



UNIVERSITÄT
WÜRZBURG

Approximation Algorithms

Lecture 1: Introduction and Vertex Cover

Part I: Organizational

Organizational

Lectures: Fri, 10:15–11:45 (ÜR I)

English/German, depending on audience.

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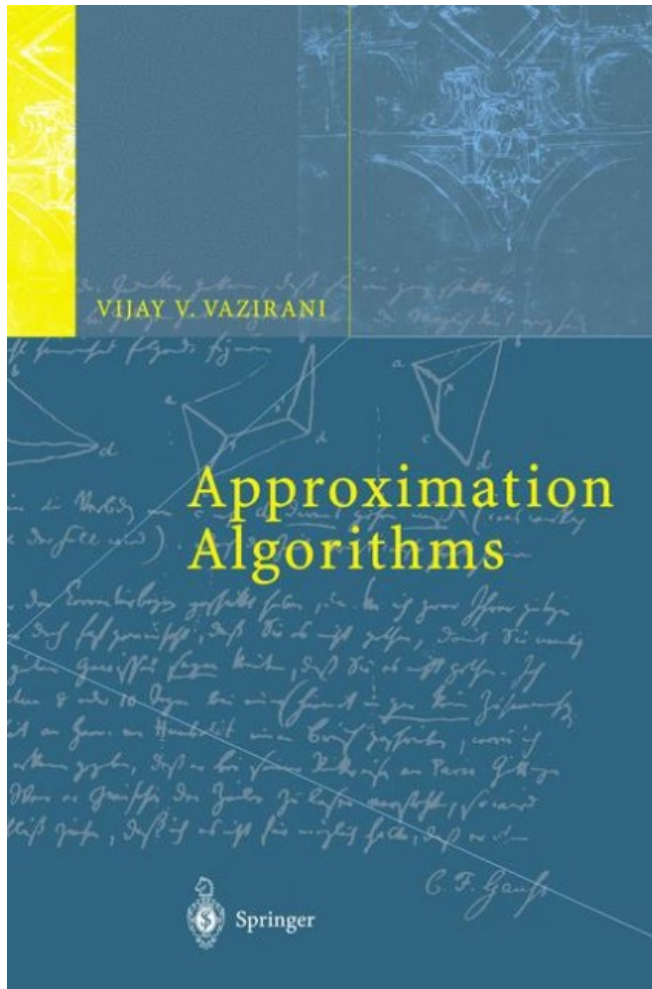
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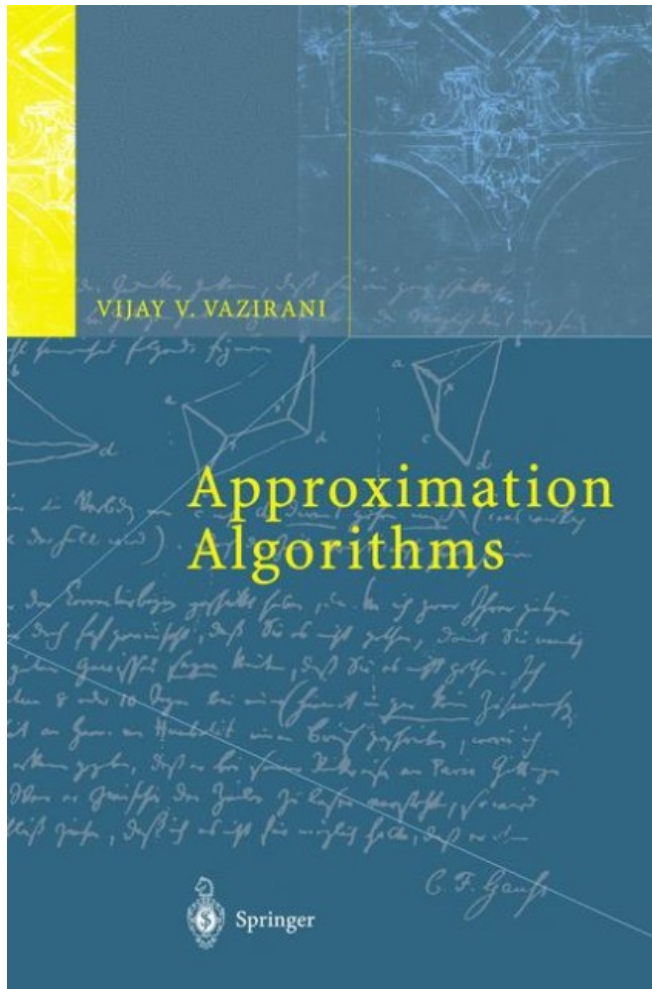
Most slides are due to Joachim Spoerhase,
polishing & colors are due to Philipp Kindermann – thanks!

Textbooks

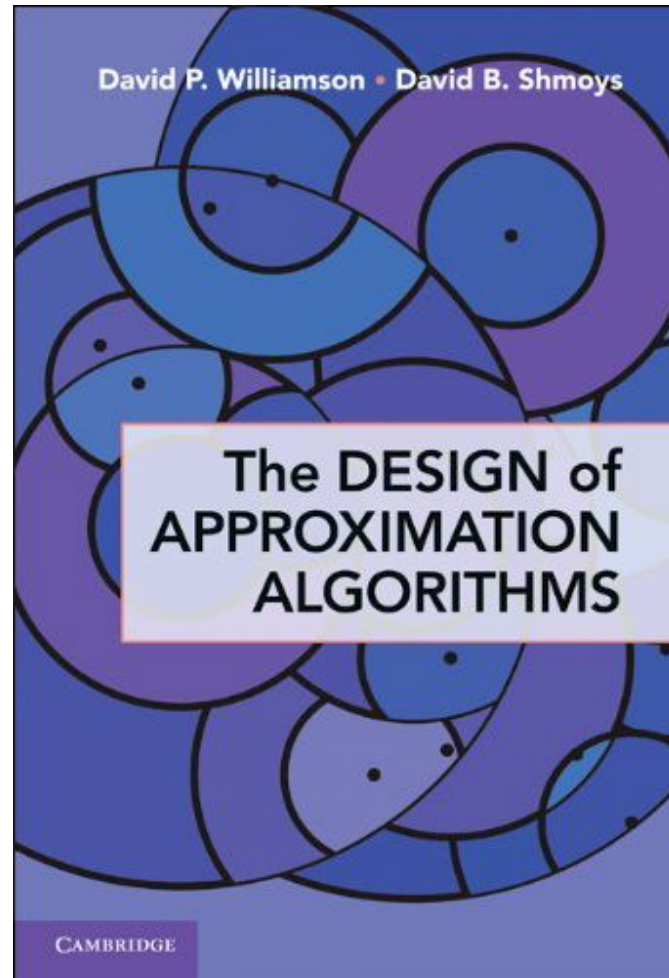


Vijay V. Vazirani:
Approximation Algorithms
Springer-Verlag, 2003.

Textbooks



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David P. Williamson & David B. Shmoys:
The Design of Approximation Algorithms
Cambridge-Verlag, 2011.

<http://www.designofapproxalgs.com/> 

Approximation Algorithms

„All exact science is dominated by the idea of approximation.“



Bertrand Russell
(1872–1970)

Approximation Algorithms

- Many optimization problems are NP-hard!
(For example, the traveling salesperson problem.)

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- Many optimization problems are NP-hard!
(For example, the traveling salesperson problem.)
- $\rightsquigarrow$ an optimal solution cannot be efficiently computed unless $P=NP$.
- However, good approximate solutions can often be found efficiently!
- **Techniques** for the design and analysis of approximation algorithms arise from studying specific optimization problems.

Overview

Combinatorial algorithms

- Introduction (Vertex Cover)
- Set Cover via Greedy
- Shortest Superstring
via reduction to SC
- Steiner Tree via MST
- Multiway Cut via Greedy
- k -Center via Parametrized Pruning
- Min-Degree Spanning Tree
and local search
- Knapsack via DP and Scaling
- Euclidean TSP via Quadrees

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LP-based algorithms

- Introduction to LP-Duality
- Set Cover via LP Rounding
- Set Cover via Primal–Dual Schema
- Maximum Satisfiability
- Scheduling und Extreme Point Solutions
- Steiner Forest via Primal–Dual

Approximation Algorithms

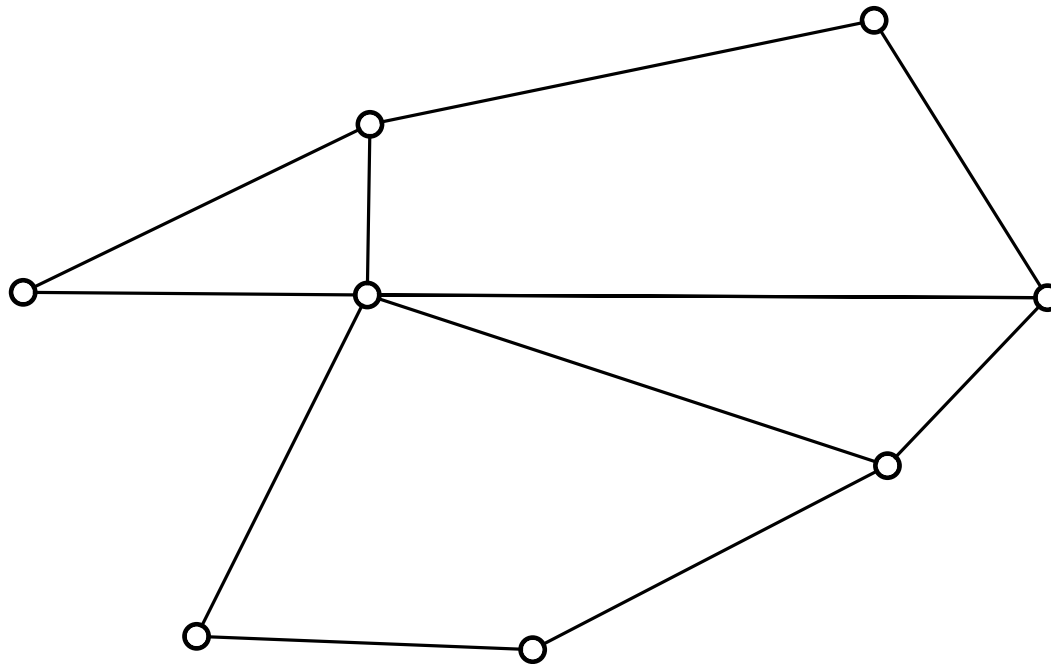
Lecture 1: Introduction and Vertex Cover

Part II: (Cardinality) Vertex Cover

VERTEXCOVER (card.)

Input: graph G

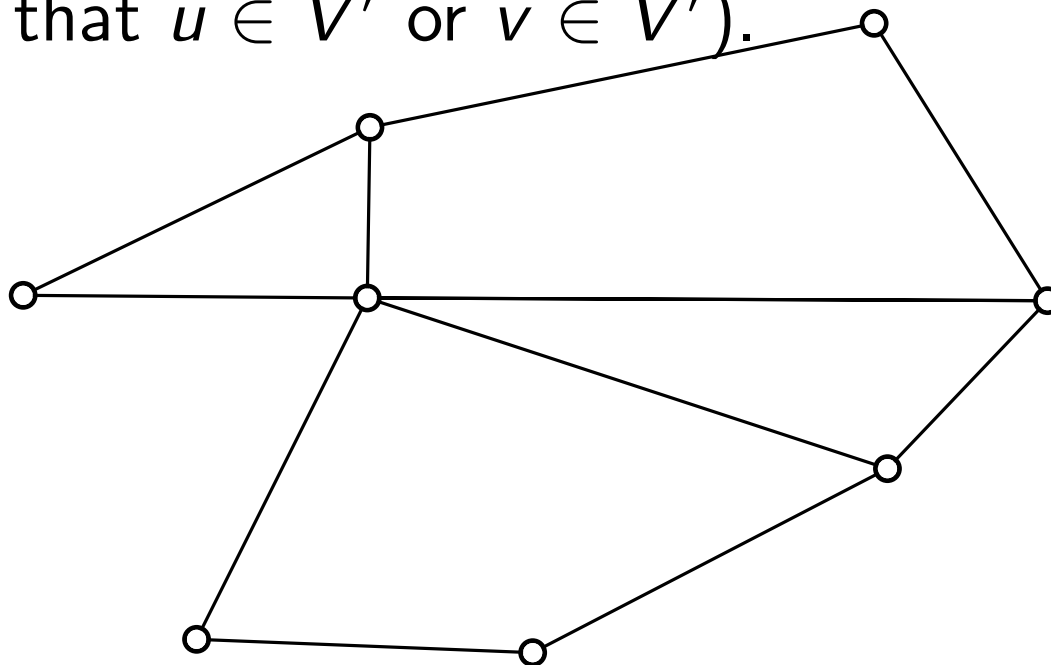
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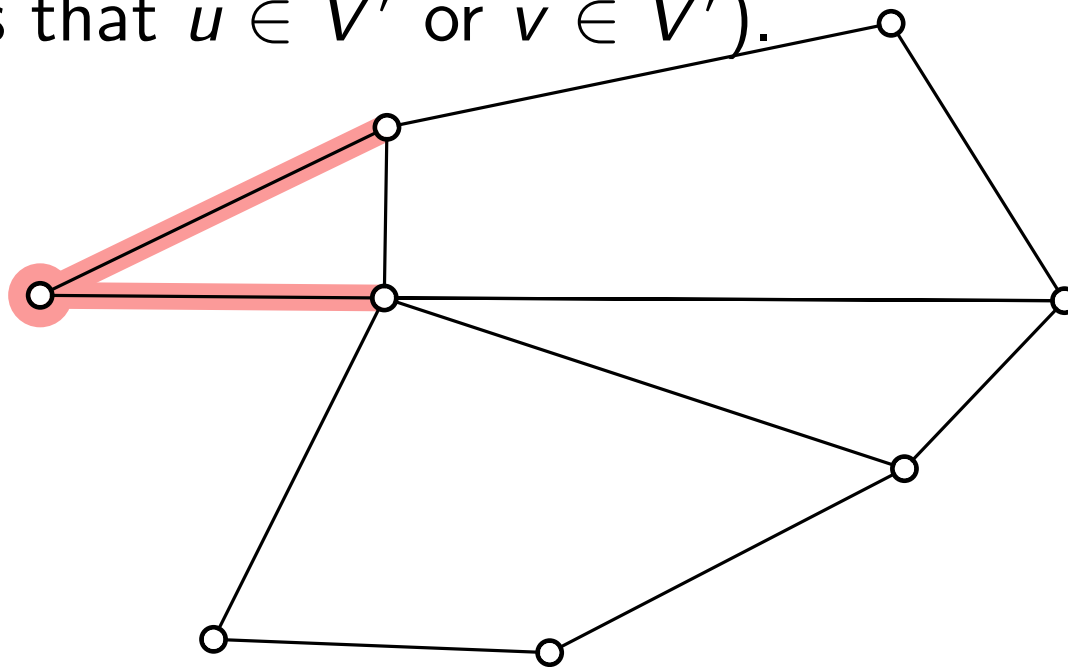
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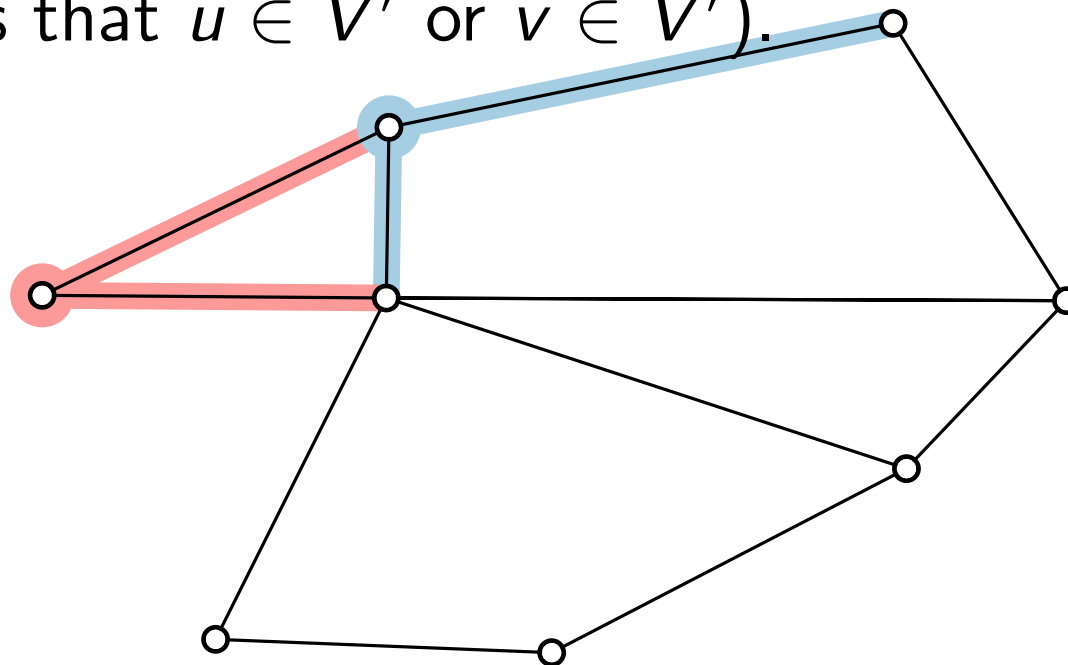
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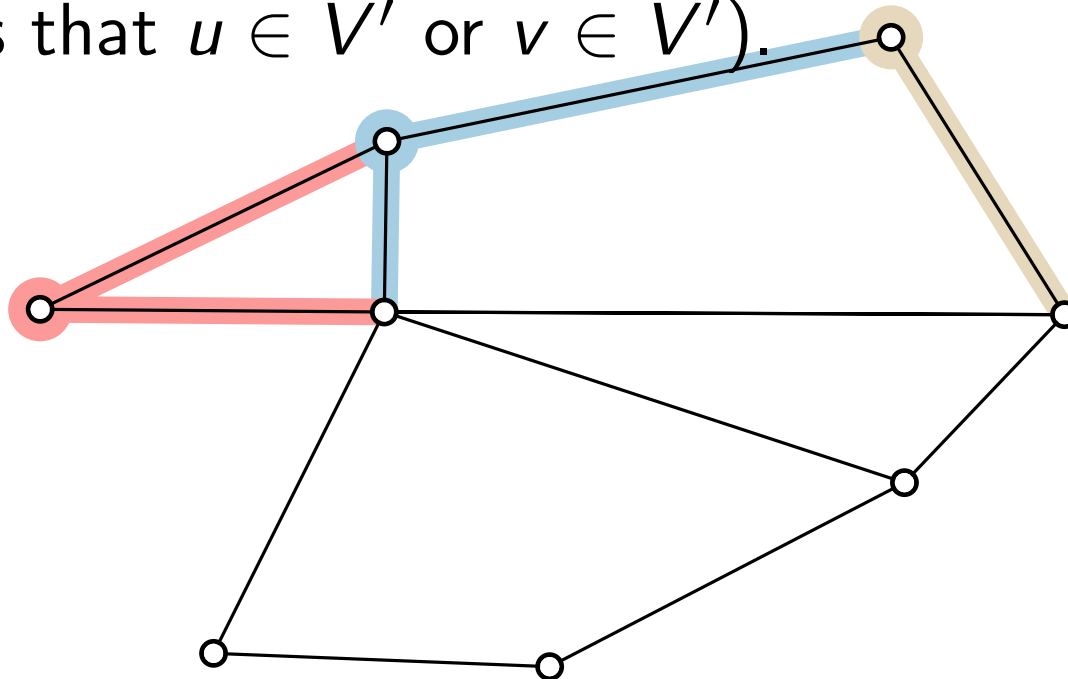
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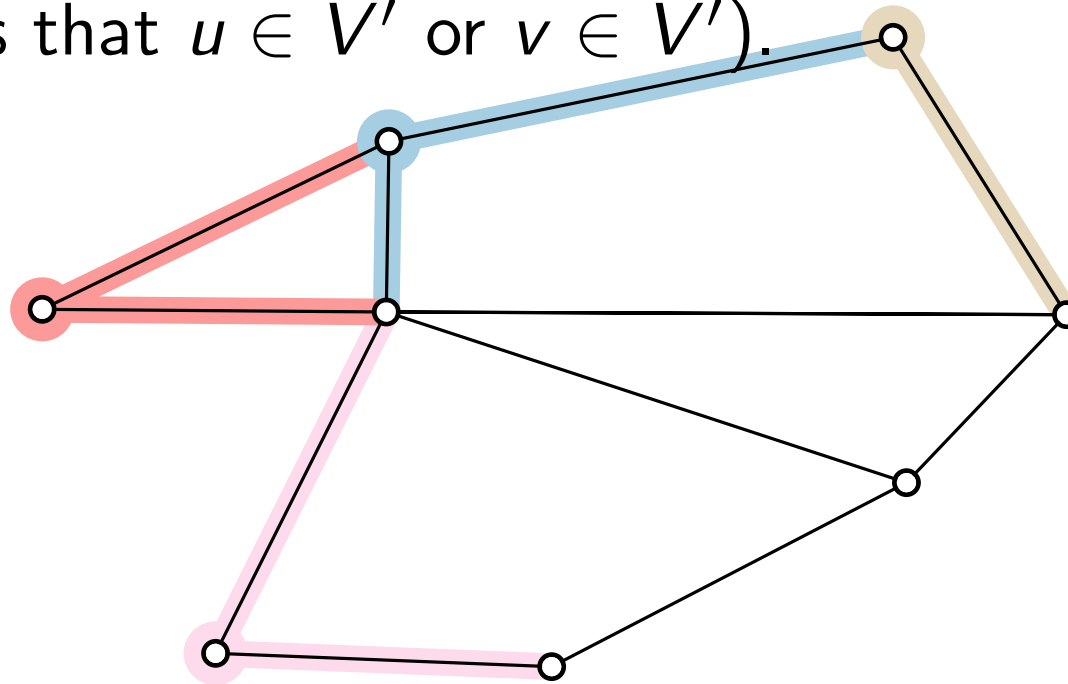
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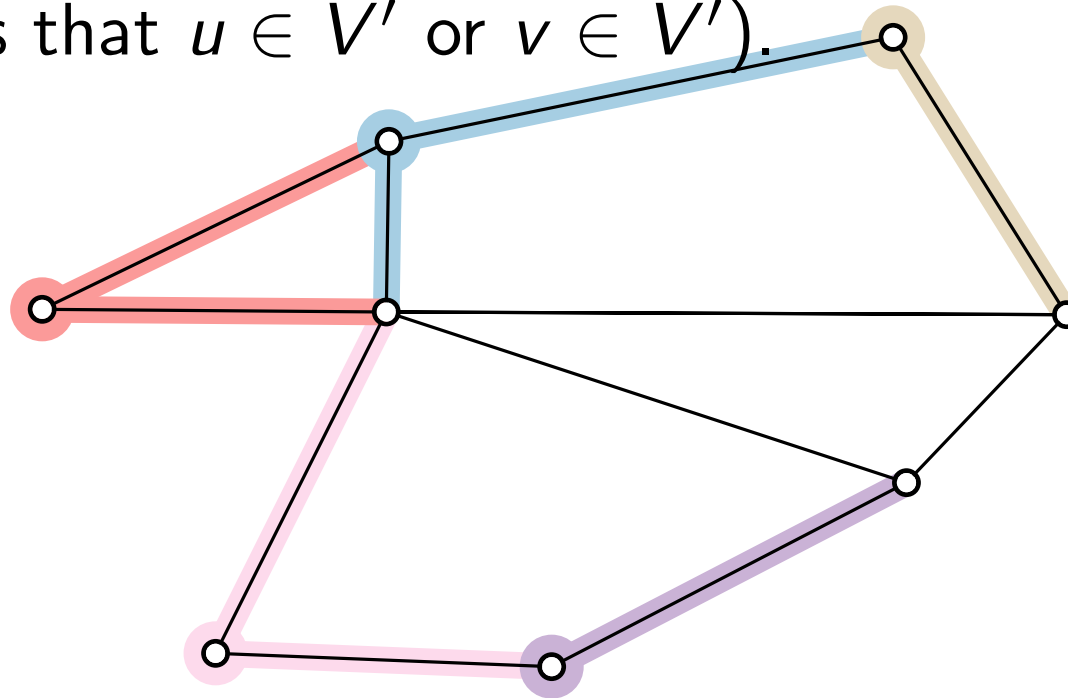
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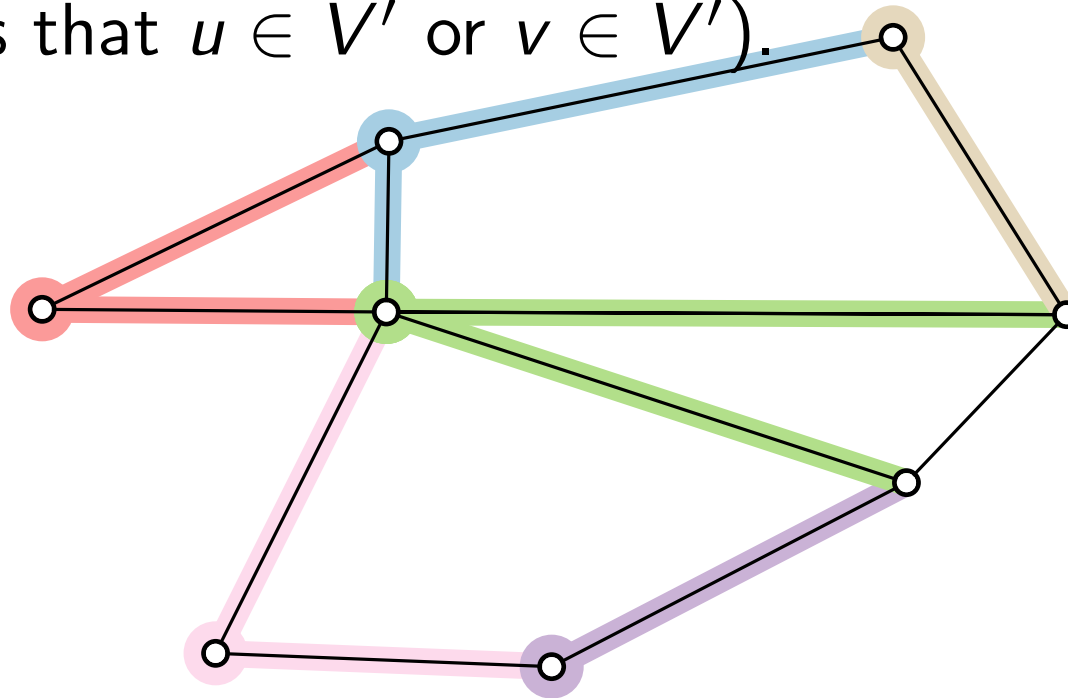
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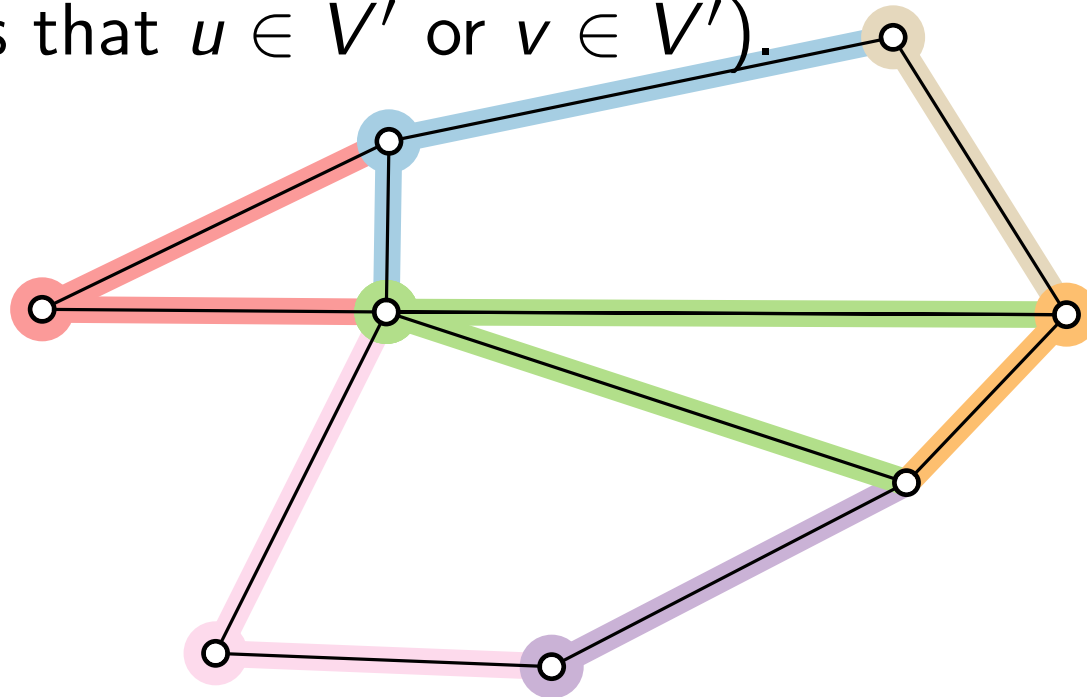
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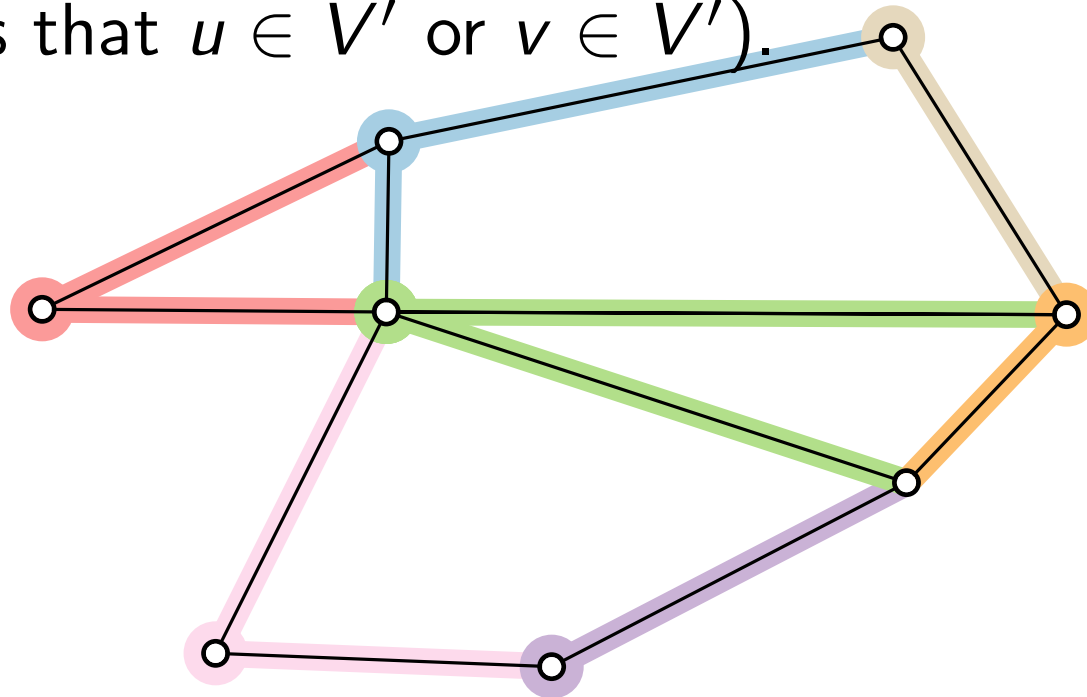
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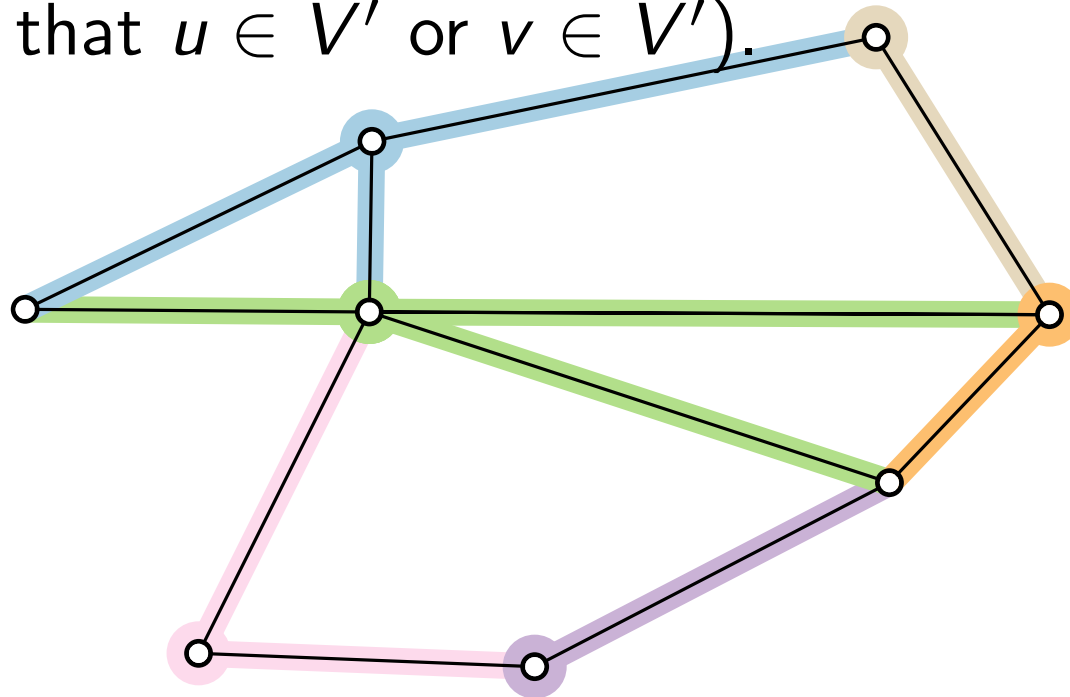


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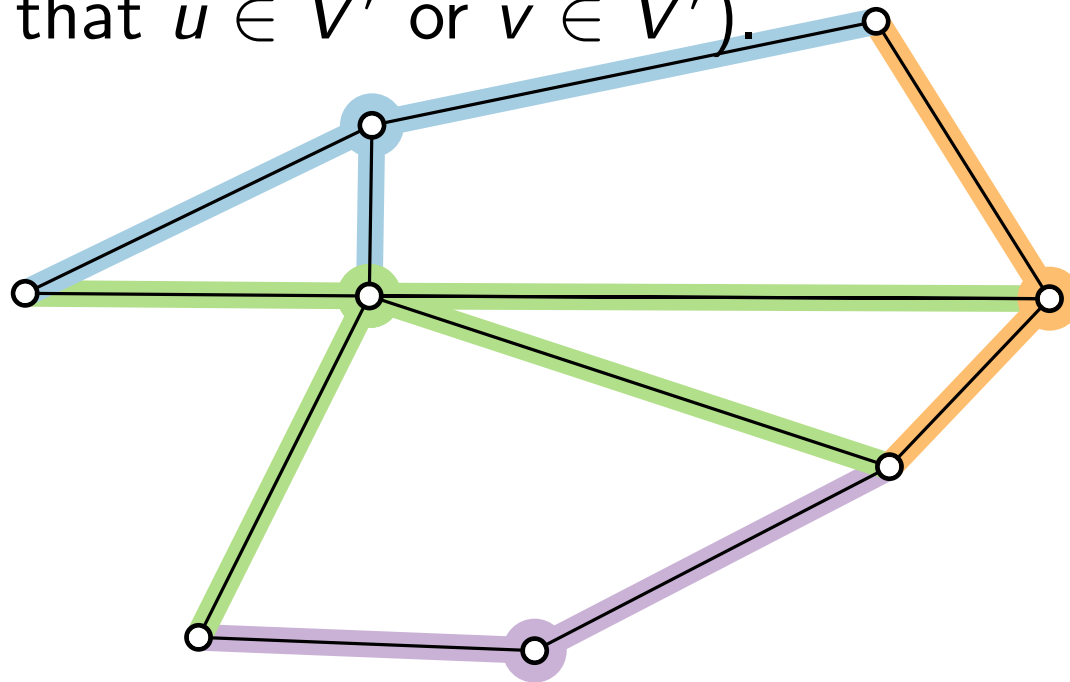


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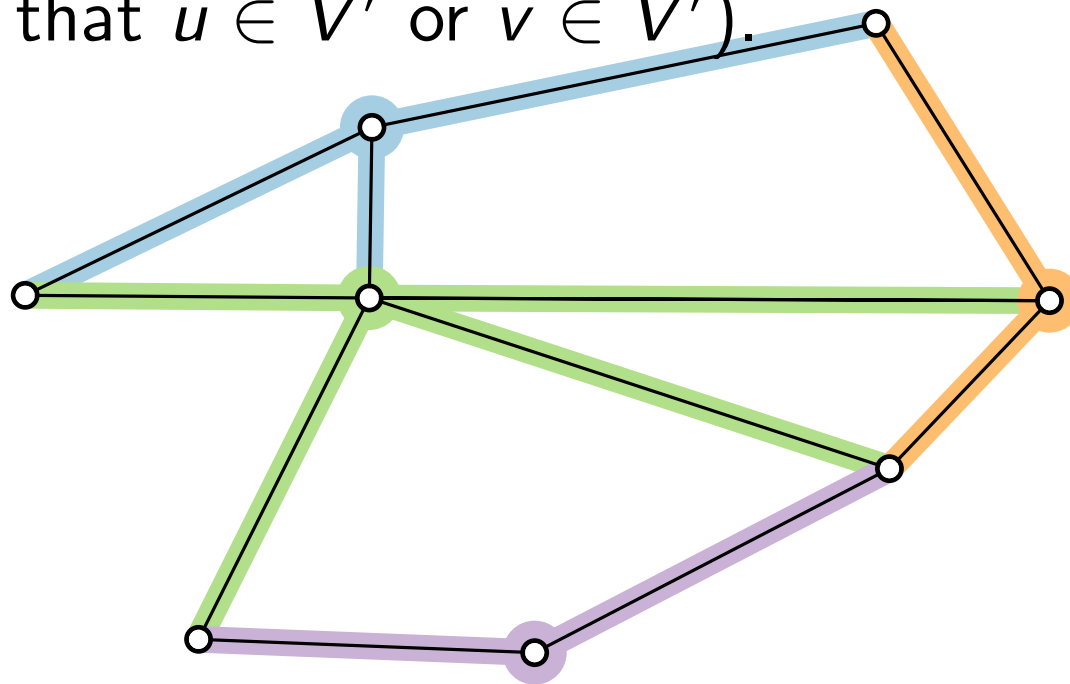


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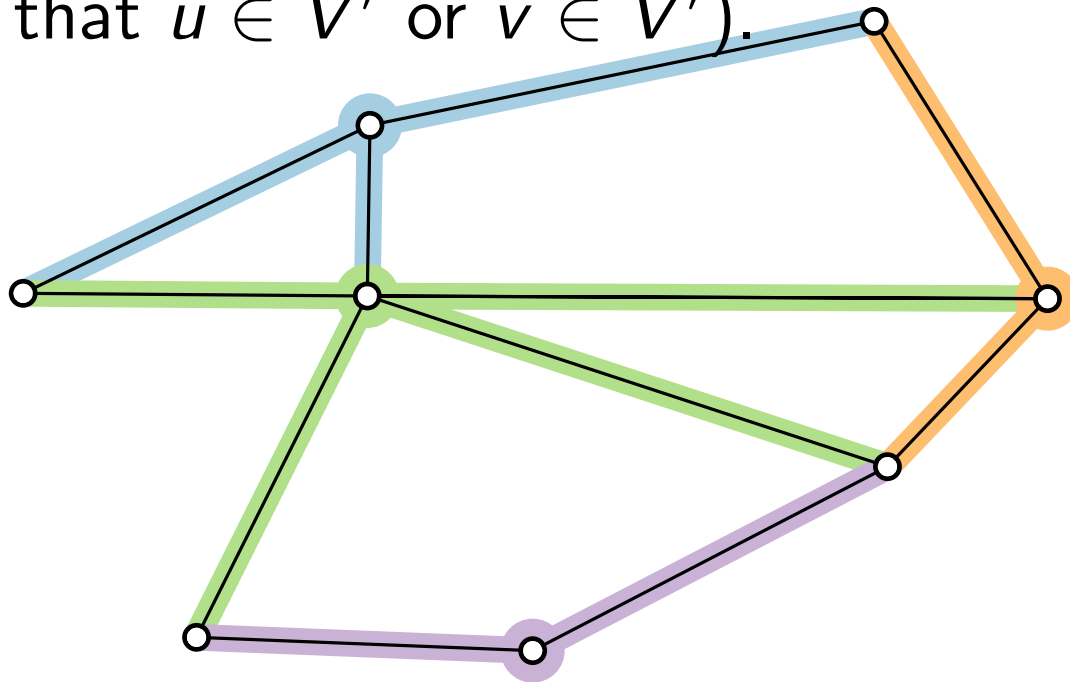


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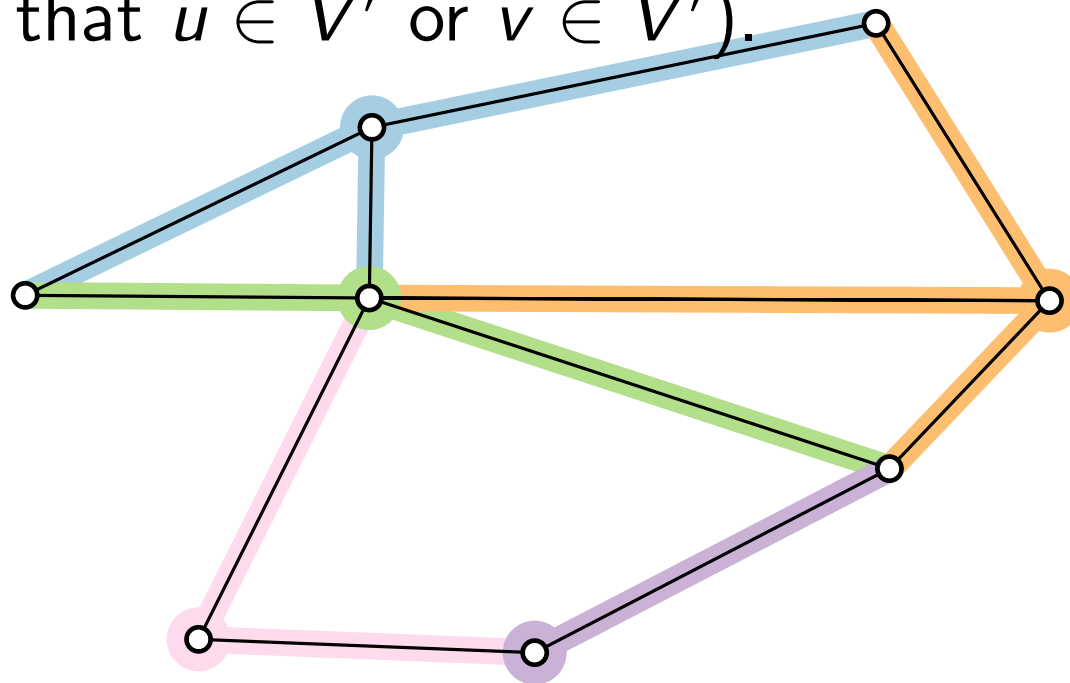


Optimum ($OPT = 4$) – but in general NP-hard to find :-)

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“good” (5/4-) approximate solution

Approximation Algorithms

Lecture 1: Introduction and Vertex Cover

Part III: NP-Optimization Problem

NP-Optimization Problem

An **NP-optimization problem** Π is given by:

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which assigns a positive objective value $\text{obj}_\Pi(I, s) \geq 0$ to
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- Π is either a minimization or maximization problem.

VERTEXCOVER: NP-Optimization Problem

Task: Fill in the gaps for $\Pi = \text{VERTEX COVER}$.

$D_\Pi =$

For $I \in D_\Pi$: $|I| =$

$S_\Pi(I) =$

- Why is $|s| \in \text{poly}(|I|)$ for every $s \in S_\Pi(I)$?
- For a given pair (s, I) , how can we efficiently decide whether $s \in S_\Pi(I)$?

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The optimal value $\text{obj}_\Pi(I, s^*)$ of the objective function is denoted by $\text{OPT}_\Pi(I)$ or simply by OPT in context.

Approximation Algorithms

Let Π be a minimization problem and $\alpha \in \mathbb{Q}^+$.

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Approximation Algorithms

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$$\frac{\text{obj}_\Pi(I, s)}{\text{OPT}_\Pi(I)} \begin{matrix} \geq \\ \leq \end{matrix} \alpha(|I|).$$

Approximation Algorithms

Lecture 1:

Introduction and Vertex Cover

Part IV:

Approximation Algorithm for VERTEXCOVER

Approximation Alg. for VERTEXCOVER

Ideas?

Approximation Alg. for VERTEXCOVER

Ideas?

- Edge-Greedy

Approximation Alg. for VERTEXCOVER

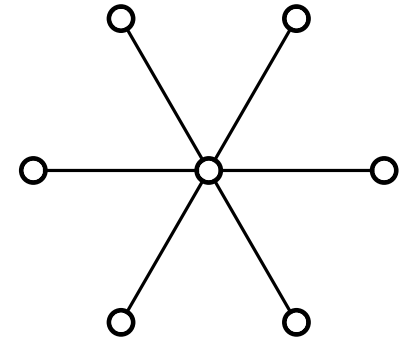
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- Edge-Greedy
- Vertex-Greedy

Approximation Alg. for VERTEXCOVER

Ideas?

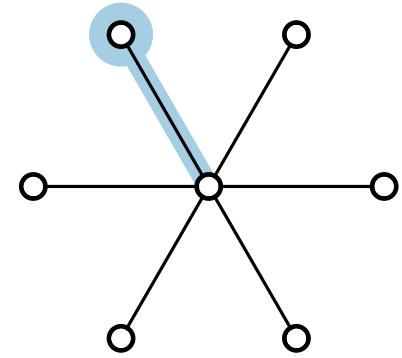
- Edge-Greedy
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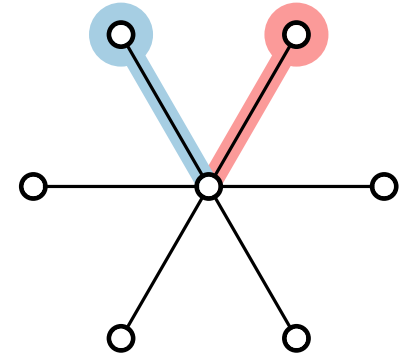
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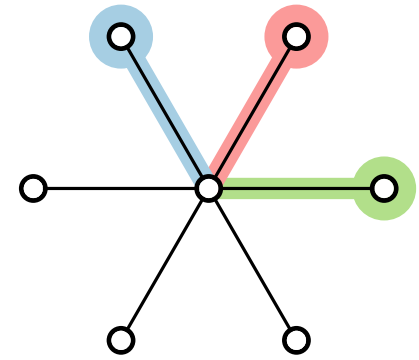
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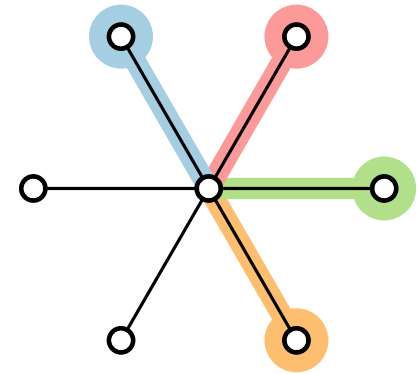
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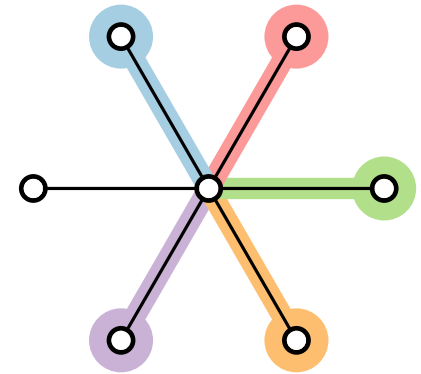
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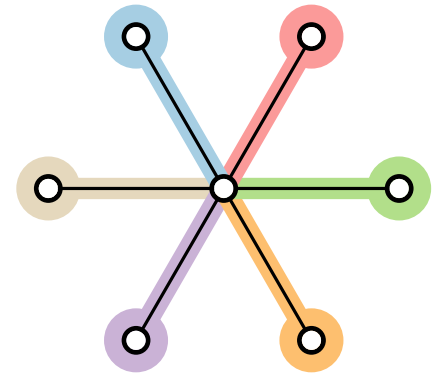
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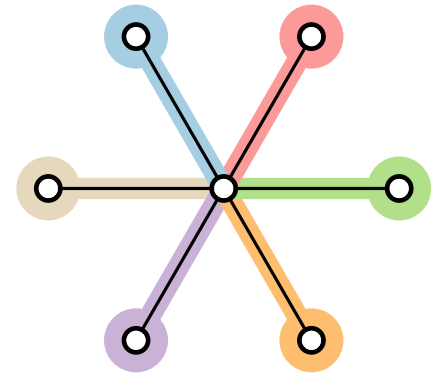
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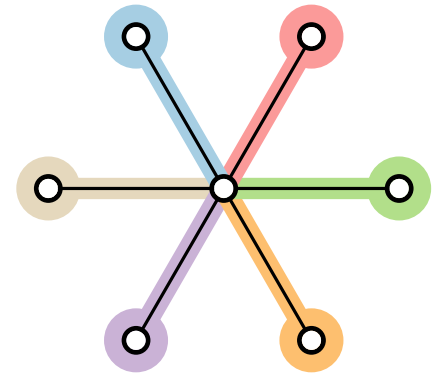


Quality?

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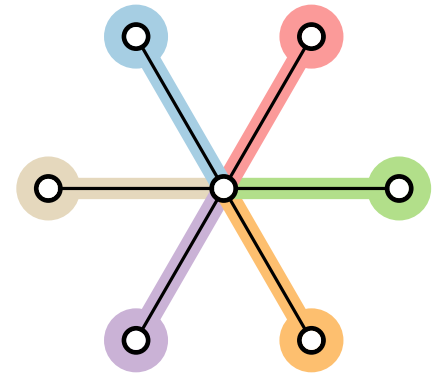
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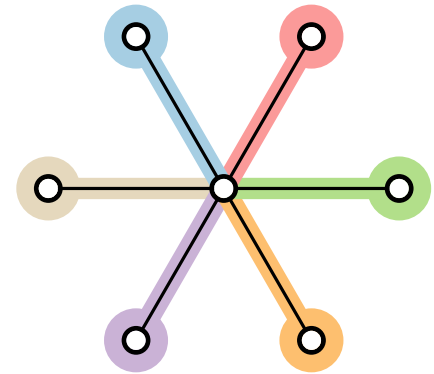
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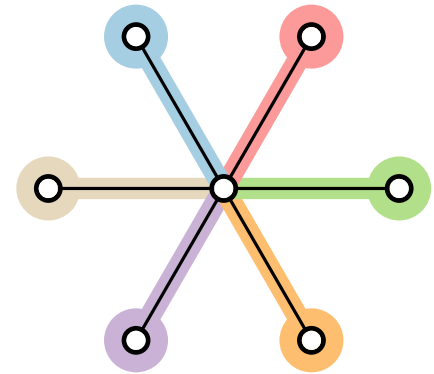
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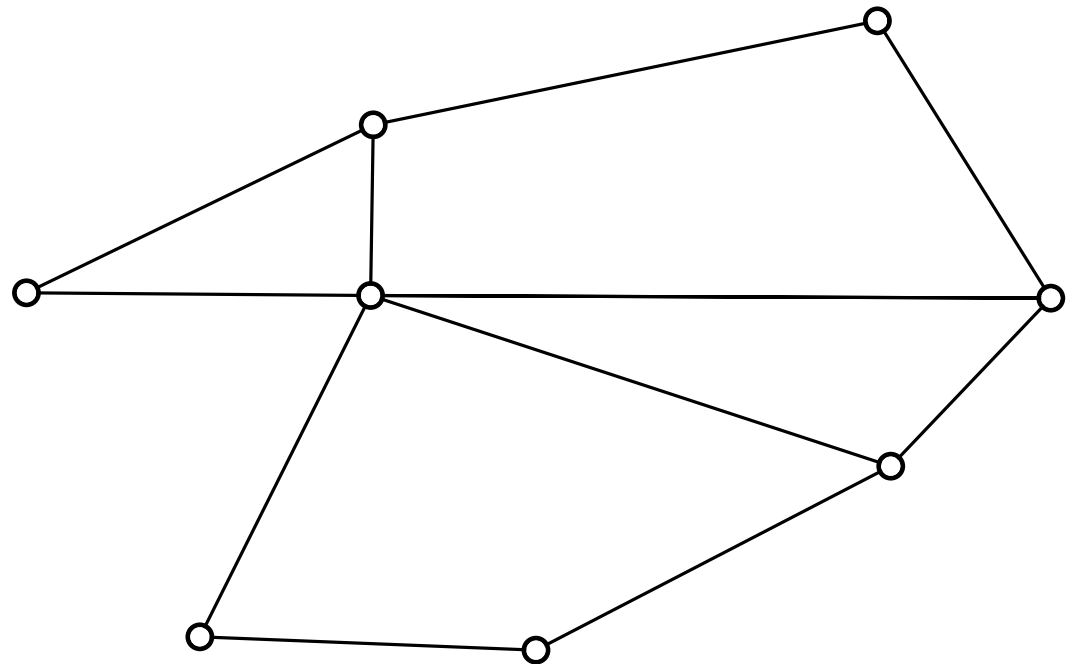
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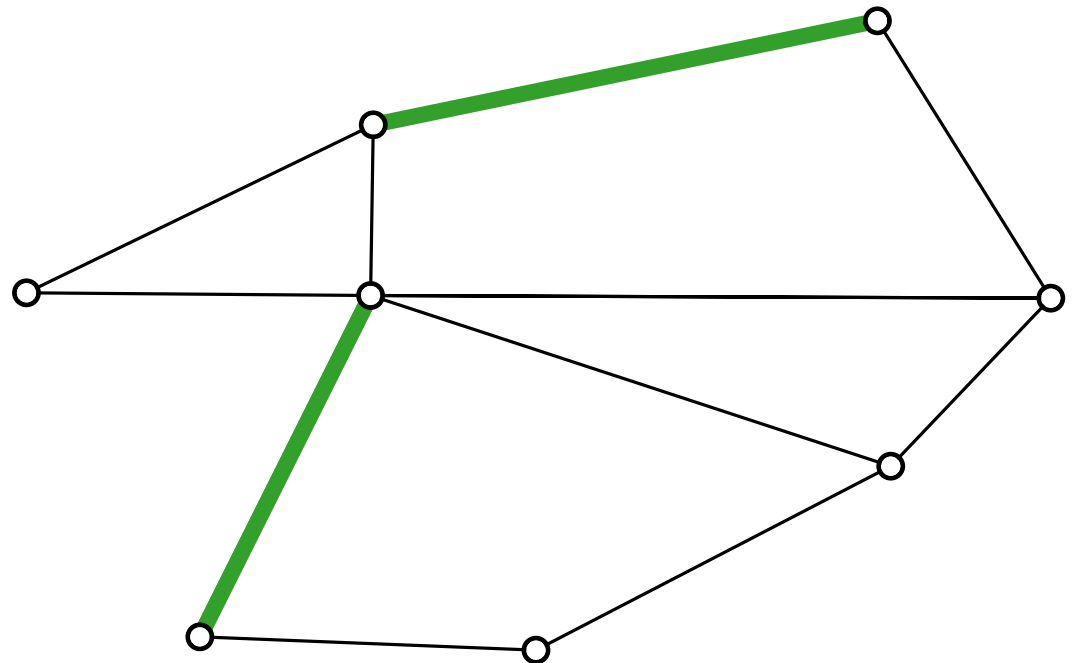
$$\frac{\text{obj}_\Pi(I, s)}{\text{OPT}} \leq \frac{\text{obj}_\Pi(I, s)}{L}$$

Lower Bound



Lower Bound by Matchings

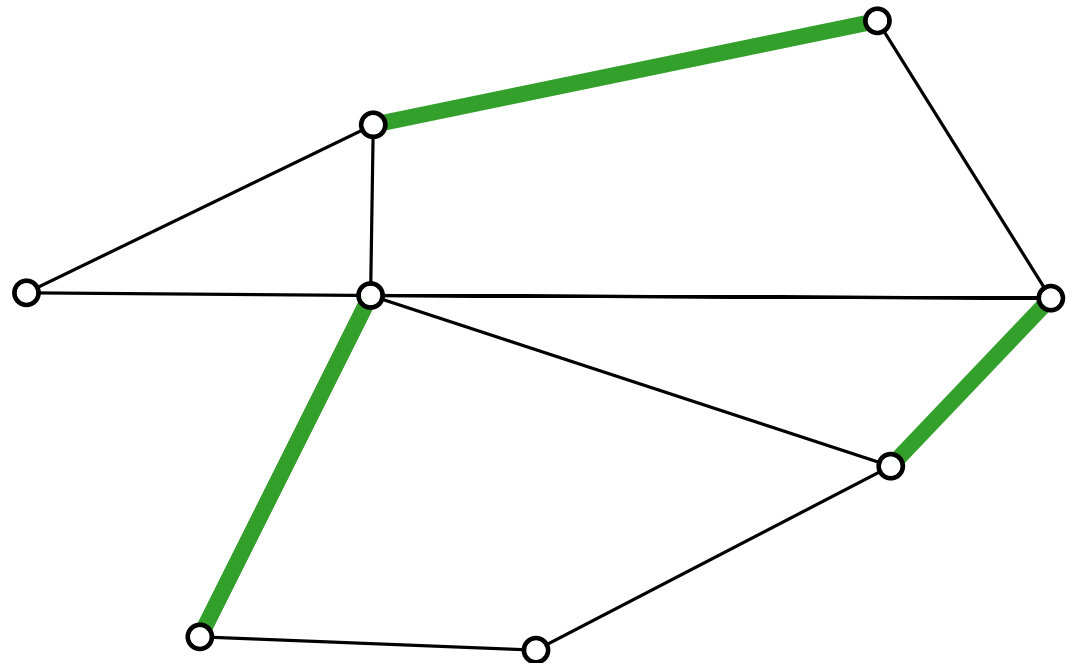
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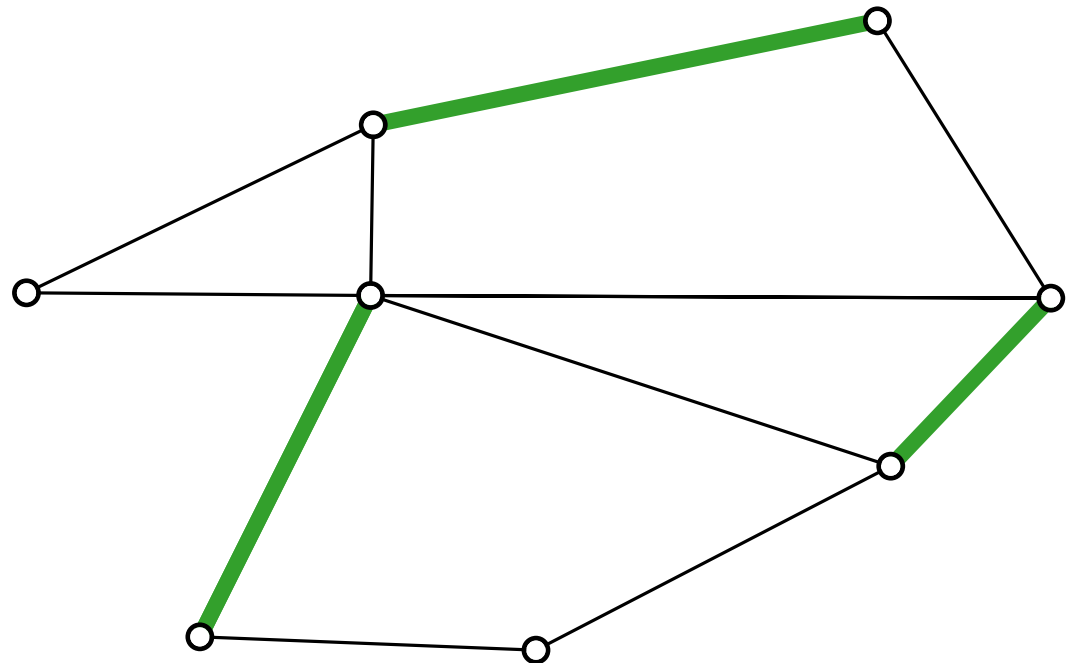


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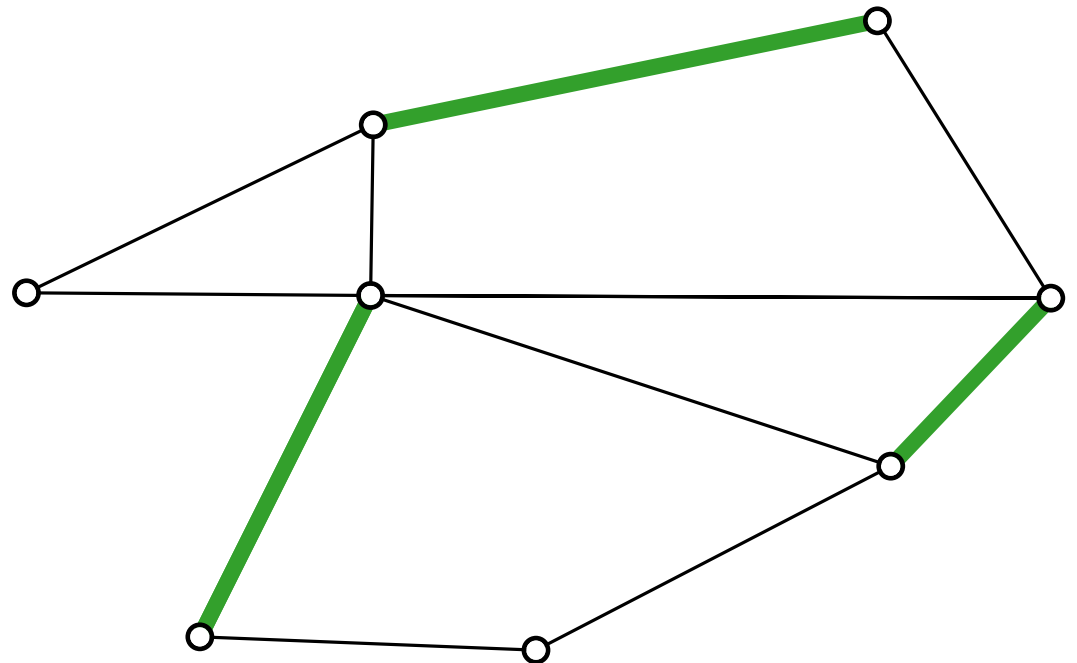


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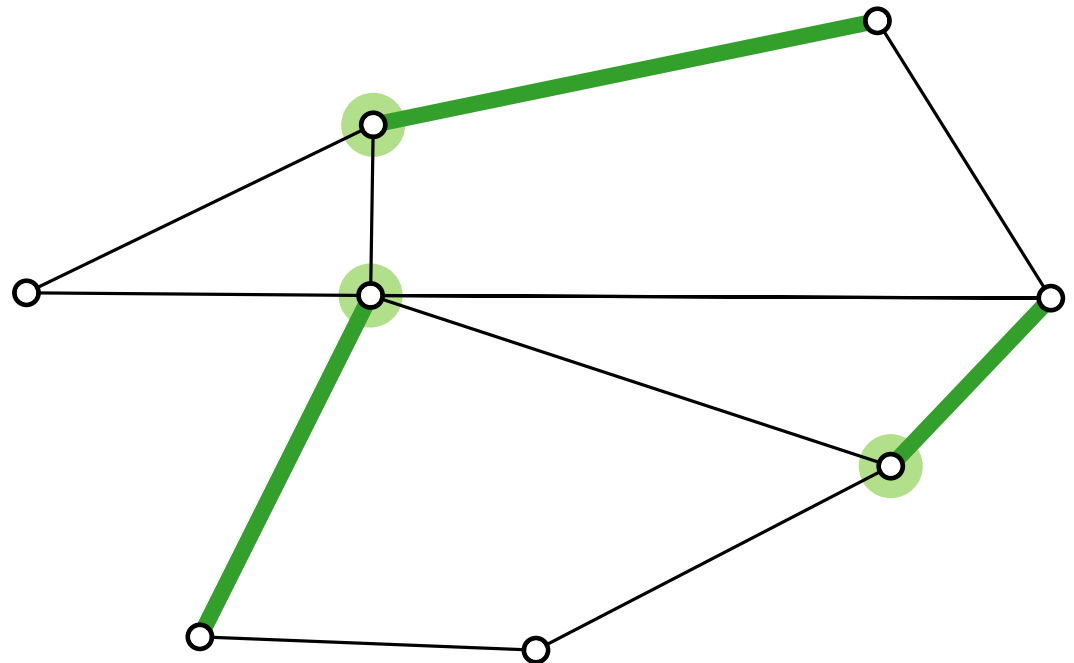
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Vertex cover of M



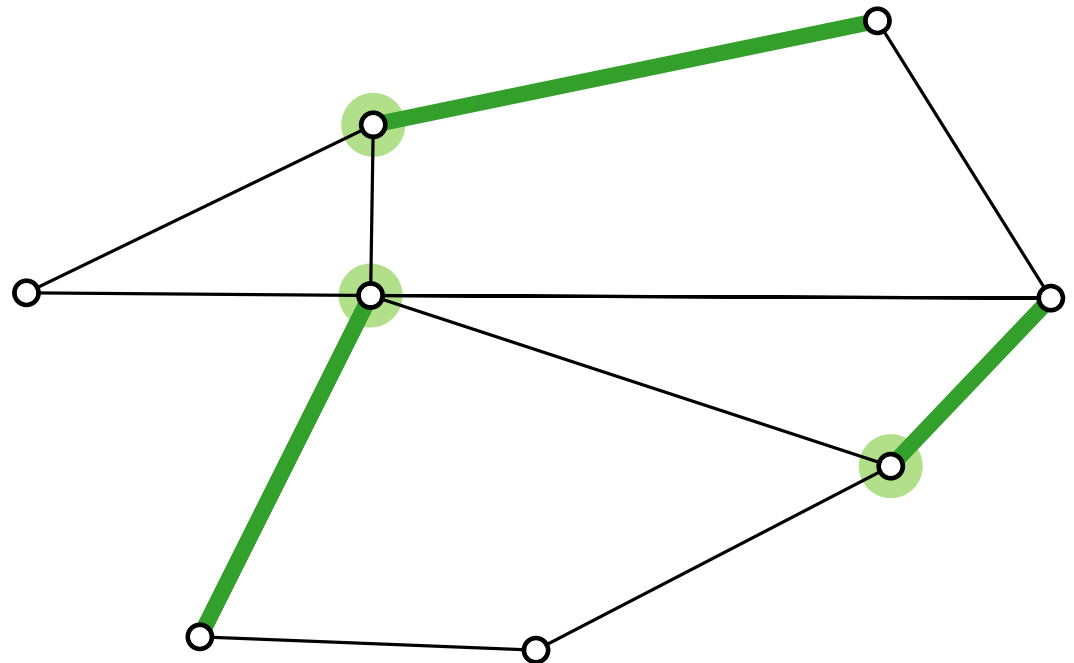
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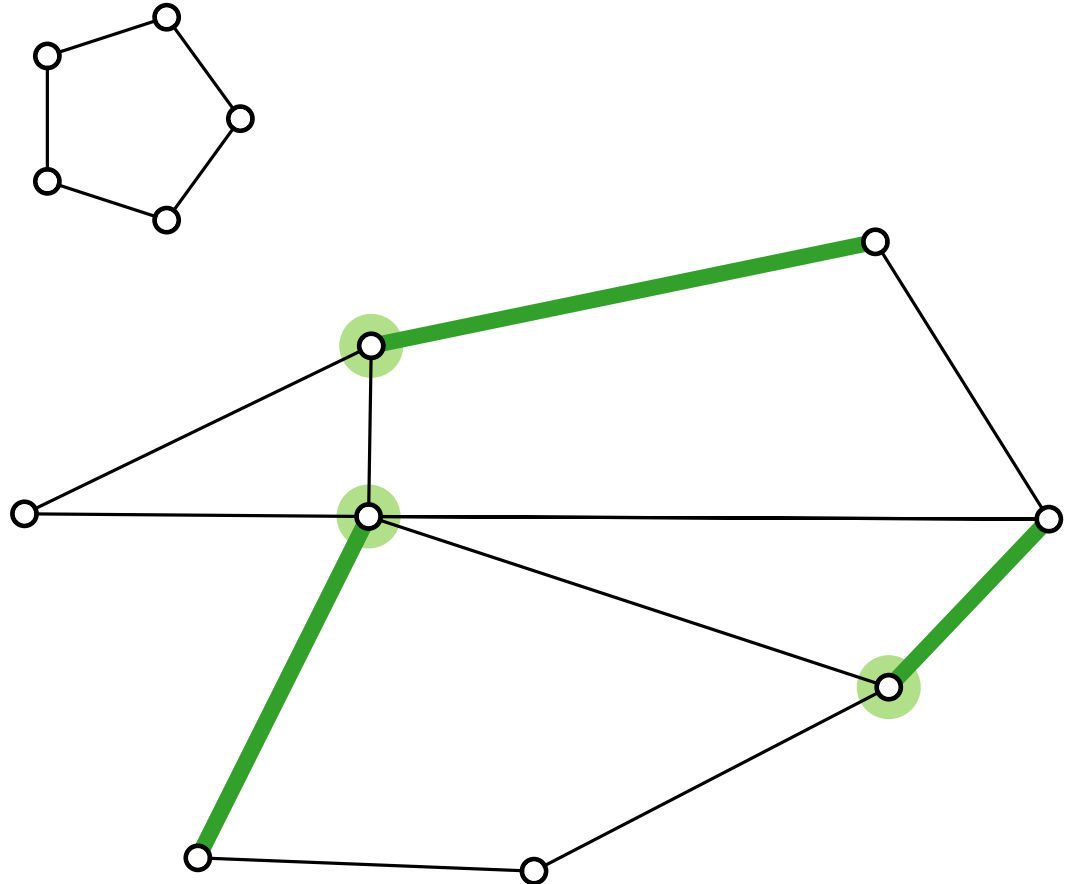
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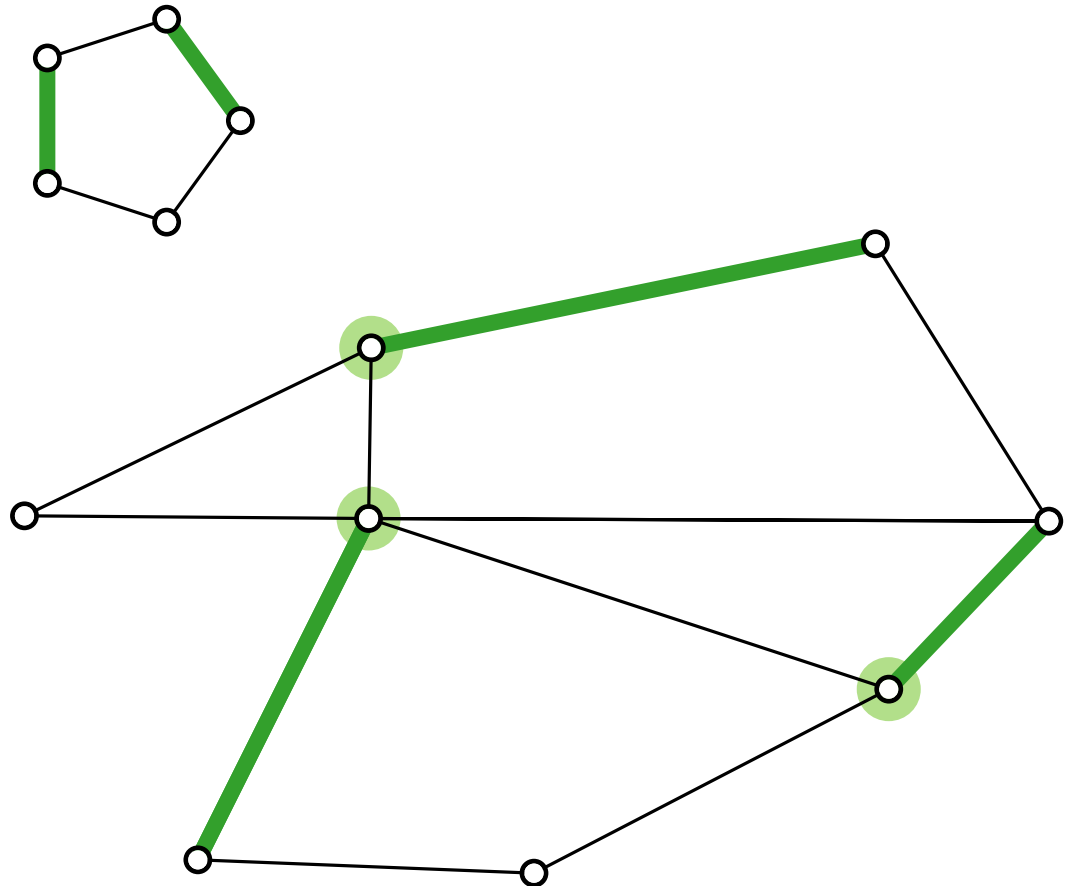
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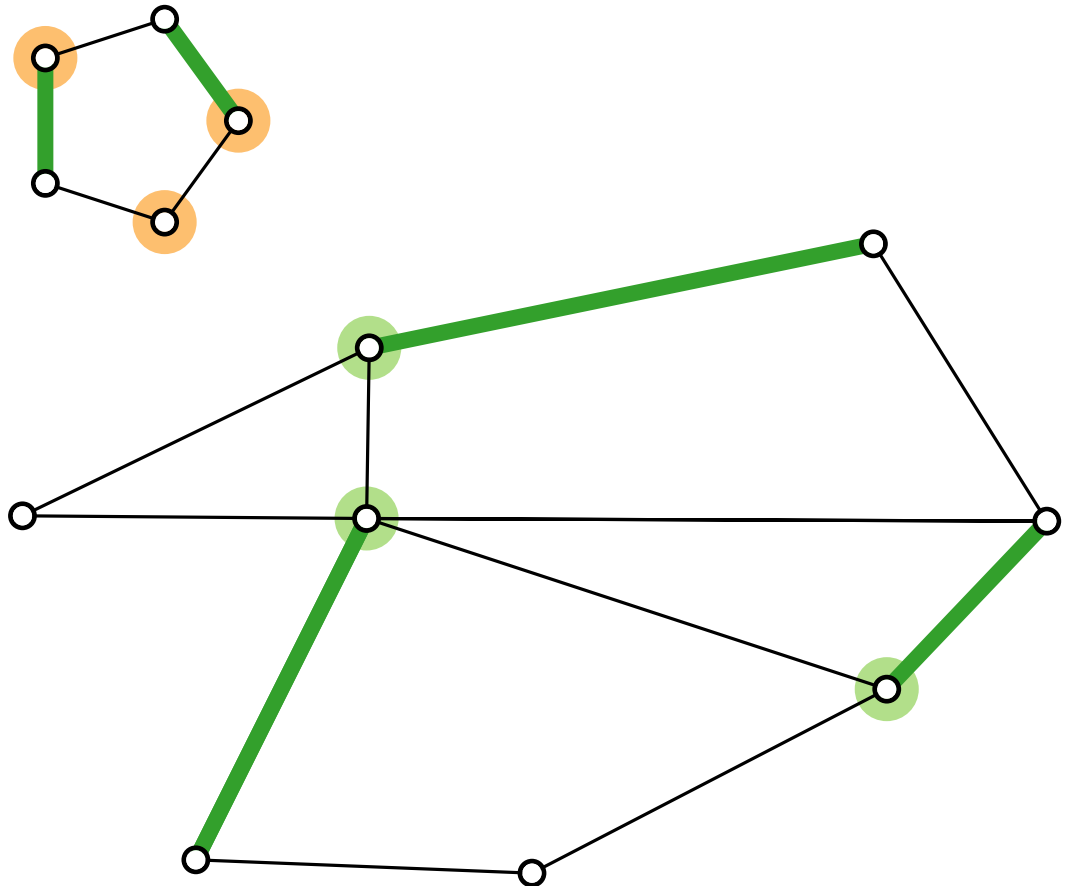
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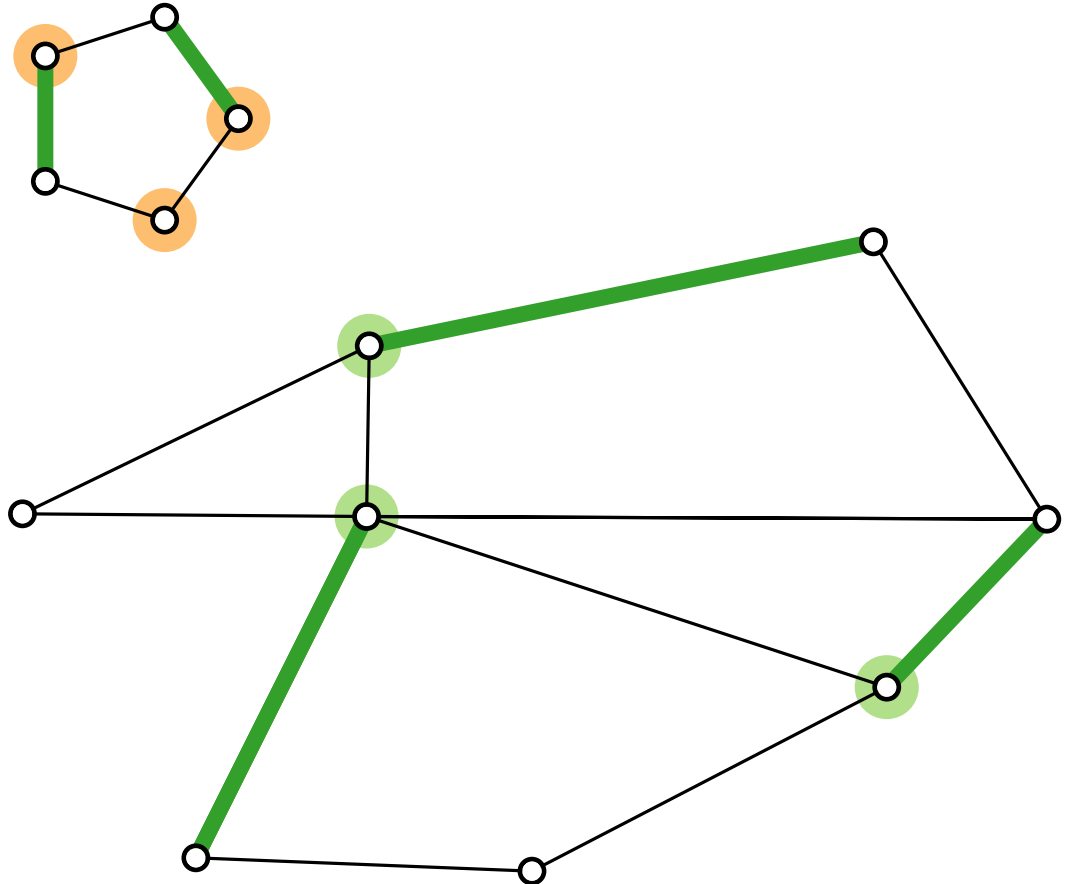
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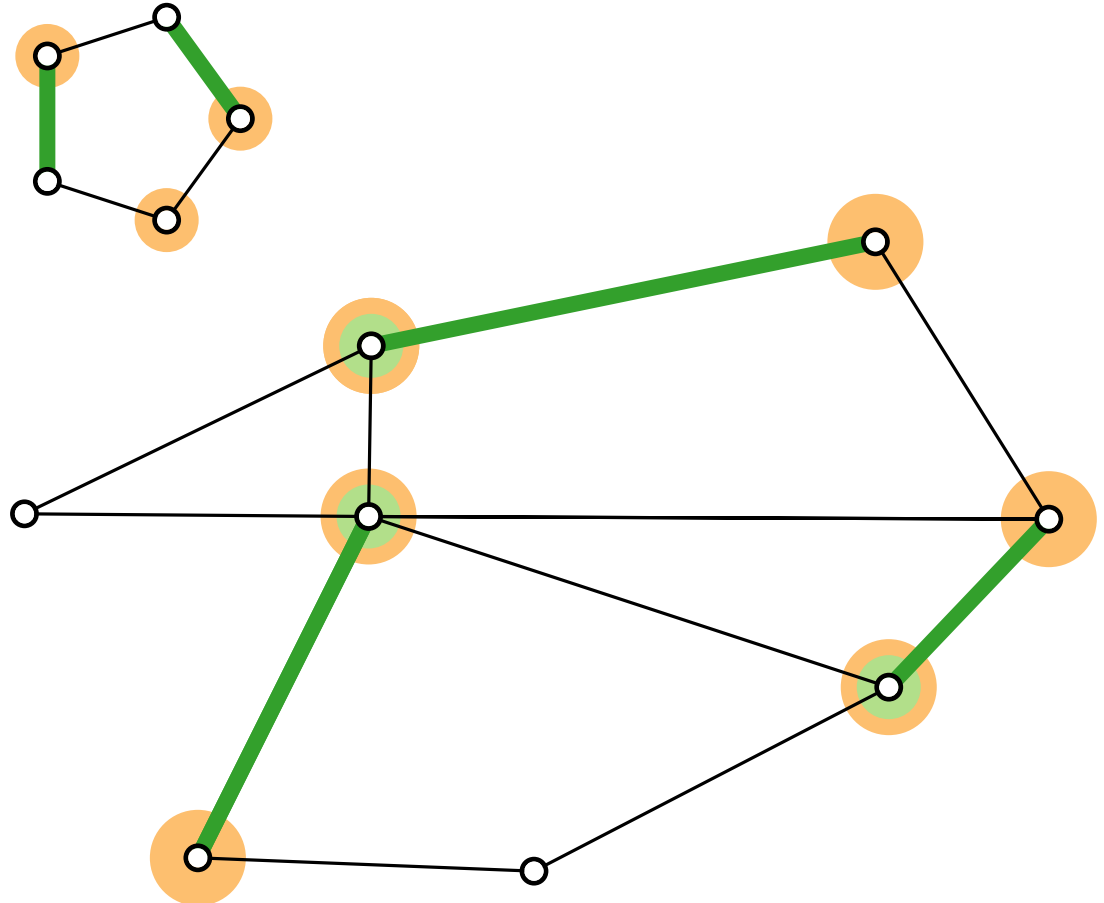
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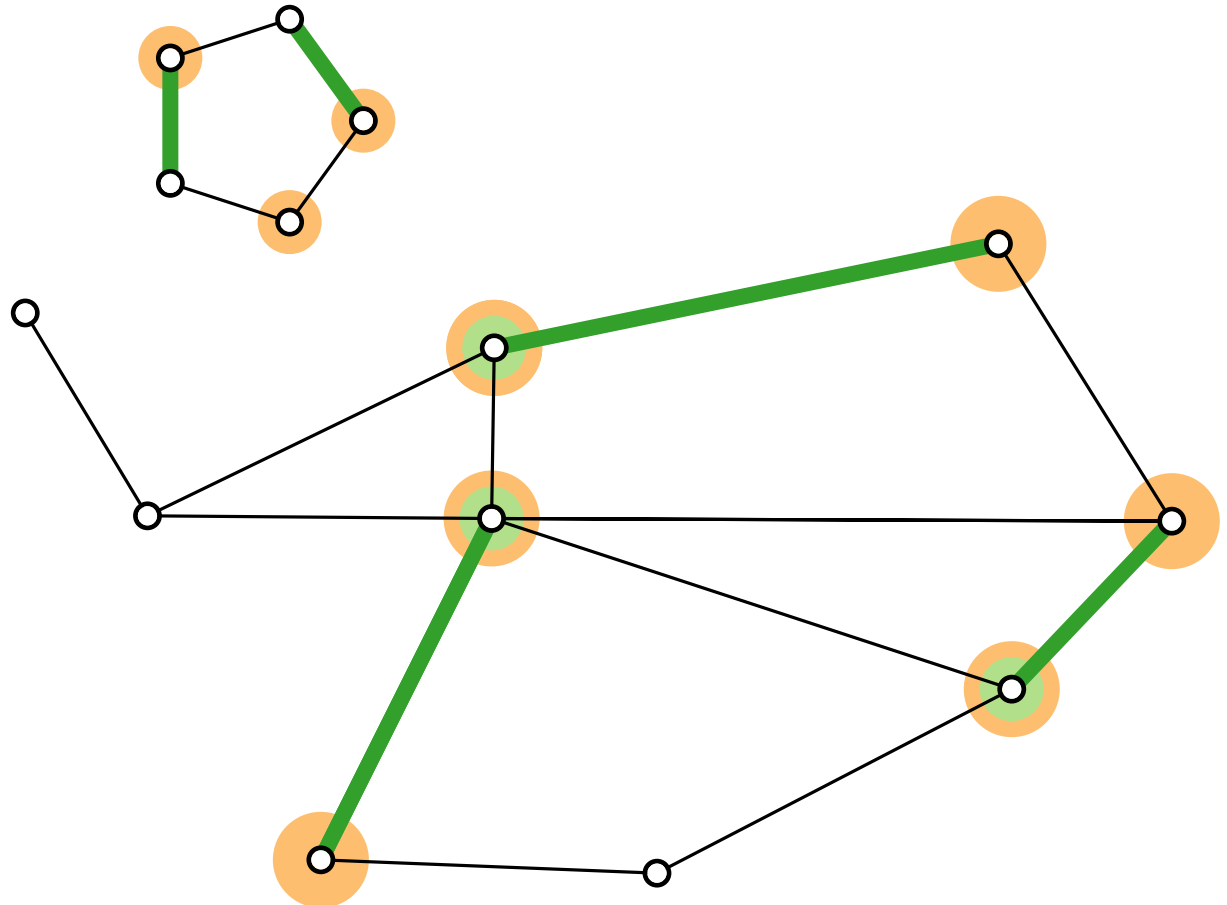
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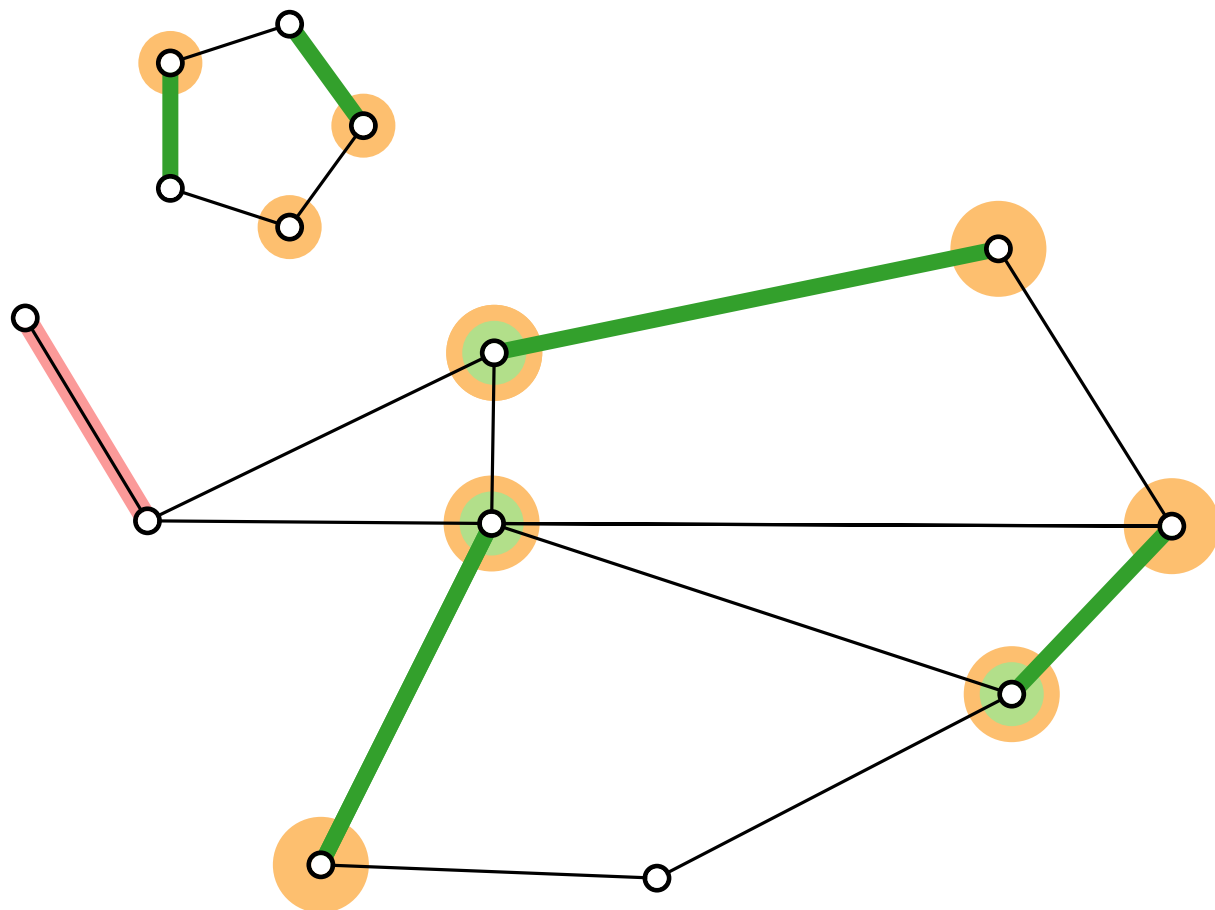
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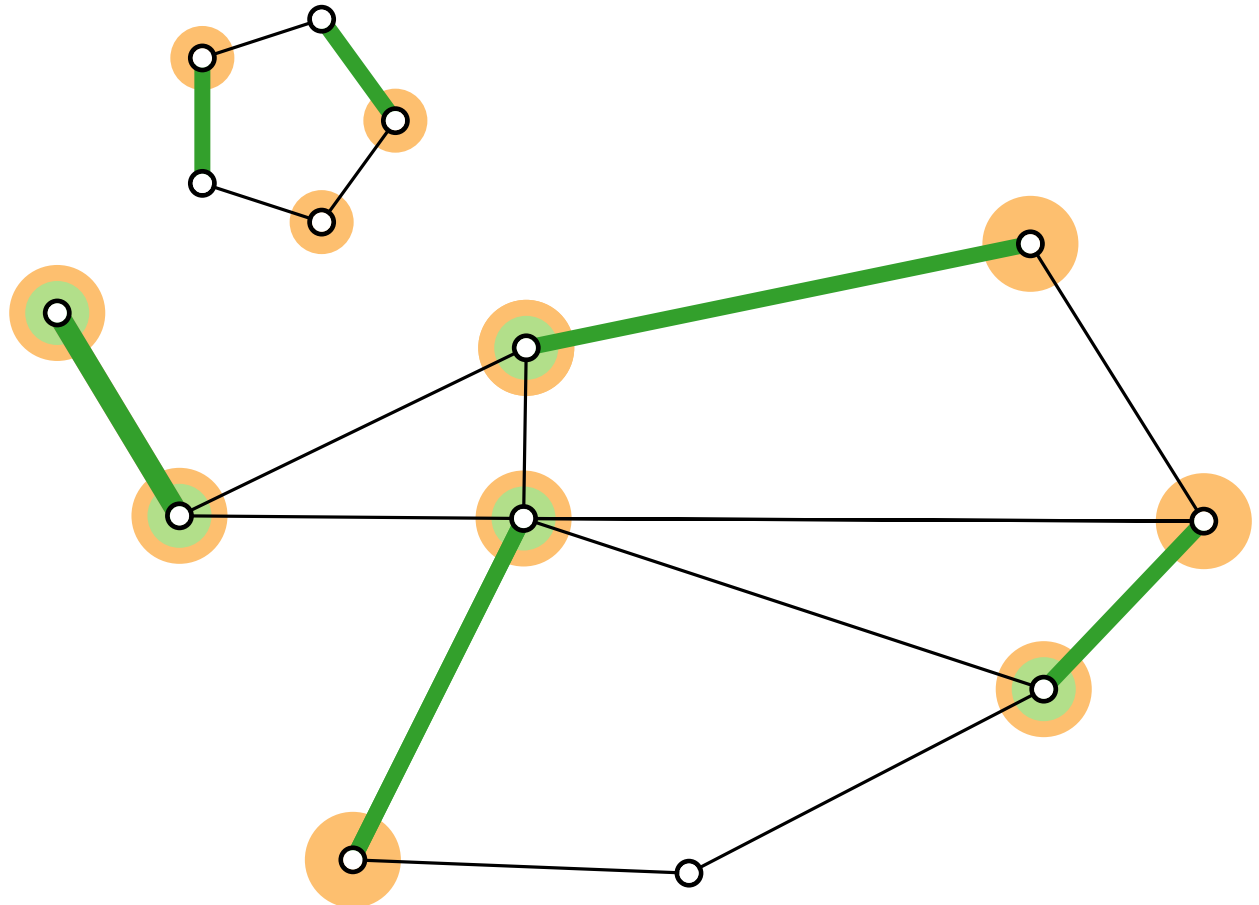
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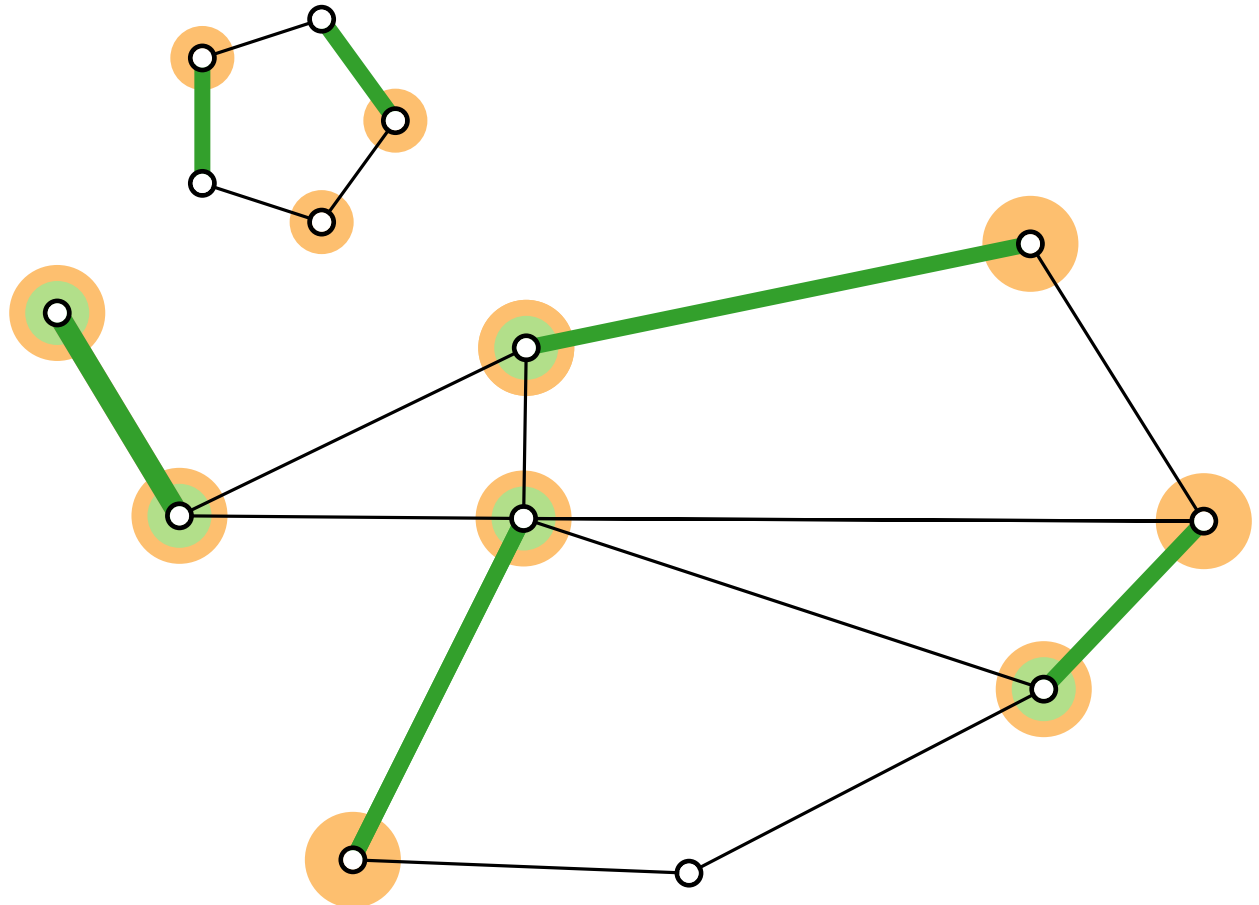
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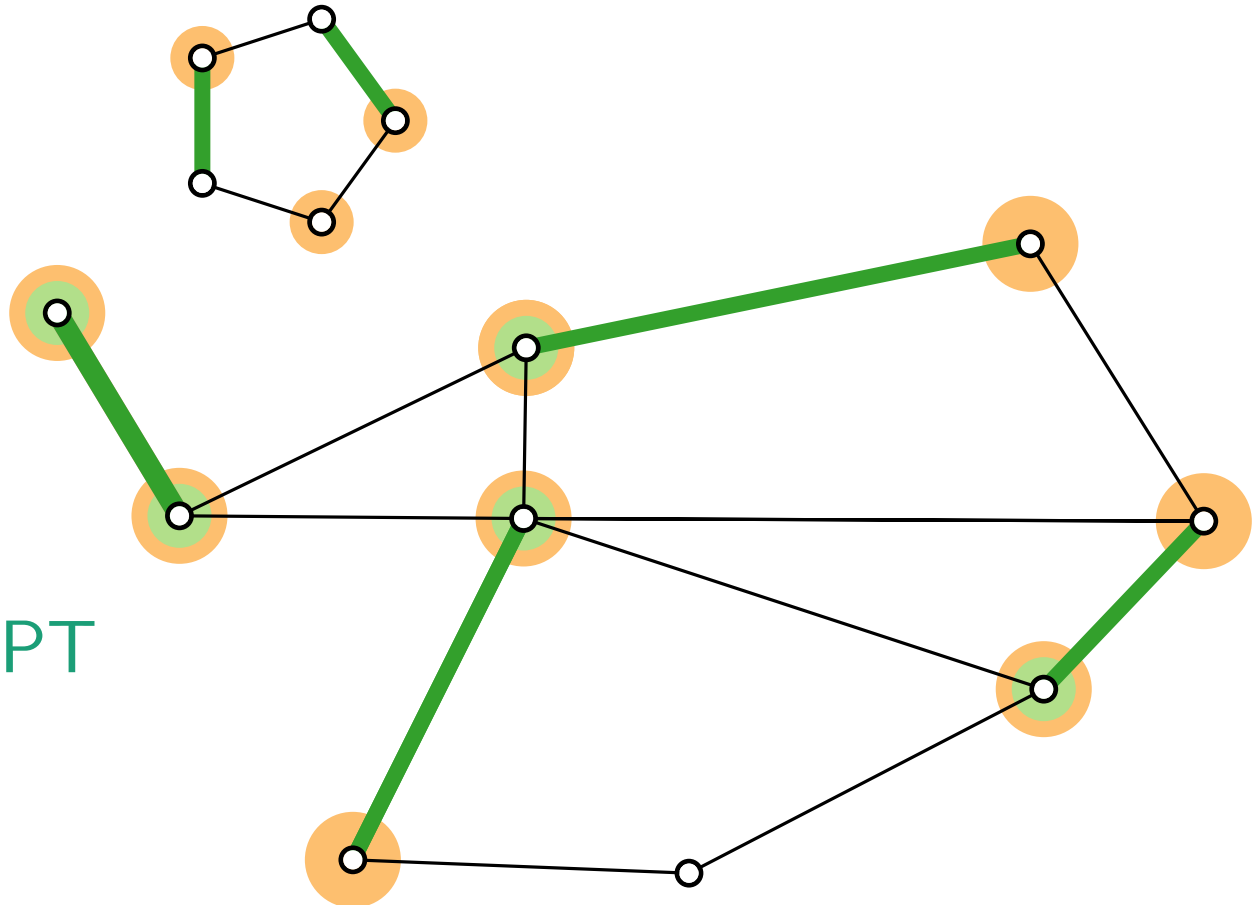
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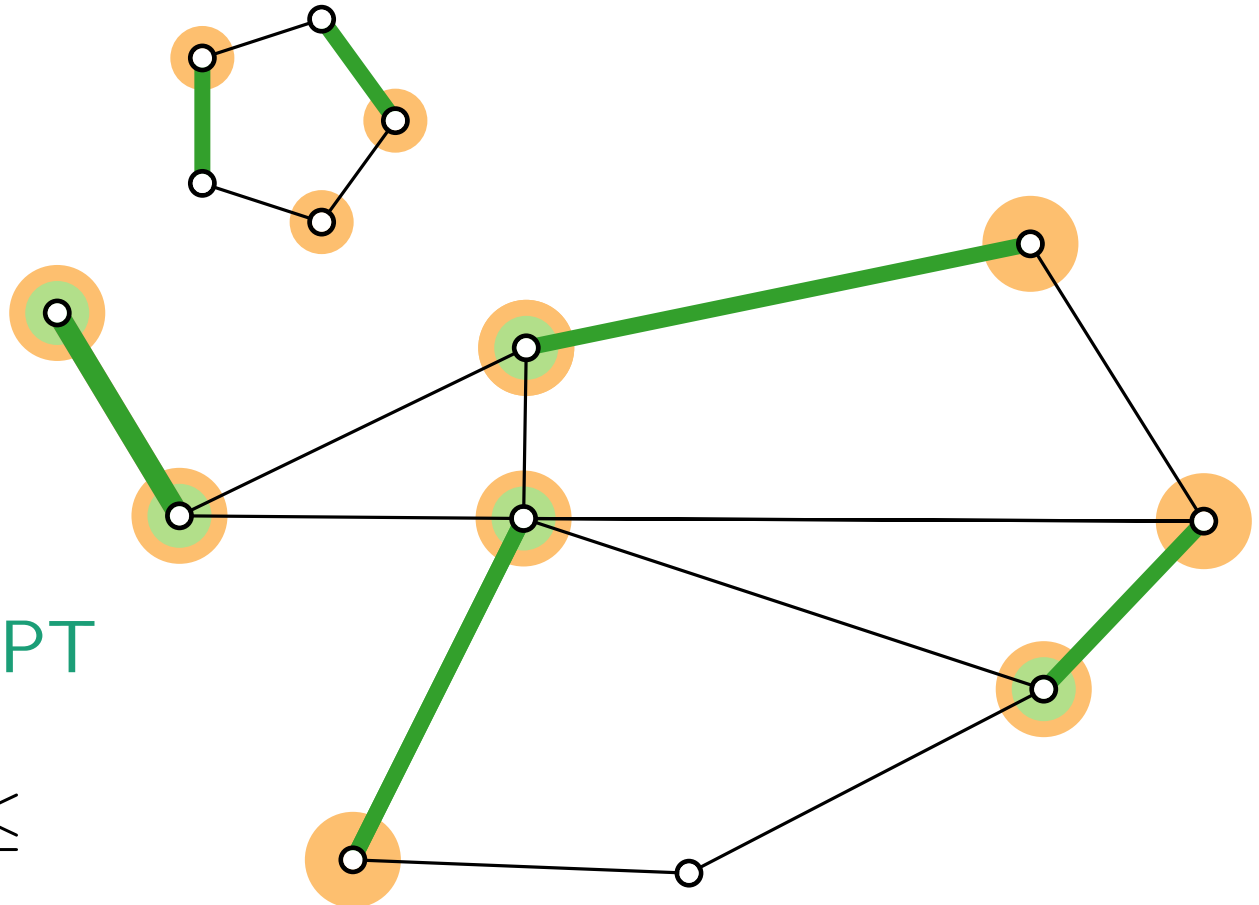
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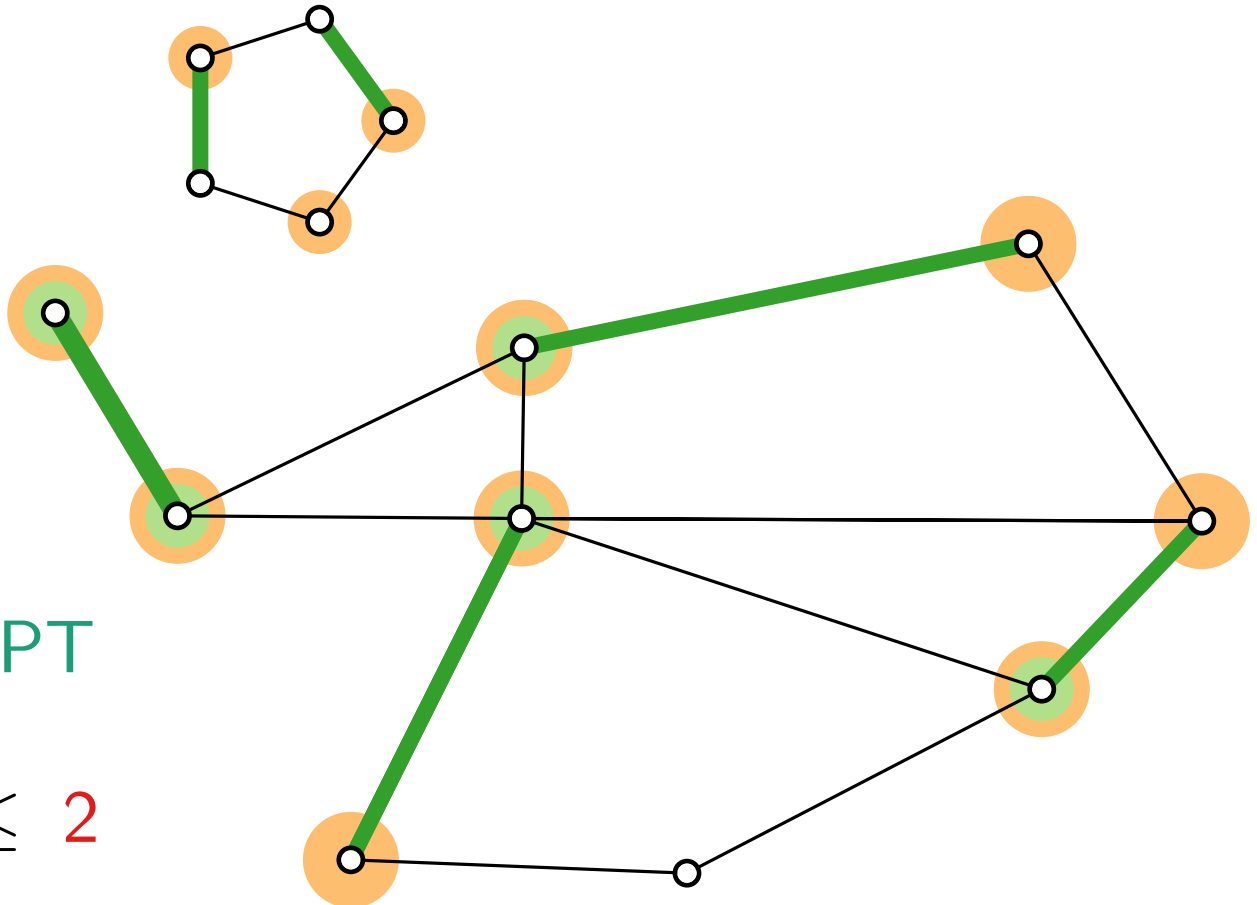
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Approximation Alg. for VERTEXCOVER

Algorithm VertexCover(G)

$M \leftarrow \emptyset$

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Theorem. The above algorithm is a factor-2 approximation algorithm for VERTEXCOVER.

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VERTEXCOVER cannot be approximated within a factor of $2 - \Theta(1)$ – if the *Unique Games Conjecture* holds.

Approximation Algorithms

Lecture 1:

Introduction and Vertex Cover

Part V:

An LP-based Algorithm for VERTEXCOVER

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$x_v \in \{0, 1\}$ for each $v \in V(G)$

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$x_v \geq 0$ ~~$x_v \in \{0, 1\}$~~ for each $v \in V(G)$

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Problem': Now we can get fractional solutions, i.e., $x_v \in (0, 1)$.

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Problem': Now we can get fractional solutions, i.e., $x_v \in (0, 1)$.

Task: Find a graph G with $\text{OPT}_{\text{LP}} \neq \text{OPT}_{\text{ILP}}$!

Solving (I)LPs

LP relaxation

minimize $\sum_{v \in V(G)} x_v$

subject to $x_u + x_v \geq 1$ for each $uv \in E(G)$

$x_v \geq 0$ ~~$x_v \in \{0, 1\}$~~ for each $v \in V(G)$

Problem: It's NP-hard to solve ILPs in general.

But: LPs can be solved efficiently (in $O(L \cdot n^{3.5})$ time),
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Problem': Now we can get fractional solutions, i.e., $x_v \in (0, 1)$.

Task: Find a graph G with $\text{OPT}_{\text{LP}} \neq \text{OPT}_{\text{ILP}}$!

Solution? Round the LP solution to get an integral solution!

Rounding the LP Solution

minimize $\sum_{v \in V(G)} x_v$

subject to $x_u + x_v \geq 1$

$x_v \geq 0$

for each $uv \in E(G)$

for each $v \in V(G)$

For each $v \in V(G)$:

Rounding the LP Solution

minimize $\sum_{v \in V(G)} x_v$

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Need to check:

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feasible!

Cost of the Solution

$$\text{minimize } \sum_{v \in V(G)} x_v$$

$$\text{subject to } x_u + x_v \geq 1$$

$$x_v \geq 0$$

$$\text{for each } uv \in E(G)$$

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Theorem. The LP rounding algorithm is a factor-2 approximation algorithm for VERTEXCOVER.

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Theorem. The LP rounding algorithm is a factor-2 approximation algorithm for **WEIGHTED VERTEX COVER**.

Cost of the Solution

$$\begin{array}{ll}\text{minimize} & \sum_{v \in V(G)} x_v \cdot w(v) \\ \text{subject to} & x_u + x_v \geq 1 \quad \text{for each } uv \in E(G) \\ & x_v \geq 0 \quad \text{for each } v \in V(G)\end{array}$$

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Approximation Algorithms

Lecture 1: Introduction and Vertex Cover

Part VI: The Vertex Cover Polytope

The Vertex Cover Polytope

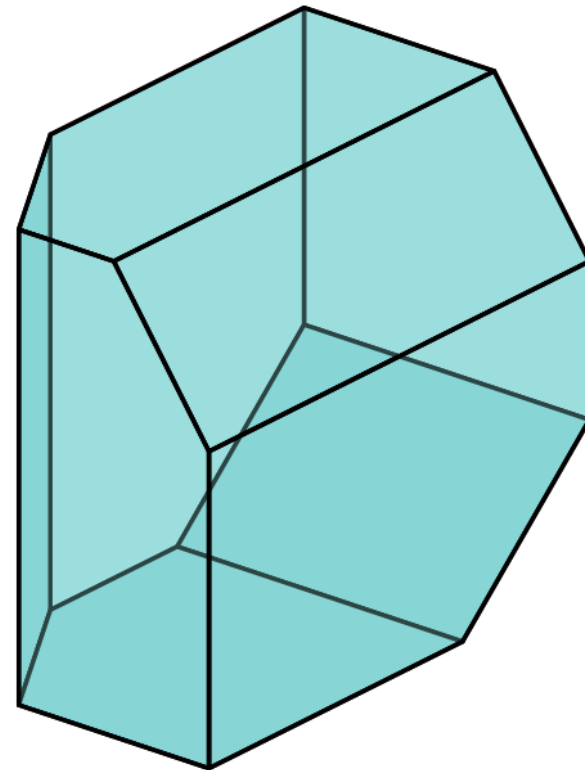
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subject to $x_u + x_v \geq 1$

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The Vertex Cover Polytope

$$\text{minimize } \sum_{v \in V(G)} x_v$$

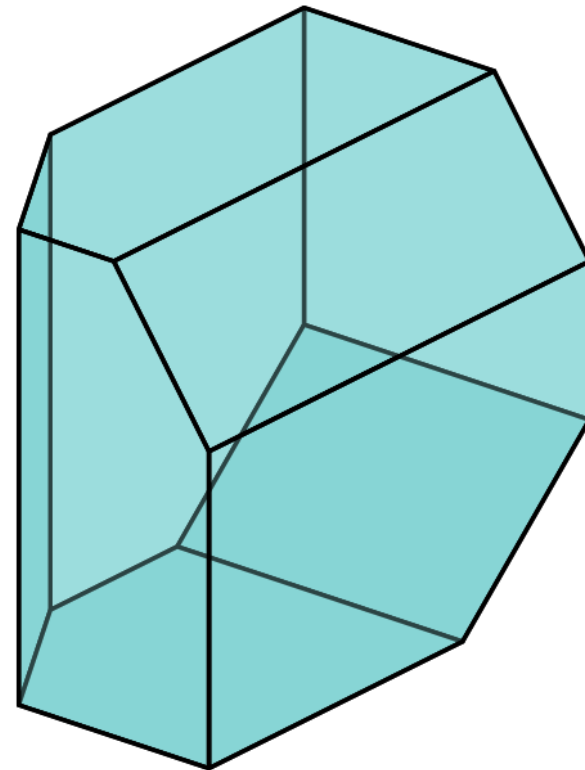
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The solution space of an LP is a *convex* polytope in $\mathbb{R}^n$, where n is the number of variables.



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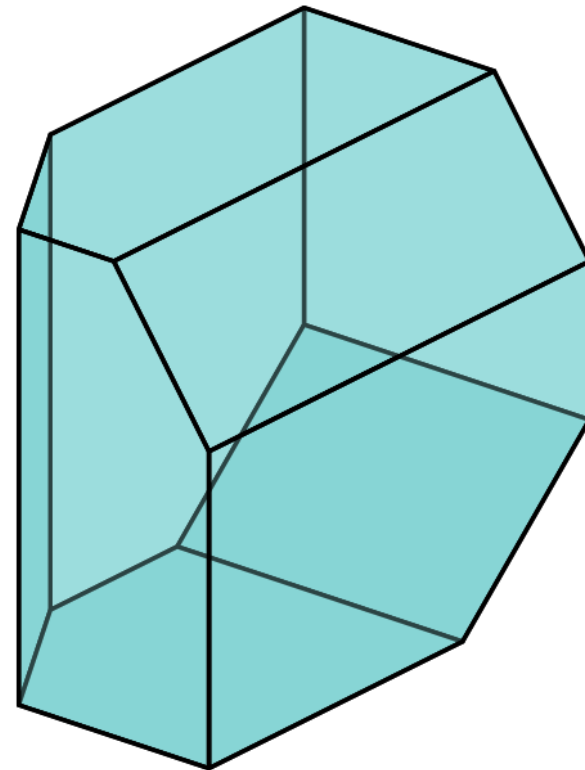
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The number of facets corresponds to the number of “relevant” constraints.



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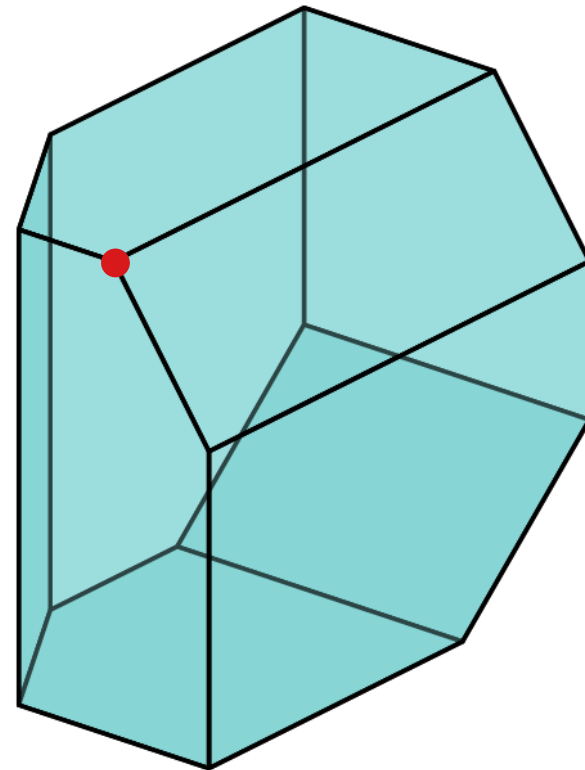
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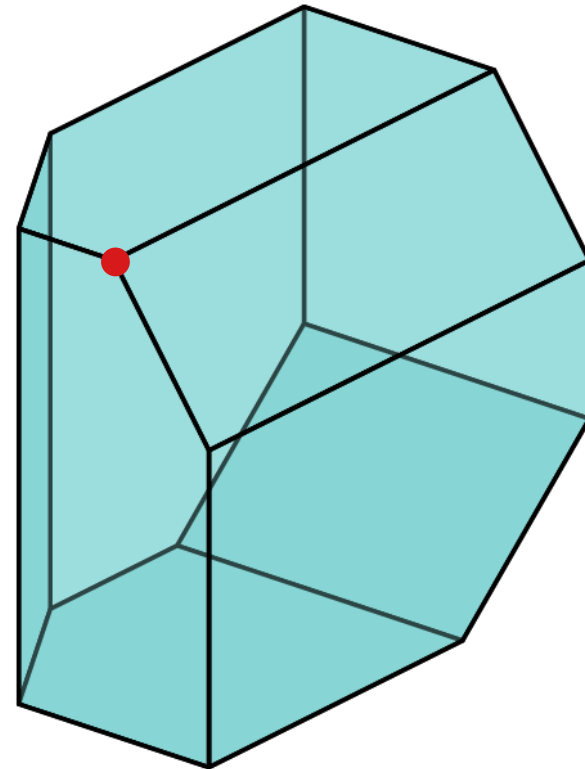
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The number of facets corresponds to the number of “relevant” constraints.

An *extreme point* of the polytope is a point that is extreme in some direction.

Most LP solvers return extreme-point solutions.



Half-Integrality

Thm. The extreme points of the vertex cover polytope are half-integral, that is, their coordinates are in $\{0, 0.5, 1\}$.

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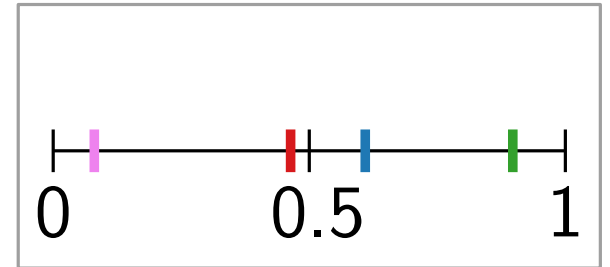
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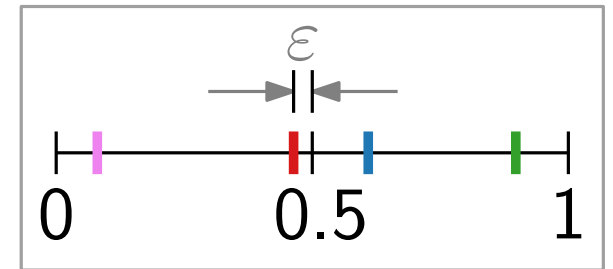
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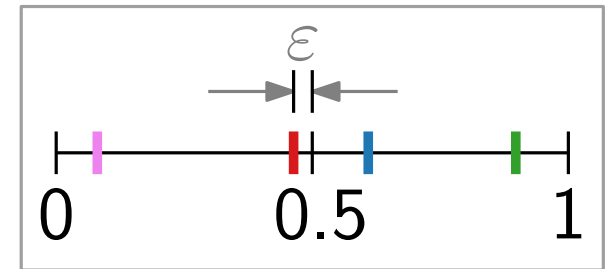
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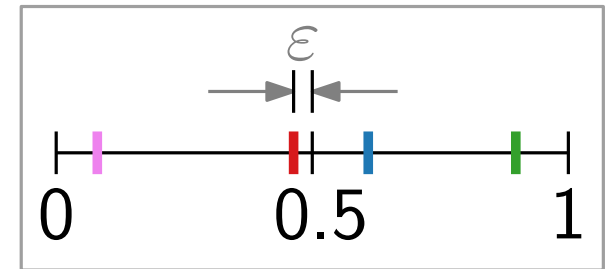
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$$x_v^+ = \begin{cases} x_v + \varepsilon & \text{if } x_v \in (0.5, 1) \\ x_v - \varepsilon & \text{if } x_v \in (0, 0.5) \\ x_v & \text{else.} \end{cases}$$



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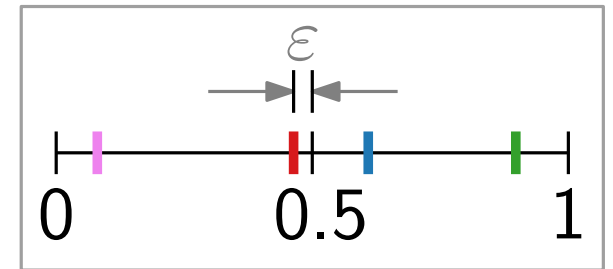
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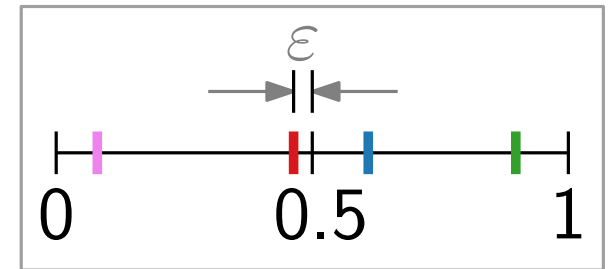
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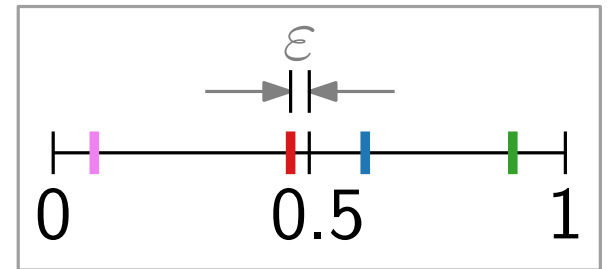
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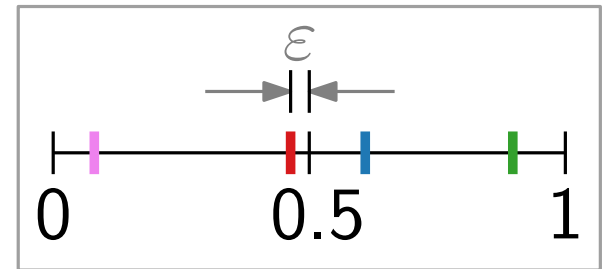
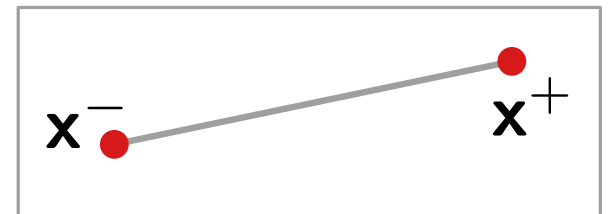
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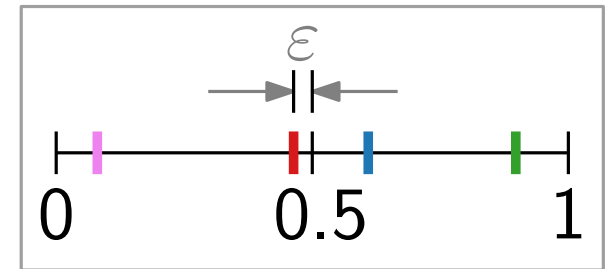
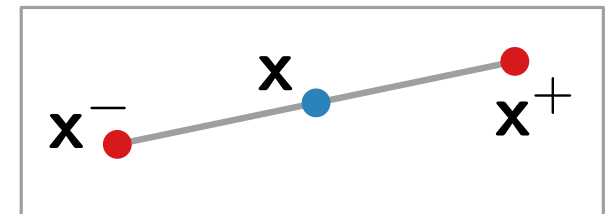
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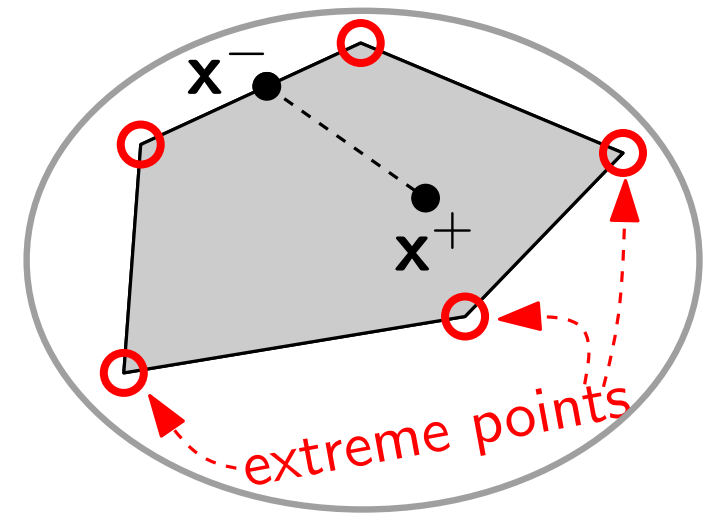
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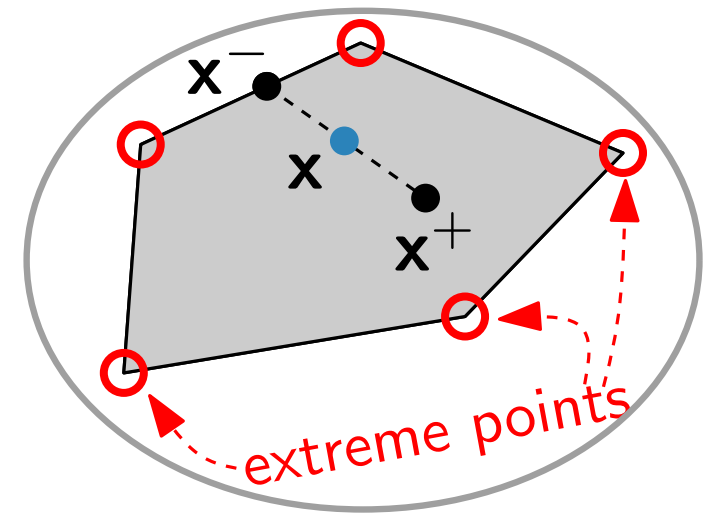
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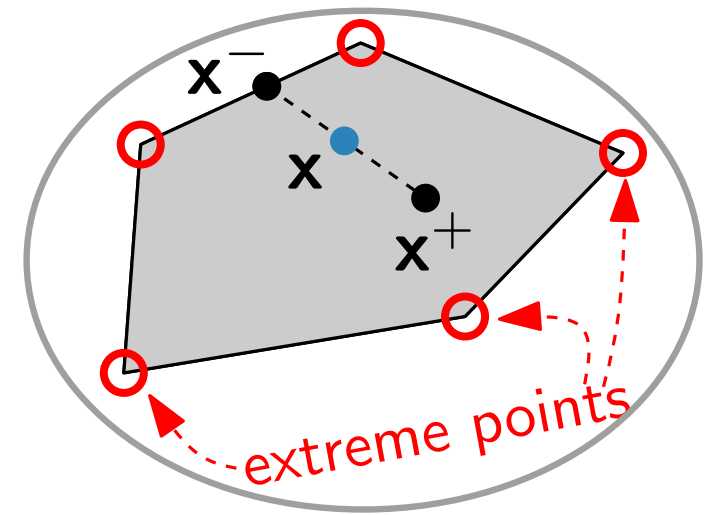
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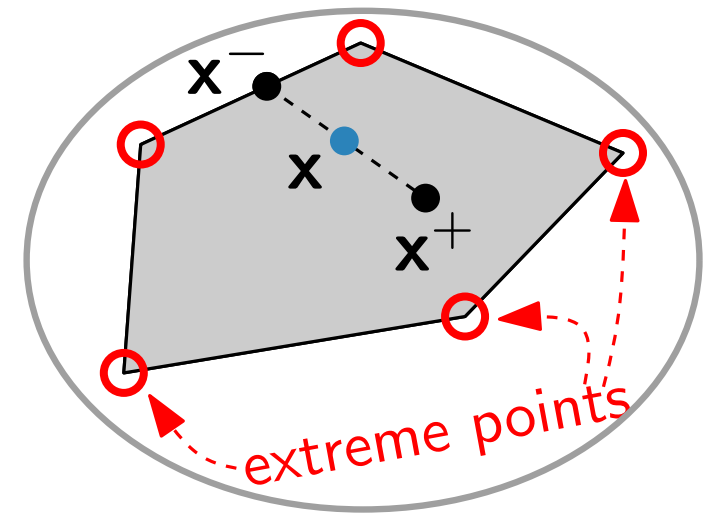
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For VC, standard LP solvers return half-integral solutions. :-)

