EUCLID'S FIRST PROOFS

The people who first used deductive reasoning in definitions, assertions and proofs for studying mathematics were Greeks. Most of what we know about Greek mathematics comes from Euclid's work *The Elements*. Unfortunately, it is not so easy to gain correct and precise information about ancient Greek mathematical works and mathematicians. In their culture oral transmission of knowledge was more popular than written records, and the few manuscripts they had were written on papyrus, a very perishable material.[1]

What is known about Euclid is that he lived in Alexandria of Egypt around 300 B.C. "a little after the time of Plato, but before Archimedes" [Hartshorne, 2000]. It is widely said also that "Euclid may not have been a first class mathematician but the long lasting nature of *The Elements* must make him the leading mathematics teacher of antiquity or perhaps of all time" [JOC/EFR, 01/1999]. In fact historians agree that "none of the theorems contained in the 13 books (chapters of *The Elements*) can with certainty be ascribed to Euclid himself. ... However, the logical organization of *The Elements*, is undoubtedly Euclid's contribution" [Anglin Lambek, 1995].

This text is the first attempt to construct all mathematics starting with only a few assertions, which we nowadays call axioms but Euclid called 'Postulates' and 'common notions'. It can seem obvious that when talking about mathematical results, one have to give axioms, which aided by definitions, suffice to prove theorems, lemmas or propositions. But in previous cultures like Babylonians and Egyptians, mathematics was not like that. It was studied only for practical reasons, so nothing similar to abstract propositions was invented.

First Thales, followed by other Greek scholars, combined calculations (Babylonians) and geometric knowledge (Egyptians) with a philosophical approach, creating the new subject which is called mathematics (from Greek: '$\mu\alpha\theta\grave{\eta}\mu\alpha\tau\alpha$' = 'that which is learned, learning subjects'). *The Elements* not only describes all previously-known geometrical and numerical properties but, what is more, gives a strong deductive foundation to mathematics. For this reason it became the starting point and the standard way of studying this subject in all later works. For example, the preface of the first (1570) English translation of the book by Billingsley testifies "Without the diligent studie of Euclides Elementes, it is impossible to attain unto perfecte knowledge of Geometrie, and consequently of any of the other Mathematical Sciences" [Hartshorne, 1997].

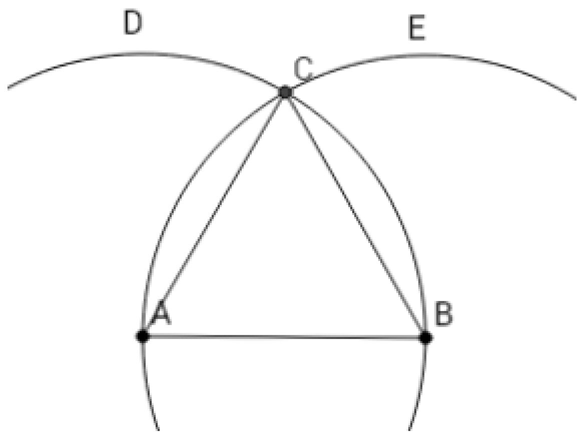
Besides its contents, "the most important contribution of Euclid's *Elements* was its innovative logical method: first, make terms explicit by forming precise definitions and so ensure mutual understanding of all words and symbols. Next make concepts explicit by stating explicit axioms or postulates (these terms are interchangeable) so that no unstated understandings or assumptions may be used. Finally, derive the logical consequences of the system employing only accepted rules of logic, applied to the axioms and to previously proved theorems" [Mlodinow, 2010].

[1] to learn more about it see [JOC/EFR, 10/1999]

The whole treatise starts with 23 definitions, 5 geometrical postulates (which allow all constructions) and 5 more general assumptions called 'common notions', such as: "Things equal to the same thing are also equal to one another"[Heiberg, 1885]. Such notion serves to proceed in a deductive way; from this foundation, Euclid goes on to prove 465 theorems.[2]

Before analyzing a prototypical example of proof, it has to be noted that Euclid made no distinction between abstract deductive reasoning and practical geometric constructions. Namely he considered both valid arguments: for him, constructing something with the ruler and compass was equivalent to proving its existence. So statements were proved both with a geometrical figure and its explanation, supported by the previous assumptions.

A clare example is given in the first proposition: "To construct an equilateral triangle on a given finite straight-line:



Let AB be the given finite straight-line. So it is required to construct an equilateral triangle on the straight-line AB. Let the circle BCD with center A and radius AB have been drawn (Postulate 3), and again let the circle ACE with center B and radius BA have been drawn (Postulate 3). And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another, to the points A and B (respectively) (Postulate 1). And since the point A is the center of the circle CDB, AC is equal to AB (Def. I.15). Again, since the point B is the center of the circle CAE, BC is equal to BA (Def. I.15). But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another (Common Notion 1). Thus, CA is also equal to CB. Thus, the three (straight-lines) CA, AB, and BC are equal to one another. Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB, (which is) the very thing it was required to do" [Heiberg, 1885].

[2]"Many authors have noted the incompleteness of Euclid's axioms in comparison to modern foundations of geometry. The most obvious point is the absence of any thought of ordering of points on a line or the concept of 'betweeness'. Euclid uses all assertions about ordering on an intuitive basis" [Artmann, 1999]. There are many other concepts which, although commonly understood by intuitive meaning, are lacking of a proper explanation as, for example, 'measure' and the act of measuring (used in books V and X), the method of 'superposition' (used to compare areas), an exact meaning of 'equal' (explaining better what is said in the first common notion), etc.

At first sight we can appreciate the rigor and sobriety of construction, but on a second, more accurate view, we have to deal with some questions.

The first one is that, despite all definitions, postulates and common notions, it is still not guaranteed that the point C, i.e. the intersection of the two circles D and E, is just a point and exists. Here are three underlying inconsistencies: one is related to the possibility of intersection of circles. While existence of intersections of lines is guaranteed in the fifth postulate, nothing is stated about intersection of circles depending on their position. Nothing is also said to guarantee the implicit assumption that two different lines cannot share a common segment, and so C is actually only one point. The third inaccuracy is related to continuity and real numbers. In fact, consider from a modern point of view that Euclid knew about and worked on a Cartesian plane over the field of rational numbers. Now, admitting that D and E intersect, if A and B are rational points, i.e. with rational coordinates (say A$=(0,0)$ and B$=(1,0)$) then C is not rational, since it has coordinates $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and therefore does not exist in that plane. Finally, even allowing that constructing figures is the same as proving their existence, there would still be an imperfection. The construction of the proof is indeed not unequivocal, because the intersecting points of the two circles are actually two distinct points, and no indication of choice is given.

Despite modern considerations, this work was very impressive and appreciated for its solidity and rigor. Through its widespread teaching, it has become the unique method used to do mathematics. As Heath wrote in his commentary, "This wonderful book, with all its imperfections, which indeed are slight enough when account is taken of the date at which it appeared, is and will doubtless remain the greatest mathematical text-book of all time" [Heath, 1921].

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