SEXAGESIMAL SYSTEM

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The sexagesimal system is a way of counting with respect to base 60, instead of the commonly used base 10. Nowadays the sexagesimal system is still used to measure time and amplitude of angles: one hour is 60 minutes each minute is 60 seconds, a round angle is 360 degrees, each degree 60' or 360", and so on.

The first populations to use this counting system were the Sumerians and later the Babylonians. The Sumerians lived in Mesopotamia around 3000 BC, but their civilization was deeply developed. One proof of that is their abstract form of writing based on cuneiform (from the Latin wedge-shaped) symbols, one of the oldest written languages. They used to write on clay tablets, which were then dried in the hot sun. A lot of them survived to the present time, letting us know a bit more about their culture.

The Sumerians had two systems of numeration: a sexagesimal for astronomical observations and a decimal for everyday use. Both systems were additive, i.e. every symbol was repeated as many times as required by the desired number. Hence, to write numbers, they combined four different signs obtained by either pressing at an angle or straight-on using two different rods, as shown in Figure 0.1.

Only with Babylonians [1] we have the first example of a sexagesimal system also using place values. They used the additive principle of the Sumerians for all numbers from 1 to 59, but they did so with only two characters: a corner sign

[1] a Semitic people who invaded Mesopotamia defeating the Sumerians and by about 1900 BC establishing their capital at Babylon

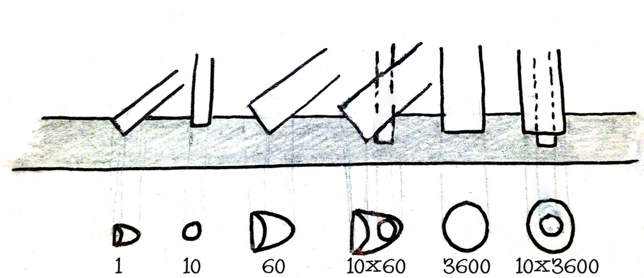

FIGURE 0.1. Sumerian digits

FIGURE 0.2. Babylonian digits

for ten and a wedge sign for a single unit (see Figure 0.2). They then used the positional principle with powers of 60. A further example see also the tablets of Figure 0.3 .

Babylonians didn't have a complete awareness of the importance of zero as number, but it seems that it was always clear from the context when a wedge symbol meant one, sixty, 60^2 or more. It would have been a bigger problem to distinguish, for example, 61 $(1 \times 60 + 1)$ from 3601 $(1 \times 60^2 + 0 \times 60 + 1)$. In that case, they used to leave more space between numbers or use some additional symbols like the ones in Figure 0.44 that only served as place horders.

Another important case where a place holder had an essential role was in the use of fractions. In a similar way as we use decimal fractionsn today, Babylonians used sexagesimal fractions. For example $\frac{3}{8}$, which in our notation is $0,375 = \frac{3}{10} + \frac{7}{10^2} + \frac{5}{10^3} = \frac{3}{8}$, was written 0 22 30, which, as sexagesimal fraction, represents $\frac{22}{60} + \frac{30}{60^2} = \frac{1320+30}{3600} = \frac{3}{8}$. Notice that if the denominator has a prime factor different from $2, 3$ or 5, than the number is not finite in base 60 fractions. In that case, a scribe would hve given an approximation of the number; for example, they would have written: $\frac{1}{13} = \frac{7}{91} = 7 \times \frac{1}{91} = approx\, 7 \times \frac{1}{90}$ and statements such as "an approximation is given since 13 does not divide". The Babylonians had no notation to indicate where the integer part ended and the fractional part began. Hence, as with the integer numbers, they used "the context makes clear" philosophy combined with symbols for place holders (as in Figure 0.5).

It is still unclear why these civilizations chose a sexagesimal system. Many historians of mathematics have tried to find an appropriate explanation, and they have come up with a few possible reasons. Theon of Alexandria in the fourth century AD suggested it was because 60 was the smallest number divisible by 1,2,3,4,5 and 6, so the number of divisors was maximised. Although this is true, it is held unrealistic, since base 12 has the same division properties, but no other culture seems to have devised number systems with that base. Exceptions are some measures like weights, money and length subdivisions as for example in old British measures[2] Following the theory based on the weights and measures, Otto Neugebauer, an Austrian historian of mathematics in the 20^{th} century, proposed that the reason of base 60 come up from the fact that the Sumerians used $1/3$ and $2/3$ as basic fractions in measures. However, although Neugebauer may be correct, remains the fact that the system of weights and measures should be a consequence of the number system rather than visa versa. Various theories have been based on astronomical events and calendars. Sixty could be the product of the number of known planets at the time besides Earth (Mercury, Venus, Mars, Jupiter, Saturn) with the number of months in the year. The historian of mathematics Moritz Cantor (1829-1920, Germany) supposed that a reason for choosing base 60 was related to the fact that the year was thought to have 360 days, although certainly Sumerians knew that the year was a little bit longer.

Other theories could be based on geometry and the fact that the equilateral triangle, which has angles of 60 degrees, was considered the fundamental geometrical building block by the Sumerians. But again, as for weights and the other measures, we can assume that degrees had been "invented" after the whole number system. As J J O'Connor and E F Robertson wrote "I feel that all of these reasons are

[2]Twelve pence were a shilling, twelve troy ounces were a troy pound, twelve inches were a foot, etc. [Enciclopediea Britannica]

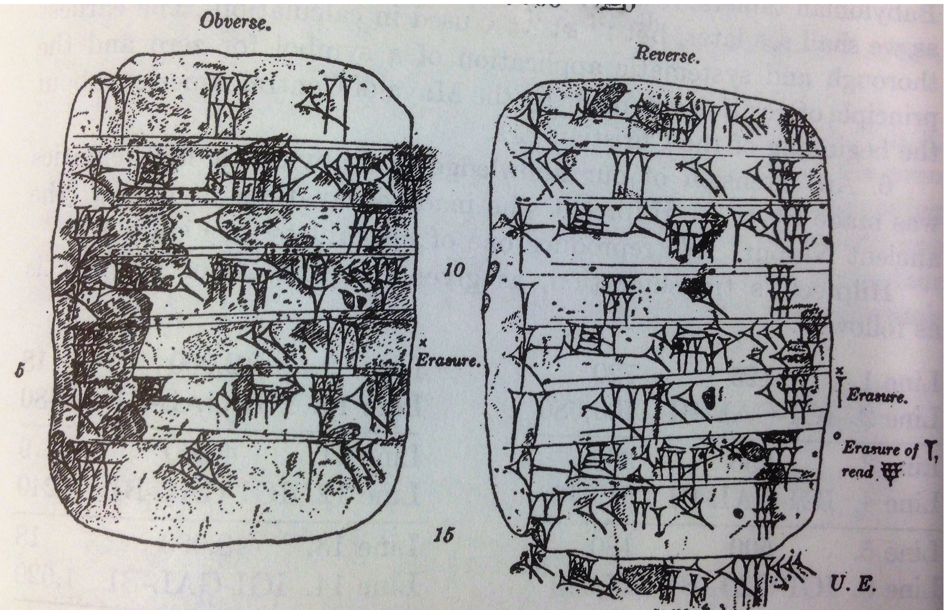

FIG. 1.—Babylonian tablets from Nippur, about 2400 B.C.

Line 1.	125	720	Line 9.	2,000	18
Line 2.	IGI-GAL-BI	103,680	Line 10.	IGI-GAL-BI	6,480
Line 3.	250	360	Line 11.	4,000	9
Line 4.	IGI-GAL-BI	51,840	Line 12.	IGI-GAL-BI	3,240
Line 5.	500	180	Line 13.	8,000	18
Line 6.	IGI-GAL-BI	25,920	Line 14.	IGI-GAL-BI	1,620
Line 7.	1,000	90	Line 15.	16,000	9
Line 8.	IGI-GAL-BI	12,960	Line 16.	IGI-GAL-BI	810

7. In further explanation, observe that in

Line 1.	$125=2\times 60+5$,	$720=12\times 60+0$
Line 2.	Its denominator,	$103{,}680=[28\times 60+48(?)]\times 60+0$
Line 3.	$250=4\times 60+10$,	$360=6\times 60+0$
Line 4.	Its denominator,	$51{,}840=[14\times 60+24]\times 60+0$
Line 5.	$500=8\times 60+20$,	$180=3\times 60+0$
Line 6.	Its denominator,	$25{,}920=[7\times 60+12]\times 60+0$
Line 7.	$1{,}000=16\times 60+40$,	$90=1\times 60+30$
Line 8.	Its denominator,	$12{,}960=[3\times 60+36]\times 60+0$

FIGURE 0.3. Tablets, about 2400 BC, and their translation

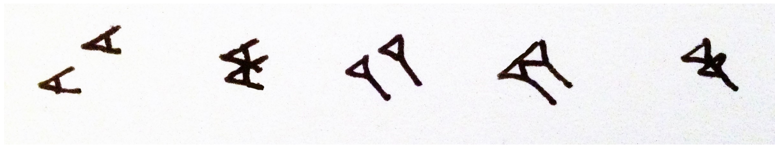

FIGURE 0.4. Different representations for the Babylonian zero

(A) $0\ 1 = \frac{0}{60^0} + \frac{1}{60^1}$

(B) $0\ 0\ 18 = \frac{0}{60^0} + \frac{0}{60^1} + \frac{18}{60^2}$

FIGURE 0.5. Babylonian fractions

really not worth considering seriously." They suggest "the reason has to involve the way that counting arose in the Sumerian civilisation, just as 10 became a base in other civilisations who began counting on their fingers, and twenty became a base for those who counted on both their fingers and toes." In fact, there exists a way to count to 60 using two hands. On the first hand using the thumb, one can count the remaining 12 phalanxes (three on each of the four fingers), keeping track of up to five rounds with the fingers of the second hand. This seems to be unnatural, until we consider that 60 probably came through the joining of two counting systems, one having base 12 and the other having base 5 or 10. Then this theory supposes that as the two peoples mixed, the two systems of counting were used by different members of the society. Through trading with each other, base 60 would naturally the common system everyone understood.

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