Pascal’s triangle

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Nowadays, students of mathematics learn about Pascal’s triangle in their first classes at university. It is often used to show them a simple example of a proof by using mathematical induction. Even historically Pascal’s triangle, or at least its proof by Blaise Pascal, marked the birth of this principle.
However, the development of this triangle goes back much further. For example, mathematicians in ancient India had already made some advancements in its discovery. Furthermore, the triangle appeared in a report by the Chinese Yáng Hui in 1261. Yáng Hui himself attributed the triangle to Jiǎ Xiàn, who lived in the eleventh century. The Chinese used the triangle to generate the binomial coefficients[1] and to compute roots, mostly square and cubic ones, so the first approach to Pascal’s triangle was a rather algebraic one. In his work, Yáng Hui described a method to create the triangle: ”Add the numbers in the two places above in order to find the number in the place below” [Katz, 2009, p. 213]. In Hui’s report, the triangle appears to a depth of six; Zhū Shìjié extended the triangle to a depth of eight in his own work from 1303 (see Figure 1), but no explicit attempt was made to expand the triangle to an arbitrary depth.

Other approaches were taken in the Islamic countries during the Middle Ages similar to the one used in China. The names al-Karajī, al-Samaw’al and Khayyam have to be mentioned, as they played a huge role in developing the binomial theorem. In Europe, Levi ben Gershon made enormous progress in 1321, when he gave an explicit formula to compute the binomial coefficient $\left(\binom{n}{k} = \frac{n!}{(n-k)!k!}\right)$. Ben Gershon also said that the number $\binom{n}{k}$ is ”the number of combinations of n things take k at a time” [Stillwell, 1989, p. 136]. In his studies, ben Gershon made a first attempt towards a new method for proving

[1] That is the coefficients in the expanding of the expression $(a+b)^n$.

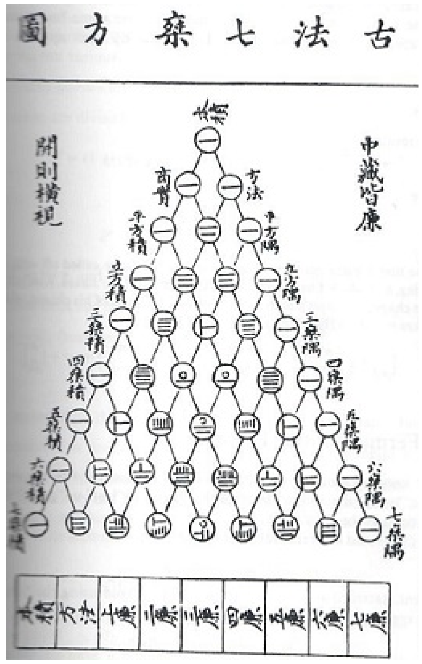

Figure 1: Chinese form of Pascal's triangle, [Stillwell, 1989, p. 137]

theorems, namely mathematical induction, but he was unable to complete the formulation of this new concept. After that, the triangle nearly fell into oblivion.

It wasn't until 1654 that Blaise Pascal rediscovered the triangle, which is named after him. In fact, he did not just rediscover it but improved upon it as well. In his *Traite du triangle arithmetique*, he combined the algebraic theories of the Chinese and Persians with the combinatorial approach. By doing so, Pascal established mathematical induction as a way of proving an assertion for an arbitrary n. Of course, the proof was not done as we would do it now. Pascal's original triangle (he called it the "arithmetique triangle") did not even look like it does today. However, if we rotate it by an angle of 45 degrees, the modern and the original triangle are the same. In Pascal's time there was nothing like double indices to specify the entries in the triangle, so Pascal instead used Greek and Latin letters to point out an explicit entry (see Figure 2). Pascal showed the regulation for the building of his triangle for an explicit case, ω (this is the fourth entry in the third row). But he mentioned that the same argumentation holds for every other element in the triangle. Furthermore, for his theorems concerning the triangle, he assumed that some assumption were true for three subtriangles and concluded that it had to be true for the fourth, surely again using some explicit examples. But that was the birth hour of the principle of mathematical induction.

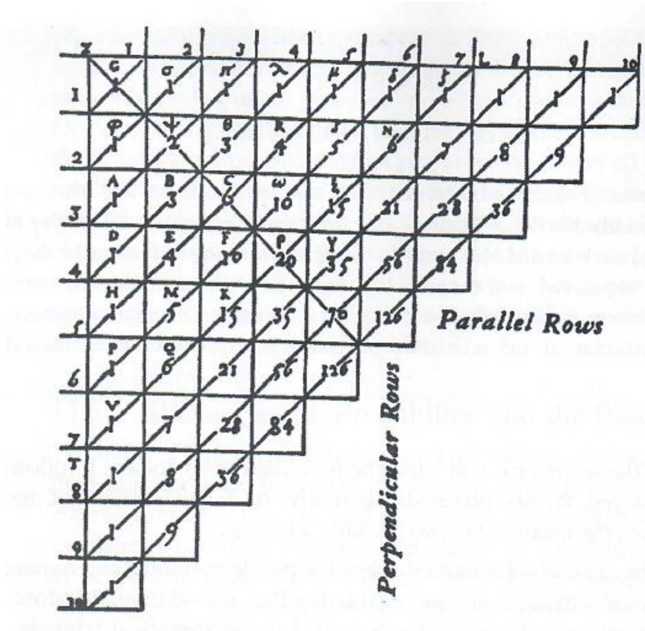

Figure 2: Pascal's original triangle, [Katz, 2009, p. 492]

After 1654, Pascal nearly withdrew from any studies in the field of mathematics. Still his ideas, especially about the triangle, were picked up later. The two British mathematicians and philosophers John Wallis and Isaac Newton recognized Pascal's triangle during their work. Even Gottfried Leibniz made use of Pascal's triangle, though he also created a new figure, the "harmonic triangle" (which is defined similarly to the "original" triangle but with quotients instead of just sums).

References

[Katz, 2009] Victor Katz, A history of Mathematics - an introduction, Addison-Wesley, Boston, 2009

[Stillwell, 1989] John Stillwell, Mathematics and its History, Springer Verlag, New York, 1989