THE THEOREM OF PYTHAGORAS

"To this day, the Theorem of Pythagoras remains the most important single theorem in the whole of mathematics" Jacob Bronowski, the Ascent of Man p.160 from preface of [Maor, 2007].

The Theorem of Pythagoras is maybe the most widely known theorem, even by non-mathematicians, who remember it from their school days. Despite the name, it would be unfair to give Pythagoras alone the credit for the discovery of the relation between sides of right triangles. In fact, there are similar results in most of the ancient cultures with advanced skills in mathematics. And although Pythagoras was the first to give a formal proof of it, no original written text about it survived to the present time, except for *the Elements*, which was written by Euclid a couple of centuries later.

The oldest evidence of some kind of knowledge about the relation between sides in right triangles and the sums of squares can be found on Babylonian tablets. The most famous examples are the Yale tablet YBC 7289 and the tablet Plimpton 322. YBC 7289 is relevant not only because it lets us know that the Babylonians were able to compute the square root of two with considerable accuracy, (while the Greeks barely accepted the existence of such numbers, calling them irrationals), but also because it was probably a kind of sample problem that explained how to compute the length of the diagonal of every square. Instead of computing the square root of the sum of the squares of the sides, one just had to multiply the side by $\sqrt{2}$. This kind of simplification shows their skill in elementary geometry.

Other similar geometrical computations can be seen in tablets like the Susa tablet and the Tell Dhibayi tablet[1]. Even disregarding the geometrical aspect, the Babylonians might have known Pythagorean triples also because of their algebraic relation, or it can be said that at least they studied some of those triples. Indeed many of the triples are written on Plimpton 322 in the collection of G.A. Plimpton in Columbia University. "As to how the Babylonian mathematicians found these triples – including such enormously large ones as (4601, 4800, 6649) – there is only one plausible explanation: they must have known an algorithm which, 1500 years later, would be formalized in Euclid's *Elements*: Let u and v be any two positive integers, with $u > v$; then the three numbers $a = 2uv$, $b = u^2 - v^2$ and $c = u^2 + v^2$ form a Pythagorean triple"[Maor, 2007]. Whether Egyptians also knew Pythagorean relations is still an open discussion; in fact, most historians point to the anciant profession of "harpedonaptai", a rope stretcher who used a rope with 3+4+5 knots as a tool for measuring right angles in constructions like the pyramids. Others like Thomas Little Heath wrote, "There seems to be no evidence that they knew that the triangle (3,4,5) is right-angled; indeed, according to the latest authority (T. Eric Peet, The Rhind Mathematical Papyrus, 1923), nothing in Egyptian mathematics suggests that Egyptians were acquainted with this or any special cases of the Pythagorean Theorem"[Little Heath, 1962].

"It is from Proclus's commentary (to the *Elements*, ca. 400 AD) that we may infer a possible proof of the Pythagorean theorem by Pythagoras, namely, a proof

[1] For more details we remand to [JOC/EFR, 2000].

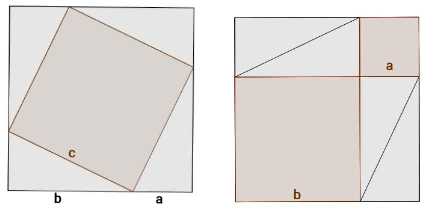


FIGURE 0.1. Pythagoras's proof by Proclus

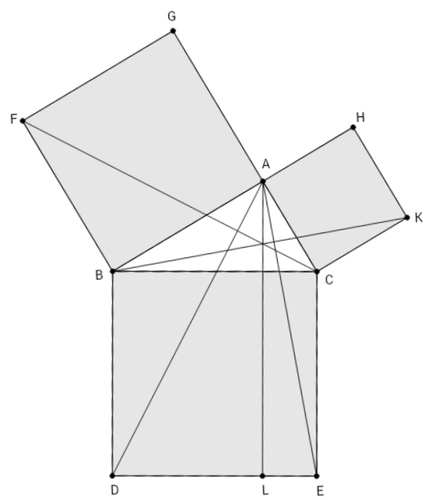
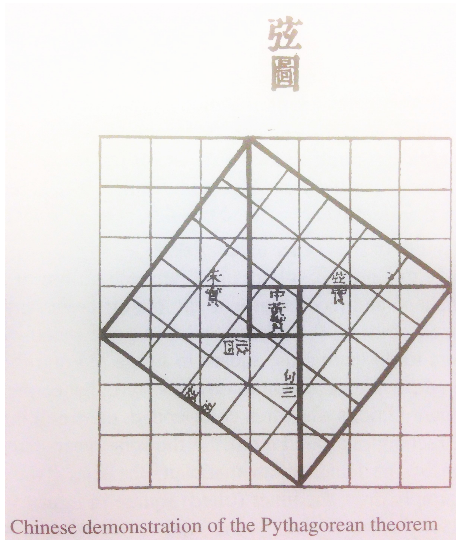


FIGURE 0.2. Pythagoras's proof in *The Elements*

by dissection: Consider a square of side a+b. Connect the dividing points between segments a and b of each side to form a tilted square, and call its side c. The original square is thus dissected into five parts – four congruent right triangles of sides a and b and hypotenuse c, and an inner square of side c. A different dissection is shown in Figure 0.1. Comparing the areas of the two figures, we have $4\frac{ab}{2} + c^2 = 4\frac{ab}{2} + a^2 + b^2$, from which we get $c^2 = a^2 + b^2$" [[Maor, 2007], p. 61]. This proof is different from the proof of the same theorem, namely Proposition 47, Book I, presented by Euclid in his *Elements* (300 BC), which used congruences of triangles and areas of constructed rectangulars (Figure 0.2).

Proclus's version is essentially the "Chinese proof" of the same theorem, which is called kou-ku, from the names kou (meaning width) and ku (meaning length). In fact, in near isolation from the West, the kingdom of China also had developed its own highly advanced culture. Unfortunately throughout Chinese history, lot of ancient works have been lost. One of the oldest Chinese mathematical works known to us is the Chao Pei Suan Ching ("The arithmetical classic of the gnomon and the circular paths of heaven"), which was written between 200 BC and 200 AD. It is an astronomical work about the calendar, but it also contains some material on earlier Chinese mathematics. Here, in the form of a dialogue between the ruler



Chinese demonstration of the Pythagorean theorem

FIGURE 0.3. from the book [Maor, 2007] page 63

Chou Kung and a scholar Shang Kao, we find the theorem of Pythagoras stated as follows: "Multiply both the height of the post [the gnomon] and the shadow length [the base] by their own values, add the squares, and take the square root." Then a diagram (Figure 0.3) is explained by the note: "Thus, let us cut a rectangle (diagonally), and make the width (kou) 3 (units) wide, and length (ku) 4 (units) long. The diagonal (ching) between the (two) corners will then be 5 (units) long. Now after drawing a square on this diagonal, circumscribe it by half-rectangles like that which has been left outside, so as to form a (square) plate. Thus the (four) outer half-rectangles of width 3, length 4, and diagonal 5, together make (techheng) two rectangles (of area 24); then (when this is subtracted from the square plate of area 49) the reminder (chang) is of area 25. This (process) is called 'piling up the rectangles' (chi chu)" [Needham]. The same diagram can be found as a "proof without words" a thousand years later in the book "The Learned" by the Hindu mathematician Bhaskara.

Even in western culture, mathematicians didn't stop proving this theorem, and a lot of different proofs have been written. It has also been conjectured by Elisha Scott Loomis[2] (1852-1940) that in the Middle Age finding a new proof was a required task for graduating with a Master's degree in mathematics. This is evidence of the "immortal" interest about this simple proposition, which is indeed not only related to right triangles but also to the concept of perpendicular segments or vectors. Therefore, this theorem continues to gain interest of mathematicians and physicists in the 21st century while studying the curvature of space.

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[2]Elisha Scott Loomis wrote a book collecting all known proofs (371 of them) and writing them up in "The Pythagorean Proposition (1927)" [Maor, 2007]

[Little Heath, 1962] The Thirteen Books of Euclid's Elements, vol. 1, Thomas Little Heath, London, Cambridge University Press, 1962, p 352; in [Maor, 2007], p.15

[Needham] [Science and Civilization, vol 3 pp 22-23, Needham; in [Maor, 2007]

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