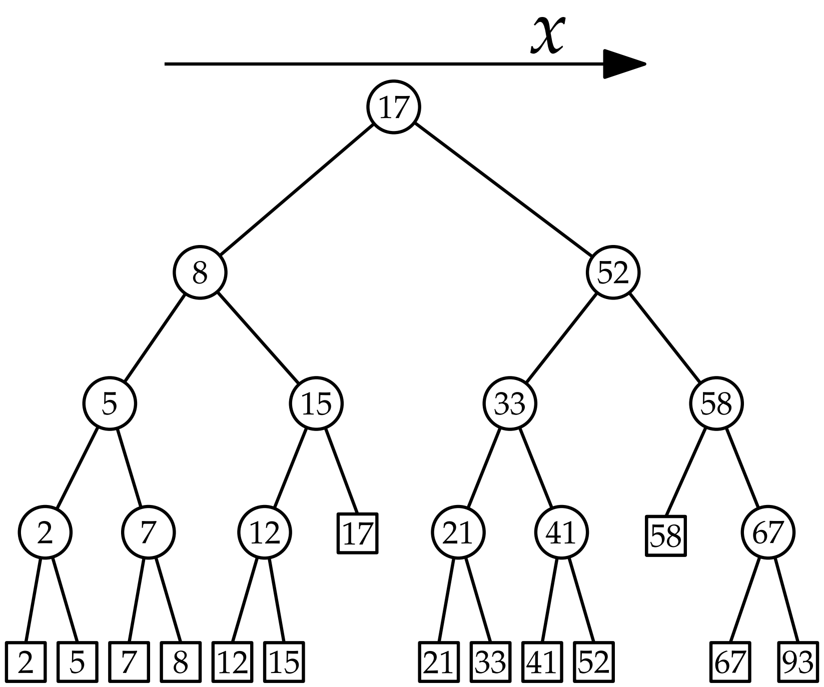

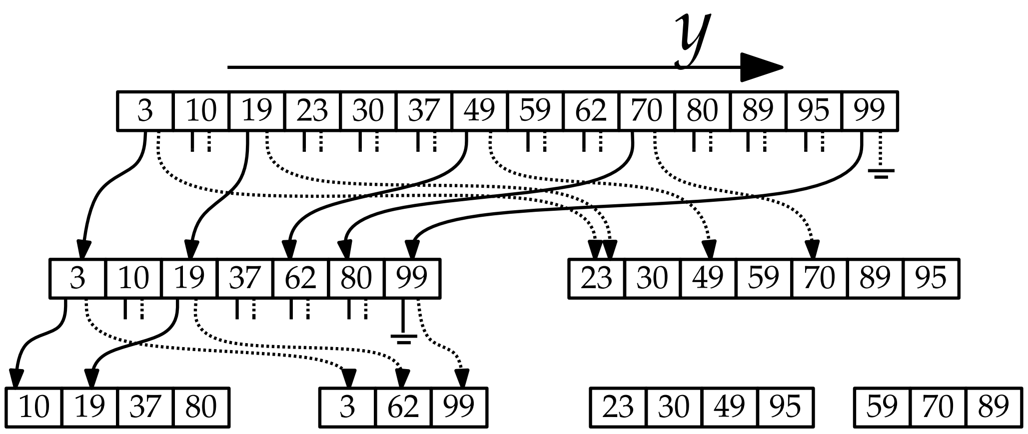

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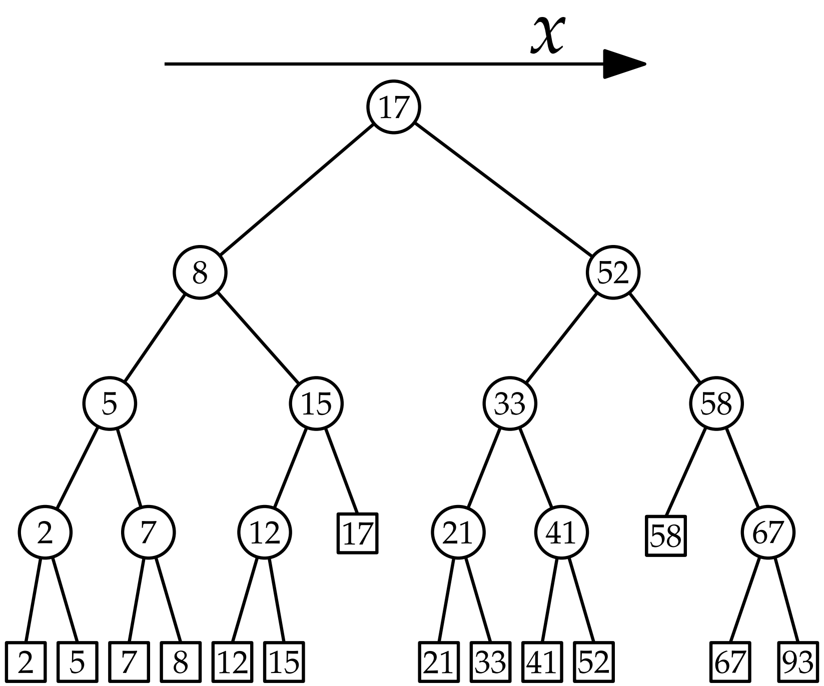

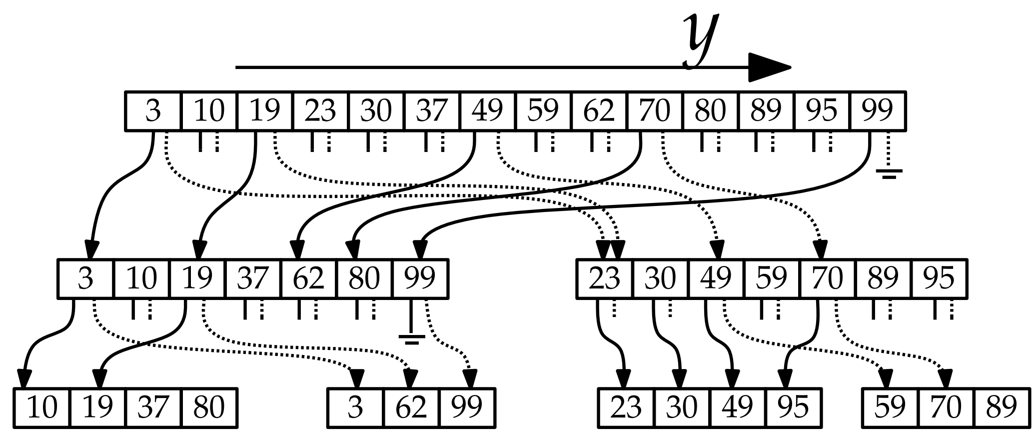

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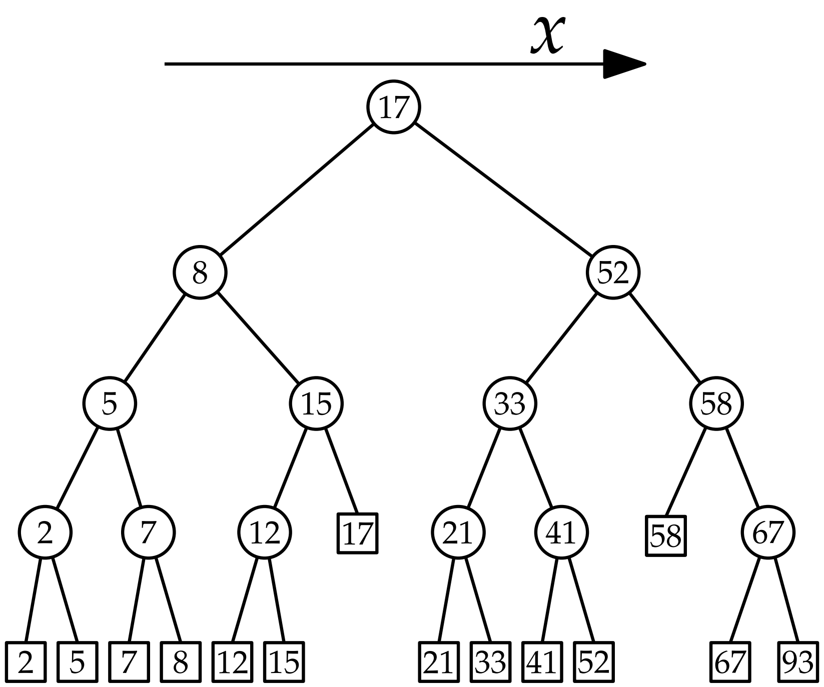

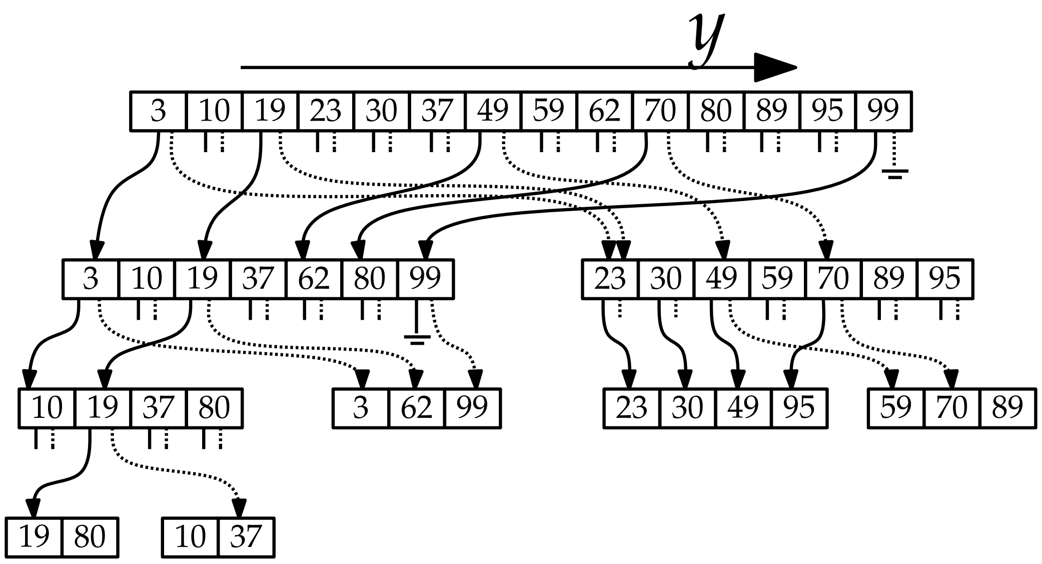

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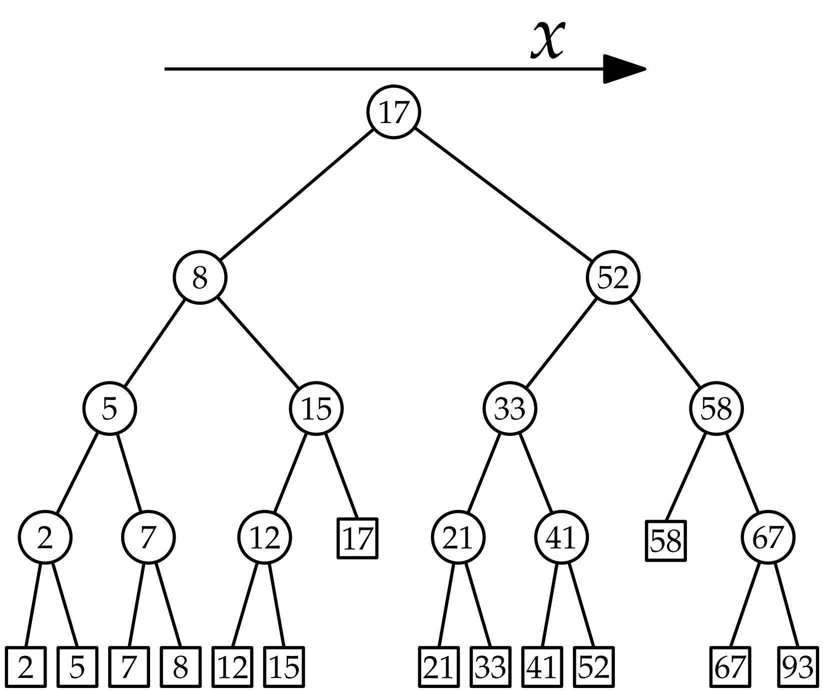

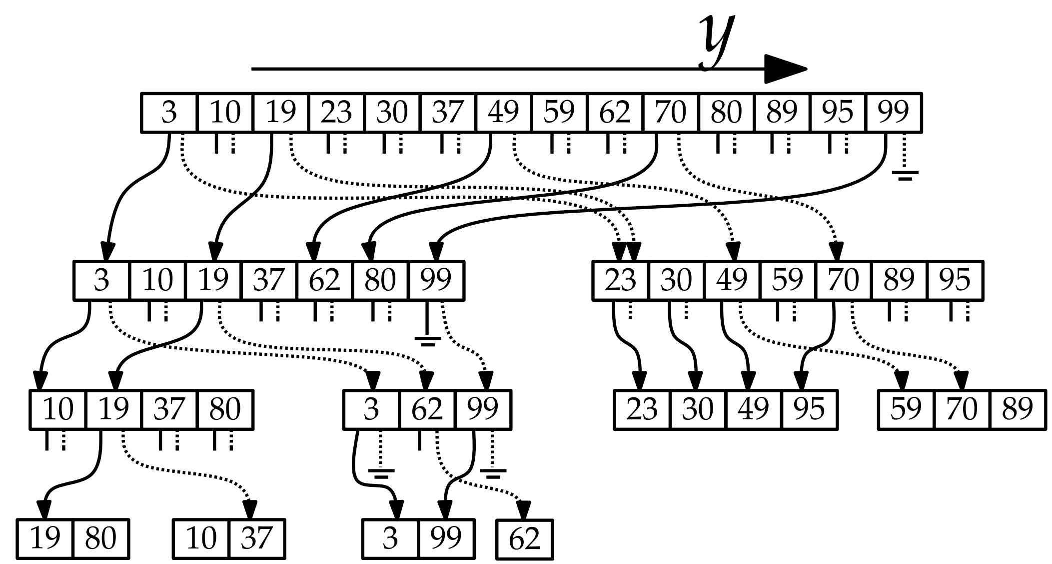

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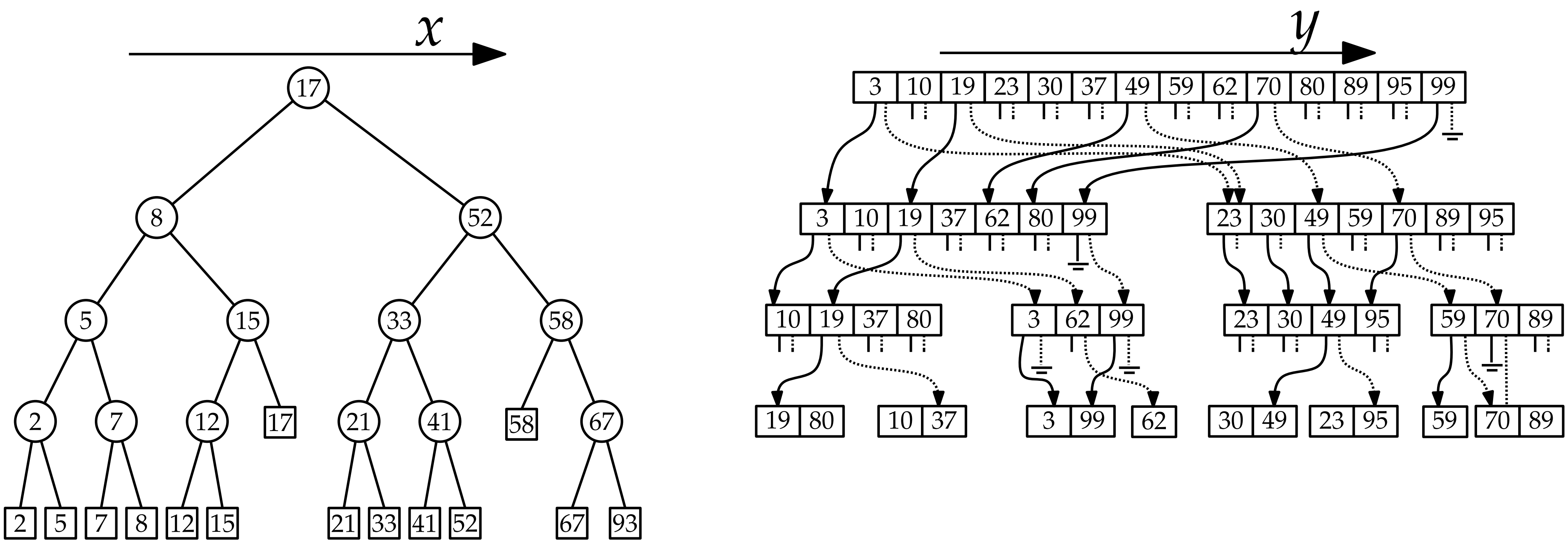

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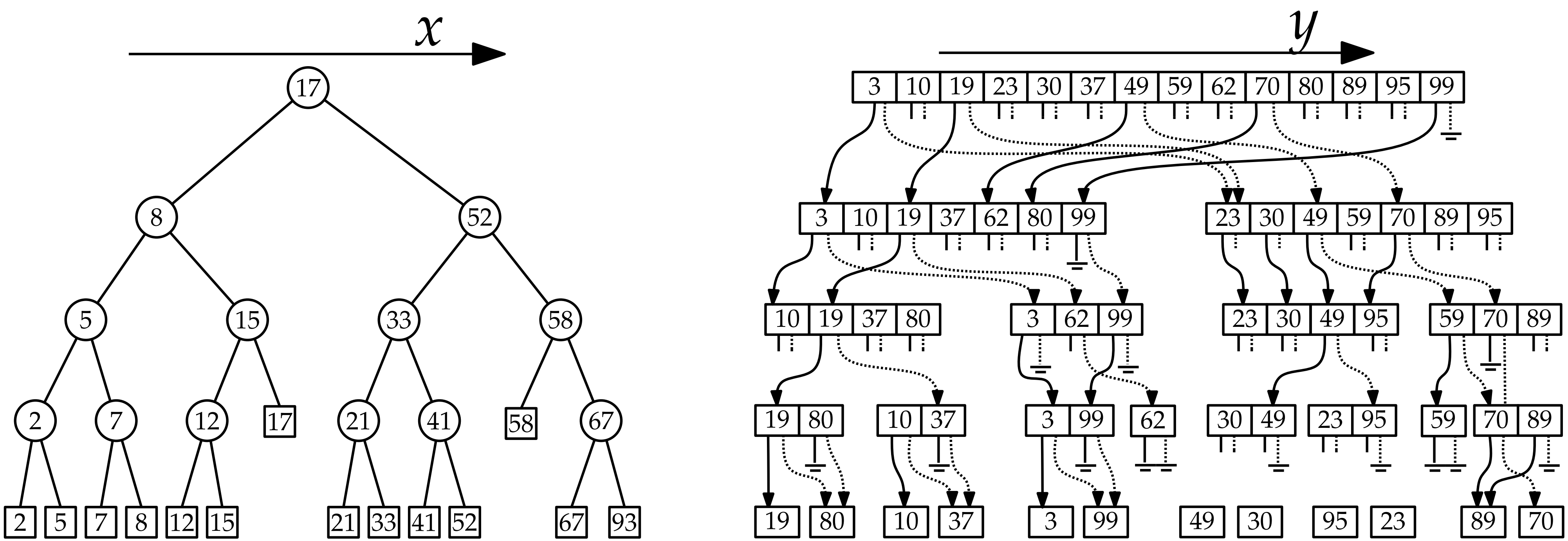

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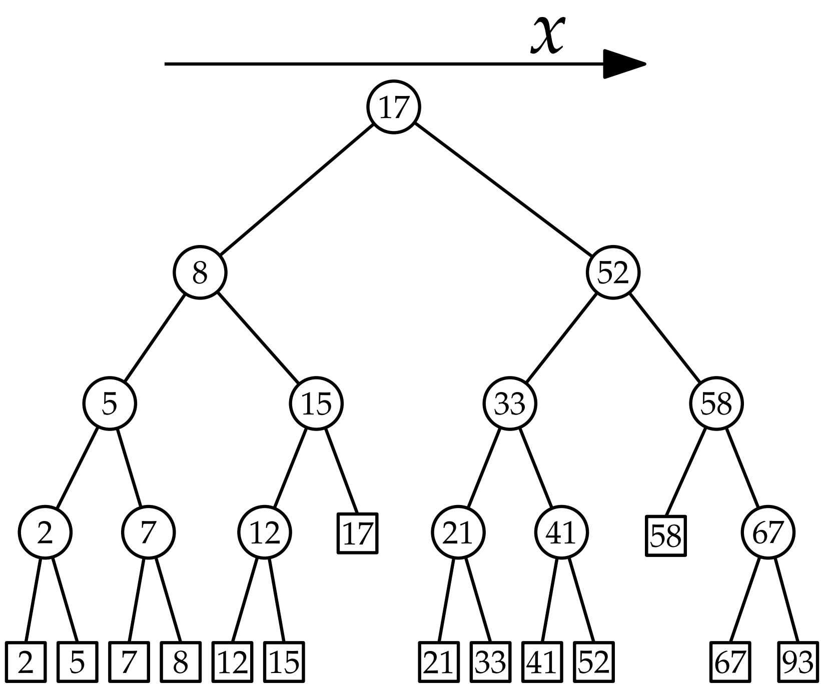

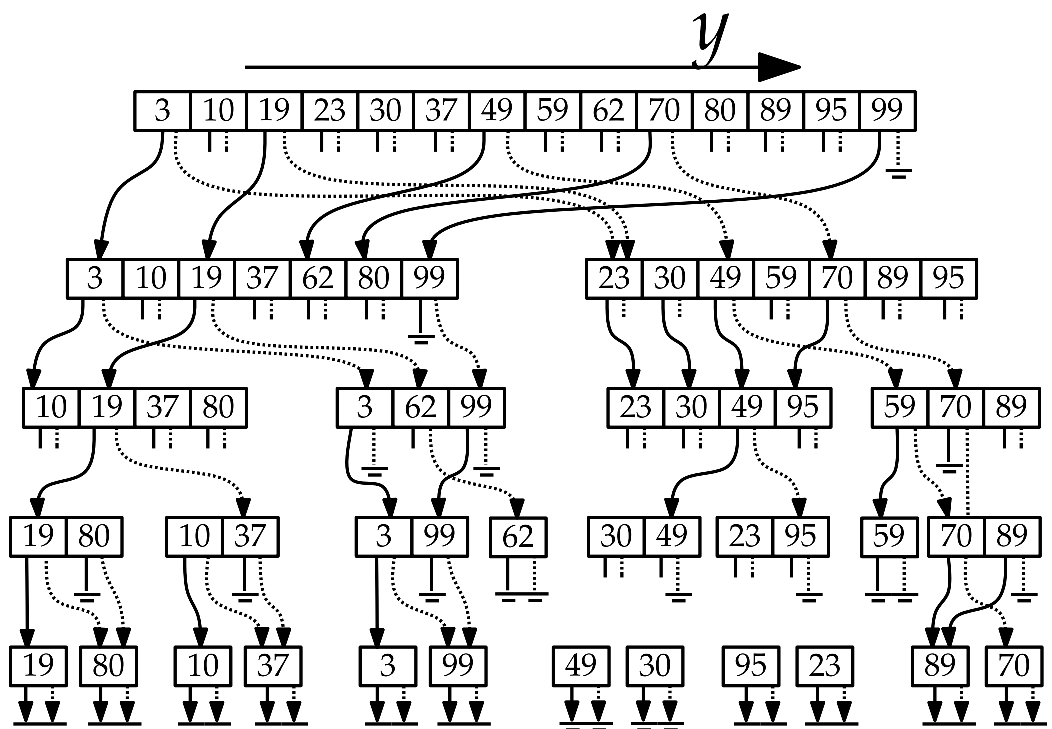

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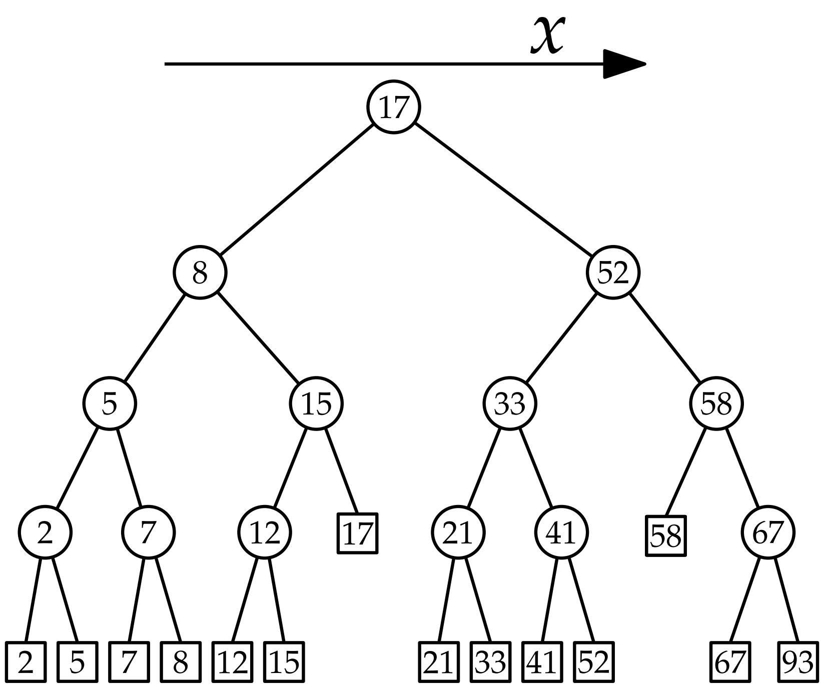
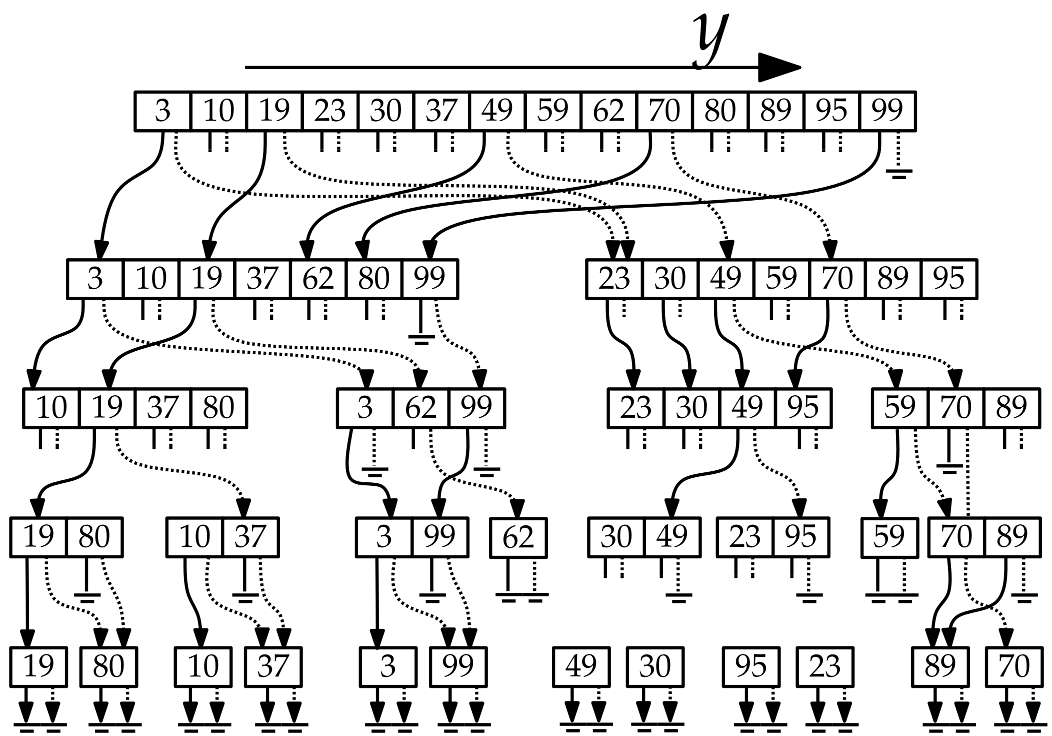
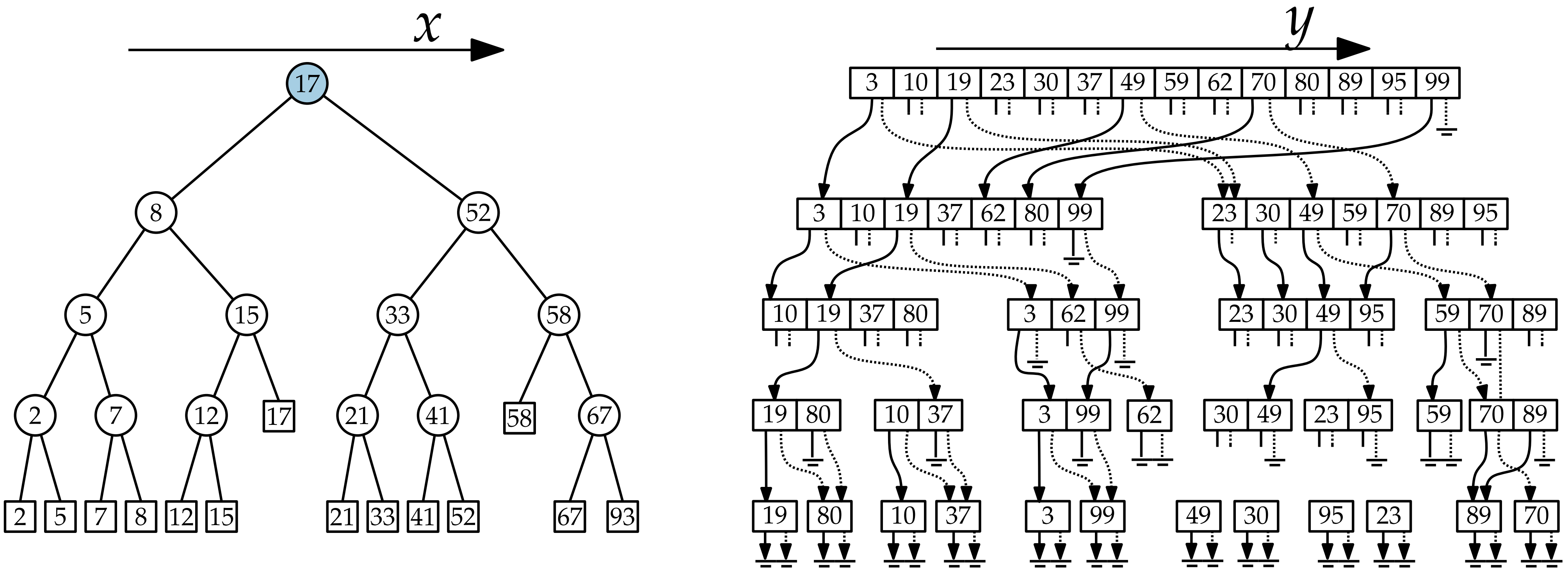




$$[16, 53] \times [18, 60]$$

Layered Range Trees



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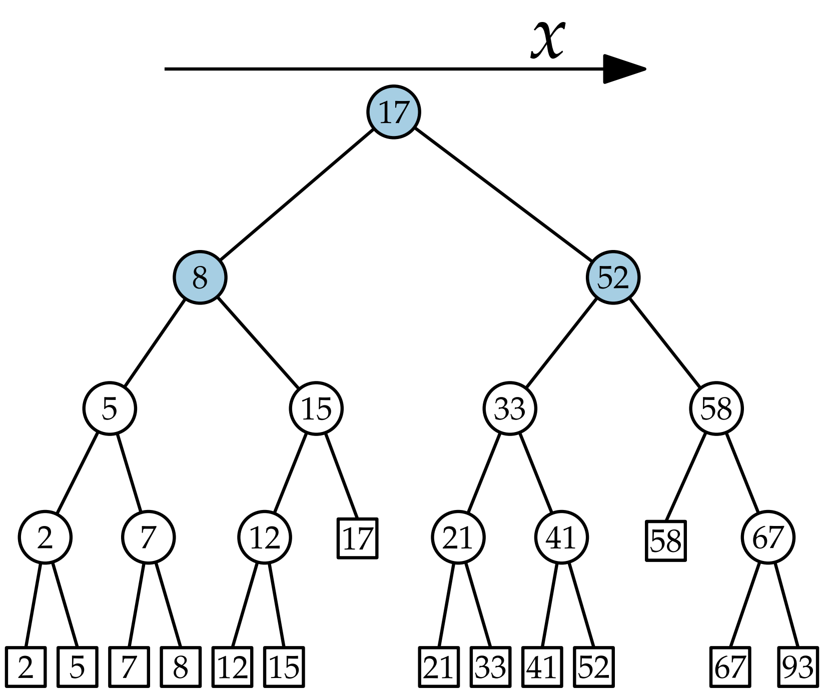

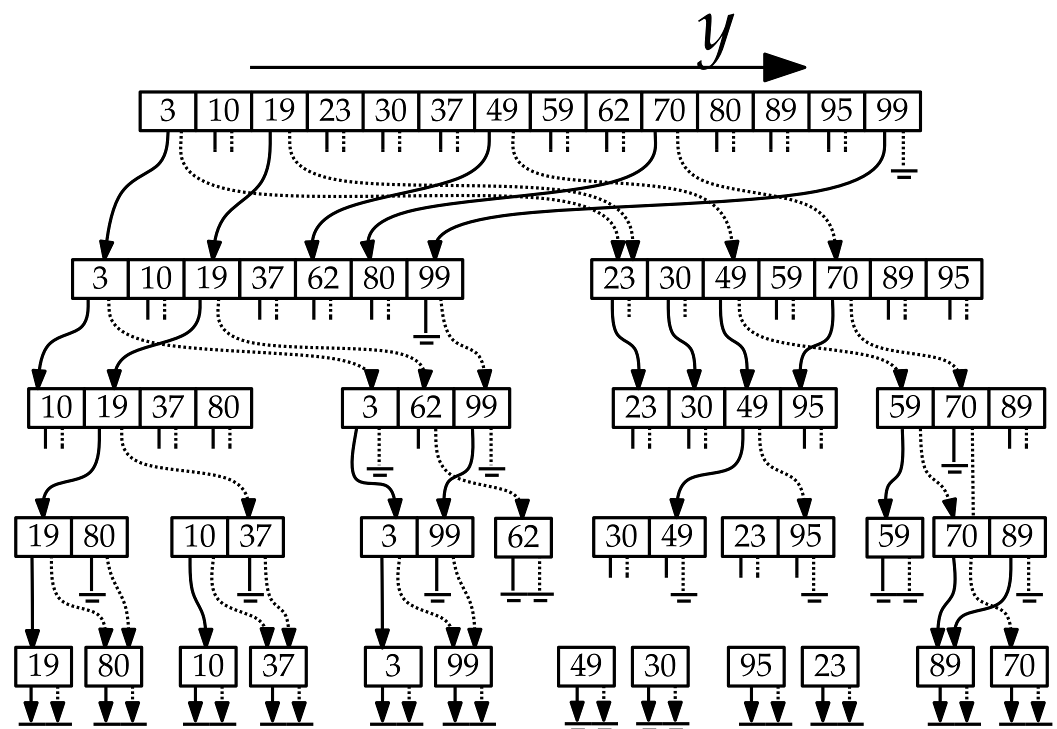

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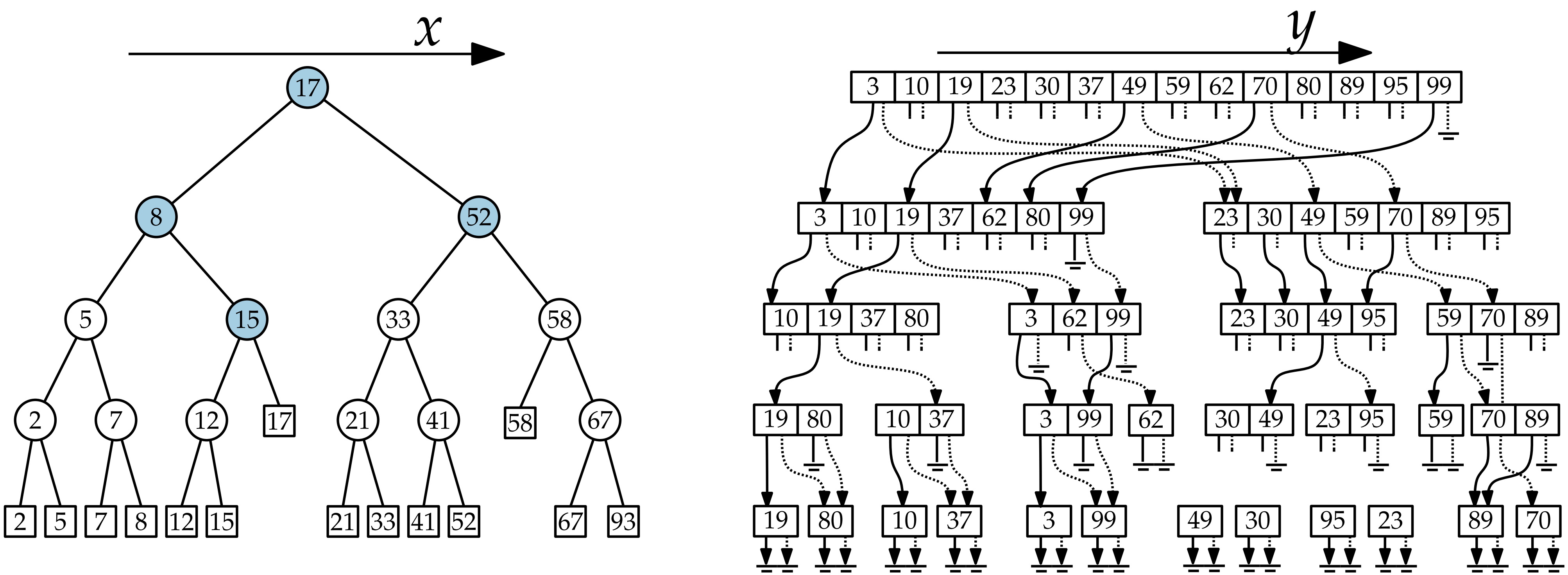

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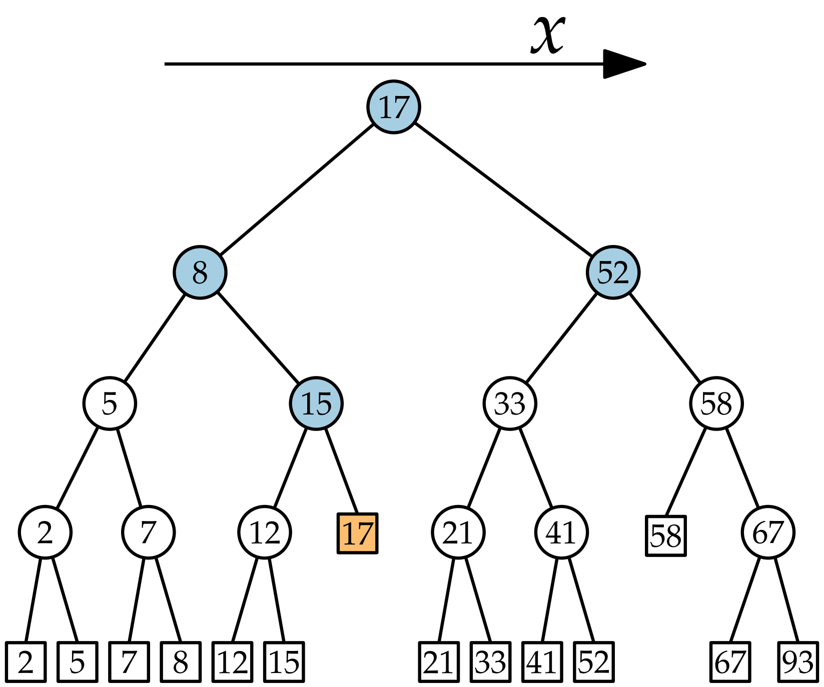

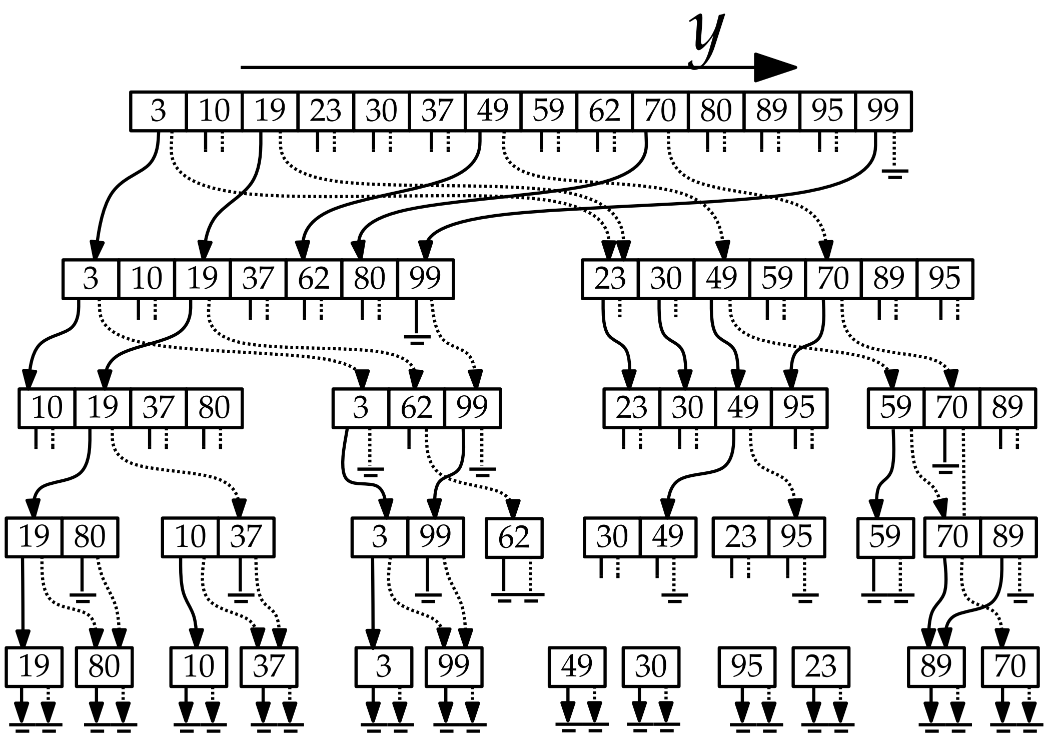

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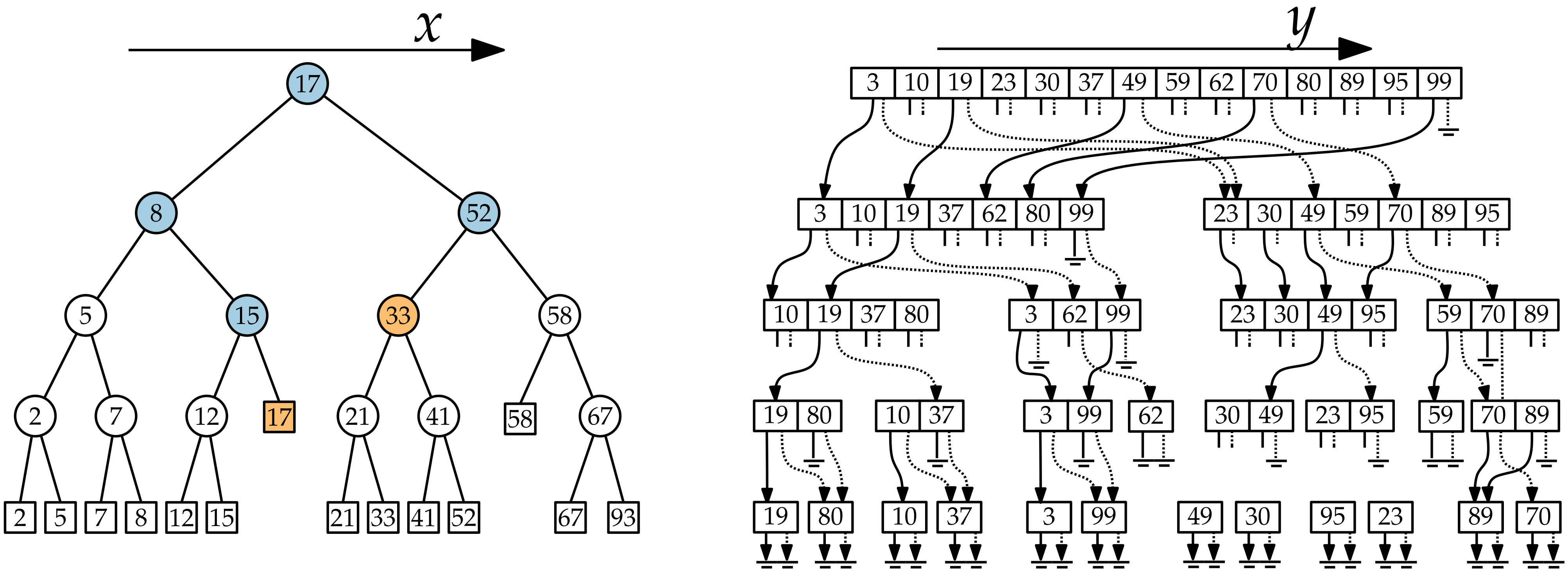

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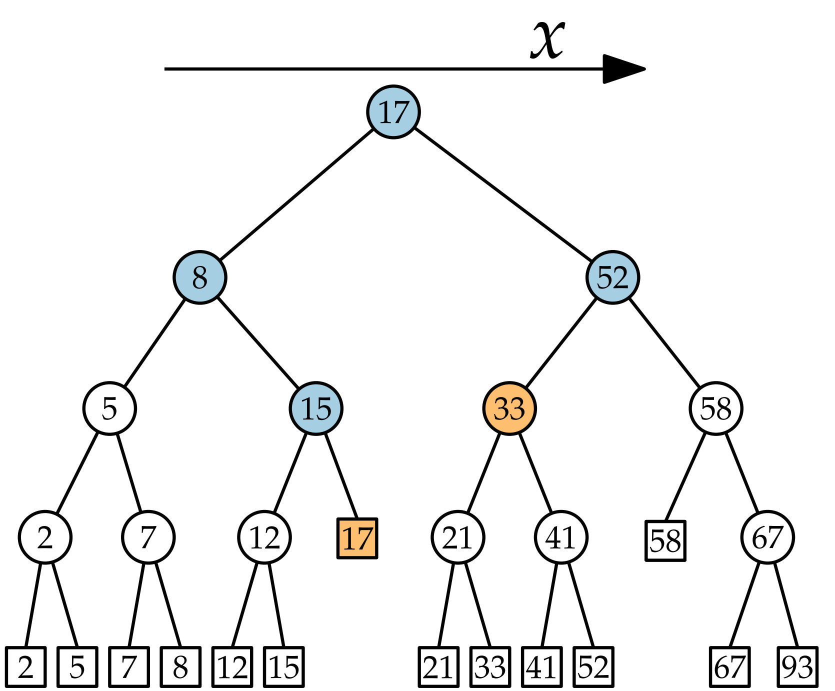

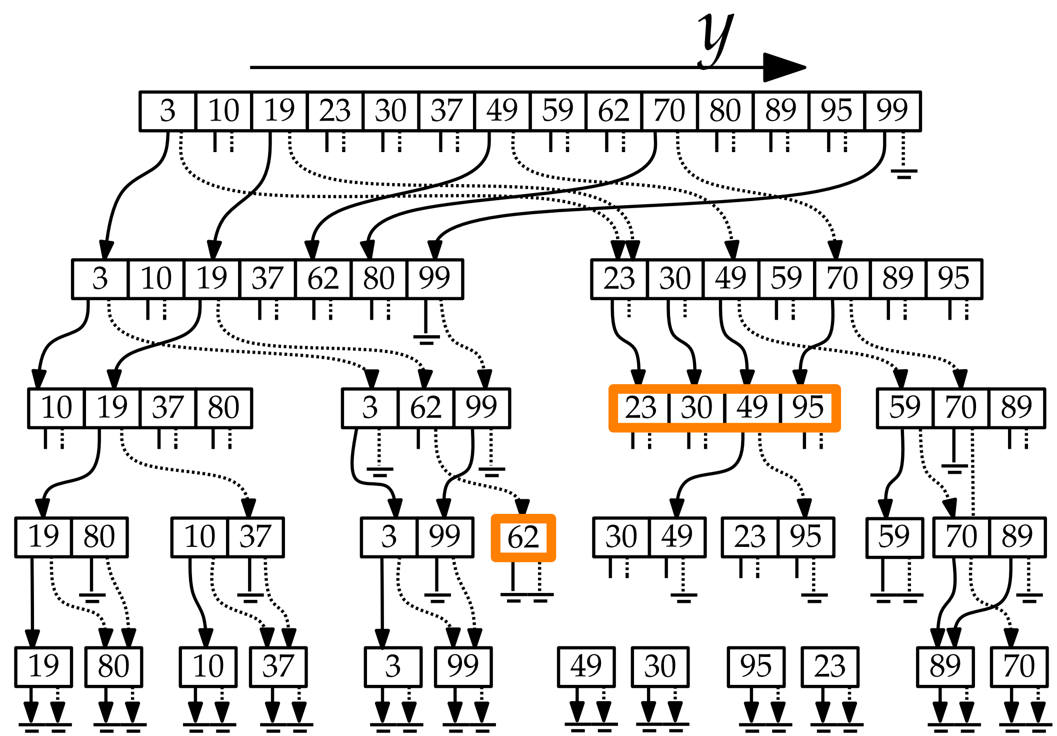

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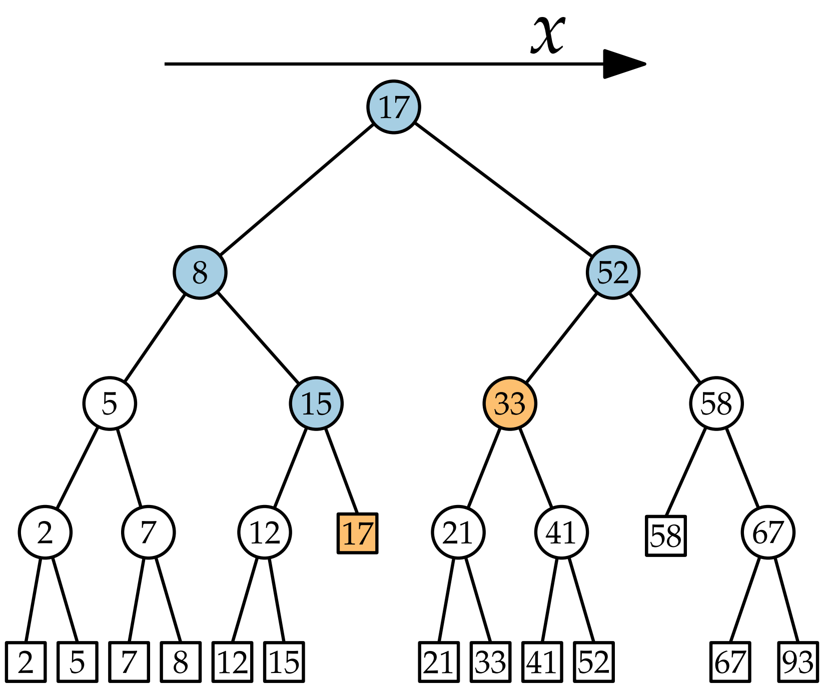

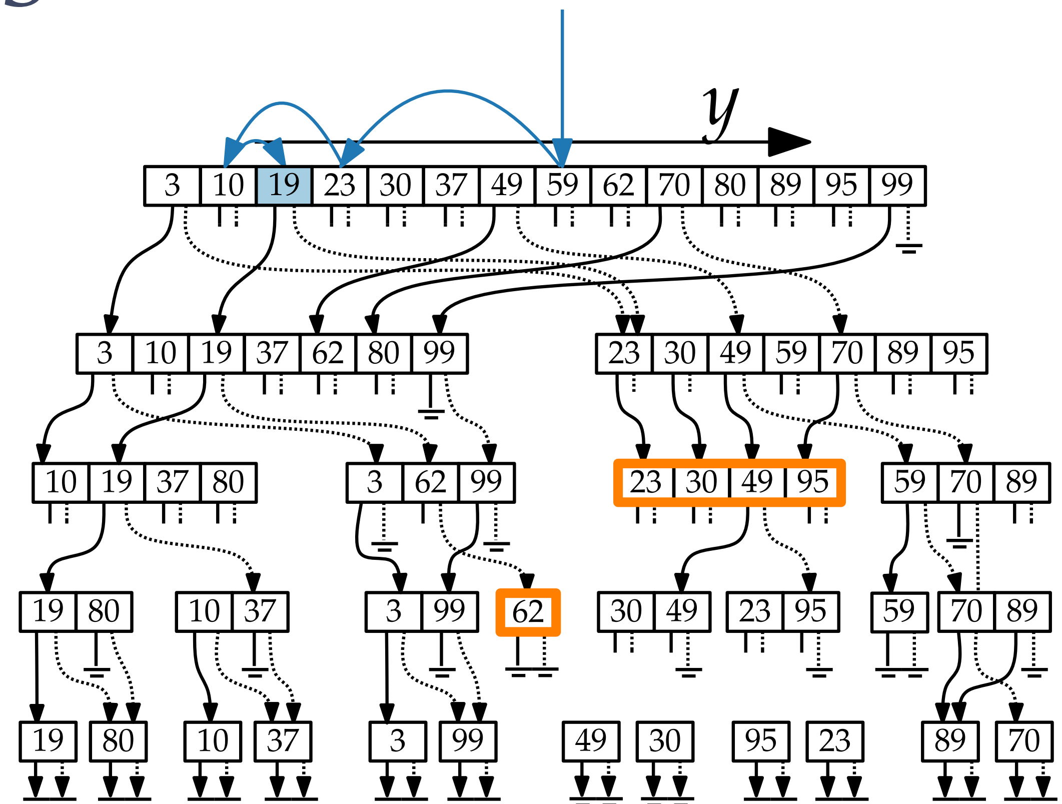

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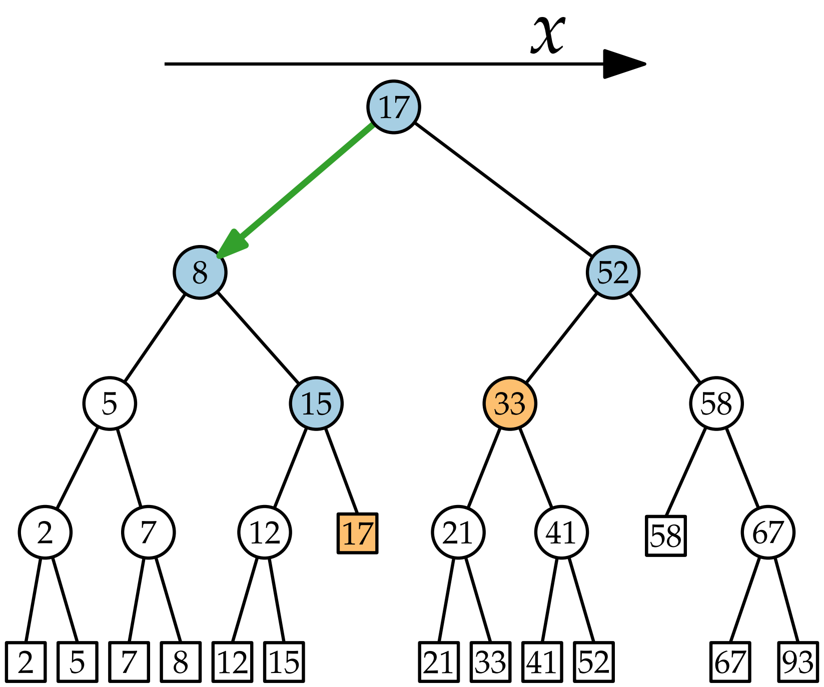

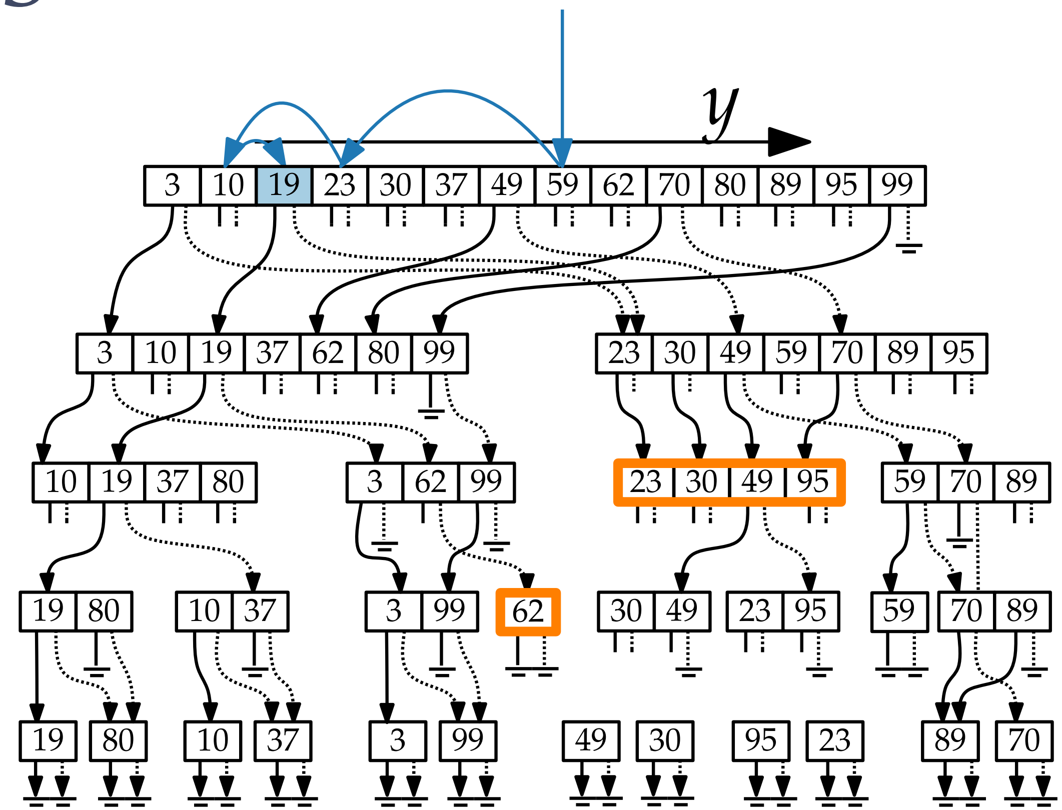

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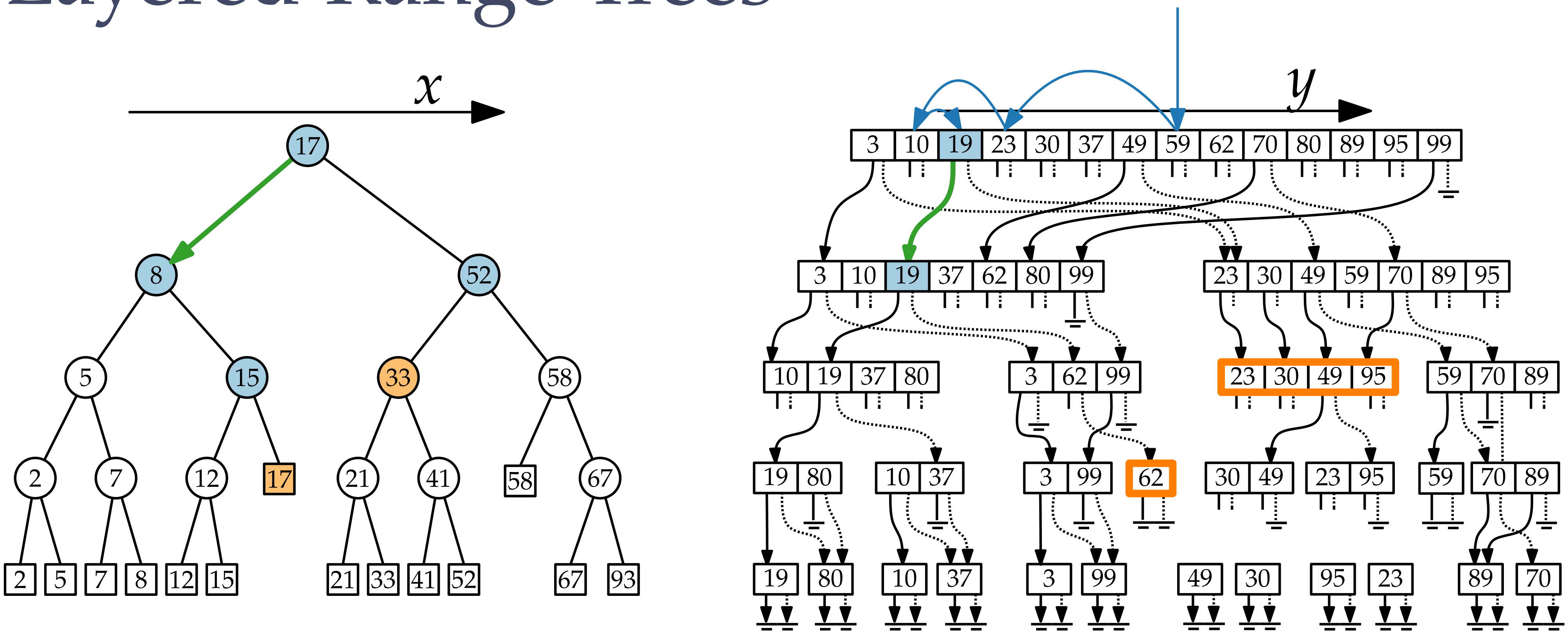

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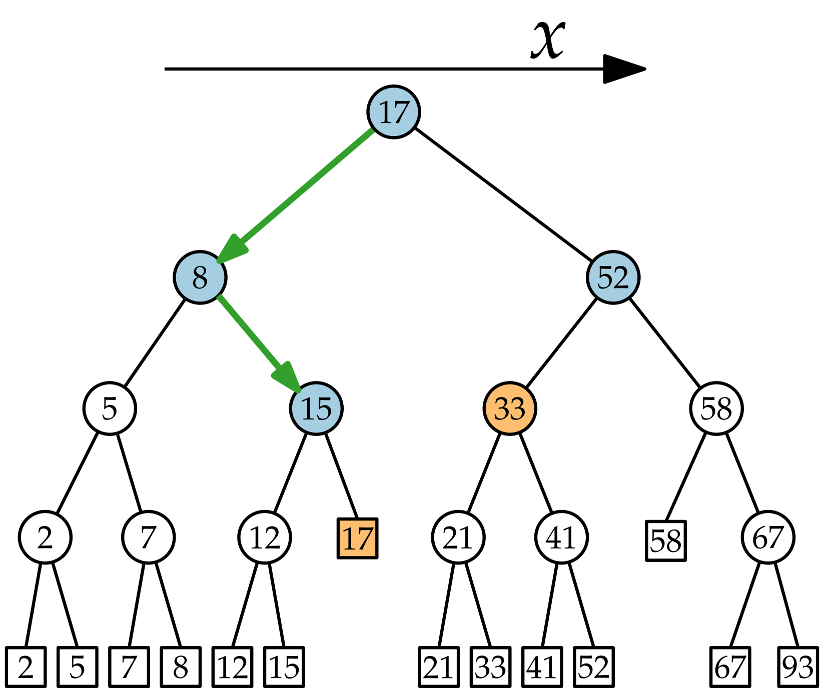

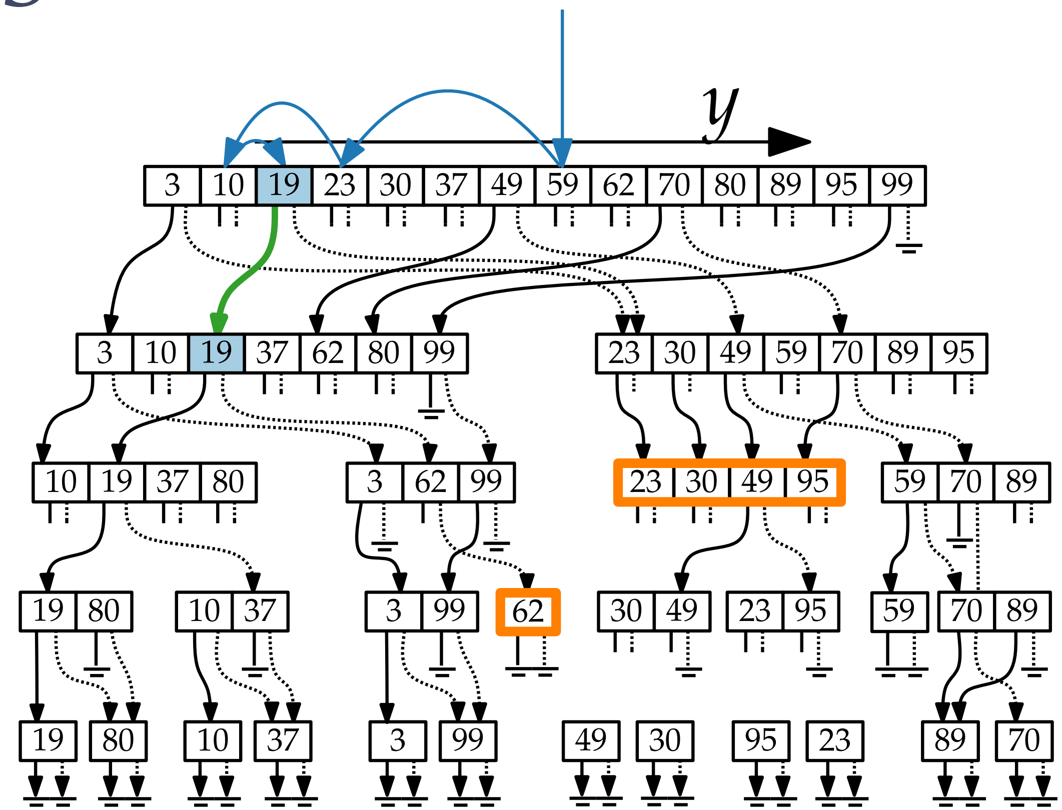

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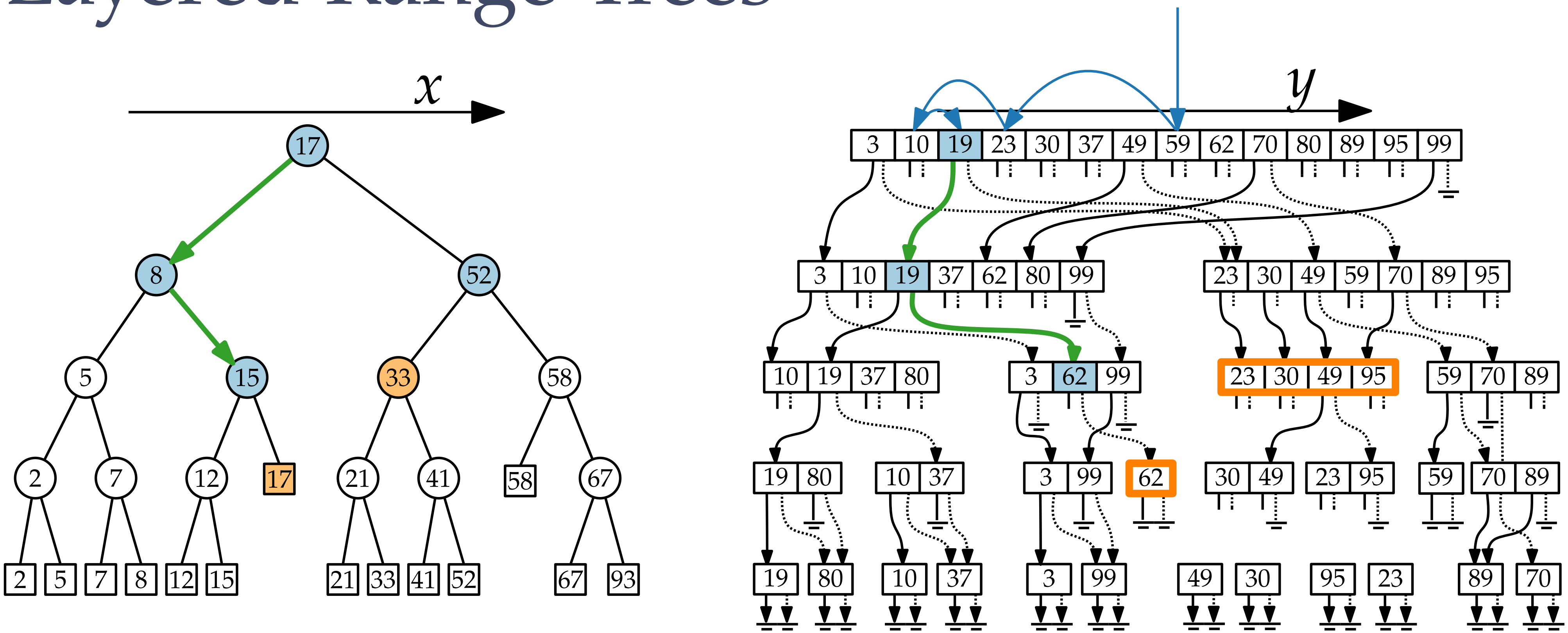

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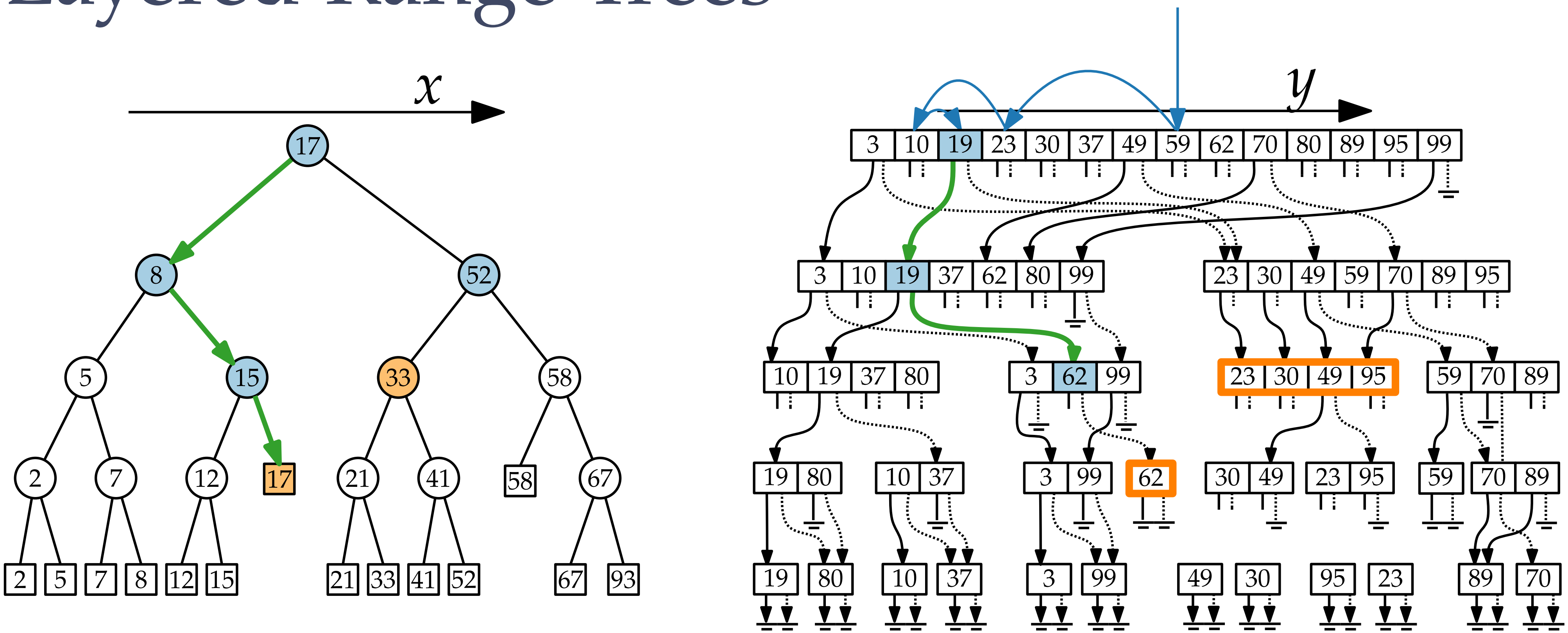

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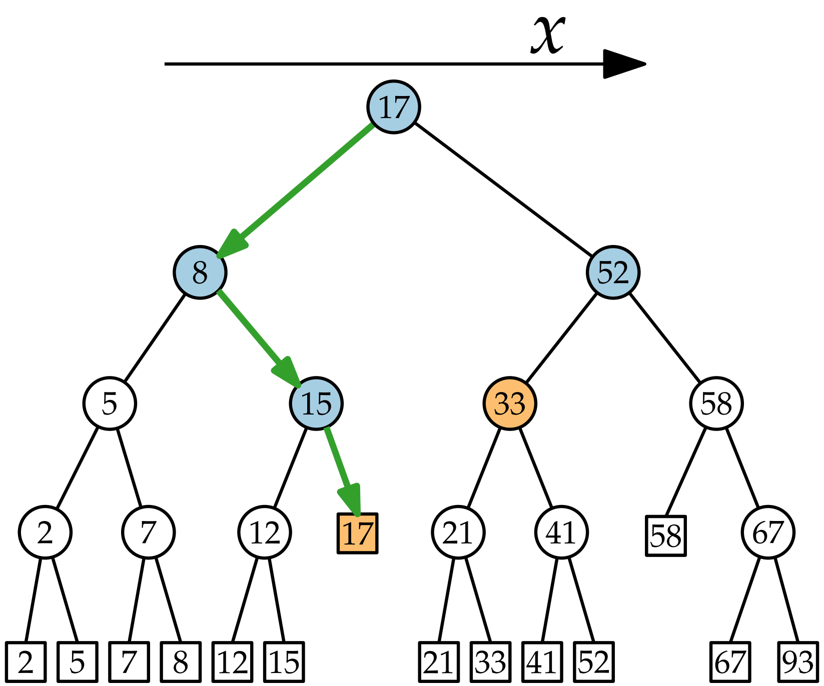

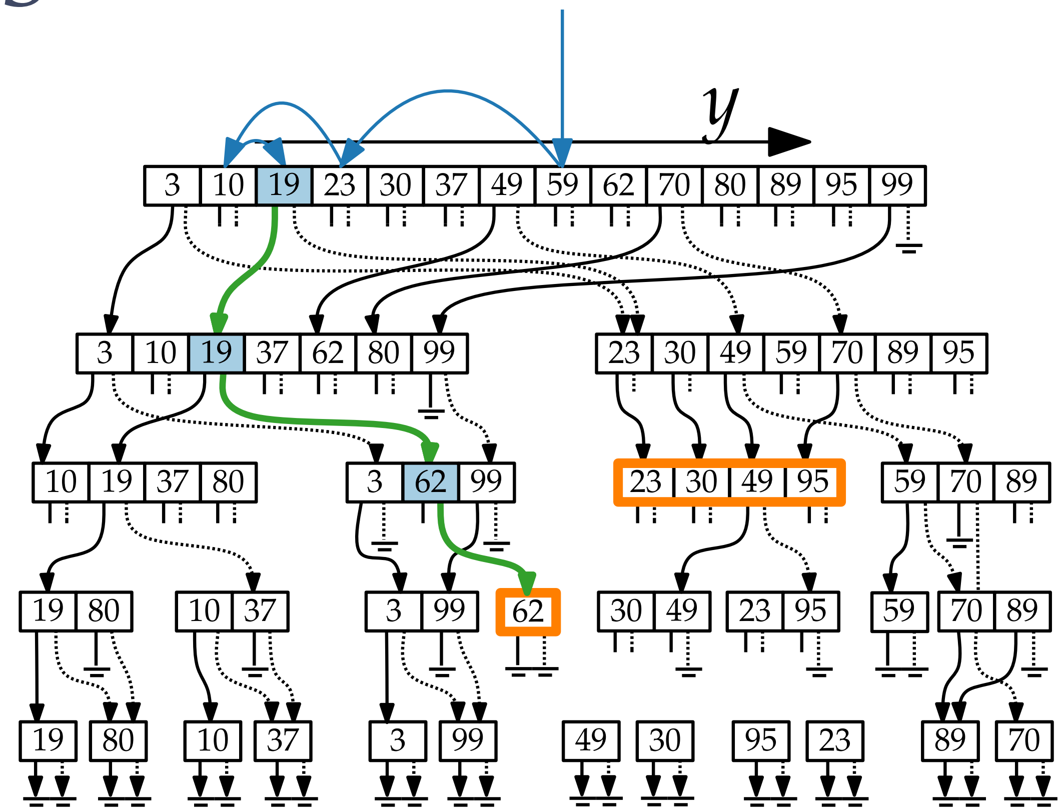

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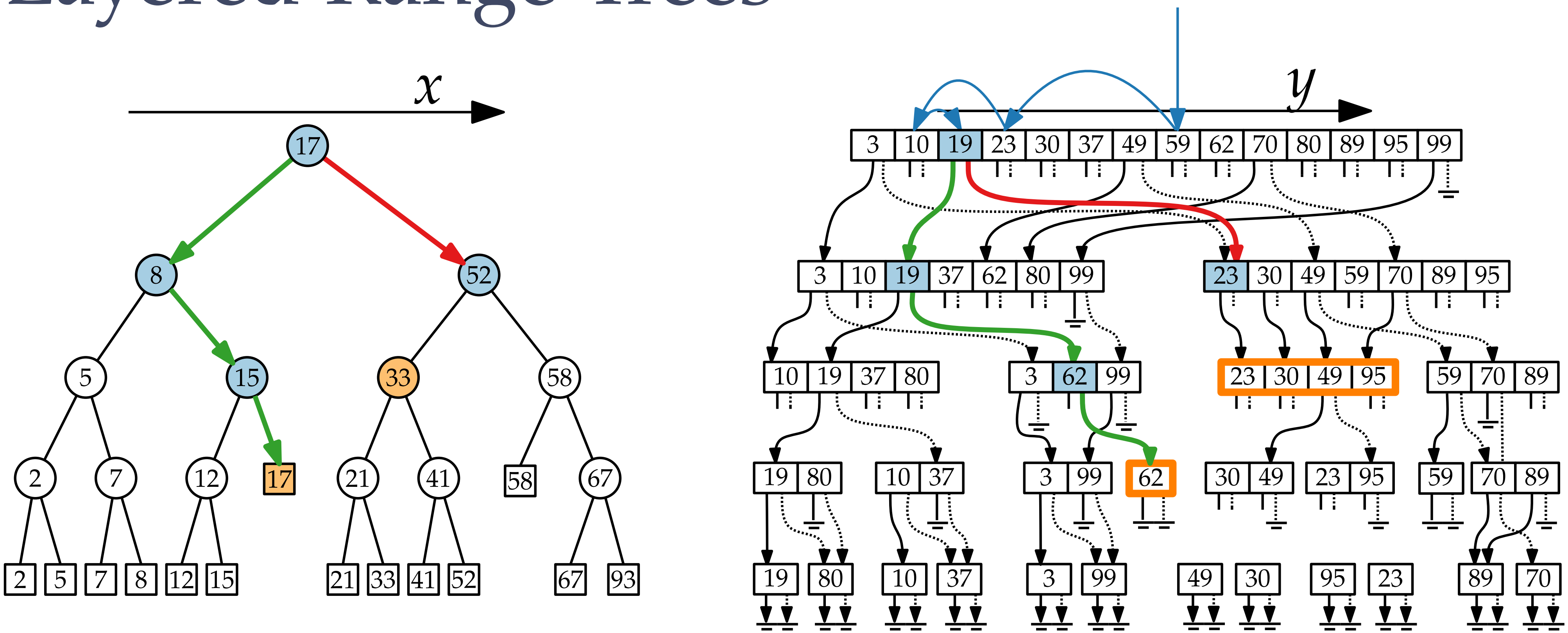

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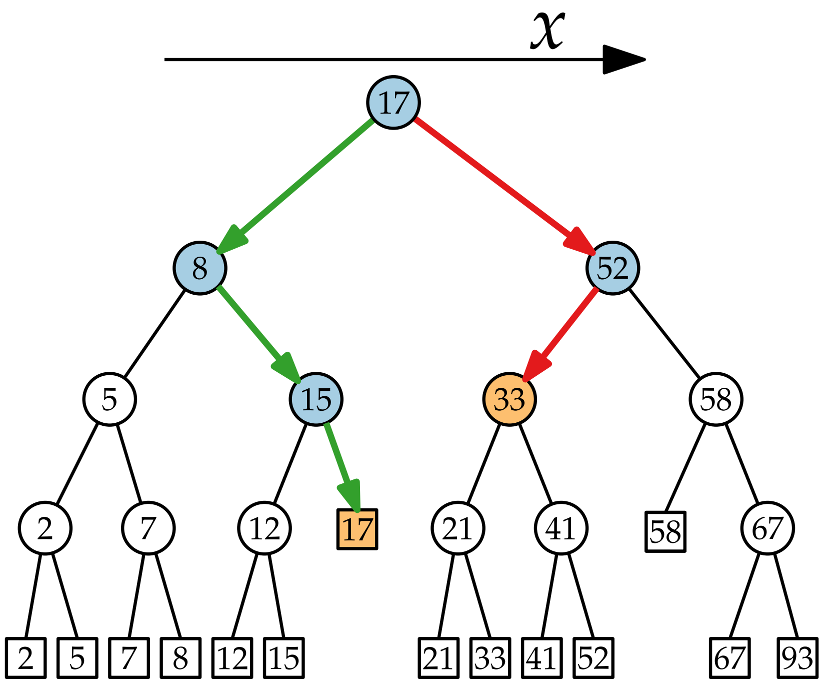

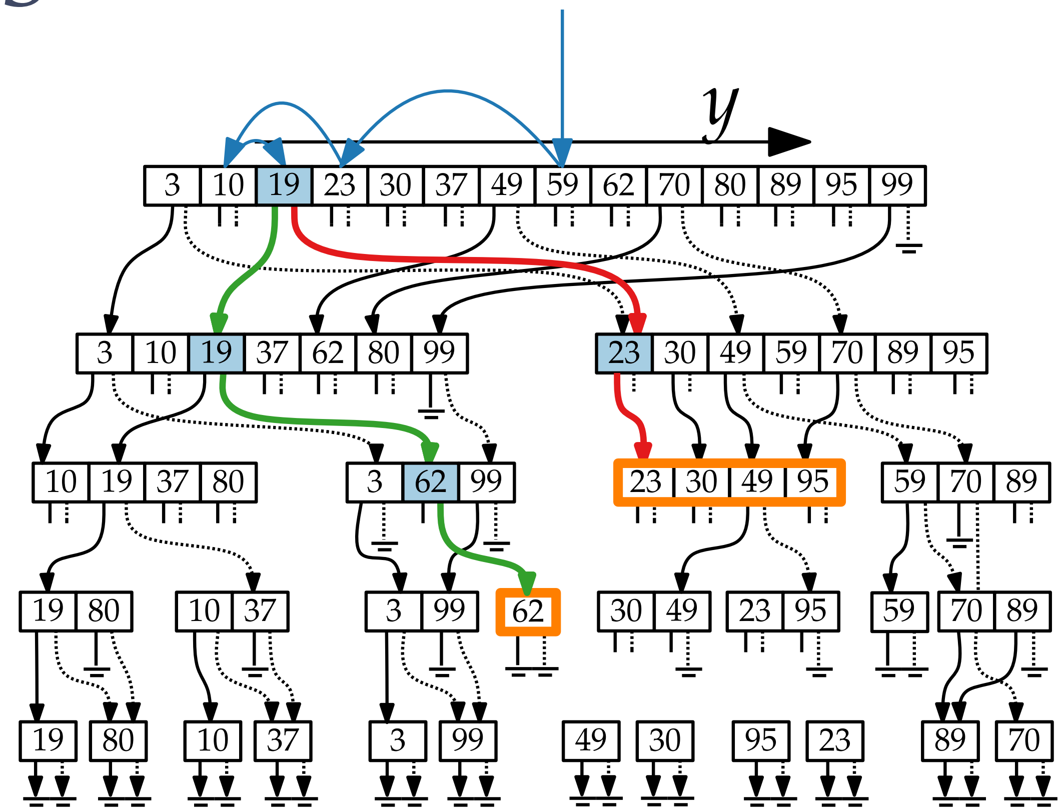

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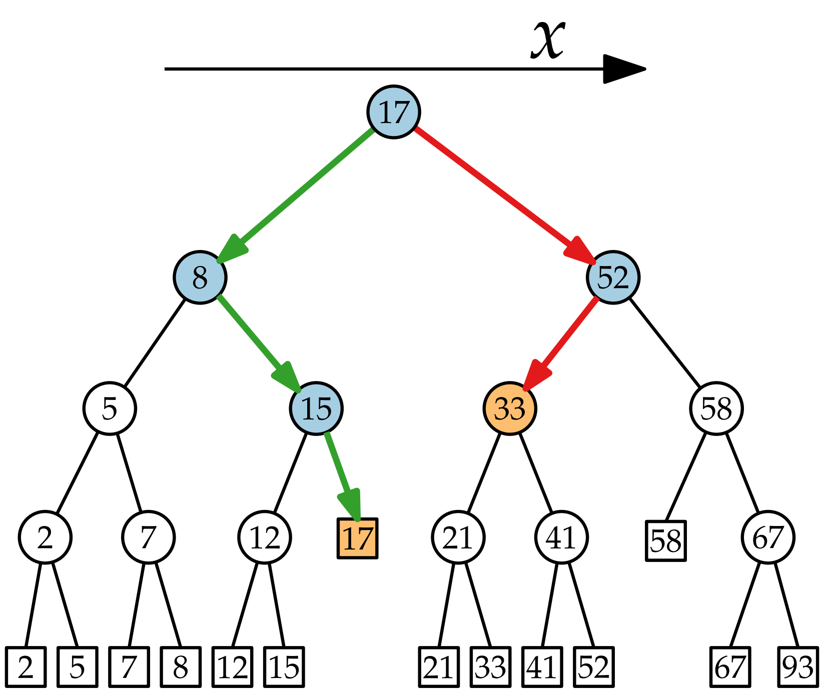

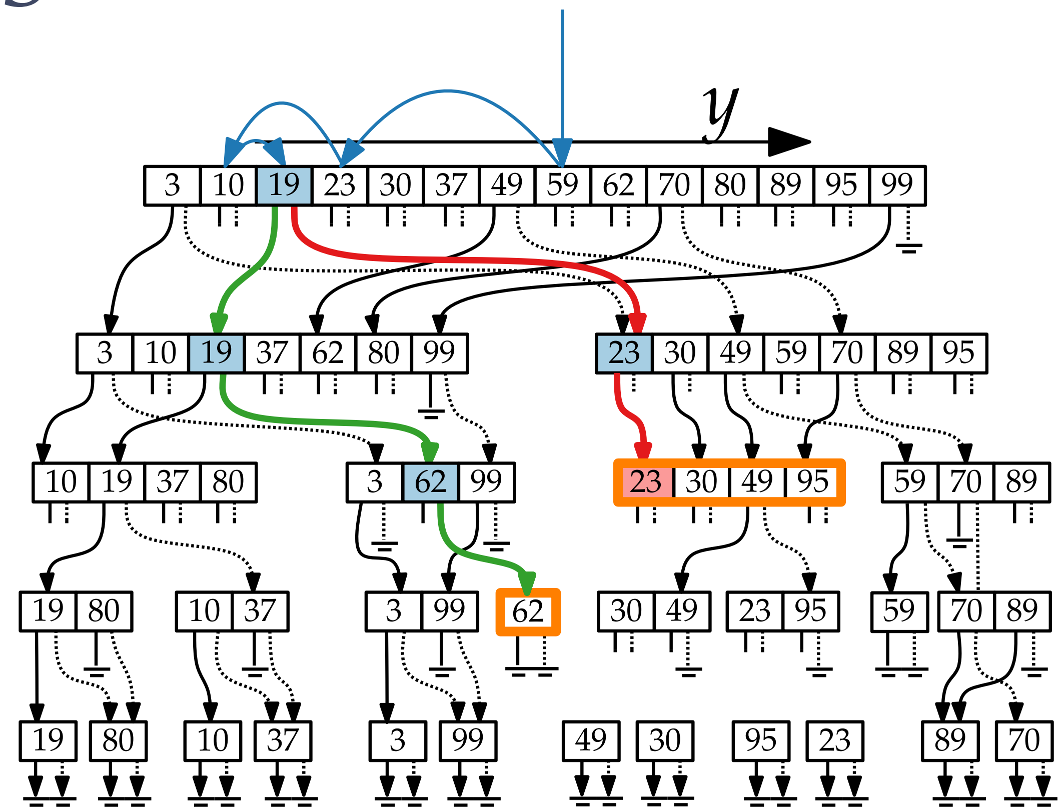

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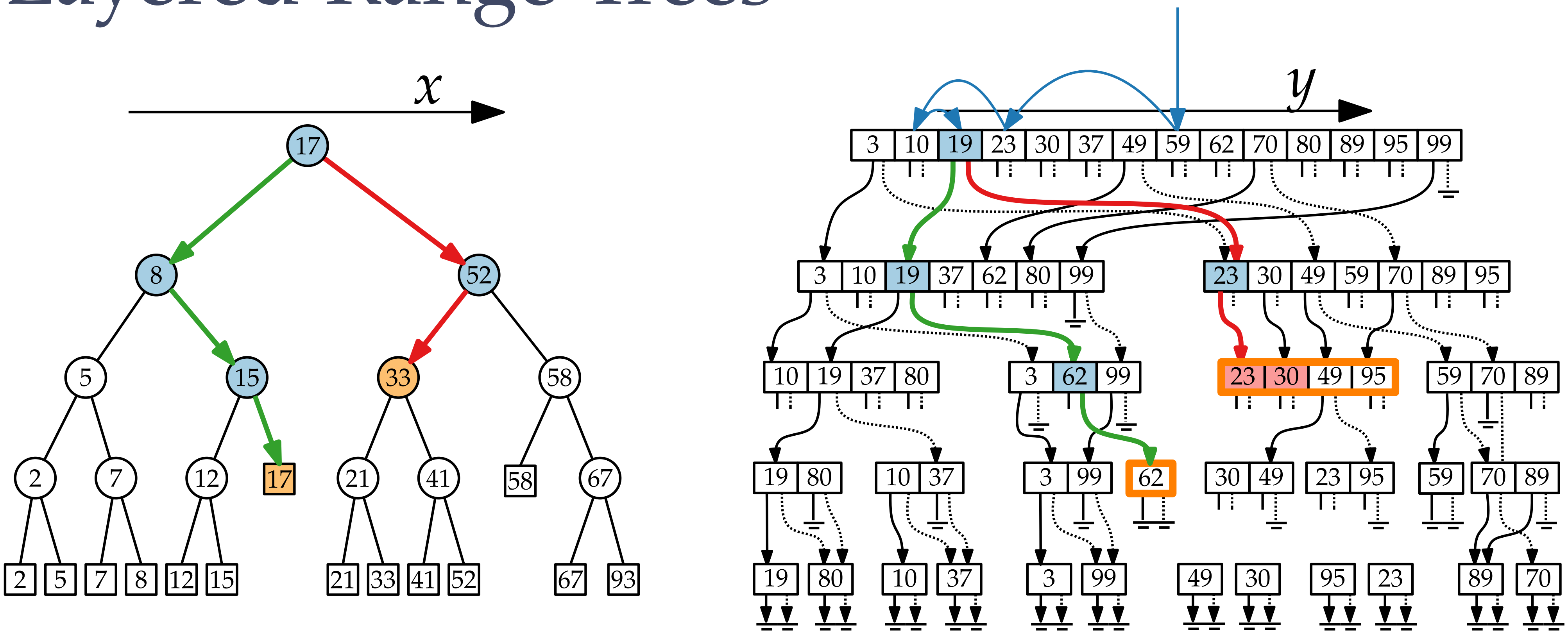

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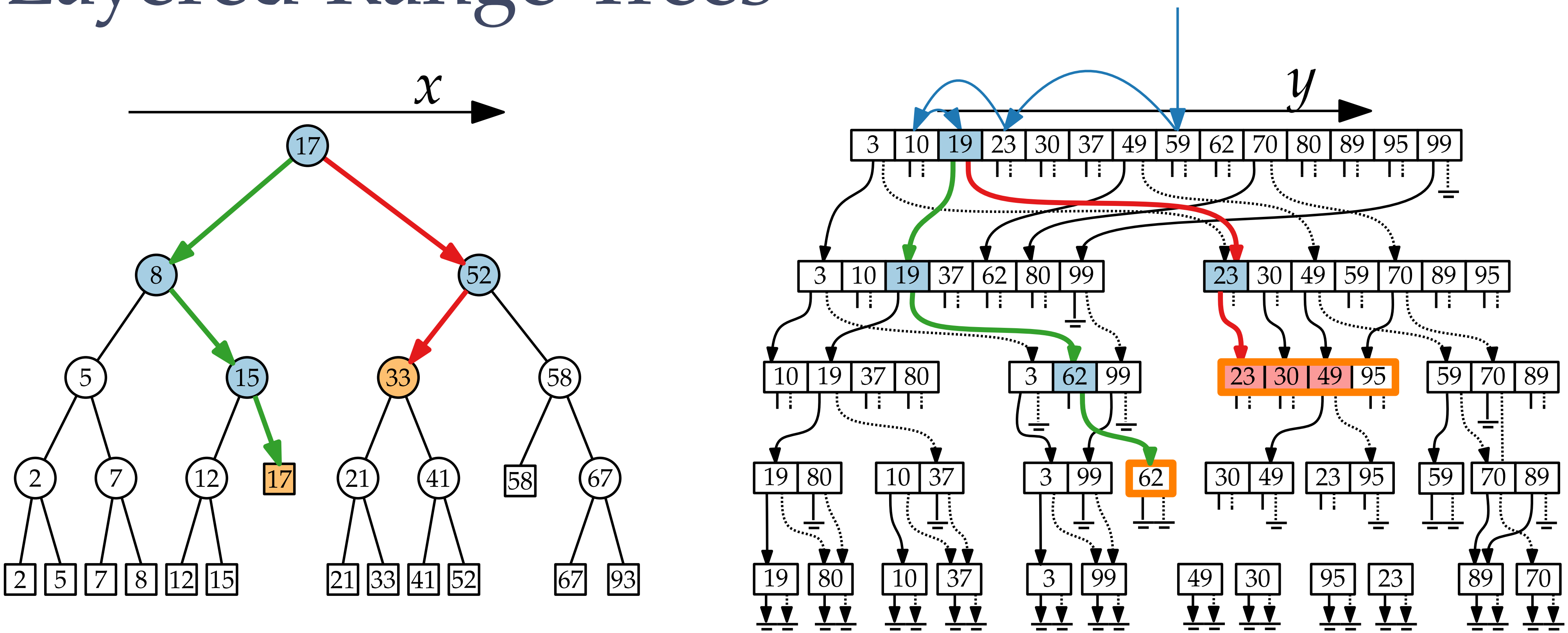

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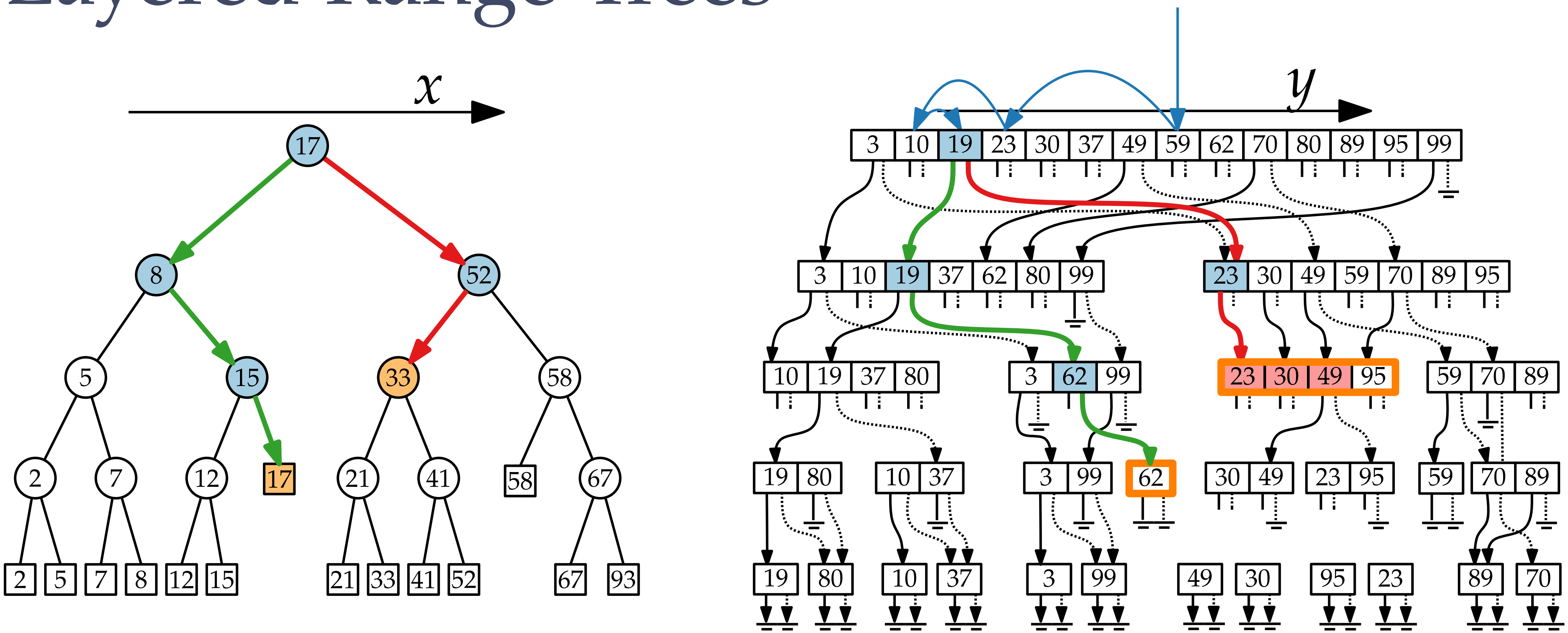
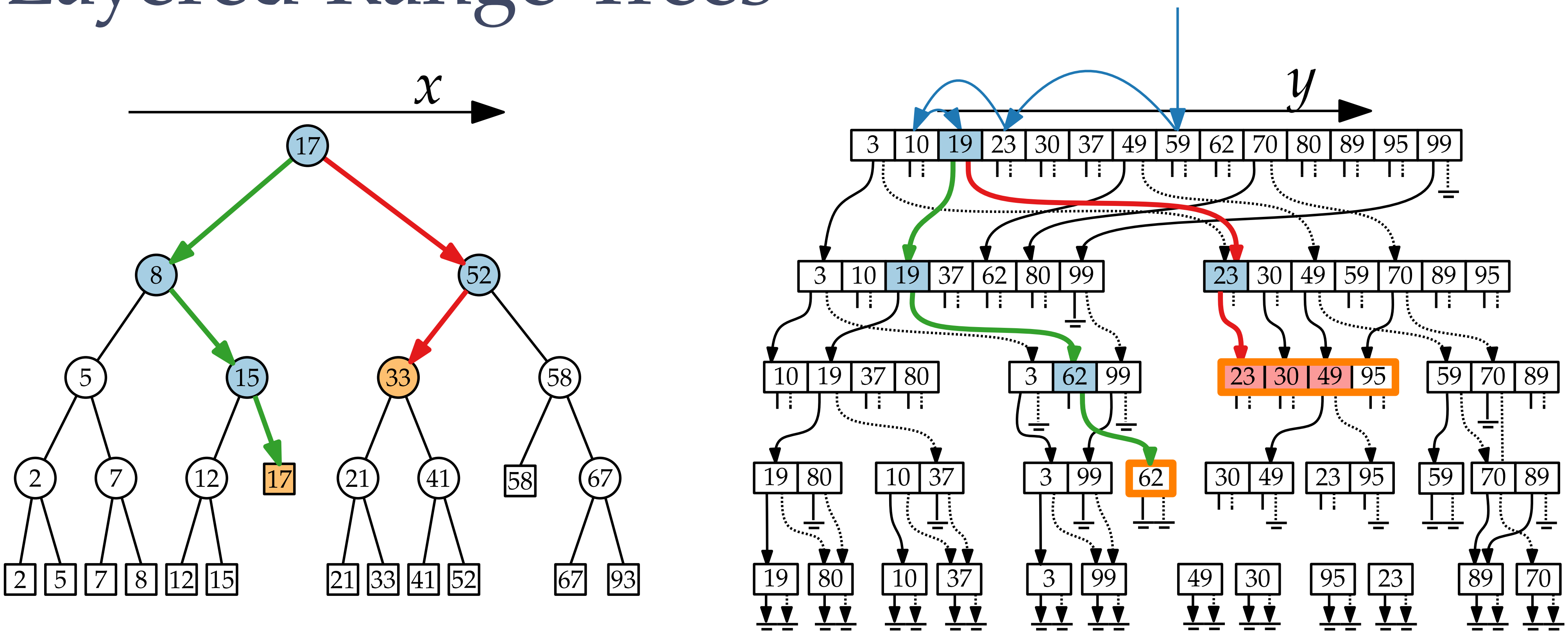


$[16, 53] \times [18, 60] \rightarrow (21, 49), (33, 30), (52, 23)$

Layered Range Trees



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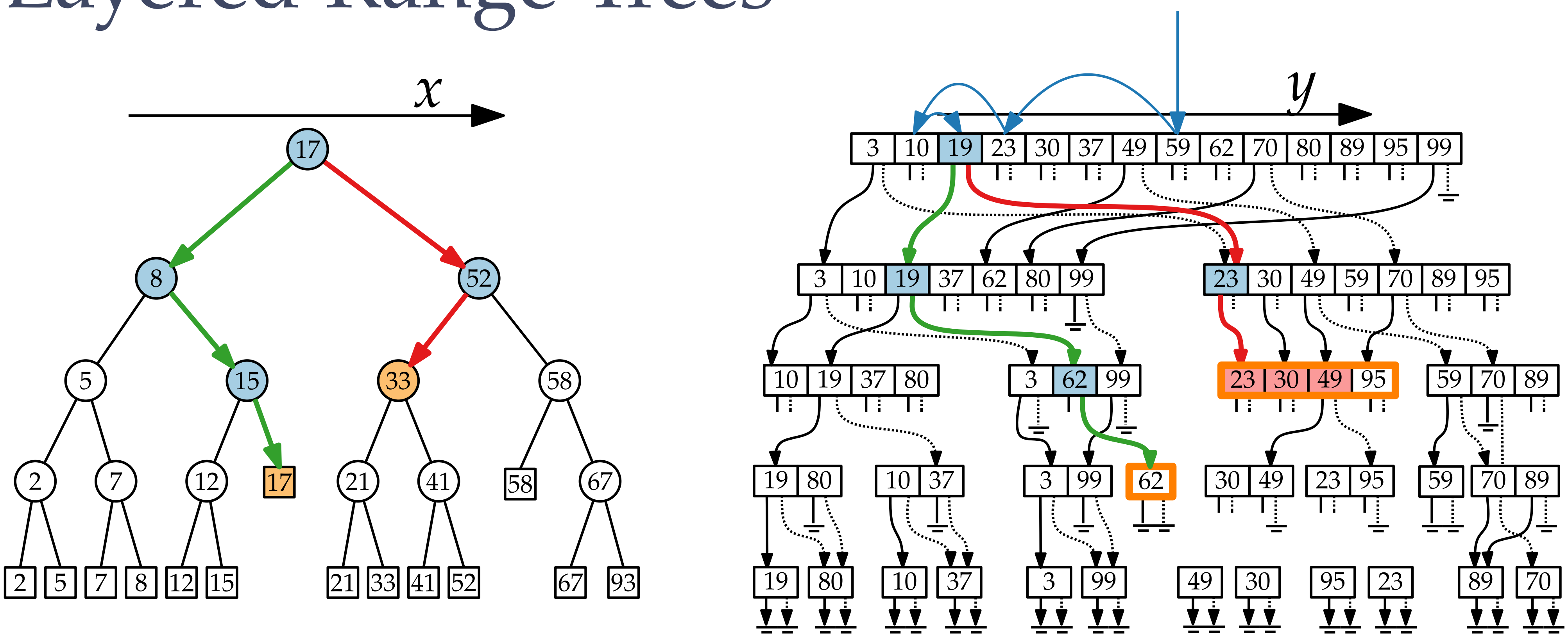
Theorem: Let $d \geq 2$ and let P be a set of n pts in $\mathbb{R}^d$. Given $O(n \log^{d-1} n)$ preprocessing time & storage, d-dim range queries on P can be answered in $O(k + \log^{d-1} n)$ time.

Layered Range Trees



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