UNIVERSITÄT WÜRZBURG

Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

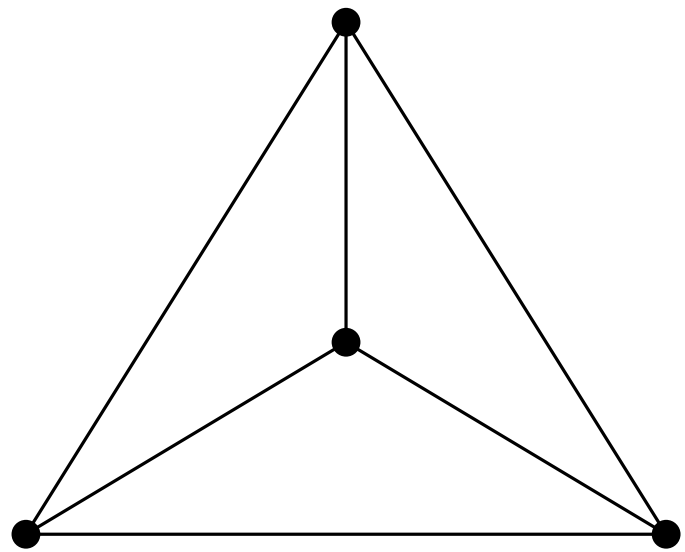
Part I:
SteinerTree

Antonio Lauerbach

Winter term 2026

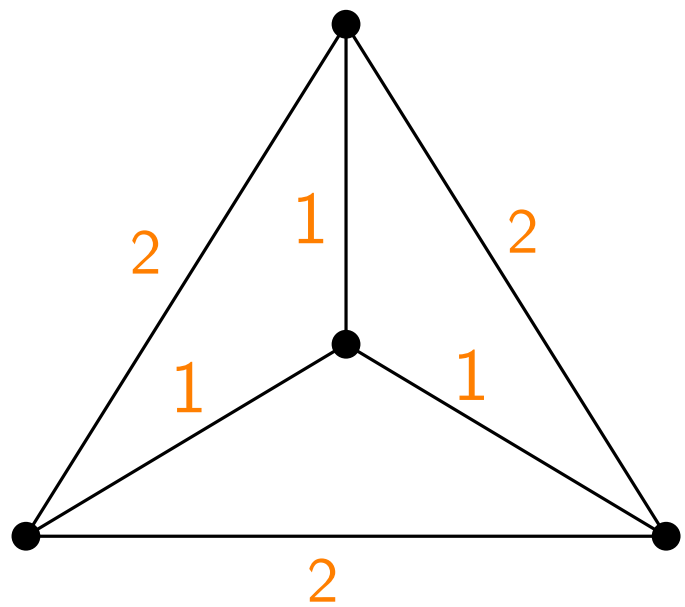
SteinerTree

Given: A graph G



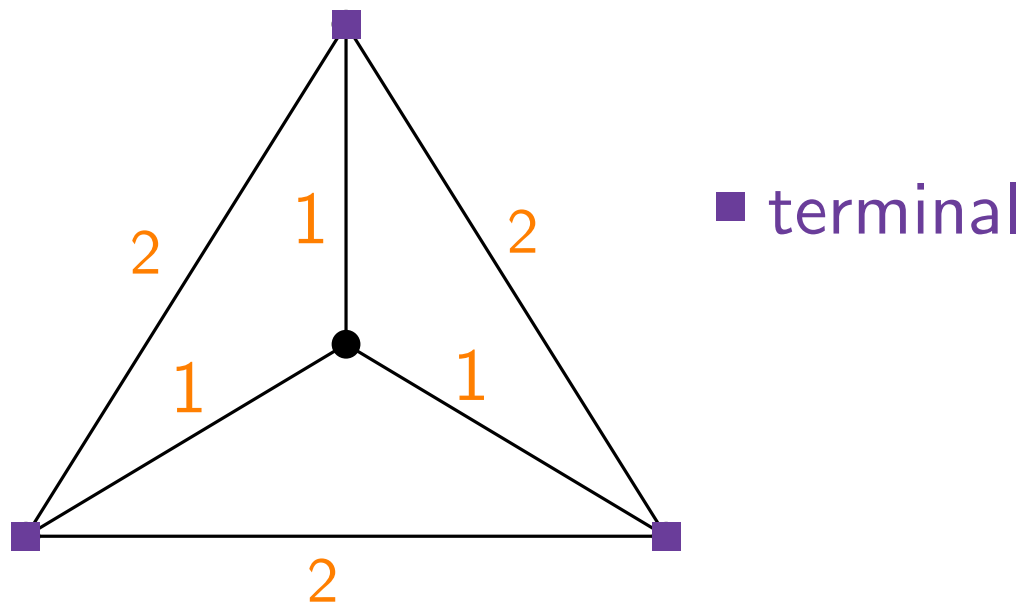
SteinerTree

Given: A graph G with edge weights $c\colon E(G) \to \mathbb{Q}^+$



SteinerTree

Given: A graph G with edge weights $c\colon E(G) \to \mathbb{Q}^+$
and a partition of $V(G)$ into a set T of **terminals**



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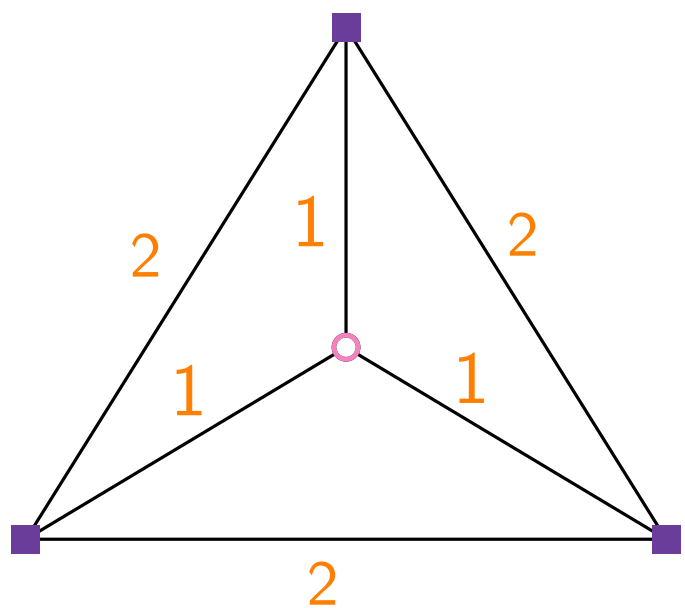

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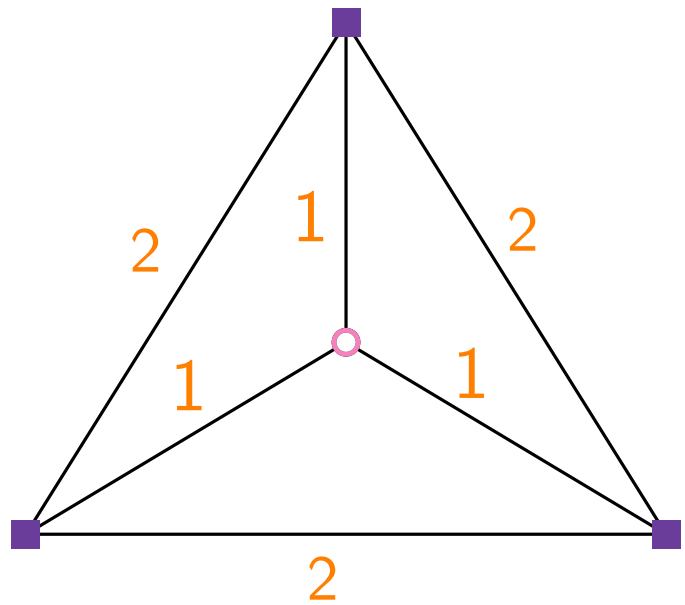

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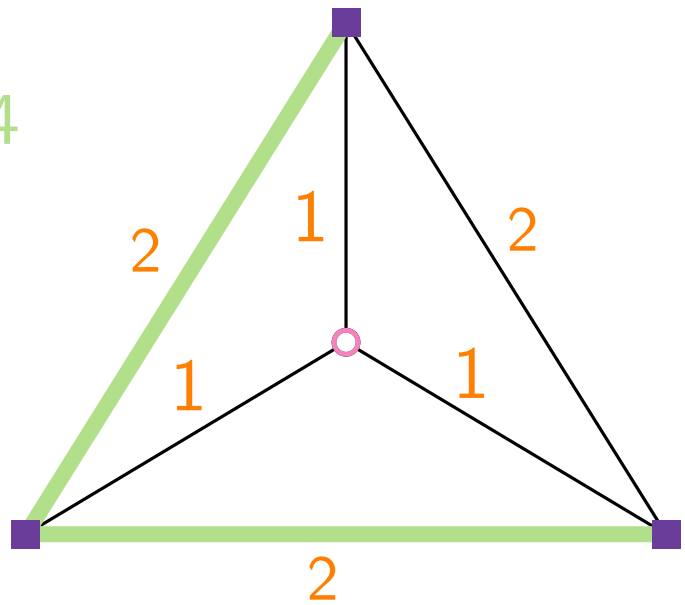

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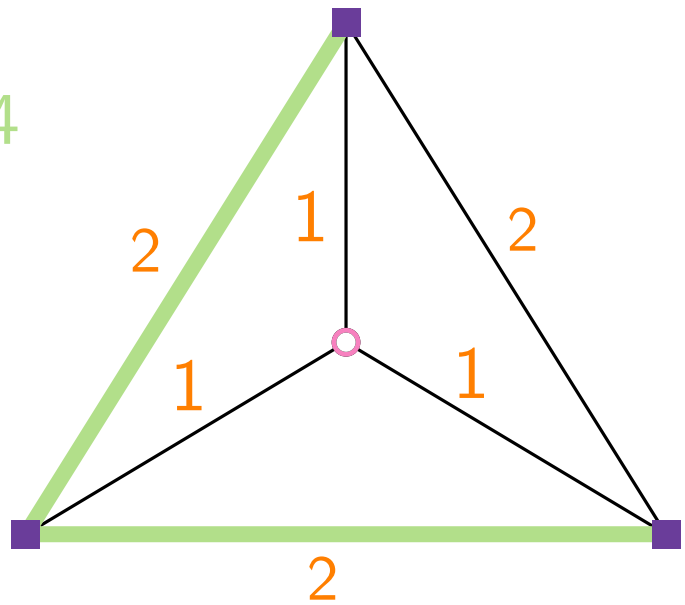

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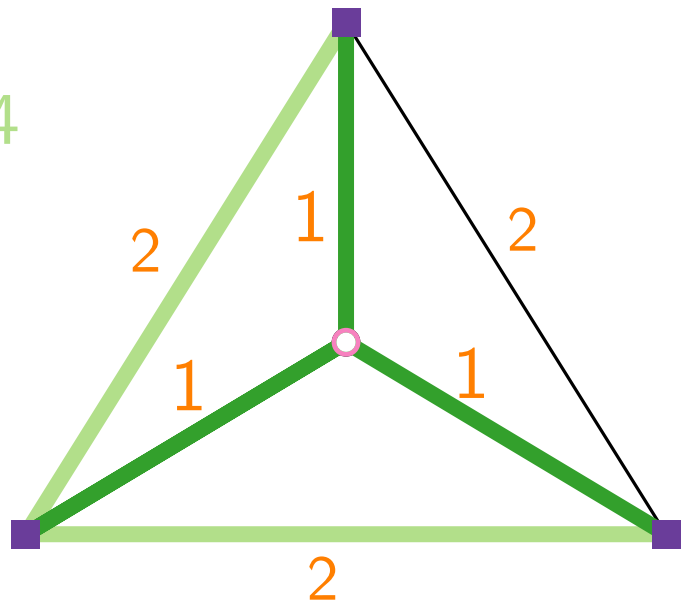

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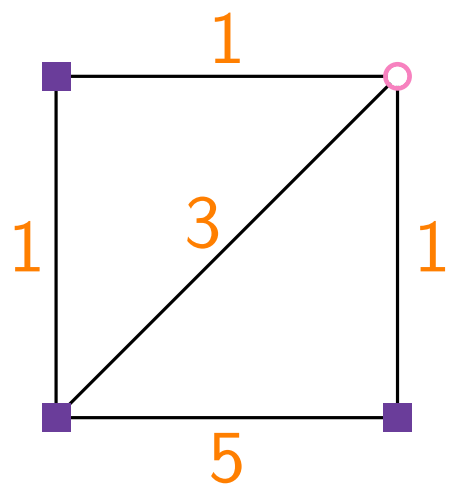

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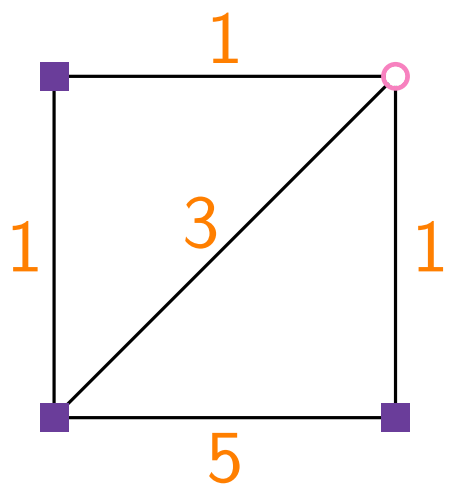

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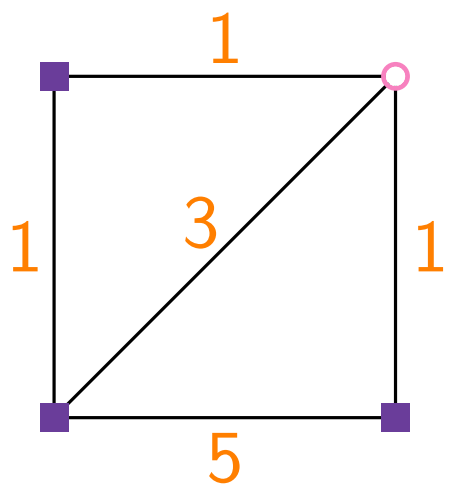

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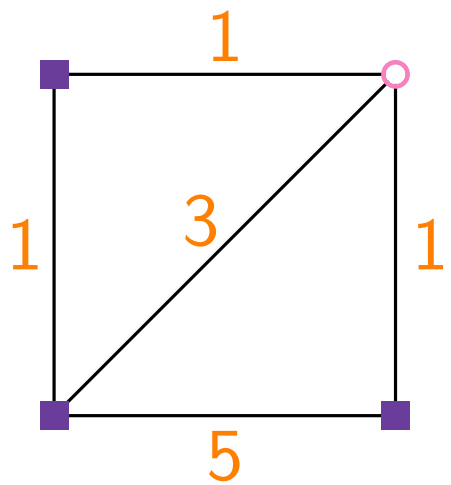

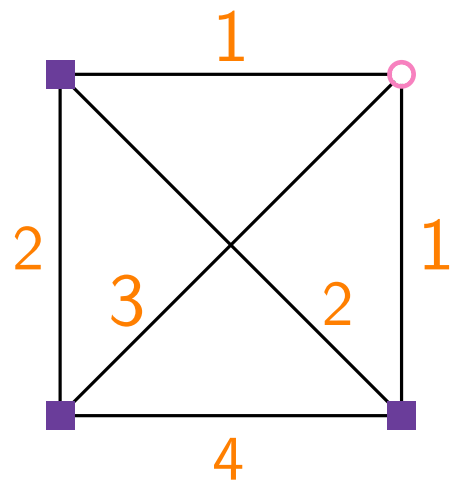

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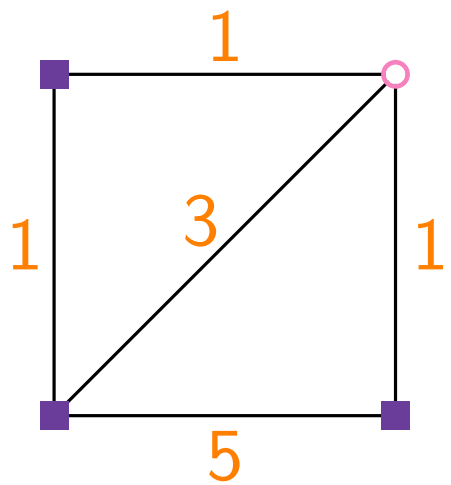

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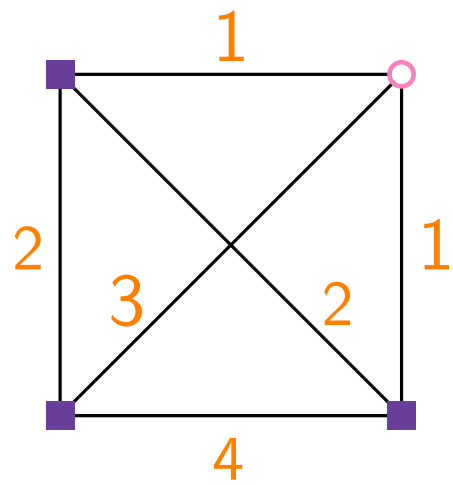

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Part II:
Approximation-Preserving Reduction

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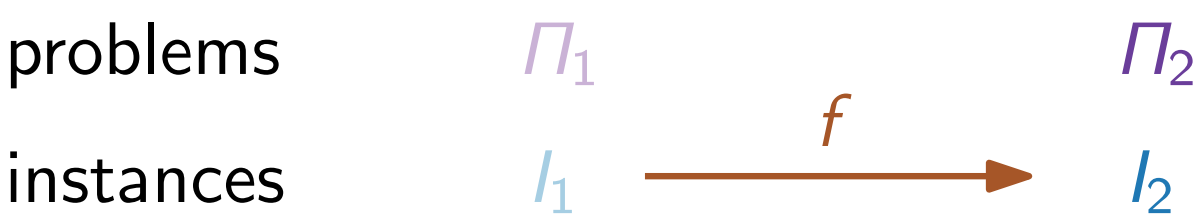

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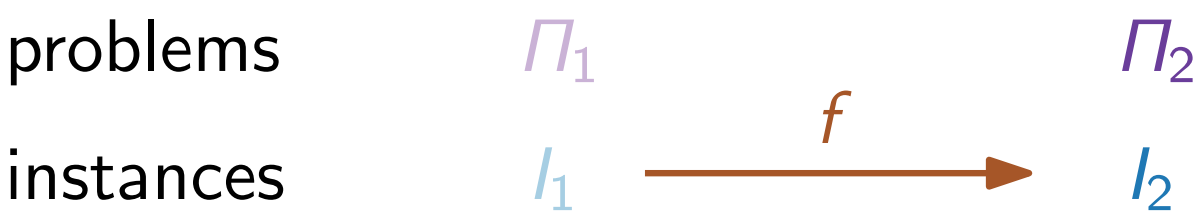

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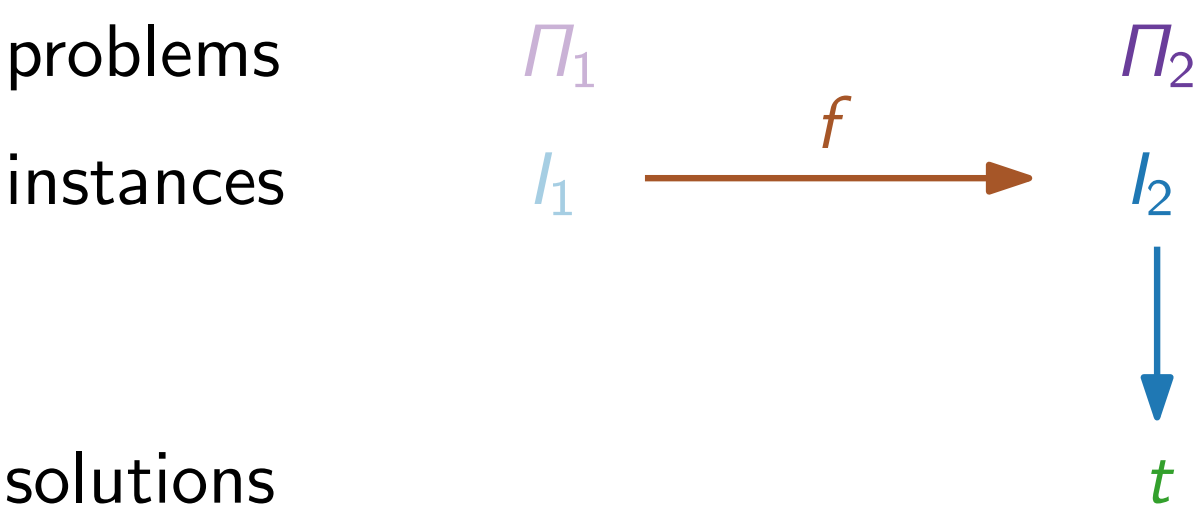

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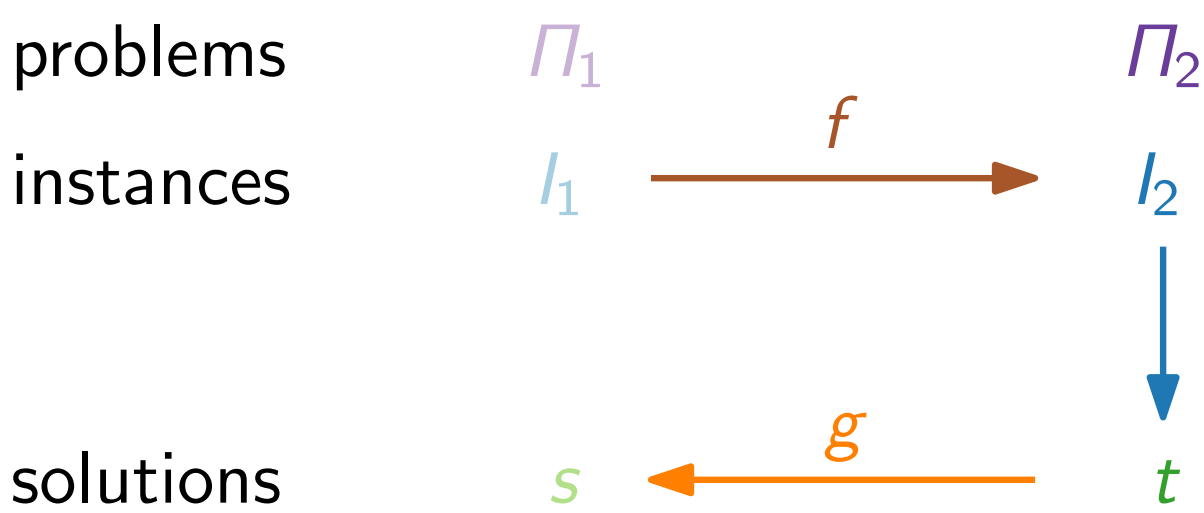

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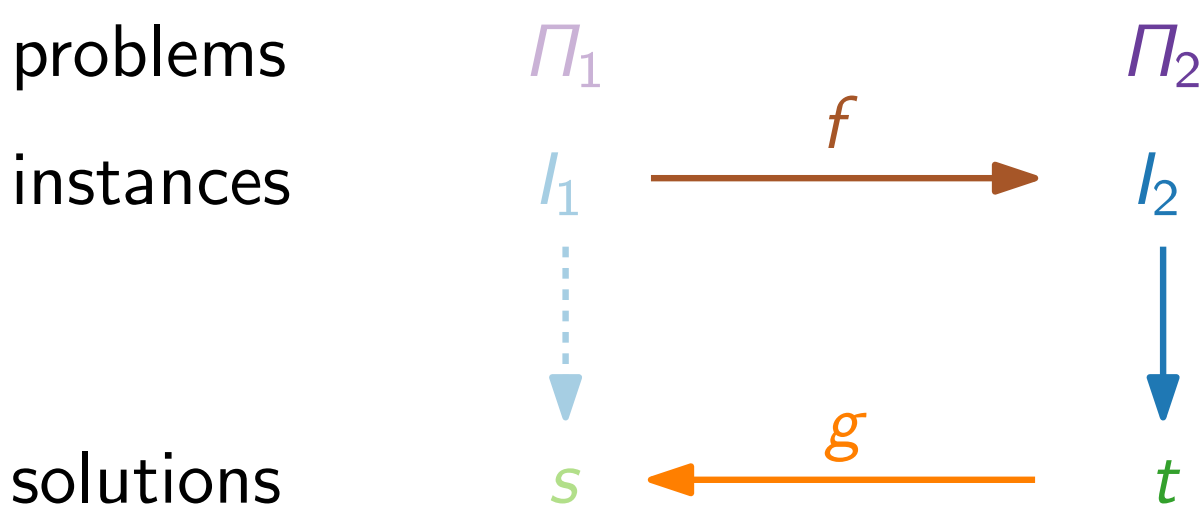

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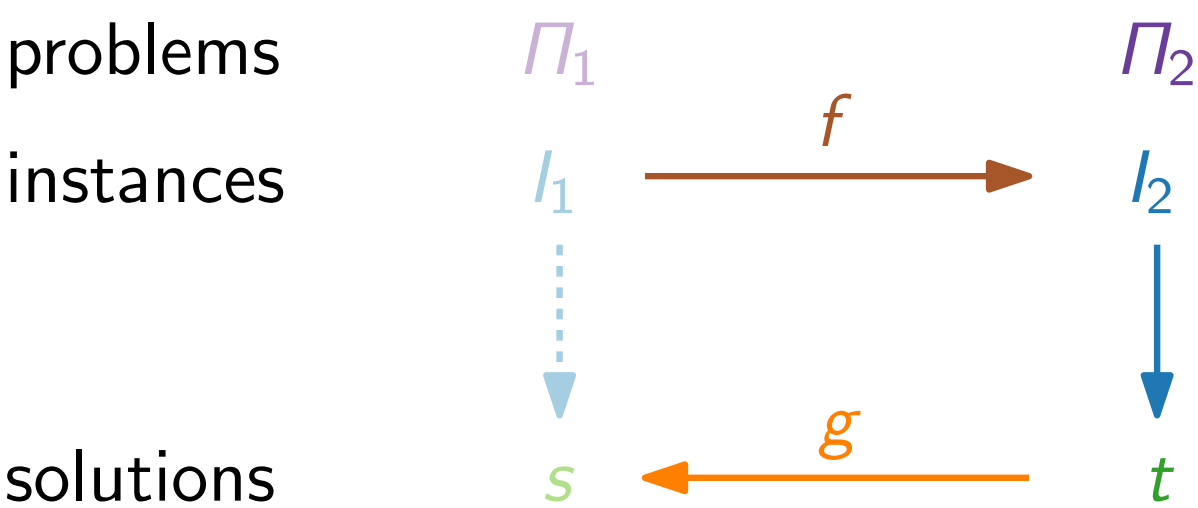
Theorem. Let Π_1, Π_2 be minimization problems with an approximation-preserving reduction (f, g) from Π_1 to Π_2.



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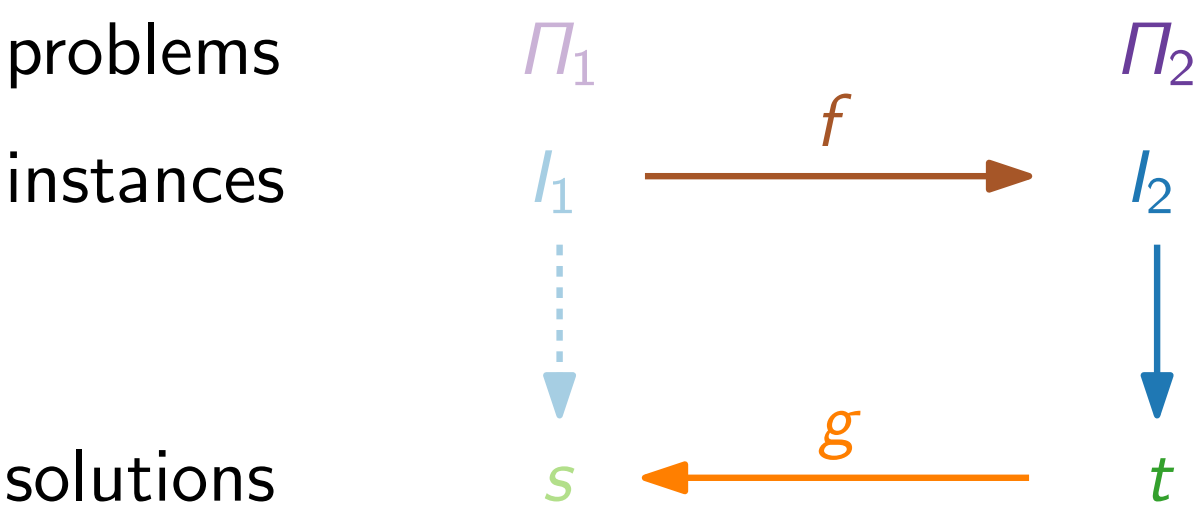

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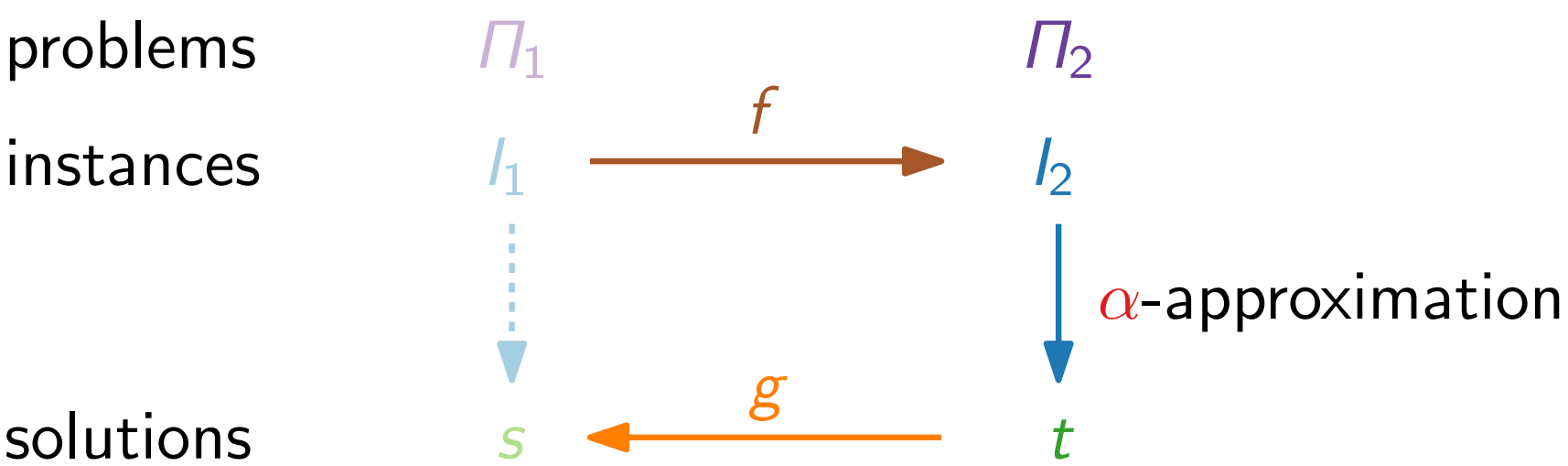

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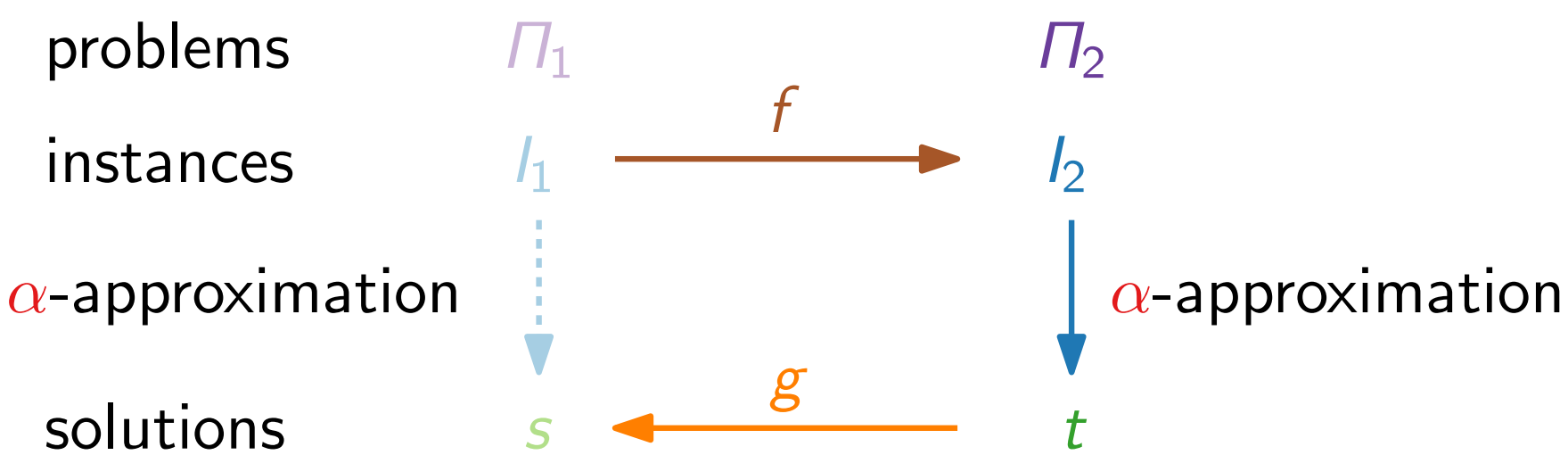

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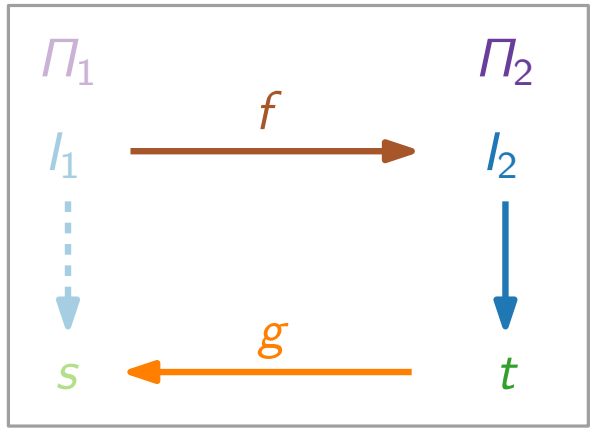

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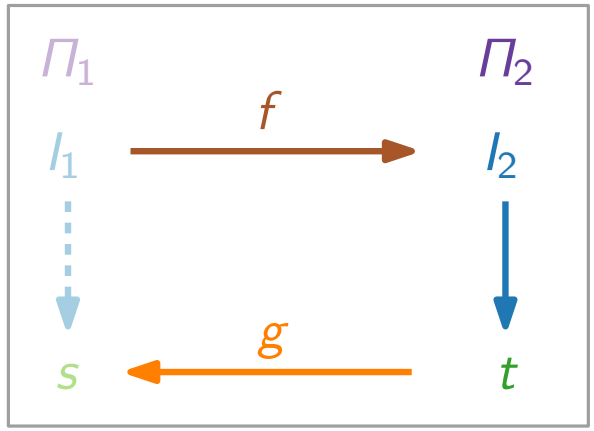

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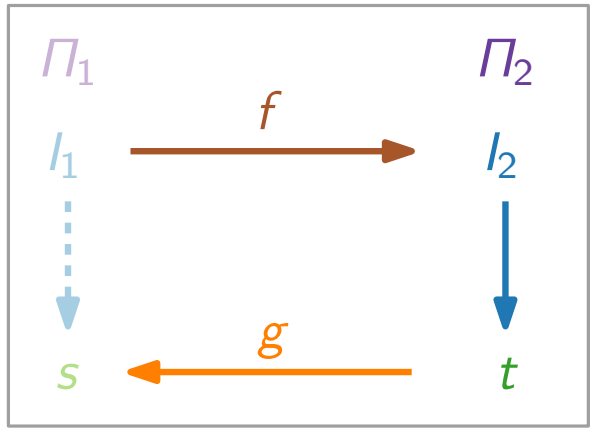

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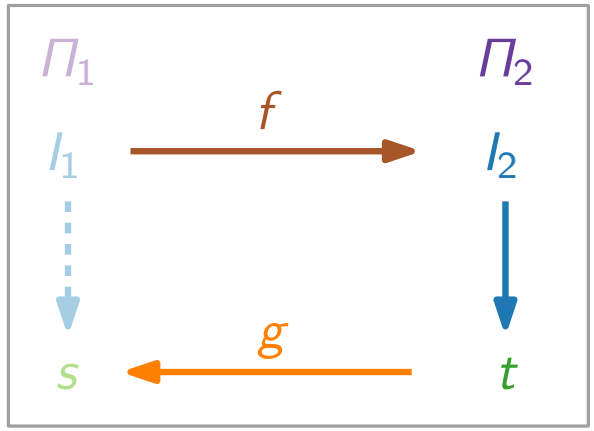

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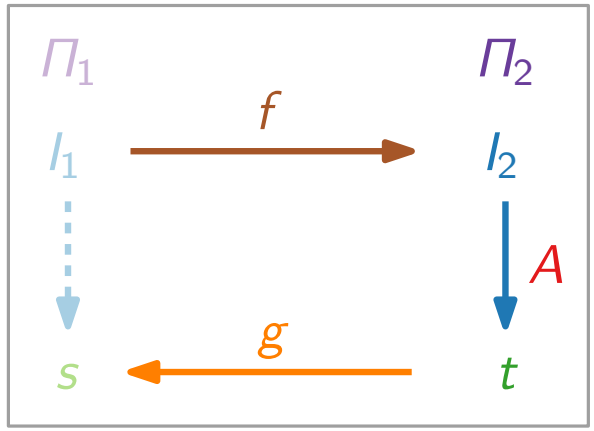

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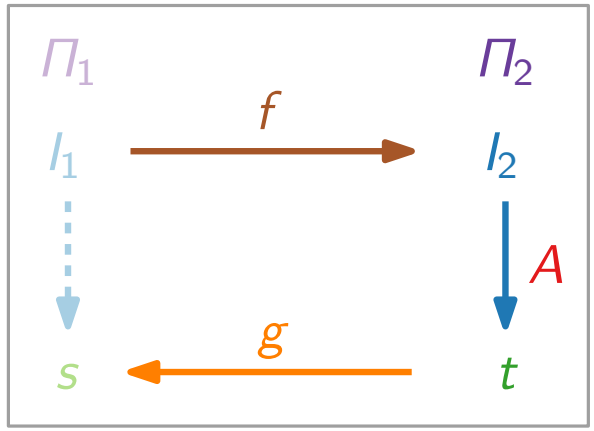

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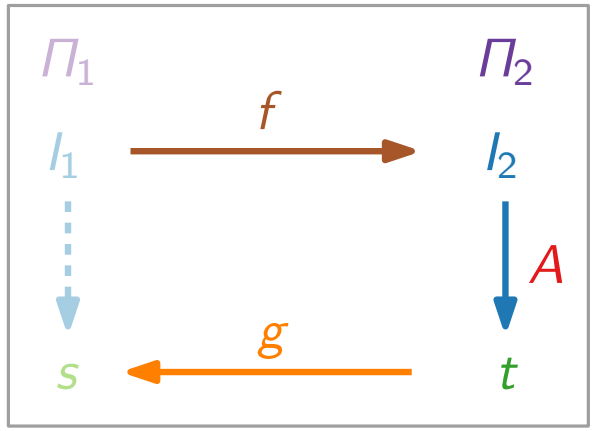

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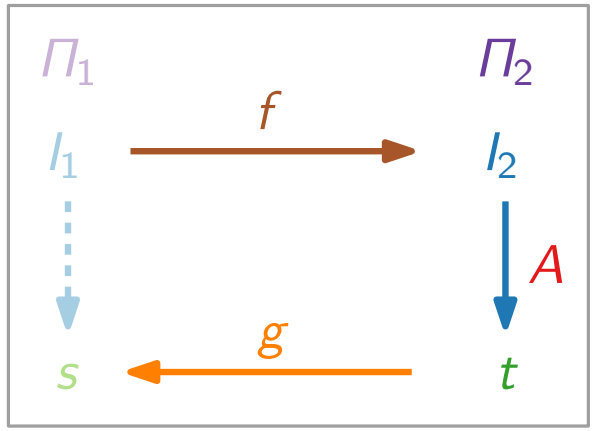

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Set $I_2 := f(I_1)$, $t := A(I_2)$ and $s := g(I_1, t)$.

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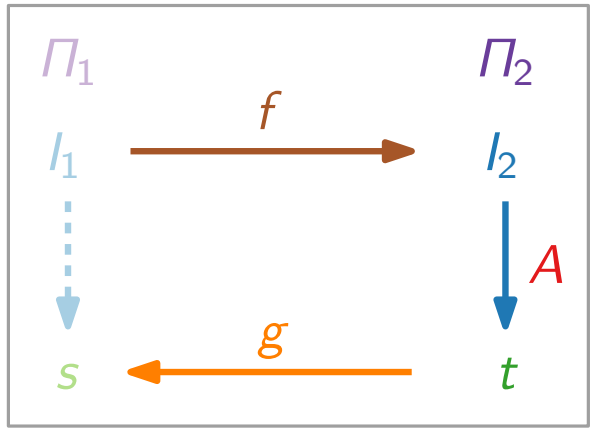
$\text{obj}_{\Pi_1}(I_1, s) \leq \text{obj}_{\Pi_2}(I_2, t) \leq$

Approximation-Preserving Reduction

Theorem. Let Π_1, Π_2 be minimization problems with an approximation-preserving reduction (f, g) from Π_1 to Π_2. If there is a factor-α approximation algorithm for Π_2, then there is a factor-α approximation algorithm for Π_1.



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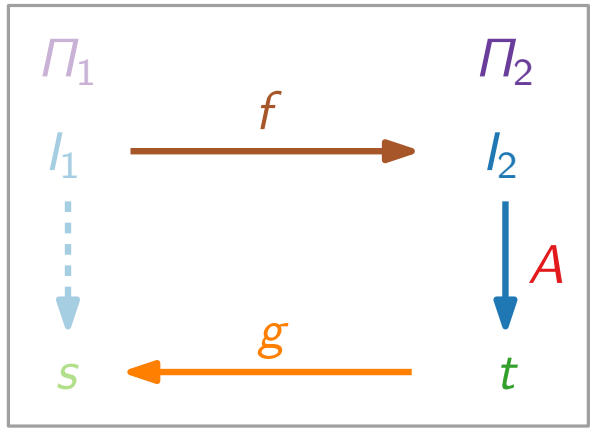

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□

Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part III:
Reduction to MetricSteinerTree

MetricSteinerTree

Theorem. There is an approximation-preserving reduction from SteinerTree to MetricSteinerTree.

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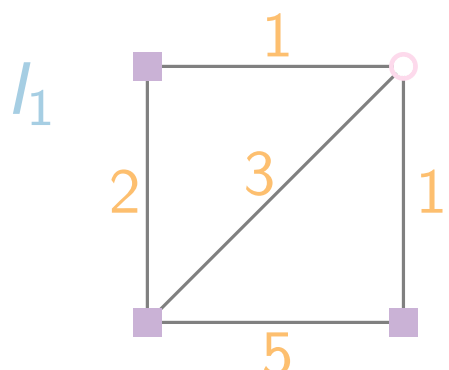

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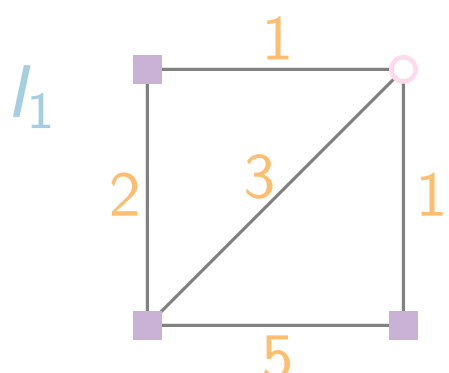

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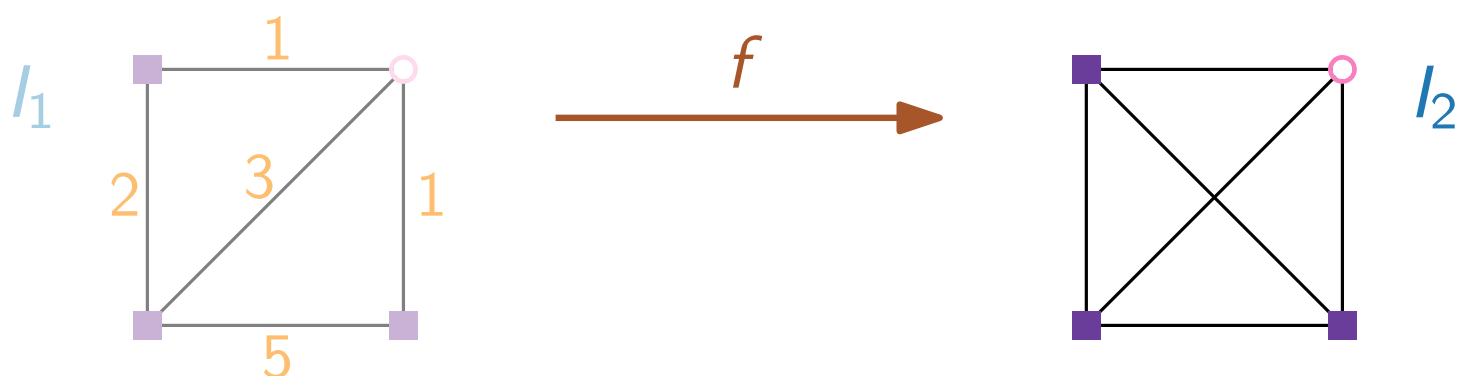

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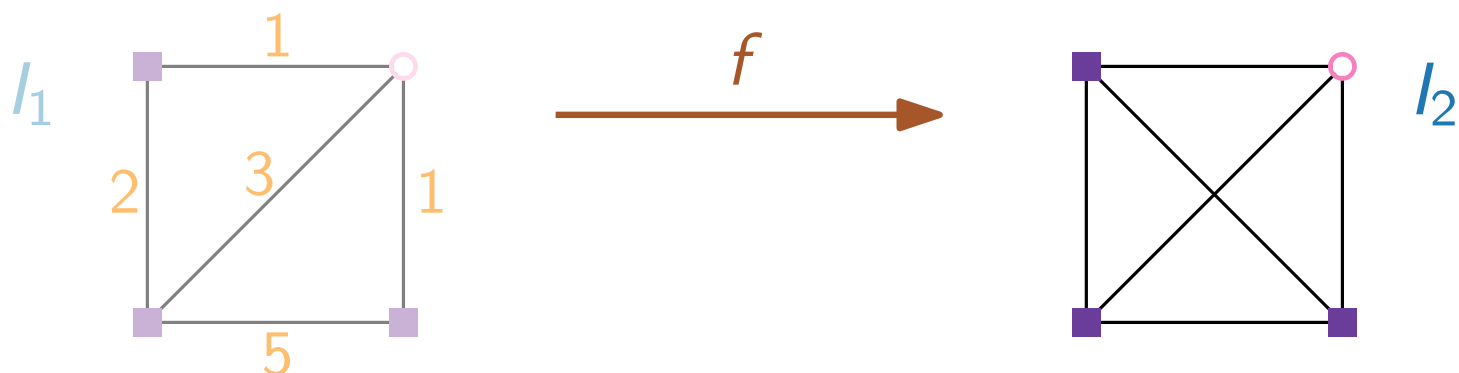

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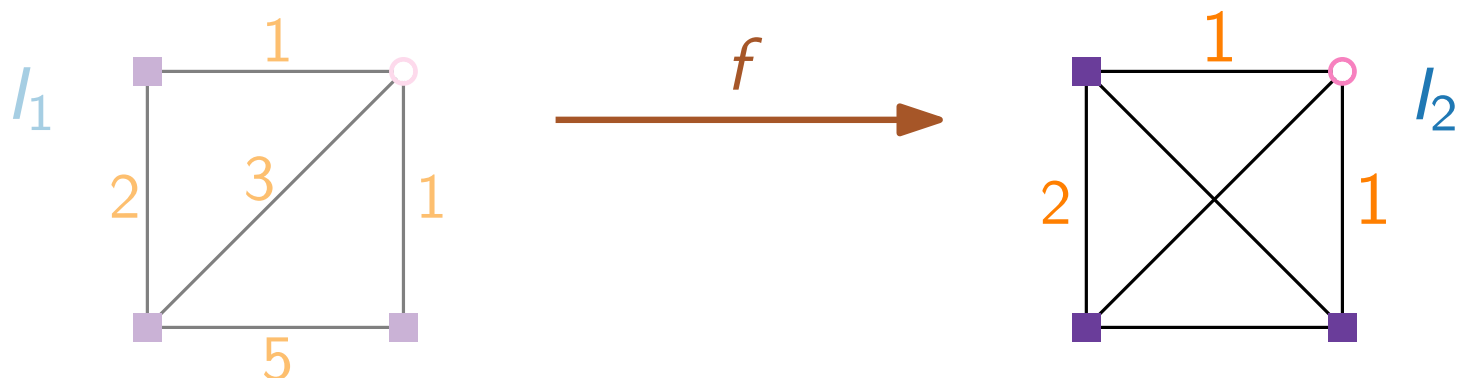

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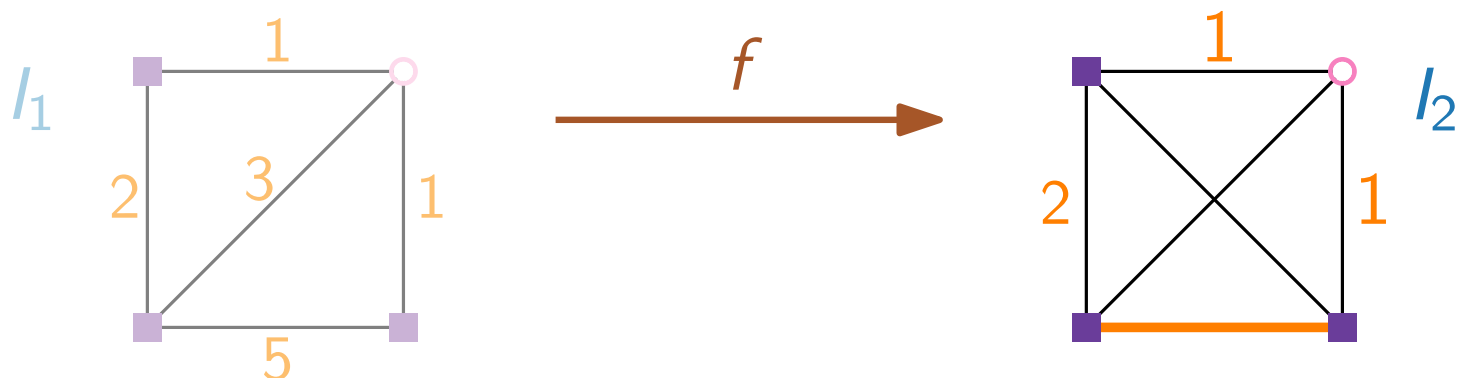

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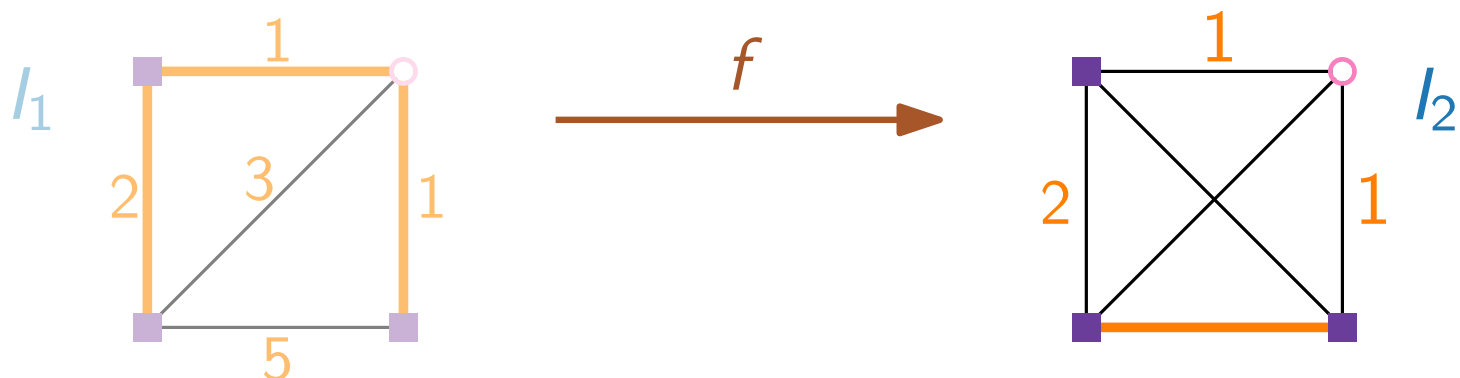

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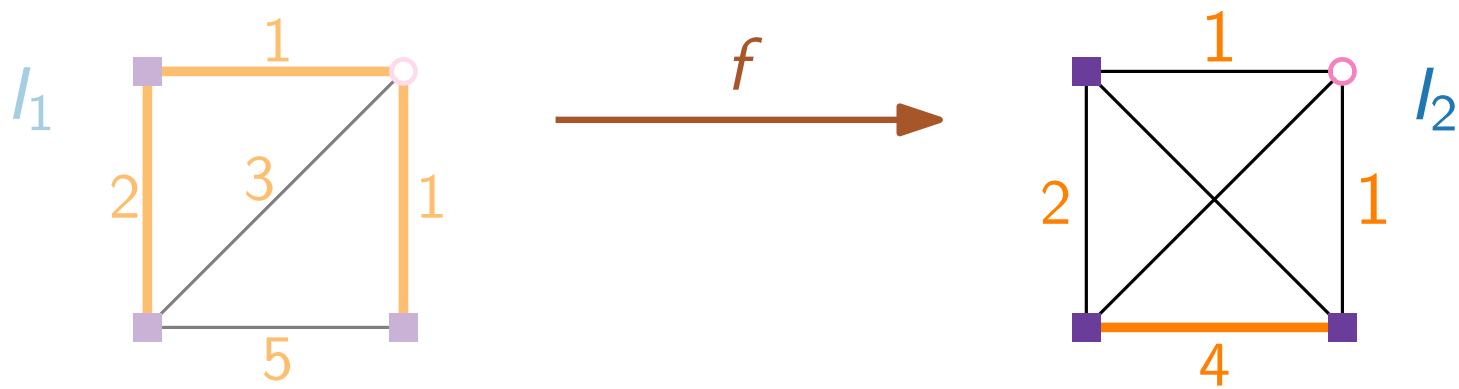

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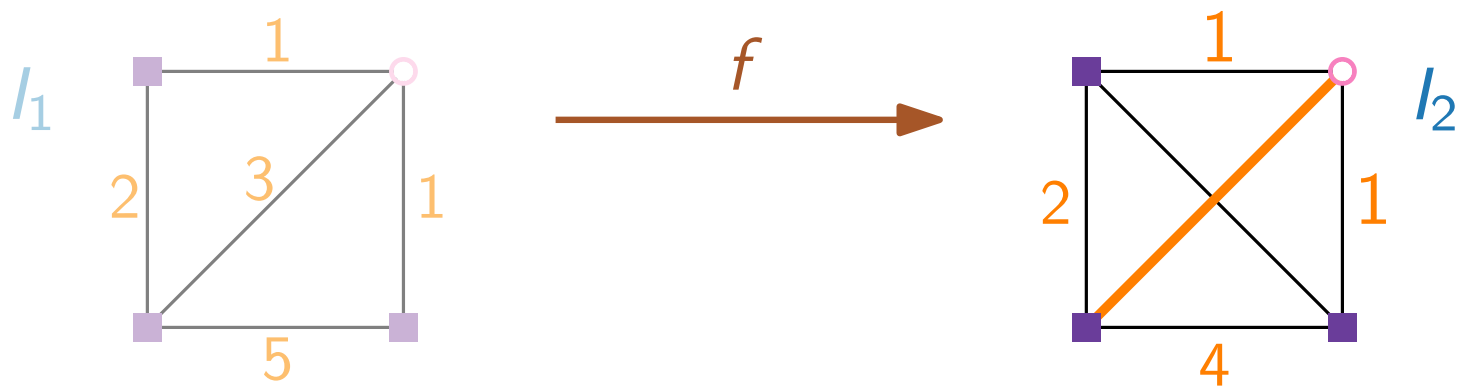

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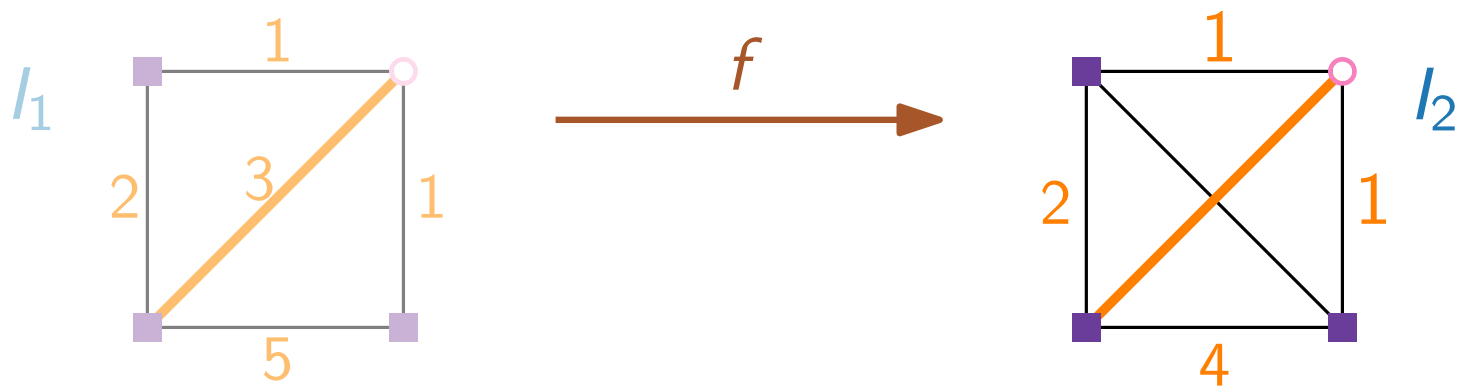

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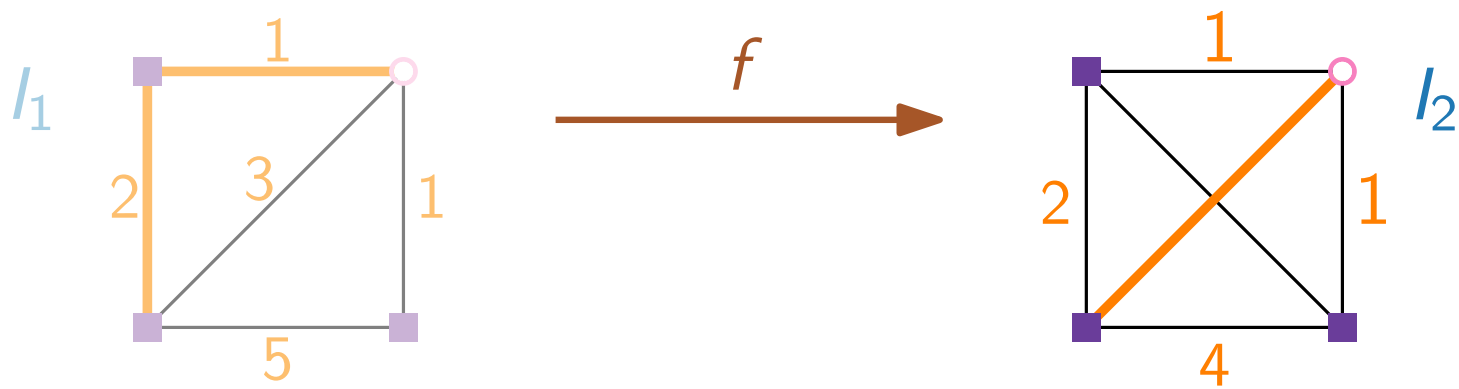

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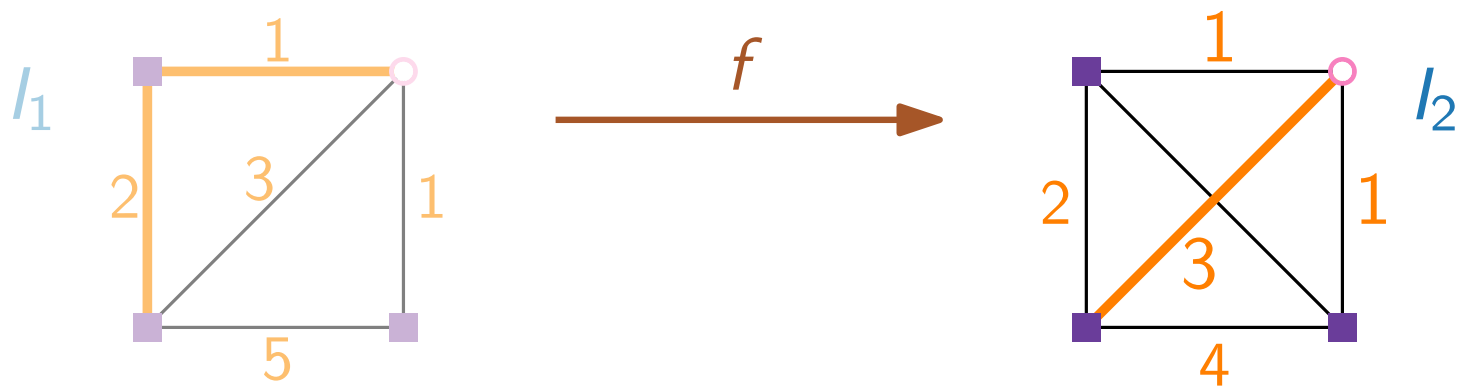

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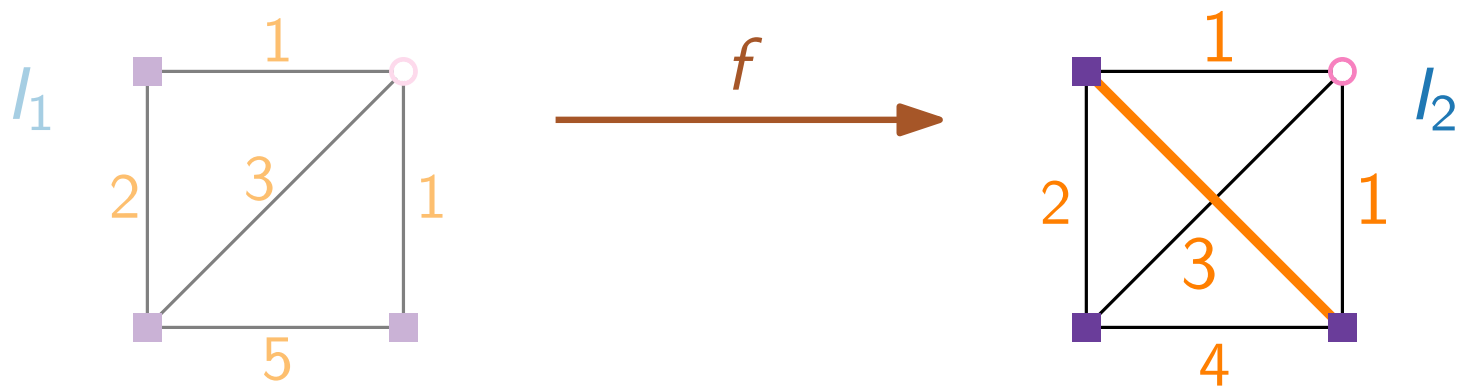

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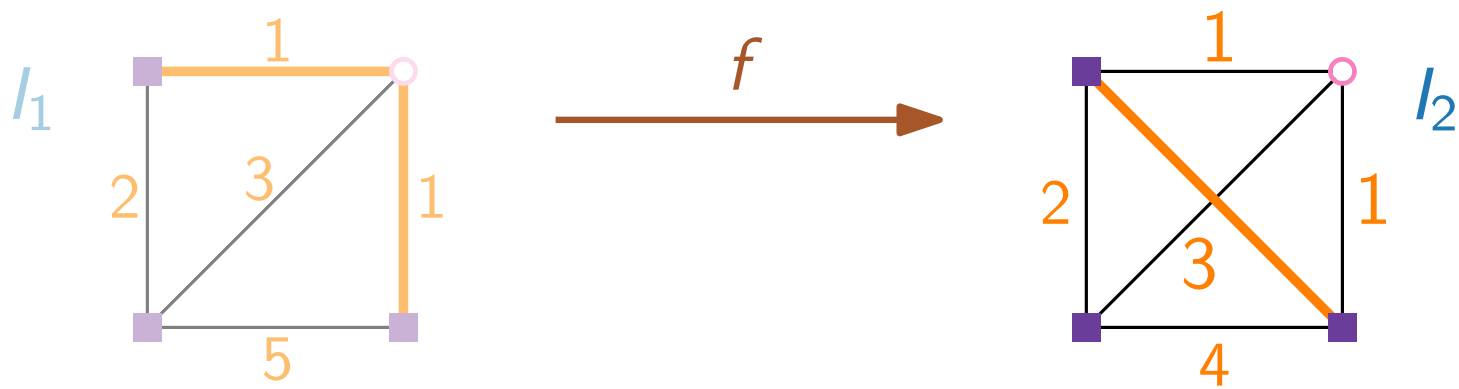

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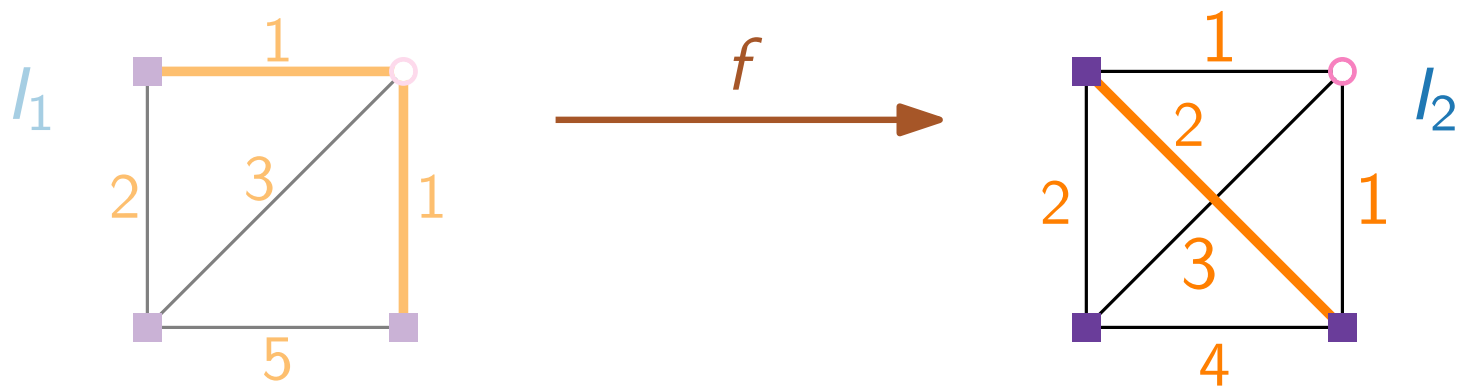

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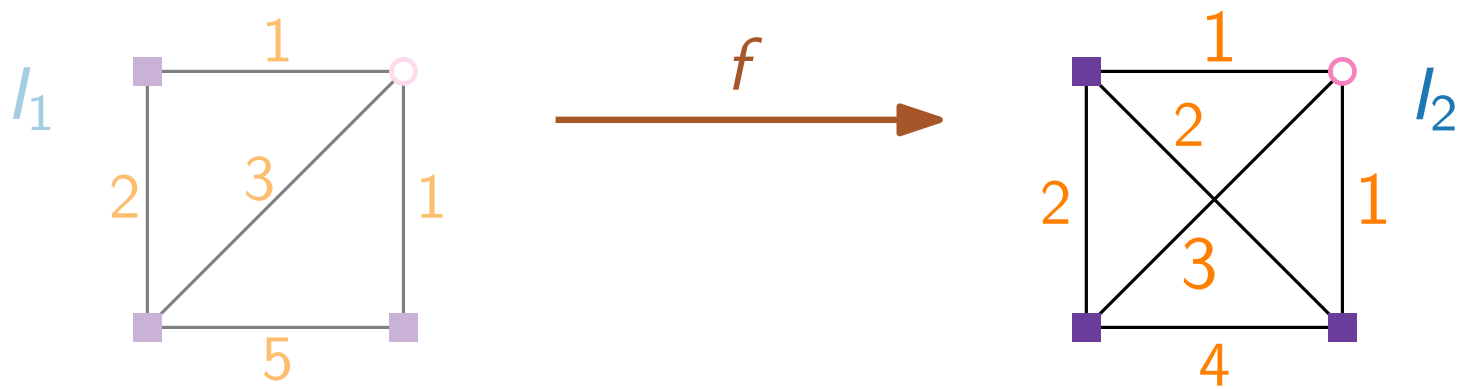

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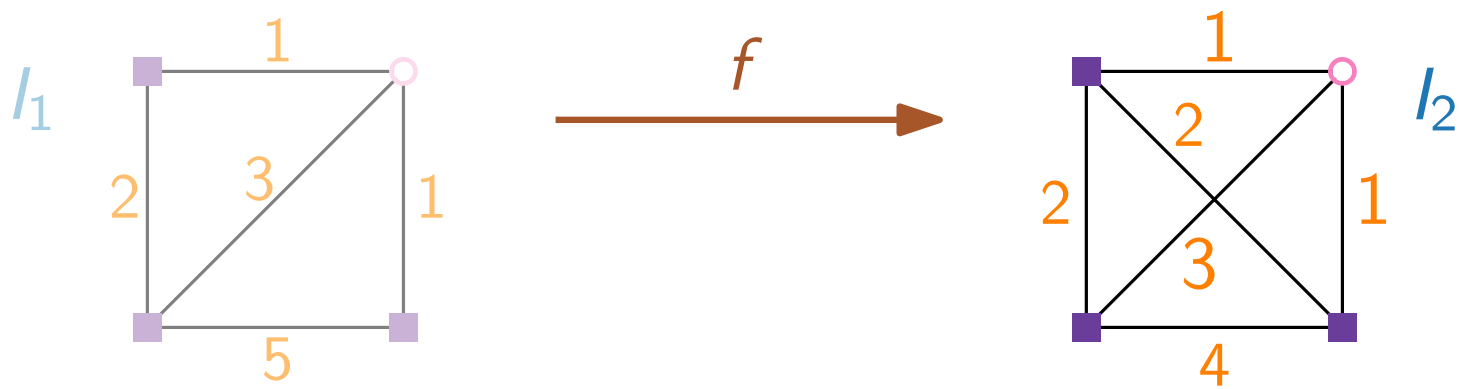

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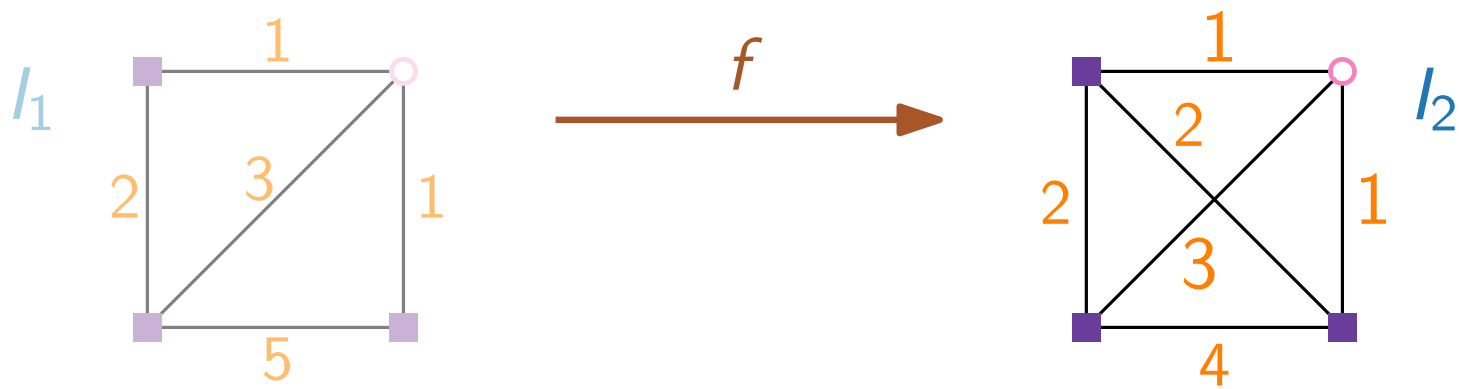

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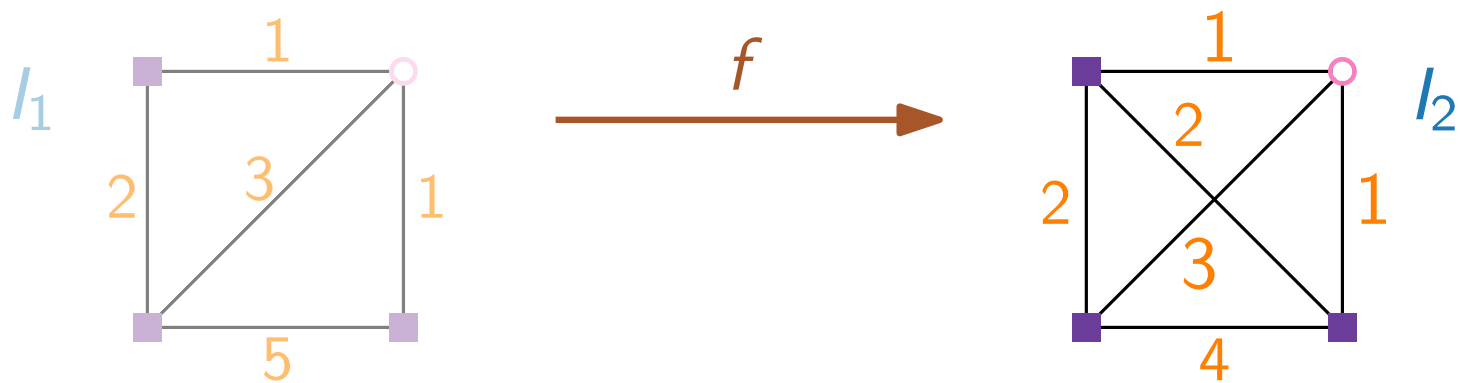

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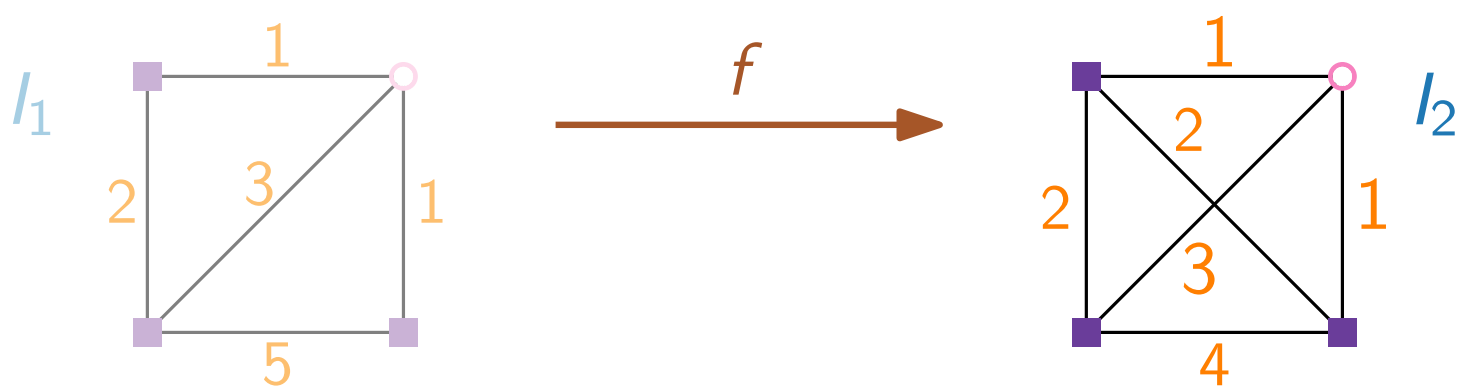

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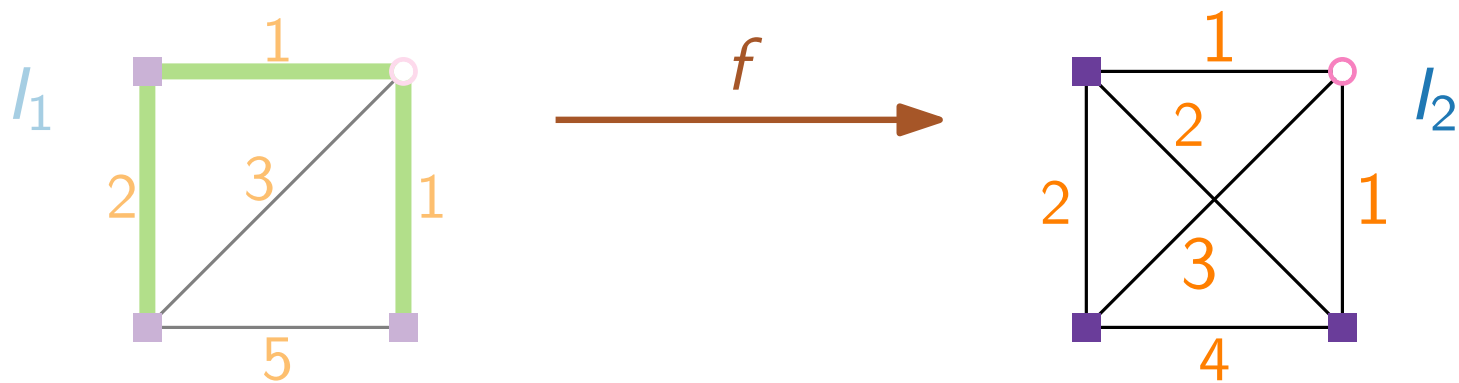

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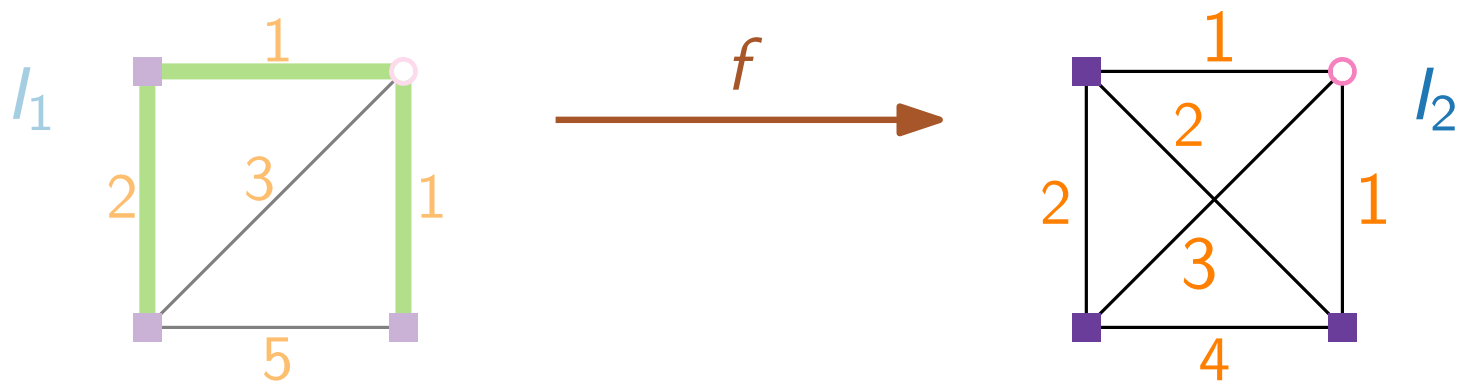

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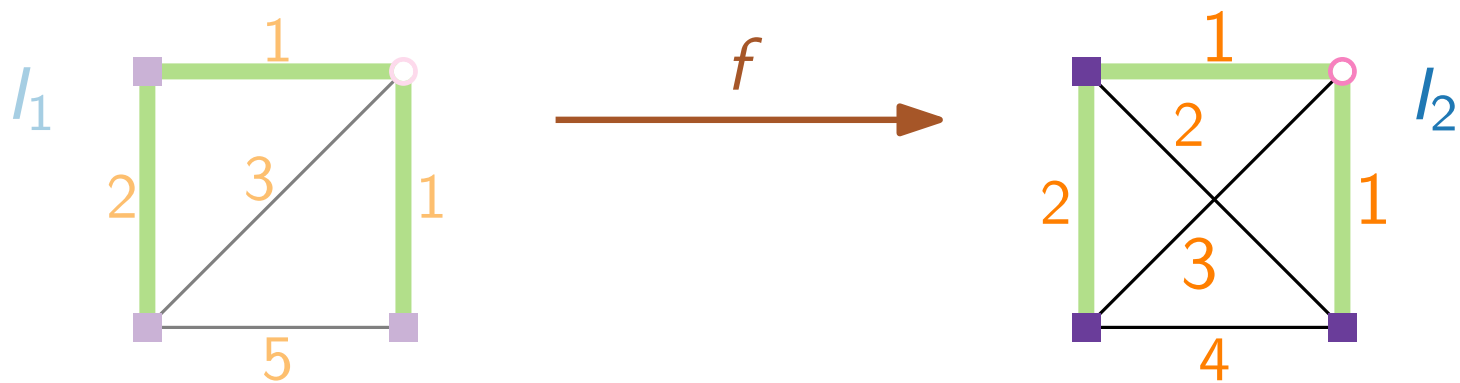

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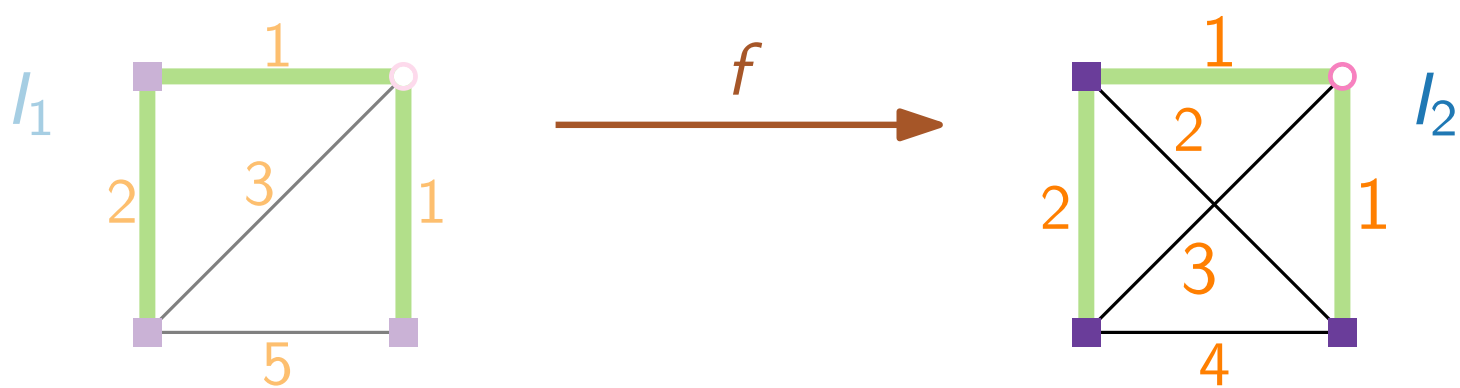

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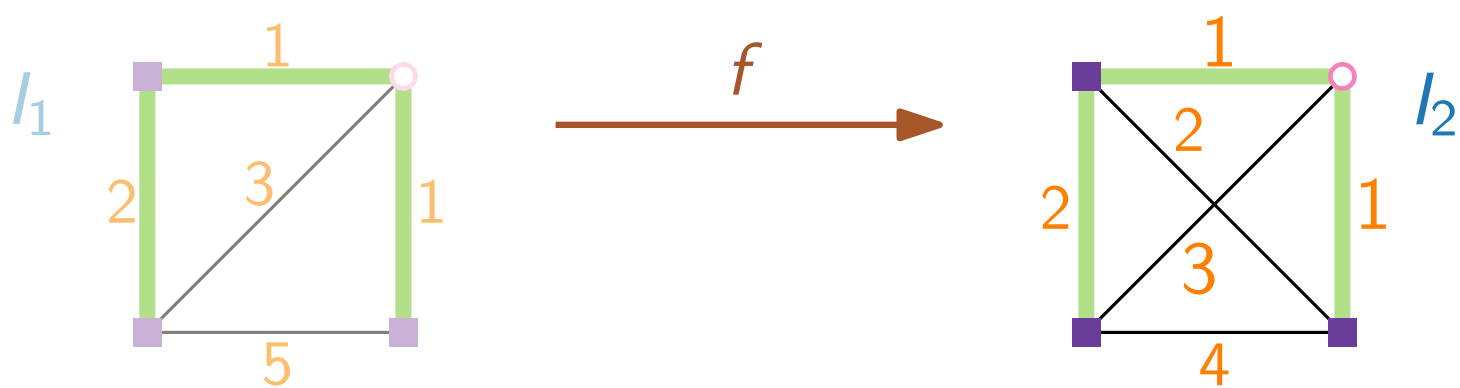

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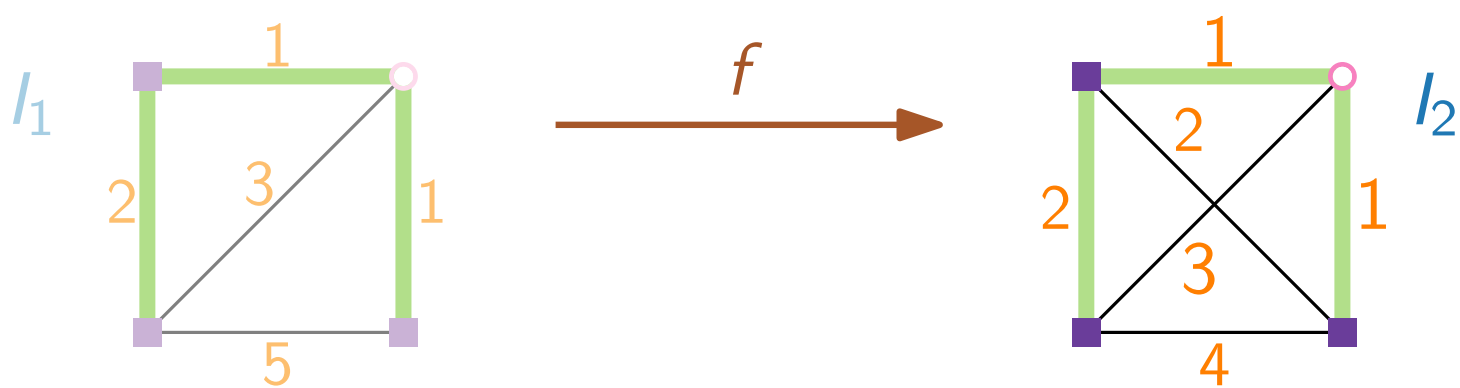


MetricSteinerTree

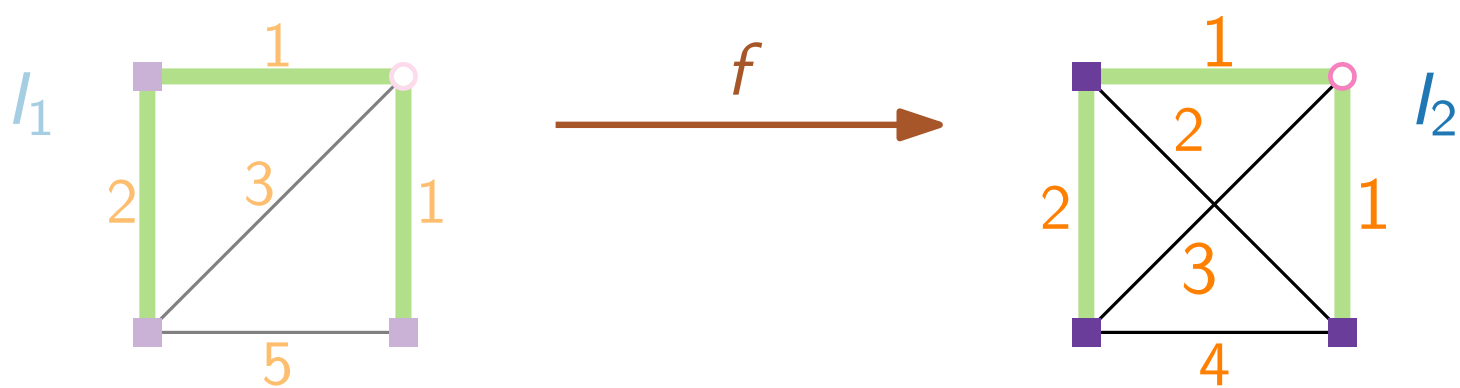
Theorem. There is an approximation-preserving reduction from SteinerTree to MetricSteinerTree.

Proof. (2) $\mathrm{OPT}(I_2) \leq \mathrm{OPT}(I_1)$

Let B^* be an optimal Steiner tree for I_1.

Note that B^* is also a feasible solution for I_2:
$E_1 \subseteq E_2$ and the vertex sets V, T, S are the same.

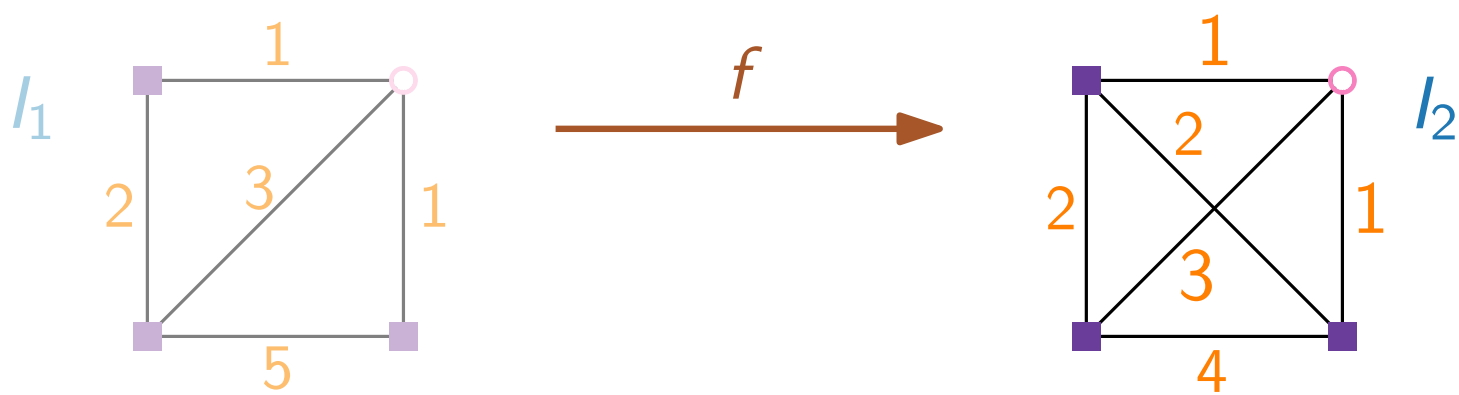
$\mathrm{OPT}(I_2) \leq c_2(B^*) \leq c_1(B^*) = \mathrm{OPT}(I_1)$



MetricSteinerTree

Theorem. There is an approximation-preserving reduction from SteinerTree to MetricSteinerTree.

Proof. (3) Mapping g $s \xleftarrow{g} t$



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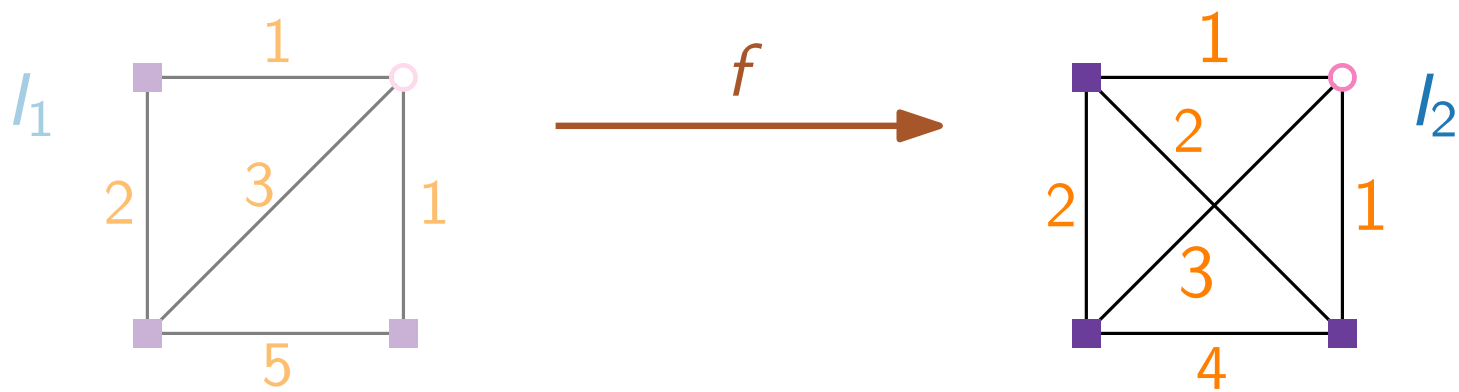

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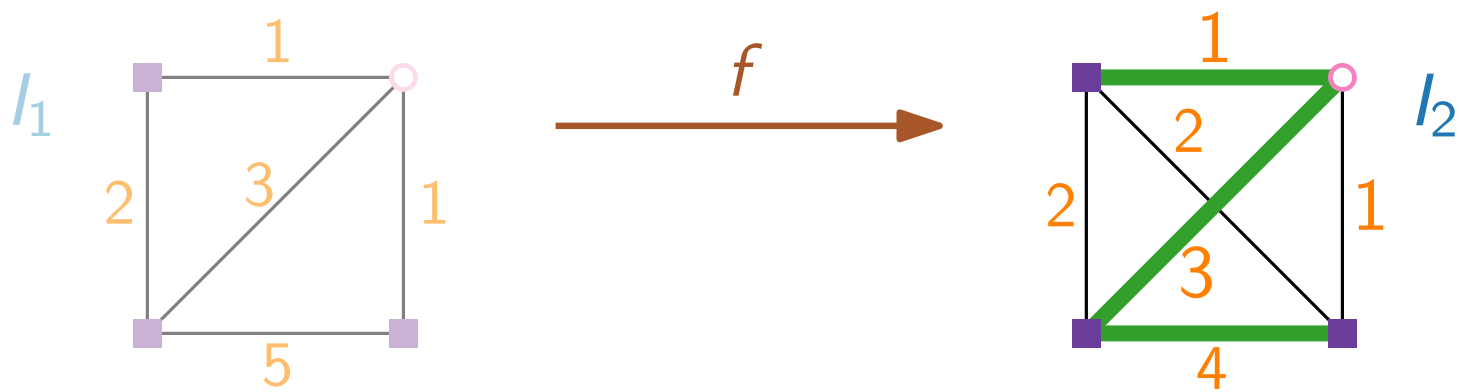

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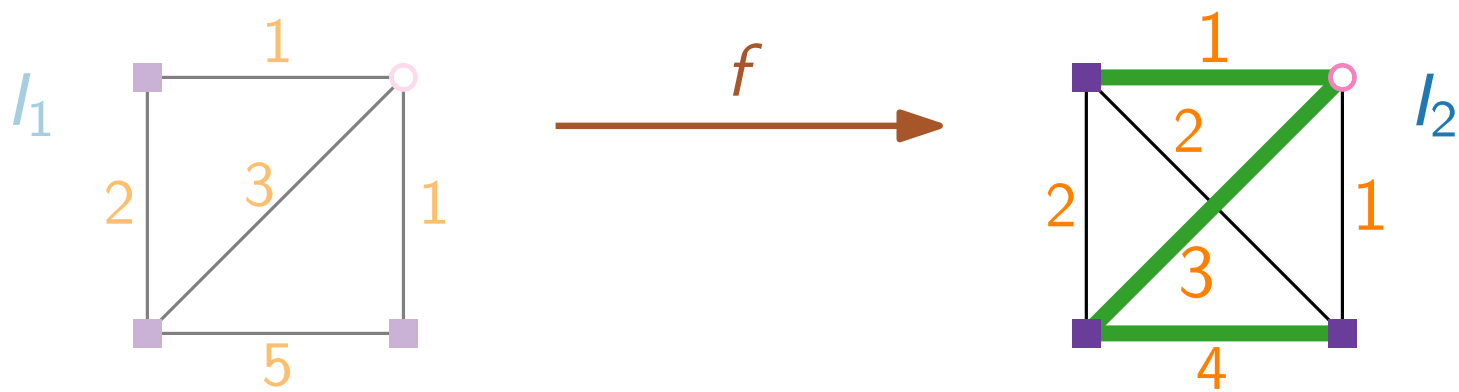

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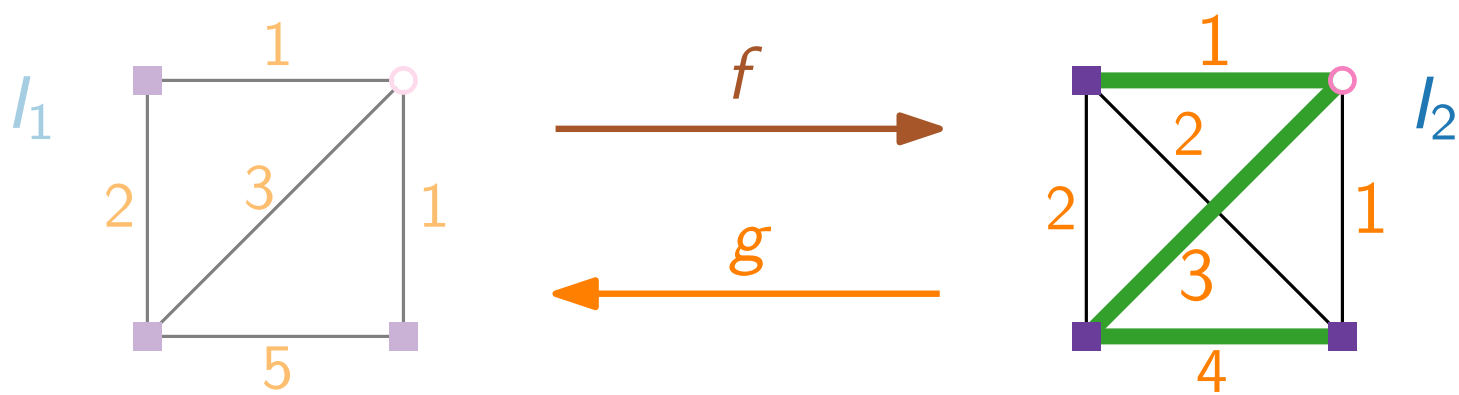

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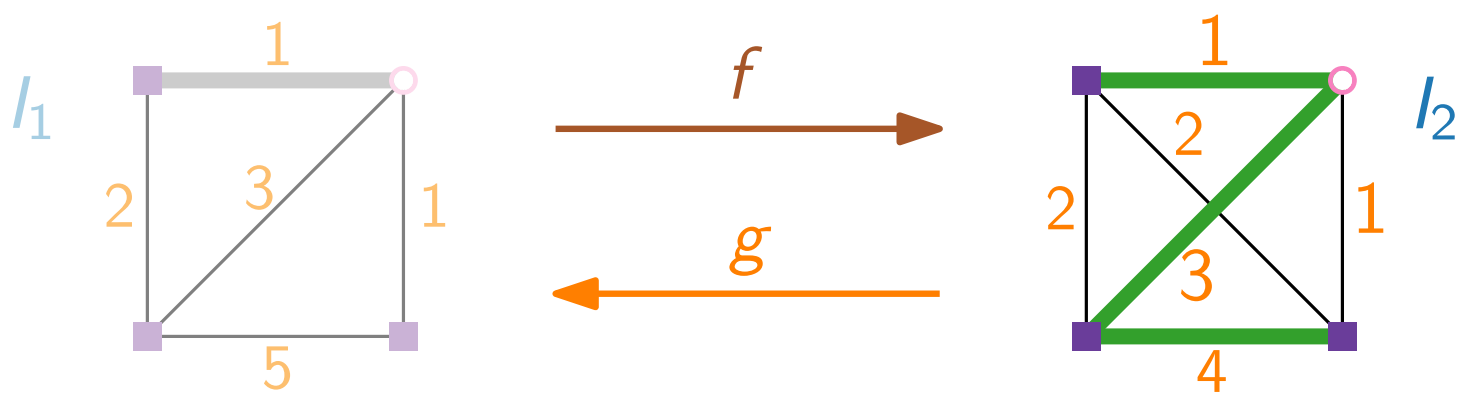

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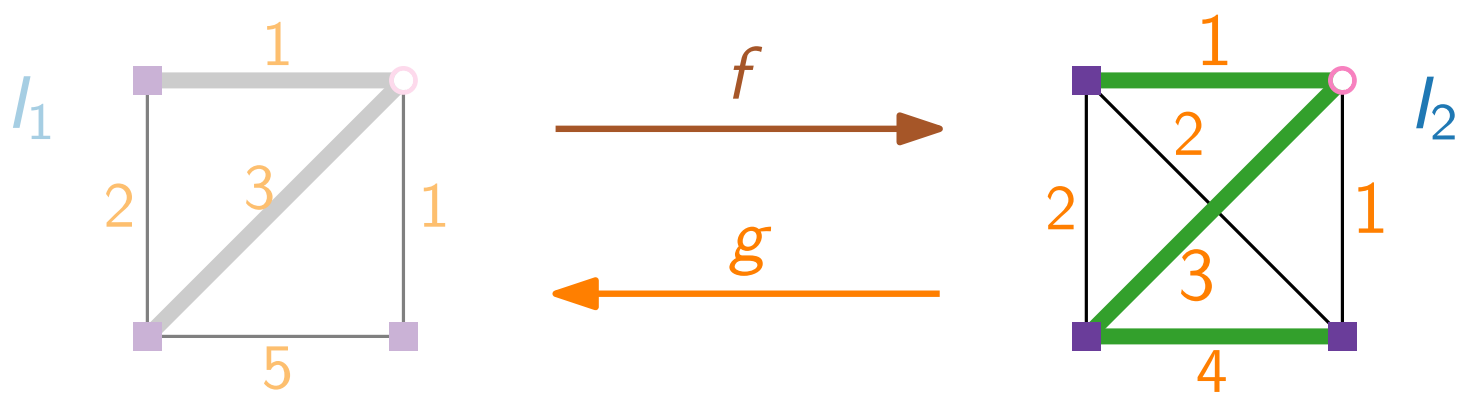

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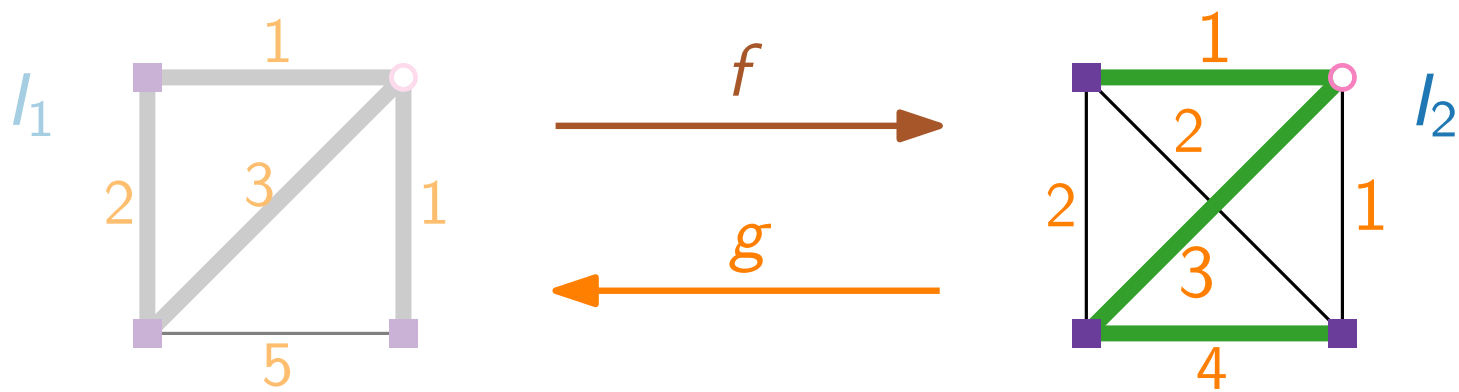

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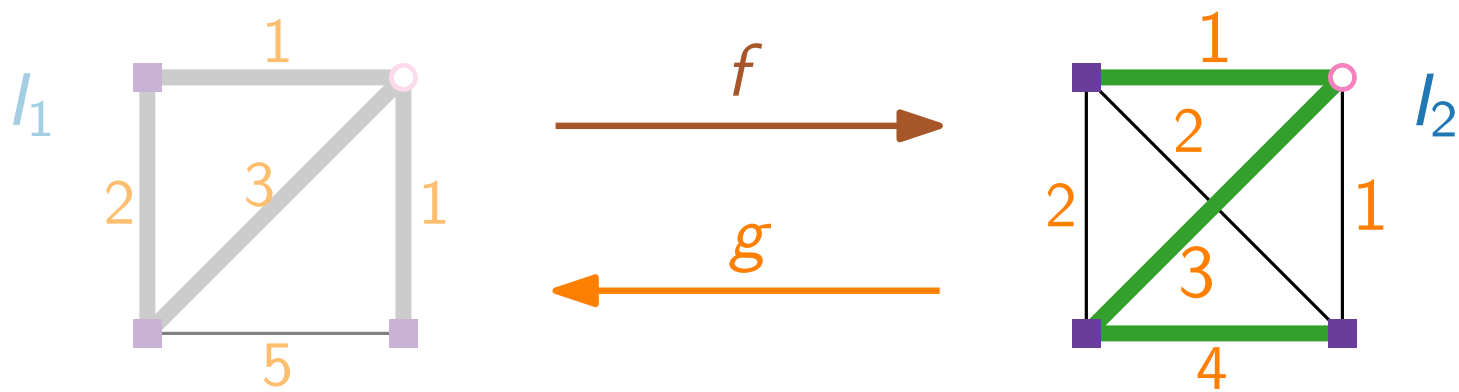

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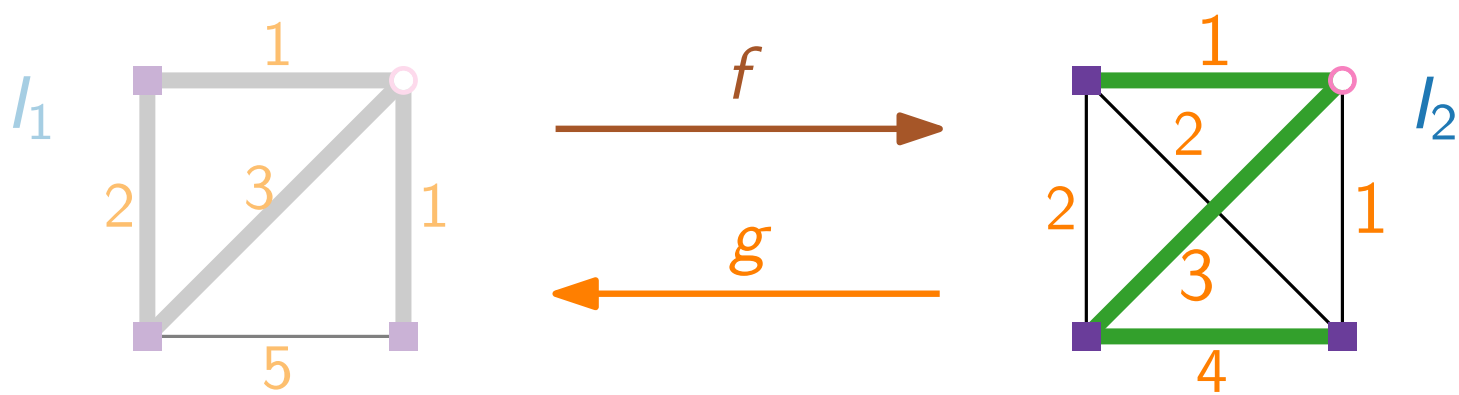
$c_1(G_1') \leq c_2(B_2)$



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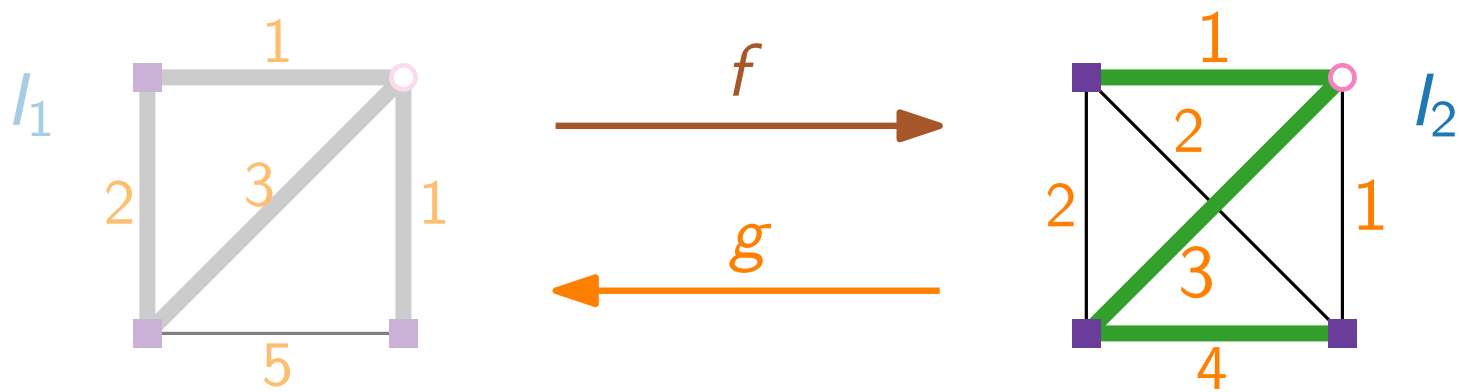

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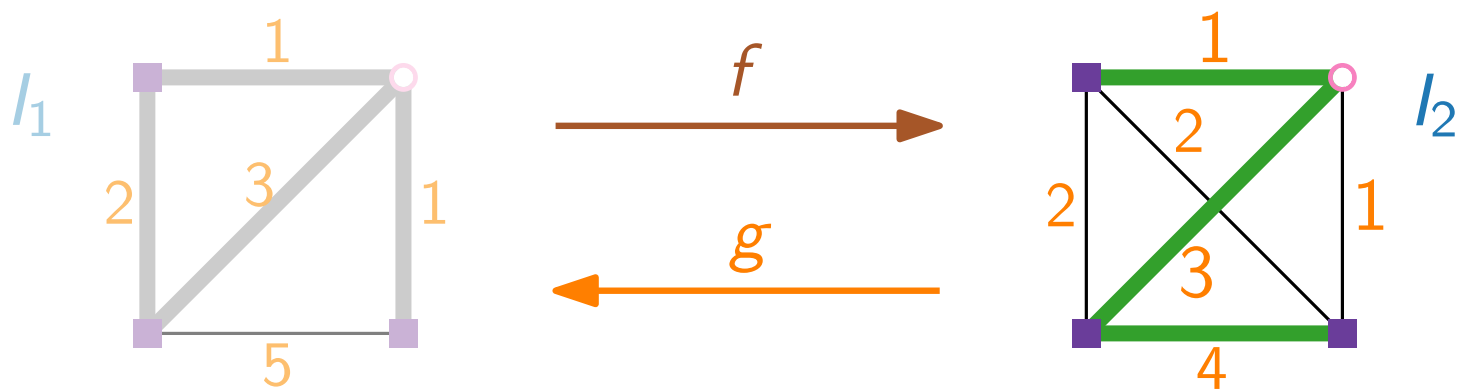

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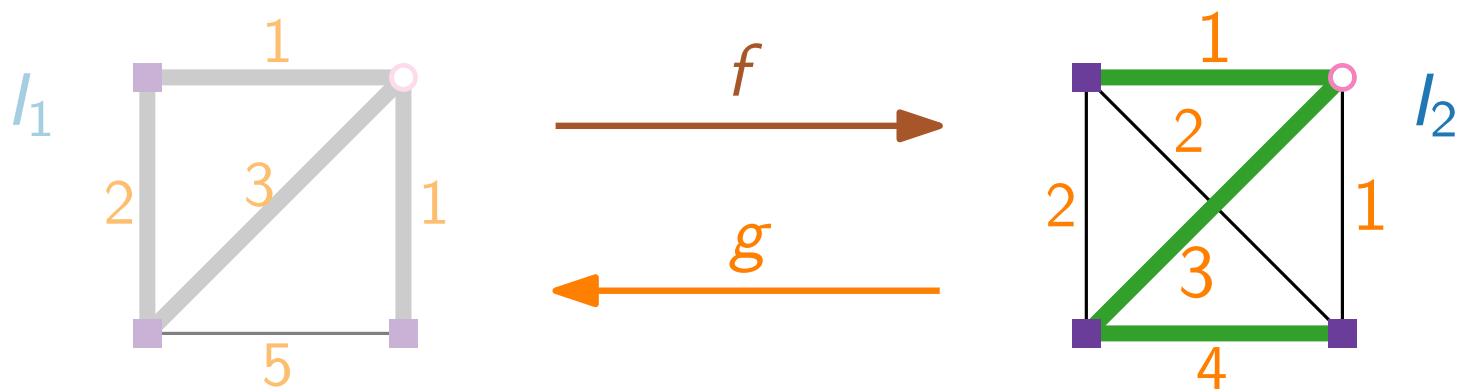

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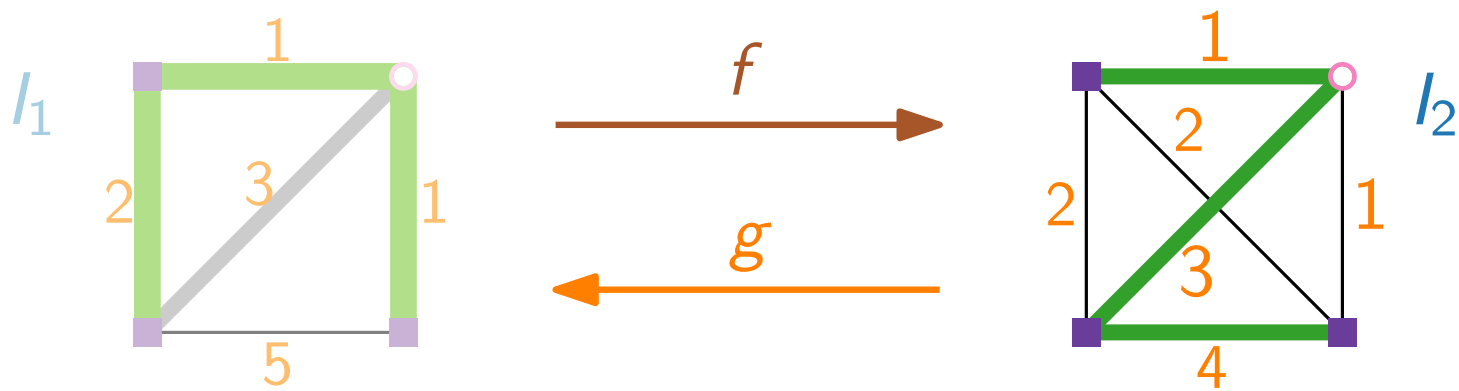

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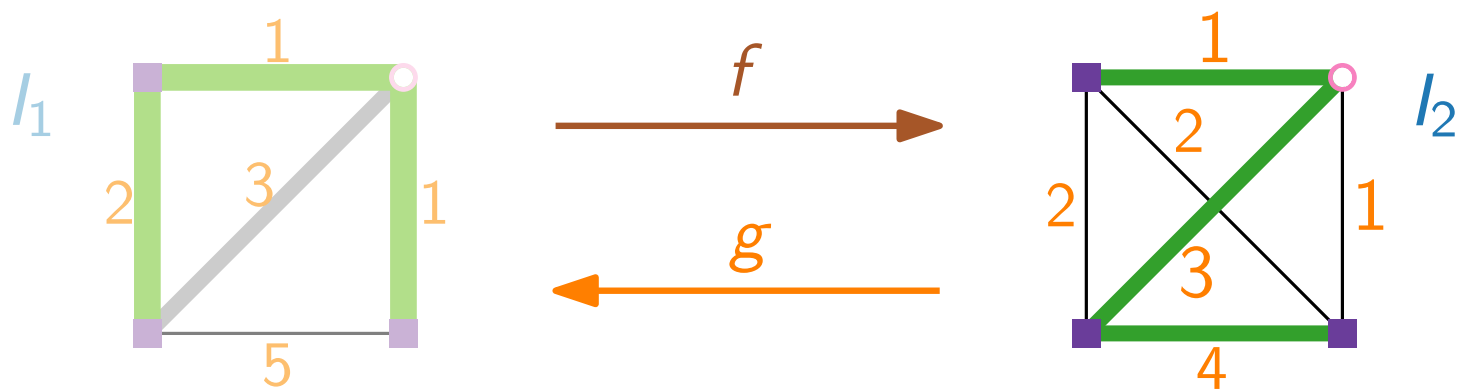

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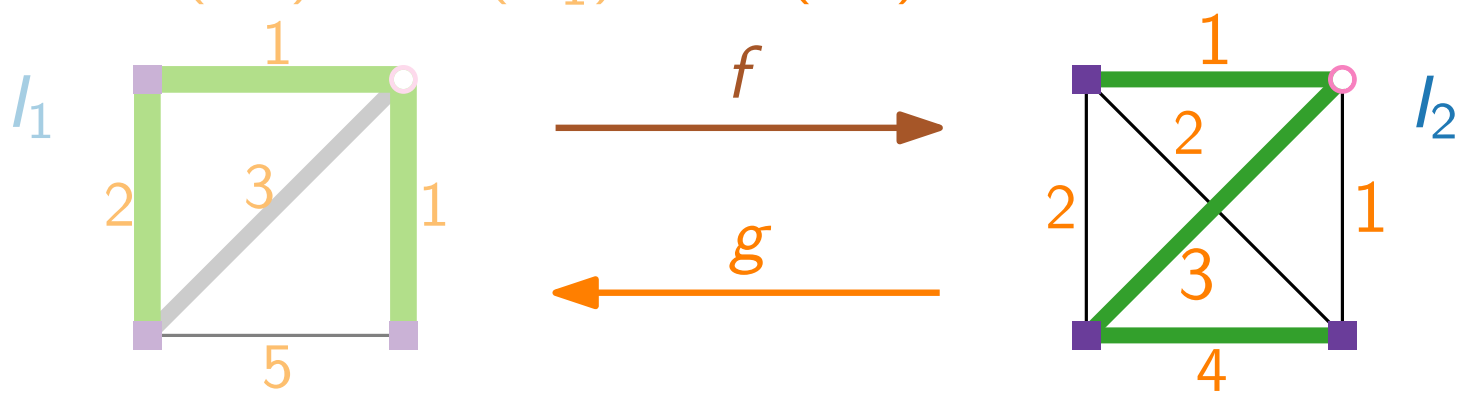
Note that $c_1(B_1) \leq c_1(G_1') \leq c_2(B_2)$.



Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part IV:
2-Approximation for MetricSteinerTree

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Theorem. For an instance of MetricSteinerTree, let B be a minimum spanning tree (MST) of the subgraph $G[T]$ induced by the terminal set T. Then $c(B) \leq 2 \cdot \mathrm{OPT}$.

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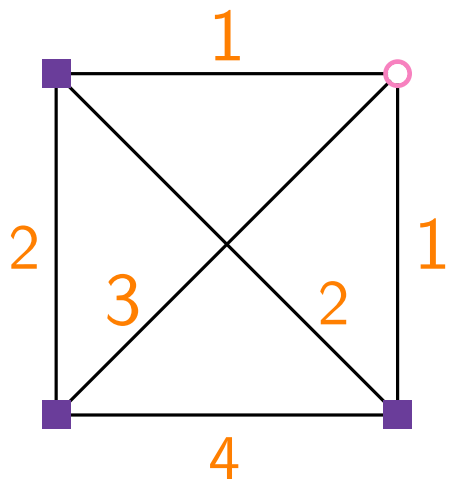

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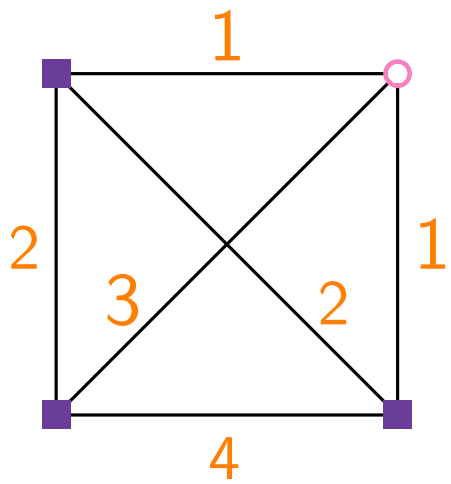

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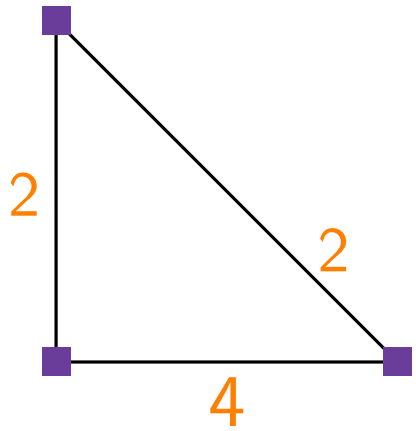

$G[T]$

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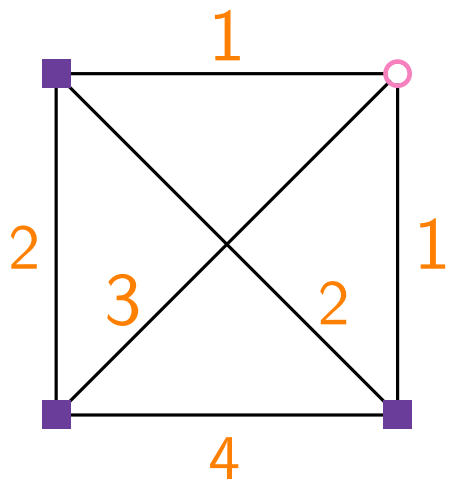

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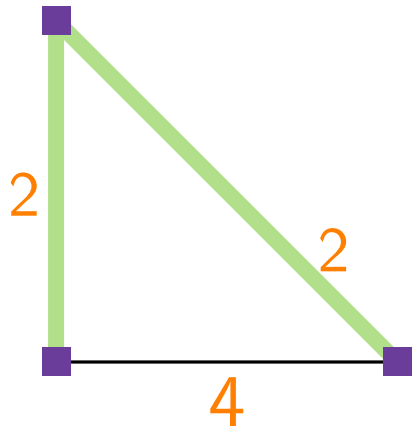

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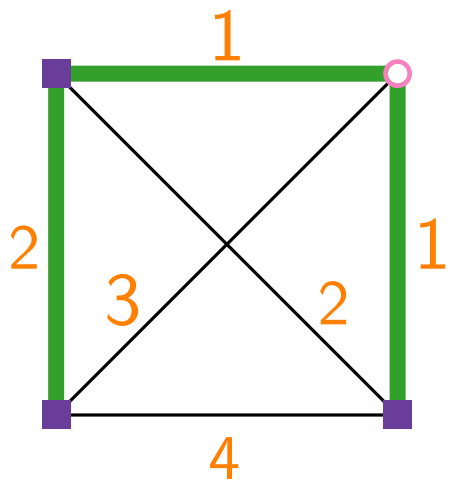

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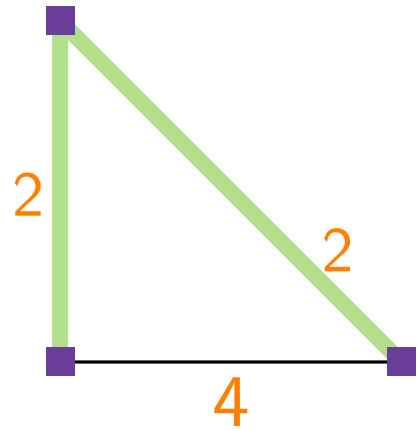

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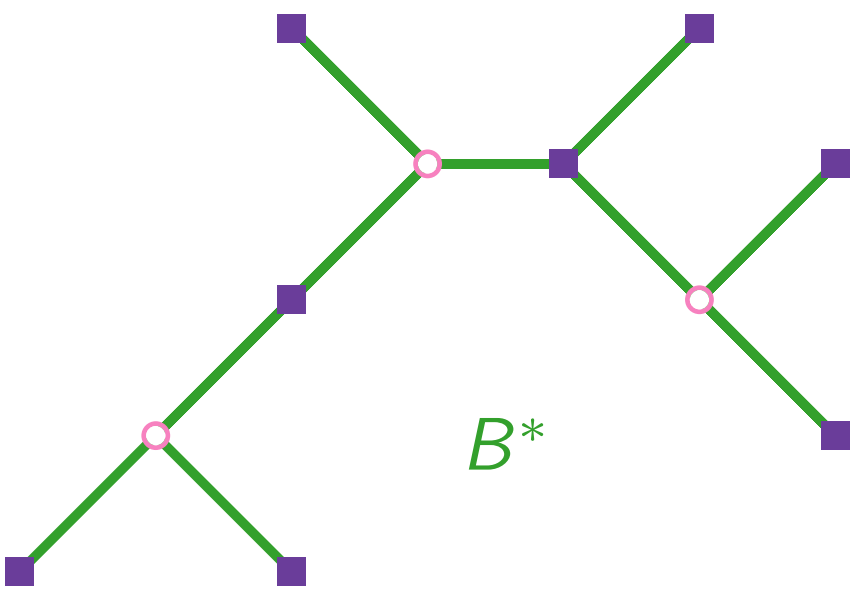

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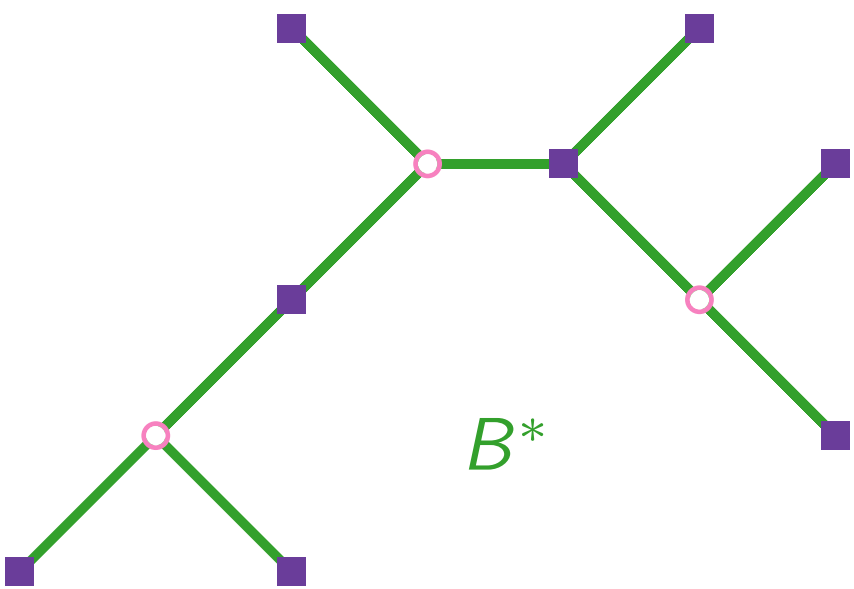
Consider an optimal Steiner tree B^*.

Duplicate all edges of B^*.

$\Rightarrow$ Eulerian (multi-)graph B' with cost $c(B') = 2 \cdot \mathrm{OPT}$.



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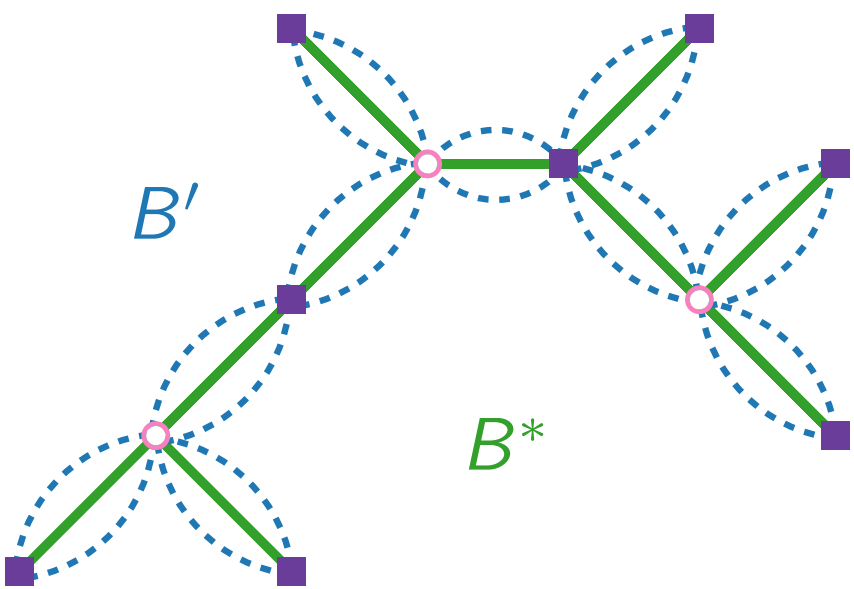

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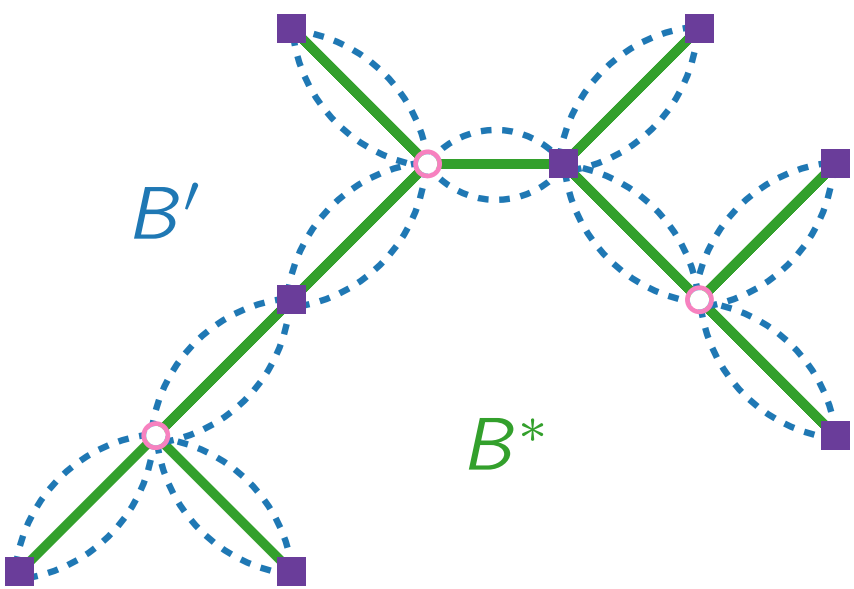

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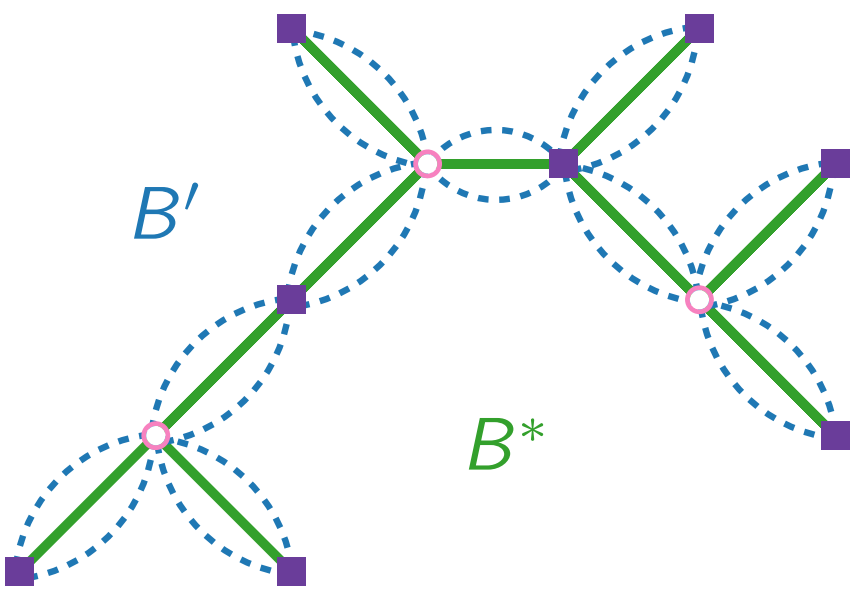

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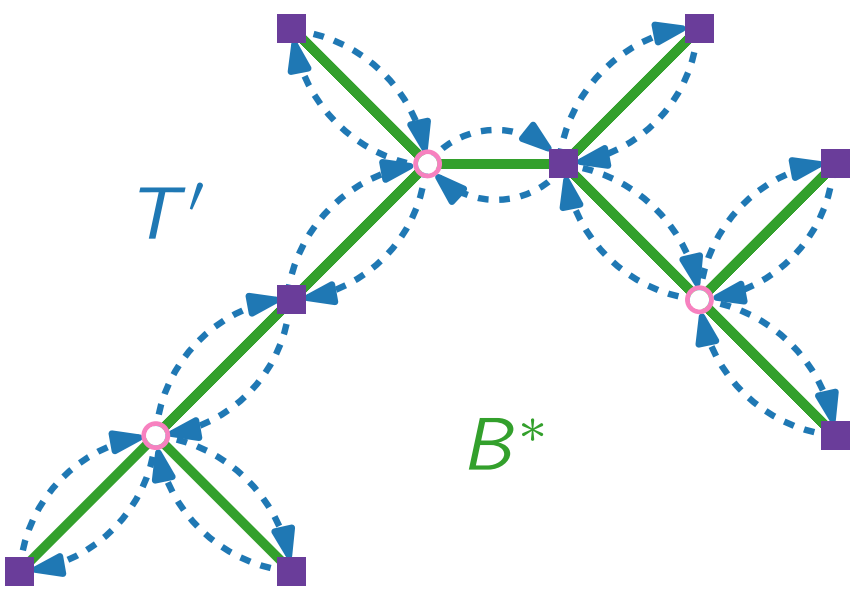

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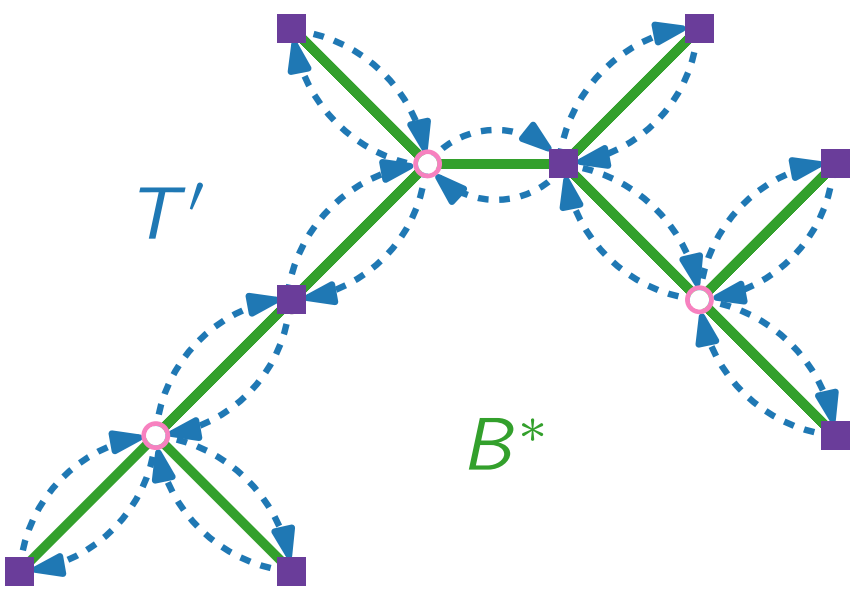

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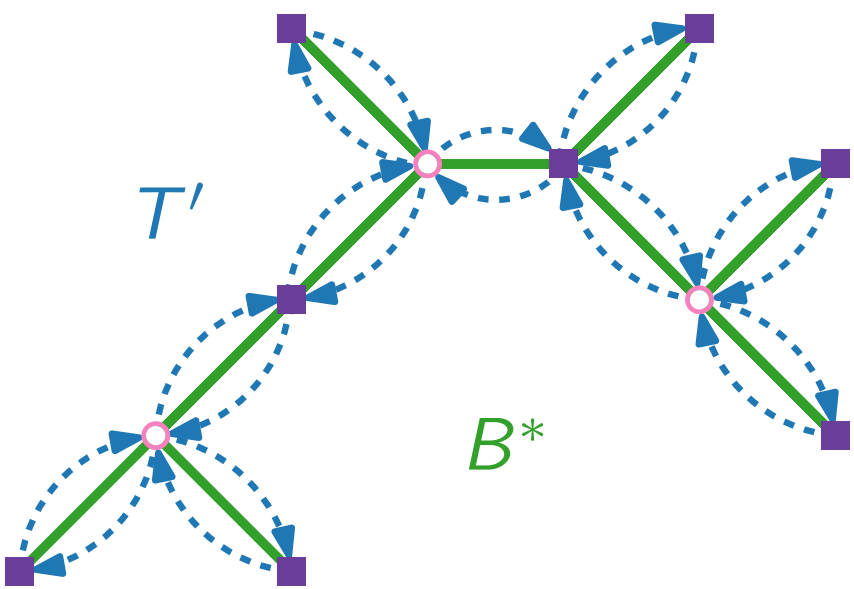

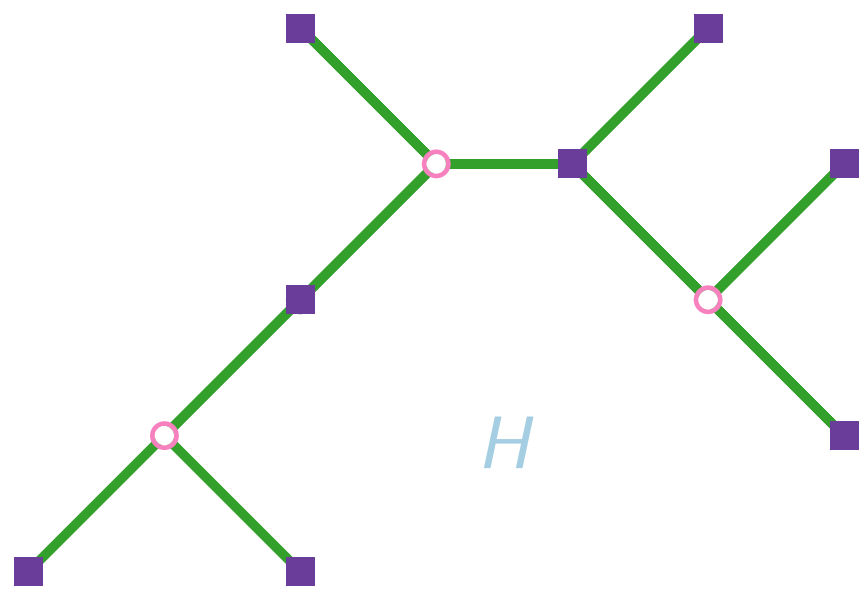

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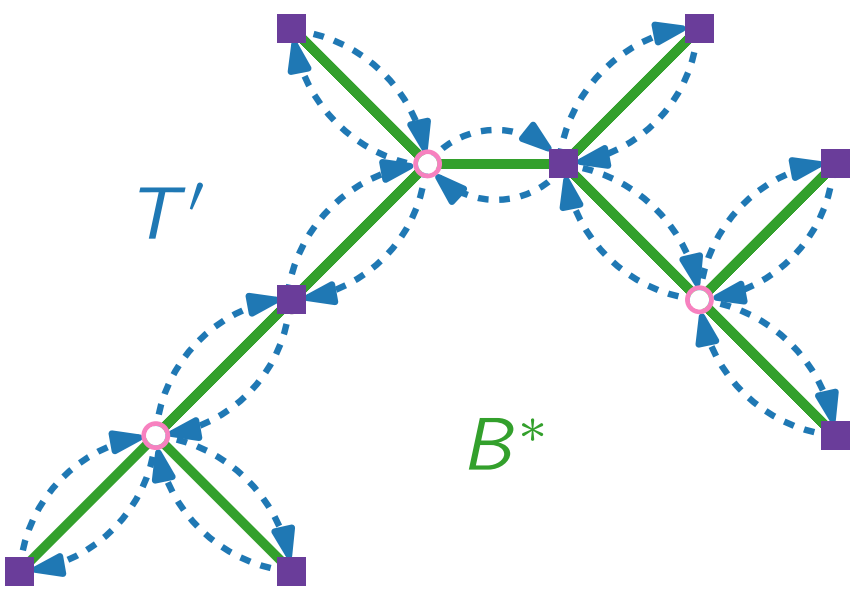

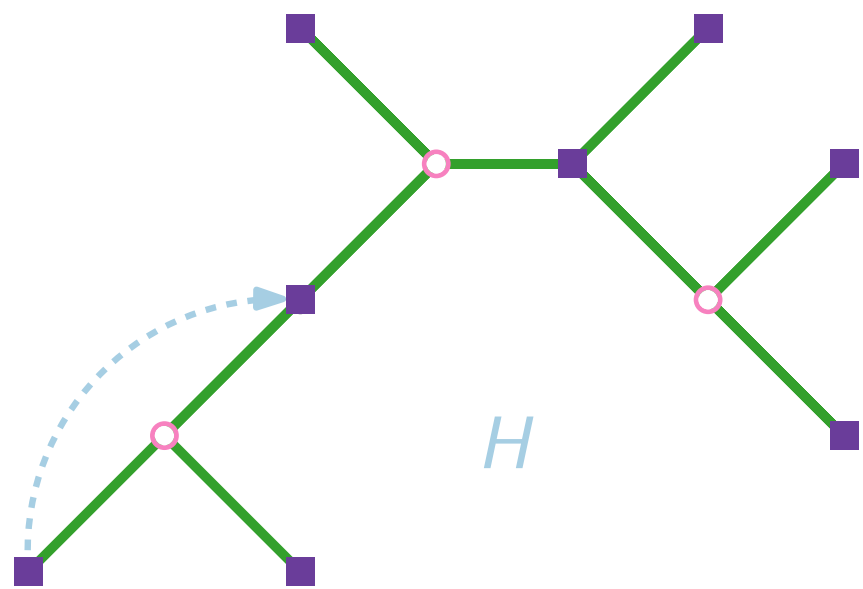

Proof of the Approximation Factor

Consider an optimal Steiner tree B^*.

Duplicate all edges of B^*.

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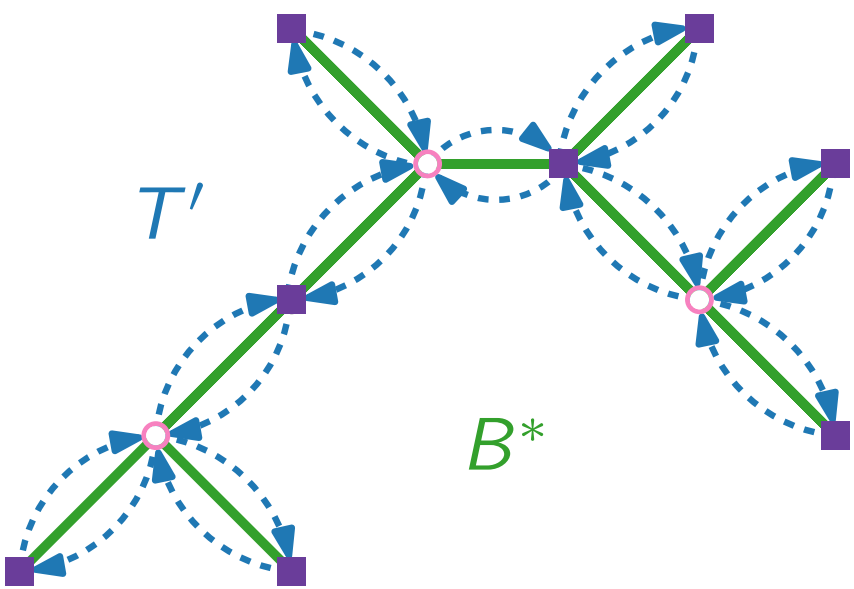

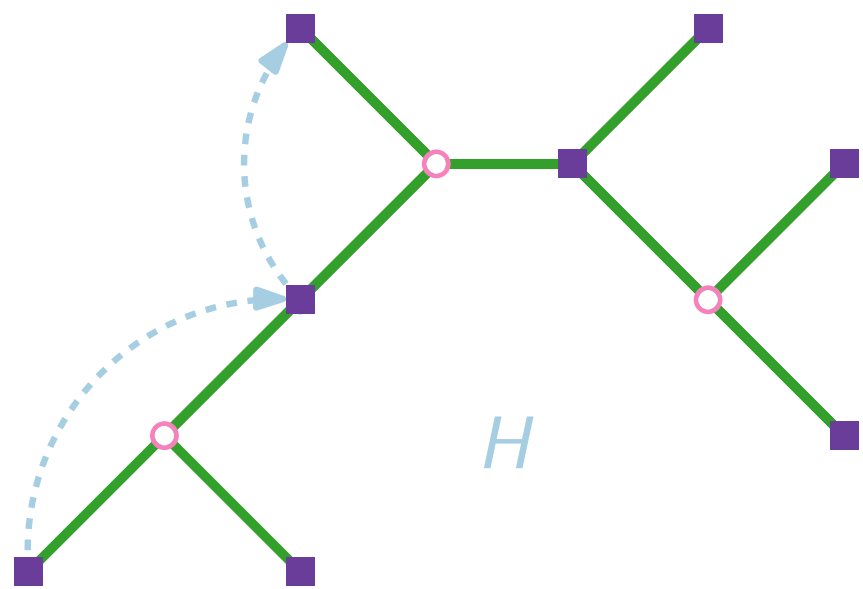

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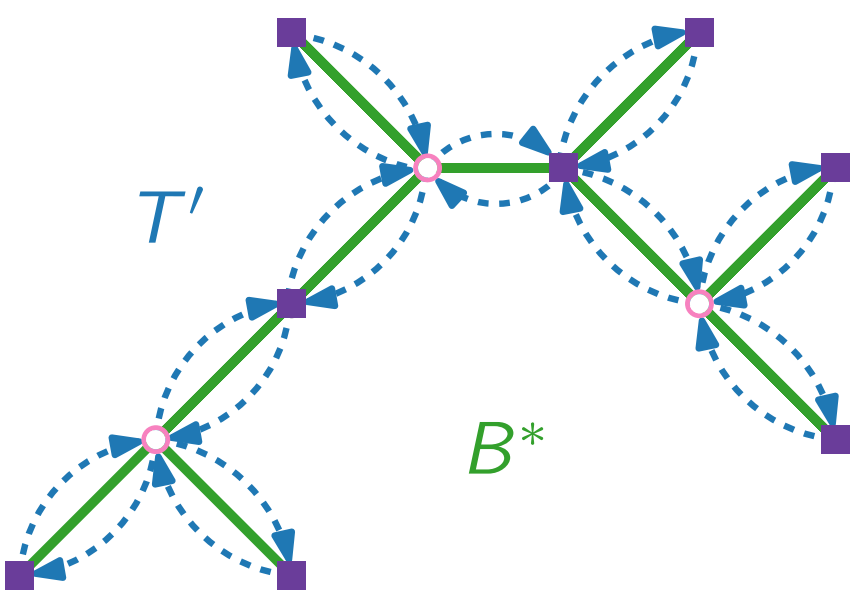

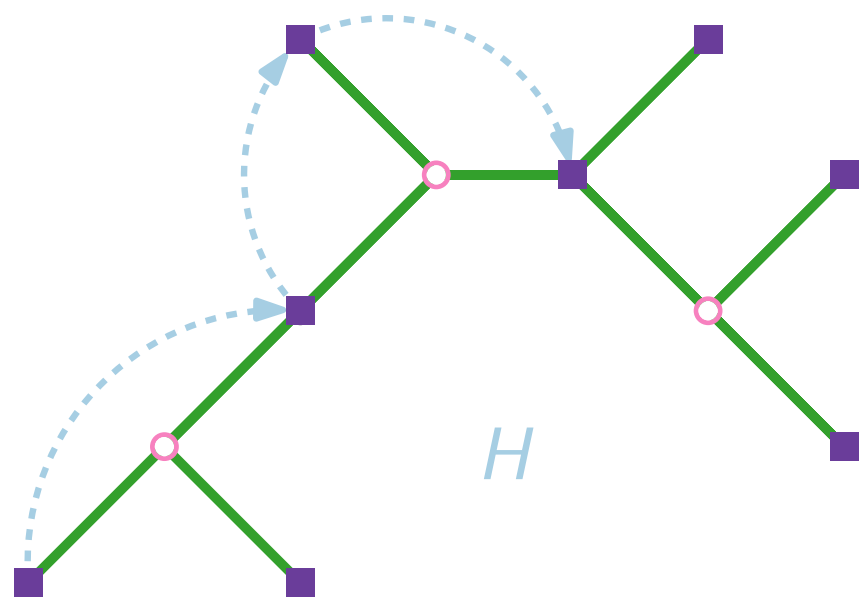

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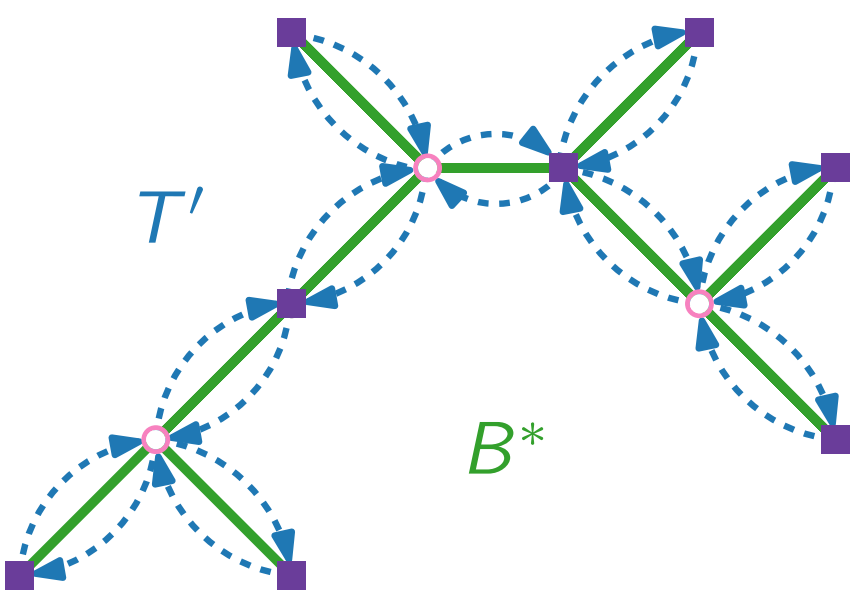

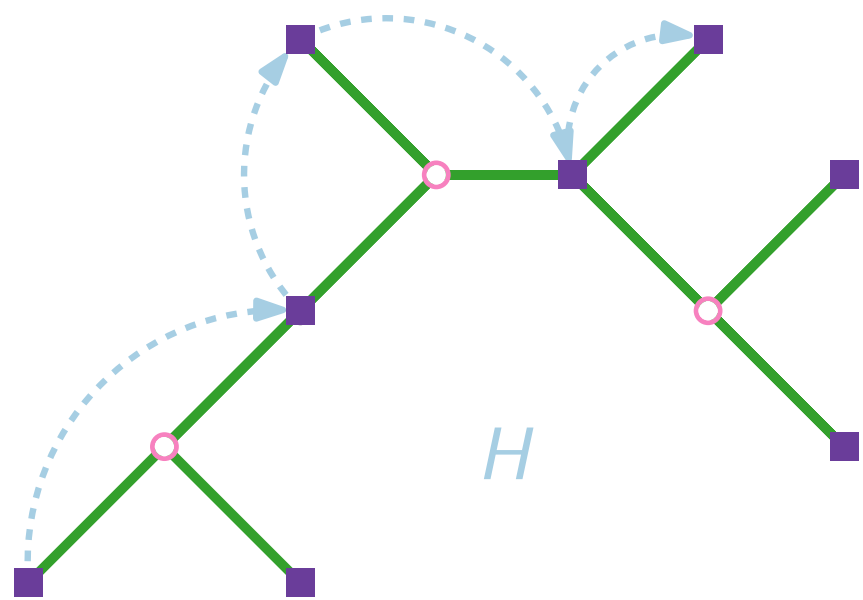

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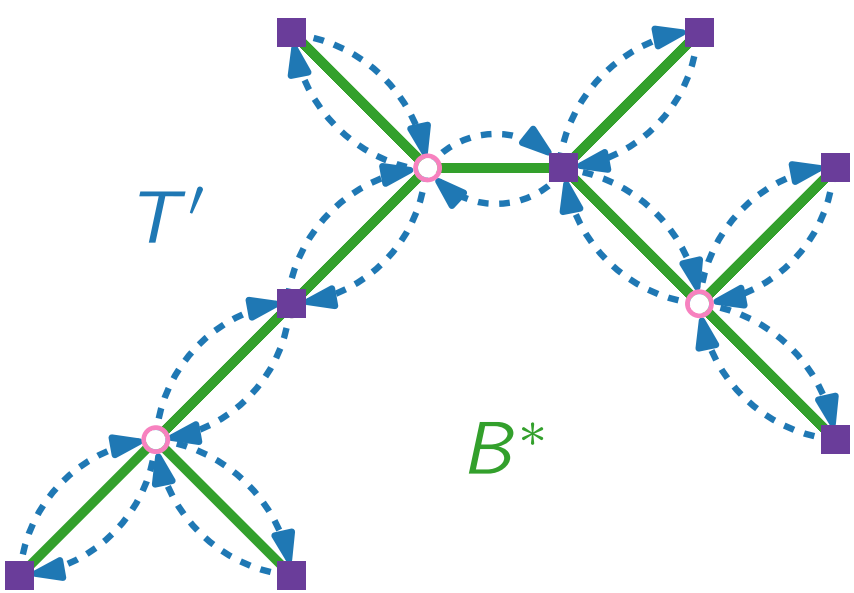

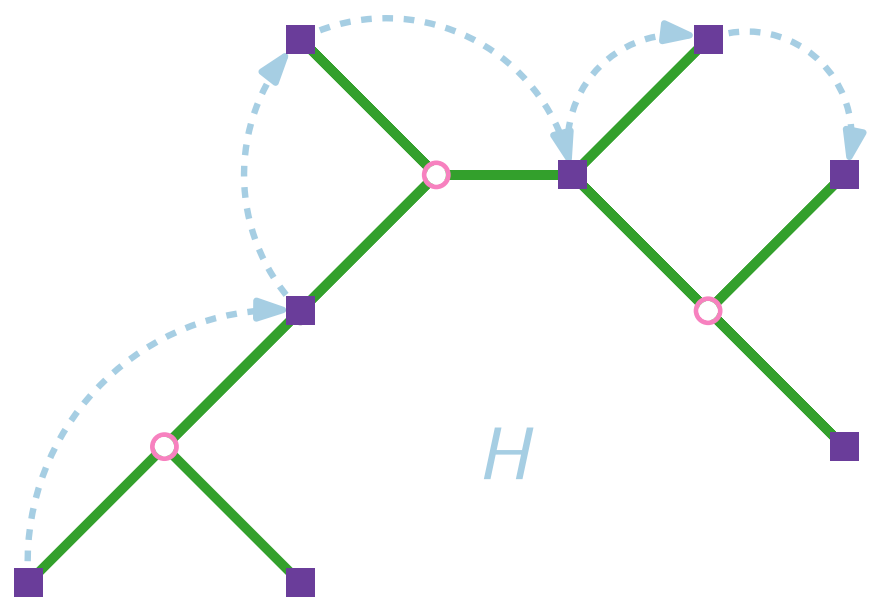

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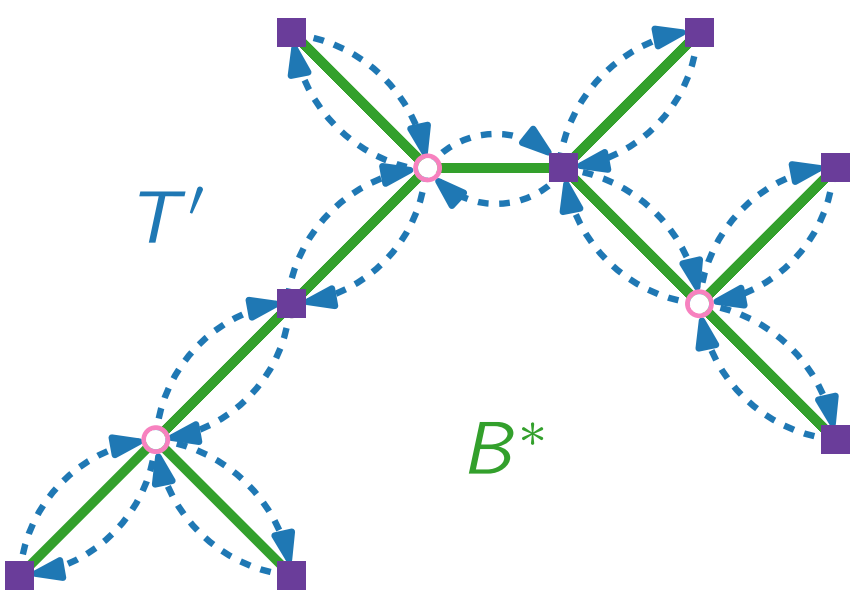

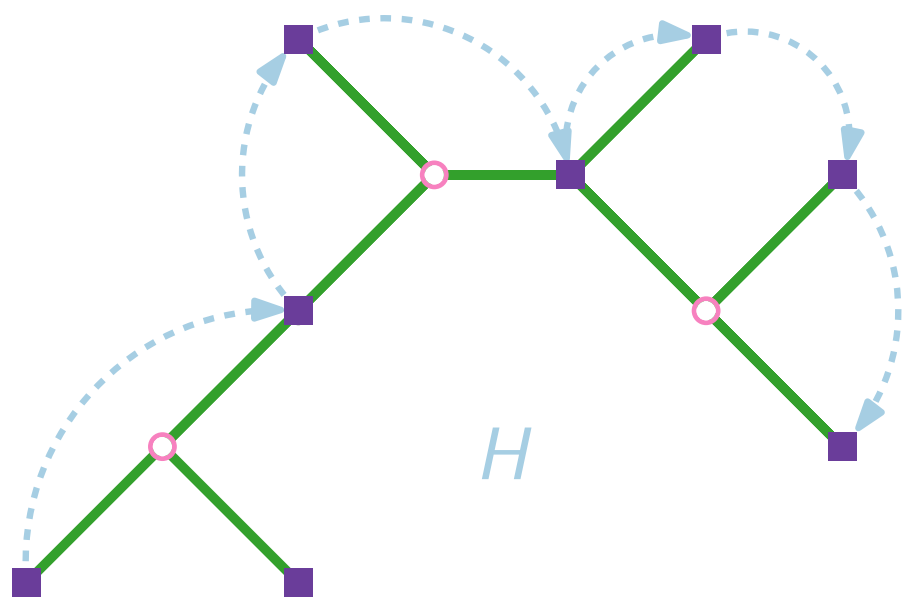

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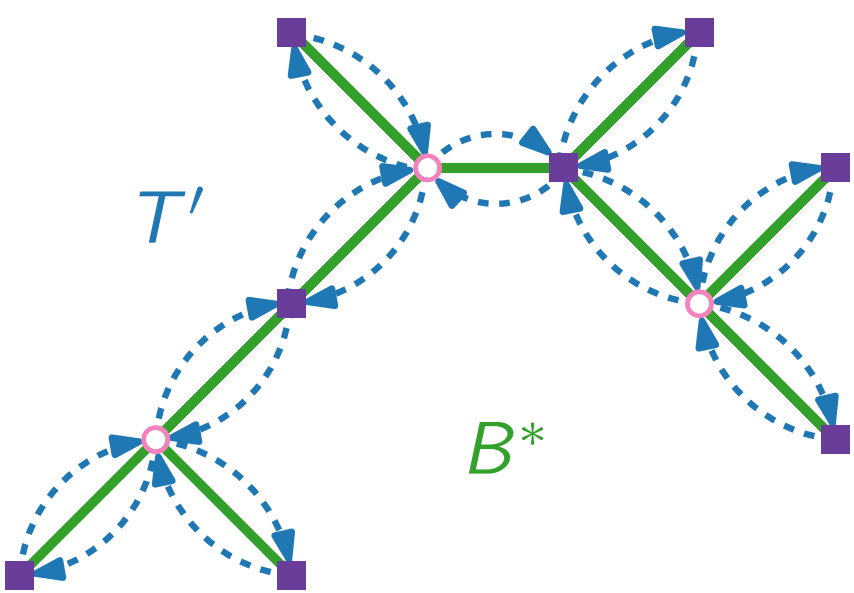

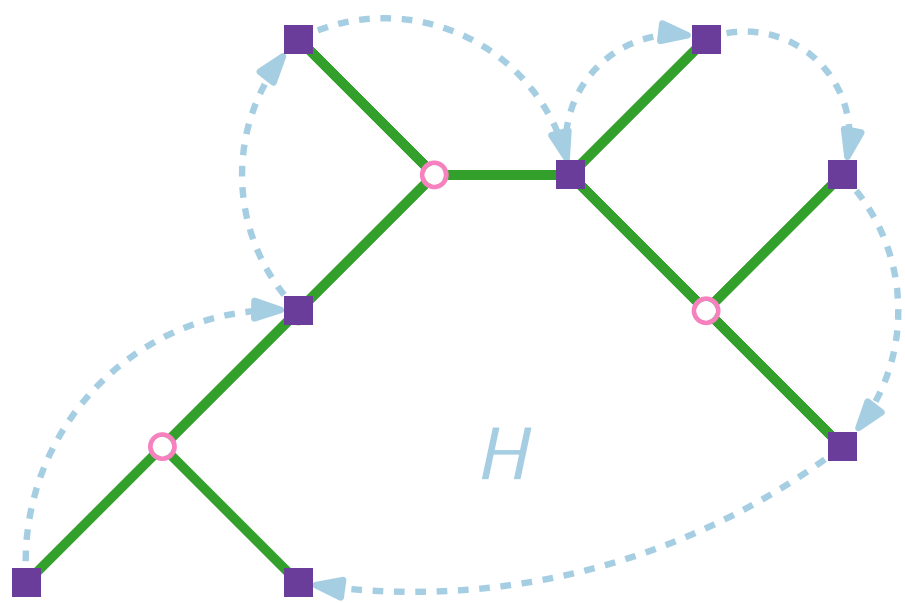

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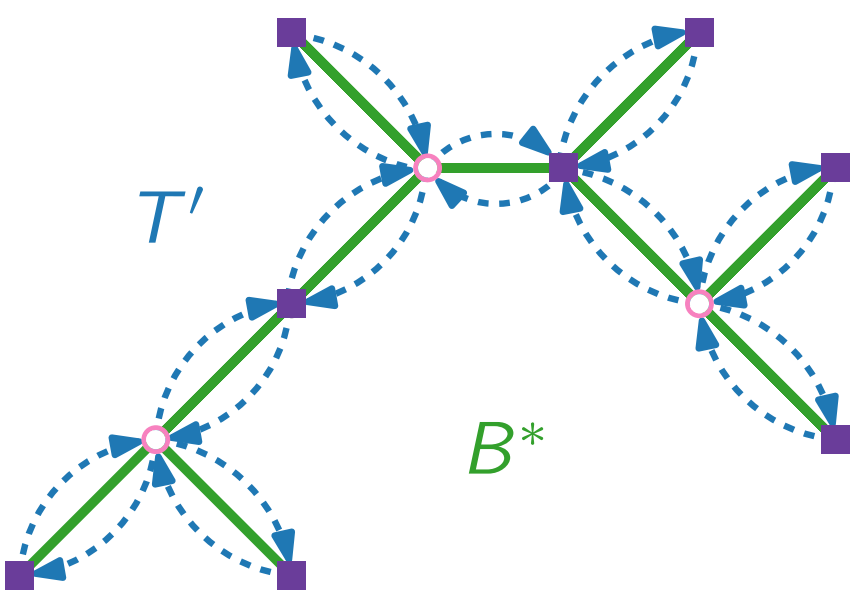
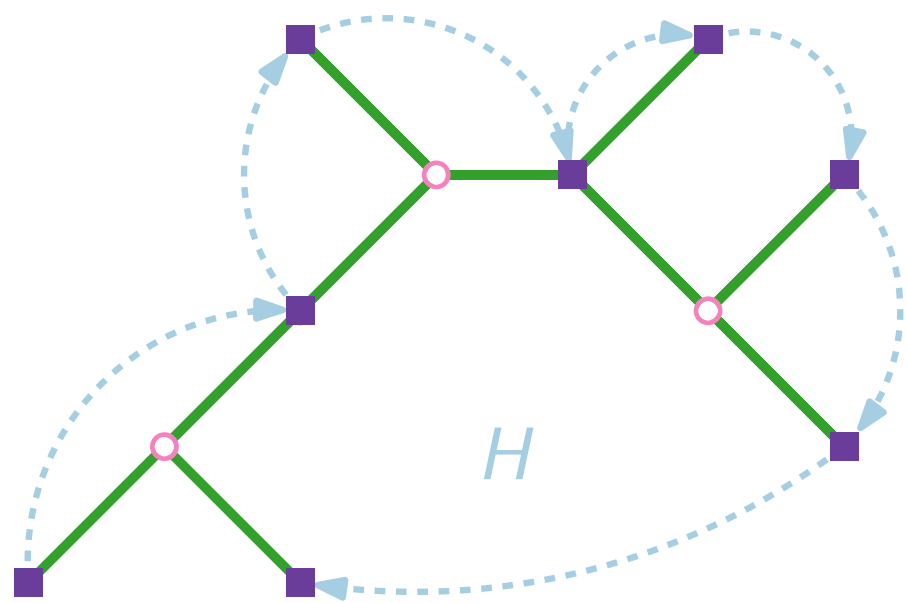
$\Rightarrow$ $c(H) \leq c(T') = 2 \cdot \mathrm{OPT}$ since G is metric.





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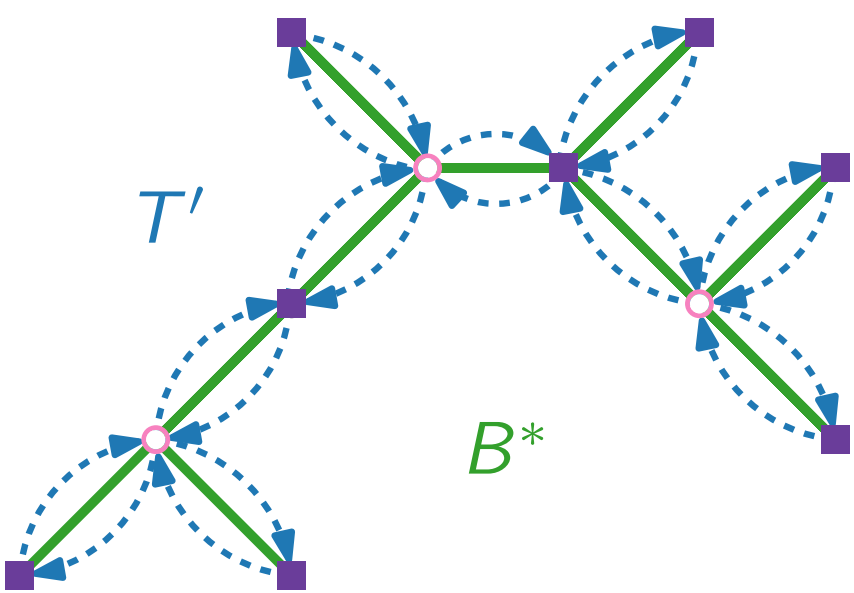
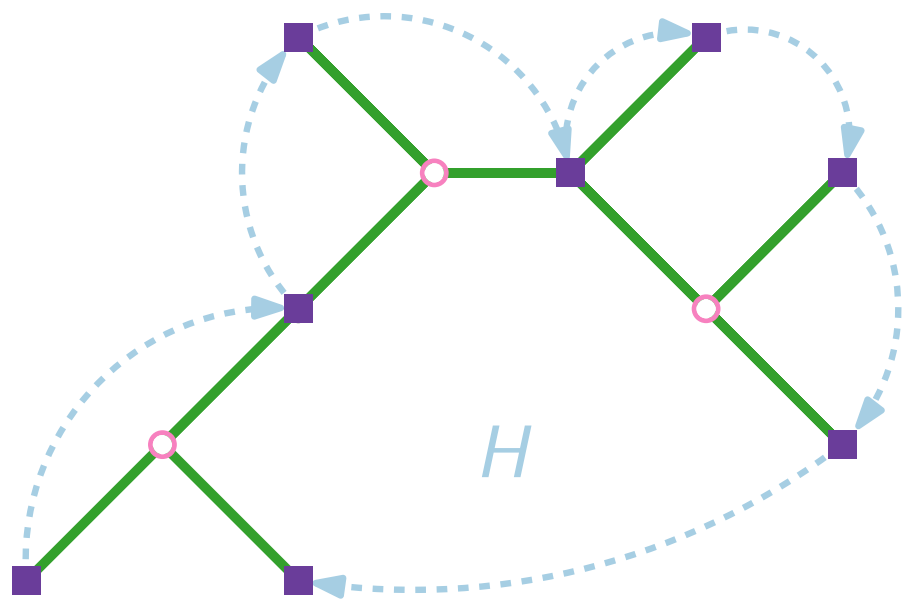
MST B of $G[T]$ costs $c(B) \leq c(H) \leq 2 \cdot \mathrm{OPT}$





Proof of the Approximation Factor

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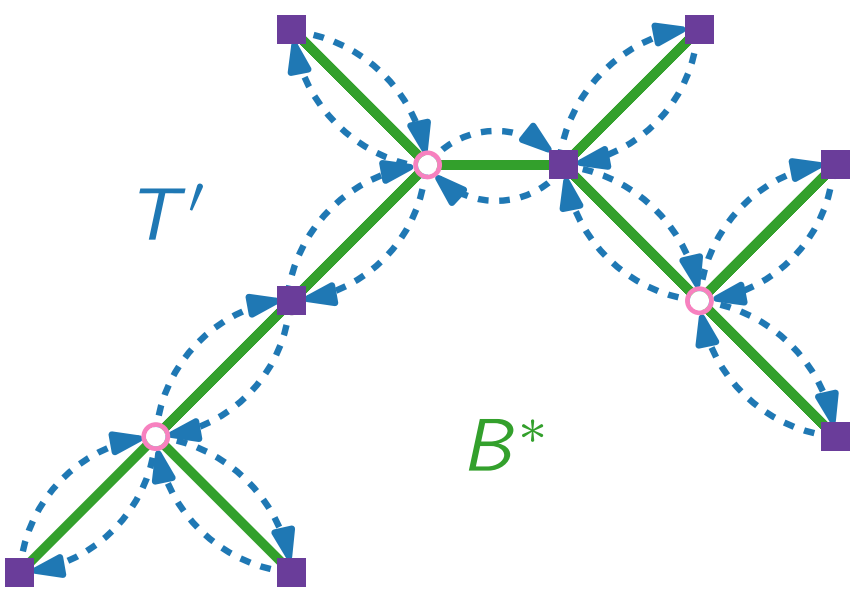
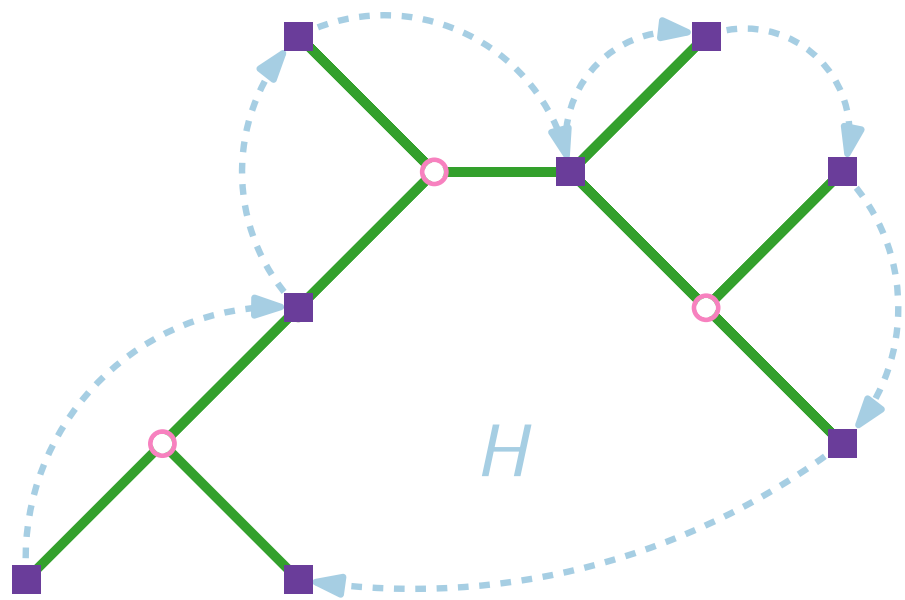
$\Rightarrow$ $c(H) \leq c(T') = 2 \cdot \mathrm{OPT}$ since G is metric.

MST B of $G[T]$ costs $c(B) \leq c(H) \leq 2 \cdot \mathrm{OPT}$ since H is a spanning tree of $G[T]$.

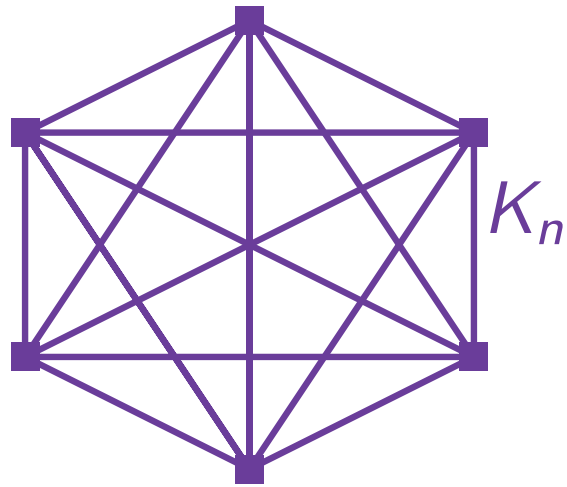




Analysis Tight?

Analysis Tight?

- terminal



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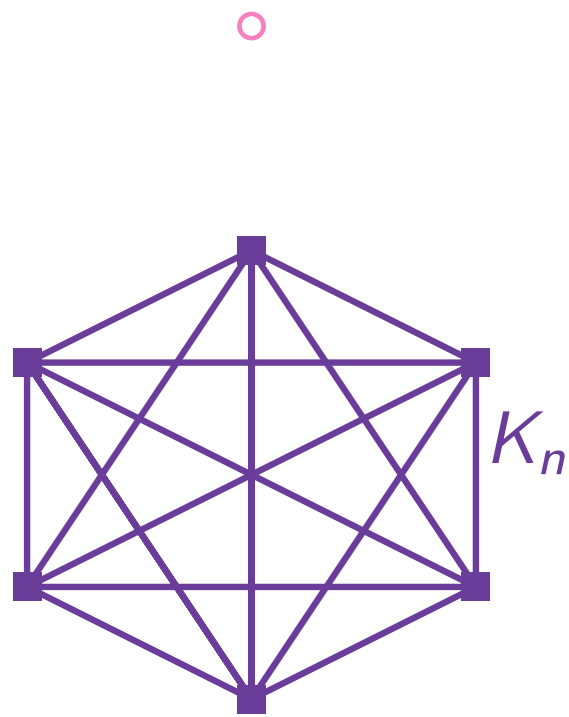
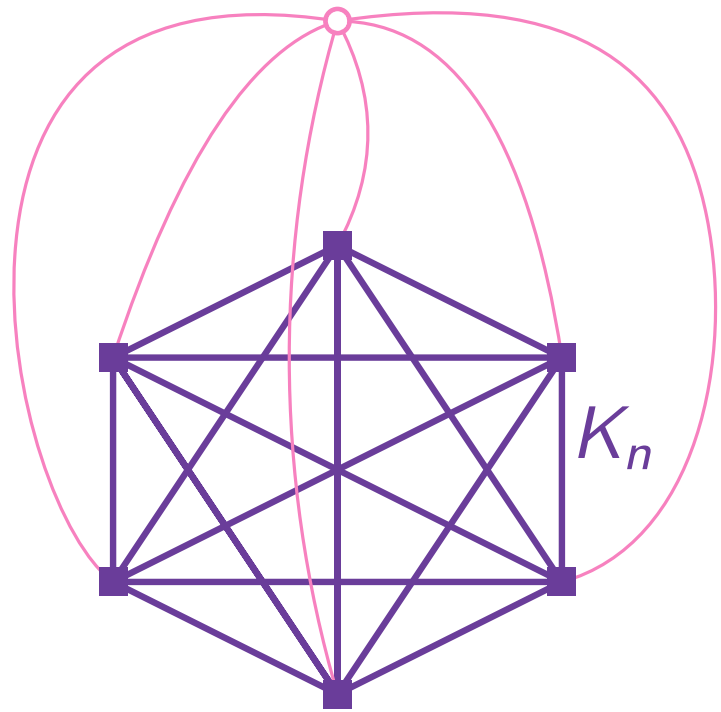


- terminal
- Steiner vertex

Analysis Tight?



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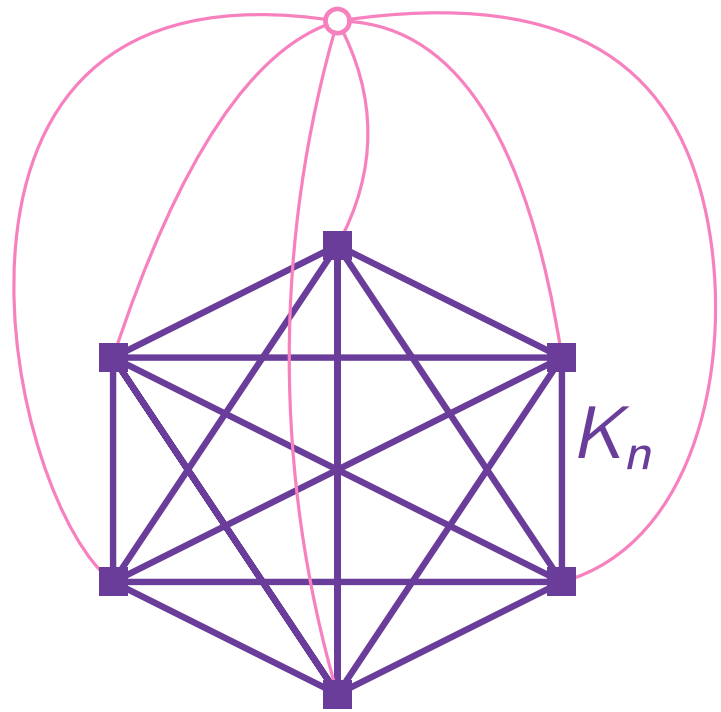


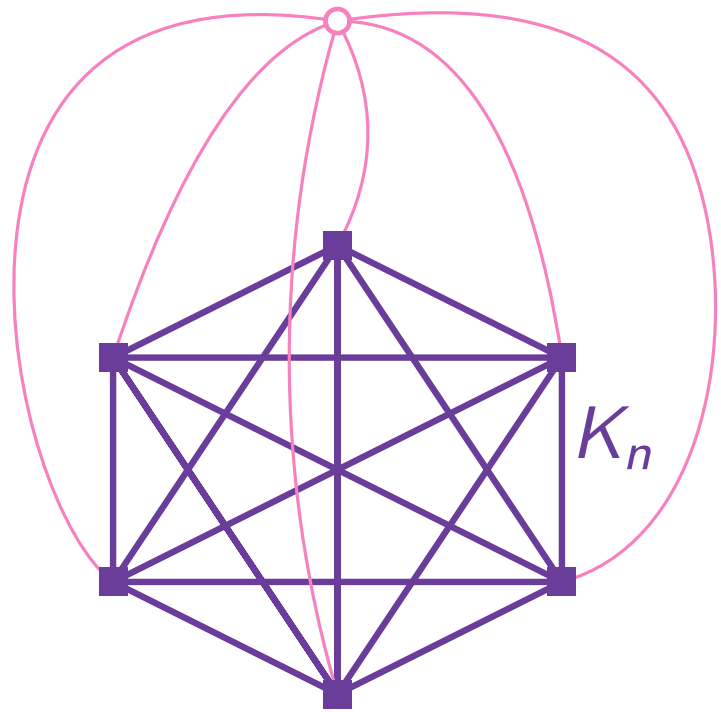
terminal

Steiner vertex

cost

cost

Analysis Tight?



terminal

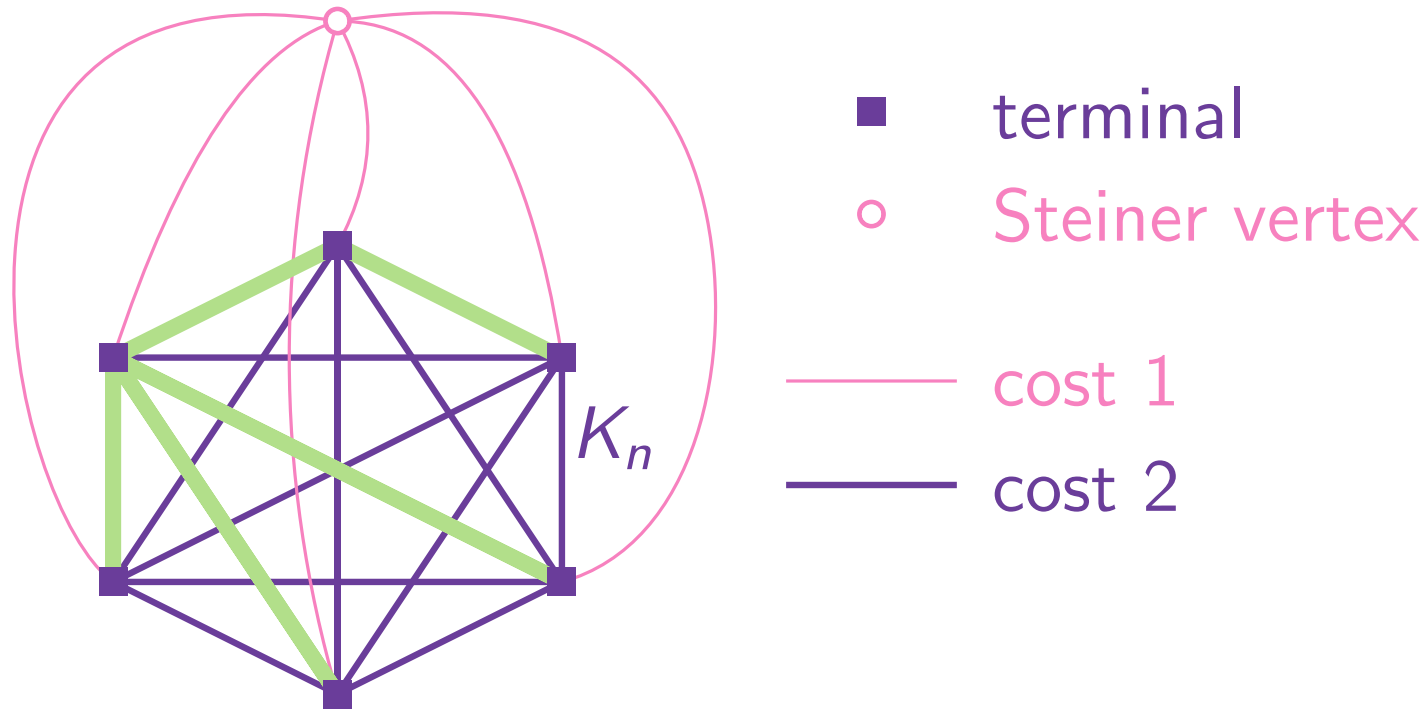
Steiner vertex

cost 1

cost 2

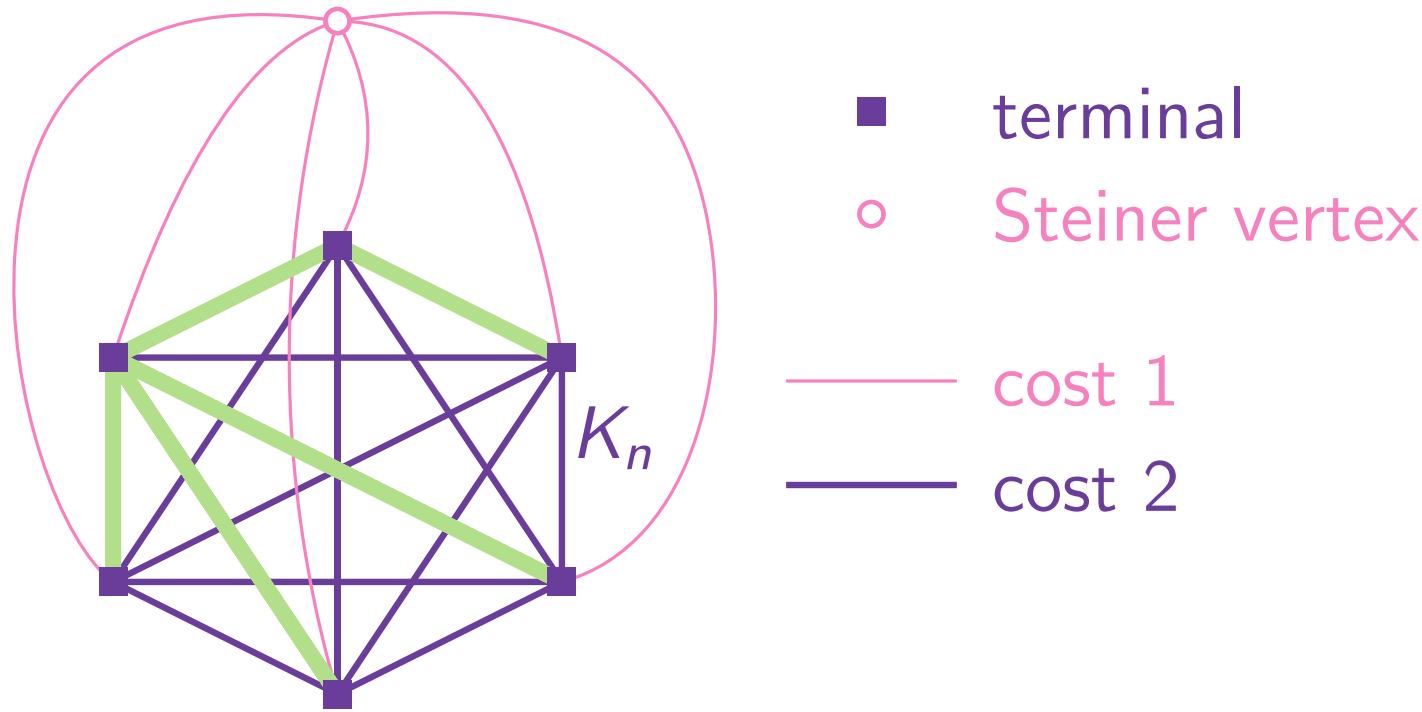
Analysis Tight?

An MST of $G[T]$ has cost



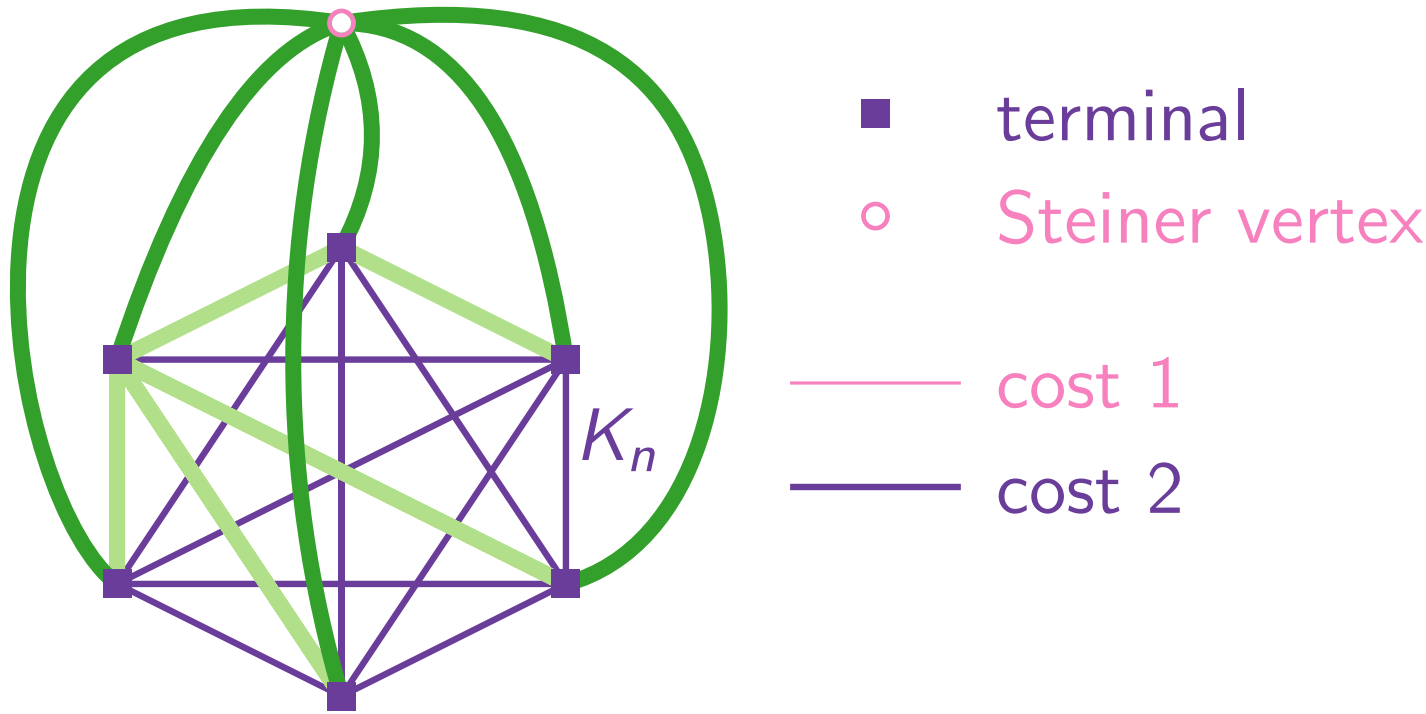
Analysis Tight?

An MST of $G[T]$ has cost $2(n-1)$.



Analysis Tight?

An MST of $G[T]$ has cost $2(n-1)$.
The optimal solution has cost n.



Analysis Tight?

An MST of $G[T]$ has cost $2(n-1)$.
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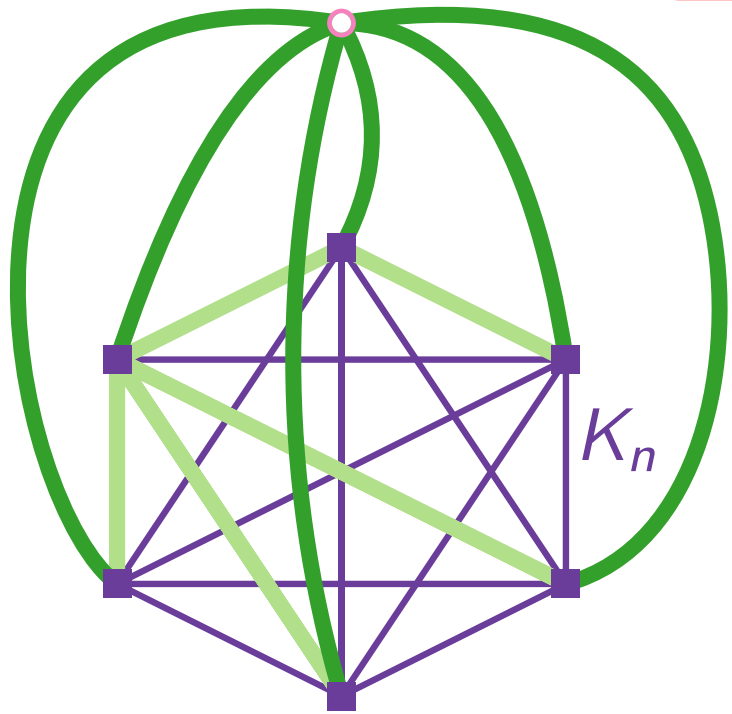
$$\frac{\text{ALG}}{\text{OPT}} = \frac{2(n-1)}{n} \to 2$$



terminal

Steiner vertex

cost 1

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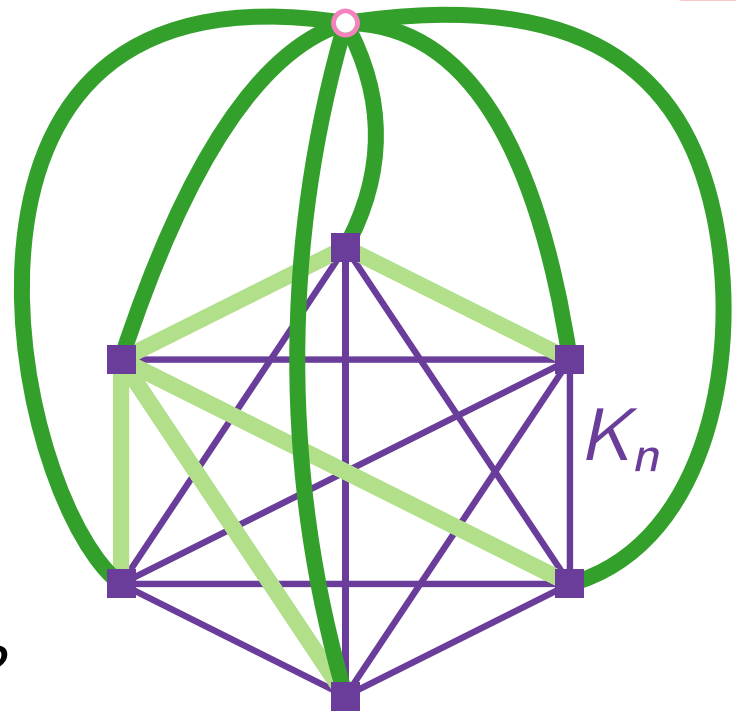

terminal
Steiner vertex
cost 1
cost 2

Can we do better?

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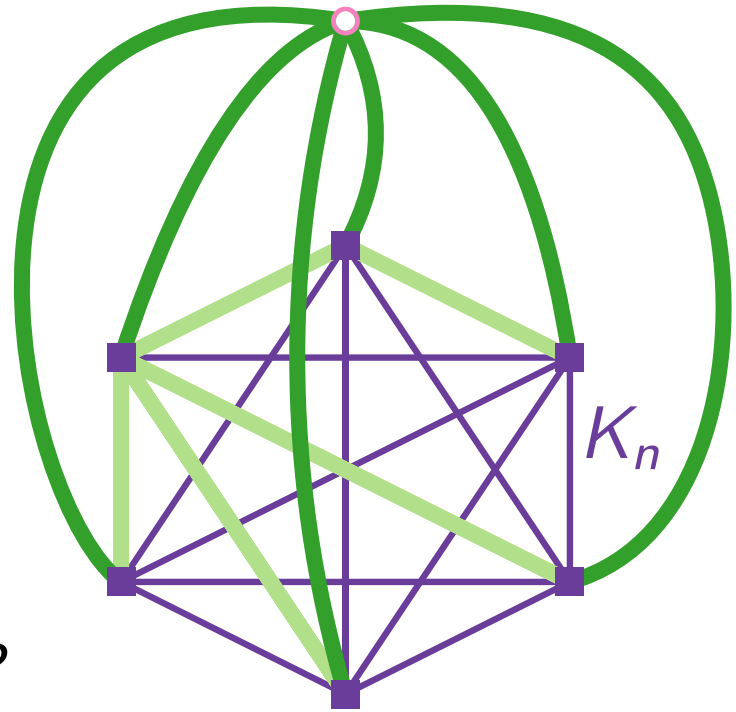

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cost 1
cost 2

Can we do better?
The best known approximation factor for SteinerTree is $\ln(4) + \varepsilon \approx 1.39$.

[Byrka, Grandoni, Rothvoß & Sanità, J. ACM'13]

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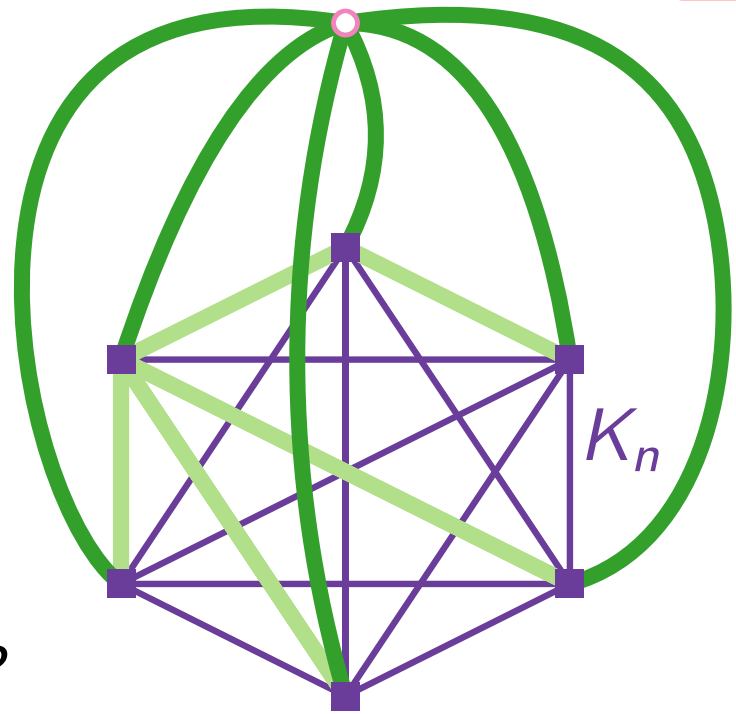

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[Byrka, Grandoni, Rothvoß & Sanità, J. ACM'13]

SteinerTree cannot be approximated within factor $\frac{96}{95} \approx 1.0105$ (unless P = NP).

[Chlebík & Chlebíková, TCS'08]

Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part V:
MultiwayCut

MultiwayCut

Given: A connected graph G

MultiwayCut

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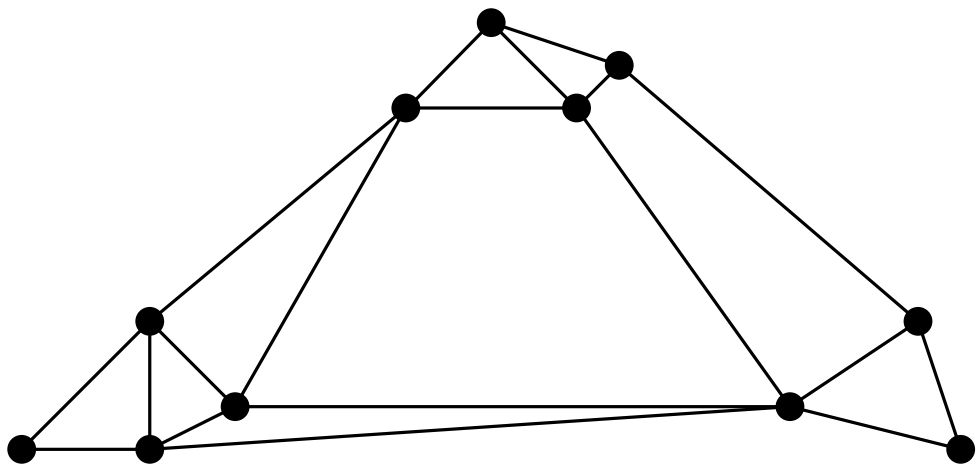
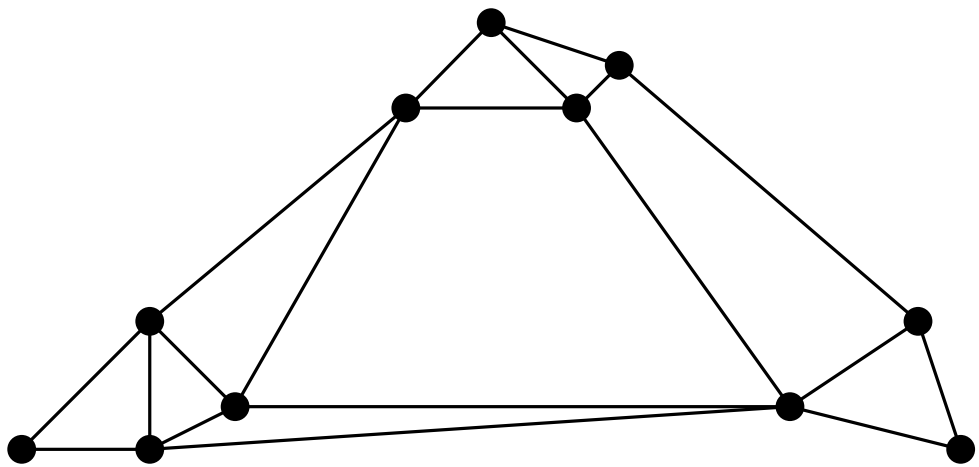


MultiwayCut

Given: A connected graph G with edge costs $c\colon E(G) \to \mathbb{Q}^+$



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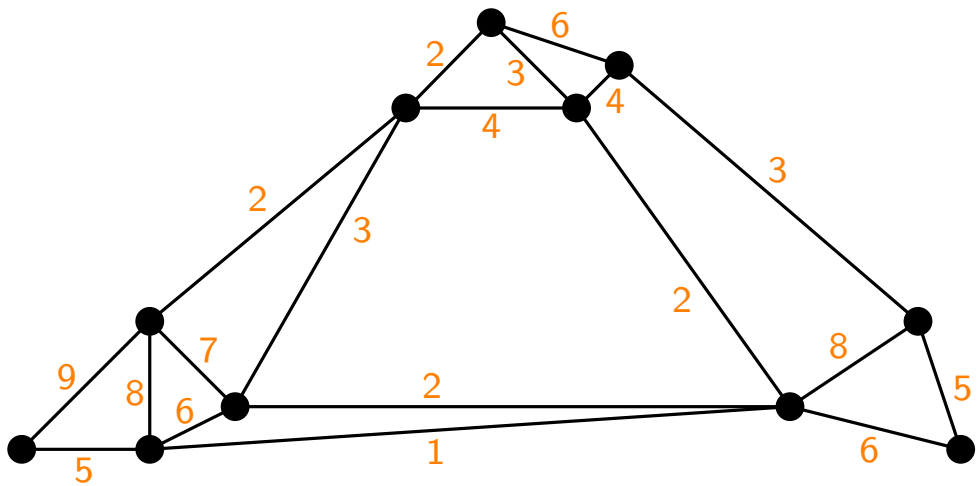
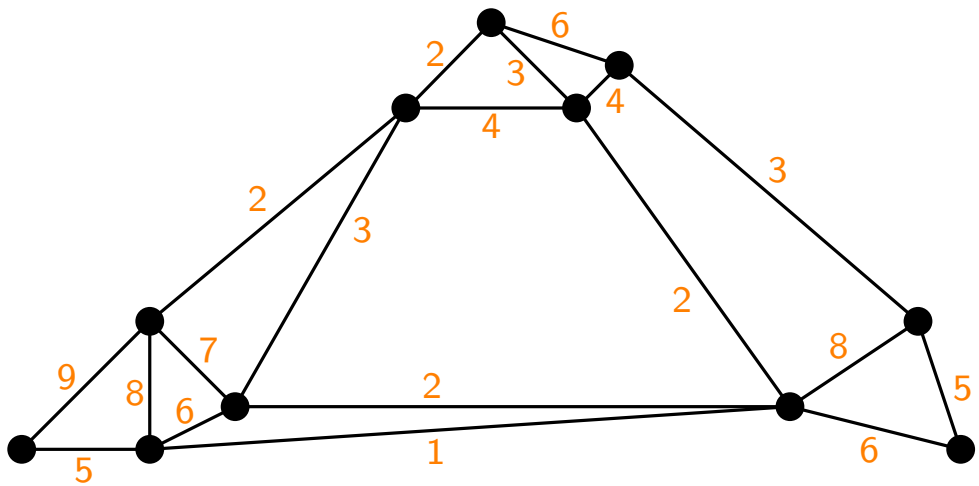


MultiwayCut

Given: A connected graph G with edge costs $c\colon E(G) \to \mathbb{Q}^+$ and a set $T = \{t_1, \ldots, t_k\} \subseteq V(G)$ of **terminals**.



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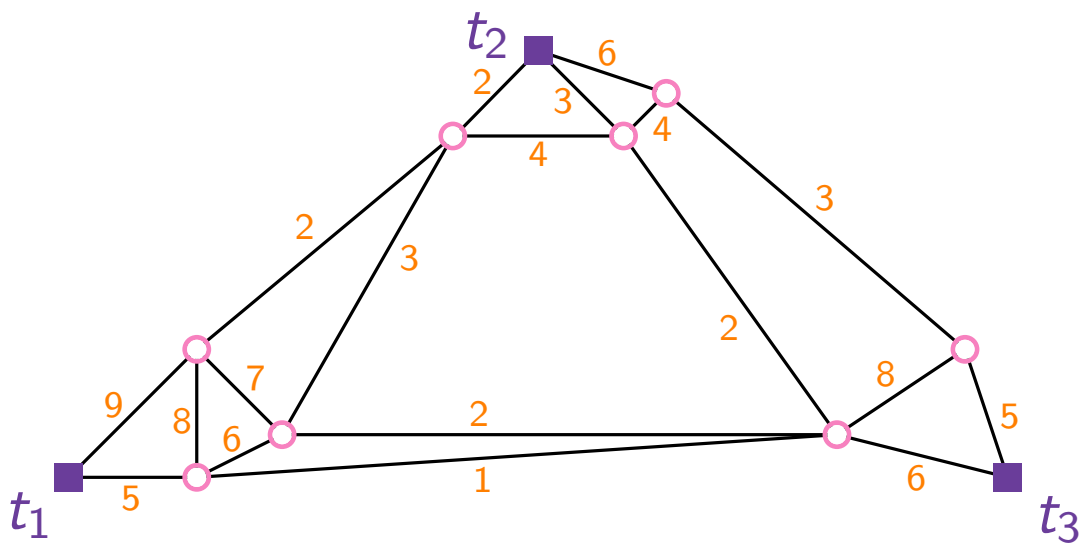

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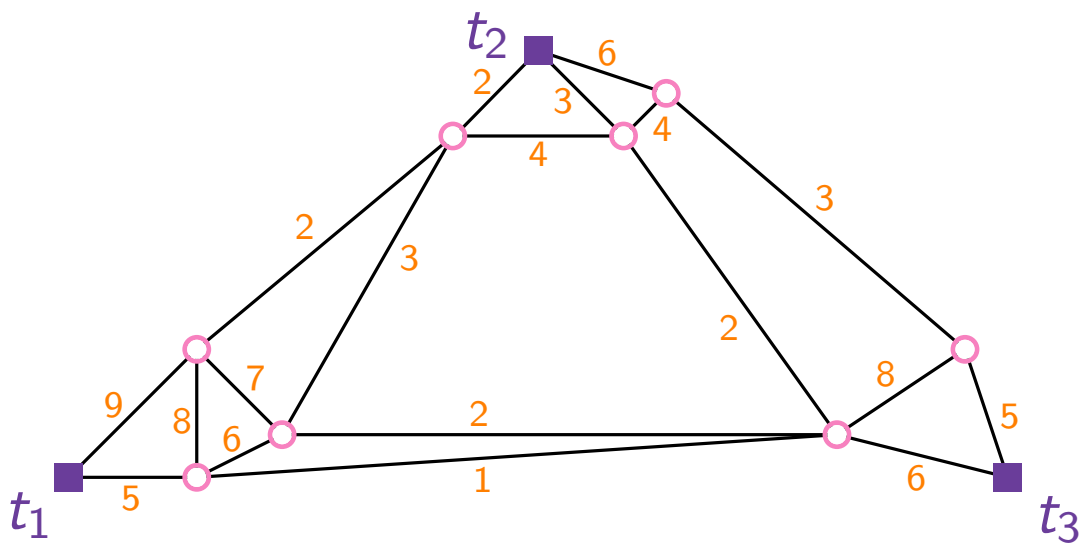
A **multiway cut** of T is a subset E' of edges such that no two terminals in the graph $(V(G), E(G) - E')$ are connected.



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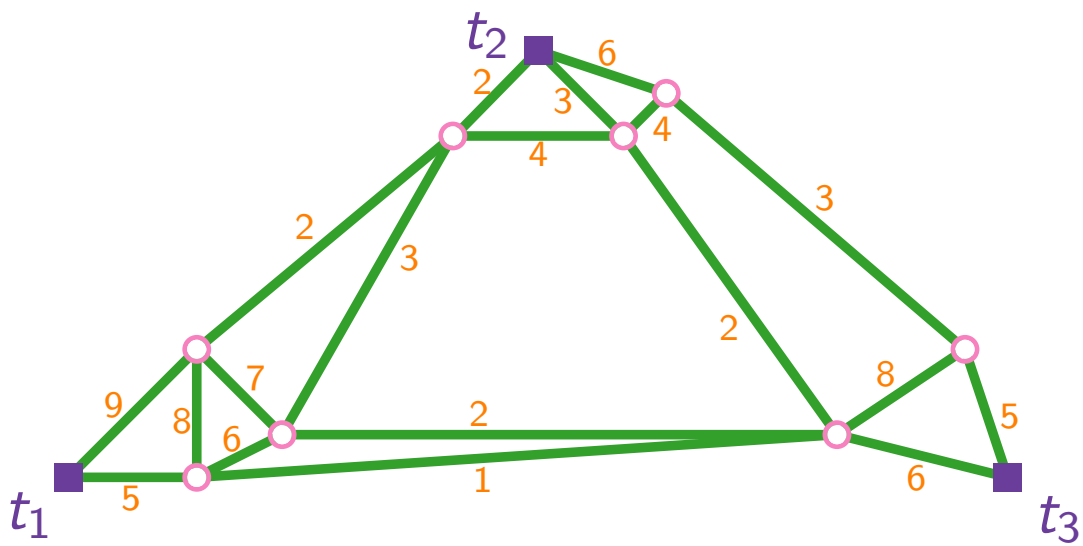

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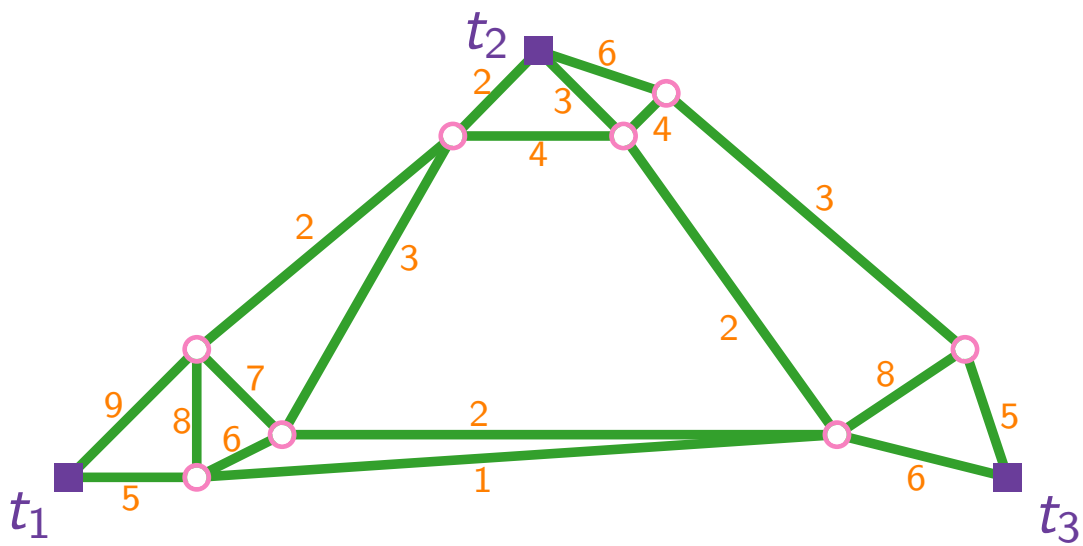
Find: A minimum-cost multiway cut of T.



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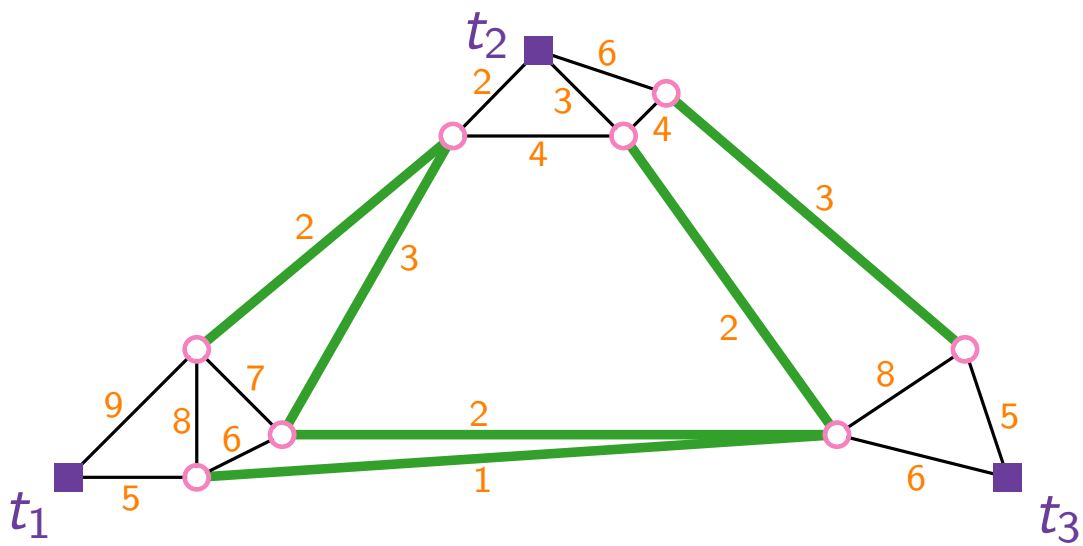

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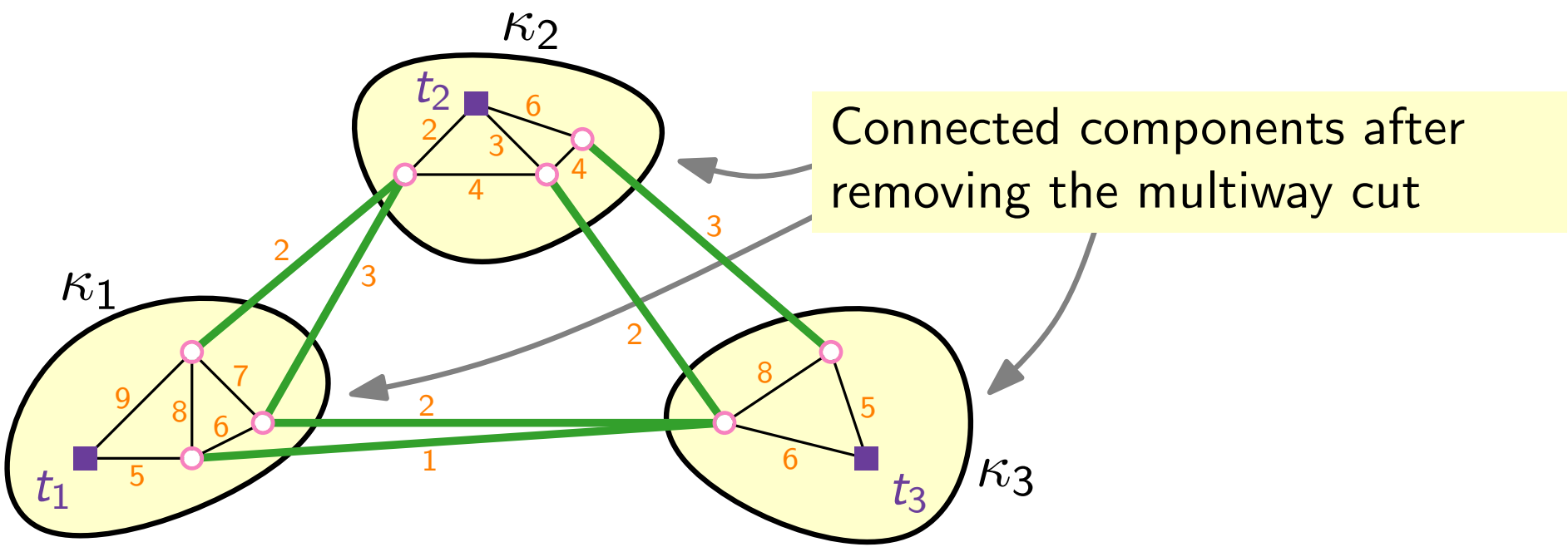

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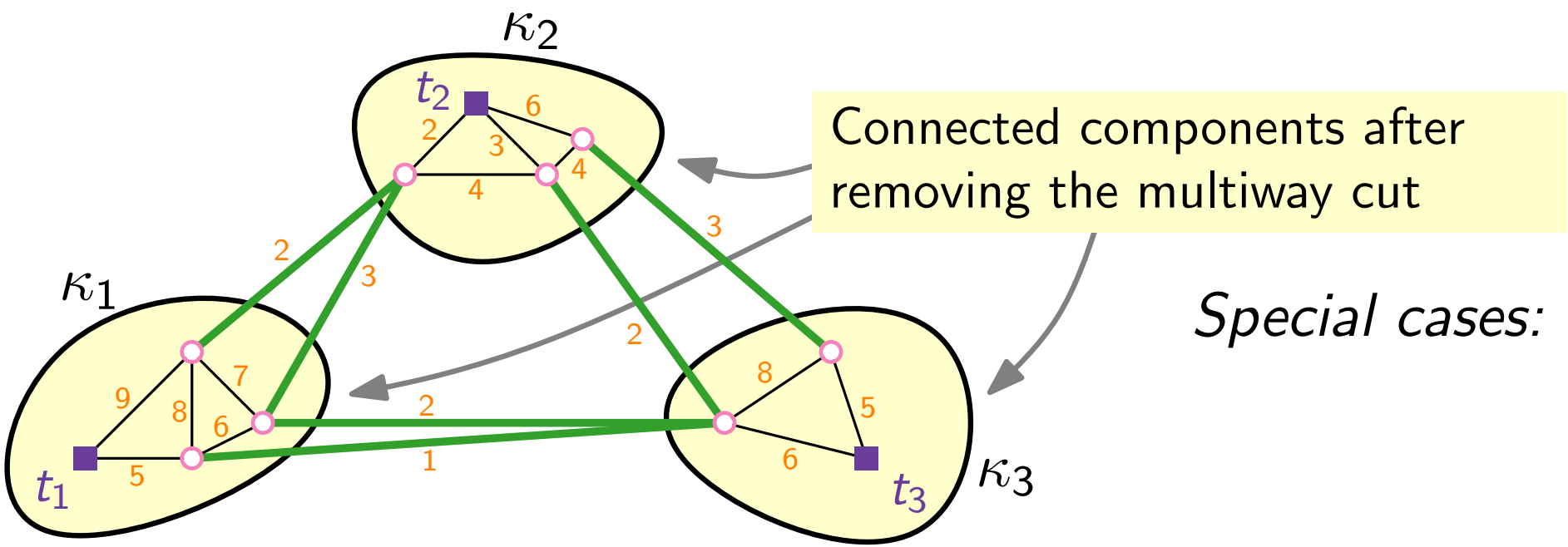

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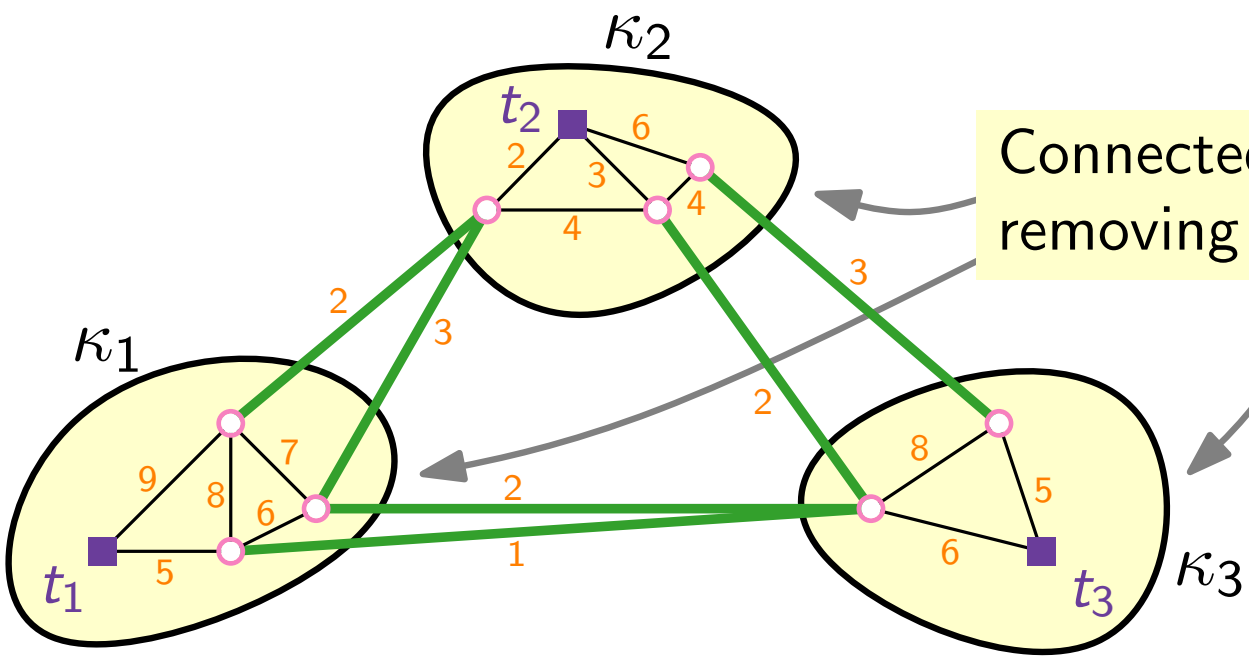

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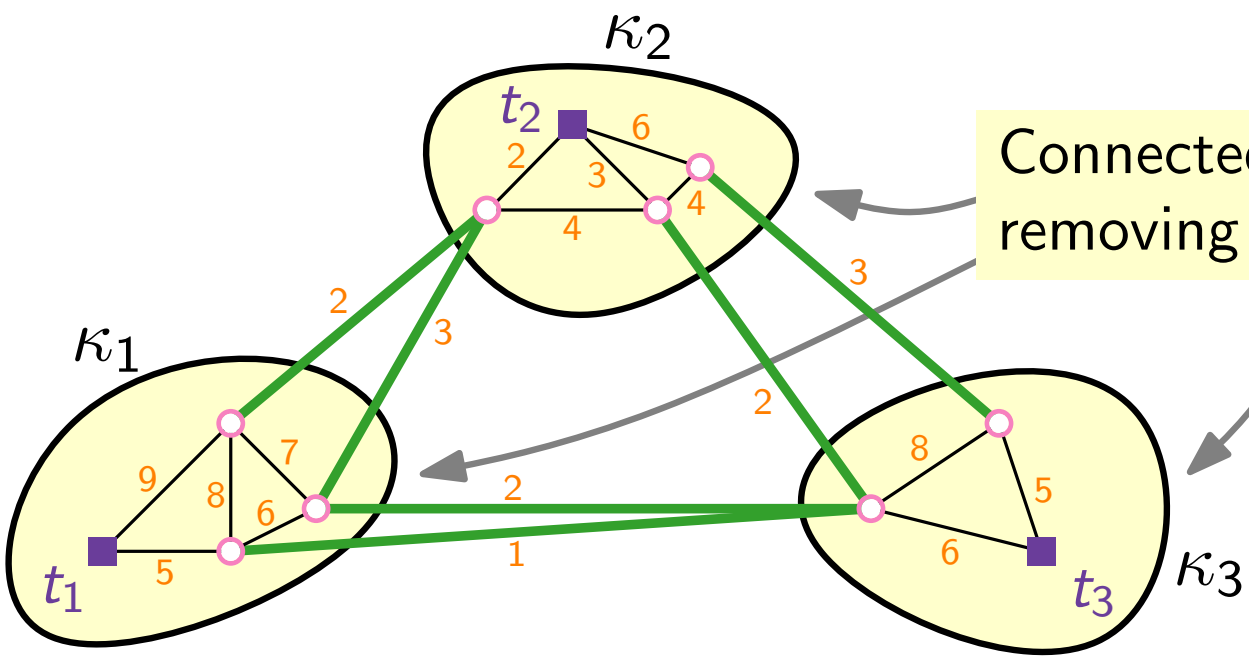

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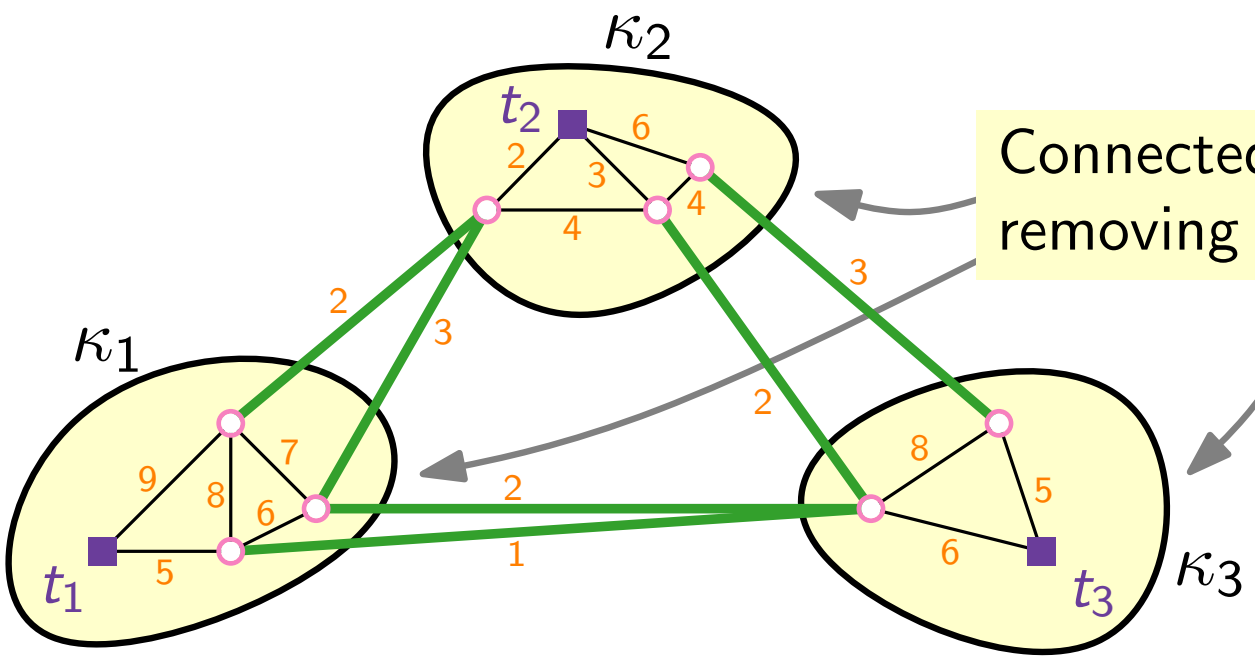

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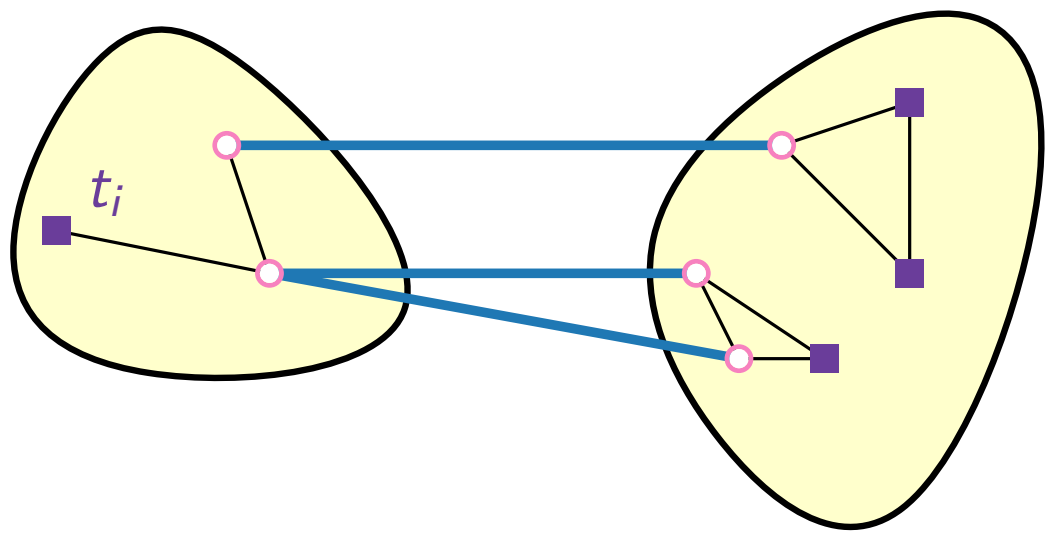

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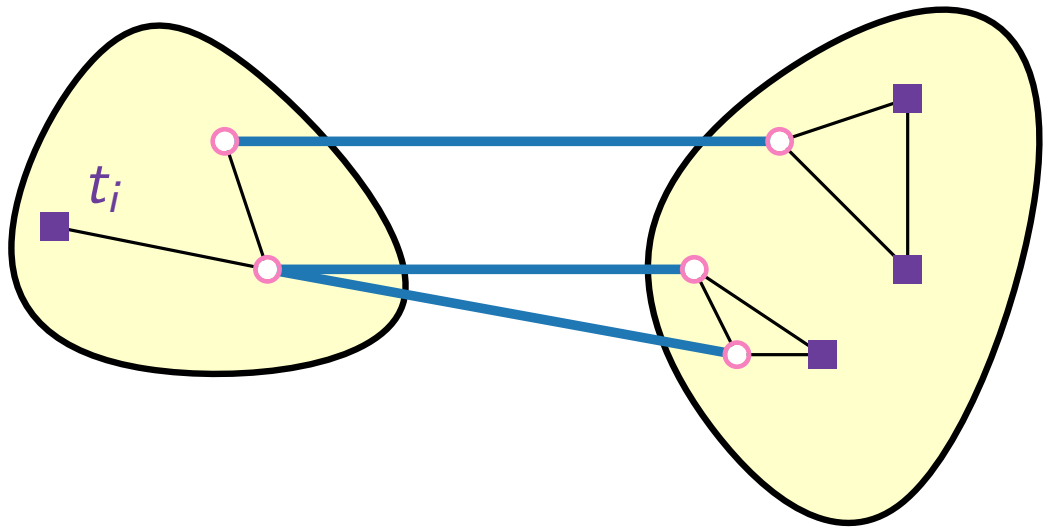

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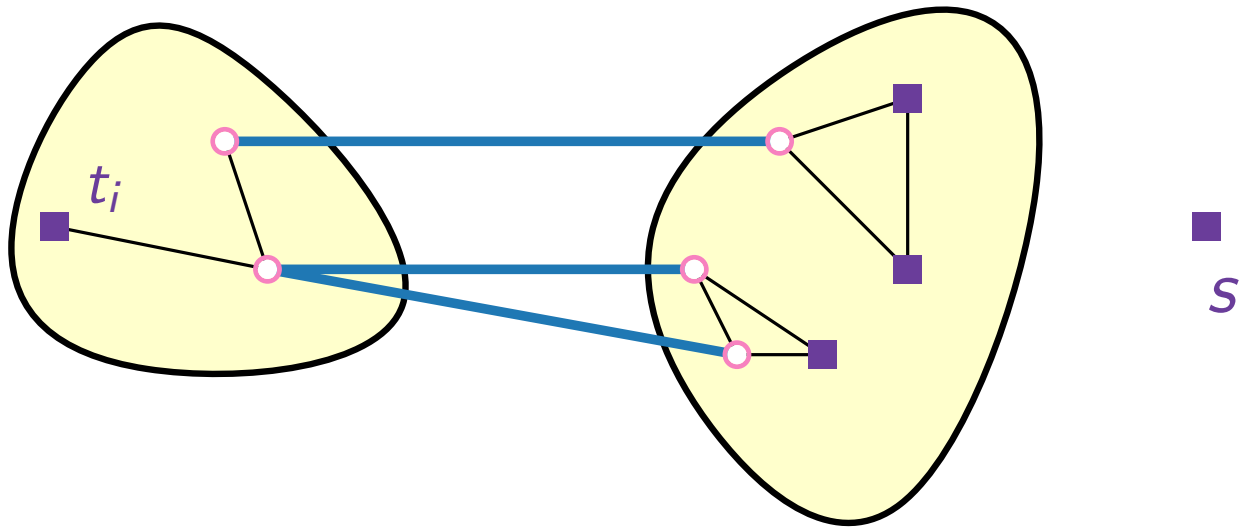

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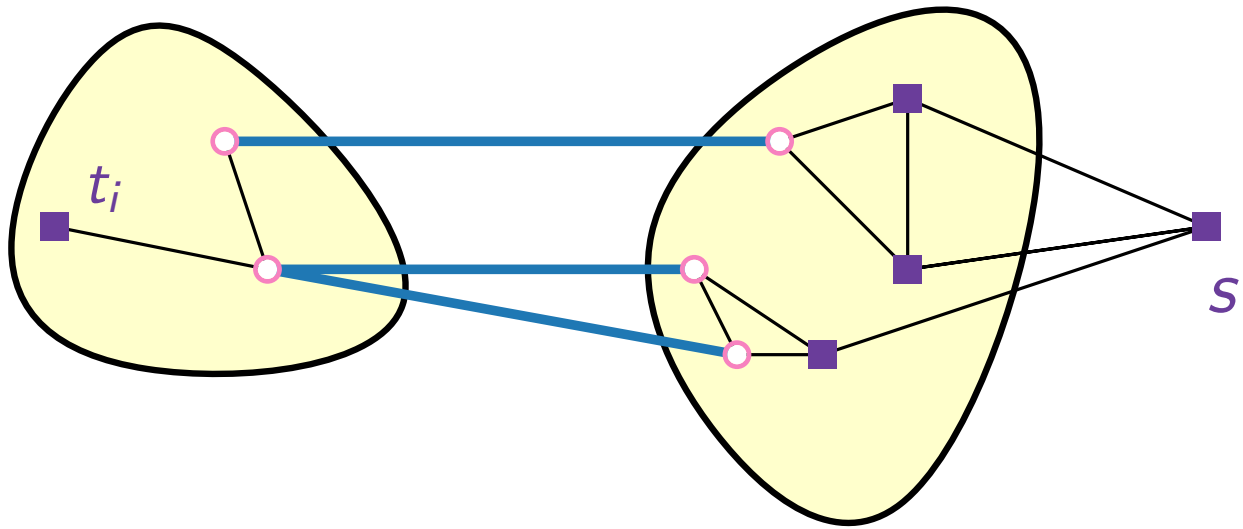

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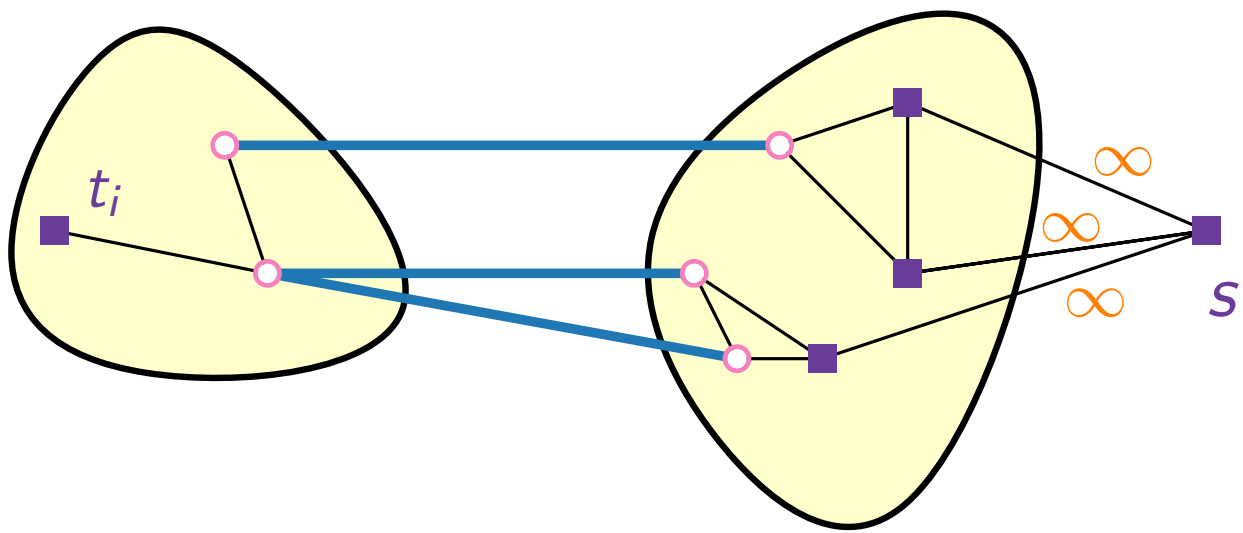

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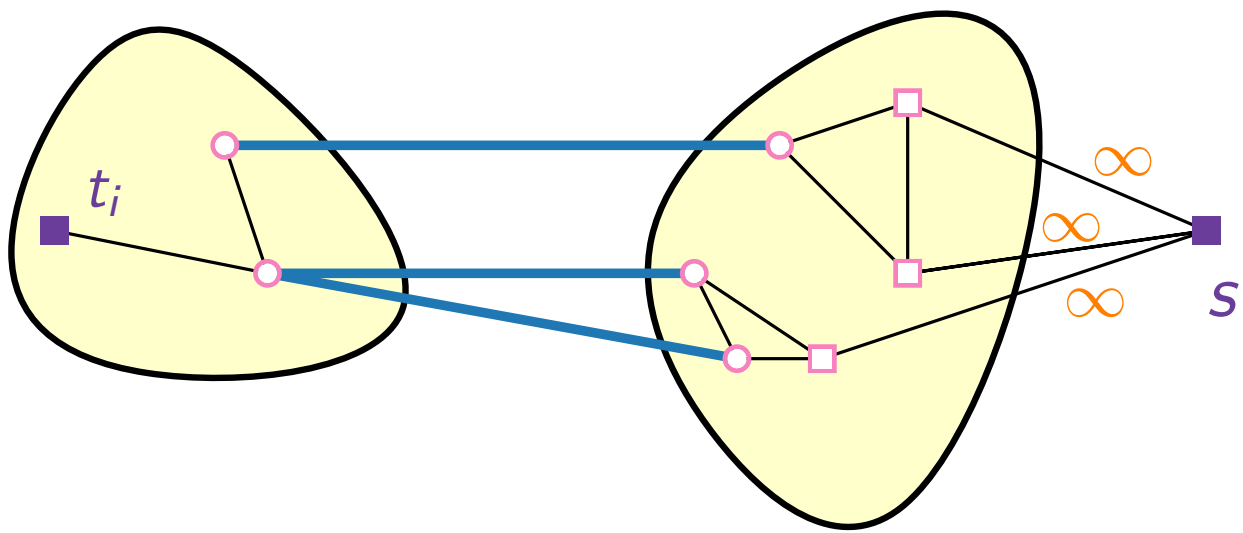

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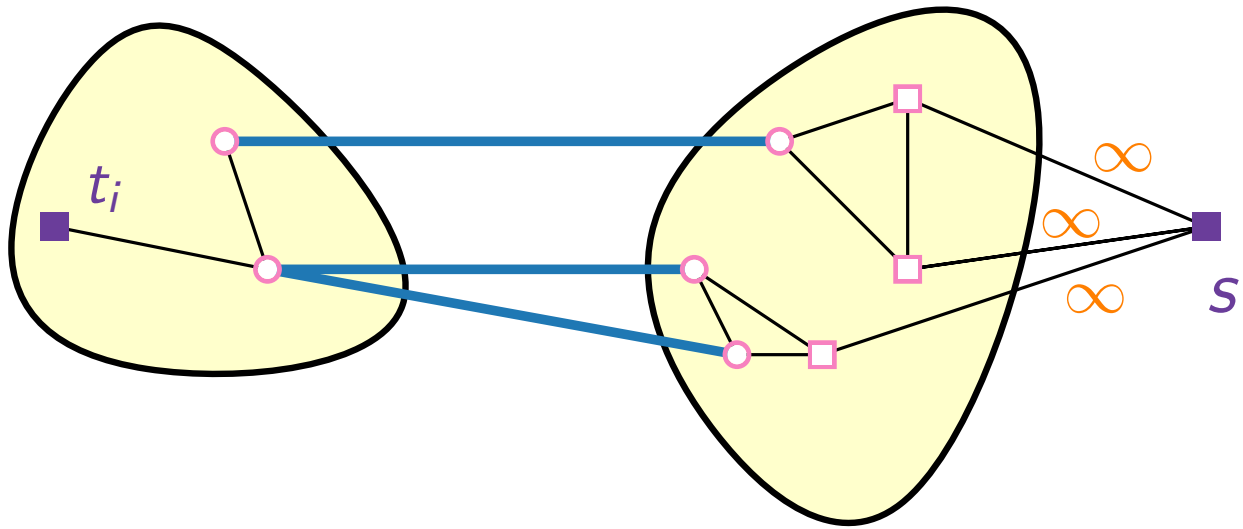

Add dummy terminal s and find a minimum-cost s–t_i cut.

Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part VI:
Algorithm for MultiwayCut

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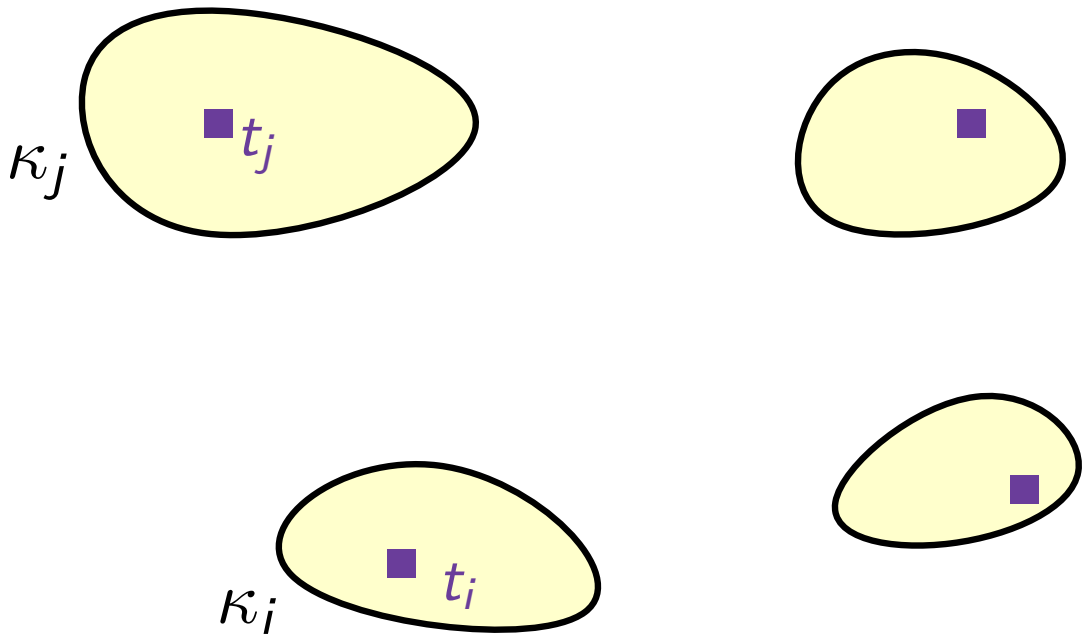

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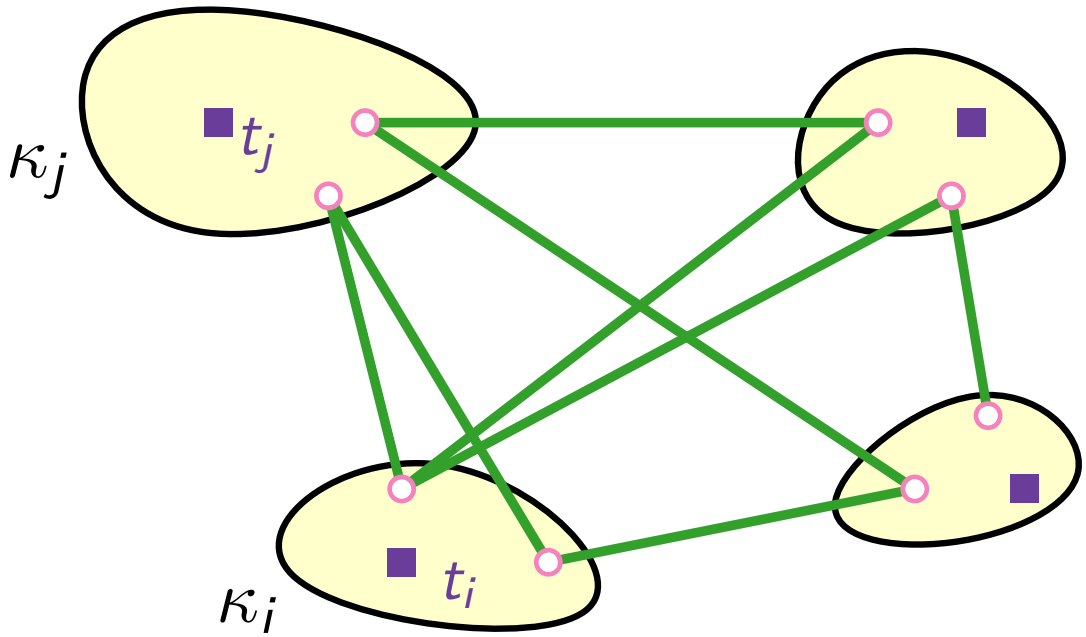

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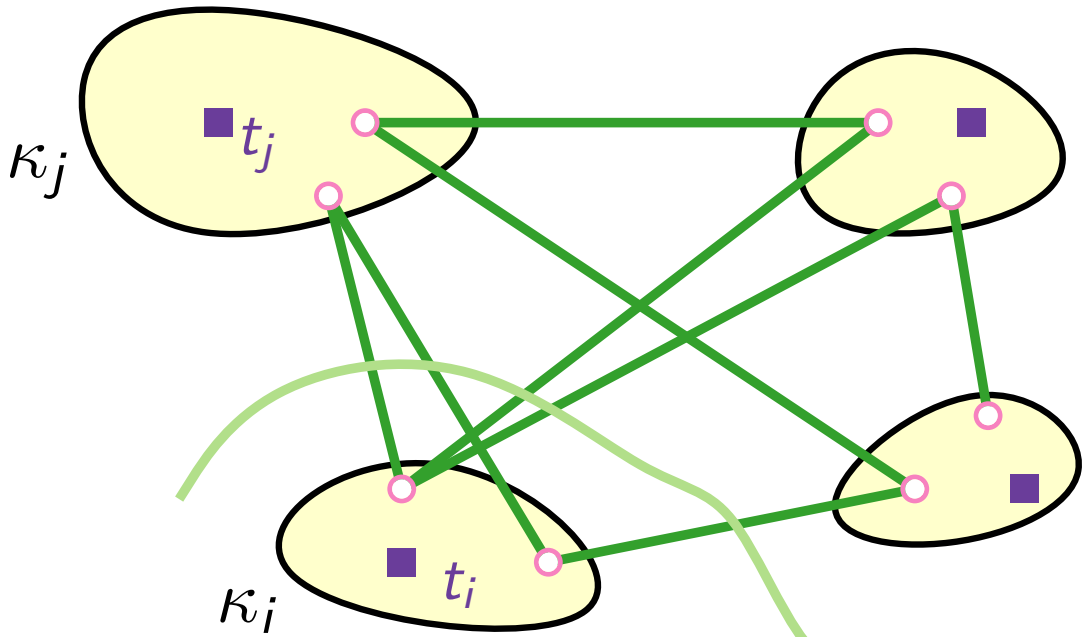

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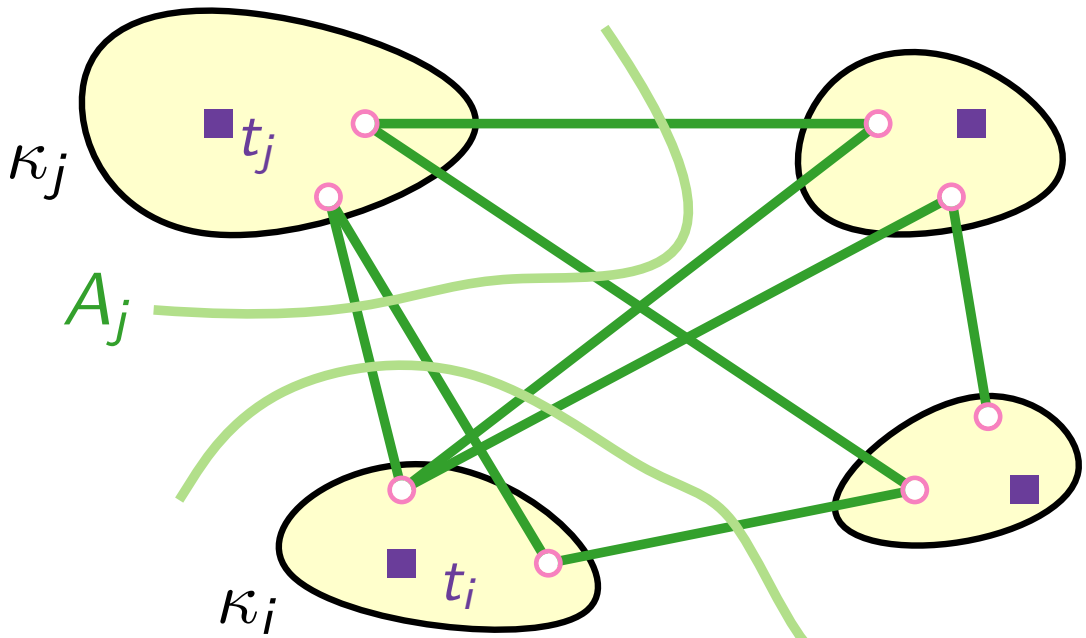

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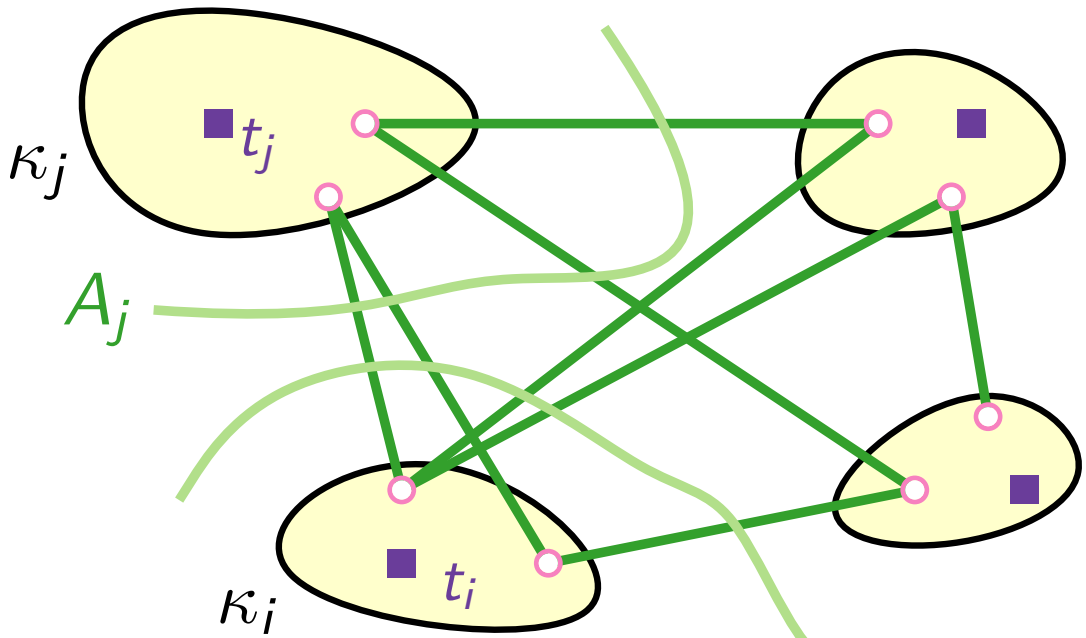

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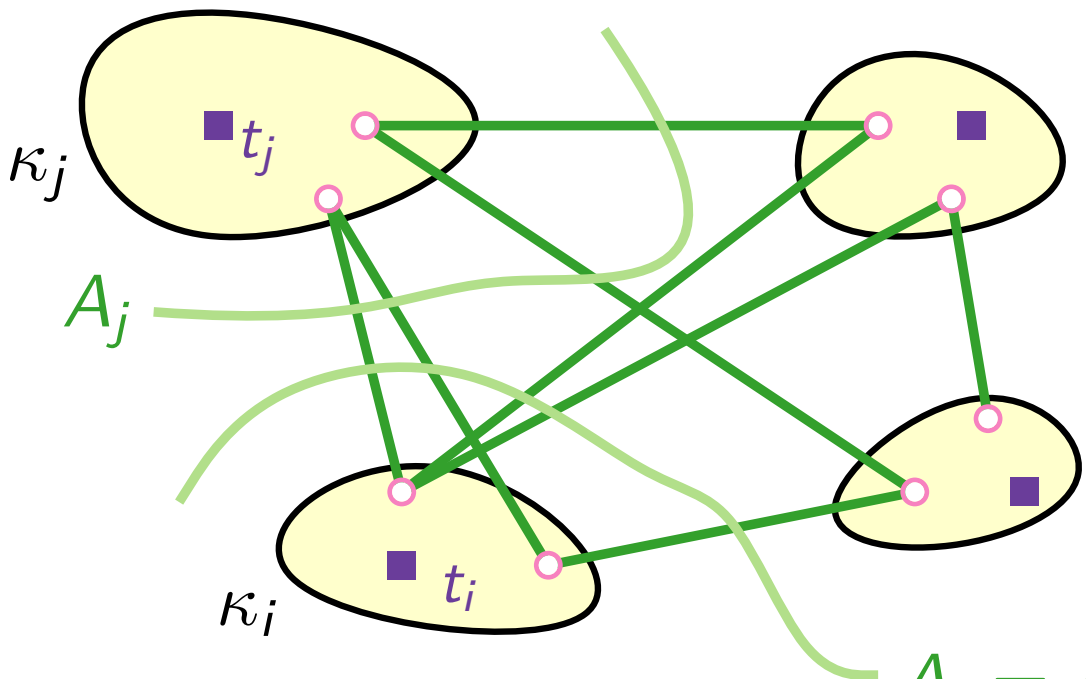

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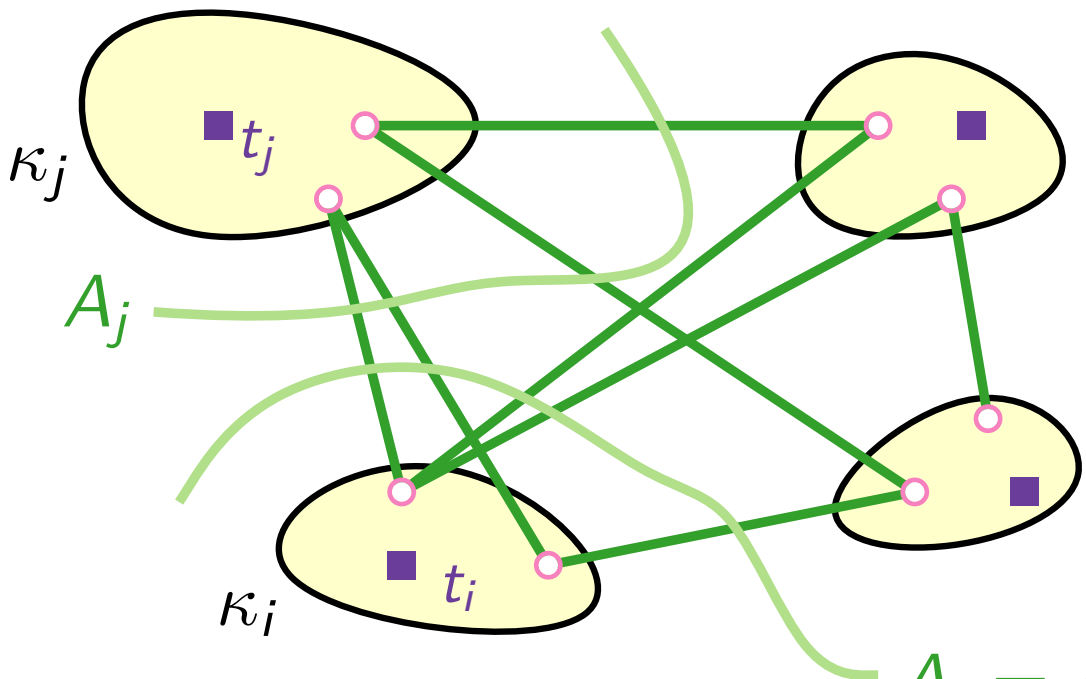

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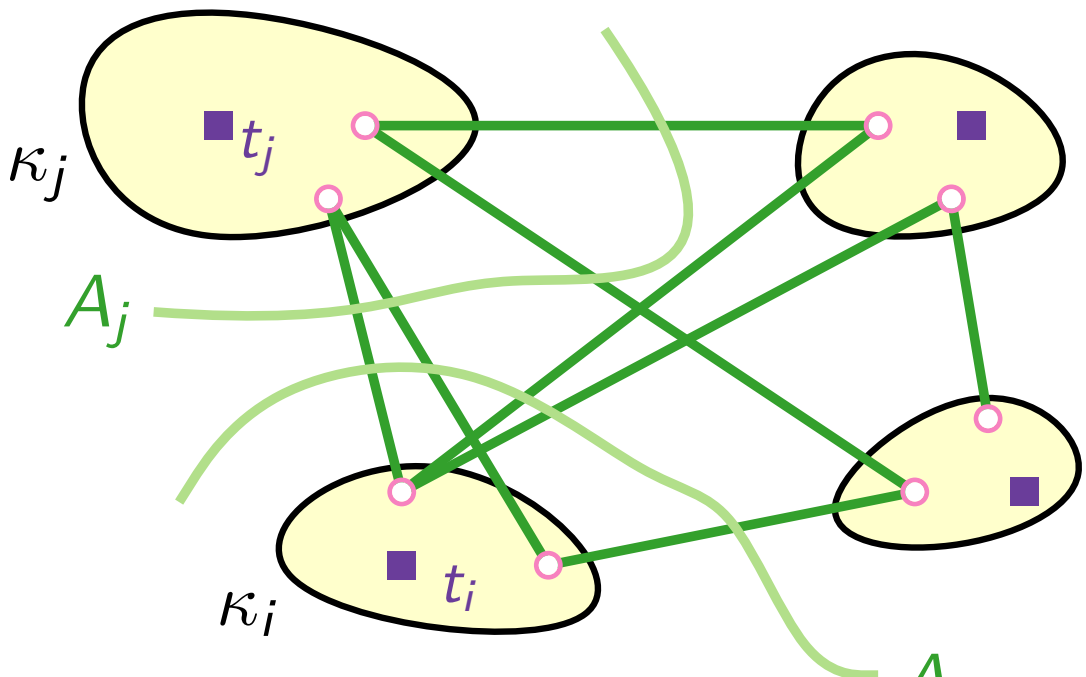

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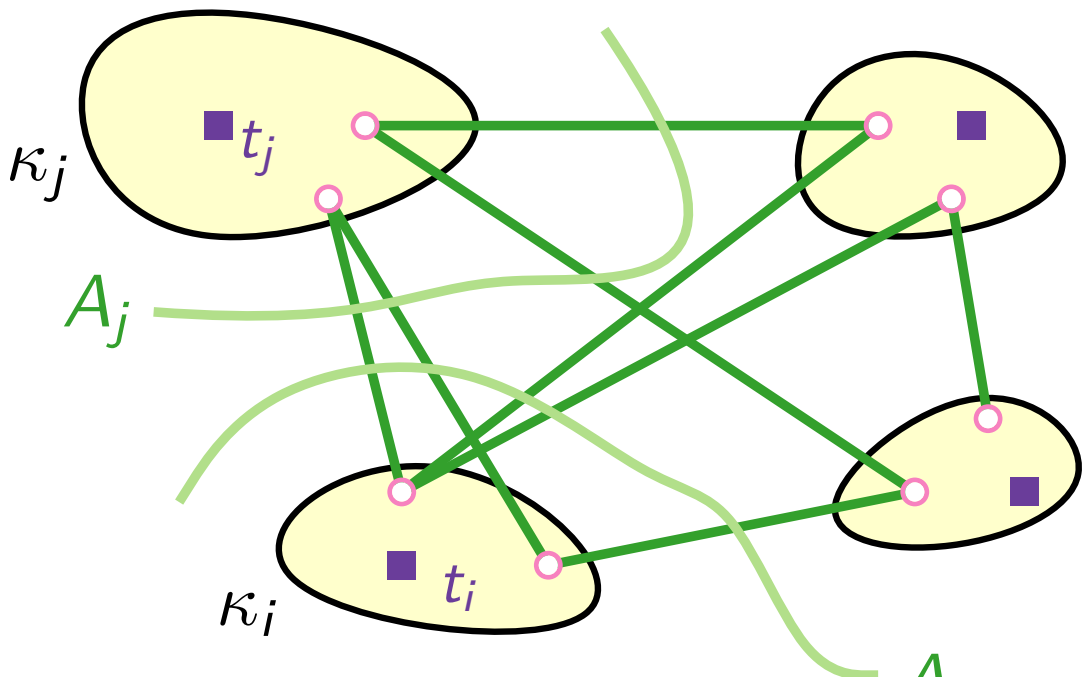

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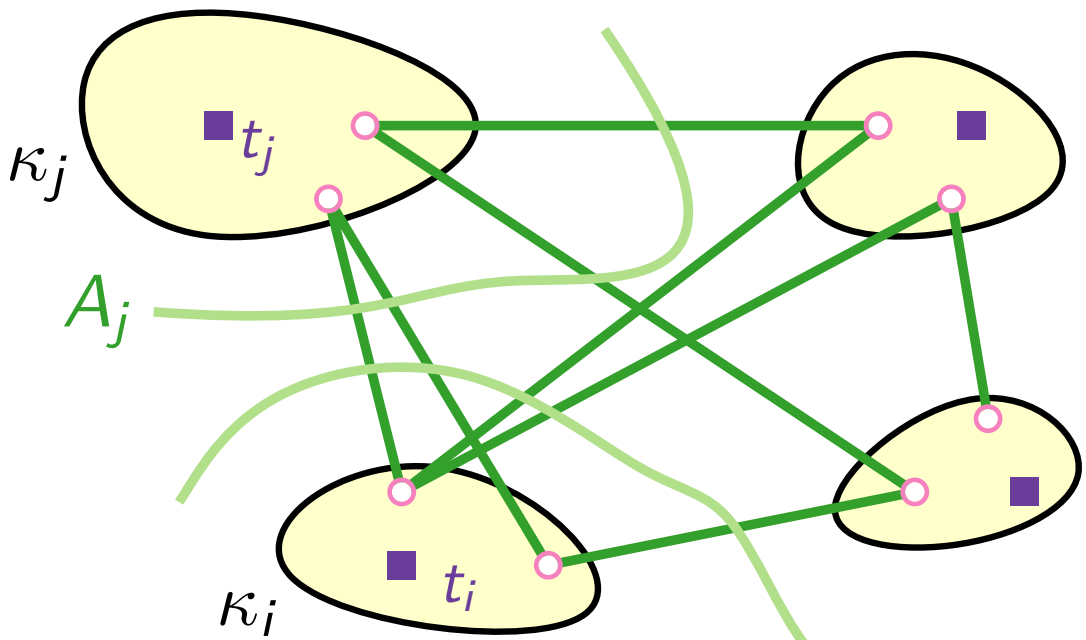

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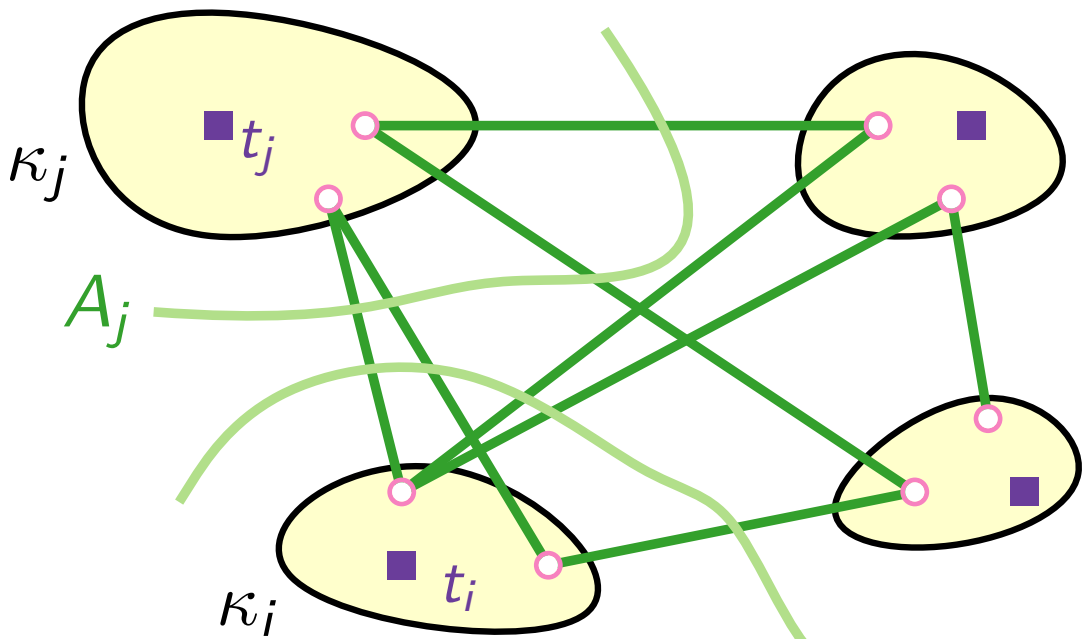

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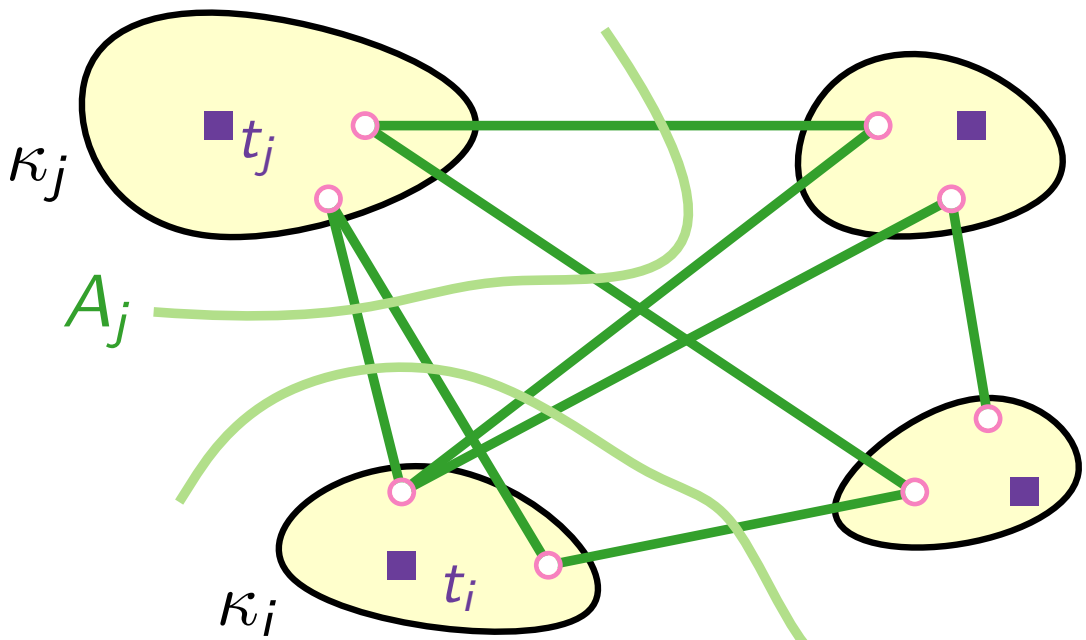

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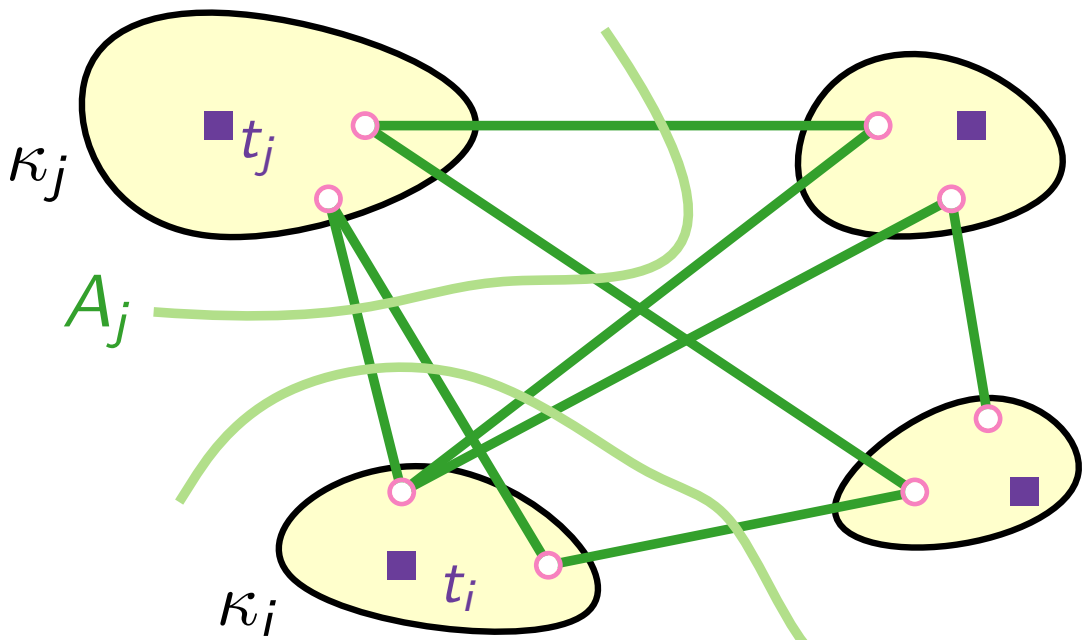

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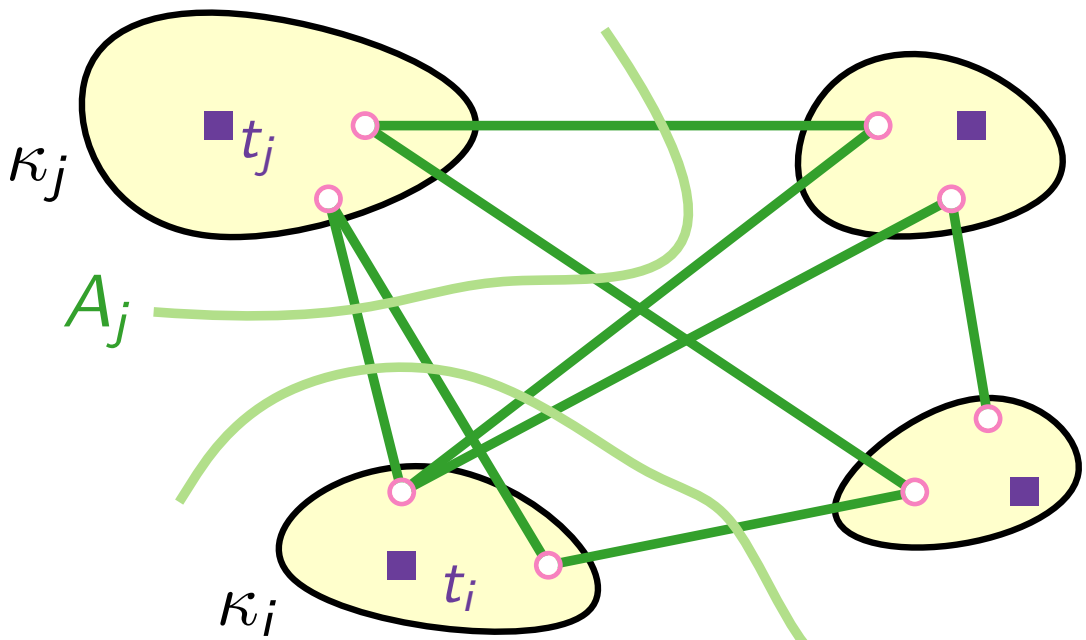

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Analysis Tight?

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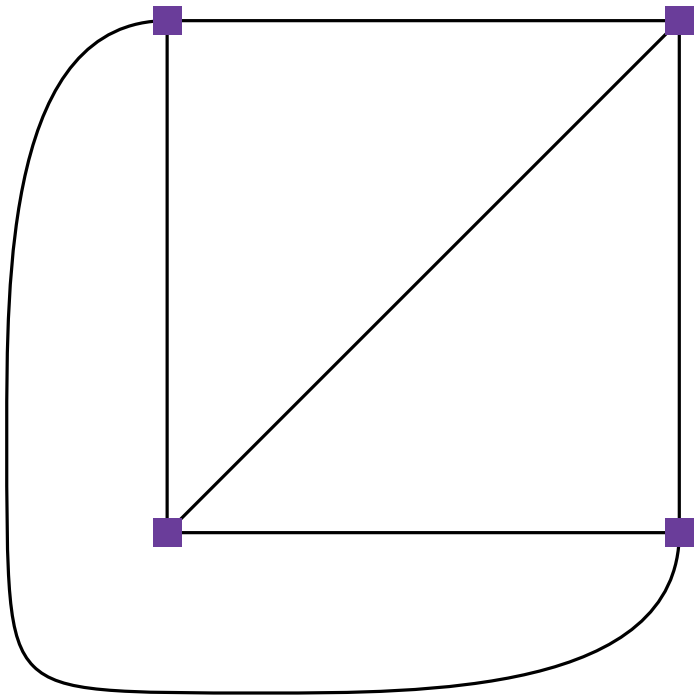
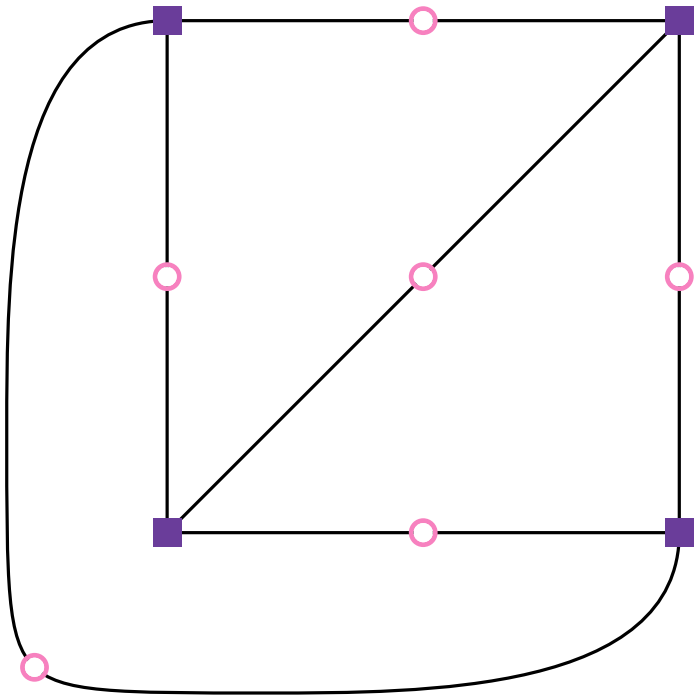
Analysis Tight?

K_k



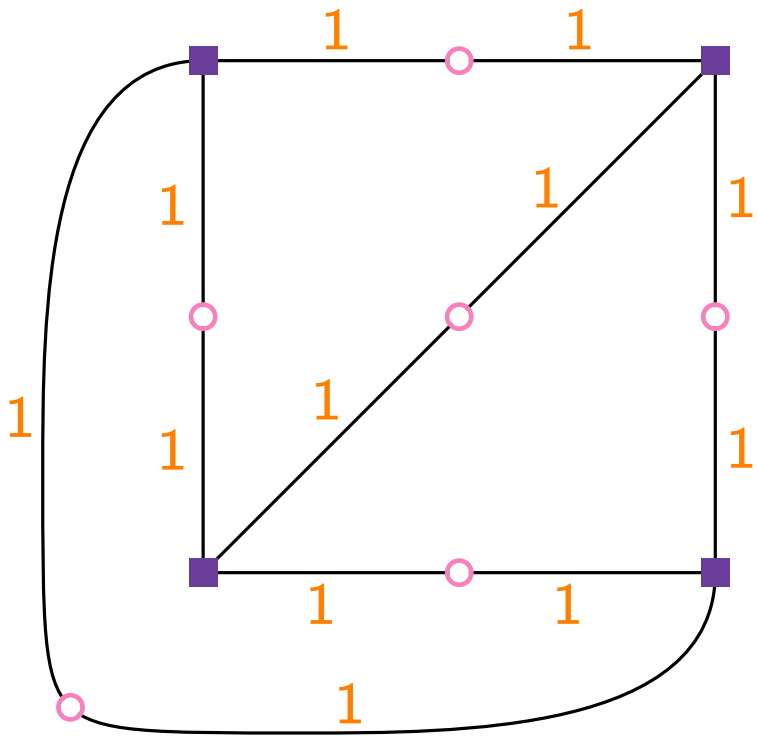
Analysis Tight?

K_k



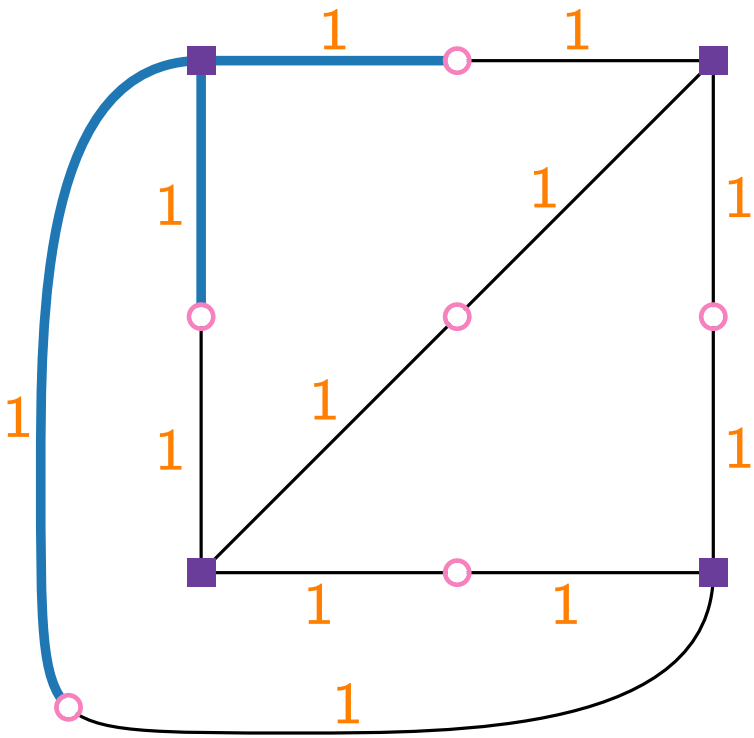
Analysis Tight?

K_k



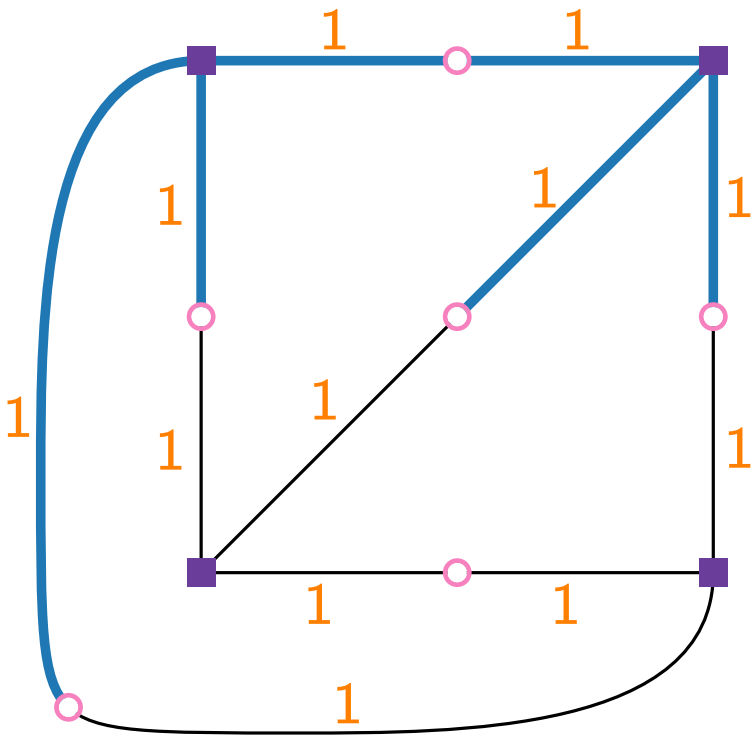
Analysis Tight?

K_k



Analysis Tight?

K_k



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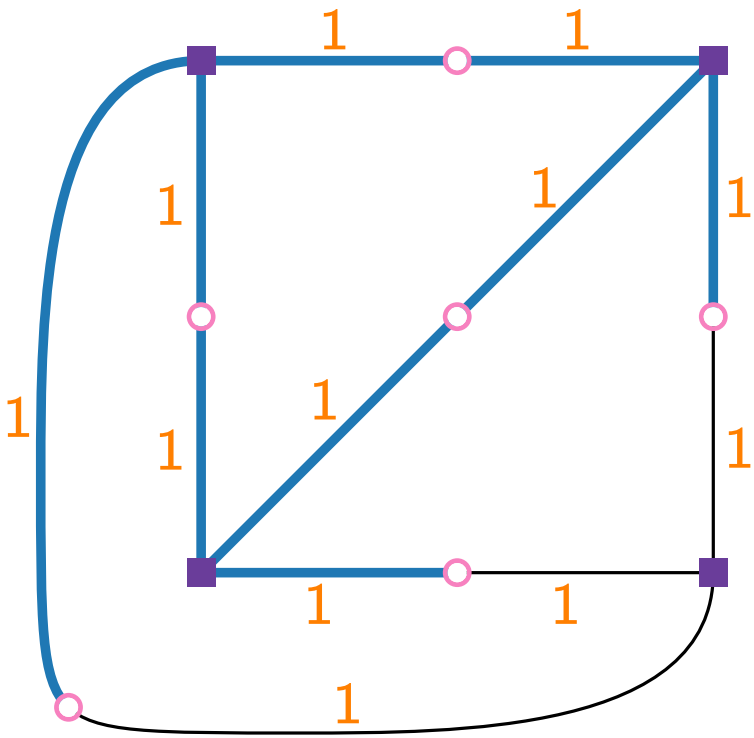
K_k



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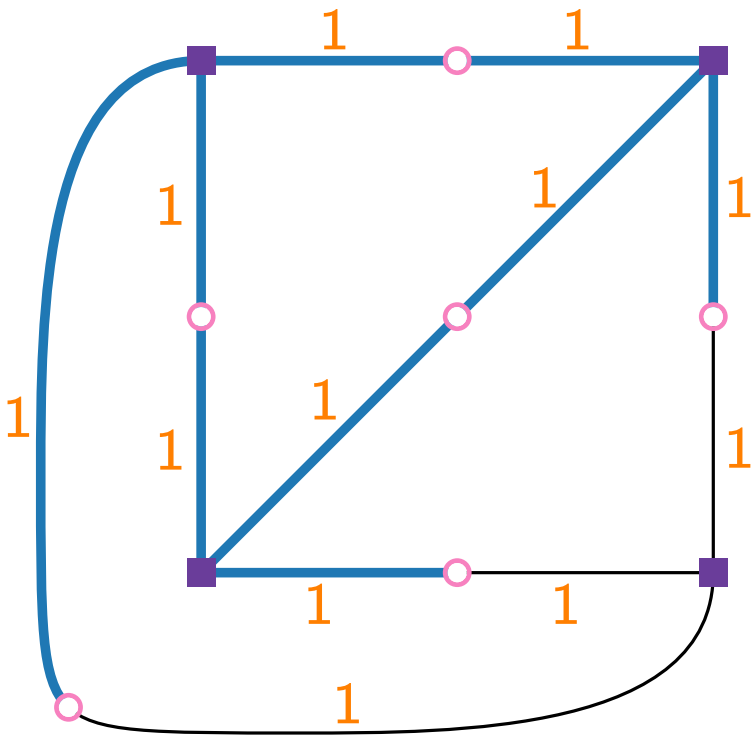
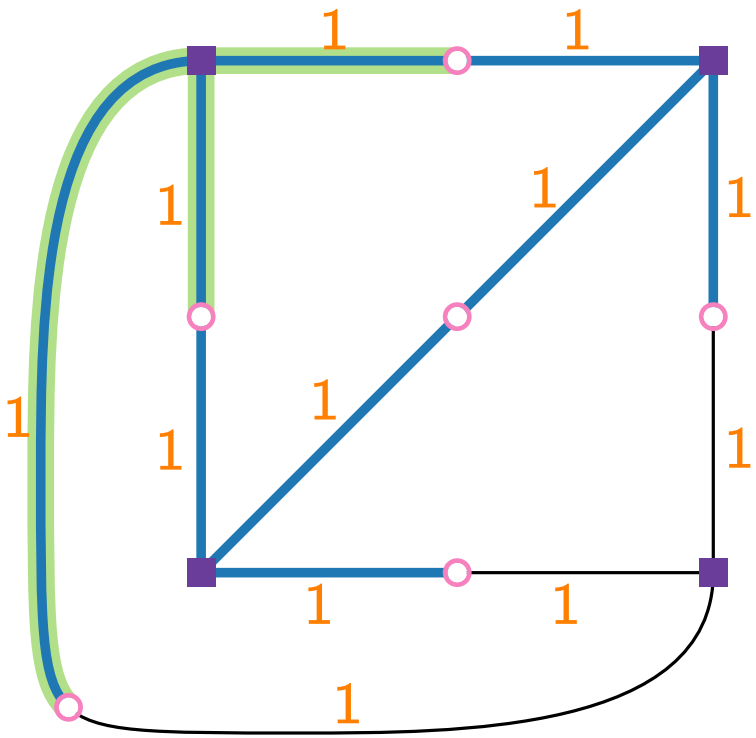
K_k



$$\text{ALG} = (k-1)(k-1)$$

Analysis Tight?

K_k



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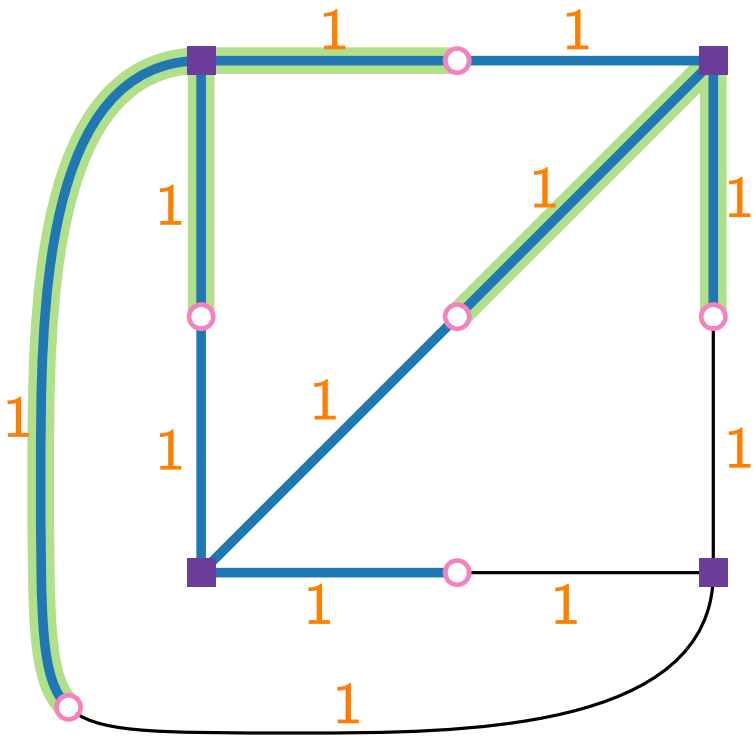

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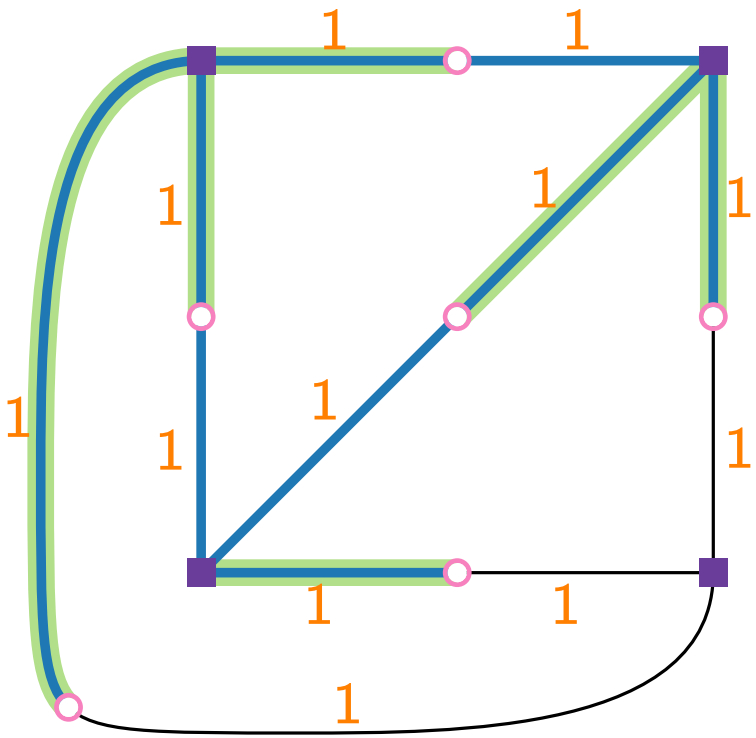
Analysis Tight?

K_k



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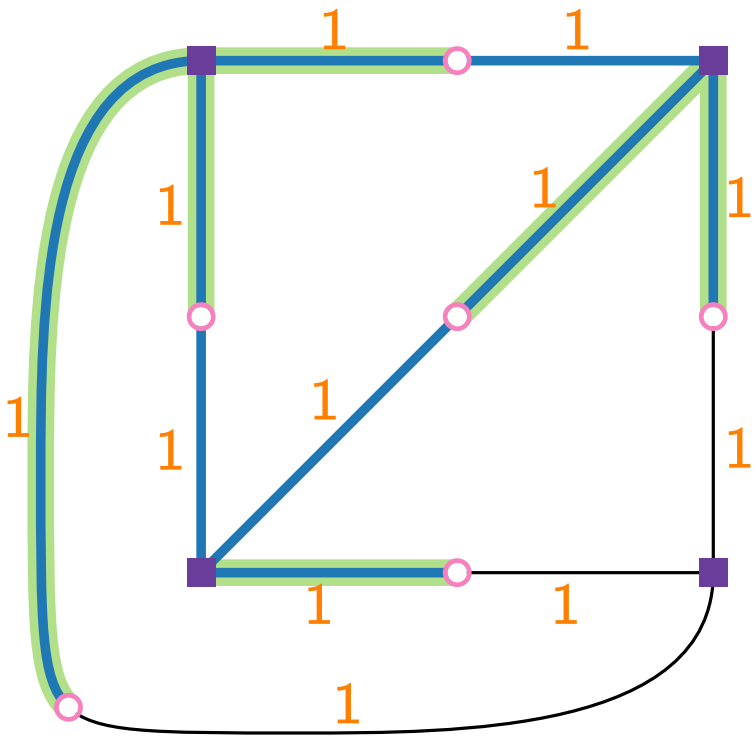
Analysis Tight?

K_k



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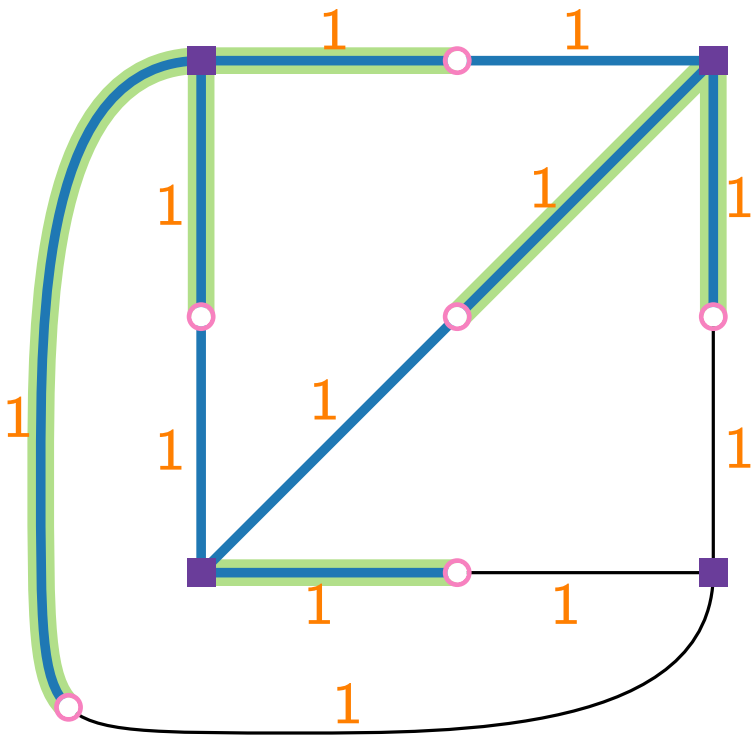
Analysis Tight?

K_k



$$\text{ALG} = (k-1)(k-1)$$

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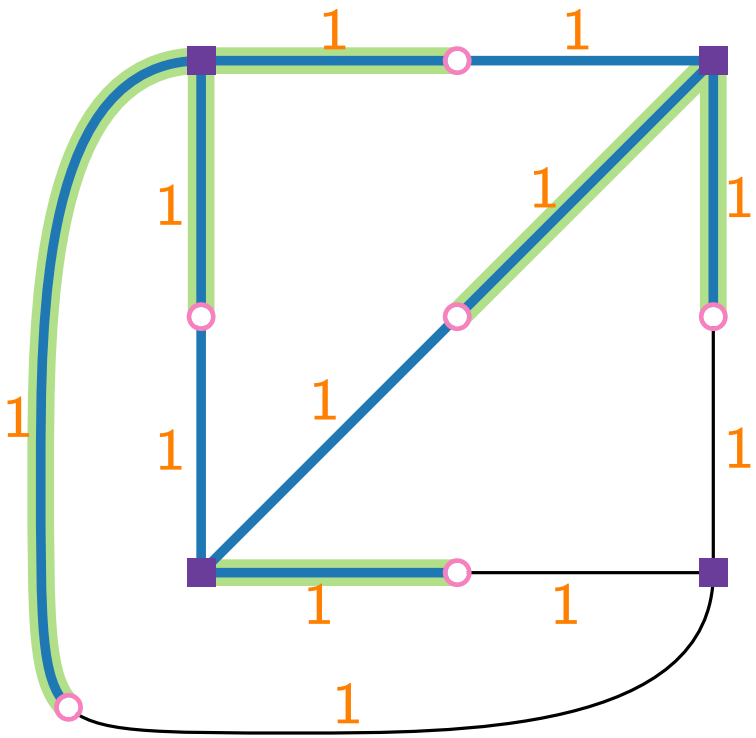
Analysis Tight?

K_k



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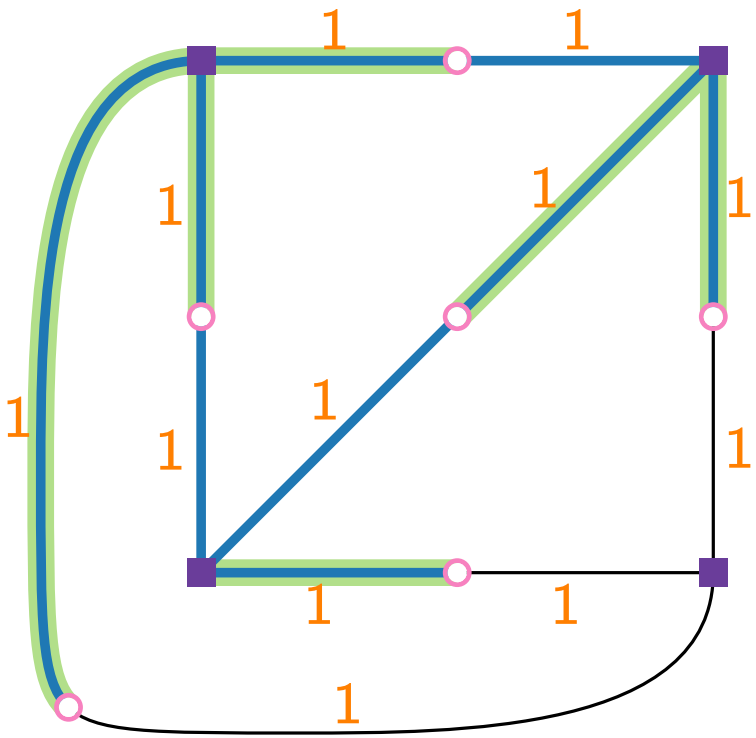
$$\text{ALG}/\text{OPT} =$$

Analysis Tight?

K_k



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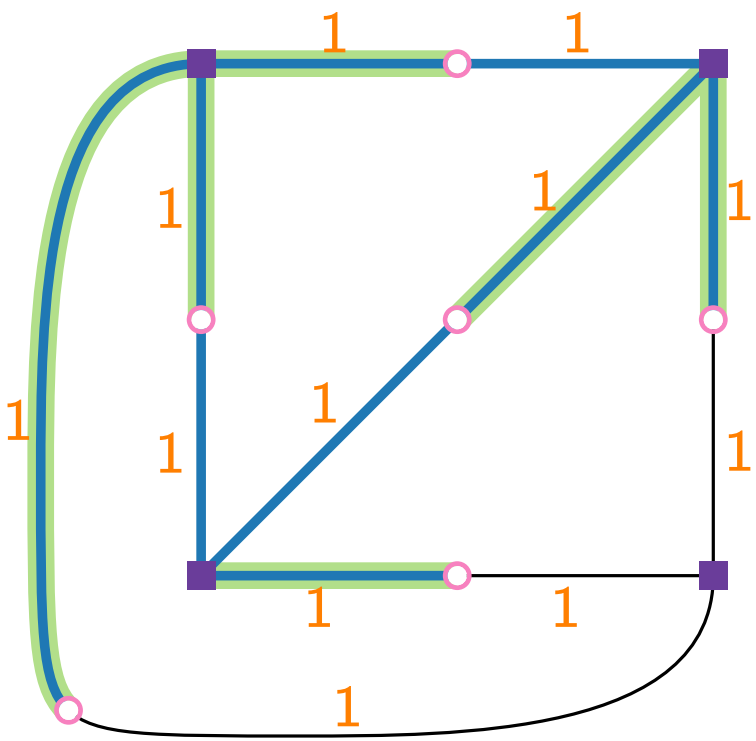
Analysis Tight?

K_k



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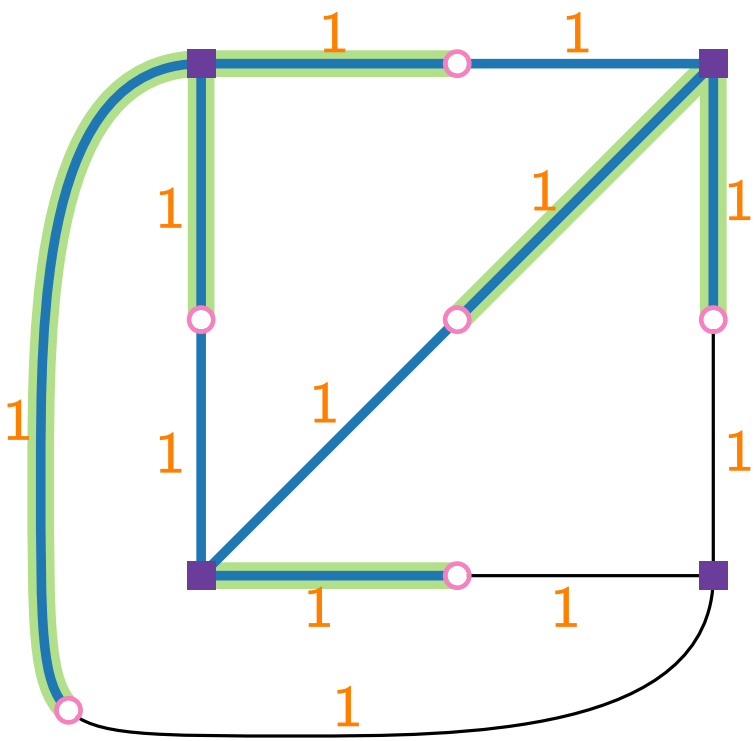
Analysis Tight?

K_k



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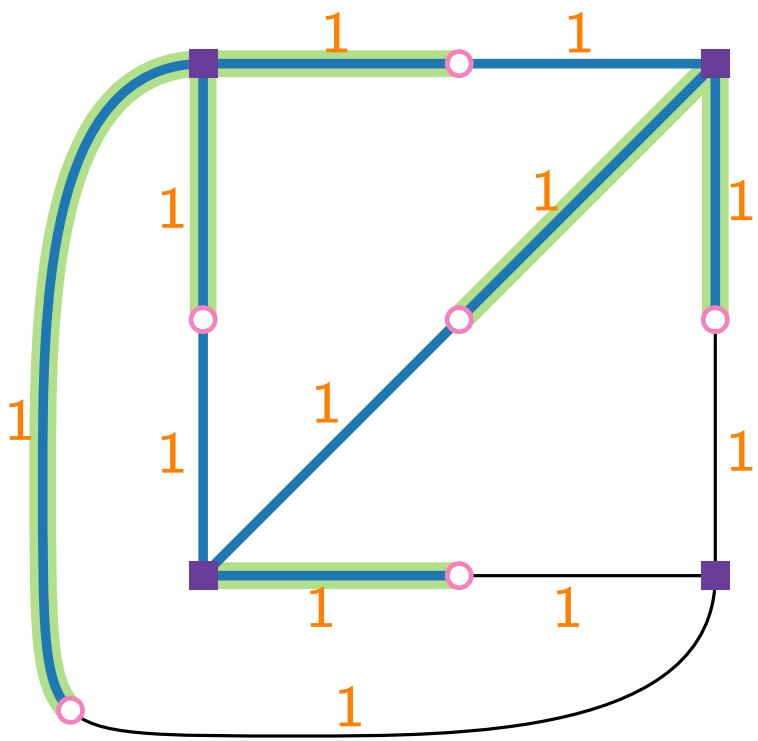
Can we do better?

Analysis Tight?

K_k



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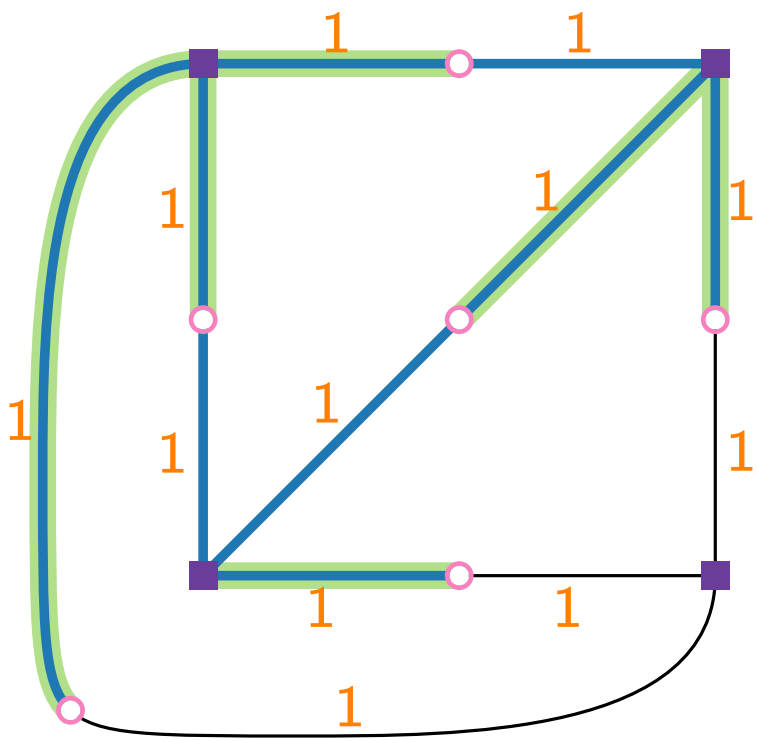
Can we do better?

The best known approximation factor for MultiwayCut is $1.2965 - \frac{1}{k}$.

[Sharma & Vondrák, STOC'14]

Analysis Tight?

K_k



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Can we do better?

The best known approximation factor for MultiwayCut is $1.2965 - \frac{1}{k}$.

[Sharma & Vondrák, STOC'14]

MultiwayCut cannot be approximated within factor $1.20016 - O(1/k)$ (unless P = NP).

[Bérczi, Chandrasekaran, Király & Madan, MP'18]