



UNIVERSITÄT
WÜRZBURG

Approximation Algorithms

Lecture 1: Introduction and Vertex Cover

Part I: Organizational

Organizational

Lectures: Fri, 10:15–11:45 (ÜR I)

English/German, depending on audience.

hands-on, with tasks/questions for the audience :-)

Tutorials: Tue, 10:15–11:45 (SE I), starting Oct. 20, 2025.

discussing old solutions and solving new tasks

roughly one exercise sheet per lecture

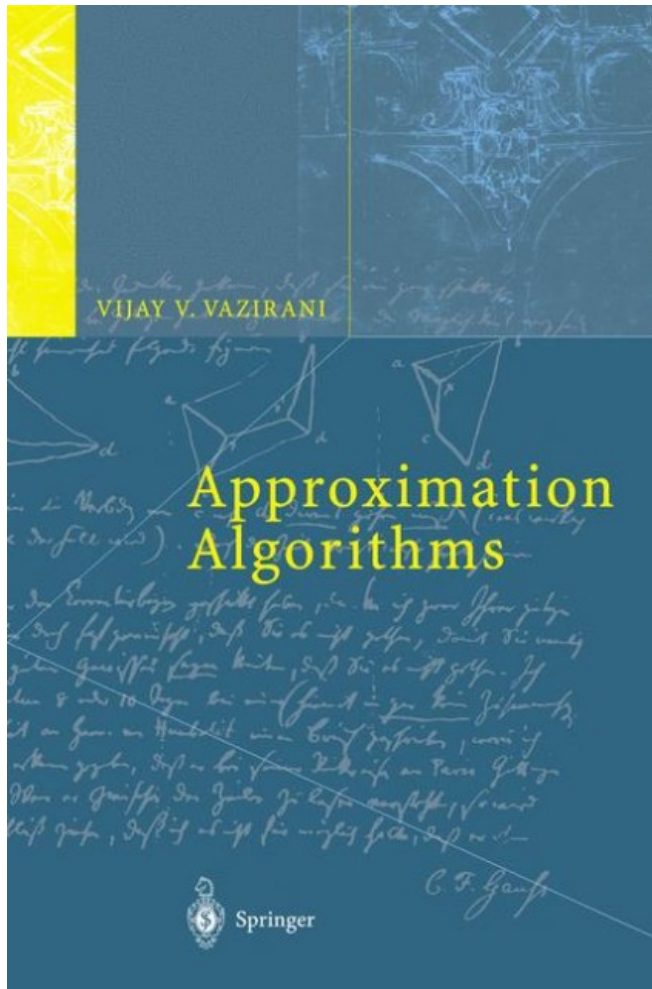
bonus (+0.3 on final grade) for $\geq 50\%$ pts & no **AI!**

Up to two students can hand in solutions together.

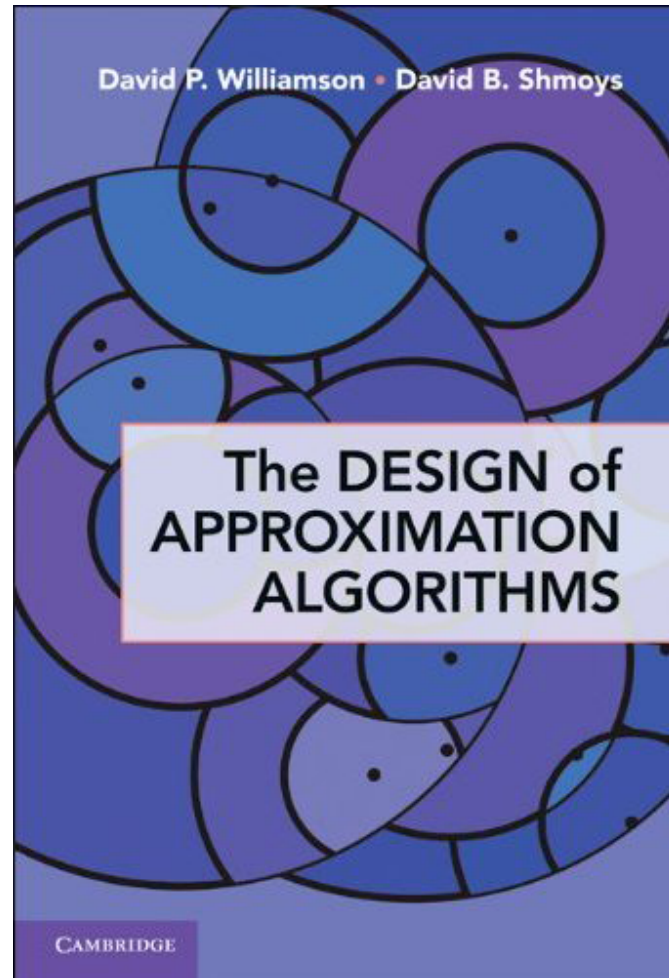
Make sure to write both names!

Most slides are due to Joachim Spoerhase,
polishing & colors are due to Philipp Kindermann – thanks!

Textbooks



Vijay V. Vazirani:
Approximation Algorithms
Springer-Verlag, 2003.



David P. Williamson & David B. Shmoys:
The Design of Approximation Algorithms
Cambridge-Verlag, 2011.

<http://www.designofapproxalgs.com/> 

Approximation Algorithms

„All exact science is dominated by the idea of approximation.“



Bertrand Russell
(1872–1970)

Approximation Algorithms

- Many optimization problems are NP-hard!
(For example, the traveling salesperson problem.)
- $\rightsquigarrow$ an optimal solution cannot be efficiently computed unless $P=NP$.
- However, good approximate solutions can often be found efficiently!
- **Techniques** for the design and analysis of approximation algorithms arise from studying specific optimization problems.

Overview

Combinatorial algorithms

- Introduction (Vertex Cover)
- Set Cover via Greedy
- Shortest Superstring via reduction to SC
- Steiner Tree via MST
- Multiway Cut via Greedy
- k -Center via Parametrized Pruning
- Min-Degree Spanning Tree and local search
- Knapsack via DP and Scaling
- Euclidean TSP via Quadtrees

LP-based algorithms

- Introduction to LP-Duality
- Set Cover via LP Rounding
- Set Cover via Primal–Dual Schema
- Maximum Satisfiability
- Scheduling und Extreme Point Solutions
- Steiner Forest via Primal–Dual

Approximation Algorithms

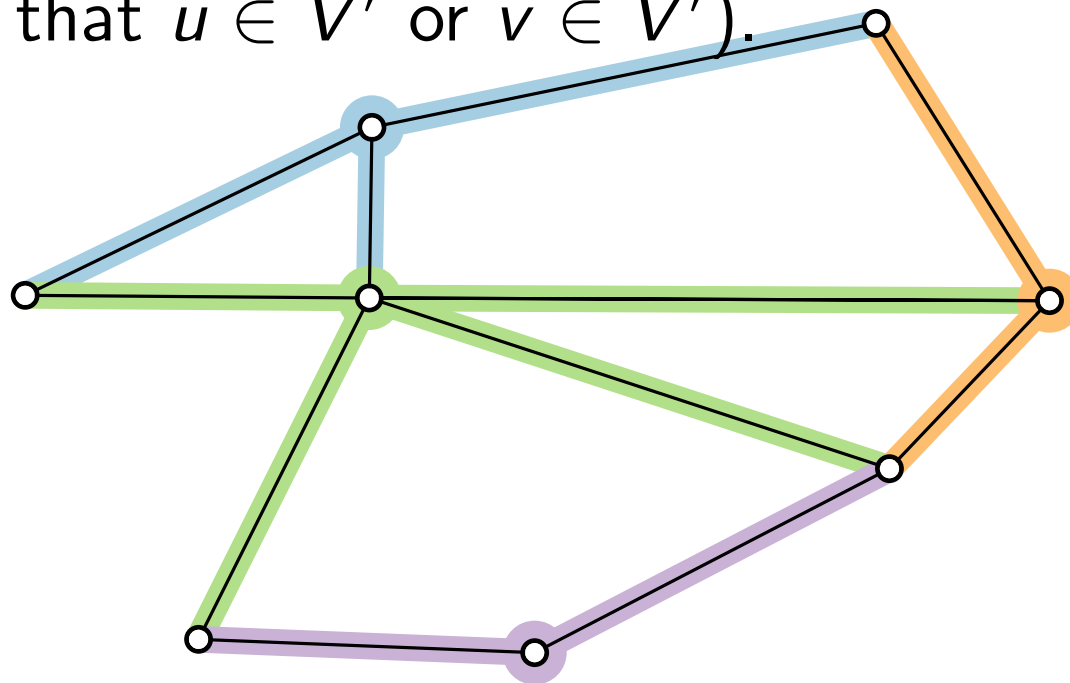
Lecture 1: Introduction and Vertex Cover

Part II: (Cardinality) Vertex Cover

VERTEXCOVER (card.)

Input: graph G

Output: a minimum **vertex cover**, that is,
a minimum-cardinality vertex set $V' \subseteq V(G)$ s. t.
every edge is **covered** (i.e., for every $uv \in E(G)$, it
holds that $u \in V'$ or $v \in V'$).



Optimum ($\text{OPT} = 4$) – but in general NP-hard to find :-)

Approximation Algorithms

Lecture 1: Introduction and Vertex Cover

Part III: NP-Optimization Problem

NP-Optimization Problem

An **NP-optimization problem** Π is given by:

- A set D_Π of **instances**.

We denote the size of an instance $I \in D_\Pi$ by $|I|$.

- For each instance $I \in D_\Pi$,
a set $S_\Pi(I) \neq \emptyset$ of **feasible solutions** for I such that:
 - for each solution $s \in S_\Pi(I)$,
its size $|s|$ is polynomially bounded in $|I|$, and
 - there is a polynomial-time algorithm that decides,
for each pair (s, I) , whether $s \in S_\Pi(I)$.
- A polynomial time computable **objective function** obj_Π
which assigns a positive objective value $\text{obj}_\Pi(I, s) \geq 0$ to
any given pair (s, I) with $s \in S_\Pi(I)$.
- Π is either a minimization or maximization problem.

VERTEXCOVER: NP-Optimization Problem

Task: Fill in the gaps for $\Pi = \text{VERTEX COVER}$.

$D_\Pi =$ set of all graphs

For $I \in D_\Pi$: $|I| =$ number of vertices of G

graph G

$S_\Pi(I) =$ set of all vertex covers of G

■ Why is $|s| \in \text{poly}(|I|)$ for every $s \in S_\Pi(I)$?

Take polynomial $p(n) = n$. $s \subseteq V(G) \Rightarrow |s| \leq p(|I|) = |V(G)|$

■ For a given pair (s, I) , how can we efficiently decide whether $s \in S_\Pi(I)$? Test whether all edges are covered.

$\text{obj}_\Pi(I, s) = |s|$

Π is a minimization problem.

Optimum and Optimal Objective Value

Let Π be a $\text{maximization problem}$ and $I \in D_\Pi$ an instance of Π .

A feasible solution $s^* \in S_\Pi(I)$ is **optimal** if $\text{obj}_\Pi(I, s^*)$ is maximum among the objective values attained by the feasible solutions of I .

The optimal value $\text{obj}_\Pi(I, s^*)$ of the objective function is denoted by $\text{OPT}_\Pi(I)$ or simply by OPT in context.

Approximation Algorithms

maximization problem

$$\alpha: \mathbb{N} \rightarrow \mathbb{Q}$$

Let Π be a minimization problem and ~~$\alpha \in \mathbb{Q}^+$~~ .

A factor- α approximation algorithm for Π is an efficient algorithm that provides, for **any** instance $I \in D_\Pi$, a feasible solution $s \in S_\Pi(I)$ such that

$$\frac{\text{obj}_\Pi(I, s)}{\text{OPT}_\Pi(I)} \begin{matrix} \geq \\ \leq \end{matrix} \alpha(|I|).$$

Approximation Algorithms

Lecture 1:

Introduction and Vertex Cover

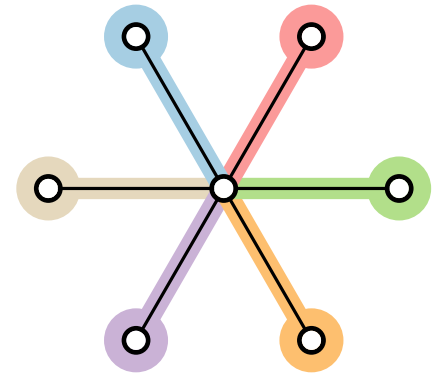
Part IV:

Approximation Algorithm for VERTEXCOVER

Approximation Alg. for VERTEXCOVER

Ideas?

- Edge-Greedy
- Vertex-Greedy



Quality?

Problem: How can we estimate $\overbrace{\text{obj}_\Pi(I, s)}^{\text{ALG}} / \text{OPT}$ – if it is hard to compute OPT ?

Idea: Find a “good” lower bound $L \leq \text{OPT}$ for OPT and compare it to our approximate solution.

$$\frac{\text{obj}_\Pi(I, s)}{\text{OPT}} \leq \frac{\text{obj}_\Pi(I, s)}{L}$$

Lower Bound by Matchings

Given a graph G , a set M of edges of G is a **matching** if no two edges of M are adjacent (i.e., share an end vertex).

M is **maximal** if there is no matching M' with $M' \supsetneq M$.

$$\text{OPT} \geq |M|$$

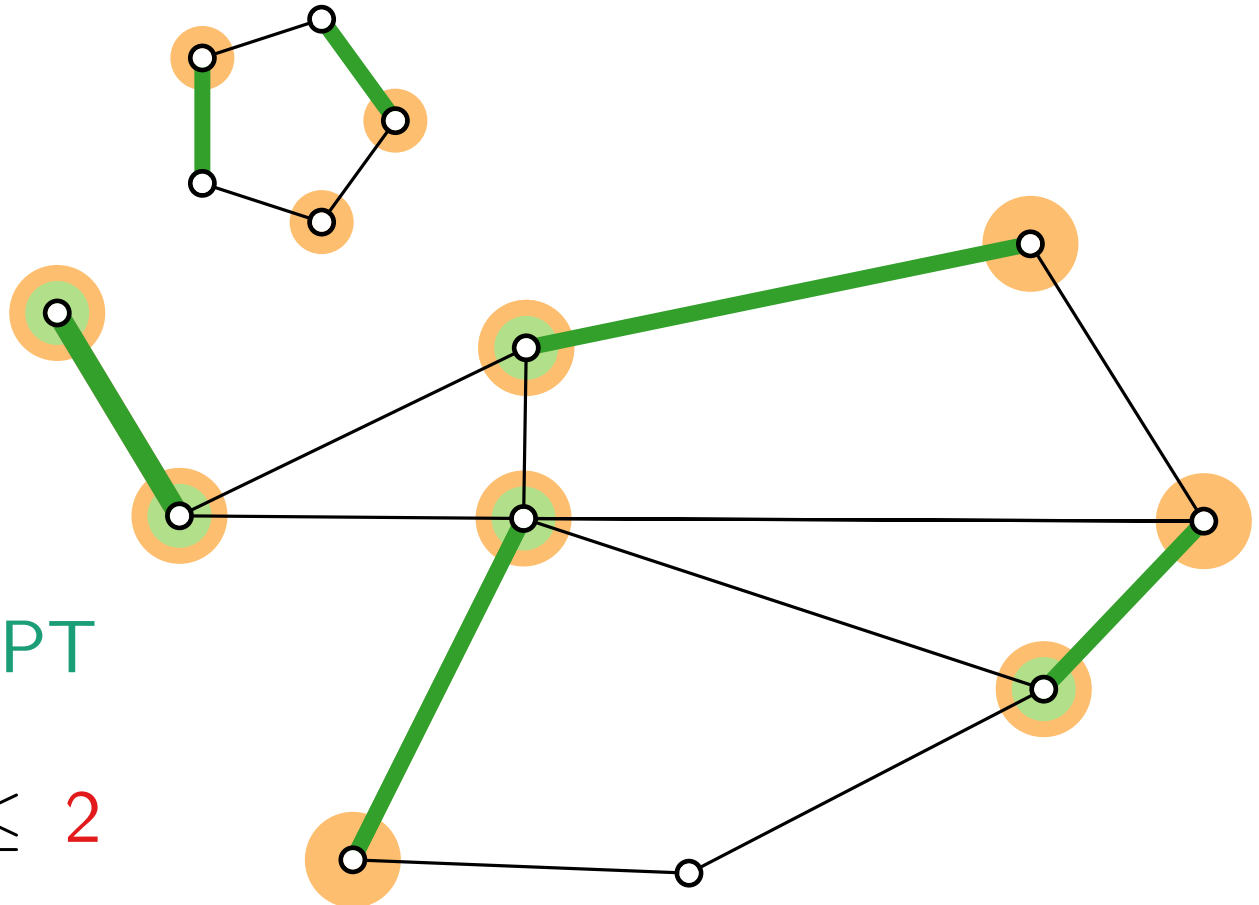
~~$$\text{OPT} = |M| ?$$~~

Vertex cover of M

Vertex cover of $E(G)$

$$\text{ALG} = 2 \cdot |M| \leq 2 \cdot \text{OPT}$$

$$\Rightarrow \frac{\text{obj}_{\Pi}(I, s)}{\text{OPT}} = \frac{\text{ALG}}{\text{OPT}} \leq 2$$



Approximation Alg. for VERTEXCOVER

Algorithm VertexCover(G)

$M \leftarrow \emptyset$

foreach $e \in E(G)$ **do**

if e is not adjacent to any edge in M **then**
 $M \leftarrow M \cup \{e\}$

return $\{u, v \mid uv \in M\}$

Theorem. The above algorithm is a factor-2 approximation algorithm for VERTEXCOVER.

Proof. $ALG = 2 \cdot |M| \leq 2 \cdot OPT$



Approximability of VERTEX COVER

The best known approximation factor for VERTEXCOVER is $2 - \Theta(1/\sqrt{\log n})$.

If $P \neq NP$, VERTEXCOVER cannot be approximated within a factor of 1.3606.

VERTEXCOVER cannot be approximated within a factor of $2 - \Theta(1)$ – if the *Unique Games Conjecture* holds.

Approximation Algorithms

Lecture 1:

Introduction and Vertex Cover

Part V:

An LP-based Algorithm for VERTEXCOVER

Task

Write an integer linear program (ILP) for VERTEXCOVER:

Using integer (and/or real) variables, express the problem using

- linear constraints and
- a linear objective function.

You can iterate over the vertices / edges of the given graph G .

Variables: for each vertex v of G , we introduce $x_v \in \{0, 1\}$.

Objective: minimize $\sum_{v \in V(G)} x_v$

v not in the solution
 v in the solution

Constraints: for each edge uv of G , we require that

$$x_u + x_v \geq 1.$$

Solving (I)LPs

LP relaxation

minimize $\sum_{v \in V(G)} x_v$

subject to $x_u + x_v \geq 1$ for each $uv \in E(G)$

$x_v \geq 0$ ~~$x_v \in \{0, 1\}$~~ for each $v \in V(G)$

Problem: It's NP-hard to solve ILPs in general.

But: LPs can be solved efficiently (in $O(L \cdot n^{3.5})$ time),
where $n = \#$ variables and $L =$ total bit complexity of coefficients.

Problem': Now we can get fractional solutions, i.e., $x_v \in (0, 1)$.

Task: Find a graph G with $\text{OPT}_{\text{LP}} \neq \text{OPT}_{\text{ILP}}$!

Solution? Round the LP solution to get an integral solution!

Rounding the LP Solution

$$\begin{array}{ll}\text{minimize} & \sum_{v \in V(G)} x_v \\ \text{subject to} & x_u + x_v \geq 1 \quad \text{for each } uv \in E(G) \\ & x_v \geq 0 \quad \text{for each } v \in V(G)\end{array}$$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

Need to check: Is $(x'_v)_{v \in V(G)}$ a feasible solution?

In other words: Is $\{v \in V(G) : x'_v = 1\}$ a vertex cover of G ?

Need to make sure that every edge uv of G is covered.

Is $x'_u = 0 = x'_v$ possible? But then $x_u < 0.5$ and $x_v < 0.5$.

This contradicts $x_u + x_v \geq 1 \Rightarrow x'_u = 1$ or $x'_v = 1 \Rightarrow (x'_v)$
feasible!

Cost of the Solution

$$\begin{array}{ll} \text{minimize} & \sum_{v \in V(G)} x_v \\ \text{subject to} & x_u + x_v \geq 1 \quad \text{for each } uv \in E(G) \\ & x_v \geq 0 \quad \text{for each } v \in V(G) \end{array}$$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

$$\text{ALG} = \sum_{v \in V(G)} x'_v \leq 2 \cdot \sum_{v \in V(G)} x_v = 2 \cdot \text{OPT}_{\text{LP}} \leq 2 \cdot \text{OPT}_{\text{ILP}}$$

Theorem. The LP rounding algorithm is a factor-2 approximation algorithm for VERTEXCOVER.

Cost of the Solution

$$\begin{array}{ll}\text{minimize} & \sum_{v \in V(G)} x_v \cdot w(v) \\ \text{subject to} & x_u + x_v \geq 1 \quad \text{for each } uv \in E(G) \\ & x_v \geq 0 \quad \text{for each } v \in V(G)\end{array}$$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

$$\text{ALG} = \sum_{v \in V(G)} x'_v \cdot w(v) \leq 2 \cdot \sum_{v \in V(G)} x_v \cdot w(v) = 2 \cdot \text{OPT}_{\text{LP}} \leq 2 \cdot \text{OPT}_{\text{ILP}}$$

Theorem. The LP rounding algorithm is a factor-2 approximation algorithm for **WEIGHTED VERTEX COVER**.

Approximation Algorithms

Lecture 1: Introduction and Vertex Cover

Part VI: The Vertex Cover Polytope

The Vertex Cover Polytope

minimize $\sum_{v \in V(G)} x_v$

subject to $x_u + x_v \geq 1$

$x_v \geq 0$

for each $uv \in E(G)$

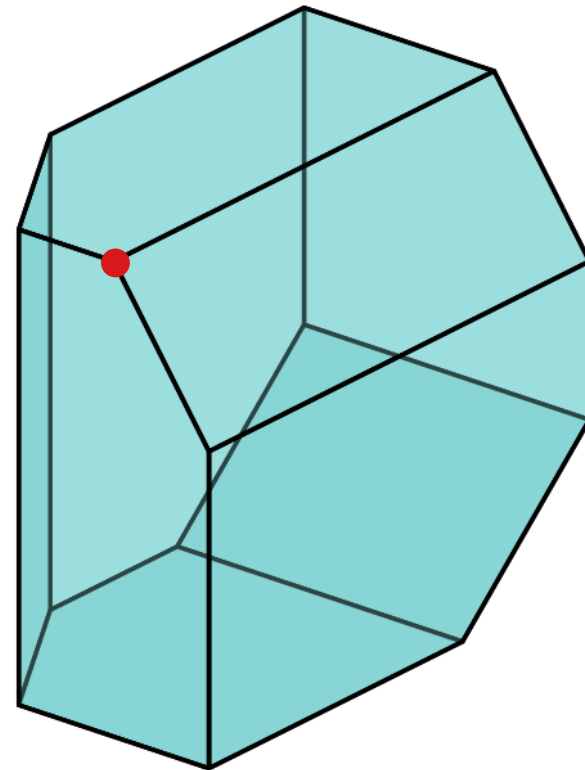
for each $v \in V(G)$

The solution space of an LP is a *convex* polytope in $\mathbb{R}^n$, where n is the number of variables.

The number of facets corresponds to the number of “relevant” constraints.

An *extreme point* of the polytope is a point that is extreme in some direction.

Most LP solvers return extreme-point solutions.



Half-Integrality

Thm. The extreme points of the vertex cover polytope are half-integral, that is, their coordinates are in $\{0, 0.5, 1\}$.

Proof. Let x be an LP solution that is not half-integral.

$$\Rightarrow \exists v \in V(G): x_v \in (0, 0.5) \cup (0.5, 1).$$

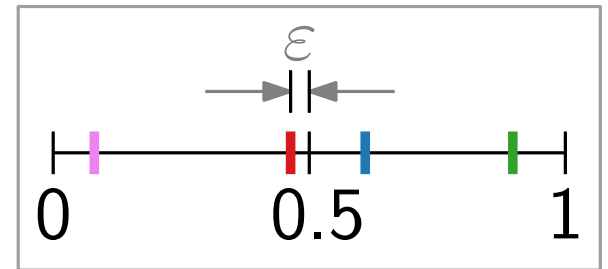
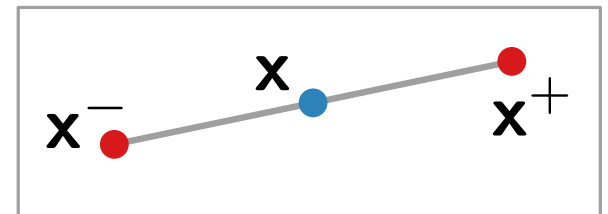
$$\text{Let } \varepsilon = \min_{u \in V(G)} \{x_u, 1 - x_u, |x_u - 0.5|\} \setminus \{0\}.$$

For every $v \in V(G)$, let

$$x_v^+ = \begin{cases} x_v + \varepsilon & \text{if } x_v \in (0.5, 1) \\ x_v - \varepsilon & \text{if } x_v \in (0, 0.5) \\ x_v & \text{else.} \end{cases}$$

$$x_v^- = \begin{cases} x_v - \varepsilon & \text{if } x_v \in (0.5, 1) \\ x_v + \varepsilon & \text{if } x_v \in (0, 0.5) \\ x_v & \text{else.} \end{cases}$$

$$\Rightarrow \frac{x^+ + x^-}{2} = x$$



Extreme Points

Claim: $\mathbf{x}^+$ is feasible.

Need to show: For every edge uv , we have $x_u^+ + x_v^+ \geq 1$.

Assume $x_u^+ < 0.5 \Rightarrow x_u \leq 0.5 \Rightarrow x_v \geq 0.5 \Rightarrow x_v^+ \geq 0.5$

$$\begin{aligned} \Rightarrow x_u^+ + x_v^+ &= (x_u - \varepsilon) + (x_v + \varepsilon) = \\ &= x_u + x_v \geq 1 \quad \checkmark \end{aligned}$$

Symmetrically: $\mathbf{x}^-$ is feasible.

We have shown:

If a solution is not half-integral, it cannot be extreme.

In other words:

All extreme-point solutions are half-integral.

For VC, standard LP solvers return half-integral solutions. :-)

