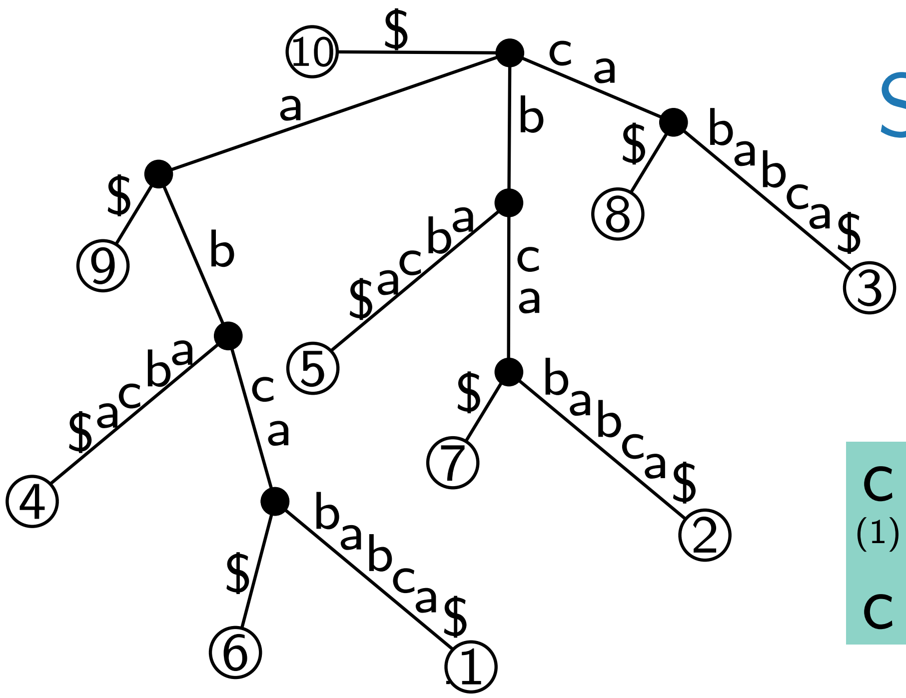
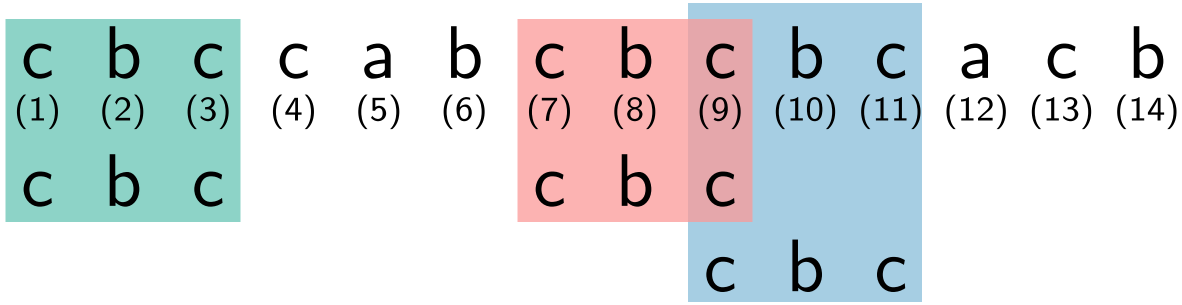
Julius-Maximilians-UNIVERSITÄT WÜRZBURG

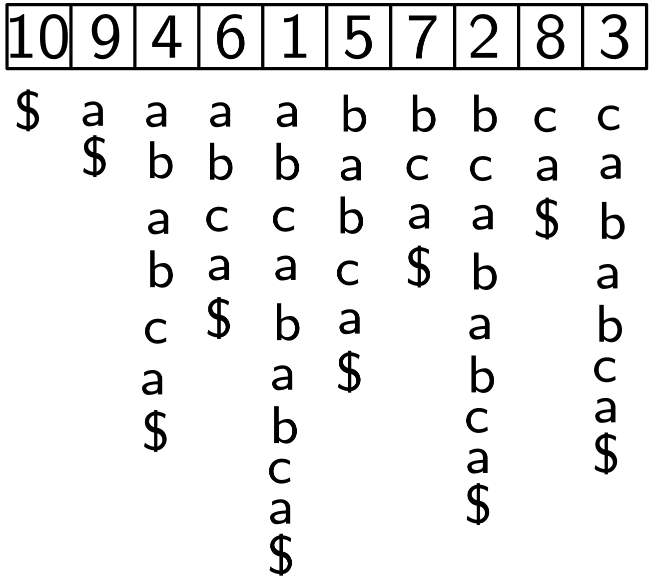
Advanced Algorithms

String Matching

Suffix Trees & Suffix Arrays







Alexander Wolff

Summer term 2026

The "Ctrl+F" Problem

STRING MATCHING

Input: Strings T (text) and P (pattern) over an alphabet Σ s.t. $|P|, |\Sigma| \leq |T|$.

Task: Find all occurrences of P in T.

The “Ctrl+F” Problem

String Matching

Input: Strings T (text) and P (pattern) over an alphabet Σ s.t. $|P|, |\Sigma| \leq |T|$.

Task: Find all occurrences of P in T.

Example:

$$\Sigma = \{\mathrm{a,b,c}\} \qquad P = \mathrm{cbc} \qquad T = \underset{(1)}{\mathrm{c}}\ \underset{(2)}{\mathrm{b}}\ \underset{(3)}{\mathrm{c}}\ \underset{(4)}{\mathrm{c}}\ \underset{(5)}{\mathrm{a}}\ \underset{(6)}{\mathrm{b}}\ \underset{(7)}{\mathrm{c}}\ \underset{(8)}{\mathrm{b}}\ \underset{(9)}{\mathrm{c}}\ \underset{(10)}{\mathrm{b}}\ \underset{(11)}{\mathrm{c}}\ \underset{(12)}{\mathrm{a}}\ \underset{(13)}{\mathrm{c}}\ \underset{(14)}{\mathrm{b}}$$

The “Ctrl+F” Problem

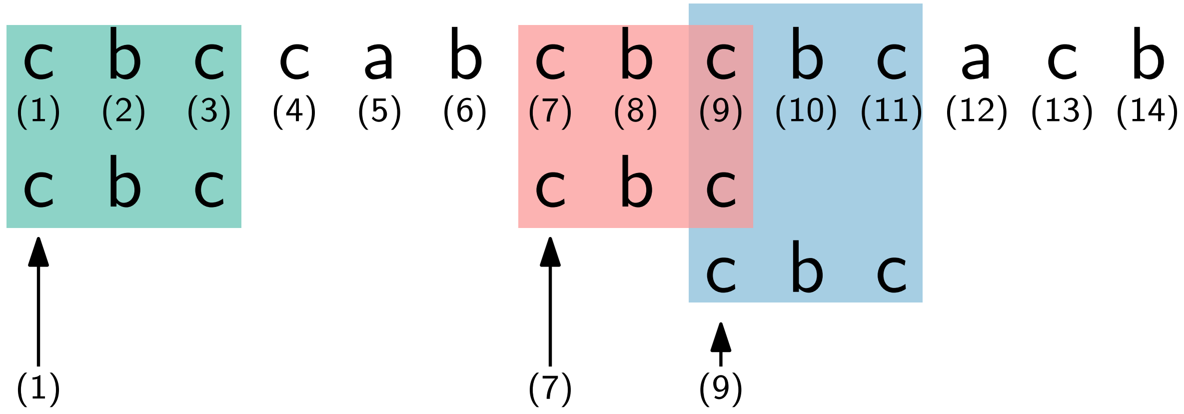
STRING MATCHING

Input: Strings T (text) and P (pattern) over an alphabet Σ s.t. $|P|, |\Sigma| \leq |T|$.

Task: Find all occurrences of P in T.

Example:

$\Sigma = \{\text{a,b,c}\}$ $P = \text{cbc}$ $T =$



P occurs in T at positions 1, 7, and 9.

The “Ctrl+F” Problem

STRING MATCHING

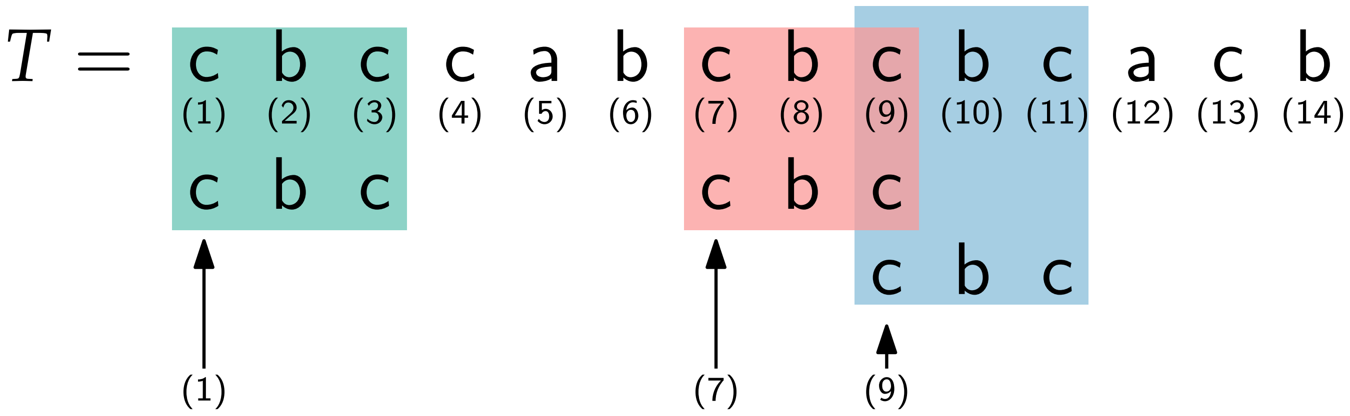
Input: Strings T (text) and P (pattern) over an alphabet Σ s.t. $|P|, |\Sigma| \leq |T|$.

Task: Find all occurrences of P in T.

Example:

$\Sigma = \{\text{a,b,c}\}$ $P = \text{cbc}$

P occurs in T at positions 1, 7, and 9.



Applications:

- Searching a text document / e-book.
- Searching a particular pattern in a DNA sequence.
- Internet search engines: determine whether a page is relavent to the user query.

Notation

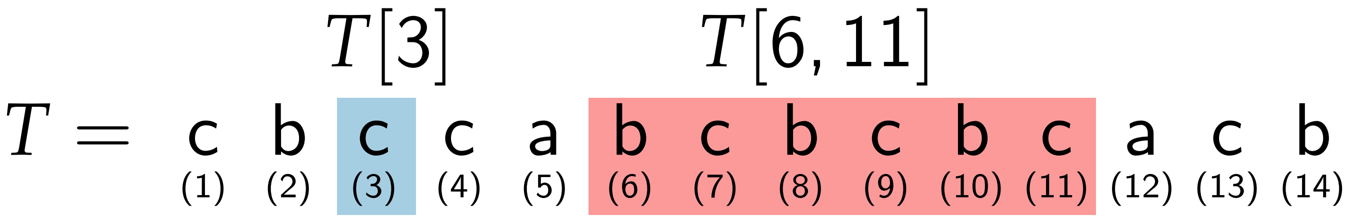
We assume T and P to be encoded as arrays with $n = |T|$ entries $T[1], T[2], \ldots, T[n]$ and $m = |P|$ entries $P[1], P[2], \ldots, P[m]$, respectively.

$T[3]$

$T =$	c	b	c	c	a	b	c	b	c	b	c	a	c	b
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)

Notation

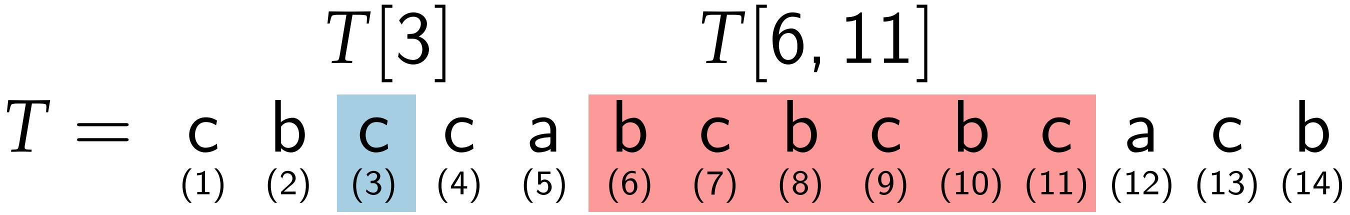
We assume T and P to be encoded as arrays with $n = |T|$ entries $T[1], T[2], \ldots, T[n]$ and $m = |P|$ entries $P[1], P[2], \ldots, P[m]$, respectively.



$T[i, j]$ with $1 \leq i \leq j \leq n$ denotes the substring of T formed by $T[i], T[i+1], \ldots, T[j]$.

Notation

We assume T and P to be encoded as arrays with $n = |T|$ entries $T[1], T[2], \dots, T[n]$ and $m = |P|$ entries $P[1], P[2], \dots, P[m]$, respectively.

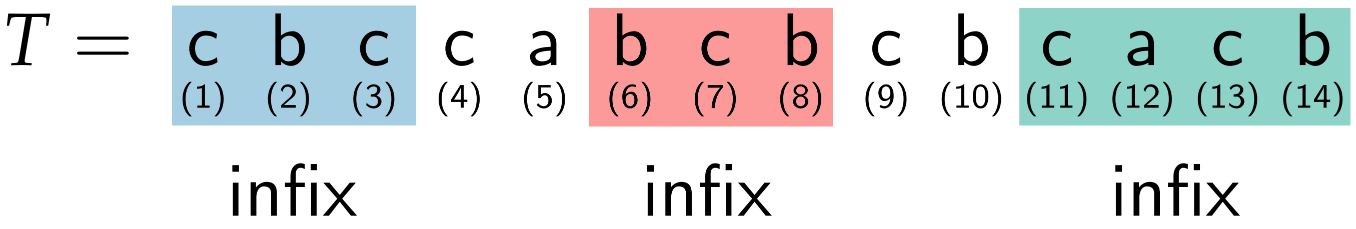


$T[3]$ $T[6, 11]$

$T =$	c	b	c	c	a	b	c	b	c	b	c	a	c	b
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)

$T[i, j]$ with $1 \leq i \leq j \leq n$ denotes the substring of T formed by $T[i], T[i+1], \dots, T[j]$.

Each substring $T[i, j]$ is called an **infix** of T.

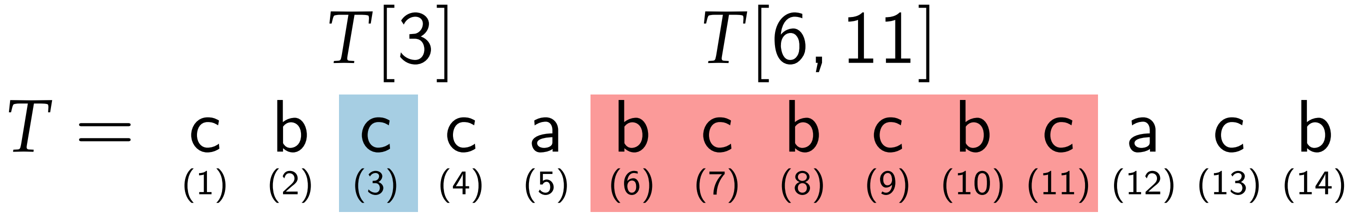


$T =$	c	b	c	c	a	b	c	b	c	b	c	a	c	b
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)

infix infix infix

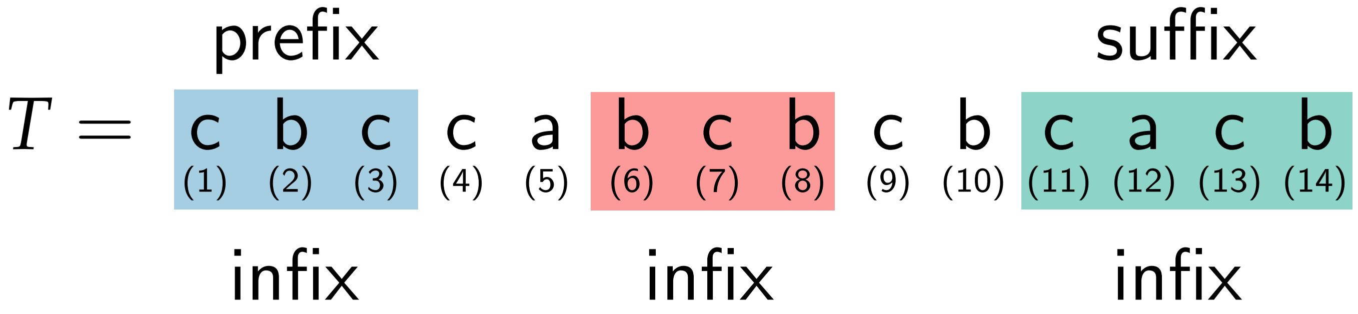
Notation

We assume T and P to be encoded as arrays with $n = |T|$ entries $T[1], T[2], \ldots, T[n]$ and $m = |P|$ entries $P[1], P[2], \ldots, P[m]$, respectively.



$T[i, j]$ with $1 \leq i \leq j \leq n$ denotes the substring of T formed by $T[i], T[i+1], \ldots, T[j]$.

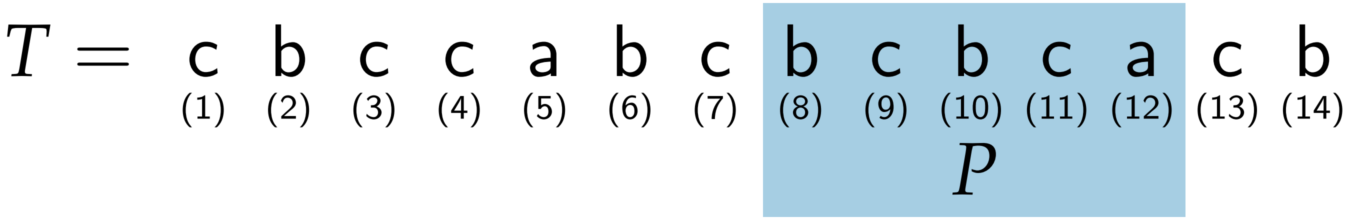
Each substring $T[i, j]$ is called an **infix** of T.
If $i = 1$, then $T[i, j]$ is also called **prefix** of T.
If $j = n$, then $T[i, j]$ is also called **suffix** of T.



Algorithmic Complexity

Occurrences of (prefixes of) P may overlap.

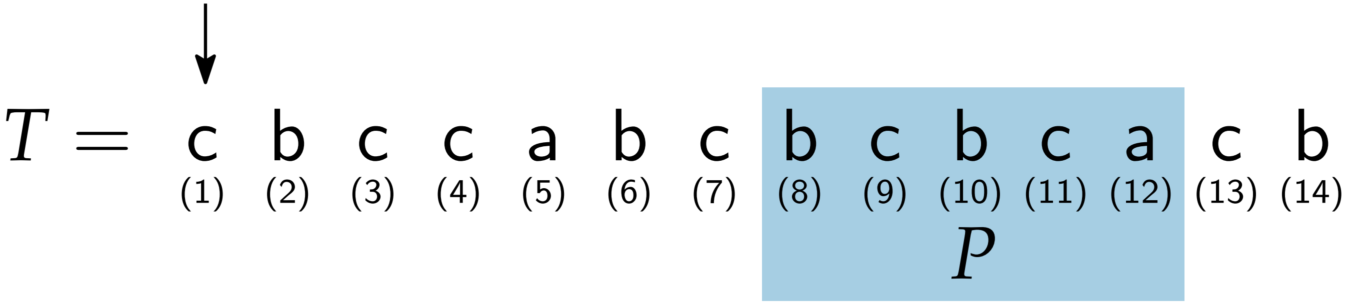
$\Rightarrow$ A simple left-to-right traversal of T is not sufficient to find all occurrences of P!



$T =$	c	b	c	c	a	b	c	b	c	b	c	a	c	b
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)

Algorithmic Complexity

Occurrences of (prefixes of) P may overlap.
$\Rightarrow$ A simple left-to-right traversal of T is not sufficient to find all occurrences of P!

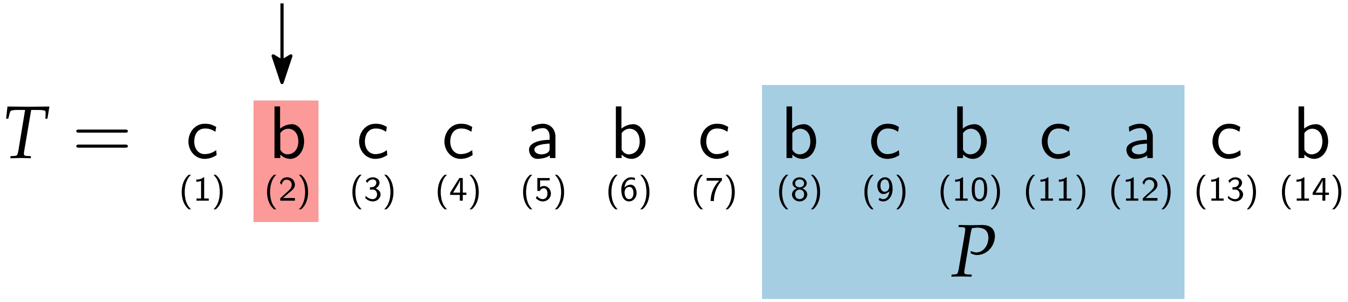


$T =$	c	b	c	c	a	b	c	b	c	b	c	a	c	b
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
										P				

Algorithmic Complexity

Occurrences of (prefixes of) P may overlap.

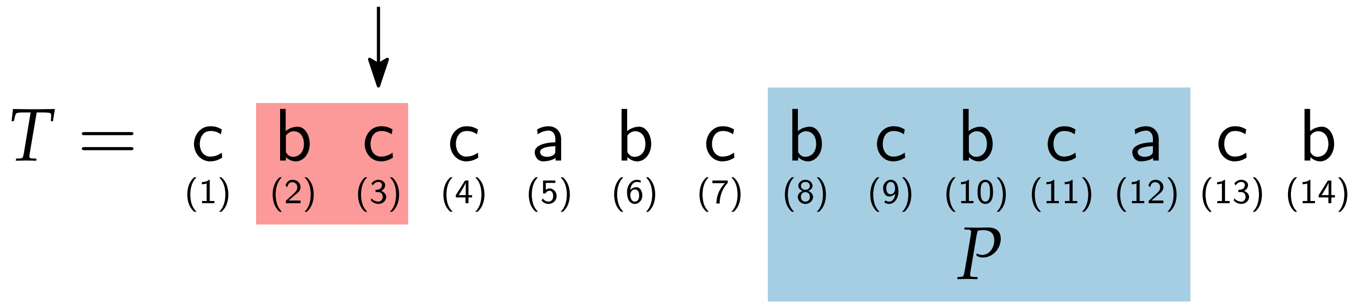
$\Rightarrow$ A simple left-to-right traversal of T is not sufficient to find all occurrences of P!



$T =$	c	b	c	c	a	b	c	b	c	b	c	a	c	b
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
										P				

Algorithmic Complexity

Occurrences of (prefixes of) P may overlap.
$\Rightarrow$ A simple left-to-right traversal of T is not sufficient to find all occurrences of P!



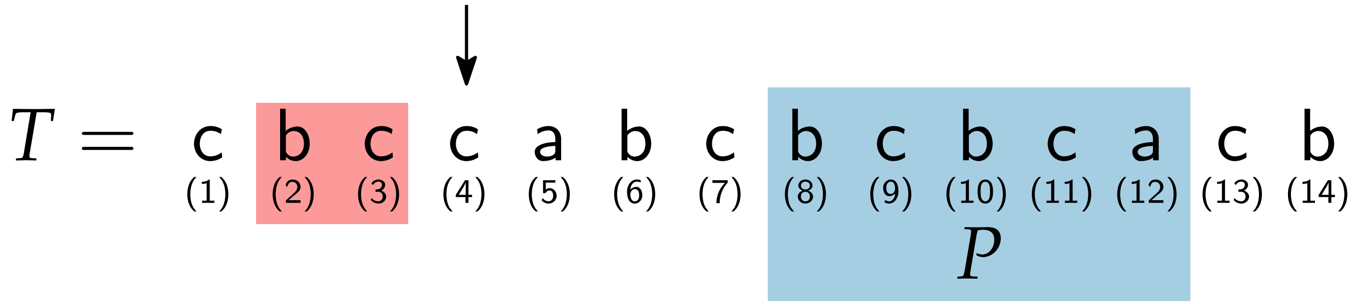
$T =$	c	b	c	c	a	b	c	b	c	b	c	a	c	b
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)

P

Algorithmic Complexity

Occurrences of (prefixes of) P may overlap.

$\Rightarrow$ A simple left-to-right traversal of T is not sufficient to find all occurrences of P!

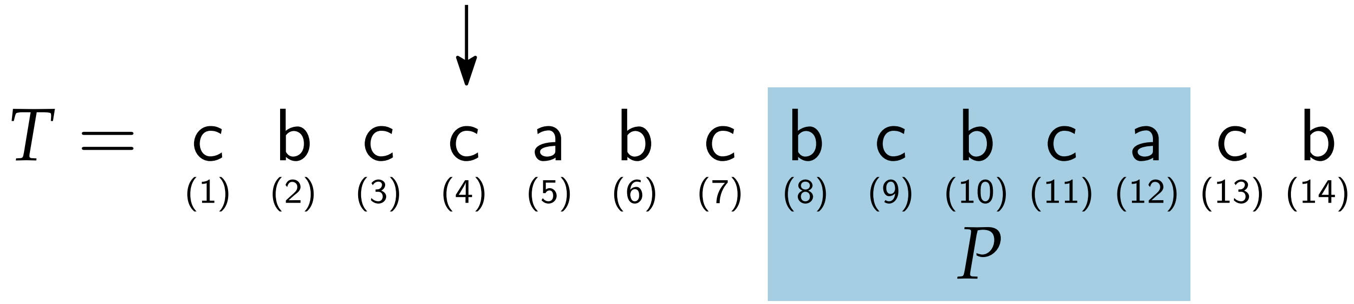


$T =$	c	b	c	c	a	b	c	b	c	b	c	a	c	b
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)

Algorithmic Complexity

Occurrences of (prefixes of) P may overlap.

$\Rightarrow$ A simple left-to-right traversal of T is not sufficient to find all occurrences of P!



$T =$ c b c c a b c b c b c a c b

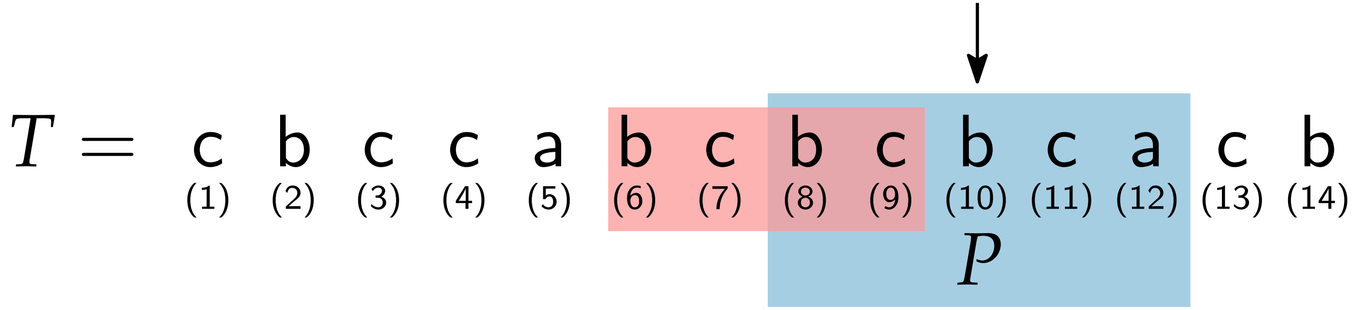
(1) (2) (3) (4) (5) (6) (7) (8) (9) (10) (11) (12) (13) (14)

P

Algorithmic Complexity

Occurrences of (prefixes of) P may overlap.

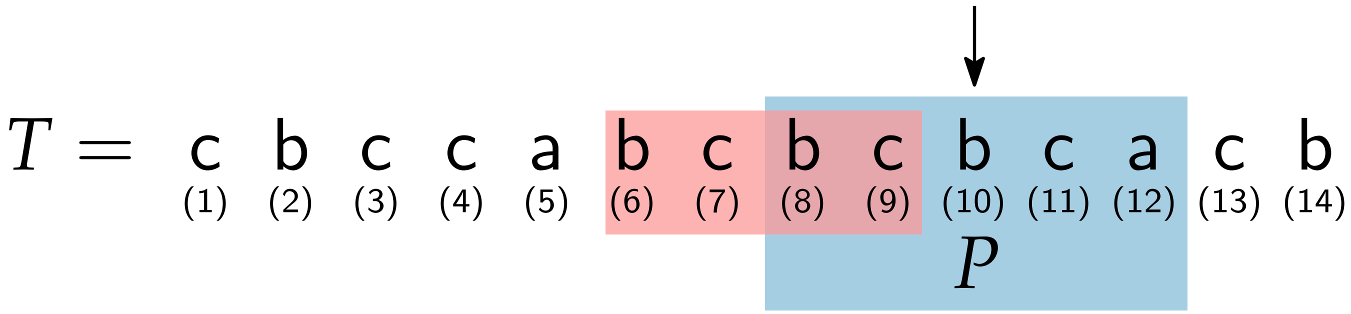
$\Rightarrow$ A simple left-to-right traversal of T is not sufficient to find all occurrences of P!



$T =$	c	b	c	c	a	b	c	b	c	b	c	a	c	b
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)

Algorithmic Complexity

Occurrences of (prefixes of) P may overlap.
$\Rightarrow$ A simple left-to-right traversal of T is not sufficient to find all occurrences of P!



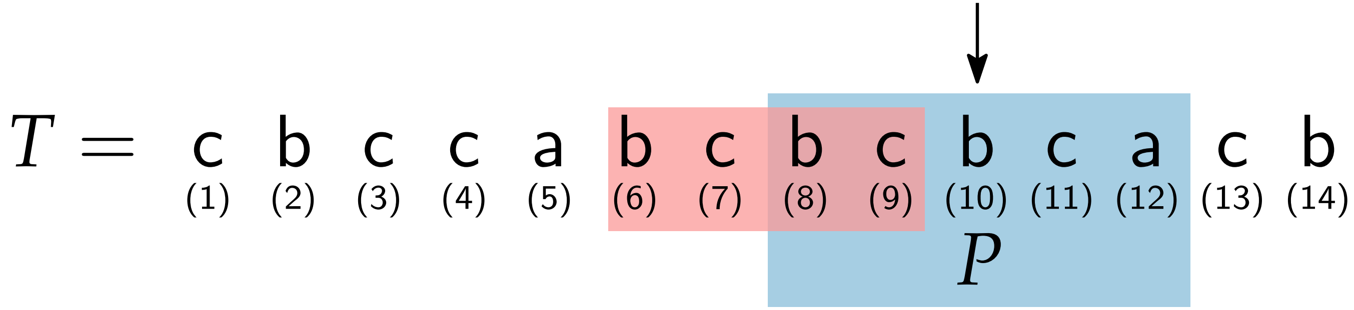
$T =$	c	b	c	c	a	b	c	b	c	b	c	a	c	b
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)

P

Observation. String Matching can be solved in $\mathcal{O}(nm)$ time.

Algorithmic Complexity

Occurrences of (prefixes of) P may overlap.
$\Rightarrow$ A simple left-to-right traversal of T is not sufficient to find all occurrences of P!



$T =$ c b c c a b c b c b c a c b

(1) (2) (3) (4) (5) (6) (7) (8) (9) (10) (11) (12) (13) (14)

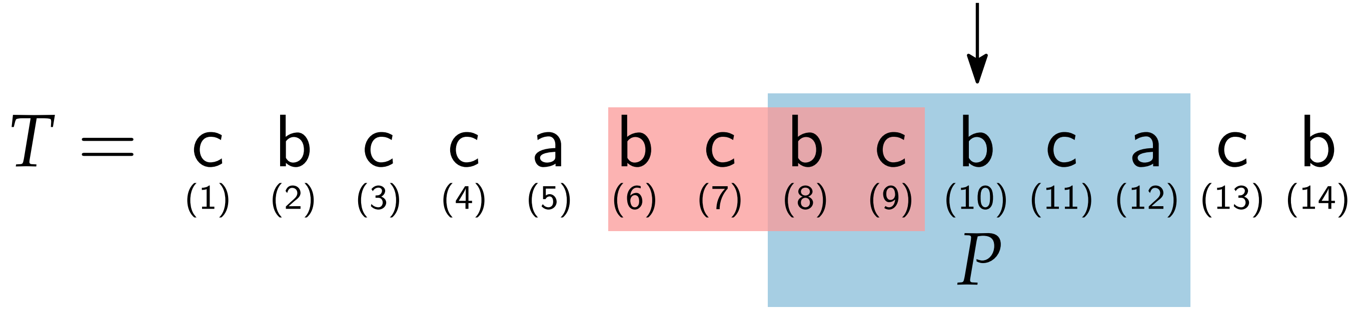
P

Observation. String Matching can be solved in $\mathcal{O}(nm)$ time.

Theorem. String Matching can be solved in $\mathcal{O}(n+m)$ time, and this time bound is optimal. [Knuth, Morris, Pratt'77]

Algorithmic Complexity

Occurrences of (prefixes of) P may overlap.
$\Rightarrow$ A simple left-to-right traversal of T is not sufficient to find all occurrences of P!



$T =$ c b c c a b c b c b c a c b

Observation. String Matching can be solved in $\mathcal{O}(nm)$ time.

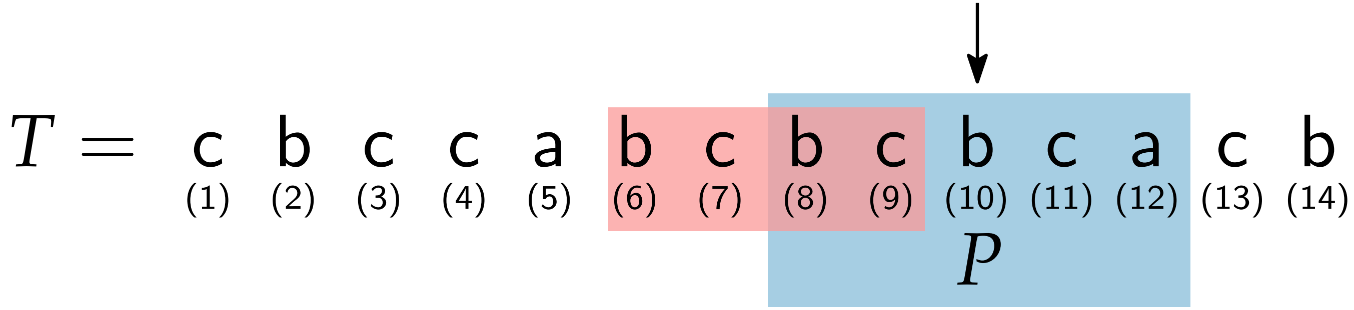
Theorem. String Matching can be solved in $\mathcal{O}(n+m)$ time, and this time bound is optimal. [Knuth, Morris, Pratt'77]

Often, many queries $P_1, P_2, P_3, \ldots$ are performed on the same text T.

Algorithmic Complexity

Occurrences of (prefixes of) P may overlap.

$\Rightarrow$ A simple left-to-right traversal of T is not sufficient to find all occurrences of P!



$T =$ c b c c a b c b c b c a c b

(1) (2) (3) (4) (5) (6) (7) (8) (9) (10) (11) (12) (13) (14)

P

Observation. String Matching can be solved in $\mathcal{O}(nm)$ time.

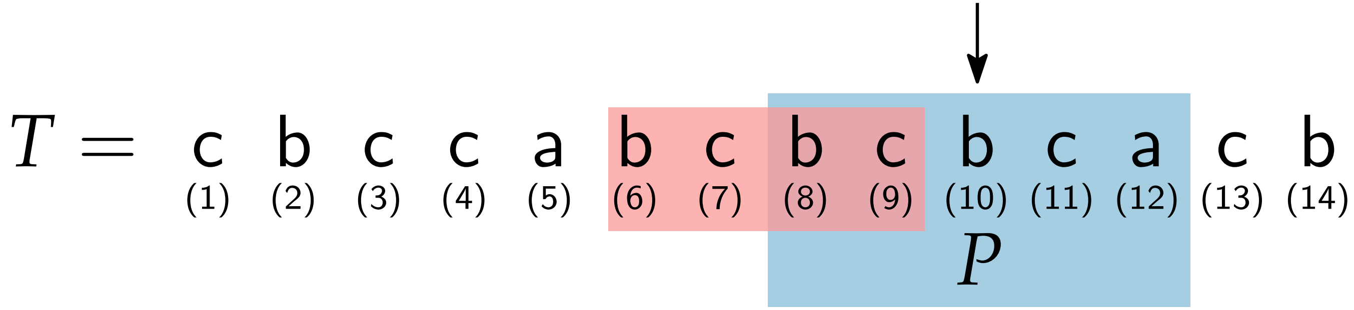
Theorem. String Matching can be solved in $\mathcal{O}(n+m)$ time, and this time bound is optimal. [Knuth, Morris, Pratt'77]

Often, many queries $P_1, P_2, P_3, \ldots$ are performed on the same text T.

Our goal: Design a data structure to store T such that each query P_i can be answered in time independent of n.

Algorithmic Complexity

Occurrences of (prefixes of) P may overlap.
$\Rightarrow$ A simple left-to-right traversal of T is not sufficient to find all occurrences of P!



$T =$	c	b	c	c	a	b	c	b	c	b	c	a	c	b
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)

Observation. String Matching can be solved in $\mathcal{O}(nm)$ time.

Theorem. String Matching can be solved in $\mathcal{O}(n+m)$ time, and this time bound is optimal. [Knuth, Morris, Pratt'77]

Often, many queries $P_1, P_2, P_3, \ldots$ are performed on the same text T.

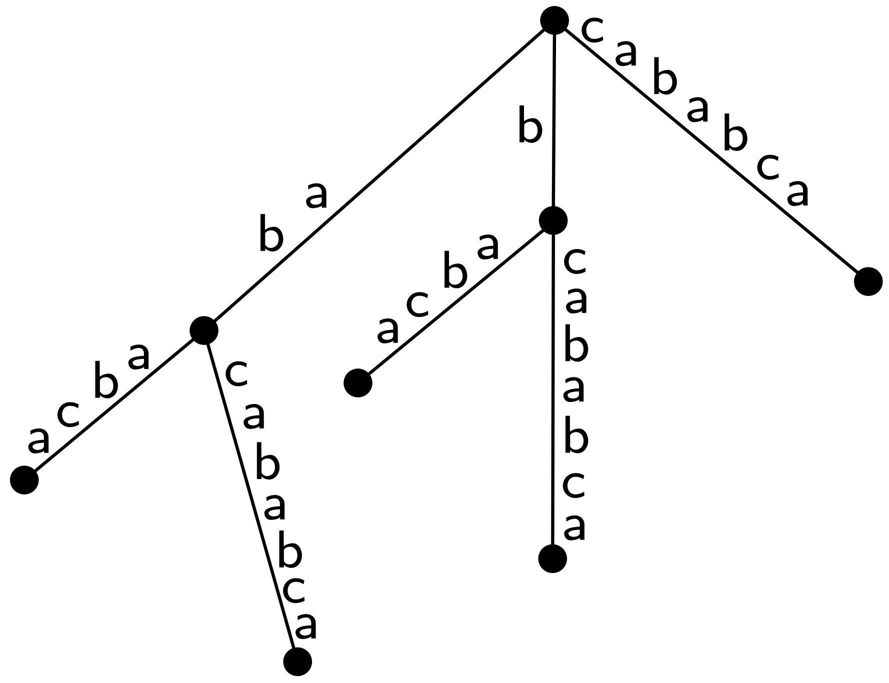
Our goal: Design a data structure to store T such that each query P_i can be answered in time independent of n.

We will see two such data structures: **suffix trees** and **suffix arrays**.

Suffix Trees (I)

Idea: Represent T as a search tree.

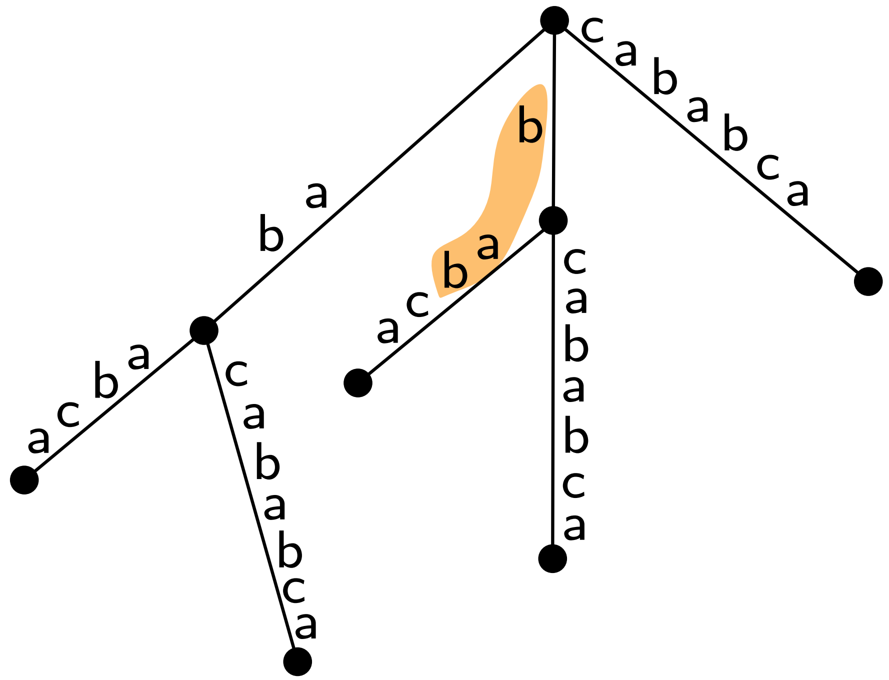
$T = \mathsf{a\ b\ c\ a\ b\ a\ b\ c\ a}$



Suffix Trees (I)

$T =$ a b c a b a b c a $\quad P =$ b a b

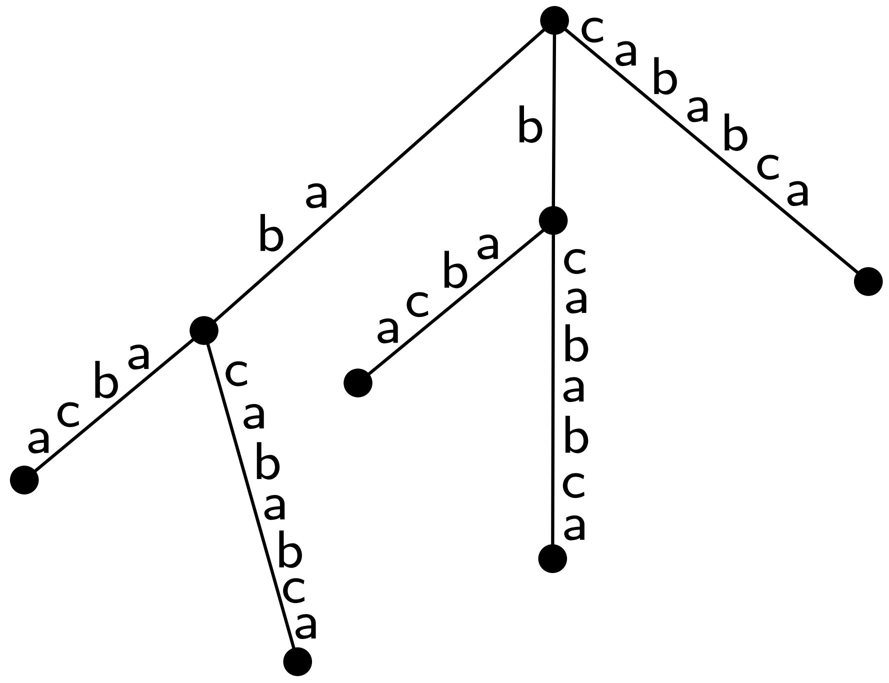
Idea: Represent T as a search tree.



Suffix Trees (I)

$T =$ a b c a b a b c a

Idea: Represent T as a search tree.



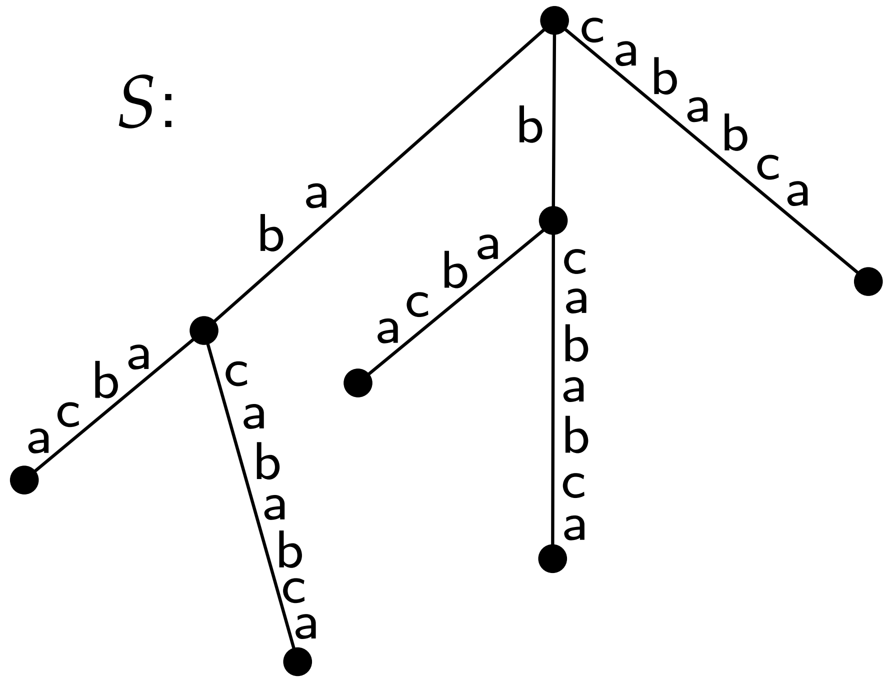
Suffix Trees (I)

$T =$ a b c a b a b c a

Idea: Represent T as a search tree.

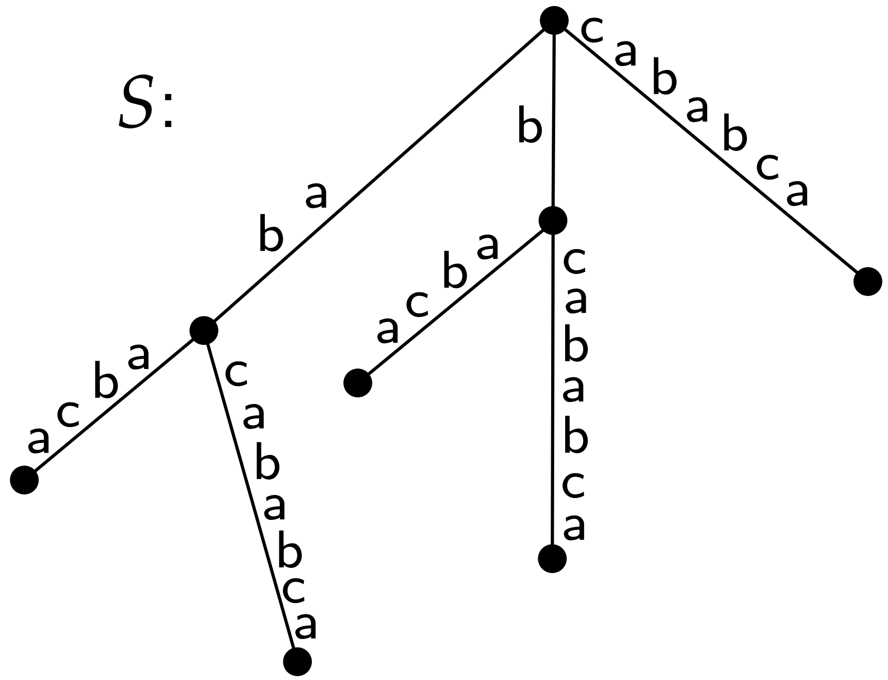
A **Σ-tree** is a rooted tree S whose edges are labeled with strings over Σ such that for each $v \in V(S)$

- the labels of the edges that lead to the children of v start with pairwise distinct elements of Σ;
- if v is not the root, then v has $\neq 1$ children.



Suffix Trees (I)

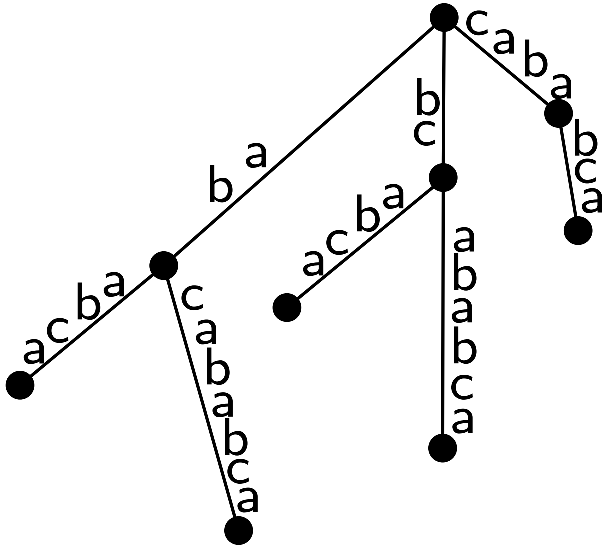
$T = \mathsf{a\ b\ c\ a\ b\ a\ b\ c\ a}$



Idea: Represent T as a search tree.

A **Σ-tree** is a rooted tree S whose edges are labeled with strings over Σ such that for each $v \in V(S)$

- the labels of the edges that lead to the children of v start with pairwise distinct elements of Σ;
- if v is not the root, then v has $\neq 1$ children.



Not a Σ-tree

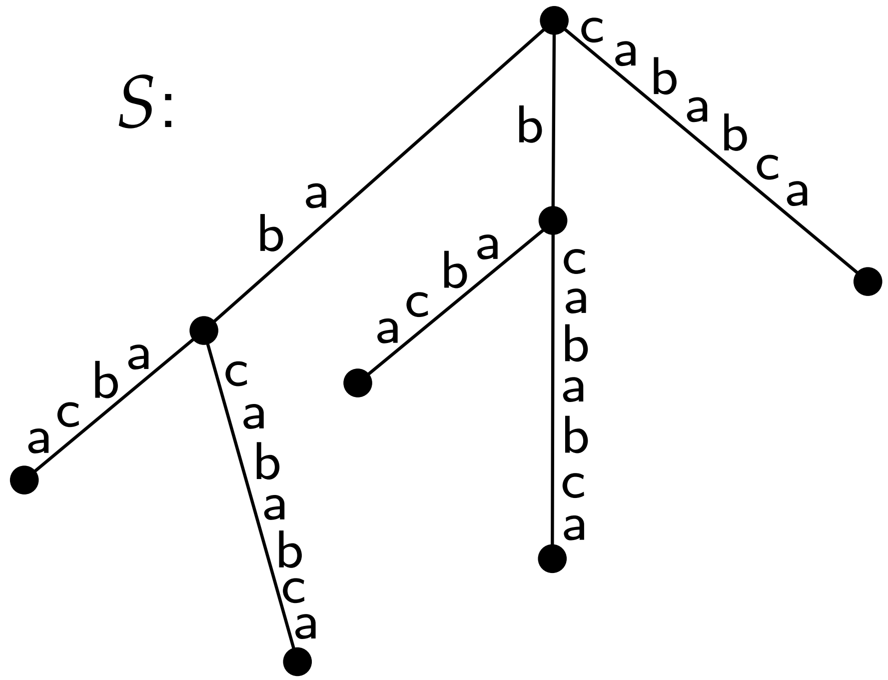
Suffix Trees (I)

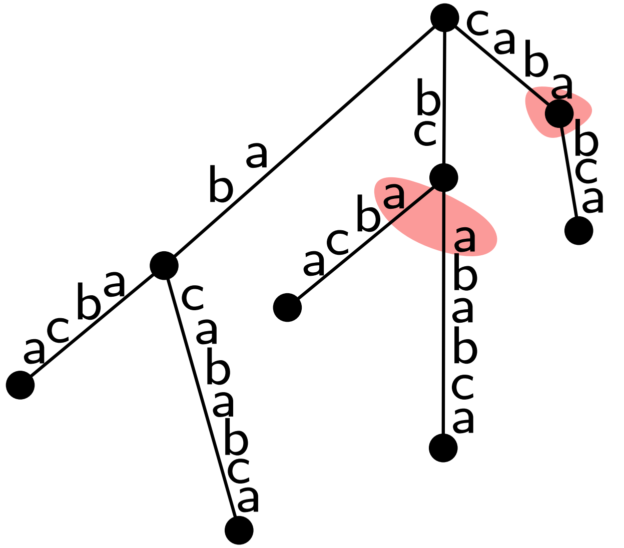
$T =$ a b c a b a b c a

Idea: Represent T as a search tree.

A **Σ-tree** is a rooted tree S whose edges are labeled with strings over Σ such that for each $v \in V(S)$

- the labels of the edges that lead to the children of v start with pairwise distinct elements of Σ;
- if v is not the root, then v has $\neq 1$ children.





Not a Σ-tree

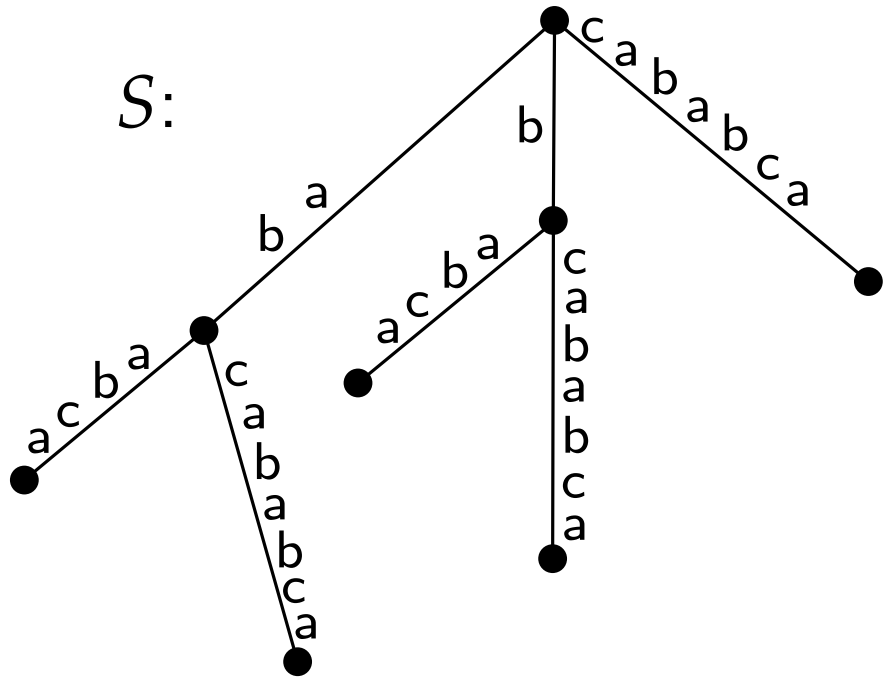
Suffix Trees (I)

$T =$ a b c a b a b c a

Idea: Represent T as a search tree.

A **Σ-tree** is a rooted tree S whose edges are labeled with strings over Σ such that for each $v \in V(S)$

- the labels of the edges that lead to the children of v start with pairwise distinct elements of Σ;
- if v is not the root, then v has $\neq 1$ children.



Suffix Trees (I)

$T =$ a b c a b a b c a

Idea: Represent T as a search tree.

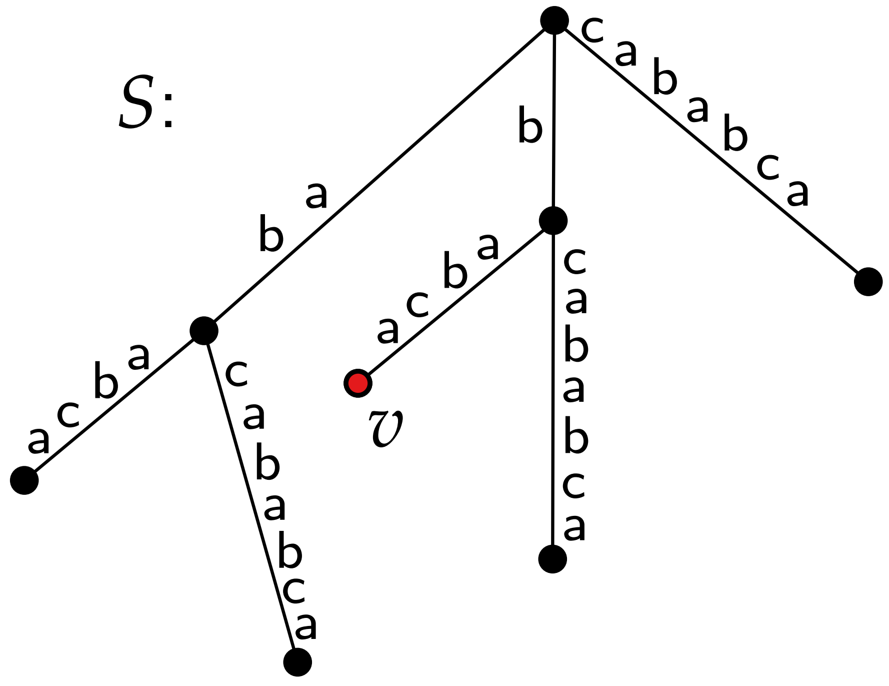
A **Σ-tree** is a rooted tree S whose edges are labeled with strings over Σ such that for each $v \in V(S)$

- the labels of the edges that lead to the children of v start with pairwise distinct elements of Σ;
- if v is not the root, then v has $\neq 1$ children.

Notation:

- $\overline{v} =$ concatenation of the labels encountered on the path from the root to v;

ıd



$\overline{v} = babca$

Suffix Trees (I)

$T =$ a b c a b a b c a

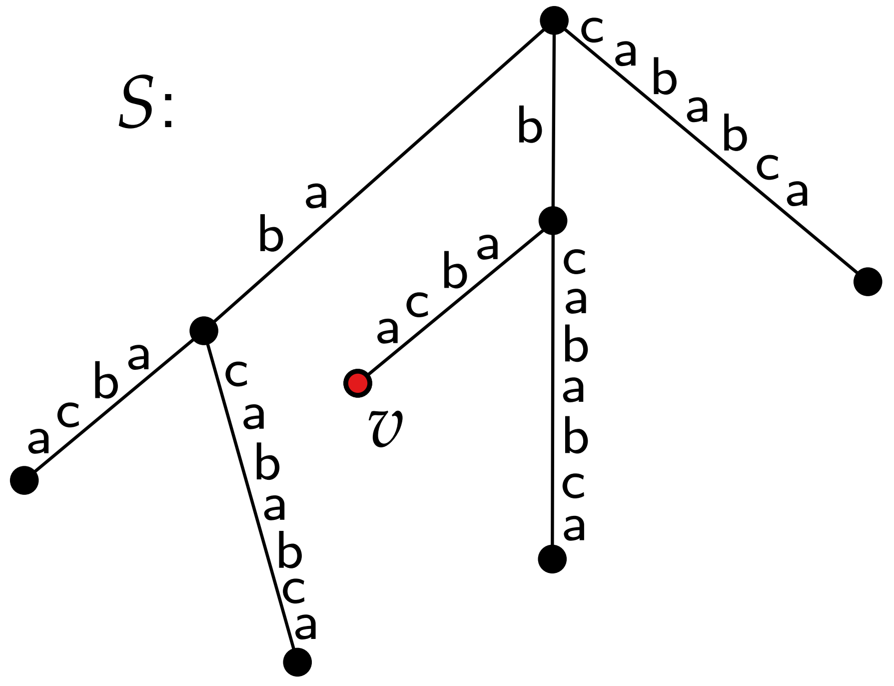
Idea: Represent T as a search tree.

A **Σ-tree** is a rooted tree S whose edges are labeled with strings over Σ such that for each $v \in V(S)$

- the labels of the edges that lead to the children of v start with pairwise distinct elements of Σ;
- if v is not the root, then v has $\neq 1$ children.

Notation:

- $\overline{v} =$ concatenation of the labels encountered on the path from the root to v;
- $d(v) = |\overline{v}|$ is the **string depth** of v;



$\overline{v} = babca$

$d(v) = |\overline{v}| = 5$

Suffix Trees (I)

$T =$ a b c a b a b c a

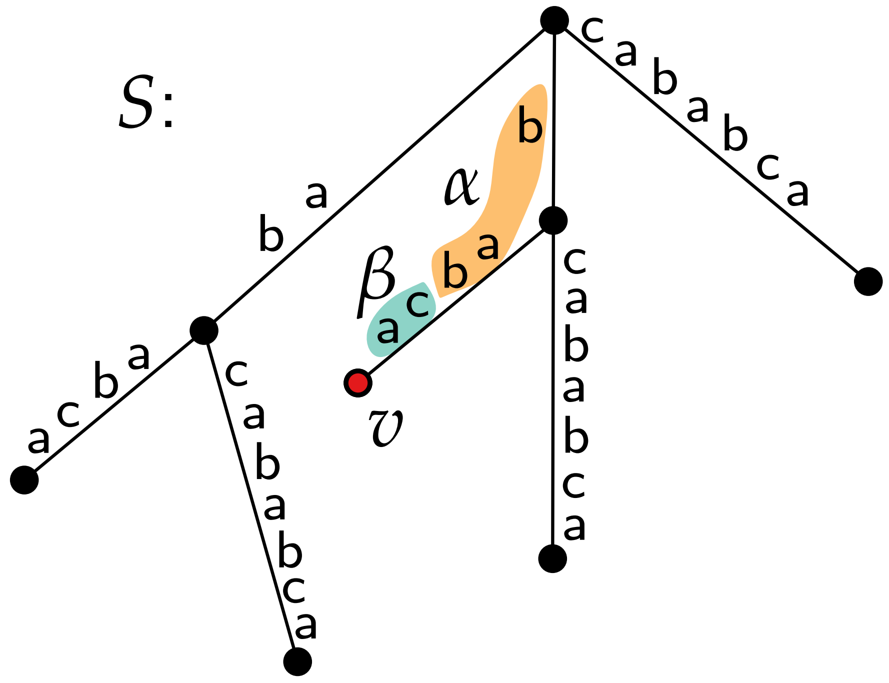
Idea: Represent T as a search tree.

A **Σ-tree** is a rooted tree S whose edges are labeled with strings over Σ such that for each $v \in V(S)$

- the labels of the edges that lead to the children of v start with pairwise distinct elements of Σ;
- if v is not the root, then v has $\neq 1$ children.

Notation:

- $\overline{v} =$ concatenation of the labels encountered on the path from the root to v;
- $d(v) = |\overline{v}|$ is the **string depth** of v;
- S **contains** a string α if there is a $v \in V(S)$ and a (maybe empty) string β such that $\overline{v} = \alpha\beta$;



$\overline{v} = babca$

$d(v) = |\overline{v}| = 5$

S contains $\alpha =$ b a b since there is a $v \in V(S)$ with $\overline{v} = \alpha\beta$ (where $\beta =$ c a).

Suffix Trees (I)

$T =$ a b c a b a b c a

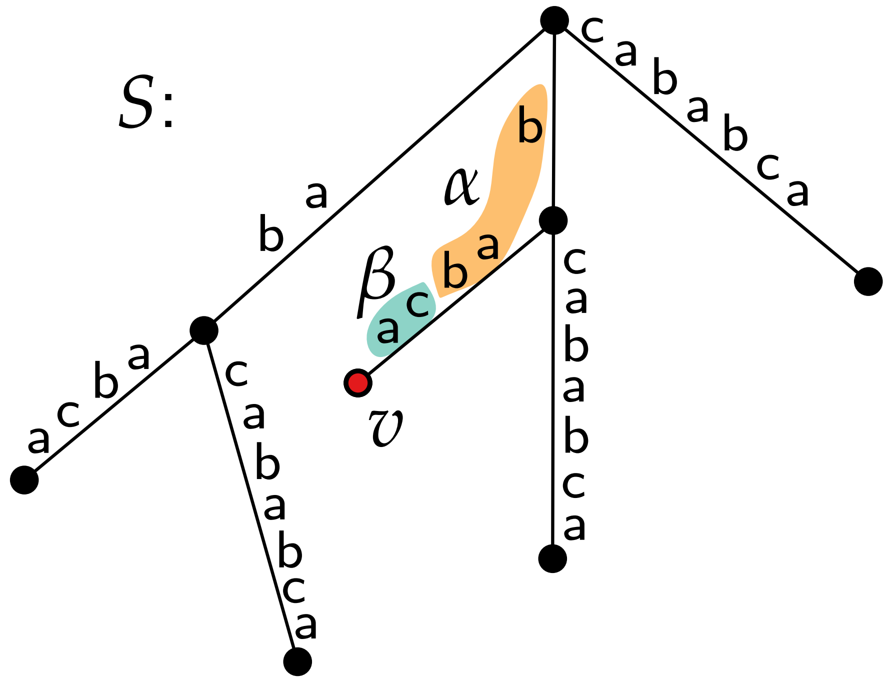
Idea: Represent T as a search tree.

A **Σ-tree** is a rooted tree S whose edges are labeled with strings over Σ such that for each $v \in V(S)$

- the labels of the edges that lead to the children of v start with pairwise distinct elements of Σ;
- if v is not the root, then v has $\neq 1$ children.

Notation:

- $\overline{v} =$ concatenation of the labels encountered on the path from the root to v;
- $d(v) = |\overline{v}|$ is the **string depth** of v;
- S **contains** a string α if there is a $v \in V(S)$ and a (maybe empty) string β such that $\overline{v} = \alpha\beta$;
- words$(S) =$ set of all strings contained in S.



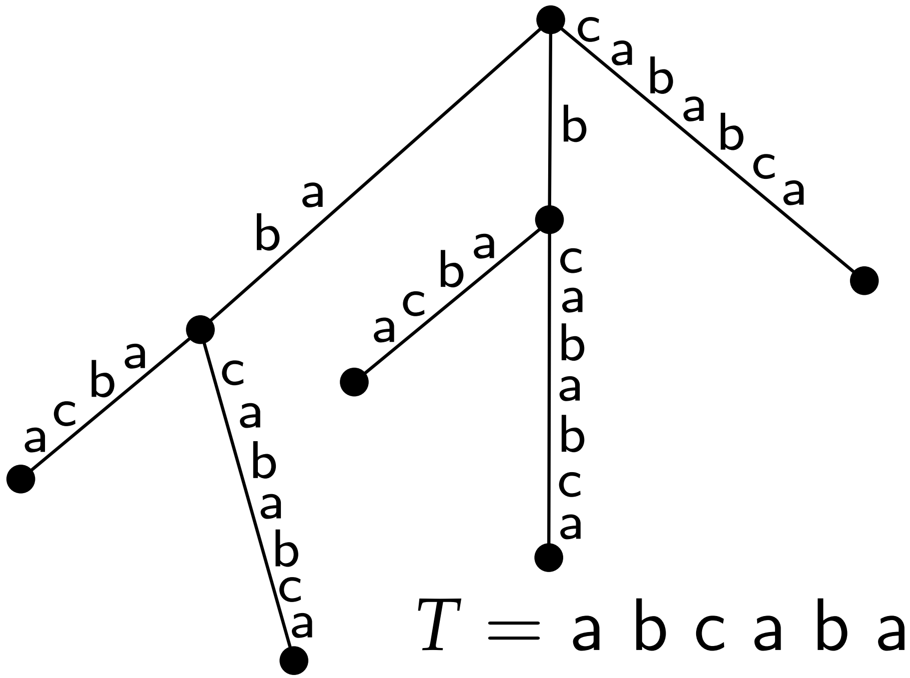
$\overline{v} = babca$

$d(v) = |\overline{v}| = 5$

S contains $\alpha =$ b a b since there is a $v \in V(S)$ with $\overline{v} = \alpha\beta$ (where $\beta =$ c a).

Suffix Trees (II)

A **suffix tree** S of T is a Σ-tree that contains exactly the infixes of T, that is, words$(S) = \{T[i,j] \mid 1 \leq i \leq j \leq n\}$.

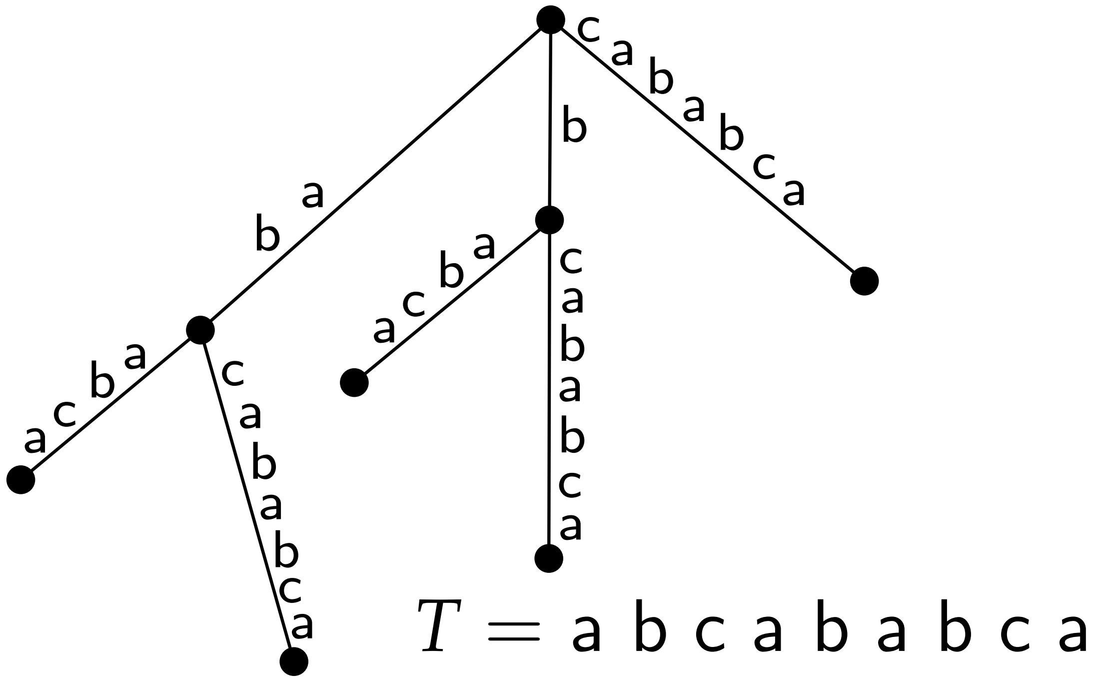


$T = \mathsf{a\ b\ c\ a\ b\ a\ b\ c\ a}$

Suffix Trees (II)

A **suffix tree** S of T is a Σ-tree that contains exactly the infixes of T, that is, $\text{words}(S) = \{T[i,j] \mid 1 \leq i \leq j \leq n\}$.

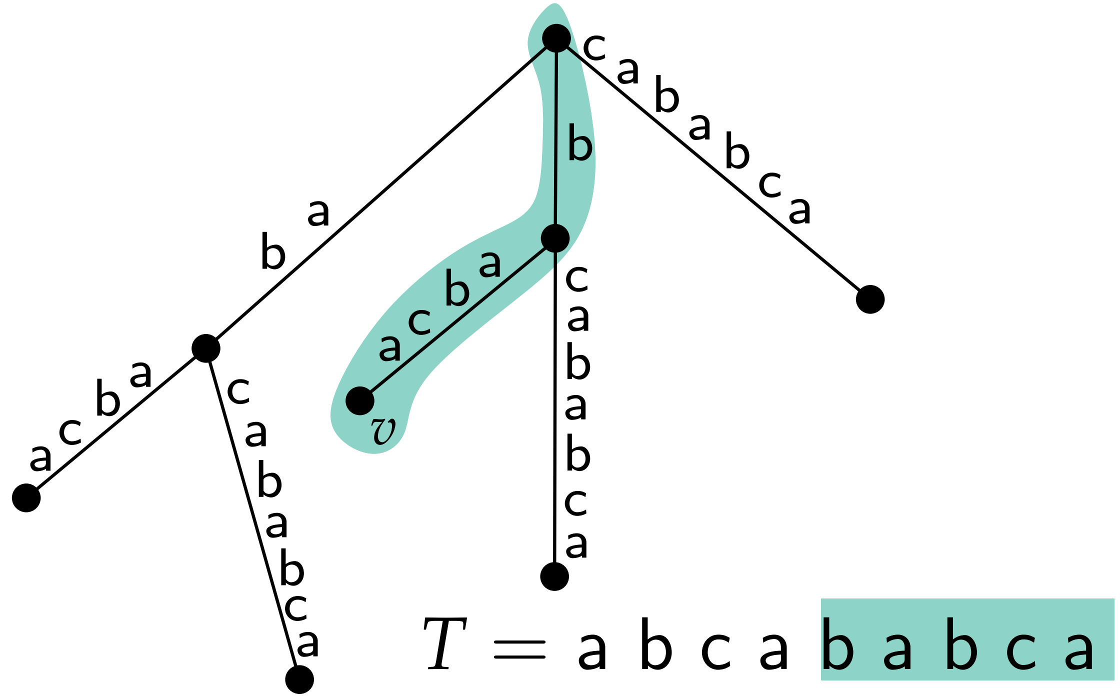
Lemma. For each leaf v of S, the infix $\overline{v}$ is a suffix of T.



Suffix Trees (II)

A **suffix tree** S of T is a Σ-tree that contains exactly the infixes of T, that is, $\text{words}(S) = \{T[i,j] \mid 1 \leq i \leq j \leq n\}$.

Lemma. For each leaf v of S, the infix $\overline{v}$ is a suffix of T.

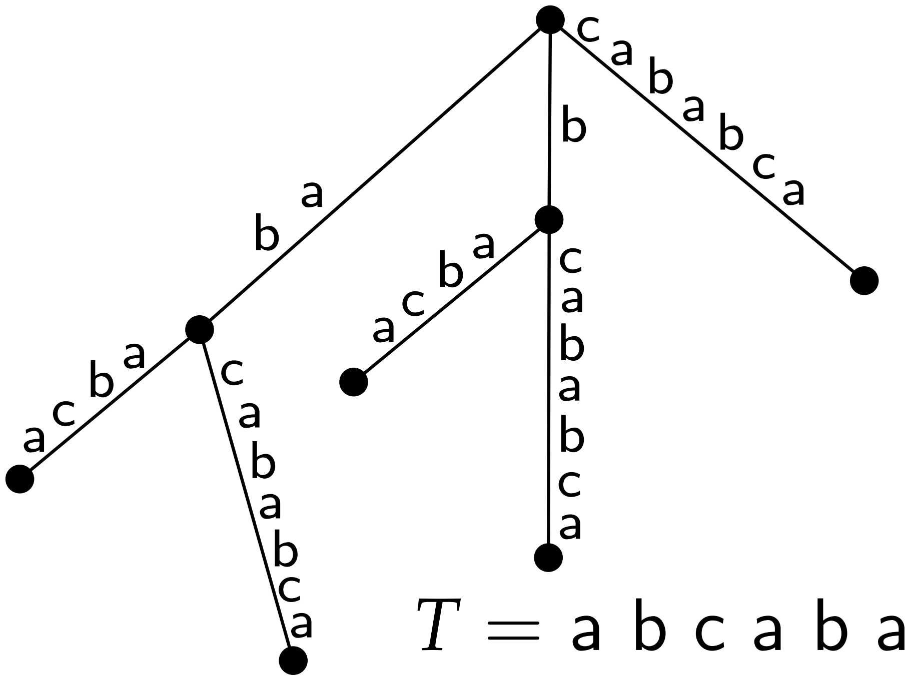


$T =$ a b c a b a b c a

Suffix Trees (II)

A **suffix tree** S of T is a Σ-tree that contains exactly the infixes of T, that is, $\text{words}(S) = \{T[i,j] \mid 1 \leq i \leq j \leq n\}$.

Lemma. For each leaf v of S, the infix $\overline{v}$ is a suffix of T.



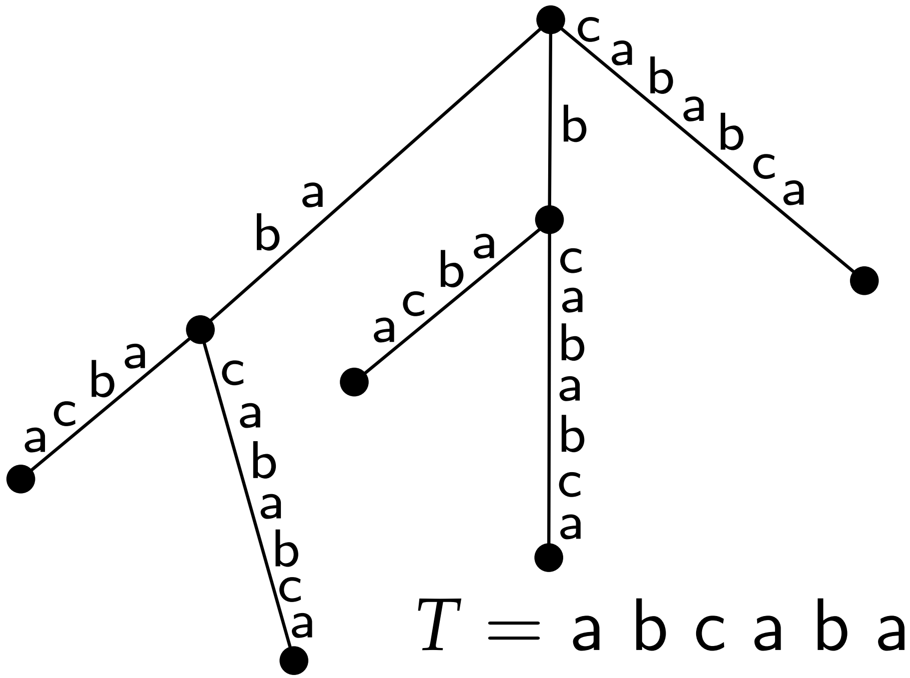
$T =$ a b c a b a b c a

Suffix Trees (II)

A **suffix tree** S of T is a Σ-tree that contains exactly the infixes of T, that is, $\text{words}(S) = \{T[i,j] \mid 1 \leq i \leq j \leq n\}$.

Lemma. For each leaf v of S, the infix $\overline{v}$ is a suffix of T.

Proof. Denote $\overline{v} = T[i,j]$ and assume $j < n$.



$T =$ a b c a b a b c a

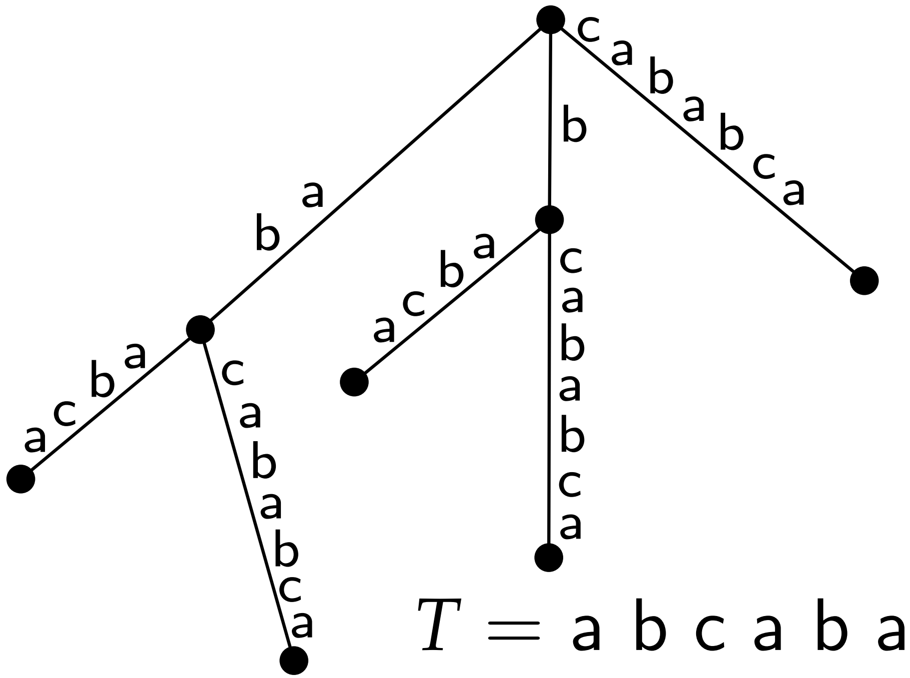
Suffix Trees (II)

A **suffix tree** S of T is a Σ-tree that contains exactly the infixes of T, that is, $\text{words}(S) = \{T[i,j] \mid 1 \leq i \leq j \leq n\}$.

Lemma. For each leaf v of S, the infix $\overline{v}$ is a suffix of T.

Proof. Denote $\overline{v} = T[i,j]$ and assume $j < n$.

$\overline{v}$ is a prefix of $T[i,n]$. Let u be a vertex such that $T[i,n]$ is a prefix of $\overline{u}$.



$T = \text{a b c a b a b c a}$

Suffix Trees (II)

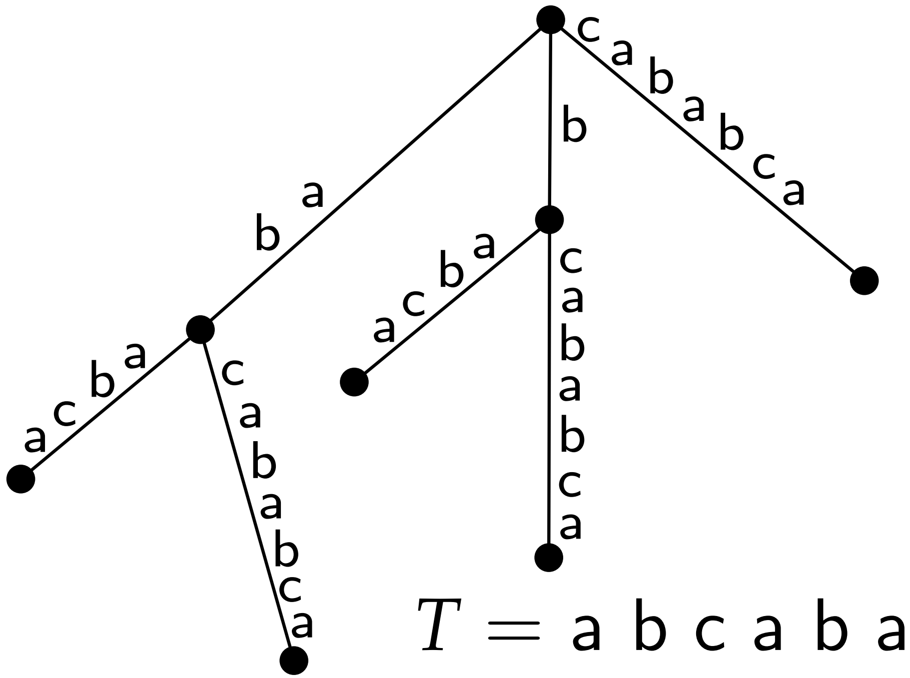
A **suffix tree** S of T is a Σ-tree that contains exactly the infixes of T, that is, $\text{words}(S) = \{T[i,j] \mid 1 \leq i \leq j \leq n\}$.

Lemma. For each leaf v of S, the infix $\overline{v}$ is a suffix of T.

Proof. Denote $\overline{v} = T[i,j]$ and assume $j < n$.

$\overline{v}$ is a prefix of $T[i,n]$. Let u be a vertex such that $T[i,n]$ is a prefix of $\overline{u}$.

$\Rightarrow$ The path from the root to v is a subpath of the path from the root to u.



$T =$ a b c a b a b c a

Suffix Trees (II)

A **suffix tree** S of T is a Σ-tree that contains exactly the infixes of T, that is, $\mathrm{words}(S) = \{T[i,j] \mid 1 \leq i \leq j \leq n\}$.

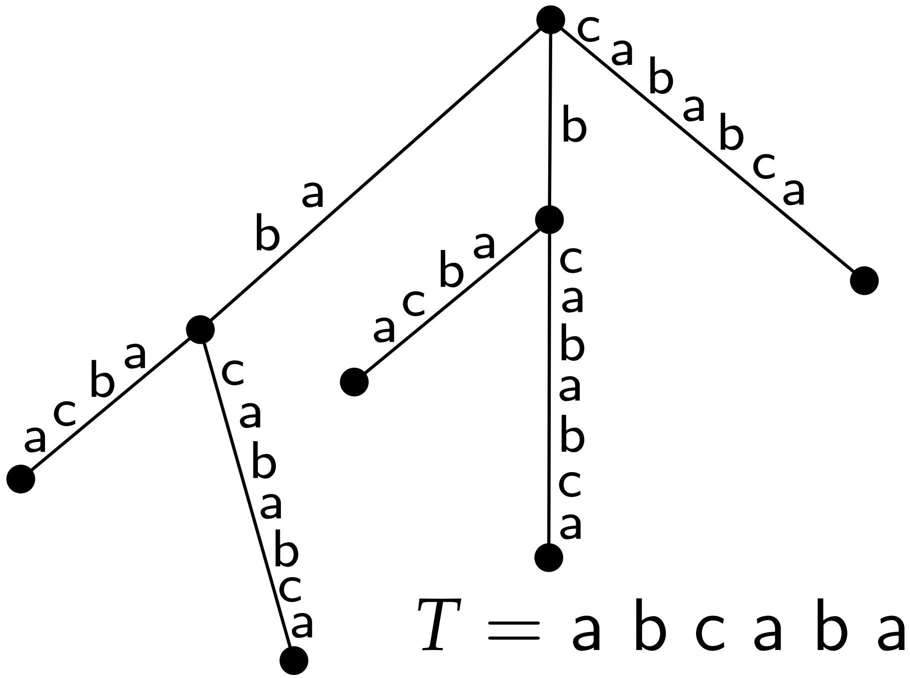
Lemma. For each leaf v of S, the infix $\overline{v}$ is a suffix of T.

Proof. Denote $\overline{v} = T[i,j]$ and assume $j < n$.

$\overline{v}$ is a prefix of $T[i,n]$. Let u be a vertex such that $T[i,n]$ is a prefix of $\overline{u}$.

$\Rightarrow$ The path from the root to v is a subpath of the path from the root to u.

$\Rightarrow$ v is not a leaf; a contradiction. $\square$

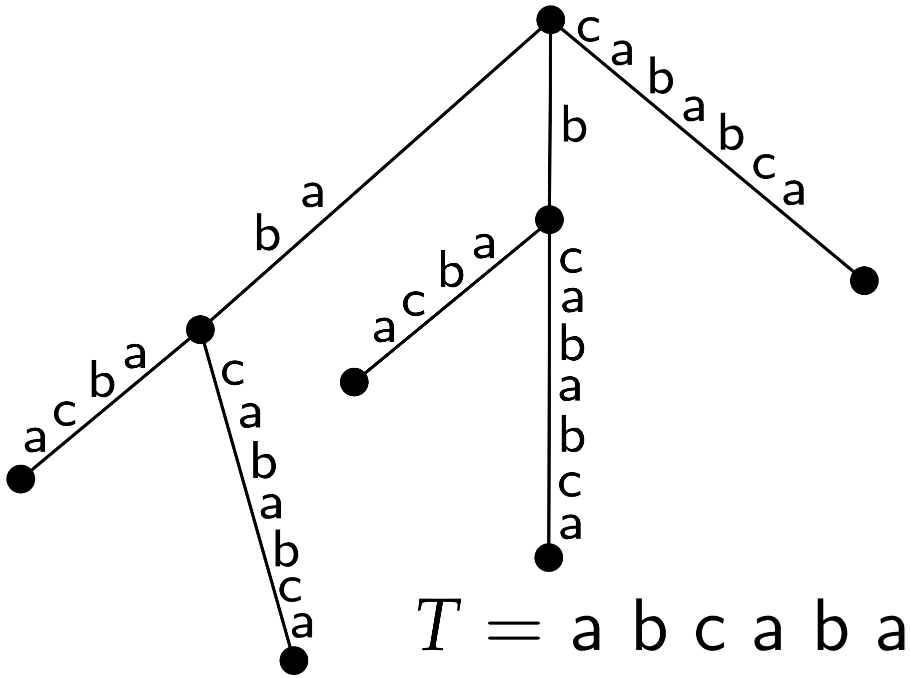


$T = \mathrm{a\ b\ c\ a\ b\ a\ b\ c\ a}$

Suffix Trees (II)

A **suffix tree** S of T is a Σ-tree that contains exactly the infixes of T, that is, $\text{words}(S) = \{T[i,j] \mid 1 \leq i \leq j \leq n\}$.

Lemma. For each leaf v of S, the infix $\overline{v}$ is a suffix of T.



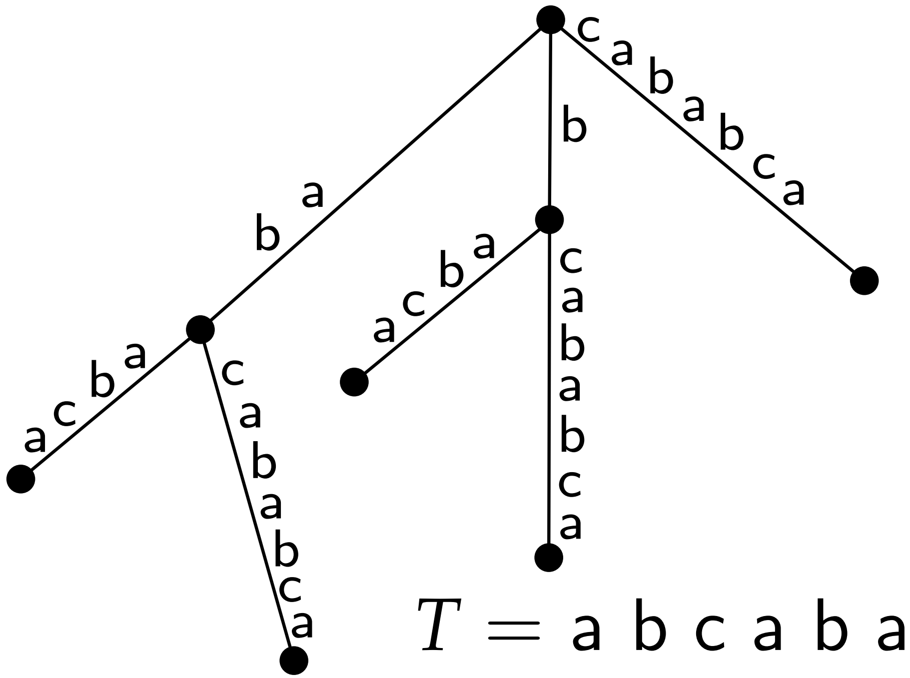
$T = \text{a b c a b a b c a}$

Suffix Trees (II)

A **suffix tree** S of T is a Σ-tree that contains exactly the infixes of T, that is, $\text{words}(S) = \{T[i,j] \mid 1 \leq i \leq j \leq n\}$.

Lemma. For each leaf v of S, the infix $\overline{v}$ is a suffix of T.

Remark. The converse is not true



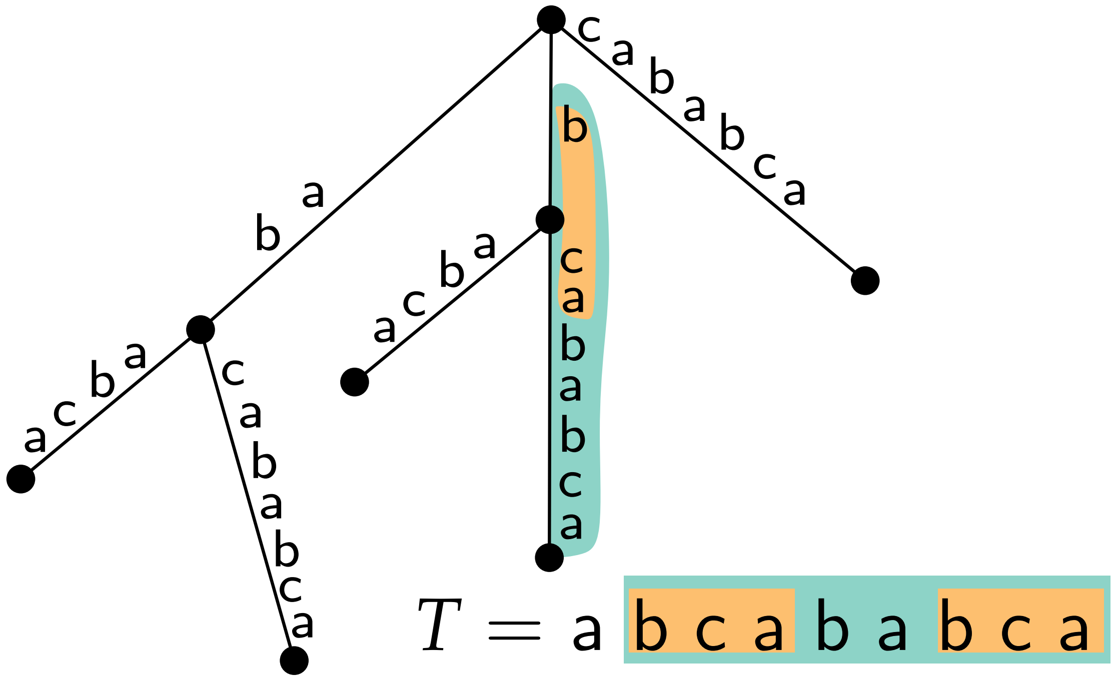
$T =$ a b c a b a b c a

Suffix Trees (II)

A **suffix tree** S of T is a Σ-tree that contains exactly the infixes of T, that is, words$(S) = \{T[i,j] \mid 1 \leq i \leq j \leq n\}$.

Lemma. For each leaf v of S, the infix $\overline{v}$ is a suffix of T.

Remark. The converse is not true since a suffix can be a prefix of another suffix.



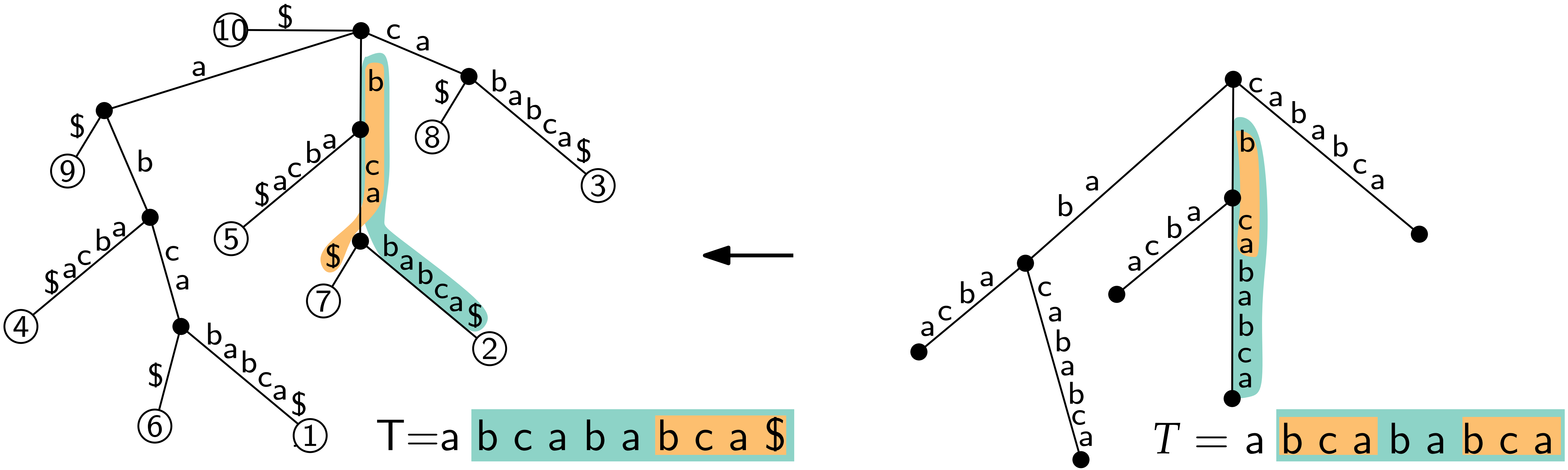
Suffix Trees (II)

A **suffix tree** S of T is a Σ-tree that contains exactly the infixes of T, that is, $\text{words}(S) = \{T[i,j] \mid 1 \le i \le j \le n\}$.

Lemma. For each leaf v of S, the infix $\overline{v}$ is a suffix of T.

Remark. The converse is not true since a suffix can be a prefix of another suffix.

Fix: Append a symbol $\$ \notin \Sigma$ to T $\Rightarrow$ the leaves correspond bijectively to the suffixes.



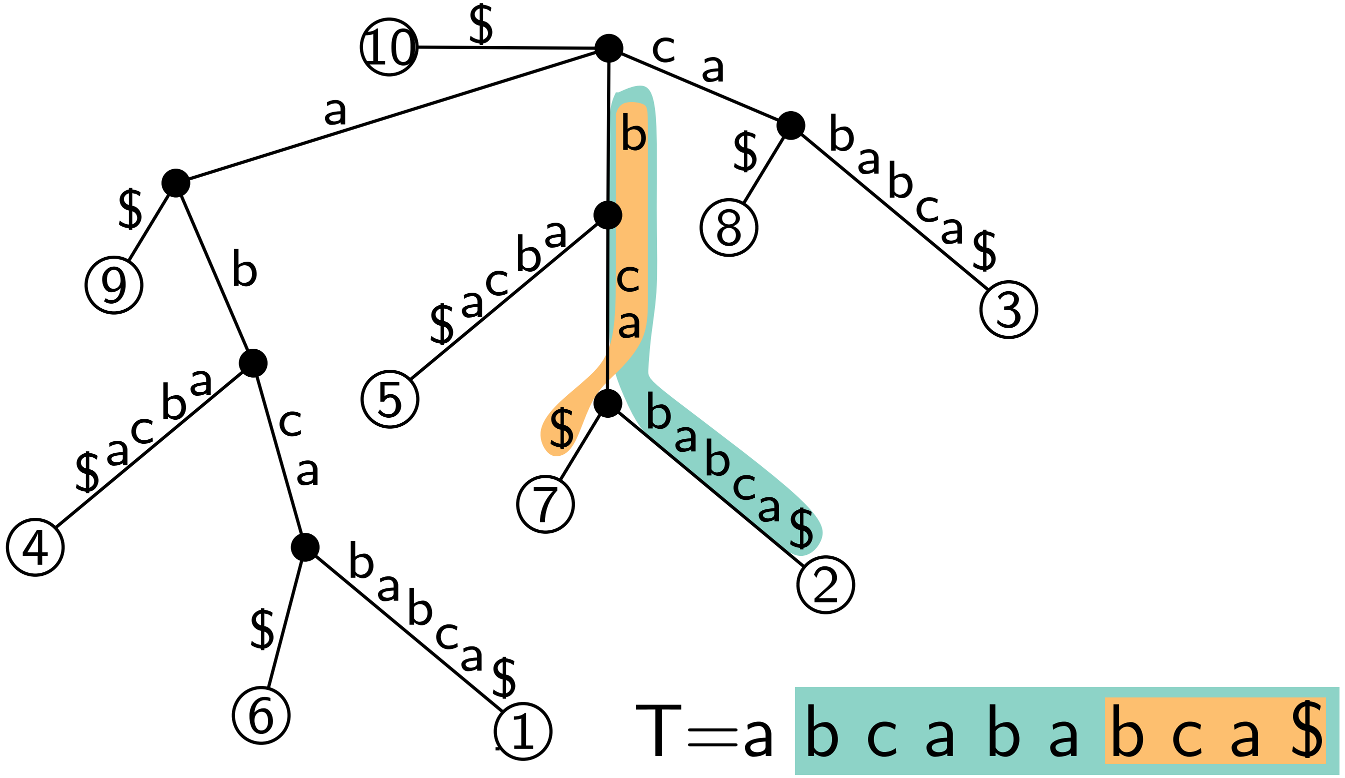
Suffix Trees (II)

A **suffix tree** S of T is a Σ-tree that contains exactly the infixes of T, that is, words$(S) = \{T[i,j] \mid 1 \leq i \leq j \leq n\}$.

Lemma. For each leaf v of S, the infix $\overline{v}$ is a suffix of T.

Remark. The converse is not true

Fix: Append a symbol $\$ \notin \Sigma$ to T $\Rightarrow$ the leaves correspond bijectively to the suffixes.



T=a b c a b a b c a $

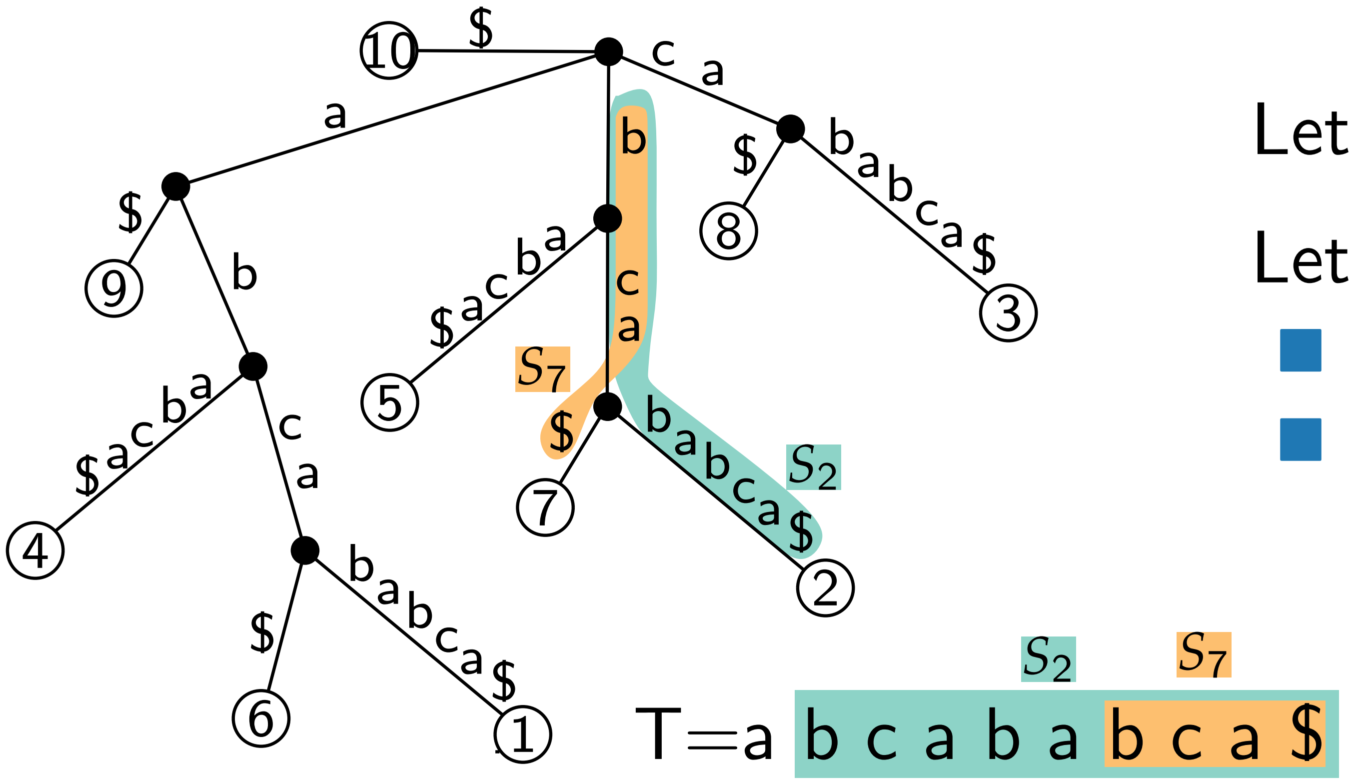
Suffix Trees (II)

A **suffix tree** S of T is a Σ-tree that contains exactly the infixes of T, that is, $\text{words}(S) = \{T[i,j] \mid 1 \leq i \leq j \leq n\}$.

Lemma. For each leaf v of S, the infix $\overline{v}$ is a suffix of T.

Remark. The converse is not true

Fix: Append a symbol $\$ \notin \Sigma$ to $T \Rightarrow$ the leaves correspond bijectively to the suffixes.



Let i denote the leaf of S where $\overline{i} = T[i,n]$.

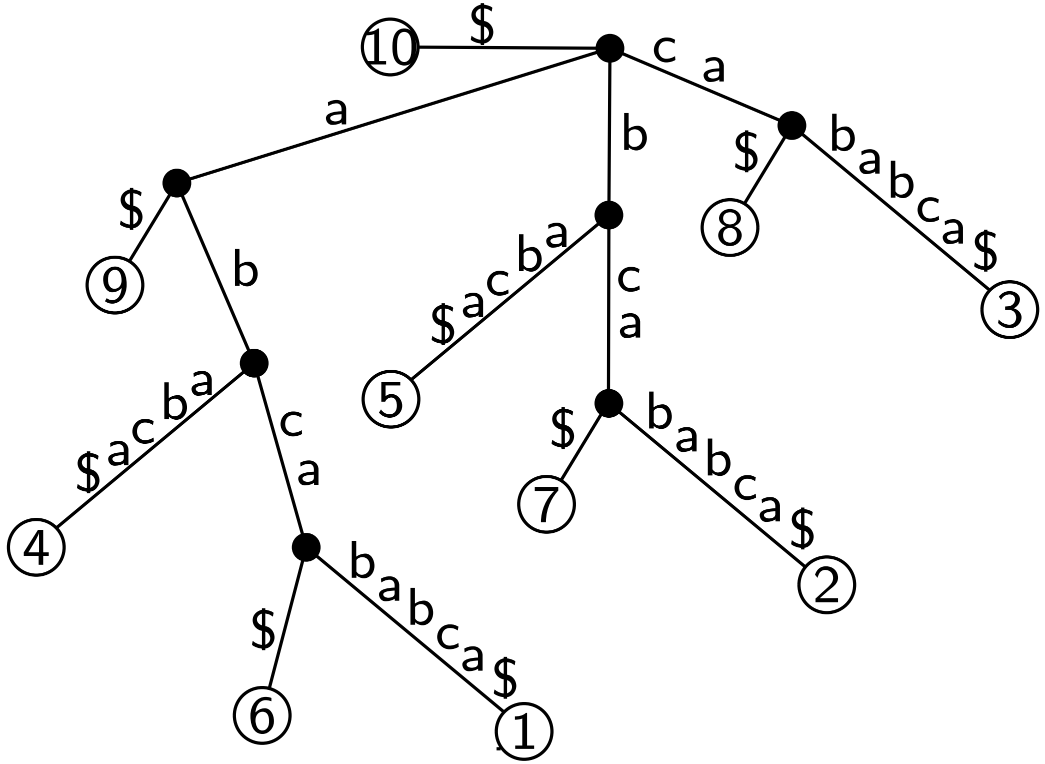
Let S_i denote

- the i-th suffix $T[i,n]$ of T;
- the path from the root of S to leaf i.

Suffix Trees (III)

Implementation details:

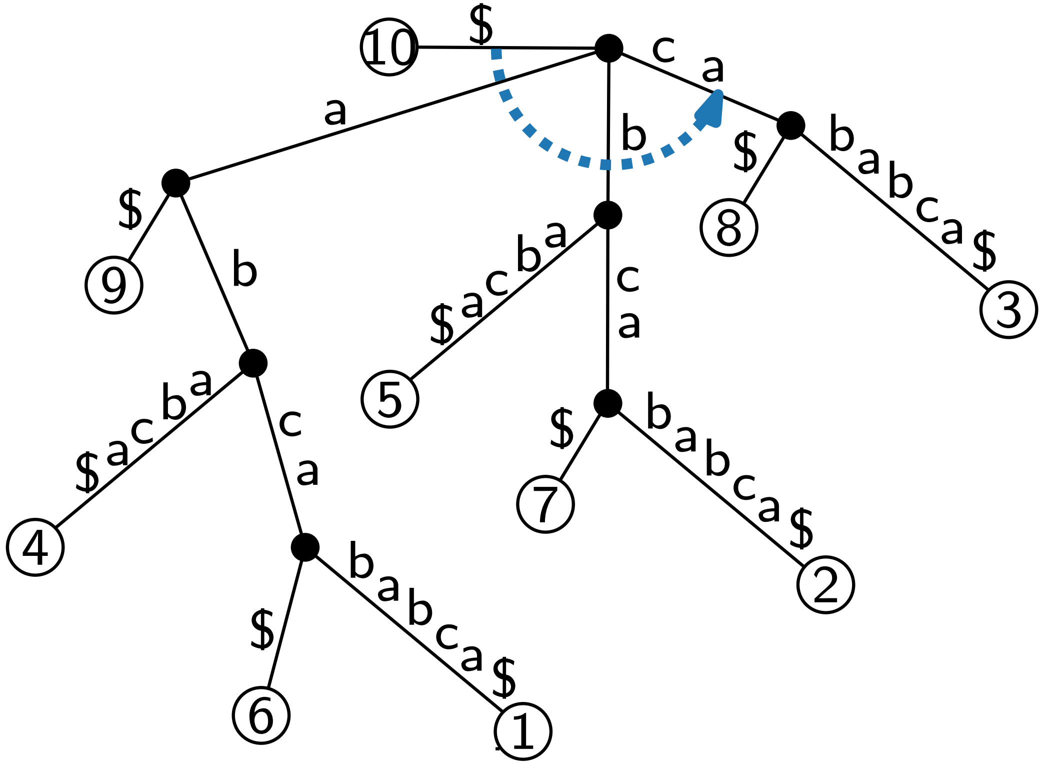
- Each edge is labeled with an infix $T[i, j]$. It suffices to store the indices i and j.
  $\Rightarrow S$ requires $\mathcal{O}(n)$ space since #leaves $=$ #suffixes $=$ n.



Suffix Trees (III)

Implementation details:

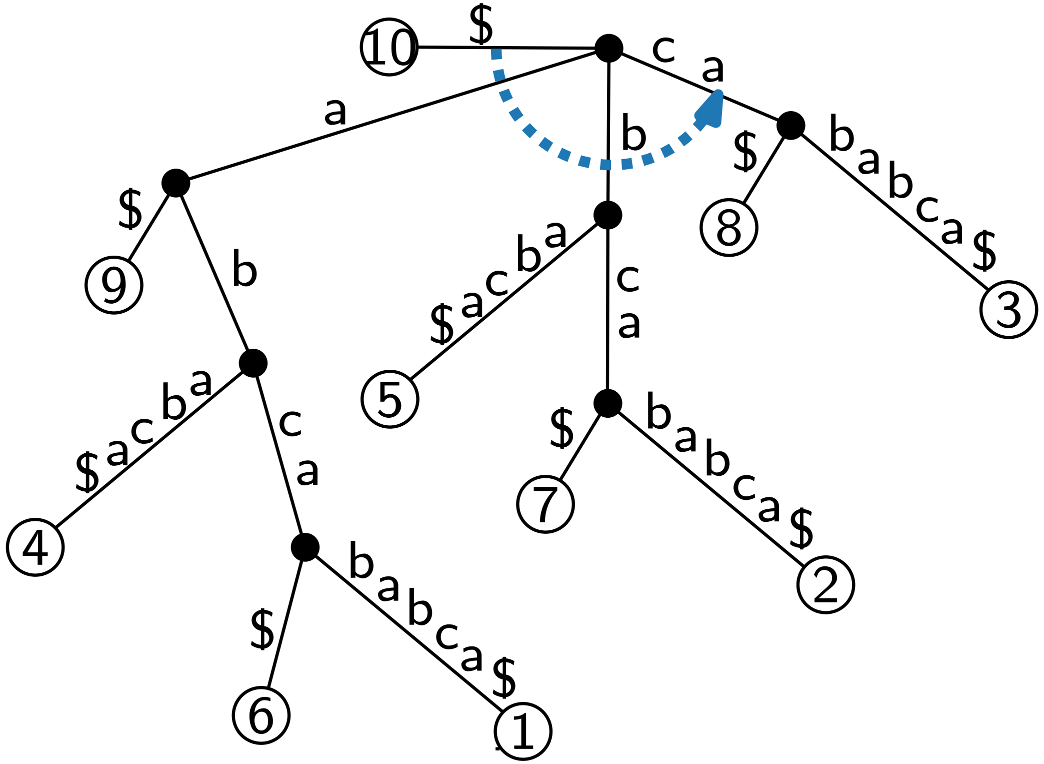
- Each edge is labeled with an infix $T[i, j]$. It suffices to store the indices i and j.
  $\Rightarrow$ S requires $\mathcal{O}(n)$ space since $\#\text{leaves} = \#\text{suffixes} = n$.
- At each vertex v with k children, the edges leading to these children are stored in an array of length k sorted by the first letter of their labels.



Suffix Trees (III)

Implementation details:

- Each edge is labeled with an infix $T[i, j]$. It suffices to store the indices i and j.
  $\Rightarrow S$ requires $\mathcal{O}(n)$ space since #leaves $=$ #suffixes $=$ n.
- At each vertex v with k children, the edges leading to these children are stored in an array of length k sorted by the first letter of their labels.

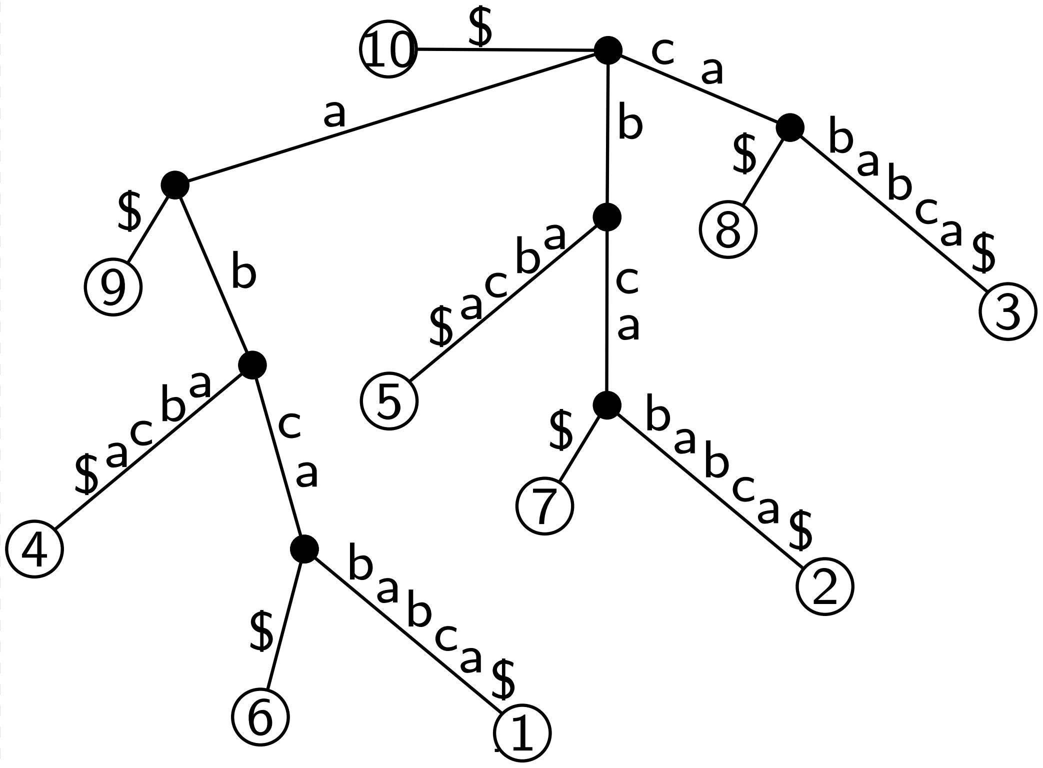


→ allows for binary search!

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.
      if $e$ does not exist then
          return "no match"
      Compare $B$ with $P[i, m]$
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return "no match"
  return "no match"
```

$T =$ a b c a b a b c a

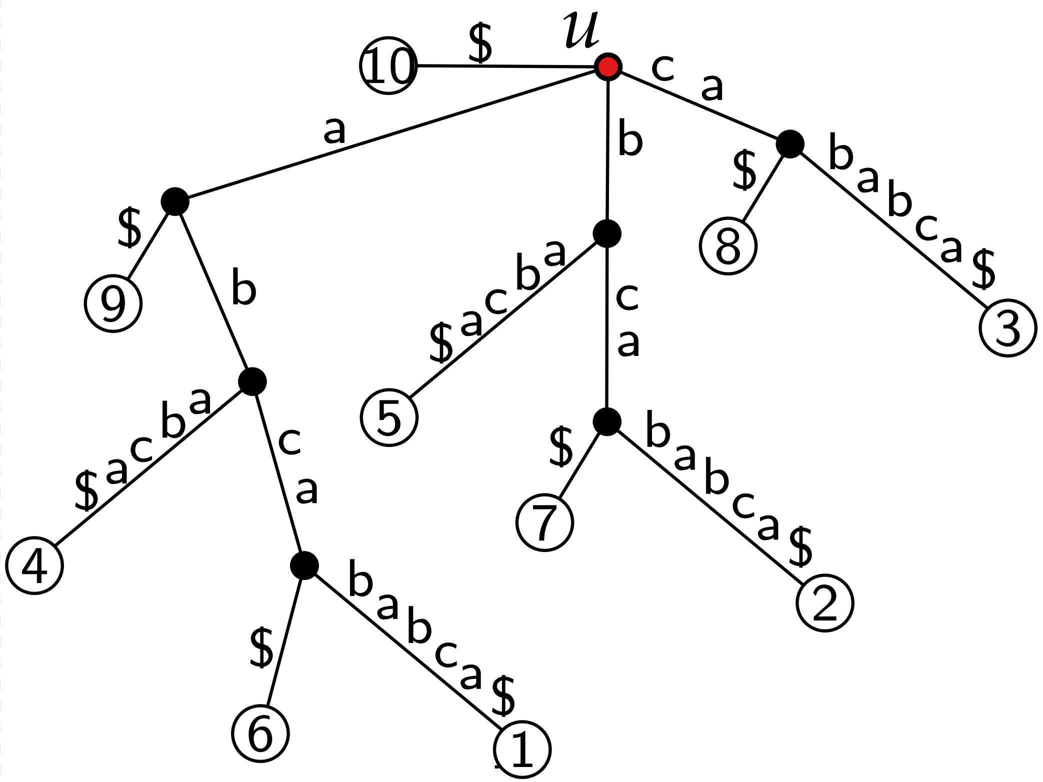


Beispiel: $P = \underset{1}{\text{a}}\ \underset{2}{\text{b}}\ \underset{3}{\text{c}}$

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.
      if $e$ does not exist then
          return “no match“
      Compare $B$ with $P[i, m]$
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return “no match“
  return “no match“
```

$T =$ a b c a b a b c a

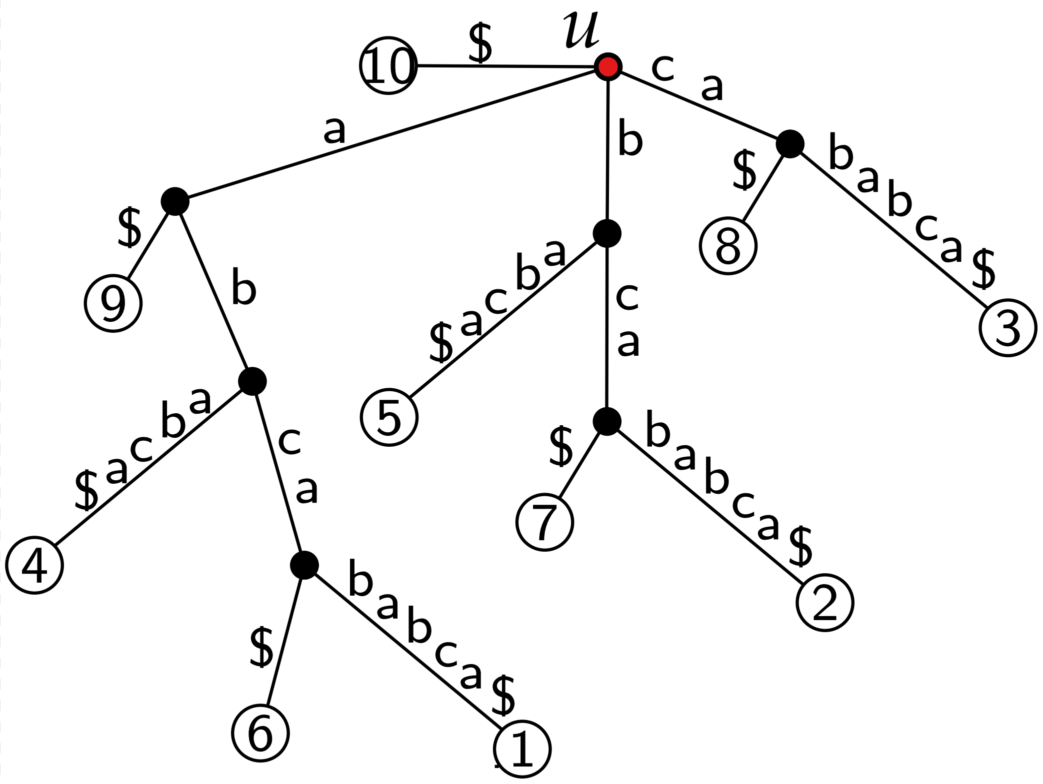


Beispiel: $P = \underset{1}{\text{a}}\ \underset{2}{\text{b}}\ \underset{3}{\text{c}}$

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.
      if $e$ does not exist then
          return “no match“
      Compare $B$ with $P[i, m]$
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return “no match“
  return “no match“
```

$T =$ a b c a b a b c a

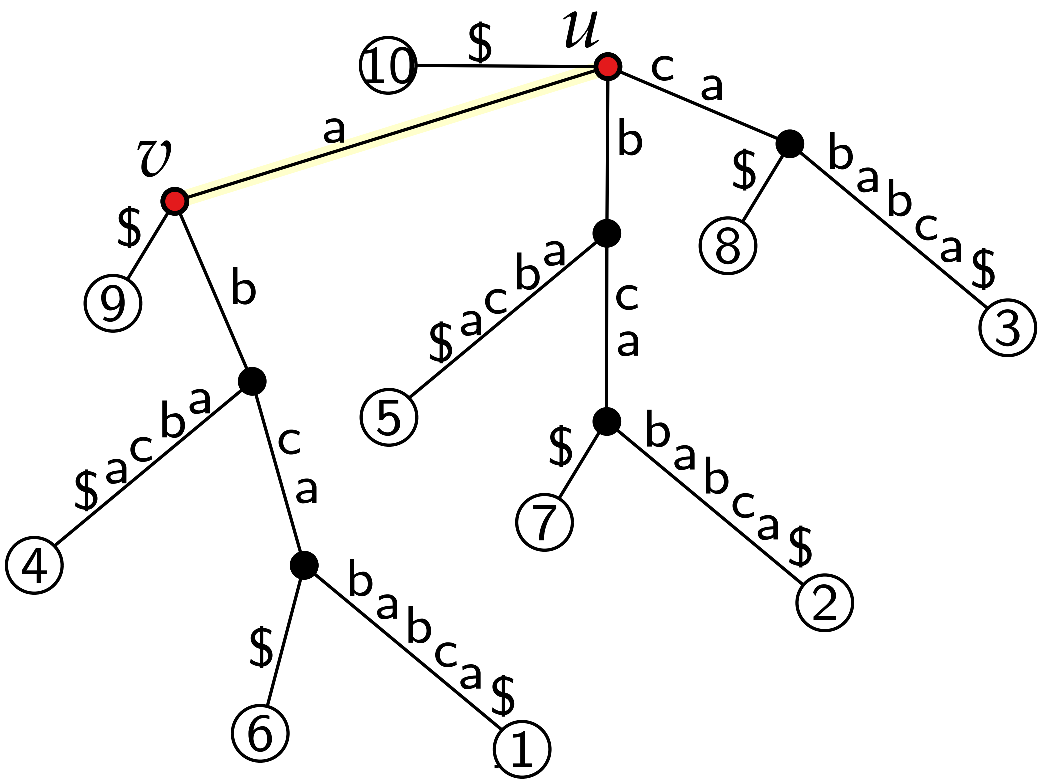


Beispiel: $P =$ a b c
1 2 3
i

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.
      if $e$ does not exist then
          return "no match"
      Compare $B$ with $P[i, m]$
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return "no match"
  return "no match"
```

$T =$ a b c a b a b c a

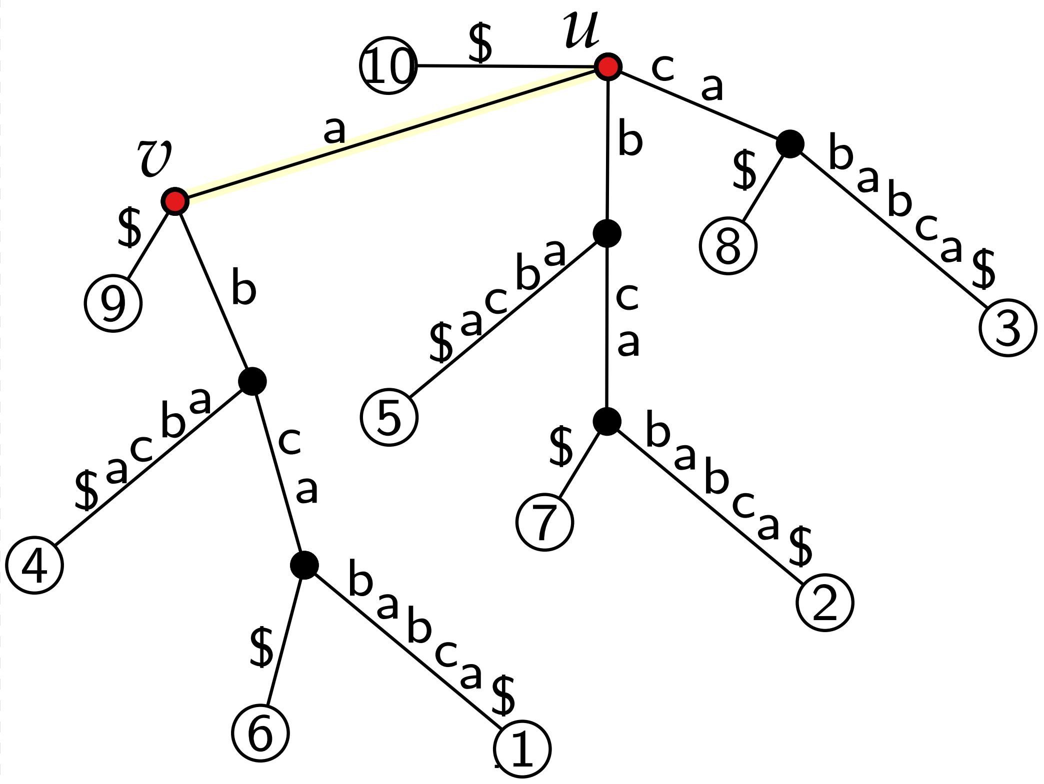


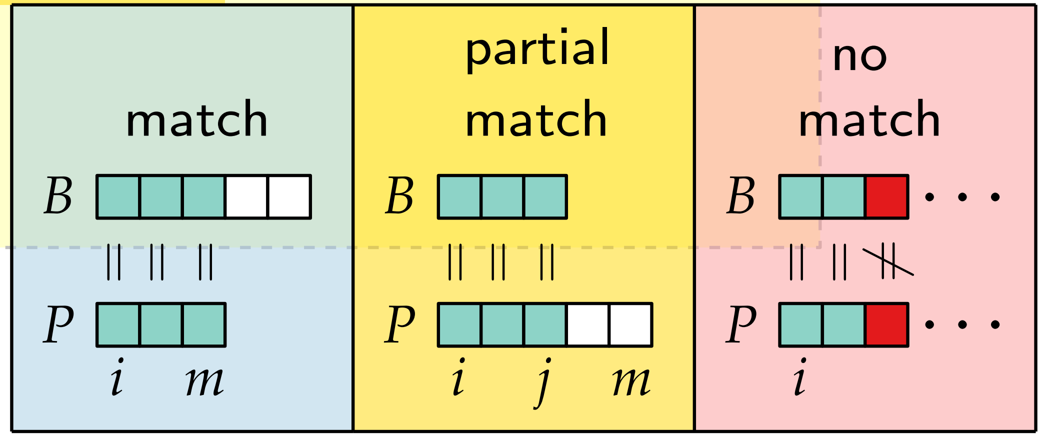
Beispiel: $P = \underset{1}{a}\ \underset{2}{b}\ \underset{3}{c}$

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.
      if $e$ does not exist then
          return “no match“
      Compare $B$ with $P[i, m]$
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return “no match“
  return “no match“
```

$T =$ a b c a b a b c a





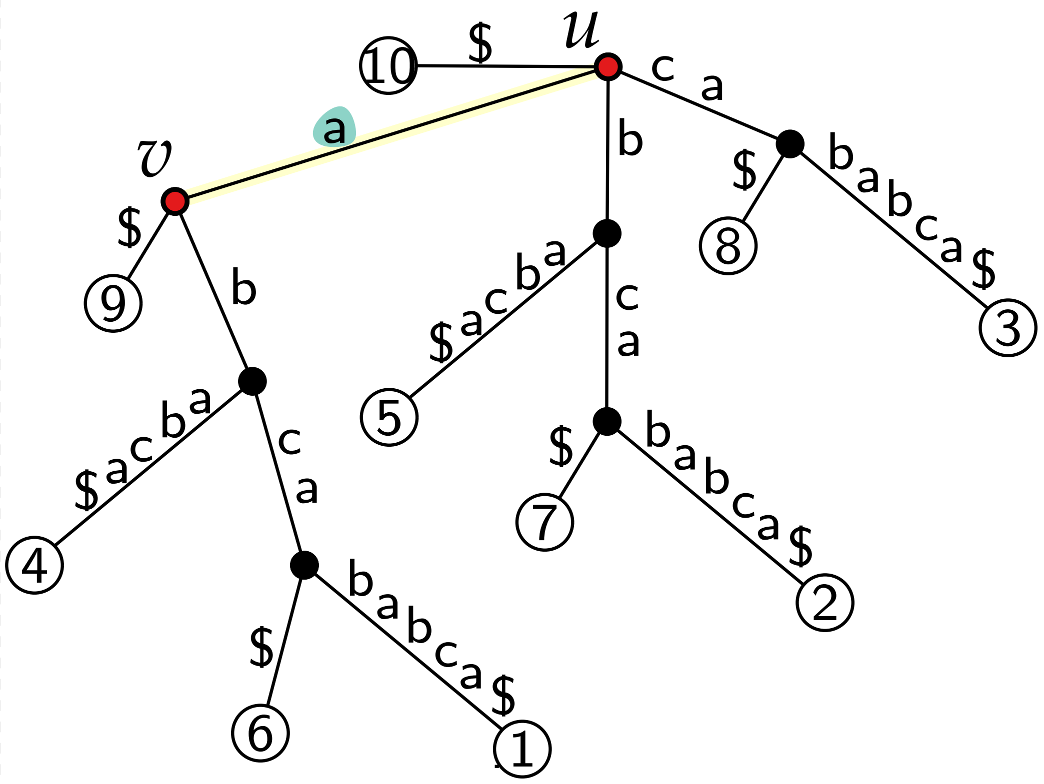
Beispiel: $P = \underset{1}{\text{a}}\ \underset{2}{\text{b}}\ \underset{3}{\text{c}}$

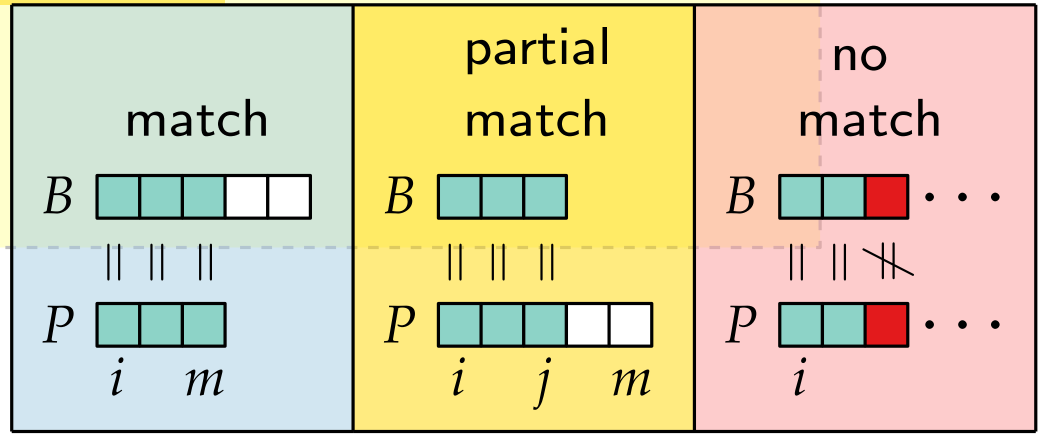
$\uparrow$
i

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
    Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.
    if $e$ does not exist then
      return “no match“
    Compare $B$ with $P[i, m]$
    if $P[i, m]$ is prefix of $B$ then
      return the indices of all leaves in the subtree rooted at $v$
    else if $P[i, j] = B$ for some $j < m$ then
      $i \leftarrow j + 1$
      $u \leftarrow v$
    else
      return “no match“
  return “no match“
```

$T =$ a b c a b a b c a



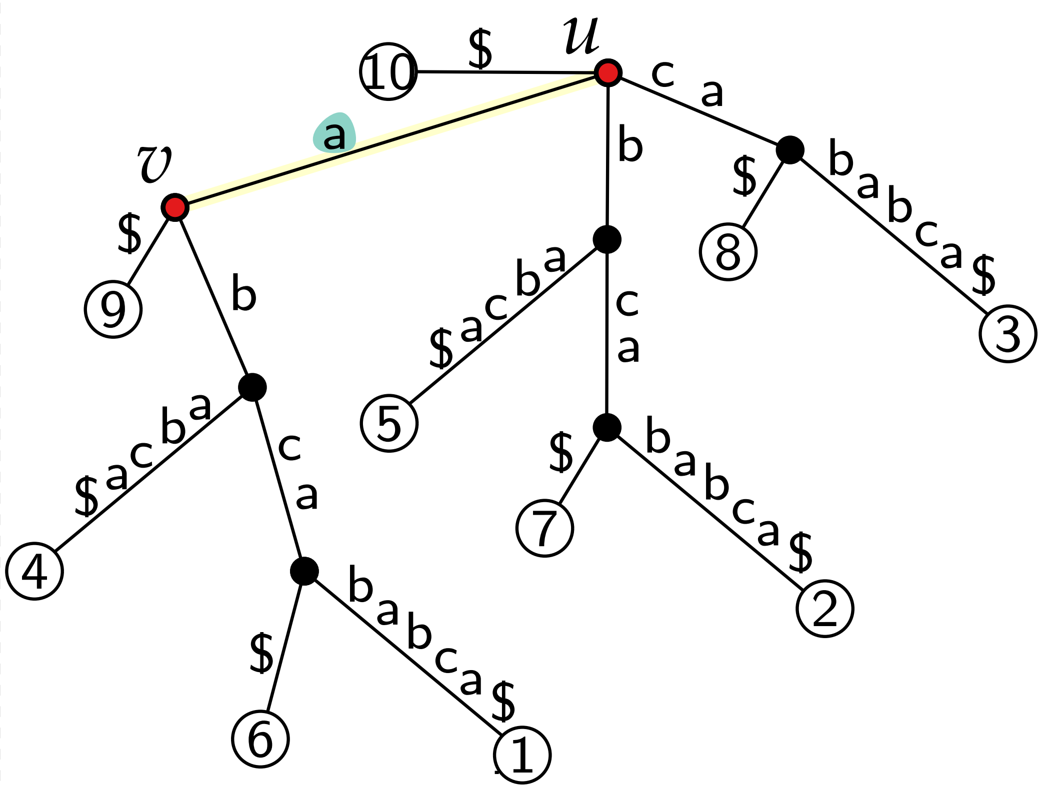


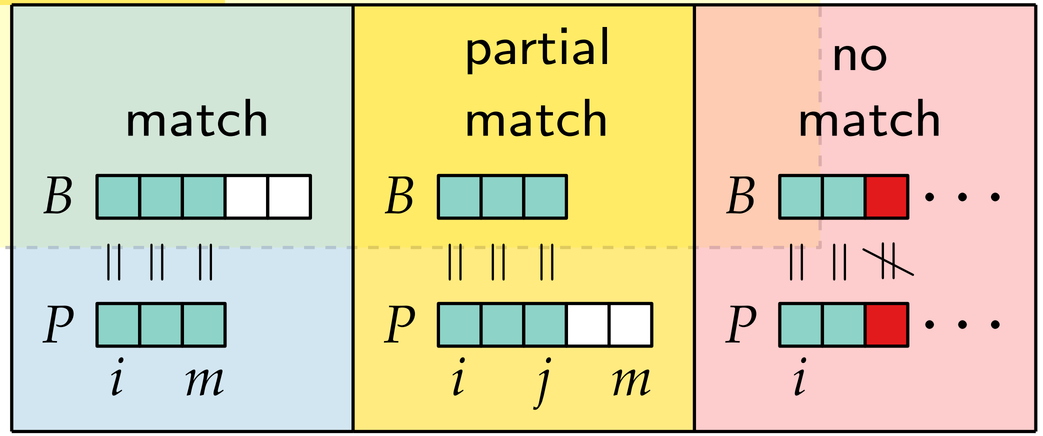
Beispiel: $P =$ a b c
1 2 3
i

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.
      if $e$ does not exist then
          return "no match"
      Compare $B$ with $P[i, m]$
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return "no match"
  return "no match"
```

$T =$ a b c a b a b c a





Beispiel: $P =$ a b c

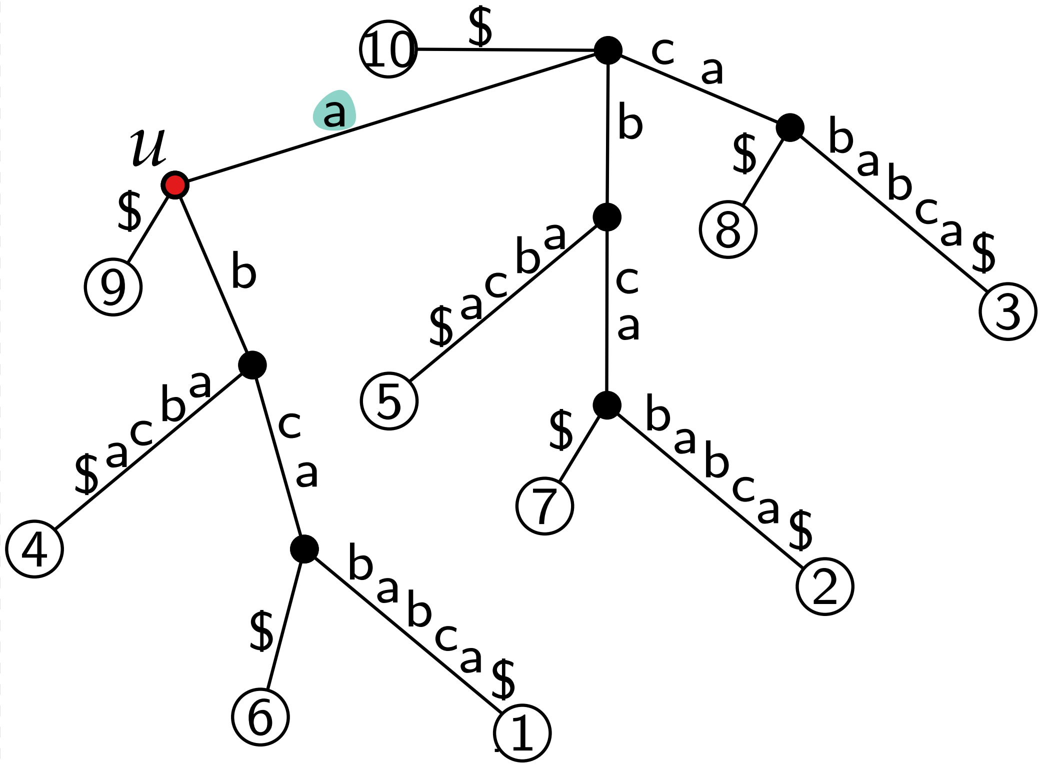
1 2 3

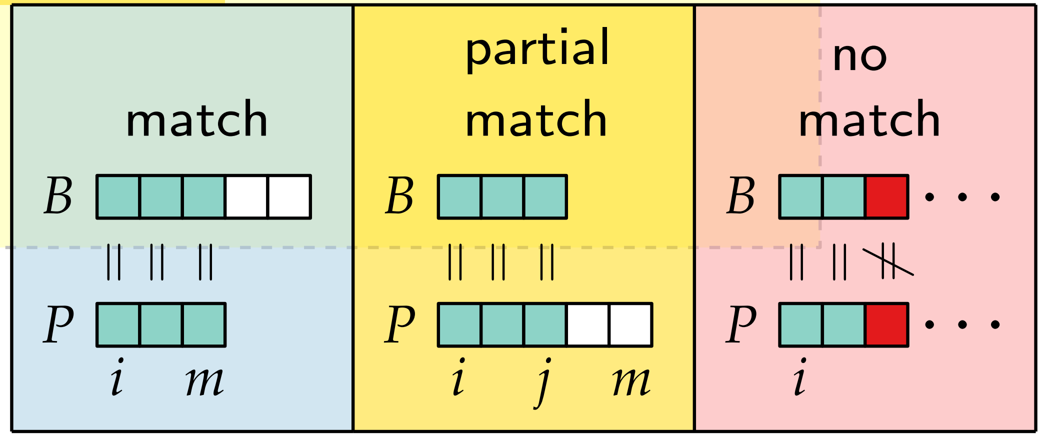
i

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.
      if $e$ does not exist then
          return "no match"
      Compare $B$ with $P[i, m]$
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return "no match"
  return "no match"
```

$T =$ a b c a b a b c a





Beispiel: $P =$ a b c

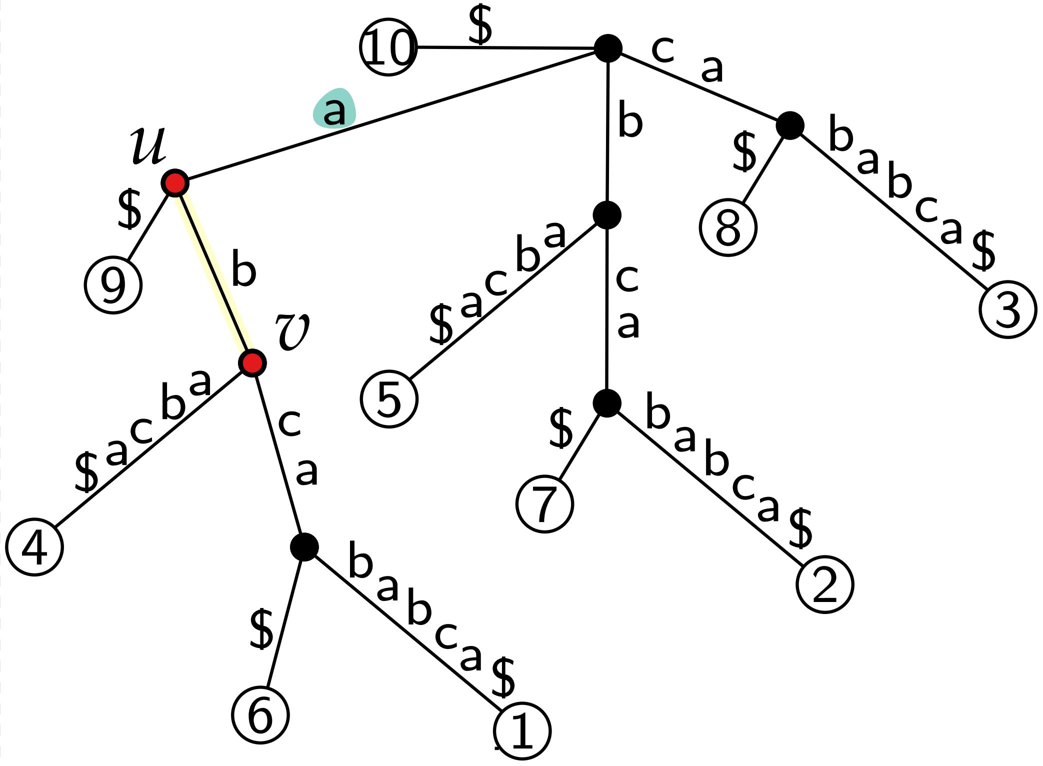
1 2 3

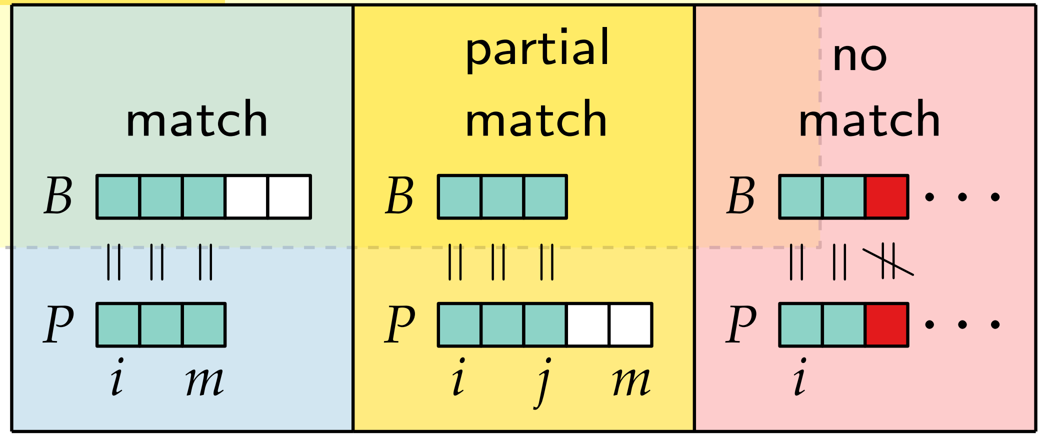
i

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.
      if $e$ does not exist then
          return "no match"
      Compare $B$ with $P[i, m]$
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return "no match"
  return "no match"
```

$T =$ a b c a b a b c a



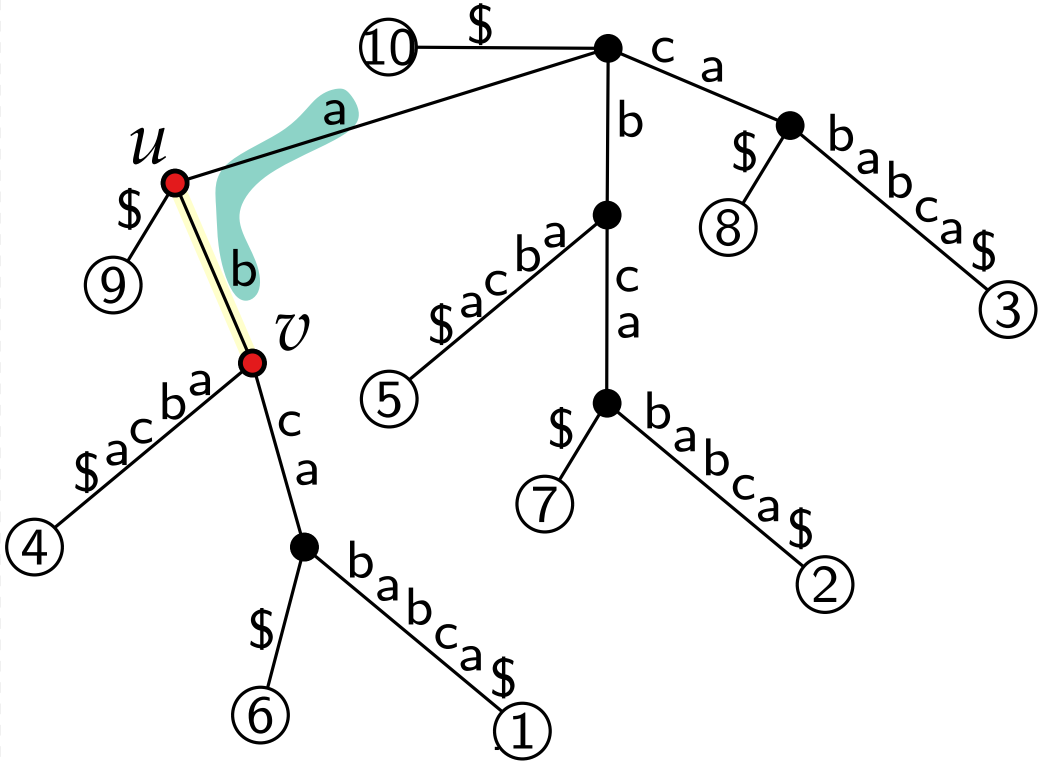


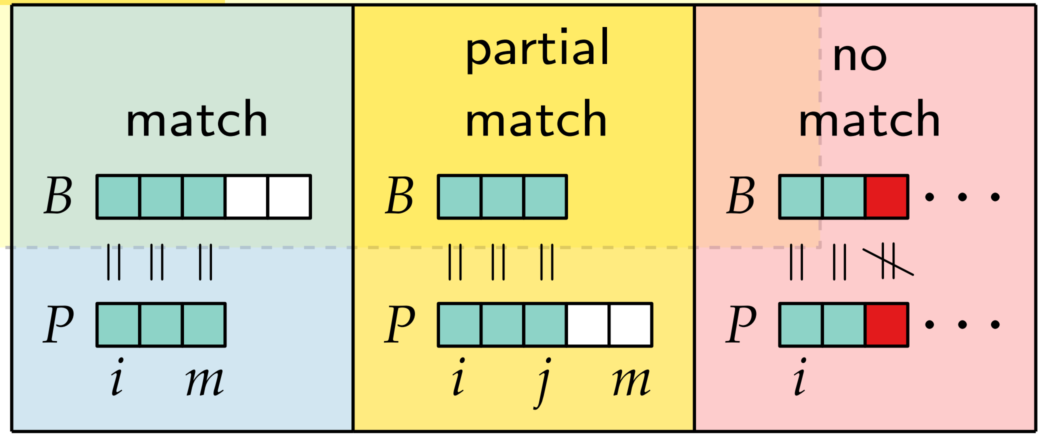
Beispiel: $P =$ a b c

Searching in Suffix Trees

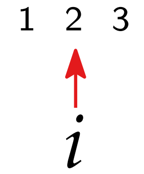
```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.
      if $e$ does not exist then
          return "no match"
      Compare $B$ with $P[i, m]$
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return "no match"
  return "no match"
```

$T =$ a b c a b a b c a





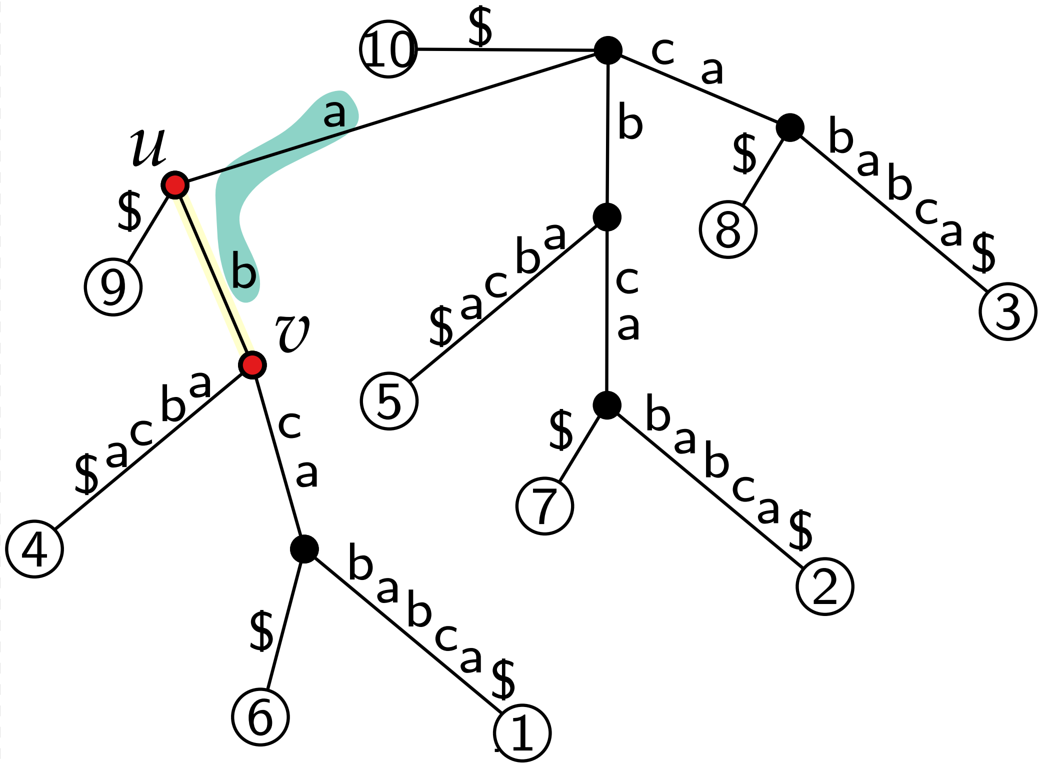
Beispiel: $P =$ a b c

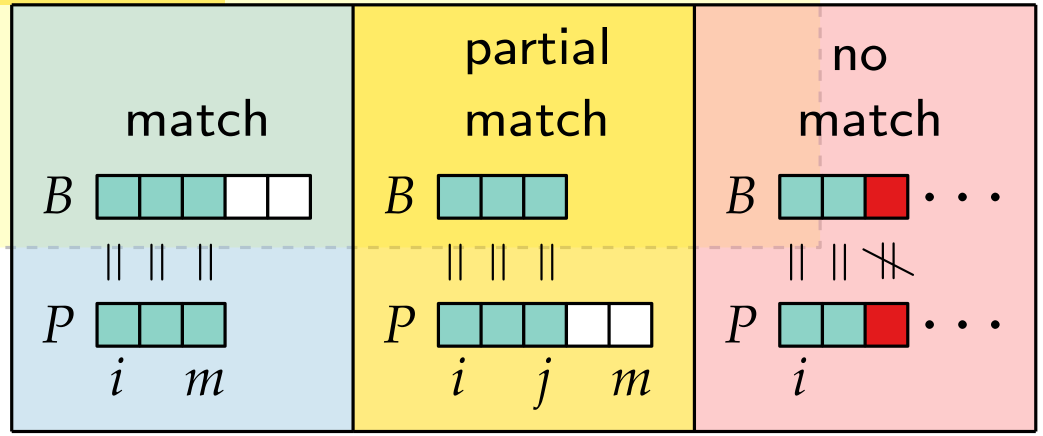


Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.
      if $e$ does not exist then
          return “no match“
      Compare $B$ with $P[i, m]$
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return “no match“
  return “no match“
```

$T =$ a b c a b a b c a





Beispiel: $P =$ a b c

1 2 3

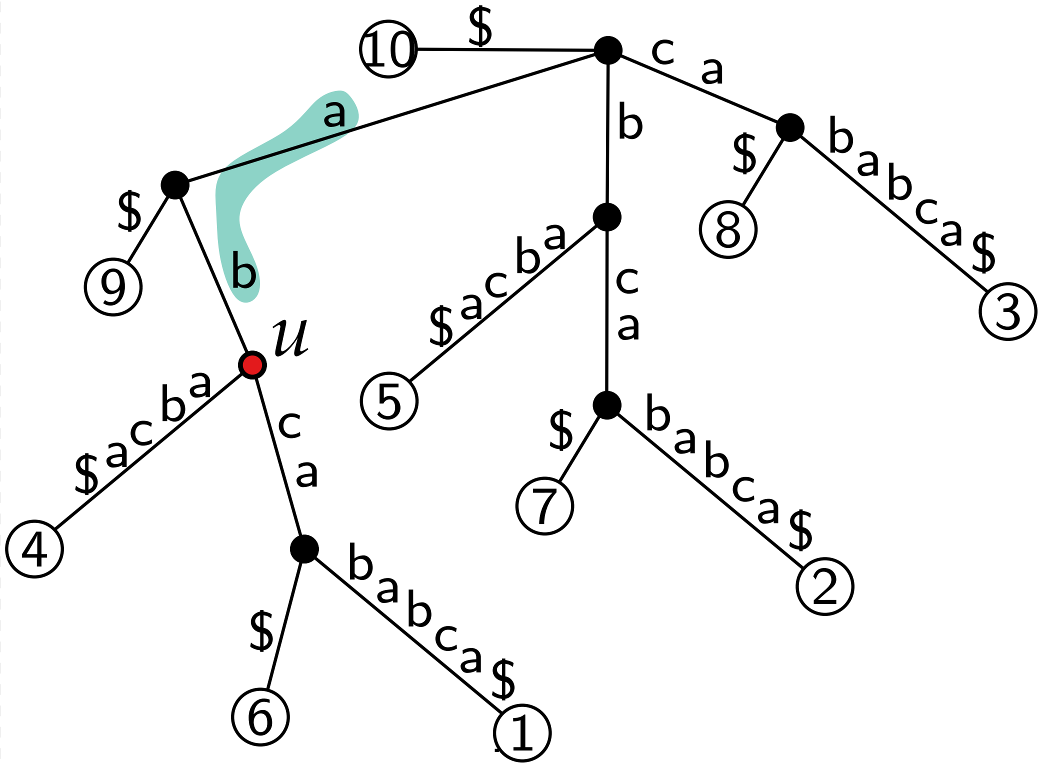
↑

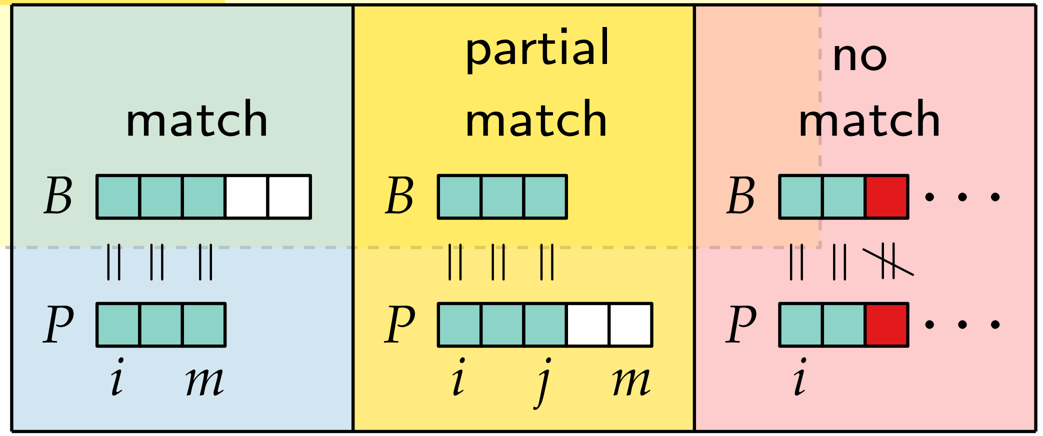
i

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.
      if $e$ does not exist then
          return "no match"
      Compare $B$ with $P[i, m]$
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return "no match"
  return "no match"
```

$T =$ a b c a b a b c a





Beispiel: $P =$ a b c

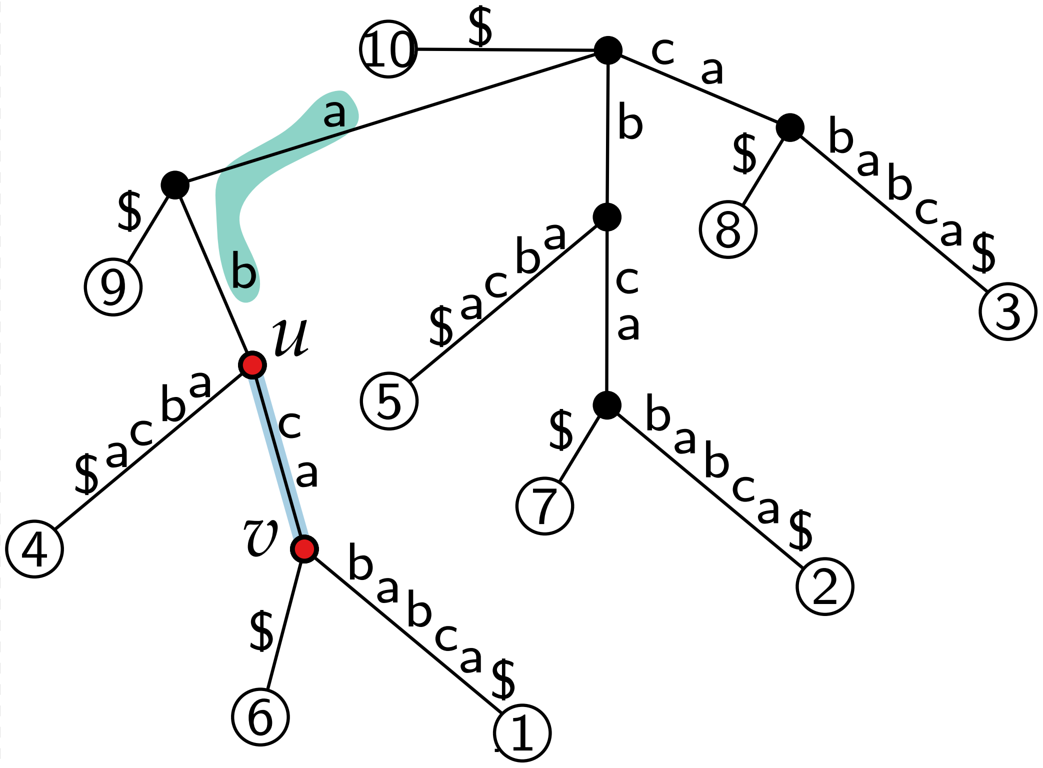
1 2 3

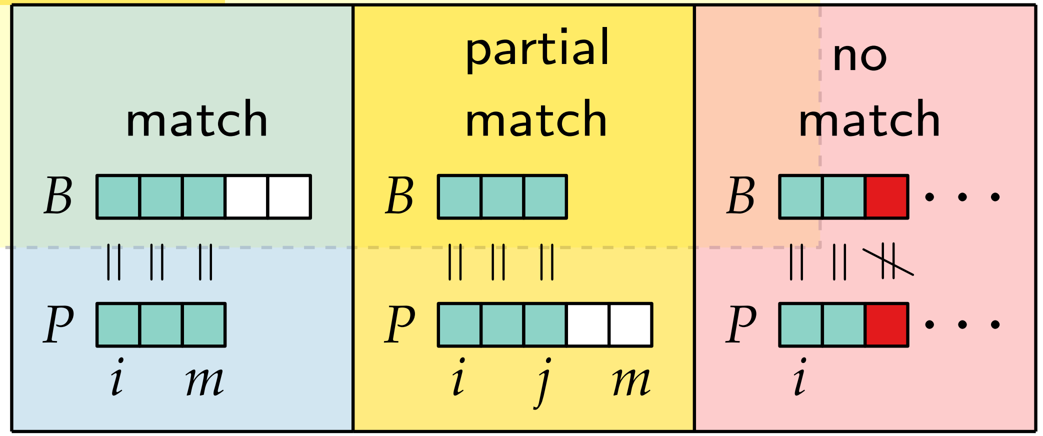
i

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.
      if $e$ does not exist then
          return “no match“
      Compare $B$ with $P[i, m]$
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return “no match“
  return “no match“
```

$T =$ a b c a b a b c a





Beispiel: $P =$ a b c

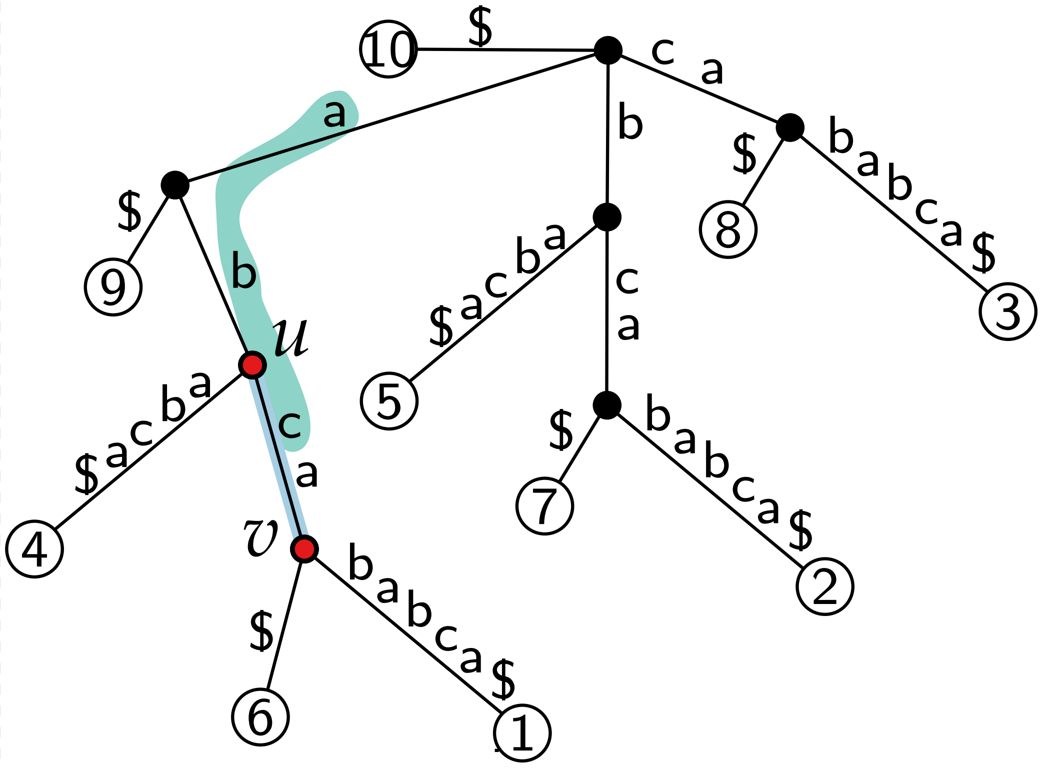
1 2 3

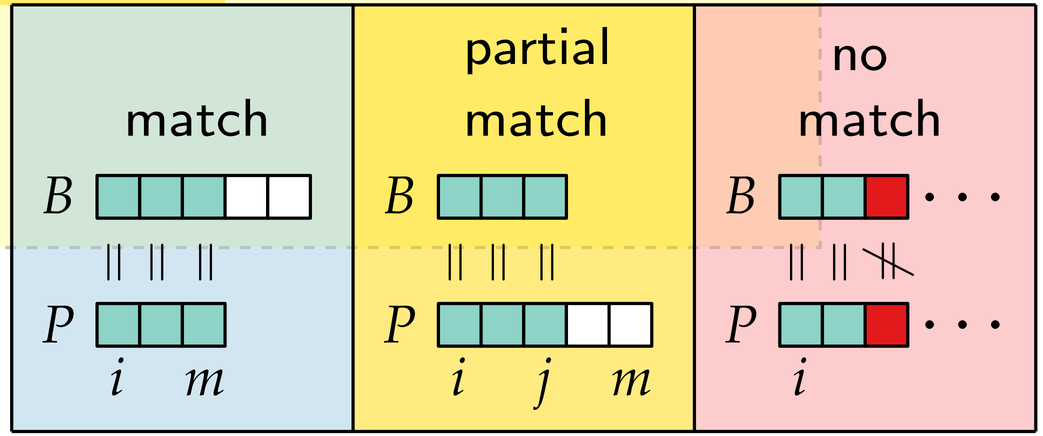
i

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.
      if $e$ does not exist then
          return “no match“
      Compare $B$ with $P[i, m]$
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return “no match“
  return “no match“
```

$T =$ a b c a b a b c a





Beispiel: $P =$ a b c

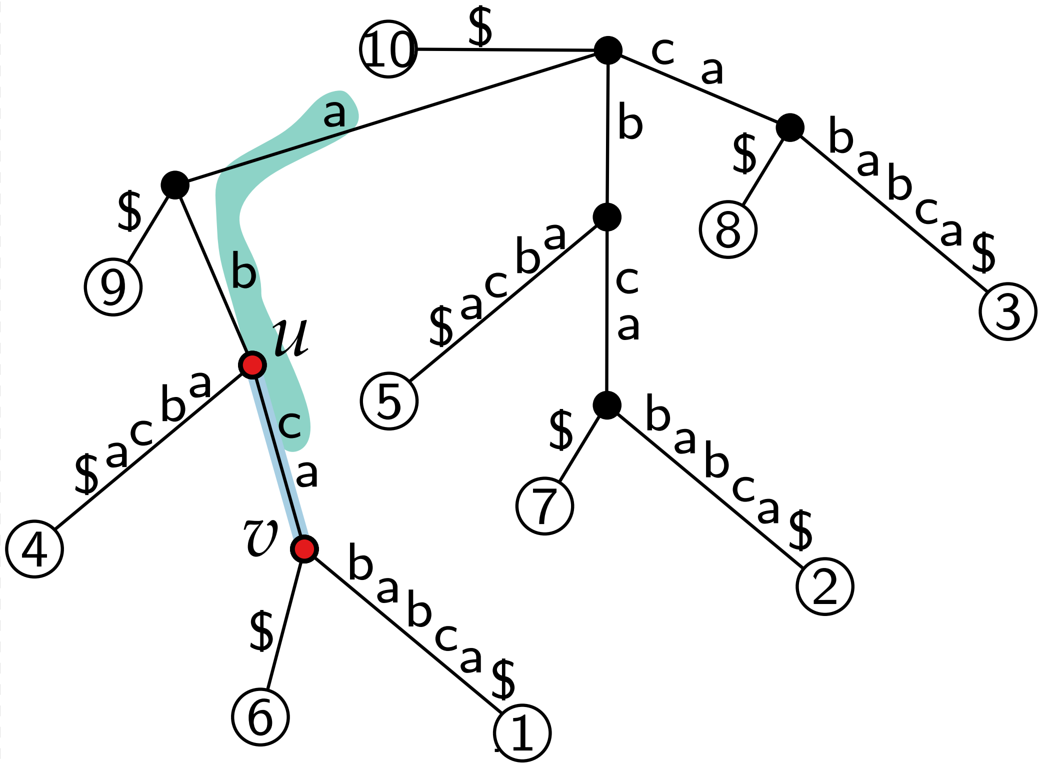
1 2 3

i

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.
      if $e$ does not exist then
          return "no match"
      Compare $B$ with $P[i, m]$
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return "no match"
  return "no match"
```

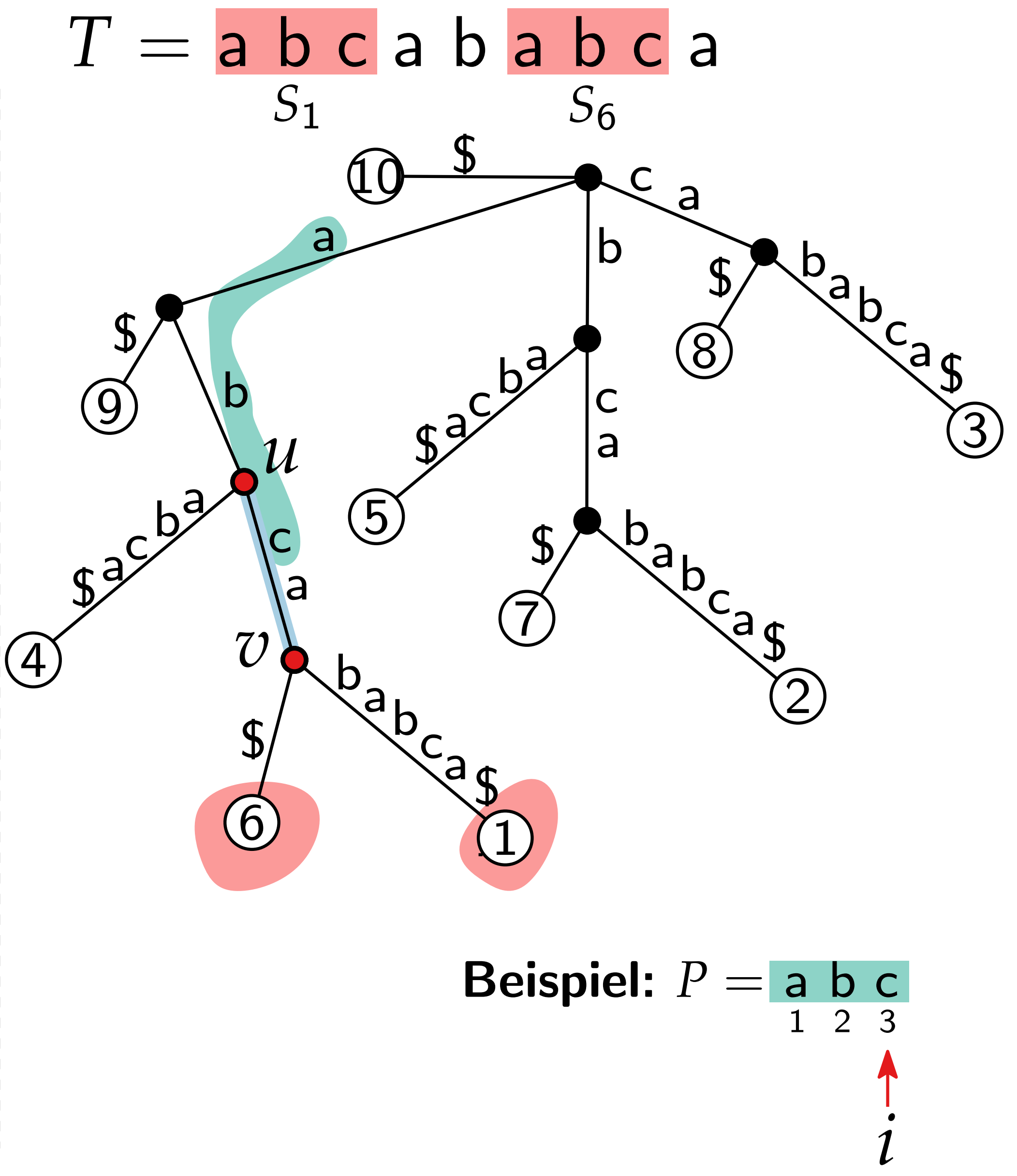
$T =$ a b c a b a b c a



Beispiel: $P =$ a b c
1 2 3
i

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.
      if $e$ does not exist then
          return "no match"
      Compare $B$ with $P[i, m]$
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return "no match"
  return "no match"
```

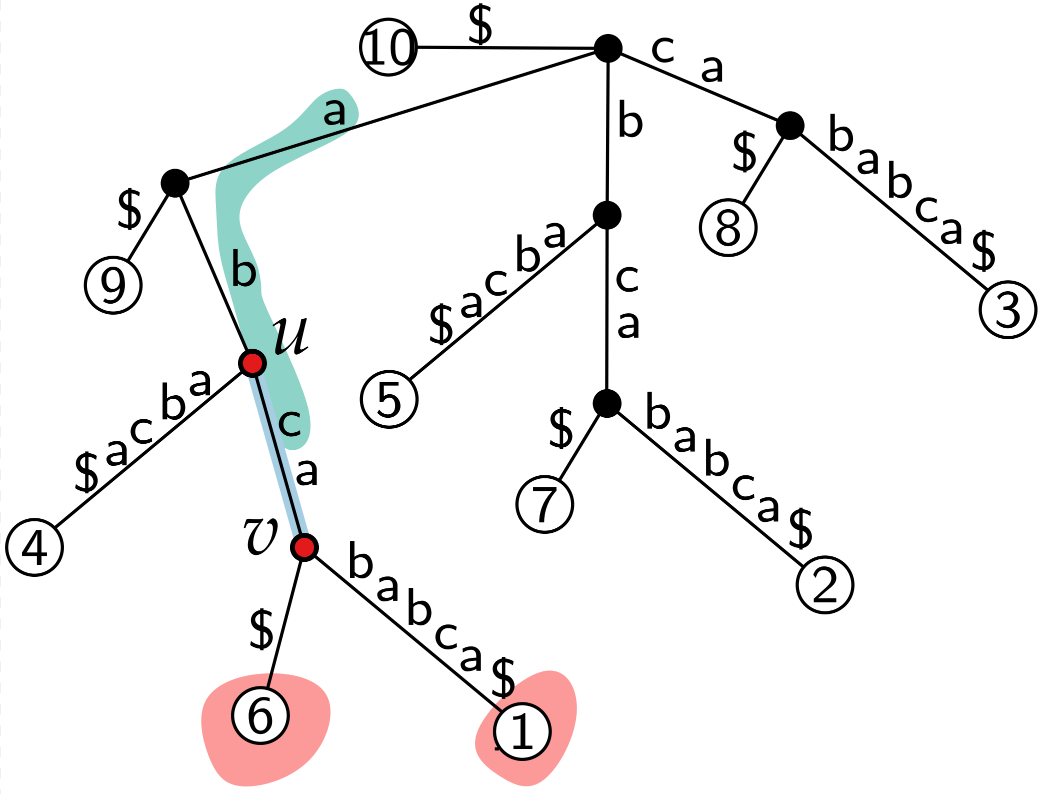


Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.
      if $e$ does not exist then
          return "no match"
      Compare $B$ with $P[i, m]$
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return "no match"
  return "no match"
```

$T =$ a b c a b a b c a

S_1 S_6



Beispiel: $P =$ a b c

1 2 3

i

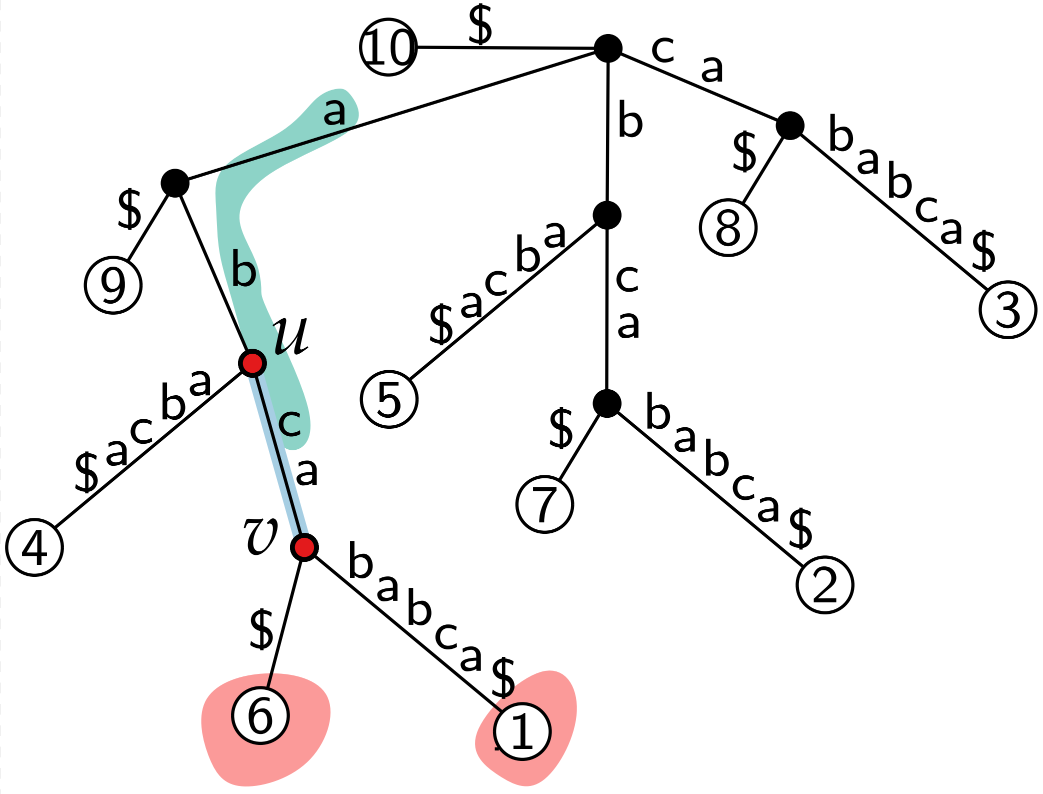
Correctness. Each occurrence of P is a prefix of exactly one suffix of T. We report all suffixes with P as a prefix.

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.
      if $e$ does not exist then
          return "no match"
      Compare $B$ with $P[i, m]$
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return "no match"
  return "no match"
```

$T =$ a b c a b a b c a

S_1 S_6



Beispiel: $P =$ a b c

1 2 3

i

Correctness. Each occurrence of P is a prefix of exactly one suffix of T. We report all suffixes with P as a prefix.

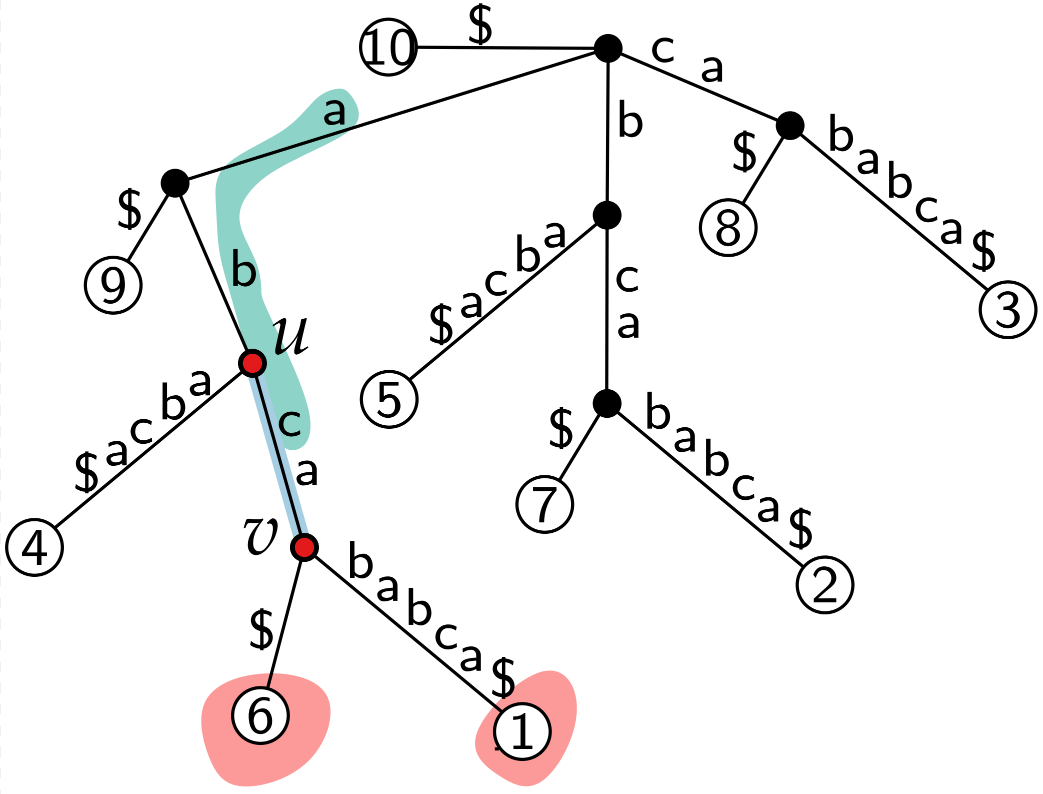
Runtime.

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$. $\mathcal{O}(\log|\Sigma|)$
      if $e$ does not exist then
          return "no match"
      Compare $B$ with $P[i, m]$
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return "no match"
  return "no match"
```

$T =$ a b c a b a b c a

S_1 S_6



Beispiel: $P =$ a b c

1 2 3

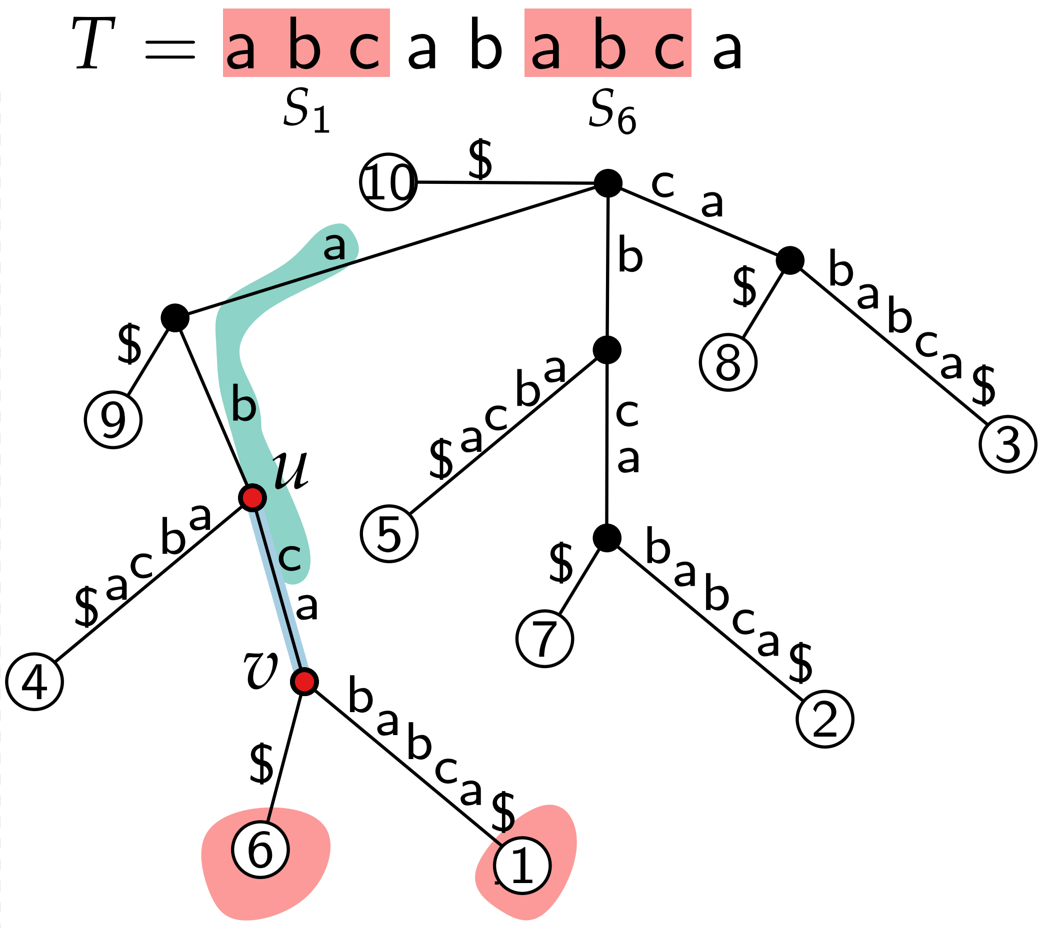
i

Correctness. Each occurrence of P is a prefix of exactly one suffix of T. We report all suffixes with P as a prefix.

Runtime.

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$. $\mathcal{O}(\log |\Sigma|)$
      if $e$ does not exist then
          return "no match"
      Compare $B$ with $P[i, m]$
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return "no match"
  return "no match"
```



Beispiel: $P =$ a b c
1 2 3
i

Correctness. Each occurrence of P is a prefix of exactly one suffix of T. We report all suffixes with P as a prefix.

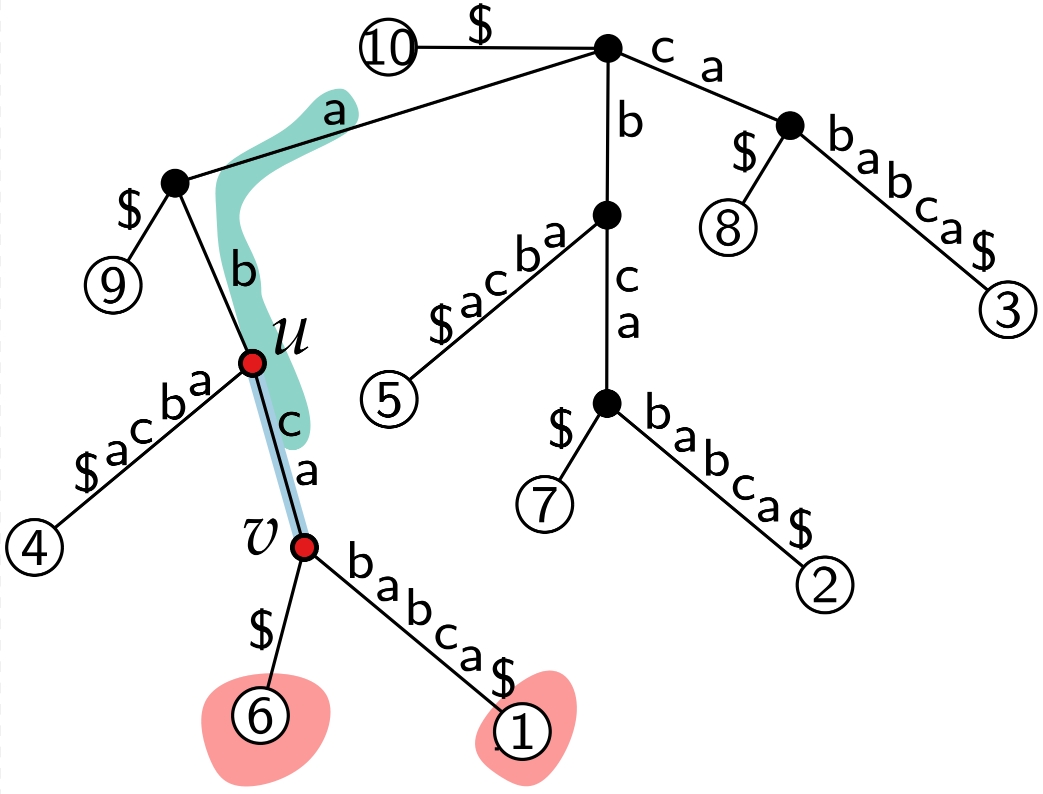
Runtime.

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.   $\mathcal{O}(\log|\Sigma|)$
      if $e$ does not exist then
          return "no match"
      Compare $B$ with $P[i, m]$                                       $m$ comparisons in total
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return "no match"
  return "no match"
```

$T =$ a b c a b a b c a

S_1 S_6



Beispiel: $P =$ a b c

1 2 3

i

Correctness. Each occurrence of P is a prefix of exactly one suffix of T. We report all suffixes with P as a prefix.

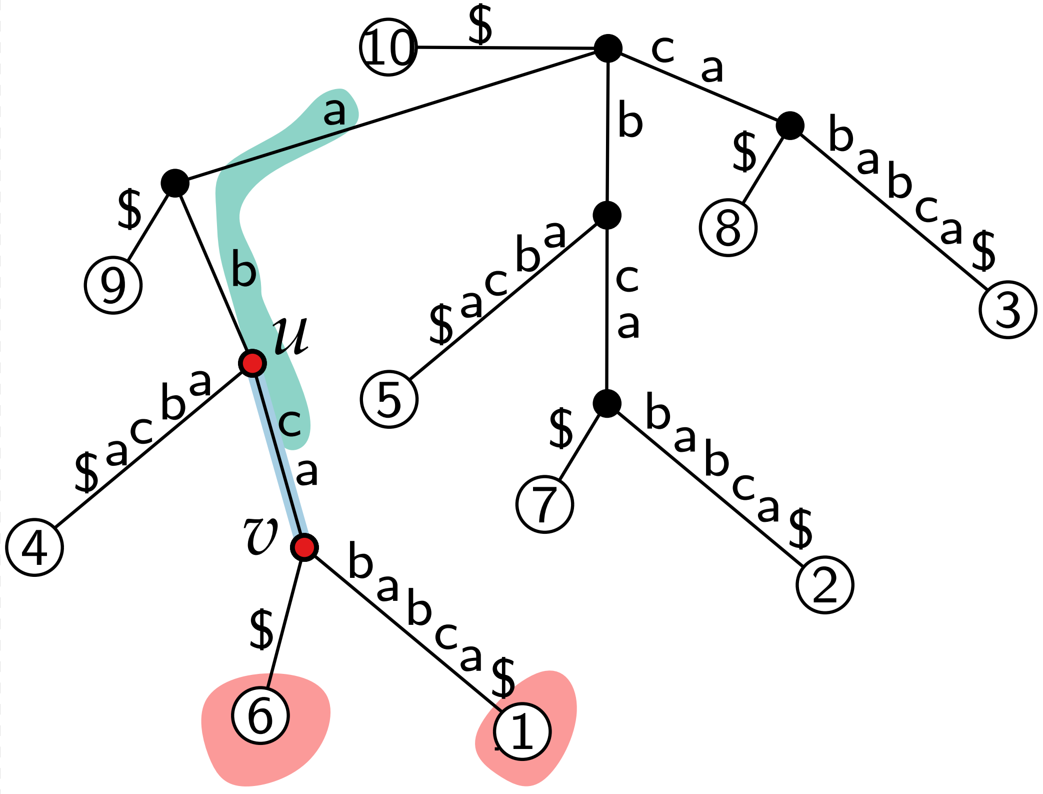
Runtime.

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.   $\mathcal{O}(\log |\Sigma|)$
      if $e$ does not exist then
          return "no match"
      Compare $B$ with $P[i, m]$                    $m$ comparisons in total
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
                                                    $\mathcal{O}(k)$ in total
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return "no match"
  return "no match"
```

$T =$ a b c a b a b c a

S_1 S_6



Beispiel: $P =$ a b c

1 2 3

i

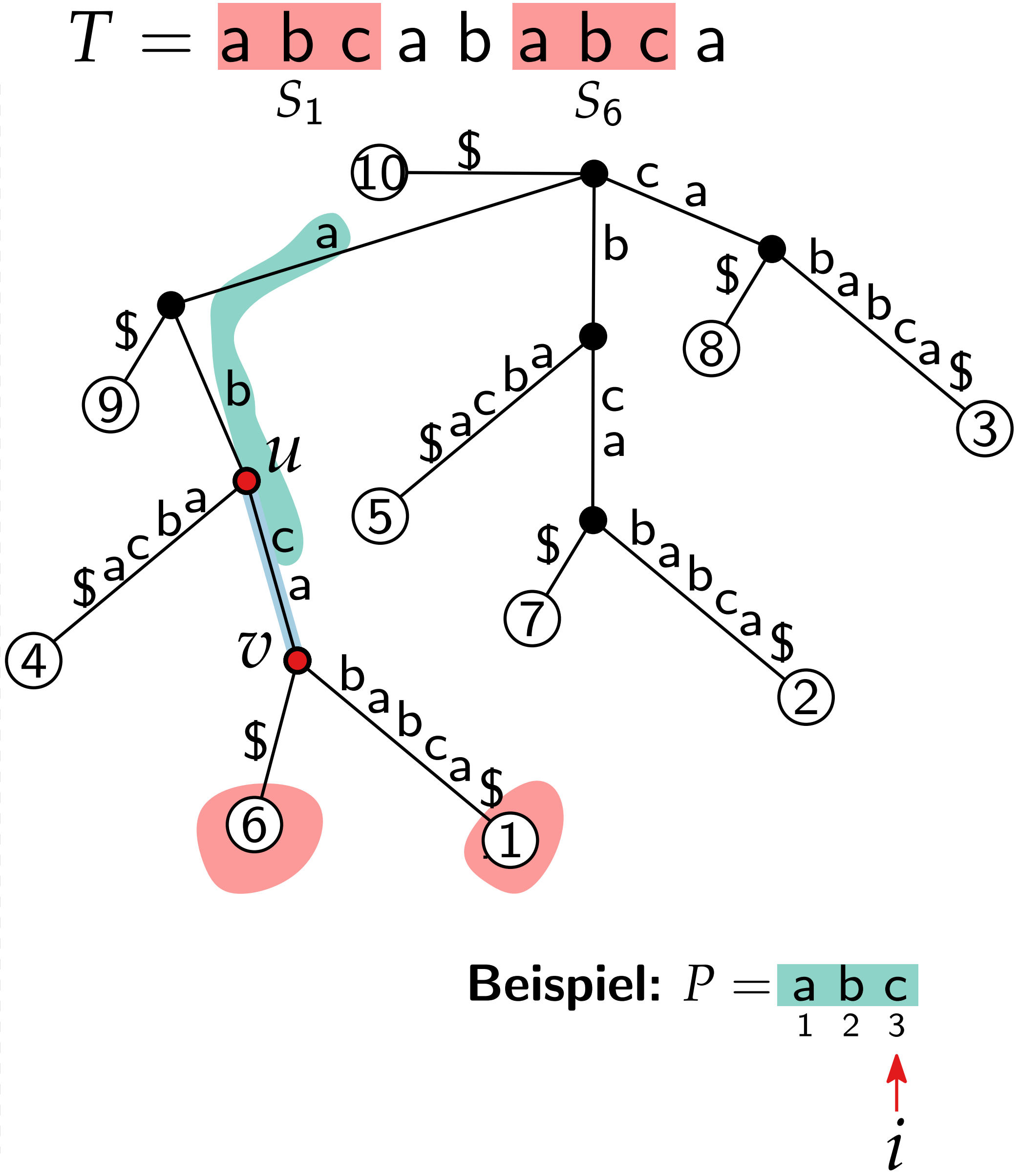
Correctness. Each occurrence of P is a prefix of exactly one suffix of T. We report all suffixes with P as a prefix.

Runtime.

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.      $\mathcal{O}(\log |\Sigma|)$
      if $e$ does not exist then
          return "no match"
      Compare $B$ with $P[i, m]$                                        $m$ comparisons in total
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
                                                                        $\mathcal{O}(k)$ in total
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return "no match"
  return "no match"
```

$\leq m$ iterations



Beispiel: $P =$ a b c
1 2 3
i

Correctness. Each occurrence of P is a prefix of exactly one suffix of T. We report all suffixes with P as a prefix.

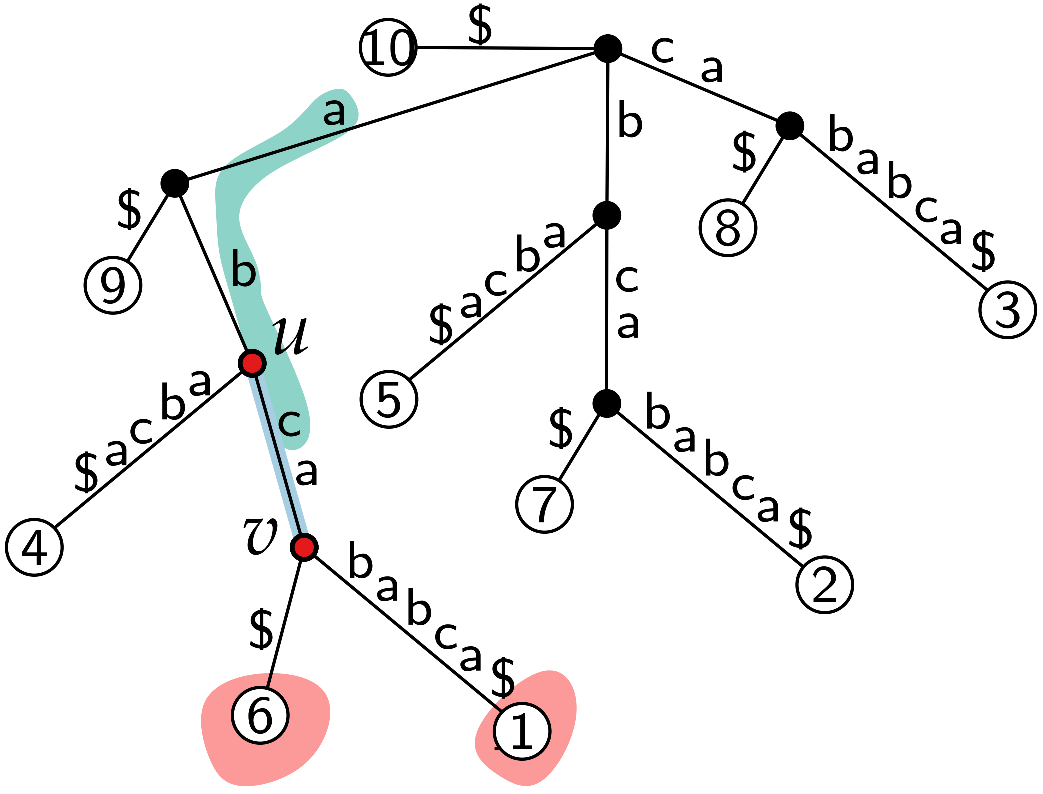
Runtime.

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do                                   $\leq m$ iterations
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.   $\mathcal{O}(\log|\Sigma|)$
      if $e$ does not exist then
          return “no match“
      Compare $B$ with $P[i, m]$                                m comparisons in total
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
                                                                $\mathcal{O}(k)$ in total
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return “no match“
  return “no match“
```

$T =$ a b c a b a b c a

S_1 S_6



Beispiel: $P =$ a b c

1 2 3

i

Correctness. Each occurrence of P is a prefix of exactly one suffix of T. We report all suffixes with P as a prefix.

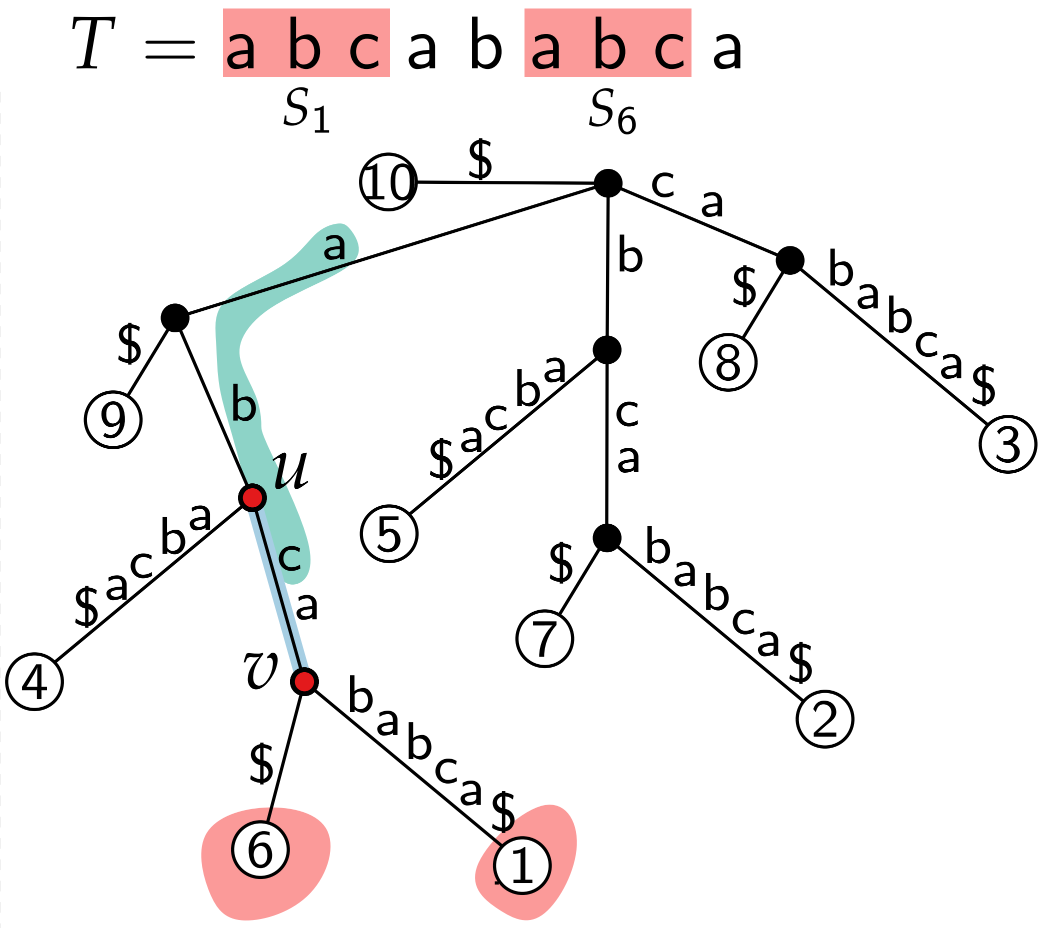
Runtime. $\mathcal{O}(k + m \log|\Sigma|)$, where k is the number of leaves in the subtree rooted at v.

Searching in Suffix Trees

```
SEARCH(suffix tree $S$, string $P$)
  $u \leftarrow$ root of $S$
  $i \leftarrow 1$
  while $u$ is not a leaf do                                        $\leq m$ iterations
      Search edge $e = (u, v)$ whose label $B$ starts with $P[i]$.  $\mathcal{O}(\log|\Sigma|)$
      if $e$ does not exist then
          return “no match“
      Compare $B$ with $P[i, m]$                                     $m$ comparisons in total
      if $P[i, m]$ is prefix of $B$ then
          return the indices of all leaves in the subtree rooted at $v$
                                                                     $\mathcal{O}(k)$ in total
      else if $P[i, j] = B$ for some $j < m$ then
          $i \leftarrow j + 1$
          $u \leftarrow v$
      else
          return “no match“
  return “no match“
```

This is a parameterized, **output-sensitive** algorithm!

$T =$ a b c a b a b c a



Beispiel: $P =$ a b c
1 2 3
i

Correctness. Each occurrence of P is a prefix of exactly one suffix of T. We report all suffixes with P as a prefix.

Runtime. $\mathcal{O}(k + m\log|\Sigma|)$, where k is the number of leaves in the subtree rooted at v.

Constructing Suffix Trees

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix tree S for T.

$T =$ a b c a b a b c a $

Constructing Suffix Trees

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix tree S for T.

Idea. For $i \in \{1, \ldots, n\}$, construct a Σ-tree N_i that contains the suffixes $S_1, \ldots, S_i$.

$T =$ a b c a b a b c a \$

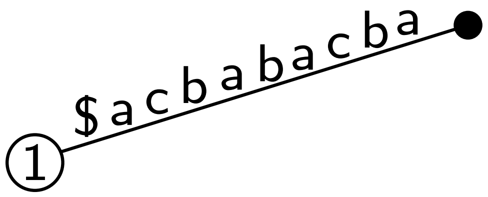
Constructing Suffix Trees

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix tree S for T.

Idea. For $i \in \{1, \ldots, n\}$, construct a Σ-tree N_i that contains the suffixes $S_1, \ldots, S_i$.

Initialization. N_1 consists of a single edge labeled S_1.

$T =$ a b c a b a b c a \$



Constructing Suffix Trees

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix tree S for T.

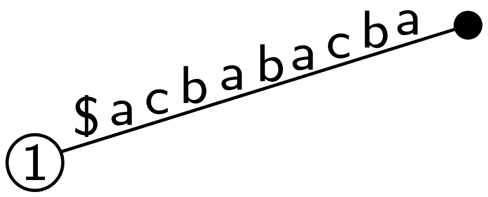
Idea. For $i \in \{1, \dots, n\}$, construct a Σ-tree N_i that contains the suffixes $S_1, \dots, S_i$.

Initialization. N_1 consists of a single edge labeled S_1.

Constructing N_{i+1} from N_i. Search the longest prefix P of S_{i+1} contained in N_i.

- **Case 1.** P ends in the middle of an edge e. Subdivide e and attach a new edge.
- **Case 2.** P ends at a vertex v. Attach a new edge, then re-sort the neighbors of v.

$T =$ a b c a b a b c a \$



Constructing Suffix Trees

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix tree S for T.

Idea. For $i \in \{1, \ldots, n\}$, construct a Σ-tree N_i that contains the suffixes $S_1, \ldots, S_i$.

Initialization. N_1 consists of a single edge labeled S_1.

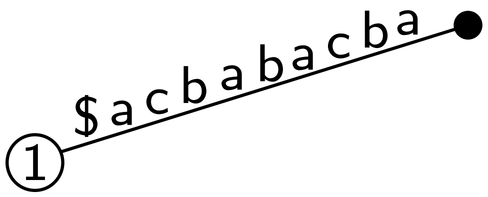
Constructing N_{i+1} from N_i. Search the longest prefix P of S_{i+1} contained in N_i.

Case 1. P ends in the middle of an edge e. Subdivide e and attach a new edge.

Case 2. P ends at a vertex v. Attach a new edge, then re-sort the neighbors of v.

$T =$ a b c a b a b c a \$

S_2



Next step:

Insert $S_2 =$ b c a b a b c a \$:

- Matching ends at the root.
- → Case 2.

Constructing Suffix Trees

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix tree S for T.

Idea. For $i \in \{1, \dots, n\}$, construct a Σ-tree N_i that contains the suffixes $S_1, \dots, S_i$.

Initialization. N_1 consists of a single edge labeled S_1.

Constructing N_{i+1} from N_i. Search the longest prefix P of S_{i+1} contained in N_i.

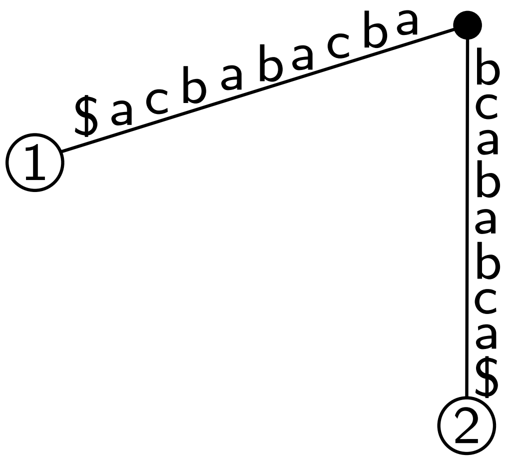
- **Case 1.** P ends in the middle of an edge e. Subdivide e and attach a new edge.
- **Case 2.** P ends at a vertex v. Attach a new edge, then re-sort the neighbors of v.

$T =$ a b c a b a b c a \$

S_3

Next step:
Insert $S_3 =$ c a b a b c a \$:

- Matching ends at the root.
- → Case 2.



Constructing Suffix Trees

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix tree S for T.

Idea. For $i \in \{1, \dots, n\}$, construct a Σ-tree N_i that contains the suffixes $S_1, \dots, S_i$.

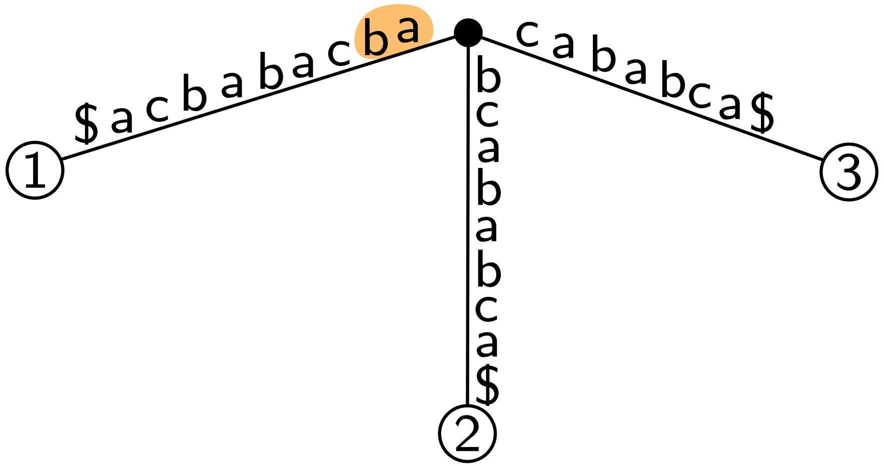
Initialization. N_1 consists of a single edge labeled S_1.

Constructing N_{i+1} from N_i. Search the longest prefix P of S_{i+1} contained in N_i.

- **Case 1.** P ends in the middle of an edge e. Subdivide e and attach a new edge.
- **Case 2.** P ends at a vertex v. Attach a new edge, then re-sort the neighbors of v.

$T =$ a b c a b a b c a $

S_4



Next step:
Insert $S_4 =$ a b a b c a $:

- Matching ends along S_1 after 2 symbols.
- → Case 1.

Constructing Suffix Trees

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix tree S for T.

Idea. For $i \in \{1, \dots, n\}$, construct a Σ-tree N_i that contains the suffixes $S_1, \dots, S_i$.

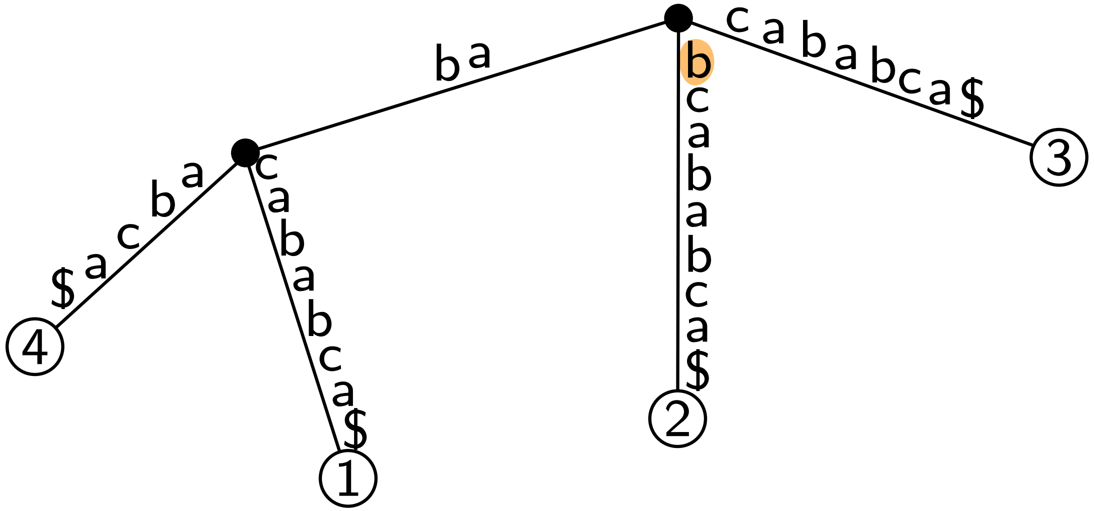
Initialization. N_1 consists of a single edge labeled S_1.

Constructing N_{i+1} from N_i. Search the longest prefix P of S_{i+1} contained in N_i.

- **Case 1.** P ends in the middle of an edge e. Subdivide e and attach a new edge.
- **Case 2.** P ends at a vertex v. Attach a new edge, then re-sort the neighbors of v.

$T =$ a b c a b a b c a \$

S_5



Next step:

Insert $S_5 =$ b a b c a \$:

- Matching ends along S_2 after 1 symbol.
- → Case 1.

Constructing Suffix Trees

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix tree S for T.

Idea. For $i \in \{1, \ldots, n\}$, construct a Σ-tree N_i that contains the suffixes $S_1, \ldots, S_i$.

Initialization. N_1 consists of a single edge labeled S_1.

Constructing N_{i+1} from N_i. Search the longest prefix P of S_{i+1} contained in N_i.

Case 1. P ends in the middle of an edge e. Subdivide e and attach a new edge.

Case 2. P ends at a vertex v. Attach a new edge, then re-sort the neighbors of v.

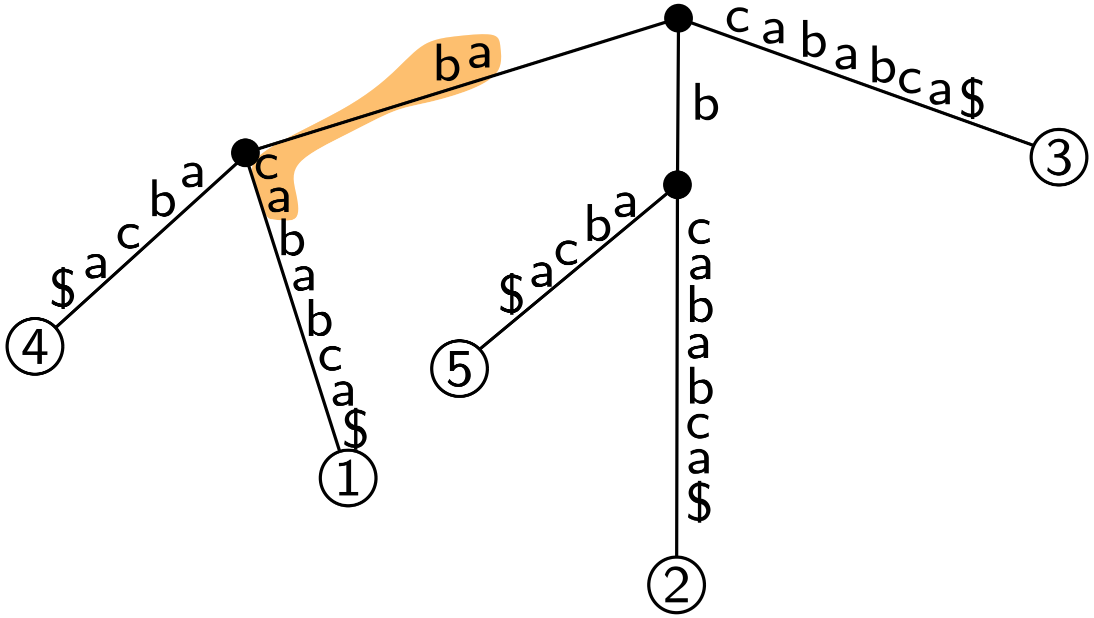
$T =$ a b c a b a b c a \$

S_6

Next step:

Insert $S_6 =$ a b c a \$:

- Matching ends along S_1 after 4 symbols.
- → Case 1.



Constructing Suffix Trees

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix tree S for T.

Idea. For $i \in \{1, \dots, n\}$, construct a Σ-tree N_i that contains the suffixes $S_1, \dots, S_i$.

Initialization. N_1 consists of a single edge labeled S_1.

Constructing N_{i+1} from N_i. Search the longest prefix P of S_{i+1} contained in N_i.

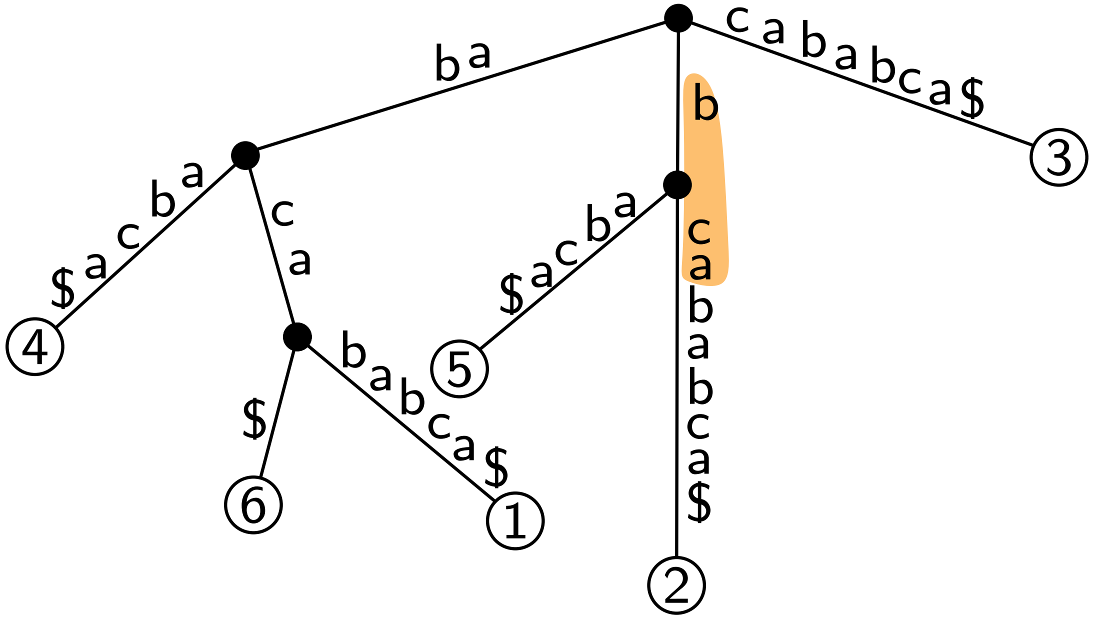
- **Case 1.** P ends in the middle of an edge e. Subdivide e and attach a new edge.
- **Case 2.** P ends at a vertex v. Attach a new edge, then re-sort the neighbors of v.

$T =$ a b c a b a b c a \$

S_7

Next step:
Insert $S_7 =$ b c a \$:

- Matching ends along S_2 after 3 symbols.
- → Case 1.



Constructing Suffix Trees

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix tree S for T.

Idea. For $i \in \{1, \dots, n\}$, construct a Σ-tree N_i that contains the suffixes $S_1, \dots, S_i$.

Initialization. N_1 consists of a single edge labeled S_1.

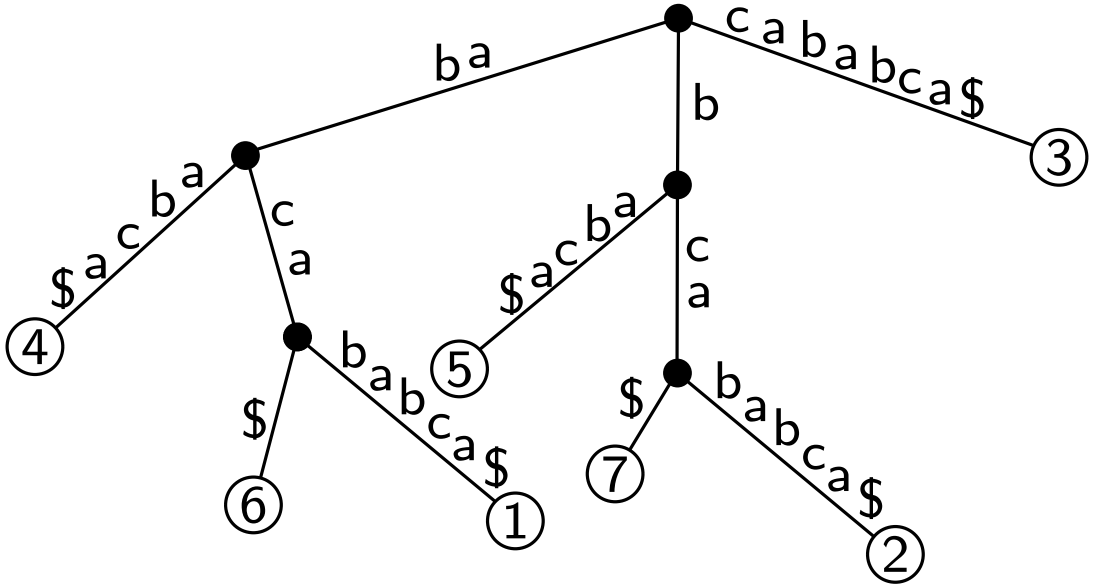
Constructing N_{i+1} from N_i. Search the longest prefix P of S_{i+1} contained in N_i.

Case 1. P ends in the middle of an edge e. Subdivide e and attach a new edge.

Case 2. P ends at a vertex v. Attach a new edge, then re-sort the neighbors of v.

$T =$ a b c a b a b c a \$

Proceed similarly with S_8, S_9, and S_{10}.



Constructing Suffix Trees

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix tree S for T.

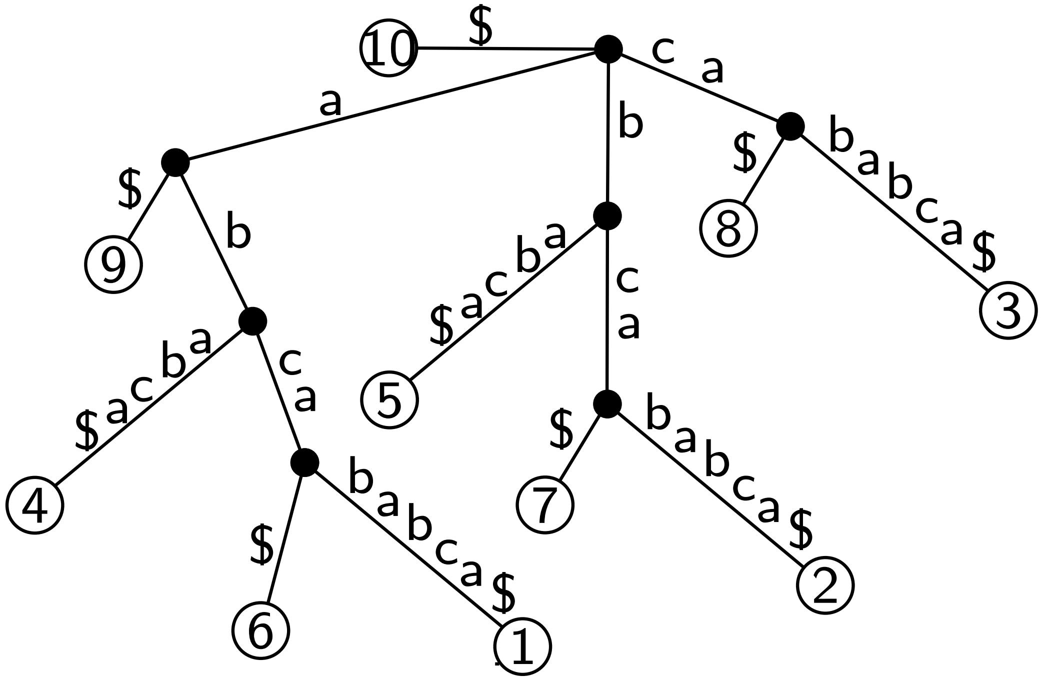
Idea. For $i \in \{1, \dots, n\}$, construct a Σ-tree N_i that contains the suffixes $S_1, \dots, S_i$.

Initialization. N_1 consists of a single edge labeled S_1.

Constructing N_{i+1} from N_i. Search the longest prefix P of S_{i+1} contained in N_i.

- **Case 1.** P ends in the middle of an edge e. Subdivide e and attach a new edge.
- **Case 2.** P ends at a vertex v. Attach a new edge, then re-sort the neighbors of v.

$T =$ a b c a b a b c a \$



Constructing Suffix Trees

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix tree S for T.

Idea. For $i \in \{1, \ldots, n\}$, construct a Σ-tree N_i that contains the suffixes $S_1, \ldots, S_i$.

Initialization. N_1 consists of a single edge labeled S_1.

Constructing N_{i+1} from N_i. Search the longest prefix P of S_{i+1} contained in N_i.

Case 1. P ends in the middle of an edge e. Subdivide e and attach a new edge.

Case 2. P ends at a vertex v. Attach a new edge, then re-sort the neighbors of v.

Running time.

$$\mathcal{O}\Bigg(\underbrace{\hspace{10cm}}_{\text{searching } P} + \underset{\nwarrow \text{ re-sorting neighbors of } v}{\hspace{2cm}}\Bigg)$$

Constructing Suffix Trees

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix tree S for T.

Idea. For $i \in \{1, \dots, n\}$, construct a Σ-tree N_i that contains the suffixes $S_1, \dots, S_i$.

Initialization. N_1 consists of a single edge labeled S_1.

Constructing N_{i+1} from N_i. Search the longest prefix P of S_{i+1} contained in N_i.

- **Case 1.** P ends in the middle of an edge e. Subdivide e and attach a new edge.
- **Case 2.** P ends at a vertex v. Attach a new edge, then re-sort the neighbors of v.

Running time.

$$\mathcal{O}\Big(\underbrace{\big((n-1)+(n-2)+\cdots+1\big)\log|\Sigma|}_{\text{searching } P} + \qquad\quad \Big)$$

re-sorting neighbors of v

Constructing Suffix Trees

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix tree S for T.

Idea. For $i \in \{1, \ldots, n\}$, construct a Σ-tree N_i that contains the suffixes $S_1, \ldots, S_i$.

Initialization. N_1 consists of a single edge labeled S_1.

Constructing N_{i+1} from N_i. Search the longest prefix P of S_{i+1} contained in N_i.

- **Case 1.** P ends in the middle of an edge e. Subdivide e and attach a new edge.
- **Case 2.** P ends at a vertex v. Attach a new edge, then re-sort the neighbors of v.

Running time.

$$\mathcal{O}\Big(\underbrace{\big((n-1) + (n-2) + \cdots + 1\big) \log |\Sigma|}_{\text{searching } P} + \qquad \Big)$$

re-sorting neighbors of v
(via Bucket Sort)

Constructing Suffix Trees

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix tree S for T.

Idea. For $i \in \{1, \ldots, n\}$, construct a Σ-tree N_i that contains the suffixes $S_1, \ldots, S_i$.

Initialization. N_1 consists of a single edge labeled S_1.

Constructing N_{i+1} from N_i. Search the longest prefix P of S_{i+1} contained in N_i.

- **Case 1.** P ends in the middle of an edge e. Subdivide e and attach a new edge.
- **Case 2.** P ends at a vertex v. Attach a new edge, then re-sort the neighbors of v.

Running time.

$$\mathcal{O}\Big(\underbrace{\big((n-1)+(n-2)+\cdots+1\big)\log|\Sigma|}_{\text{searching } P}+n|\Sigma|\Big)$$

re-sorting neighbors of v
(via Bucket Sort)

Constructing Suffix Trees

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix tree S for T.

Idea. For $i \in \{1, \dots, n\}$, construct a Σ-tree N_i that contains the suffixes $S_1, \dots, S_i$.

Initialization. N_1 consists of a single edge labeled S_1.

Constructing N_{i+1} from N_i. Search the longest prefix P of S_{i+1} contained in N_i.

- **Case 1.** P ends in the middle of an edge e. Subdivide e and attach a new edge.
- **Case 2.** P ends at a vertex v. Attach a new edge, then re-sort the neighbors of v.

Running time.

$$\mathcal{O}\Big(\underbrace{\big((n-1)+(n-2)+\cdots+1\big)\log|\Sigma|}_{\text{searching } P} + n|\Sigma|\Big) \subseteq \mathcal{O}(\qquad\qquad)$$

re-sorting neighbors of v
(via Bucket Sort)

Constructing Suffix Trees

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix tree S for T.

Idea. For $i \in \{1, \dots, n\}$, construct a Σ-tree N_i that contains the suffixes $S_1, \dots, S_i$.

Initialization. N_1 consists of a single edge labeled S_1.

Constructing N_{i+1} from N_i. Search the longest prefix P of S_{i+1} contained in N_i.

- **Case 1.** P ends in the middle of an edge e. Subdivide e and attach a new edge.
- **Case 2.** P ends at a vertex v. Attach a new edge, then re-sort the neighbors of v.

Running time.

$$\mathcal{O}\Big(\underbrace{((n-1)+(n-2)+\dots+1)\log|\Sigma|}_{\text{searching } P} + n|\Sigma|\Big) \subseteq \mathcal{O}(n^2 \log|\Sigma|)$$

re-sorting neighbors of v
(via Bucket Sort)

Constructing Suffix Trees

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix tree S for T.

Idea. For $i \in \{1, \dots, n\}$, construct a Σ-tree N_i that contains the suffixes $S_1, \dots, S_i$.

Initialization. N_1 consists of a single edge labeled S_1.

Constructing N_{i+1} from N_i. Search the longest prefix P of S_{i+1} contained in N_i.

Case 1. P ends in the middle of an edge e. Subdivide e and attach a new edge.

Case 2. P ends at a vertex v. Attach a new edge, then re-sort the neighbors of v.

Running time.

$$\mathcal{O}\Big(\big((n-1)+(n-2)+\dots+1\big)\log|\Sigma| + n|\Sigma|\Big) \subseteq \mathcal{O}(n^2 \log|\Sigma|)$$

It is possible to construct suffix trees in $\mathcal{O}(n)$ time, either

- directly, e.g., with an algorithm by Farach (1997); or
- indirectly, by first constructing a **suffix array**, e.g., with an algorithm by Kärkkäinen and Sanders (2003).

Suffix Arrays

$T = \text{a b c a b a b c a \$}$

$A =$	10	9	4	6	1	5	7	2	8	3
	$	a$	ababca$	abca$	abcababca$	babca$	bca$	bcababca$	ca$	cababca$

A **suffix array** A of a text T with $n = |T|$ stores a permutation of the indices $\{1, 2, \ldots, n\}$ s.t. $S_{A[i]}$ **is the** i**-th suffix** of T in lexicographical order.

Suffix Arrays

$T =$ a b c a b a b c a $

$A =$	10	9	4	6	1	5	7	2	8	3
	$	a$	ababca$	abca$	abcababca$	babca$	bca$	bcababca$	ca$	cababca$

A **suffix array** A of a text T with $n = |T|$ stores a permutation of the indices $\{1, 2, \ldots, n\}$ s.t. $S_{A[i]}$ **is the i-th suffix** of T in lexicographical order.

$S_{A[i-1]} < S_{A[i]}$ for each $1 < i \leq n$

Suffix Arrays

$T = $ a b c a b a b c a $

$A = $

10	9	4	6	1	5	7	2	8	3
$	a $	a b a b c a $	a b c a $	a b c a b a b c a $	b a b c a $	b c a $	b c a b a b c a $	c a $	c a b a b c a $

A **suffix array** A of a text T with $n = |T|$ stores a permutation of the indices $\{1, 2, \ldots, n\}$ s.t. $S_{A[i]}$ **is the i-th suffix** of T in lexicographical order.

$S_{A[i-1]} < S_{A[i]}$ for each $1 < i \le n$

Convention. $ is the smallest letter.

Suffix Arrays

$T =$ a b c a b a b c a \$

$A =$	10	9	4	6	1	5	7	2	8	3
	\$	a\$	ababca\$	abca\$	abcababca\$	babca\$	bca\$	bcababca\$	ca\$	cababca\$

A **suffix array** A of a text T with $n = |T|$ stores a permutation of the indices $\{1, 2, \ldots, n\}$ s.t. $S_{A[i]}$ **is the i-th suffix** of T in lexicographical order.

$S_{A[i-1]} < S_{A[i]}$ for each $1 < i \leq n$

Convention. \$ is the smallest letter.

Properties.

- The entries of A correspond to a lexicographical sorting of the suffixes of T.

Suffix Arrays

$T = \text{a b c a b a b c a \$}$

$A =$	10	9	4	6	1	5	7	2	8	3
	$	a$	ababca$	abca$	abcababca$	babca$	bca$	bcababca$	ca$	cababca$

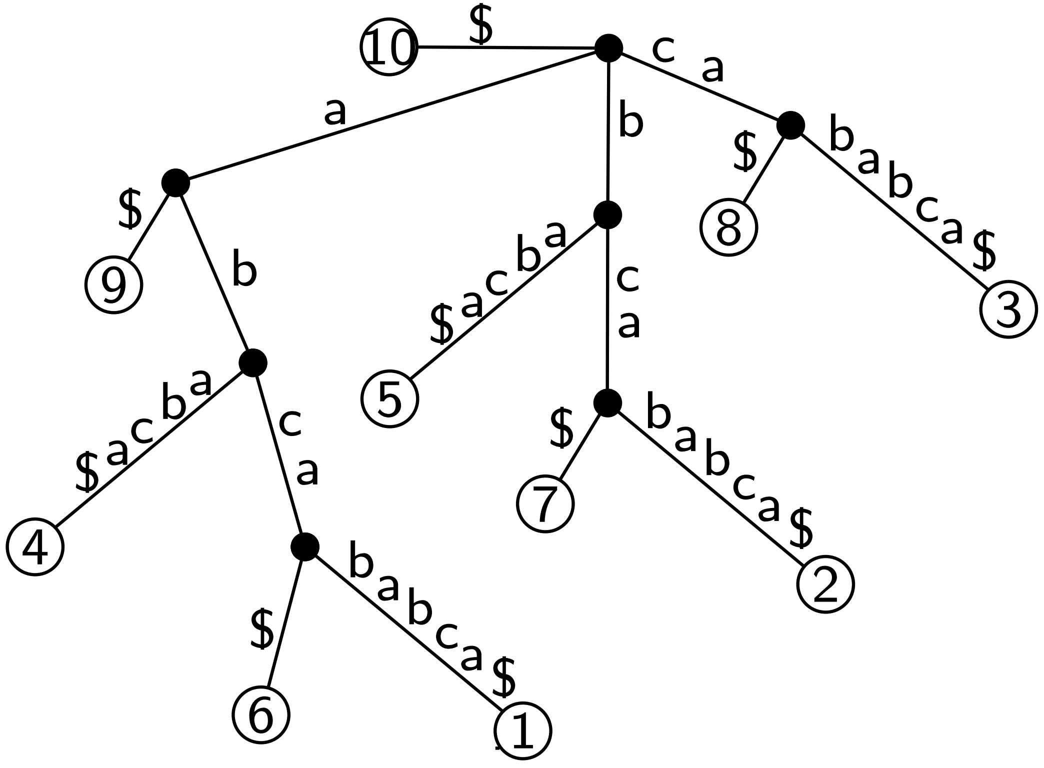
A **suffix array** A of a text T with $n = |T|$ stores a permutation of the indices $\{1, 2, \ldots, n\}$ s.t. $S_{A[i]}$ **is the** i**-th suffix** of T in lexicographical order.

$S_{A[i-1]} < S_{A[i]}$ for each $1 < i \leq n$

Convention. $ is the smallest letter.

Properties.

- The entries of A correspond to a lexicographical sorting of the suffixes of T.
- The entries of A corresponds to the order in which the leaves of a suffix tree S of T are encoutered by a DFS that chooses the next edge according to the lexicographical order.



Suffix Arrays

$T = \text{a b c a b a b c a \$}$

$A =$

10	9	4	6	1	5	7	2	8	3
$	a$	ababca$	abca$	abcababca$	babca$	bca$	bcababca$	ca$	cababca$

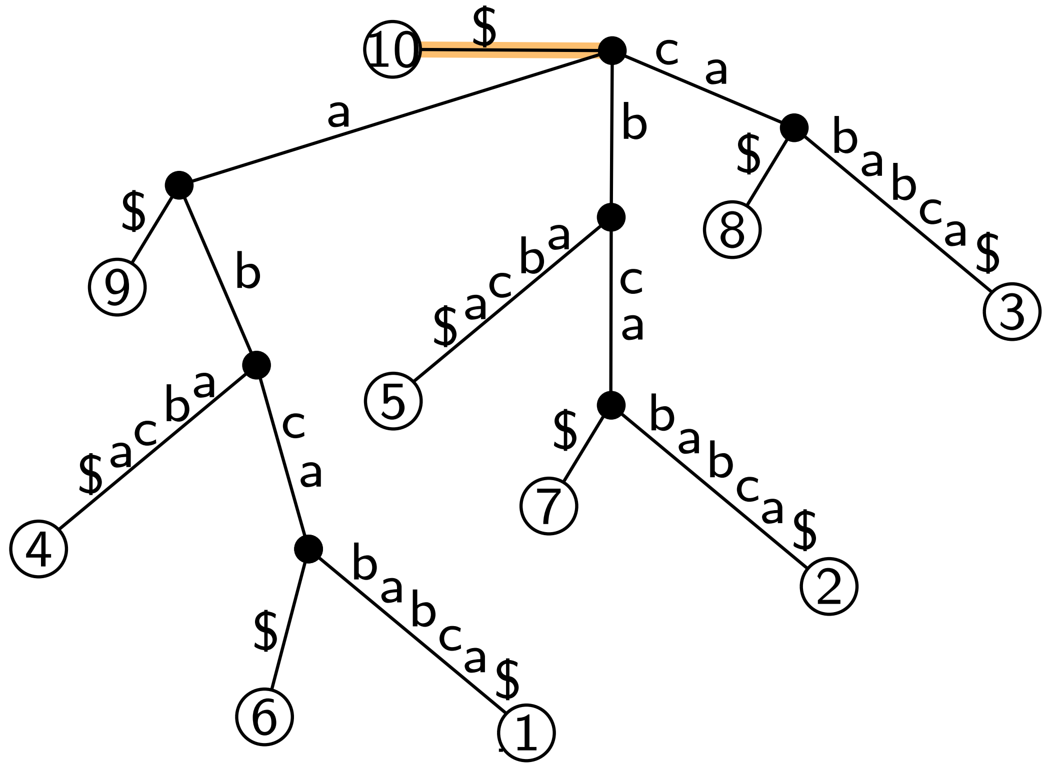
A **suffix array** A of a text T with $n = |T|$ stores a permutation of the indices $\{1, 2, \dots, n\}$ s.t. $S_{A[i]}$ **is the i-th suffix** of T in lexicographical order.

$S_{A[i-1]} < S_{A[i]}$ for each $1 < i \le n$

Convention. $ is the smallest letter.

Properties.

- The entries of A correspond to a lexicographical sorting of the suffixes of T.
- The entries of A corresponds to the order in which the leaves of a suffix tree S of T are encoutered by a DFS that chooses the next edge according to the lexicographical order.



Suffix Arrays

$T = \text{a b c a b a b c a \$}$

$A =$

10	9	4	6	1	5	7	2	8	3
\$	a\$	ababca\$	abca\$	abcababca\$	babca\$	bca\$	bcababca\$	ca\$	cababca\$

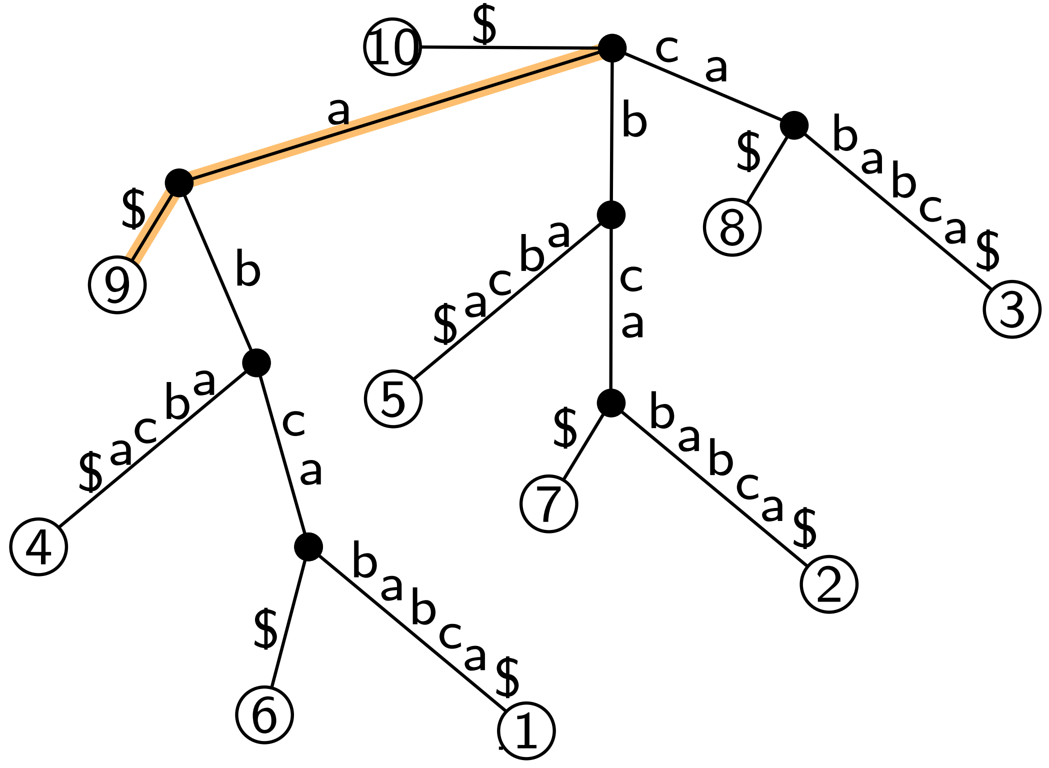
A **suffix array** A of a text T with $n = |T|$ stores a permutation of the indices $\{1, 2, \dots, n\}$ s.t. $S_{A[i]}$ **is the i-th suffix** of T in lexicographical order.

$S_{A[i-1]} < S_{A[i]}$ for each $1 < i \leq n$

Convention. \$ is the smallest letter.

Properties.

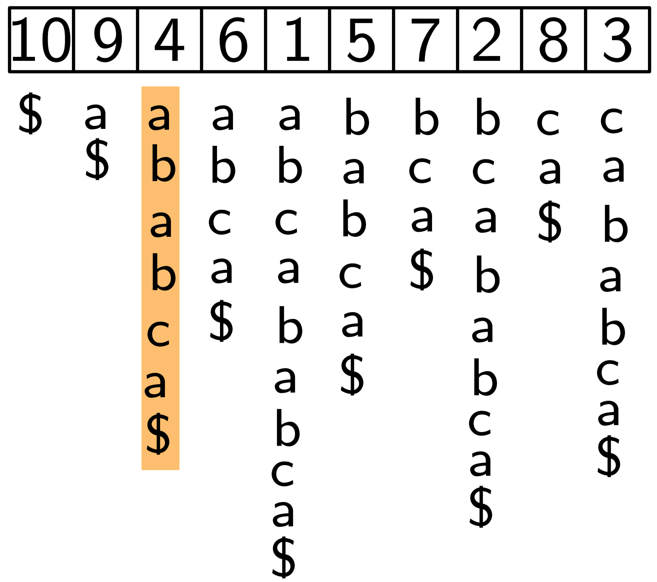
- The entries of A correspond to a lexicographical sorting of the suffixes of T.
- The entries of A corresponds to the order in which the leaves of a suffix tree S of T are encoutered by a DFS that chooses the next edge according to the lexicographical order.



Suffix Arrays

$T =$ a b c a b a b c a \$

$A =$ 10 | 9 | 4 | 6 | 1 | 5 | 7 | 2 | 8 | 3



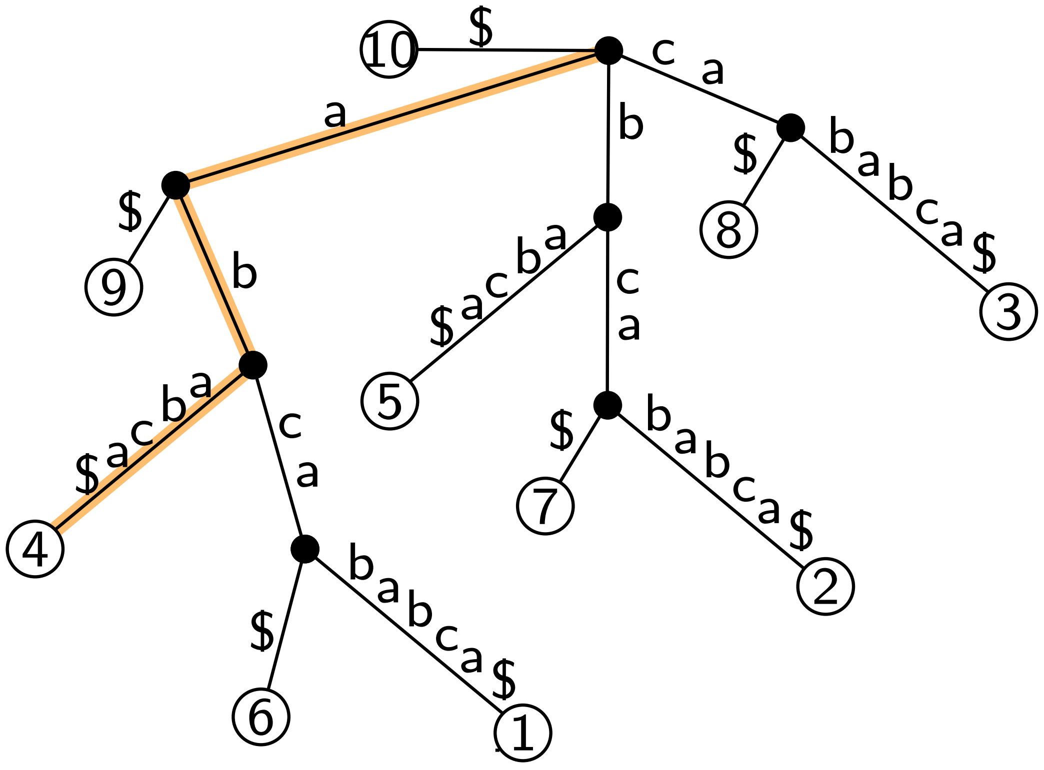
A **suffix array** A of a text T with $n = |T|$ stores a permutation of the indices $\{1, 2, \ldots, n\}$ s.t. $S_{A[i]}$ **is the i-th suffix** of T in lexicographical order.

$S_{A[i-1]} < S_{A[i]}$ for each $1 < i \leq n$

Convention. \$ is the smallest letter.

Properties.

- The entries of A correspond to a lexicographical sorting of the suffixes of T.
- The entries of A corresponds to the order in which the leaves of a suffix tree S of T are encoutered by a DFS that chooses the next edge according to the lexicographical order.



Suffix Arrays

$T = \text{a b c a b a b c a \$}$

$A =$	10	9	4	6	1	5	7	2	8	3
	\$	a\$	ababca\$	abca\$	abcababca\$	babca\$	bca\$	bcababca\$	ca\$	cababca\$

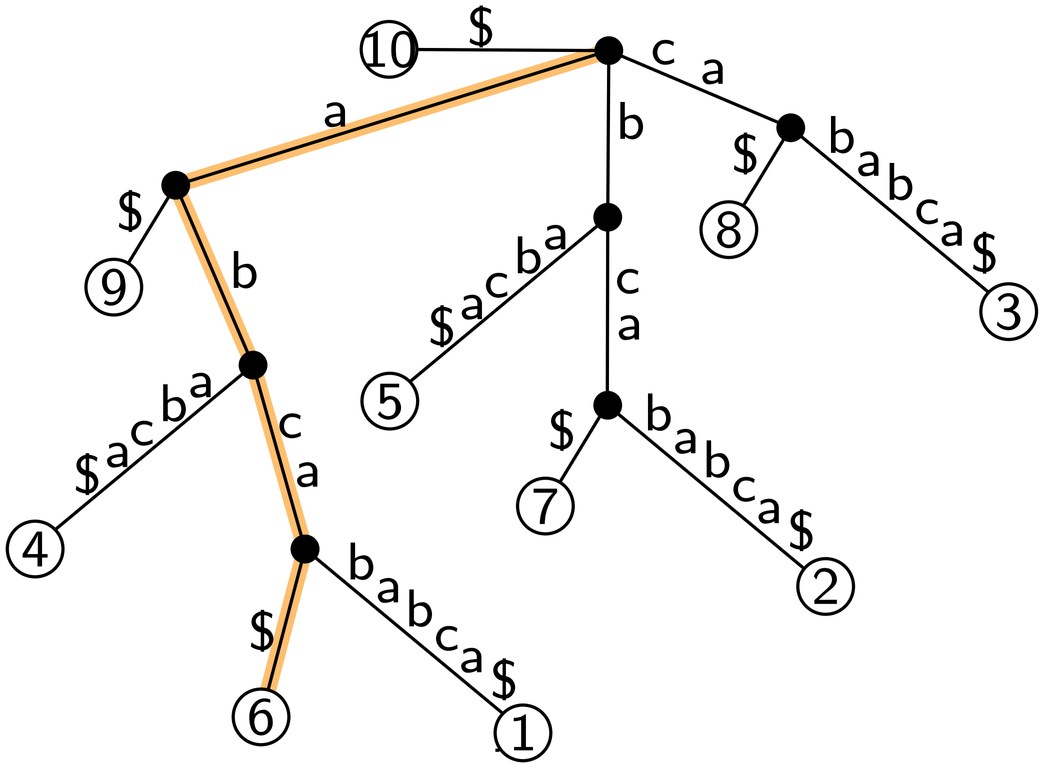
A **suffix array** A of a text T with $n = |T|$ stores a permutation of the indices $\{1, 2, \ldots, n\}$ s.t. $S_{A[i]}$ **is the i-th suffix** of T in lexicographical order.

$S_{A[i-1]} < S_{A[i]}$ for each $1 < i \le n$

Convention. \$ is the smallest letter.

Properties.

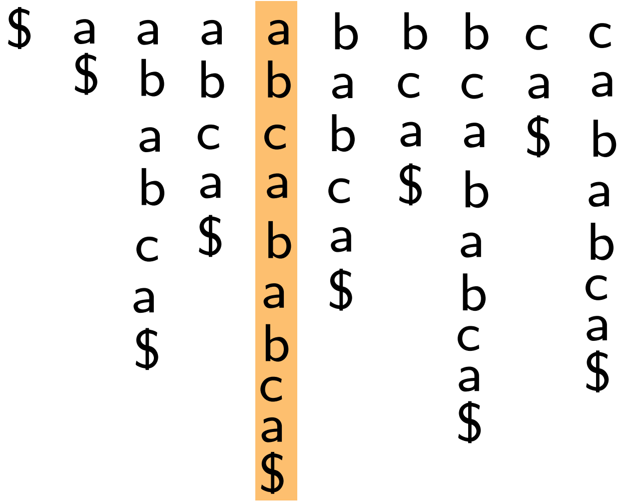
- The entries of A correspond to a lexicographical sorting of the suffixes of T.
- The entries of A corresponds to the order in which the leaves of a suffix tree S of T are encoutered by a DFS that chooses the next edge according to the lexicographical order.



Suffix Arrays

$T = \text{a b c a b a b c a \$}$

$A = \boxed{10}\boxed{9}\boxed{4}\boxed{6}\boxed{1}\boxed{5}\boxed{7}\boxed{2}\boxed{8}\boxed{3}$



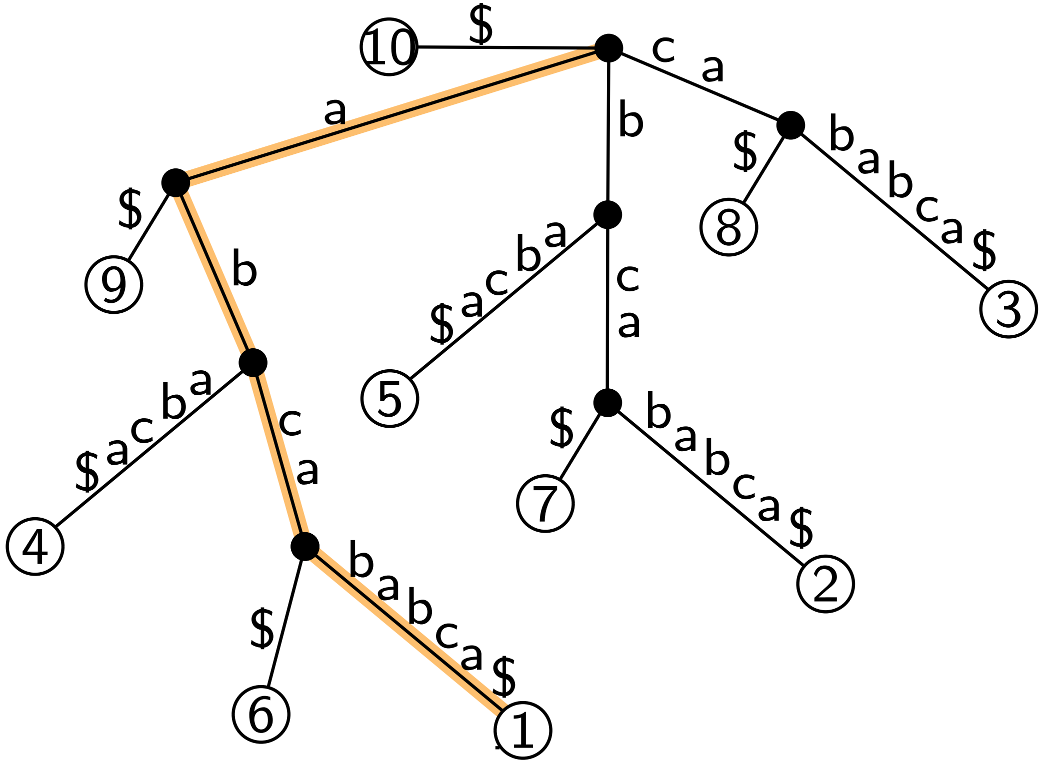
A **suffix array** A of a text T with $n = |T|$ stores a permutation of the indices $\{1, 2, \ldots, n\}$ s.t. $S_{A[i]}$ **is the i-th suffix** of T in lexicographical order.

$S_{A[i-1]} < S_{A[i]}$ for each $1 < i \leq n$

Convention. \$ is the smallest letter.

Properties.

- The entries of A correspond to a lexicographical sorting of the suffixes of T.
- The entries of A corresponds to the order in which the leaves of a suffix tree S of T are encoutered by a DFS that chooses the next edge according to the lexicographical order.



Suffix Arrays

$T = \text{a b c a b a b c a \$}$

$A =$

10	9	4	6	1	5	7	2	8	3
$	a $	a b a b c a $	a b c a $	a b c a b a b c a $	b a b c a $	b c a $	b c a b a b c a $	c a $	c a b a b c a $

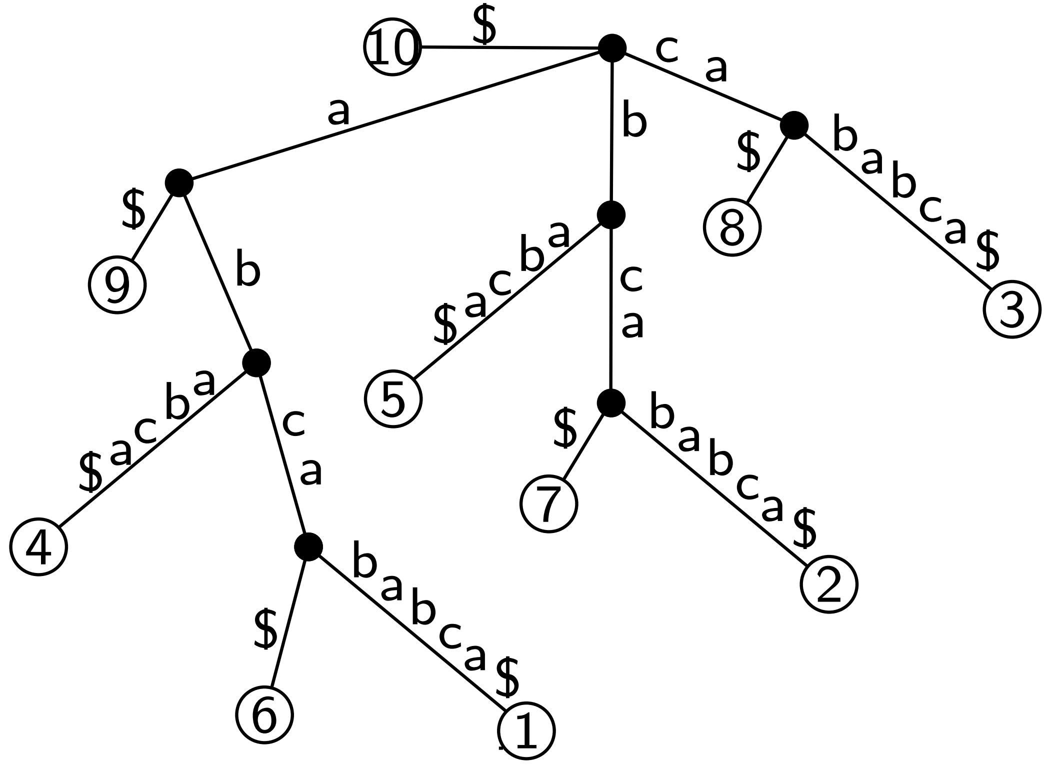
A **suffix array** A of a text T with $n = |T|$ stores a permutation of the indices $\{1, 2, \ldots, n\}$ s.t. $S_{A[i]}$ **is the i-th suffix** of T in lexicographical order.

$S_{A[i-1]} < S_{A[i]}$ for each $1 < i \leq n$

Convention. $ is the smallest letter.

Properties.

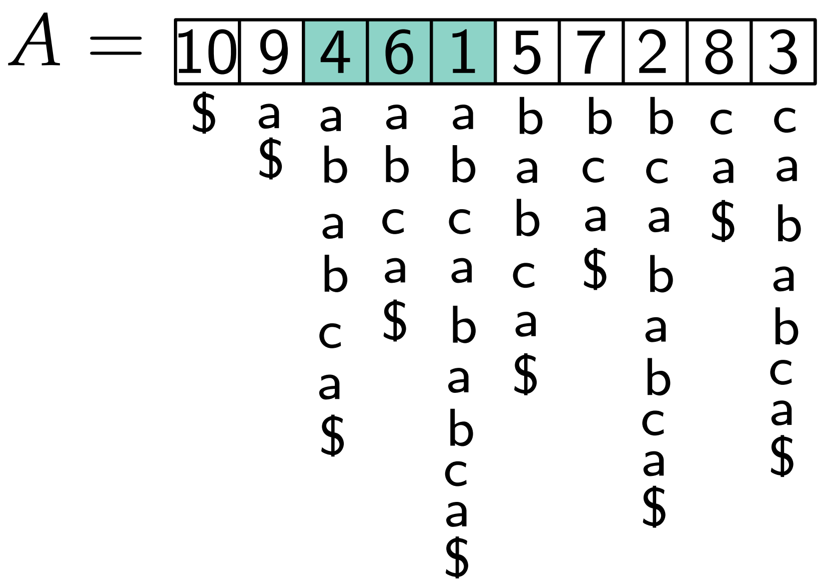
- The entries of A correspond to a lexicographical sorting of the suffixes of T.
- The entries of A corresponds to the order in which the leaves of a suffix tree S of T are encoutered by a DFS that chooses the next edge according to the lexicographical order.



Searching in Suffix Arrays

Observation. The occurrences of a pattern P in T form an interval in A.

$T =$ a b c a b a b c a \$



$A =$									
10	9	4	6	1	5	7	2	8	3
\$	a\$	ababca\$	abca\$	abcababca\$	babca\$	bca\$	bcababca\$	ca\$	cababca\$

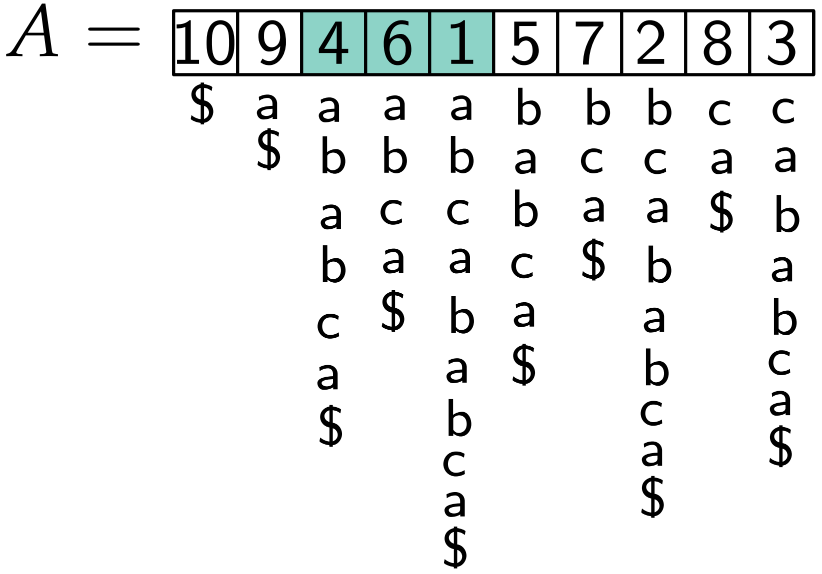
$P =$ a b

Searching in Suffix Arrays

Observation. The occurrences of a pattern P in T form an interval in A.

Idea. Find the left and the right boundary of the interval via two binary searches.
Report all entries in the interval!

$T = \text{a b c a b a b c a \$}$



$A =$

10	9	4	6	1	5	7	2	8	3
$	a$	ababca$	abca$	abcababca$	babca$	bca$	bcababca$	ca$	cababca$

$P = \text{a b}$

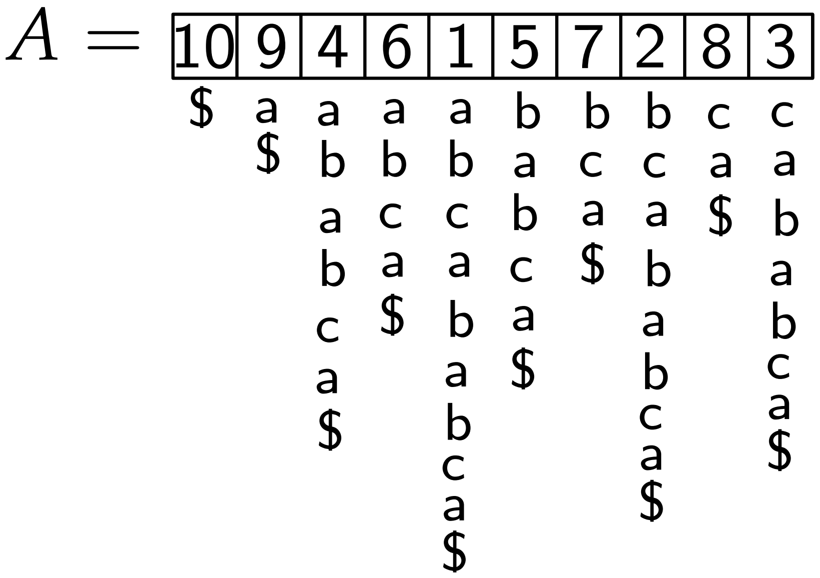
Searching in Suffix Arrays

Observation. The occurrences of a pattern P in T form an interval in A.

Idea. Find the left and the right boundary of the interval via two binary searches.
Report all entries in the interval!

```
FINDLEFTBOUNDARY(suffix array $A$, string $P$)
  $\ell \leftarrow 1$ // left index of candidates
  $r \leftarrow A$.length // right index of candidates
  while $\ell < r$ do
      $i \leftarrow \ell + \lfloor (r - \ell)/2 \rfloor$
      if $P > S_{A[i]}[1, m]$ then
          $\ell \leftarrow i + 1$ // continue with right half
      else
          $r \leftarrow i$ // continue with left half
  if $P$ is no prefix of $A[\ell]$ then
      return “no match“
  return $\ell$
```

$T =$ a b c a b a b c a \$



$A =$	10	9	4	6	1	5	7	2	8	3
	\$	a\$	ababca\$	abca\$	abcababca\$	babca\$	bca\$	bcababca\$	ca\$	cababca\$

$P =$ a b

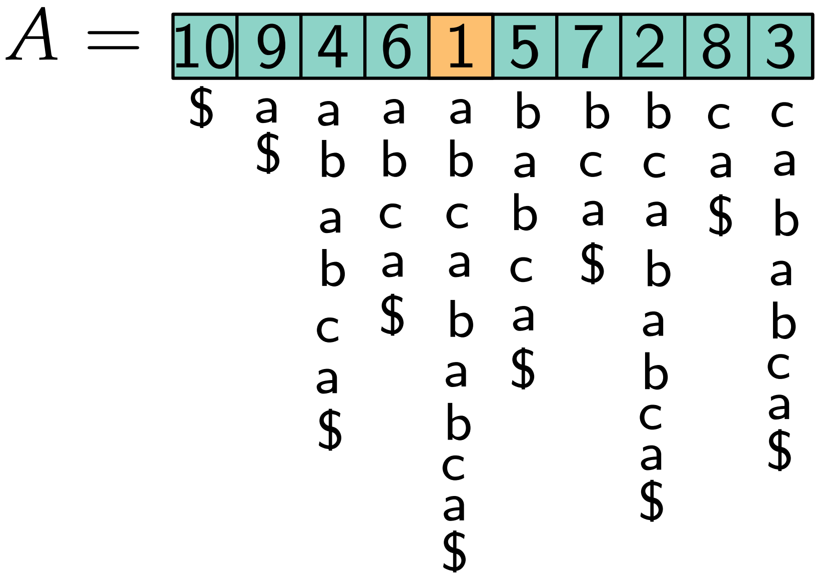
Searching in Suffix Arrays

Observation. The occurrences of a pattern P in T form an interval in A.

Idea. Find the left and the right boundary of the interval via two binary searches.
Report all entries in the interval!

```
FindLeftBoundary(suffix array $A$, string $P$)
  $\ell \leftarrow 1$ // left index of candidates
  $r \leftarrow A$.length // right index of candidates
  while $\ell < r$ do
      $i \leftarrow \ell + \lfloor (r - \ell)/2 \rfloor$
      if $P > S_{A[i]}[1, m]$ then
          $\ell \leftarrow i + 1$ // continue with right half
      else
          $r \leftarrow i$ // continue with left half
  if $P$ is no prefix of $A[\ell]$ then
      return “no match“
  return $\ell$
```

$T =$ a b c a b a b c a $



$A =$

10	9	4	6	1	5	7	2	8	3
$	a$	ababca$	abca$	abcababca$	babca$	bca$	bcababca$	ca$	cababca$

$P =$ a b

Searching in Suffix Arrays

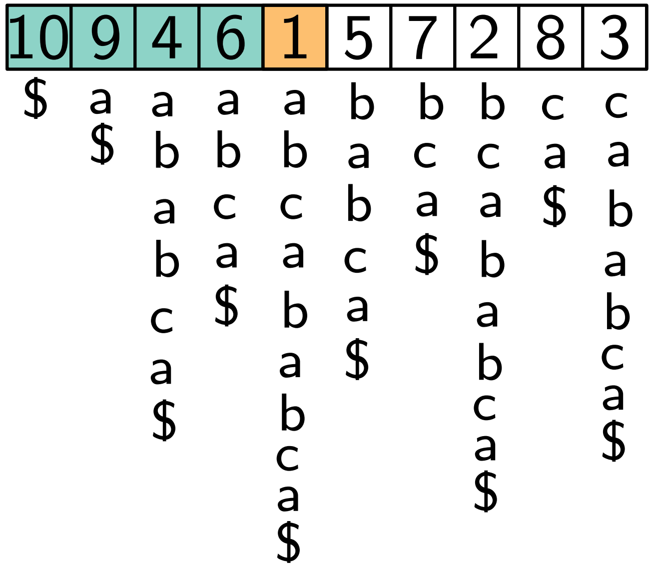
Observation. The occurrences of a pattern P in T form an interval in A.

Idea. Find the left and the right boundary of the interval via two binary searches.
Report all entries in the interval!

$T =$ a b c a b a b c a $

```
FindLeftBoundary(suffix array $A$, string $P$)
  $\ell \leftarrow 1$ // left index of candidates
  $r \leftarrow A$.length // right index of candidates
  while $\ell < r$ do
      $i \leftarrow \ell + \lfloor (r - \ell)/2 \rfloor$
      if $P > S_{A[i]}[1, m]$ then
          $\ell \leftarrow i + 1$ // continue with right half
      else
          $r \leftarrow i$ // continue with left half
  if $P$ is no prefix of $A[\ell]$ then
      return “no match“
  return $\ell$
```

$A =$



10	9	4	6	1	5	7	2	8	3
$	a$	ababca$	abca$	abcababca$	babca$	bca$	bcababca$	ca$	cababca$

$P =$ a b

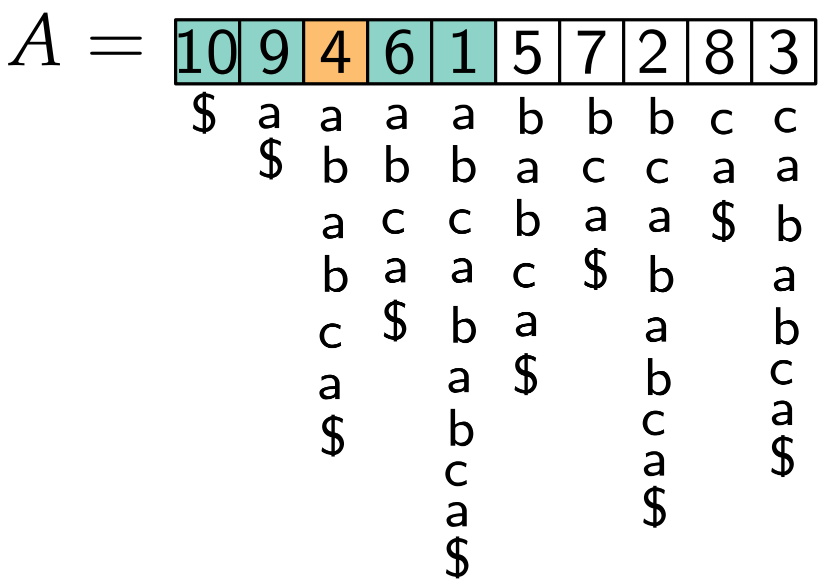
Searching in Suffix Arrays

Observation. The occurrences of a pattern P in T form an interval in A.

Idea. Find the left and the right boundary of the interval via two binary searches.
Report all entries in the interval!

```
FindLeftBoundary(suffix array $A$, string $P$)
  $\ell \leftarrow 1$ // left index of candidates
  $r \leftarrow A$.length // right index of candidates
  while $\ell < r$ do
      $i \leftarrow \ell + \lfloor (r - \ell)/2 \rfloor$
      if $P > S_{A[i]}[1, m]$ then
          $\ell \leftarrow i + 1$ // continue with right half
      else
          $r \leftarrow i$ // continue with left half
  if $P$ is no prefix of $A[\ell]$ then
      return "no match"
  return $\ell$
```

$T =$ a b c a b a b c a $



$A =$

10	9	4	6	1	5	7	2	8	3
$	a$	ababca$	abca$	abcababca$	babca$	bca$	bcababca$	ca$	cababca$

$P =$ a b

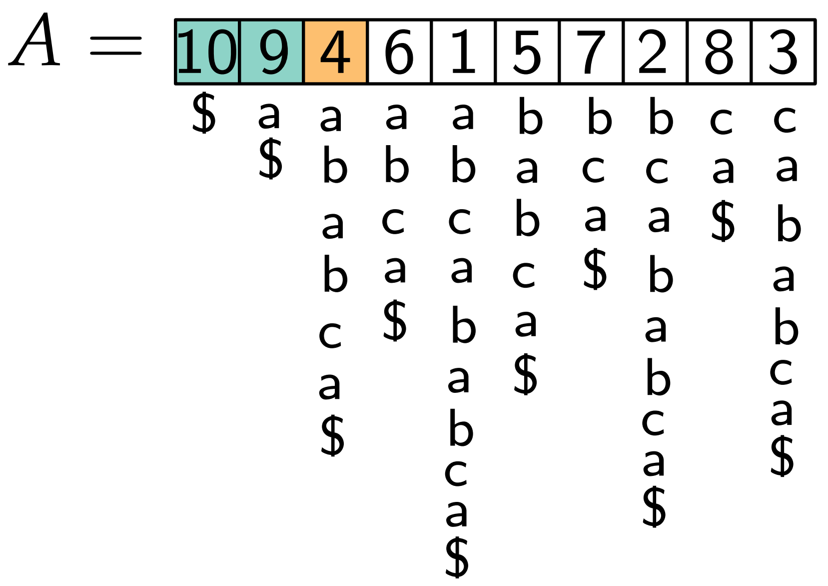
Searching in Suffix Arrays

Observation. The occurrences of a pattern P in T form an interval in A.

Idea. Find the left and the right boundary of the interval via two binary searches.
Report all entries in the interval!

```
FindLeftBoundary(suffix array $A$, string $P$)
  $\ell \leftarrow 1$ // left index of candidates
  $r \leftarrow A$.length // right index of candidates
  while $\ell < r$ do
      $i \leftarrow \ell + \lfloor (r - \ell)/2 \rfloor$
      if $P > S_{A[i]}[1, m]$ then
          $\ell \leftarrow i + 1$ // continue with right half
      else
          $r \leftarrow i$ // continue with left half
  if $P$ is no prefix of $A[\ell]$ then
      return “no match“
  return $\ell$
```

$T =$ a b c a b a b c a $



$P =$ a b

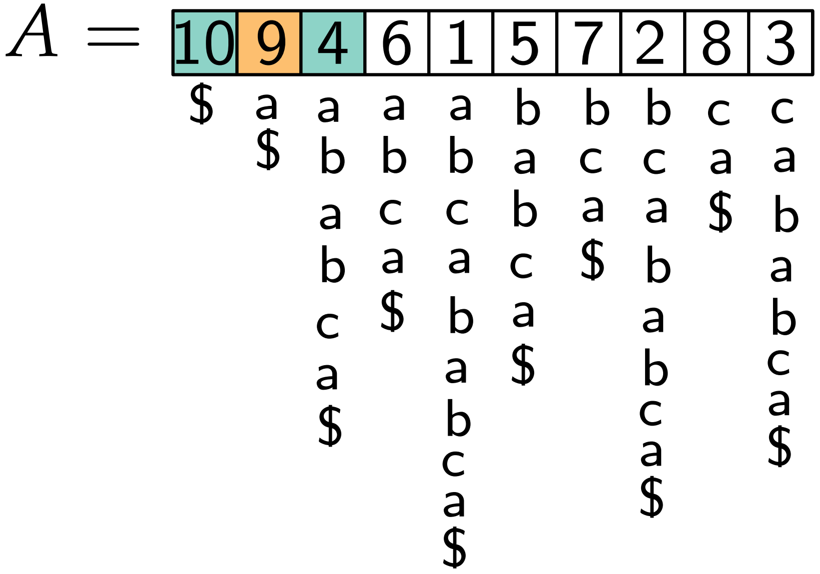
Searching in Suffix Arrays

Observation. The occurrences of a pattern P in T form an interval in A.

Idea. Find the left and the right boundary of the interval via two binary searches. Report all entries in the interval!

```
FindLeftBoundary(suffix array $A$, string $P$)
  $\ell \leftarrow 1$ // left index of candidates
  $r \leftarrow A$.length // right index of candidates
  while $\ell < r$ do
      $i \leftarrow \ell + \lfloor (r - \ell)/2 \rfloor$
      if $P > S_{A[i]}[1, m]$ then
          $\ell \leftarrow i + 1$ // continue with right half
      else
          $r \leftarrow i$ // continue with left half
  if $P$ is no prefix of $A[\ell]$ then
      return "no match"
  return $\ell$
```

$T =$ a b c a b a b c a $



$A =$	10	9	4	6	1	5	7	2	8	3
	$	a$	ababca$	abca$	abcababca$	babca$	bca$	bcababca$	ca$	cababca$

$P =$ a b

Searching in Suffix Arrays

Observation. The occurrences of a pattern P in T form an interval in A.

Idea. Find the left and the right boundary of the interval via two binary searches.
Report all entries in the interval!

```
FindLeftBoundary(suffix array $A$, string $P$)
  $\ell \leftarrow 1$ // left index of candidates
  $r \leftarrow A$.length // right index of candidates
  while $\ell < r$ do
      $i \leftarrow \ell + \lfloor (r - \ell)/2 \rfloor$
      if $P > S_{A[i]}[1, m]$ then
          $\ell \leftarrow i + 1$ // continue with right half
      else
          $r \leftarrow i$ // continue with left half
  if $P$ is no prefix of $A[\ell]$ then
      return "no match"
  return $\ell$
```

$T =$ a b c a b a b c a $

$A =$

10	9	4	6	1	5	7	2	8	3
$	a $	a b a b c a $	a b c a $	a b c a b a b c a $	b a b c a $	b c a $	b c a b a b c a $	c a $	c a b a b c a $

$P =$ a b

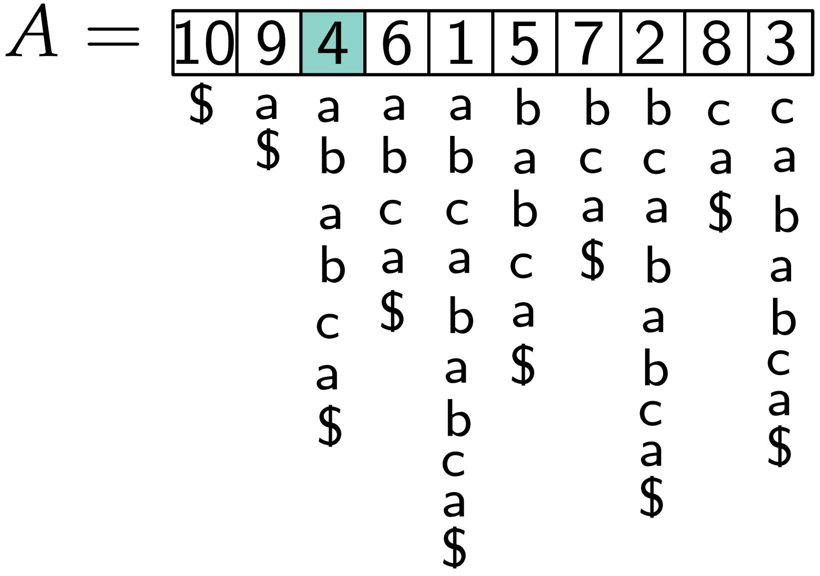
Searching in Suffix Arrays

Observation. The occurrences of a pattern P in T form an interval in A.

Idea. Find the left and the right boundary of the interval via two binary searches.
Report all entries in the interval!

```
FindLeftBoundary(suffix array $A$, string $P$)
  $\ell \leftarrow 1$ // left index of candidates
  $r \leftarrow A$.length // right index of candidates
  while $\ell < r$ do
      $i \leftarrow \ell + \lfloor (r - \ell)/2 \rfloor$
      if $P > S_{A[i]}[1, m]$ then
          $\ell \leftarrow i + 1$ // continue with right half
      else
          $r \leftarrow i$ // continue with left half
  if $P$ is no prefix of $A[\ell]$ then
      return "no match"
  return $\ell$
```

$T =$ a b c a b a b c a $



$A =$	10	9	4	6	1	5	7	2	8	3
	$	a$	ababca$	abca$	abcababca$	babca$	bca$	bcababca$	ca$	cababca$

$P =$ a b

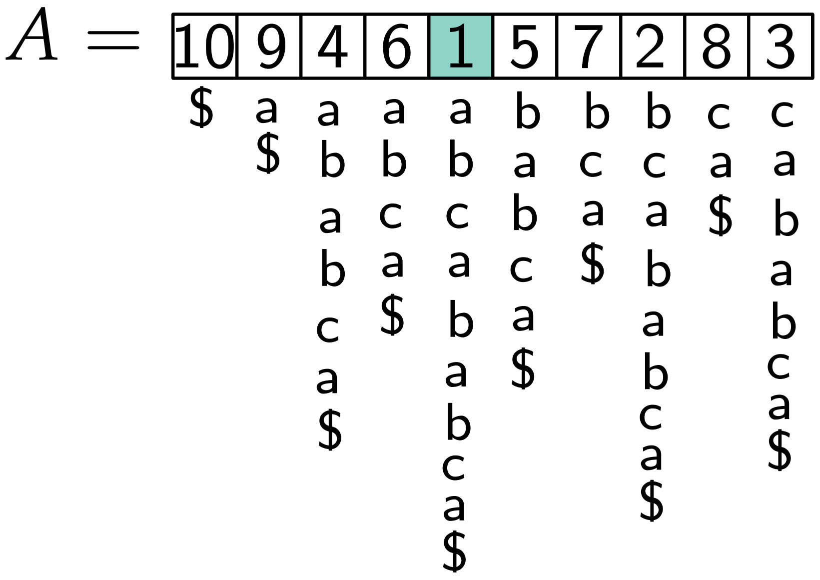
Searching in Suffix Arrays

Observation. The occurrences of a pattern P in T form an interval in A.

Idea. Find the left and the right boundary of the interval via two binary searches.
Report all entries in the interval!

```
FindRightBoundary(suffix array $A$, string $P$)
  $\ell \leftarrow 1$ // left index of candidates
  $r \leftarrow A$.length // right index of candidates
  while $r > \ell$ do
      $i \leftarrow \ell + \lceil (r - \ell)/2 \rceil$
      if $P < S_{A[i]}[1, m]$ then
          $r \leftarrow i - 1$ // continue with left half
      else
          $\ell \leftarrow i$ // continue with right half
  if $P$ is no prefix of $A[r]$ then
      return “no match“
  return $r$
```

$T =$ a b c a b a b c a $



$A =$	10	9	4	6	1	5	7	2	8	3
	$	a$	ababca$	abca$	abcababca$	babca$	bca$	bcababca$	ca$	cababca$

$P =$ a b

Searching in Suffix Arrays

Observation. The occurrences of a pattern P in T form an interval in A.

Idea. Find the left and the right boundary of the interval via two binary searches.
Report all entries in the interval!

$T =$ a b c a b a b c a $

```
FindRightBoundary(suffix array $A$, string $P$)
  $\ell \leftarrow 1$ // left index of candidates
  $r \leftarrow A$.length // right index of candidates
  while $r > \ell$ do
    $i \leftarrow \ell + \lceil (r - \ell)/2 \rceil$
    if $P < S_{A[i]}[1, m]$ then
      $r \leftarrow i - 1$ // continue with left half
    else
      $\ell \leftarrow i$ // continue with right half
  if $P$ is no prefix of $A[r]$ then
    return “no match“
  return $r$
```

$A =$	10	9	4	6	1	5	7	2	8	3
	$	a $	a b a b c a $	a b c a $	a b c a b a b c a $	b a b c a $	b c a $	b c a b a b c a $	c a $	c a b a b c a $

$P =$ a b

Each lexicographic comparison can be done in $\mathcal{O}(\quad)$ time.

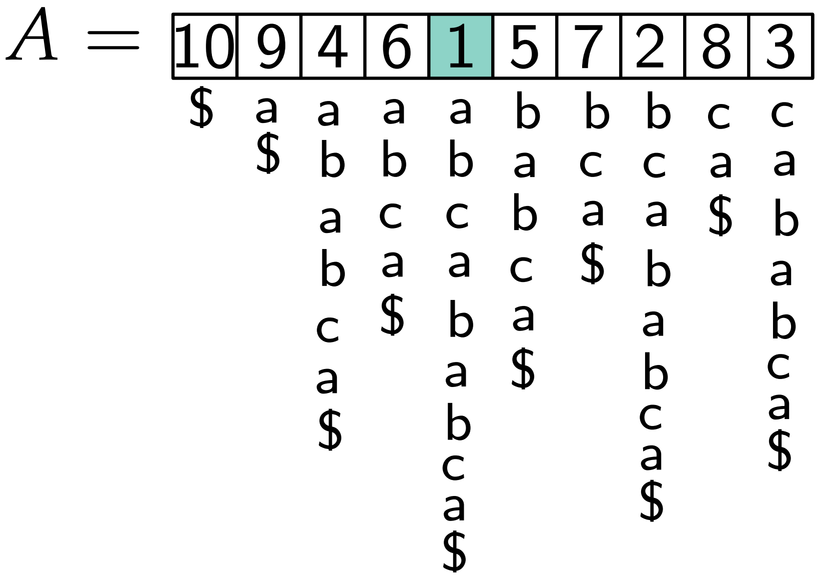
Searching in Suffix Arrays

Observation. The occurrences of a pattern P in T form an interval in A.

Idea. Find the left and the right boundary of the interval via two binary searches.
Report all entries in the interval!

```
FindRightBoundary(suffix array $A$, string $P$)
    $\ell \leftarrow 1$ // left index of candidates
    $r \leftarrow A$.length // right index of candidates
    while $r > \ell$ do
        $i \leftarrow \ell + \lceil (r - \ell)/2 \rceil$
        if $P < S_{A[i]}[1, m]$ then
            $r \leftarrow i - 1$ // continue with left half
        else
            $\ell \leftarrow i$ // continue with right half
    if $P$ is no prefix of $A[r]$ then
        return "no match"
    return $r$
```

$T =$ a b c a b a b c a \$



$A =$	10	9	4	6	1	5	7	2	8	3
	\$	a \$	a b a b c a \$	a b c a \$	a b c a b a b c a \$	b a b c a \$	b c a \$	b c a b a b c a \$	c a \$	c a b a b c a \$

$P =$ a b

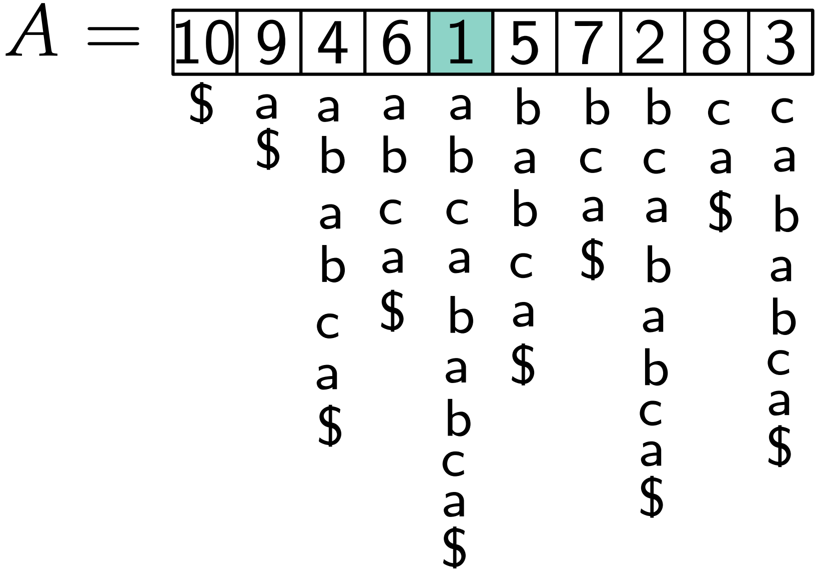
Each lexicographic comparison can be done in $\mathcal{O}(m)$ time.

Searching in Suffix Arrays

Observation. The occurrences of a pattern P in T form an interval in A.

Idea. Find the left and the right boundary of the interval via two binary searches.
Report all entries in the interval!

$T =$ a b c a b a b c a $



$A =$

10	9	4	6	1	5	7	2	8	3
$	a$	ababca$	abca$	abcababca$	babca$	bca$	bcababca$	ca$	cababca$

$P =$ a b

```
FindRightBoundary(suffix array $A$, string $P$)
  $\ell \leftarrow 1$ // left index of candidates
  $r \leftarrow A$.length // right index of candidates
  while $r > \ell$ do
      $i \leftarrow \ell + \lceil (r - \ell)/2 \rceil$
      if $P < S_{A[i]}[1, m]$ then
          $r \leftarrow i - 1$ // continue with left half
      else
          $\ell \leftarrow i$ // continue with right half
  if $P$ is no prefix of $A[r]$ then
      return “no match“
  return $r$
```

Each lexicographic comparison can be done in $\mathcal{O}(m)$ time.

$\Rightarrow$ The k occurrences of P can be found in $\mathcal{O}(\qquad\qquad)$ time.

Searching in Suffix Arrays

Observation. The occurrences of a pattern P in T form an interval in A.

Idea. Find the left and the right boundary of the interval via two binary searches.
Report all entries in the interval!

```
FindRightBoundary(suffix array $A$, string $P$)
  $\ell \leftarrow 1$ // left index of candidates
  $r \leftarrow A$.length // right index of candidates
  while $r > \ell$ do
      $i \leftarrow \ell + \lceil (r - \ell)/2 \rceil$
      if $P < S_{A[i]}[1, m]$ then
          $r \leftarrow i - 1$ // continue with left half
      else
          $\ell \leftarrow i$ // continue with right half
  if $P$ is no prefix of $A[r]$ then
      return "no match"
  return $r$
```

$T =$ a b c a b a b c a $

$A =$	10	9	4	6	1	5	7	2	8	3
	$	a$	ababca$	abca$	abcababca$	babca$	bca$	bcababca$	ca$	cababca$

$P =$ a b

Each lexicographic comparison can be done in $\mathcal{O}(m)$ time.

$\Rightarrow$ The k occurrences of P can be found in $\mathcal{O}(k + m \log n)$ time.

Constructing Suffix Arrays – First Attempt

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix array A for T.

Constructing Suffix Arrays – First Attempt

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix array A for T.

Idea.

- If $n \in \mathcal{O}(1)$, then use brute-force.
- Otherwise, dissect T into triplets.
- Interpret the triplets as letters over an alphabet $\Sigma' \subseteq \Sigma^3$.
- Interpret T as a string R over Σ' with $|R| = \lceil n/3 \rceil$.
- Recurse!

Constructing Suffix Arrays – First Attempt

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix array A for T.

Idea.

- If $n \in \mathcal{O}(1)$, then use brute-force.
- Otherwise, dissect T into triplets.
- Interpret the triplets as letters over an alphabet $\Sigma' \subseteq \Sigma^3$.
- Interpret T as a string R over Σ' with $|R| = \lceil n/3 \rceil$.
- Recurse!

$T =$ y a b b a d a b b a

Constructing Suffix Arrays – First Attempt

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix array A for T.

Idea.

- If $n \in \mathcal{O}(1)$, then use brute-force.
- Otherwise, dissect T into triplets.
- Interpret the triplets as letters over an alphabet $\Sigma' \subseteq \Sigma^3$.
- Interpret T as a string R over Σ' with $|R| = \lceil n/3 \rceil$.
- Recurse!

padding

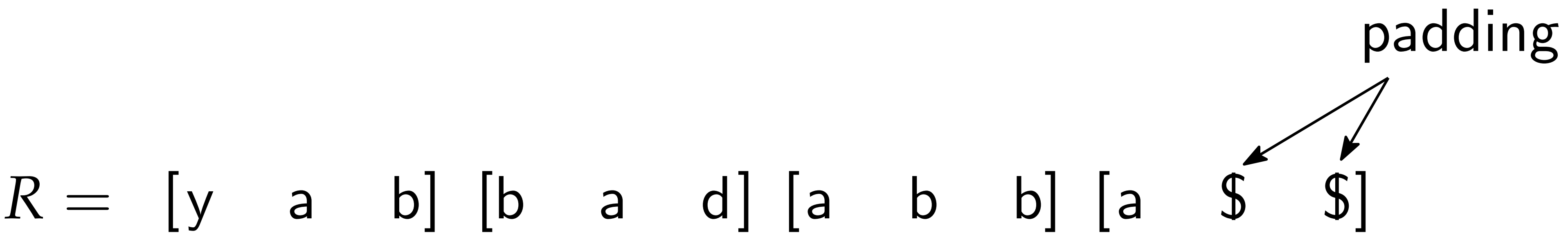
$$R = [\text{y a b}]\ [\text{b a d}]\ [\text{a b b}]\ [\text{a \$ \$}]$$

Constructing Suffix Arrays – First Attempt

Task. Given a string T with $n = |T|$ over alphabet Σ, construct a suffix array A for T.

Idea.

- If $n \in \mathcal{O}(1)$, then use brute-force.
- Otherwise, dissect T into triplets.
- Interpret the triplets as letters over an alphabet $\Sigma' \subseteq \Sigma^3$.
- Interpret T as a string R over Σ' with $|R| = \lceil n/3 \rceil$.
- Recurse!



padding

$$R = [\text{y a b}]\ [\text{b a d}]\ [\text{a b b}]\ [\text{a \$ \$}]$$

Problem. But how can a suffix array for R be used to create a suffix array for T?

Constructing Suffix Arrays – Overview

Shortened notation: $T = t_0 t_1 \dots t_{n-1}$

	0	1	2	3	4	5	6	7	8	9	10	11
T=	y	a	b	b	a	d	a	b	b	a	d	o

$\mathcal{S}(T)$ = suffixes of T =

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Constructing Suffix Arrays – Overview

Shortened notation: $T = t_0 t_1 \dots t_{n-1}$

	0	1	2	3	4	5	6	7	8	9	10	11
$T=$	y	a	b	b	a	d	a	b	b	a	d	o

$\mathcal{S}(T)$ = suffixes of T =

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Constructing Suffix Arrays – Overview

Shortened notation: $T = t_0 t_1 \ldots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for x mod $y = z$.

	0	1	2	3	4	5	6	7	8	9	10	11
$T=$	y	a	b	b	a	d	a	b	b	a	d	o

$\mathcal{S}_0 =$ suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1 =$ suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2 =$ suffixes with index $i \equiv 2(3)$

$\mathcal{S}(T) =$ suffixes of $T =$

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Constructing Suffix Arrays – Overview

Shortened notation: $T = t_0 t_1 \dots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for $x \bmod y = z$.

	0	1	2	3	4	5	6	7	8	9	10	11
$T=$	y	a	b	b	a	d	a	b	b	a	d	o

$\mathcal{S}_0$ = suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1$ = suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2$ = suffixes with index $i \equiv 2(3)$

```
ConstructSuffixArray(string $T$)
    if $n \in \mathcal{O}(1)$ then
        construct $A$ in $\mathcal{O}(1)$ time.
    else
        sort $\mathcal{S}_1 \cup \mathcal{S}_2$ into an array $A_{12}$
        use $A_{12}$ to sort $\mathcal{S}_0$ into an array $A_0$
        merge $A_{12}$ with $A_0$
```

$\mathcal{S}(T)$ = suffixes of T =

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Constructing Suffix Arrays – Overview

Shortened notation: $T = t_0 t_1 \dots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for x mod $y = z$.

	0	1	2	3	4	5	6	7	8	9	10	11
$T=$	y	a	b	b	a	d	a	b	b	a	d	o

$\mathcal{S}_0 =$ suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1 =$ suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2 =$ suffixes with index $i \equiv 2(3)$

```
ConstructSuffixArray(string $T$)
    if $n \in \mathcal{O}(1)$ then
        construct $A$ in $\mathcal{O}(1)$ time.
    else
        sort $\mathcal{S}_1 \cup \mathcal{S}_2$ into an array $A_{12}$
        use $A_{12}$ to sort $\mathcal{S}_0$ into an array $A_0$
        merge $A_{12}$ with $A_0$
```

$\mathcal{S}(T) =$ suffixes of $T =$

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

For simplicity, we assume $n \equiv 0(3)$.

Constructing Suffix Arrays – Overview

Shortened notation: $T = t_0 t_1 \dots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for $x \bmod y = z$.

	0	1	2	3	4	5	6	7	8	9	10	11
$T=$	y	a	b	b	a	d	a	b	b	a	d	o

$\mathcal{S}_0$ = suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1$ = suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2$ = suffixes with index $i \equiv 2(3)$

```
CONSTRUCTSUFFIXARRAY(string $T$)
    if $n \in \mathcal{O}(1)$ then
        construct $A$ in $\mathcal{O}(1)$ time.
    else
        sort $\mathcal{S}_1 \cup \mathcal{S}_2$ into an array $A_{12}$
        use $A_{12}$ to sort $\mathcal{S}_0$ into an array $A_0$
        merge $A_{12}$ with $A_0$
```

using the idea from the previous slide!

$\mathcal{S}(T)$ = suffixes of T =

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

For simplicity, we assume $n \equiv 0(3)$.

Step 1: Sorting $\mathcal{S}_1 \cup \mathcal{S}_2$

Shortened notation: $T = t_0 t_1 \ldots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for $x \bmod y = z$.

$\mathcal{S}_0 =$ suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1 =$ suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2 =$ suffixes with index $i \equiv 2(3)$

$\mathcal{S}(T) =$ suffixes of $T =$

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Step 1: Sorting $\mathcal{S}_1 \cup \mathcal{S}_2$

Shortened notation: $T = t_0 t_1 \ldots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for x mod $y = z$.

Partition $\mathcal{S}_1$ and $\mathcal{S}_2$ into triplets and concatenate them:

$\mathcal{S}_0$ = suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1$ = suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2$ = suffixes with index $i \equiv 2(3)$

$$R_1 = [t_1 t_2 t_3][t_4 t_5 t_6] \ldots = [abb][ada][bba][do\$]$$

$$R_2 = [t_2 t_3 t_4][t_5 t_6 t_7] \ldots = [bba][dab][bad][o\$\$]$$

$\mathcal{S}(T)$ = suffixes of T =

S_0	yabbadabbado
S_1	abbadabbado
S_2	bbadabbado
S_3	badabbado
S_4	adabbado
S_5	dabbado
S_6	abbado
S_7	bbado
S_8	bado
S_9	ado
S_{10}	do
S_{11}	o

Step 1: Sorting $\mathcal{S}_1 \cup \mathcal{S}_2$

Shortened notation: $T = t_0 t_1 \ldots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for x mod $y = z$.

Partition $\mathcal{S}_1$ and $\mathcal{S}_2$ into triplets and concatenate them:

$\mathcal{S}_0$ = suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1$ = suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2$ = suffixes with index $i \equiv 2(3)$

$R =$ [abb][ada][bba][do\$][bba][dab][bad][o\$\$]

$$R_1 = [t_1 t_2 t_3][t_4 t_5 t_6] \ldots = [abb][ada][bba][do\$]$$

$$R_2 = [t_2 t_3 t_4][t_5 t_6 t_7] \ldots = [bba][dab][bad][o\$\$]$$

$\mathcal{S}(T)$ = suffixes of T =

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Step 1: Sorting $\mathcal{S}_1 \cup \mathcal{S}_2$

Shortened notation: $T = t_0 t_1 \dots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for x mod $y = z$.

Partition $\mathcal{S}_1$ and $\mathcal{S}_2$ into triplets and concatenate them:

$R=$ [abb][ada][bba][do\$][bba][dab][bad][o\$\$]

$\mathcal{S}_0$ = suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1$ = suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2$ = suffixes with index $i \equiv 2(3)$

$\mathcal{S}(T)$ = suffixes of T =

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Step 1: Sorting $\mathcal{S}_1 \cup \mathcal{S}_2$

Shortened notation: $T = t_0 t_1 \ldots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for x mod $y = z$.

Partition $\mathcal{S}_1$ and $\mathcal{S}_2$ into triplets and concatenate them:

$R=$ [abb][ada][bba][do\$][bba][dab][bad][o\$\$]

$\mathcal{S}(R)=$

$S_1(R)$	[abb][ada][bba][do\$][bba][dab][bad][o\$\$]
$S_2(R)$	[ada][bba][do\$][bba][dab][bad][o\$\$]
$S_3(R)$	[bba][do\$][bba][dab][bad][o\$\$]
$S_4(R)$	[do\$][bba][dab][bad][o\$\$]
$S_5(R)$	[bba][dab][bad][o\$\$]
$S_6(R)$	[dab][bad][o\$\$]
$S_7(R)$	[bad][o\$\$]
$S_8(R)$	[o\$\$]

$\mathcal{S}_0$ = suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1$ = suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2$ = suffixes with index $i \equiv 2(3)$

$\mathcal{S}(T)$ = suffixes of T =

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Step 1: Sorting $\mathcal{S}_1 \cup \mathcal{S}_2$

Shortened notation: $T = t_0 t_1 \dots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for x mod $y = z$.

Partition $\mathcal{S}_1$ and $\mathcal{S}_2$ into triplets and concatenate them:

$R=$ [abb][ada][bba][do$][bba][dab][bad][o$$]

$\mathcal{S}(R)=$

$S_1(R)$	[abb][ada][bba][do$][bba][dab][bad][o$$]
$S_2(R)$	[ada][bba][do$][bba][dab][bad][o$$]
$S_3(R)$	[bba][do$][bba][dab][bad][o$$]
$S_4(R)$	[do$][bba][dab][bad][o$$]
$S_5(R)$	[bba][dab][bad][o$$]
$S_6(R)$	[dab][bad][o$$]
$S_7(R)$	[bad][o$$]
$S_8(R)$	[o$$]

Observation. $\mathcal{S}(R)$ corresponds bijectively to $\mathcal{S}_1 \cup \mathcal{S}_2$

$\mathcal{S}_0$ = suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1$ = suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2$ = suffixes with index $i \equiv 2(3)$

$\mathcal{S}(T)$ = suffixes of T =

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Step 1: Sorting $\mathcal{S}_1 \cup \mathcal{S}_2$

Shortened notation: $T = t_0 t_1 \ldots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for x mod $y = z$.

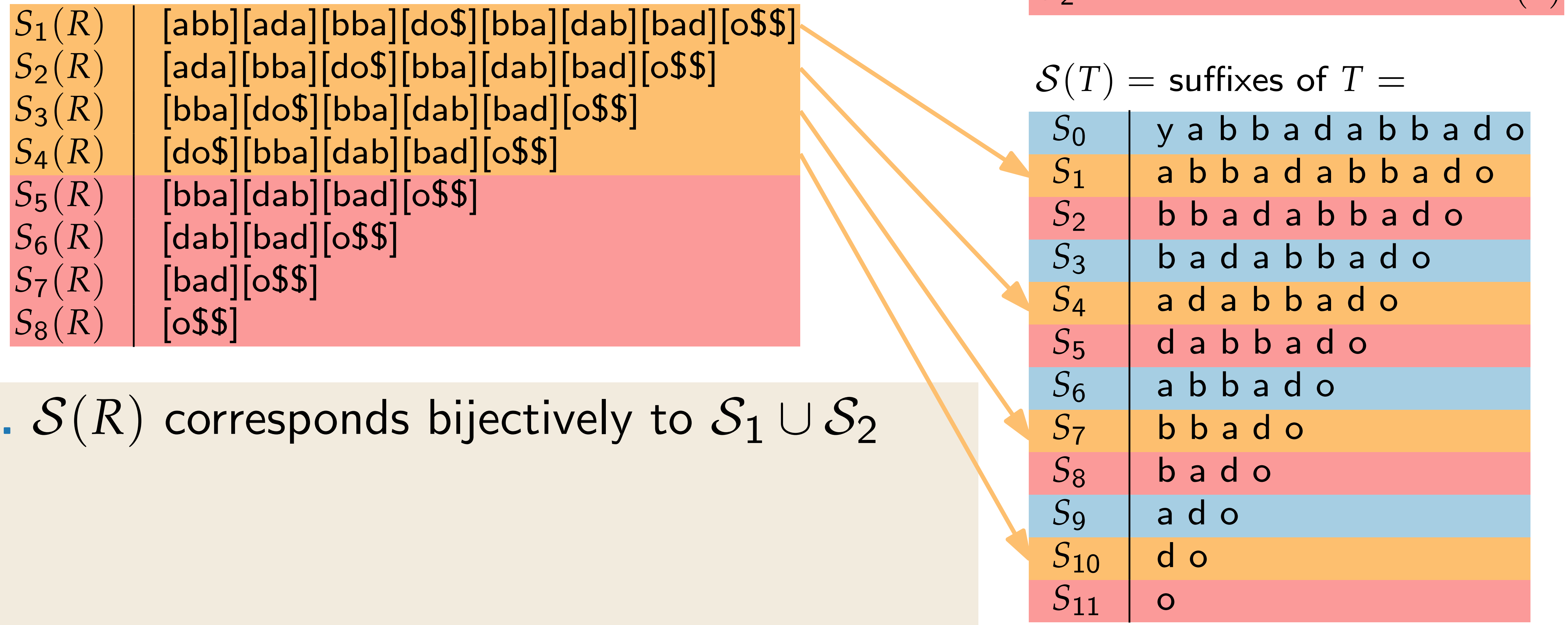
Partition $\mathcal{S}_1$ and $\mathcal{S}_2$ into triplets and concatenate them:

$\mathcal{S}_0$ = suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1$ = suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2$ = suffixes with index $i \equiv 2(3)$

$R =$ [abb][ada][bba][do\$][bba][dab][bad][o\$\$]



$\mathcal{S}(R) =$

$S_1(R)$	[abb][ada][bba][do\$][bba][dab][bad][o\$\$]
$S_2(R)$	[ada][bba][do\$][bba][dab][bad][o\$\$]
$S_3(R)$	[bba][do\$][bba][dab][bad][o\$\$]
$S_4(R)$	[do\$][bba][dab][bad][o\$\$]
$S_5(R)$	[bba][dab][bad][o\$\$]
$S_6(R)$	[dab][bad][o\$\$]
$S_7(R)$	[bad][o\$\$]
$S_8(R)$	[o\$\$]

$\mathcal{S}(T)$ = suffixes of T =

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Observation. $\mathcal{S}(R)$ corresponds bijectively to $\mathcal{S}_1 \cup \mathcal{S}_2$

Step 1: Sorting $\mathcal{S}_1 \cup \mathcal{S}_2$

Shortened notation: $T = t_0 t_1 \dots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for $x \bmod y = z$.

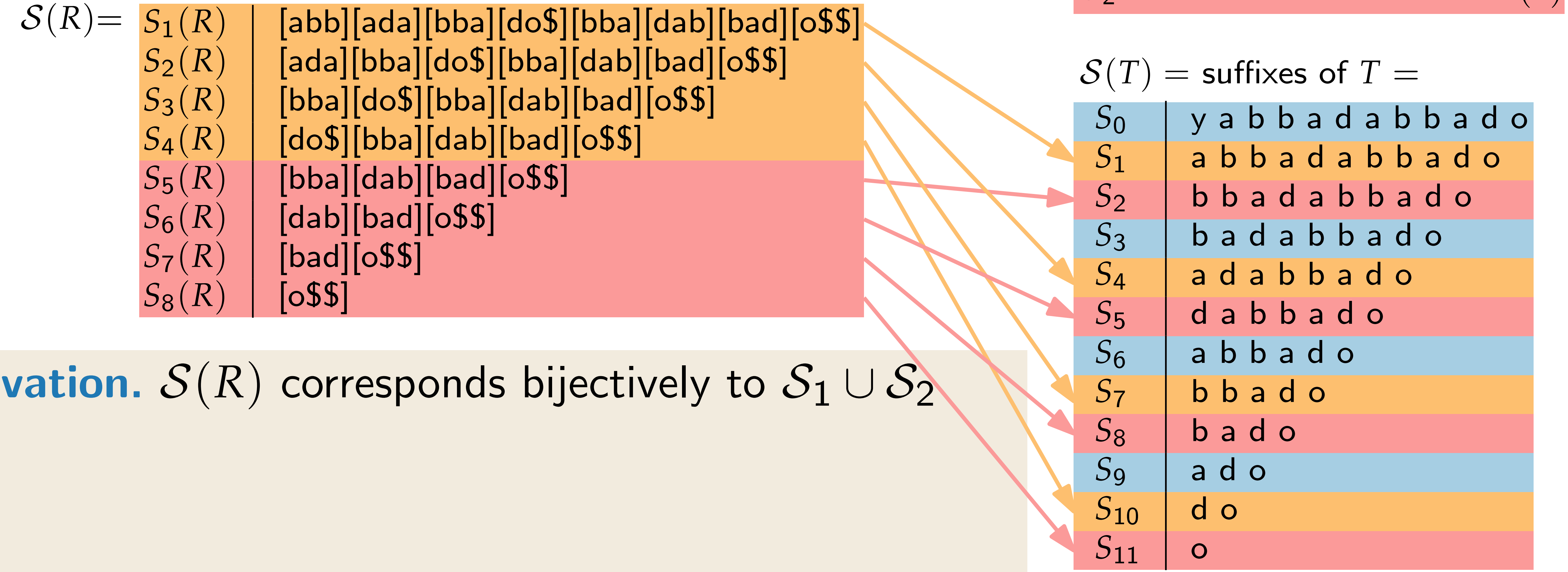
$\mathcal{S}_0 =$ suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1 =$ suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2 =$ suffixes with index $i \equiv 2(3)$

Partition $\mathcal{S}_1$ and $\mathcal{S}_2$ into triplets and concatenate them:

$R=$ [abb][ada][bba][do\$][bba][dab][bad][o\$\$]



$\mathcal{S}(R)=$

$S_1(R)$	[abb][ada][bba][do\$][bba][dab][bad][o\$\$]
$S_2(R)$	[ada][bba][do\$][bba][dab][bad][o\$\$]
$S_3(R)$	[bba][do\$][bba][dab][bad][o\$\$]
$S_4(R)$	[do\$][bba][dab][bad][o\$\$]
$S_5(R)$	[bba][dab][bad][o\$\$]
$S_6(R)$	[dab][bad][o\$\$]
$S_7(R)$	[bad][o\$\$]
$S_8(R)$	[o\$\$]

$\mathcal{S}(T) =$ suffixes of $T =$

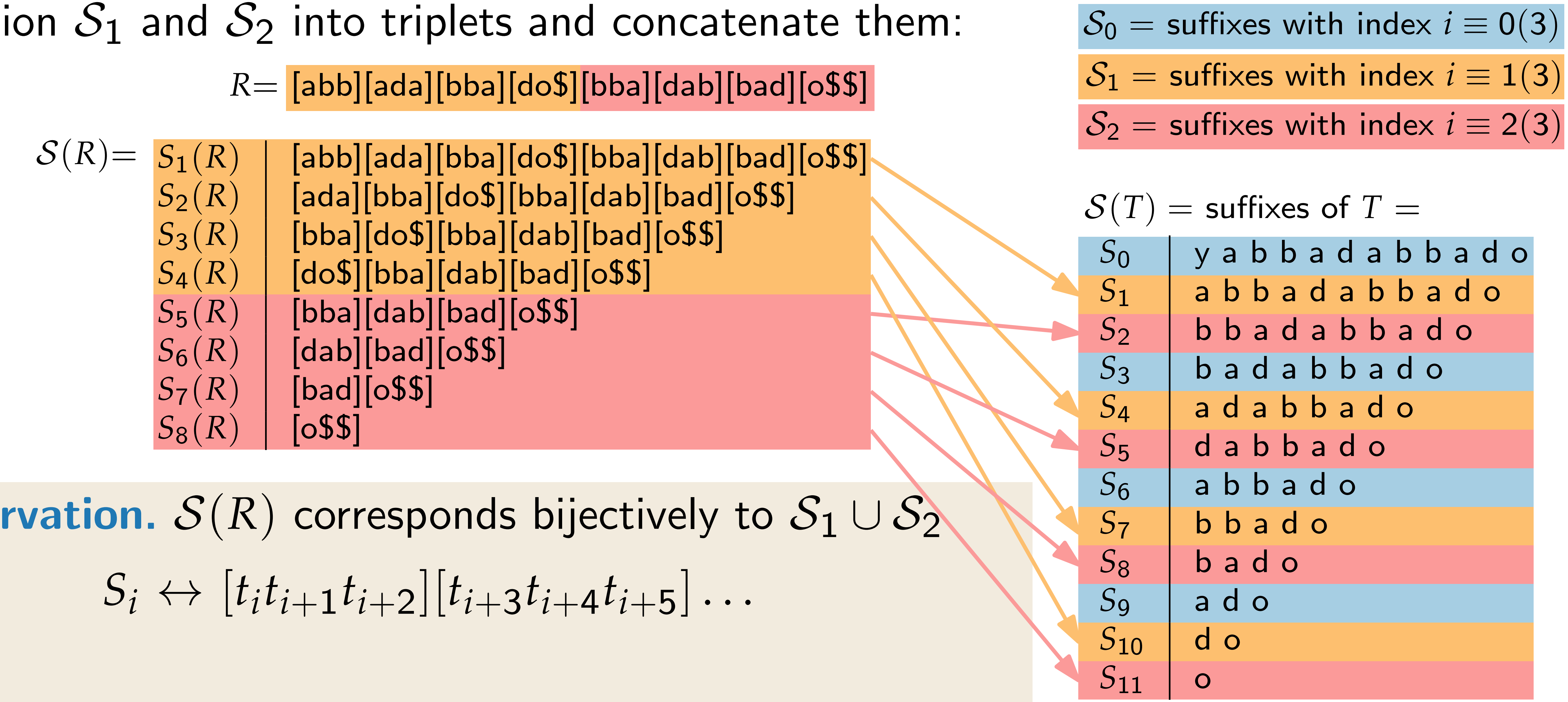
S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Observation. $\mathcal{S}(R)$ corresponds bijectively to $\mathcal{S}_1 \cup \mathcal{S}_2$

Step 1: Sorting $\mathcal{S}_1 \cup \mathcal{S}_2$

Shortened notation: $T = t_0 t_1 \dots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for x mod $y = z$.

Partition $\mathcal{S}_1$ and $\mathcal{S}_2$ into triplets and concatenate them:



$\mathcal{S}_0$ = suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1$ = suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2$ = suffixes with index $i \equiv 2(3)$

$R=$ [abb][ada][bba][do$][bba][dab][bad][o$$]

$\mathcal{S}(R)=$

$S_1(R)$	[abb][ada][bba][do$][bba][dab][bad][o$$]
$S_2(R)$	[ada][bba][do$][bba][dab][bad][o$$]
$S_3(R)$	[bba][do$][bba][dab][bad][o$$]
$S_4(R)$	[do$][bba][dab][bad][o$$]
$S_5(R)$	[bba][dab][bad][o$$]
$S_6(R)$	[dab][bad][o$$]
$S_7(R)$	[bad][o$$]
$S_8(R)$	[o$$]

$\mathcal{S}(T)$ = suffixes of T =

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Observation. $\mathcal{S}(R)$ corresponds bijectively to $\mathcal{S}_1 \cup \mathcal{S}_2$

$$S_i \leftrightarrow [t_i t_{i+1} t_{i+2}][t_{i+3} t_{i+4} t_{i+5}] \dots$$

Step 1: Sorting $\mathcal{S}_1 \cup \mathcal{S}_2$

Shortened notation: $T = t_0 t_1 \ldots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for x mod $y = z$.

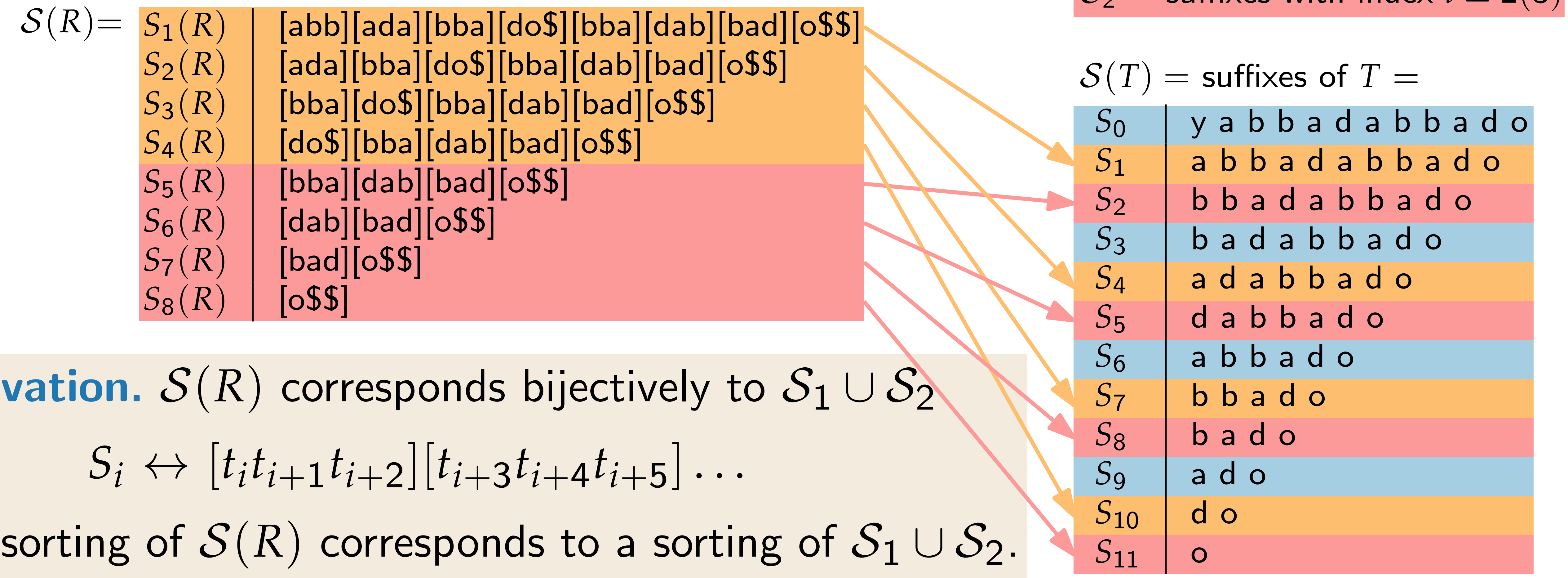
Partition $\mathcal{S}_1$ and $\mathcal{S}_2$ into triplets and concatenate them:

$\mathcal{S}_0$ = suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1$ = suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2$ = suffixes with index $i \equiv 2(3)$

$R=$ [abb][ada][bba][do$][bba][dab][bad][o$$]



$\mathcal{S}(R)=$

$S_1(R)$	[abb][ada][bba][do$][bba][dab][bad][o$$]
$S_2(R)$	[ada][bba][do$][bba][dab][bad][o$$]
$S_3(R)$	[bba][do$][bba][dab][bad][o$$]
$S_4(R)$	[do$][bba][dab][bad][o$$]
$S_5(R)$	[bba][dab][bad][o$$]
$S_6(R)$	[dab][bad][o$$]
$S_7(R)$	[bad][o$$]
$S_8(R)$	[o$$]

$\mathcal{S}(T)$ = suffixes of T =

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Observation. $\mathcal{S}(R)$ corresponds bijectively to $\mathcal{S}_1 \cup \mathcal{S}_2$

$$S_i \leftrightarrow [t_i t_{i+1} t_{i+2}][t_{i+3} t_{i+4} t_{i+5}] \ldots$$

and a sorting of $\mathcal{S}(R)$ corresponds to a sorting of $\mathcal{S}_1 \cup \mathcal{S}_2$.

Step 1: Sorting $\mathcal{S}_1 \cup \mathcal{S}_2$

$S_i < S_j \quad \Leftrightarrow \quad S_i\$ < S_j\$ \quad \Leftrightarrow \quad S_i\$\ldots < S_j\$\ldots$ since the positions of the first \$ symbols in the strings $S_k(R)$ are pairwise distinct.

Shortened notation: $T = t_0 t_1 \ldots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for $x \bmod y = z$.

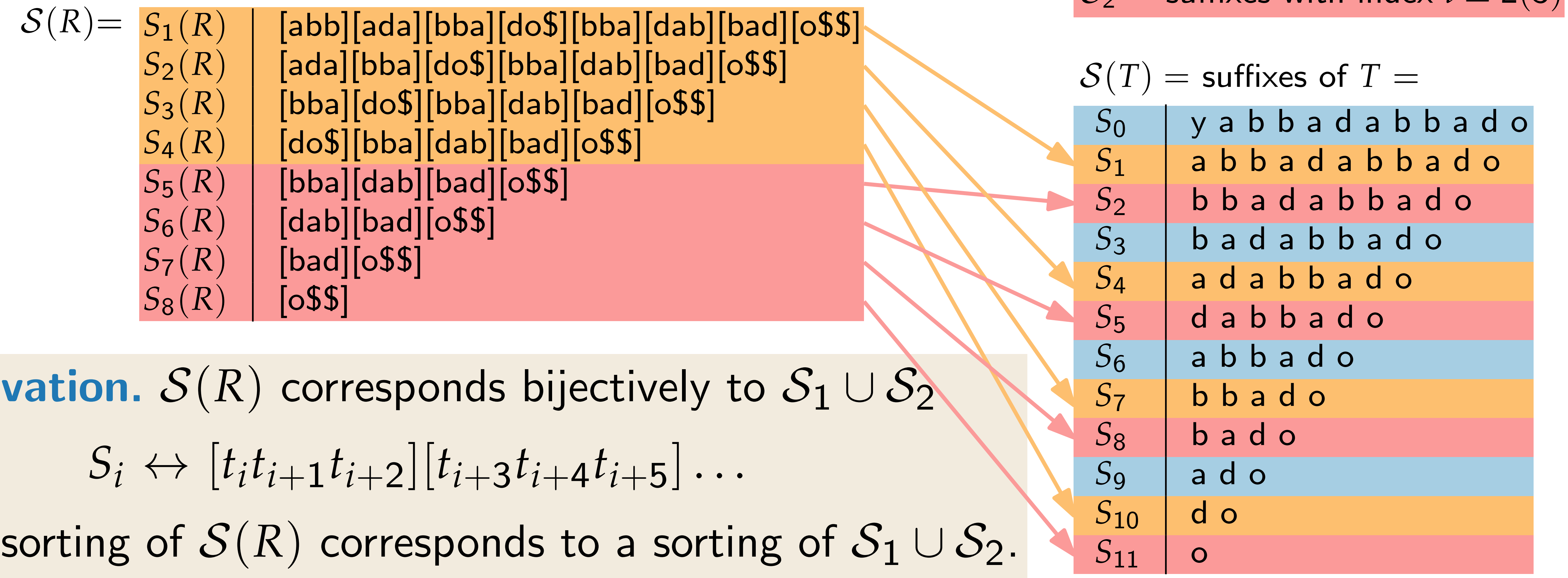
Partition $\mathcal{S}_1$ and $\mathcal{S}_2$ into triplets and concatenate them:

$\mathcal{S}_0$ = suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1$ = suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2$ = suffixes with index $i \equiv 2(3)$

$R=$ [abb][ada][bba][do\$][bba][dab][bad][o\$\$]



$\mathcal{S}(R)=$

$S_1(R)$	[abb][ada][bba][do\$][bba][dab][bad][o\$\$]
$S_2(R)$	[ada][bba][do\$][bba][dab][bad][o\$\$]
$S_3(R)$	[bba][do\$][bba][dab][bad][o\$\$]
$S_4(R)$	[do\$][bba][dab][bad][o\$\$]
$S_5(R)$	[bba][dab][bad][o\$\$]
$S_6(R)$	[dab][bad][o\$\$]
$S_7(R)$	[bad][o\$\$]
$S_8(R)$	[o\$\$]

$\mathcal{S}(T)$ = suffixes of T =

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Observation. $\mathcal{S}(R)$ corresponds bijectively to $\mathcal{S}_1 \cup \mathcal{S}_2$

$$S_i \leftrightarrow [t_i t_{i+1} t_{i+2}][t_{i+3} t_{i+4} t_{i+5}]\ldots$$

and a sorting of $\mathcal{S}(R)$ corresponds to a sorting of $\mathcal{S}_1 \cup \mathcal{S}_2$.

Sorting $\mathcal{S}(R)$

Sort the "letters" $(=$ triplets) of R via RADIXSORT. This can be done in time

$$\mathcal{O}\left(3\left(\tfrac{2}{3}n + |\Sigma|\right)\right) \subseteq \mathcal{O}(n)$$

$R=$ [abb][ada][bba][do$][bba][dab][bad][o$$]

Rank	triple
1	[abb]
2	[ada]
3	[bad]
4	[bba]
5	[dab]
6	[do$]
7	[o$$]

$\mathcal{S}(R)=$

$S_1(R)$	[abb][ada][bba][do$][bba][dab][bad][o$$]
$S_2(R)$	[ada][bba][do$][bba][dab][bad][o$$]
$S_3(R)$	[bba][do$][bba][dab][bad][o$$]
$S_4(R)$	[do$][bba][dab][bad][o$$]
$S_5(R)$	[bba][dab][bad][o$$]
$S_6(R)$	[dab][bad][o$$]
$S_7(R)$	[bad][o$$]
$S_8(R)$	[o$$]

Sorting $\mathcal{S}(R)$

Sort the "letters" (= triplets) of R via RADIXSORT. This can be done in time

$$\mathcal{O}(3(\tfrac{2}{3}n + |\Sigma|)) \subseteq \mathcal{O}(n)$$

#digits

$R=$ [abb][ada][bba][do$][bba][dab][bad][o$$]

Rank	**triple**
1	[abb]
2	[ada]
3	[bad]
4	[bba]
5	[dab]
6	[do$]
7	[o$$]

$\mathcal{S}(R)=$

$S_1(R)$	[abb][ada][bba][do$][bba][dab][bad][o$$]
$S_2(R)$	[ada][bba][do$][bba][dab][bad][o$$]
$S_3(R)$	[bba][do$][bba][dab][bad][o$$]
$S_4(R)$	[do$][bba][dab][bad][o$$]
$S_5(R)$	[bba][dab][bad][o$$]
$S_6(R)$	[dab][bad][o$$]
$S_7(R)$	[bad][o$$]
$S_8(R)$	[o$$]

Sorting $\mathcal{S}(R)$

Sort the "letters" (= triplets) of R via RadixSort. This can be done in time

$$\mathcal{O}(\underset{\#\text{digits}}{3}(\underset{\#\text{objects}}{\tfrac{2}{3}n} + |\Sigma|)) \subseteq \mathcal{O}(n)$$

$R=$ [abb][ada][bba][do\$][bba][dab][bad][o\$\$]

Rank	triple
1	[abb]
2	[ada]
3	[bad]
4	[bba]
5	[dab]
6	[do\$]
7	[o\$\$]

$\mathcal{S}(R)=$

$S_1(R)$	[abb][ada][bba][do\$][bba][dab][bad][o\$\$]
$S_2(R)$	[ada][bba][do\$][bba][dab][bad][o\$\$]
$S_3(R)$	[bba][do\$][bba][dab][bad][o\$\$]
$S_4(R)$	[do\$][bba][dab][bad][o\$\$]
$S_5(R)$	[bba][dab][bad][o\$\$]
$S_6(R)$	[dab][bad][o\$\$]
$S_7(R)$	[bad][o\$\$]
$S_8(R)$	[o\$\$]

Sorting $\mathcal{S}(R)$

Sort the "letters" (= triplets) of R via RadixSort. This can be done in time

$$\mathcal{O}(3(\tfrac{2}{3}n + |\Sigma|)) \subseteq \mathcal{O}(n)$$

#digits #objects alphabet size

$R=$ [abb][ada][bba][do$][bba][dab][bad][o$$]

Rank	triple
1	[abb]
2	[ada]
3	[bad]
4	[bba]
5	[dab]
6	[do$]
7	[o$$]

$\mathcal{S}(R)=$

$S_1(R)$	[abb][ada][bba][do$][bba][dab][bad][o$$]
$S_2(R)$	[ada][bba][do$][bba][dab][bad][o$$]
$S_3(R)$	[bba][do$][bba][dab][bad][o$$]
$S_4(R)$	[do$][bba][dab][bad][o$$]
$S_5(R)$	[bba][dab][bad][o$$]
$S_6(R)$	[dab][bad][o$$]
$S_7(R)$	[bad][o$$]
$S_8(R)$	[o$$]

Sorting $\mathcal{S}(R)$

Sort the “letters” (= triplets) of R via RADIXSORT. This can be done in time

$$\mathcal{O}(\underset{\#\text{digits}}{3}(\underset{\#\text{objects}}{\tfrac{2}{3}n} + \underset{\text{alphabet size}}{|\Sigma|})) \subseteq \mathcal{O}(n)$$

Replace each triplet of R by its rank → string R' with alphabet size $\leq \frac{2}{3}n \leq n$.

$R=$ [abb][ada][bba][do\$][bba][dab][bad][o\$\$]

$R'=$ 1 2 4 6 4 5 3 7

Rank	**triple**
1	[abb]
2	[ada]
3	[bad]
4	[bba]
5	[dab]
6	[do\$]
7	[o\$\$]

$\mathcal{S}(R)=$

$S_1(R)$	[abb][ada][bba][do\$][bba][dab][bad][o\$\$]
$S_2(R)$	[ada][bba][do\$][bba][dab][bad][o\$\$]
$S_3(R)$	[bba][do\$][bba][dab][bad][o\$\$]
$S_4(R)$	[do\$][bba][dab][bad][o\$\$]
$S_5(R)$	[bba][dab][bad][o\$\$]
$S_6(R)$	[dab][bad][o\$\$]
$S_7(R)$	[bad][o\$\$]
$S_8(R)$	[o\$\$]

Sorting $\mathcal{S}(R)$

Sort the "letters" (= triplets) of R via RadixSort. This can be done in time

$$\mathcal{O}(\underset{\#\text{digits}}{3}(\underset{\#\text{objects}}{\tfrac{2}{3}n} + \underset{\text{alphabet size}}{|\Sigma|})) \subseteq \mathcal{O}(n)$$

Replace each triplet of R by its rank → string R' with alphabet size $\leq \frac{2}{3}n \leq n$.

$R =$ [abb][ada][bba][do$][bba][dab][bad][o$$]

$R' =$ 1 2 4 6 4 5 3 7

Rank	**triple**
1	[abb]
2	[ada]
3	[bad]
4	[bba]
5	[dab]
6	[do$]
7	[o$$]

$\mathcal{S}(R) =$

$S_1(R)$	[abb][ada][bba][do$][bba][dab][bad][o$$]
$S_2(R)$	[ada][bba][do$][bba][dab][bad][o$$]
$S_3(R)$	[bba][do$][bba][dab][bad][o$$]
$S_4(R)$	[do$][bba][dab][bad][o$$]
$S_5(R)$	[bba][dab][bad][o$$]
$S_6(R)$	[dab][bad][o$$]
$S_7(R)$	[bad][o$$]
$S_8(R)$	[o$$]

$\mathcal{S}(R') =$

$S_1(R')$	1 2 4 6 4 5 3 7
$S_2(R')$	2 4 6 4 5 3 7
$S_3(R')$	4 6 4 5 3 7
$S_4(R')$	6 4 5 3 7
$S_5(R')$	4 5 3 7
$S_6(R')$	5 3 7
$S_7(R')$	3 7
$S_8(R')$	7

Sorting $\mathcal{S}(R)$

Sort the "letters" (= triplets) of R via RADIXSORT. This can be done in time

$$\mathcal{O}(3(\tfrac{2}{3}n + |\Sigma|)) \subseteq \mathcal{O}(n)$$

#digits #objects alphabet size

Replace each triplet of R by its rank → string R' with alphabet size $\leq \frac{2}{3}n \leq n$.

A sorting of $\mathcal{S}(R')$ corresponds to a sorting of $\mathcal{S}(R)$ and can be obtained recursively.

$R =$ [abb][ada][bba][do$][bba][dab][bad][o$$]

$R' =$ 1 2 4 6 4 5 3 7

Rank	triple
1	[abb]
2	[ada]
3	[bad]
4	[bba]
5	[dab]
6	[do$]
7	[o$$]

$\mathcal{S}(R) =$

$S_1(R)$	[abb][ada][bba][do$][bba][dab][bad][o$$]
$S_2(R)$	[ada][bba][do$][bba][dab][bad][o$$]
$S_3(R)$	[bba][do$][bba][dab][bad][o$$]
$S_4(R)$	[do$][bba][dab][bad][o$$]
$S_5(R)$	[bba][dab][bad][o$$]
$S_6(R)$	[dab][bad][o$$]
$S_7(R)$	[bad][o$$]
$S_8(R)$	[o$$]

$\mathcal{S}(R') =$

$S_1(R')$	1 2 4 6 4 5 3 7
$S_2(R')$	2 4 6 4 5 3 7
$S_3(R')$	4 6 4 5 3 7
$S_4(R')$	6 4 5 3 7
$S_5(R')$	4 5 3 7
$S_6(R')$	5 3 7
$S_7(R')$	3 7
$S_8(R')$	7

Sorting $\mathcal{S}(R)$

Sort the“letters” (= triplets) of R via RadixSort. This can be done in time

$$\mathcal{O}(3(\tfrac{2}{3}n + |\Sigma|)) \subseteq \mathcal{O}(n)$$

#digits #objects alphabet size

ConstructSuffixArray(R')

Replace each triplet of R by its rank → string R' with alphabet size $\leq \frac{2}{3}n \leq n$.

A sorting of $\mathcal{S}(R')$ corresponds to a sorting of $\mathcal{S}(R)$ and can be obtained recursively.

$R=$ [abb][ada][bba][do\$][bba][dab][bad][o\$\$]

$R'=$ 1 2 4 6 4 5 3 7

Rank	triple
1	[abb]
2	[ada]
3	[bad]
4	[bba]
5	[dab]
6	[do\$]
7	[o\$\$]

$\mathcal{S}(R)=$

$S_1(R)$	[abb][ada][bba][do\$][bba][dab][bad][o\$\$]
$S_2(R)$	[ada][bba][do\$][bba][dab][bad][o\$\$]
$S_3(R)$	[bba][do\$][bba][dab][bad][o\$\$]
$S_4(R)$	[do\$][bba][dab][bad][o\$\$]
$S_5(R)$	[bba][dab][bad][o\$\$]
$S_6(R)$	[dab][bad][o\$\$]
$S_7(R)$	[bad][o\$\$]
$S_8(R)$	[o\$\$]

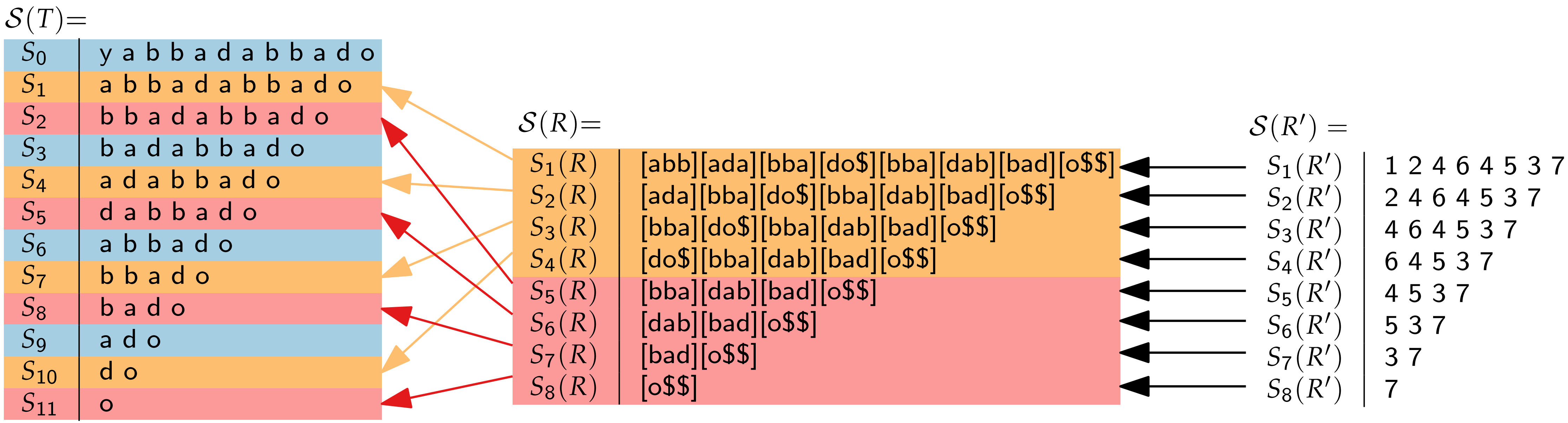
$\mathcal{S}(R') =$

$S_1(R')$	1 2 4 6 4 5 3 7
$S_2(R')$	2 4 6 4 5 3 7
$S_3(R')$	4 6 4 5 3 7
$S_4(R')$	6 4 5 3 7
$S_5(R')$	4 5 3 7
$S_6(R')$	5 3 7
$S_7(R')$	3 7
$S_8(R')$	7

Summary of Step 1

Rank	triple
1	[abb]
2	[ada]
3	[bad]
4	[bba]
5	[dab]
6	[do$]
7	[o$$]

Full example.



$\mathcal{S}(T)=$

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

$\mathcal{S}(R)=$

$S_1(R)$	[abb][ada][bba][do$][bba][dab][bad][o$$]
$S_2(R)$	[ada][bba][do$][bba][dab][bad][o$$]
$S_3(R)$	[bba][do$][bba][dab][bad][o$$]
$S_4(R)$	[do$][bba][dab][bad][o$$]
$S_5(R)$	[bba][dab][bad][o$$]
$S_6(R)$	[dab][bad][o$$]
$S_7(R)$	[bad][o$$]
$S_8(R)$	[o$$]

$\mathcal{S}(R') =$

$S_1(R')$	1 2 4 6 4 5 3 7
$S_2(R')$	2 4 6 4 5 3 7
$S_3(R')$	4 6 4 5 3 7
$S_4(R')$	6 4 5 3 7
$S_5(R')$	4 5 3 7
$S_6(R')$	5 3 7
$S_7(R')$	3 7
$S_8(R')$	7

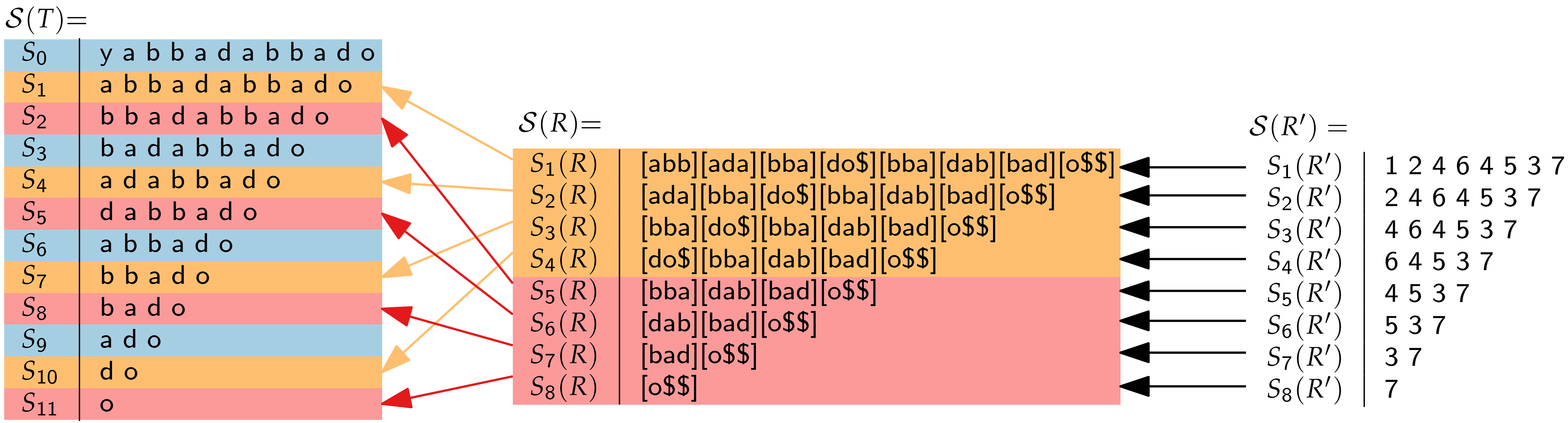
A_{12}

1	S_1	a b b a d a b b a d o	$S_1(R')$	1 2 4 6 4 5 3 7
2	S_4	a d a b b a d o	$S_2(R')$	2 4 6 4 5 3 7
3	S_8	b a d o	$S_7(R')$	3 7
4	S_2	b b a d a b b a d o	$S_5(R')$	4 5 3 7
5	S_7	b b a d o	$S_3(R')$	4 6 4 5 3 7
6	S_5	d a b b a d o	$S_6(R')$	5 3 7
7	S_{10}	d o	$S_4(R')$	6 4 5 3 7
8	S_{11}	o	$S_8(R')$	7

Summary of Step 1

Rank	triple
1	[abb]
2	[ada]
3	[bad]
4	[bba]
5	[dab]
6	[do$]
7	[o$$]

Full example.



$\mathcal{S}(T)=$

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

$\mathcal{S}(R)=$

$S_1(R)$	[abb][ada][bba][do$][bba][dab][bad][o$$]
$S_2(R)$	[ada][bba][do$][bba][dab][bad][o$$]
$S_3(R)$	[bba][do$][bba][dab][bad][o$$]
$S_4(R)$	[do$][bba][dab][bad][o$$]
$S_5(R)$	[bba][dab][bad][o$$]
$S_6(R)$	[dab][bad][o$$]
$S_7(R)$	[bad][o$$]
$S_8(R)$	[o$$]

$\mathcal{S}(R')=$

$S_1(R')$	1 2 4 6 4 5 3 7
$S_2(R')$	2 4 6 4 5 3 7
$S_3(R')$	4 6 4 5 3 7
$S_4(R')$	6 4 5 3 7
$S_5(R')$	4 5 3 7
$S_6(R')$	5 3 7
$S_7(R')$	3 7
$S_8(R')$	7

A_{12}

1	S_1	a b b a d a b b a d o	$S_1(R')$	1 2 4 6 4 5 3 7
2	S_4	a d a b b a d o	$S_2(R')$	2 4 6 4 5 3 7
3	S_8	b a d o	$S_7(R')$	3 7
4	S_2	b b a d a b b a d o	$S_5(R')$	4 5 3 7
5	S_7	b b a d o	$S_3(R')$	4 6 4 5 3 7
6	S_5	d a b b a d o	$S_6(R')$	5 3 7
7	S_{10}	d o	$S_4(R')$	6 4 5 3 7
8	S_{11}	o	$S_8(R')$	7

Running time of Step 1.

$$Z_1(n) = \mathcal{O}(n) + Z(\tfrac{2}{3}n)$$

where $Z(n)$ is the time to execute ConstructSuffixArray on a string of length n.

Construction of Suffix Arrays – Overview

Shortened notation: $T = t_0 t_1 \dots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for x mod $y = z$.

	0	1	2	3	4	5	6	7	8	9	10	11
$T=$	y	a	b	b	a	d	a	b	b	a	d	o

$\mathcal{S}_0$ = suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1$ = suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2$ = suffixes with index $i \equiv 2(3)$

```
ConstructSuffixArray(string $T$)
    if $n \in \mathcal{O}(1)$ then
        construct $A$ in $\mathcal{O}(1)$ time.
    else
        sort $\mathcal{S}_1 \cup \mathcal{S}_2$ into an array $A_{12}$
        use $A_{12}$ to sort $\mathcal{S}_0$ into an array $A_0$
        merge $A_{12}$ with $A_0$
```

For simplicity, we assume $n \equiv 0(3)$.

$\mathcal{S}(T)$ = suffixes of T =

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Step 2: Sorting $\mathcal{S}_0$

Shortened notation: $T = t_0 t_1 \ldots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for x mod $y = z$.

	0	1	2	3	4	5	6	7	8	9	10	11
$T=$	y	a	b	b	a	d	a	b	b	a	d	o

$\mathcal{S}_0 =$ suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1 =$ suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2 =$ suffixes with index $i \equiv 2(3)$

$\mathcal{S}(T) =$ suffixes of $T =$

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Step 2: Sorting $\mathcal{S}_0$

Shortened notation: $T = t_0 t_1 \dots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for $x \bmod y = z$.

	0	1	2	3	4	5	6	7	8	9	10	11
$T=$	y	a	b	b	a	d	a	b	b	a	d	o

$\mathcal{S}_0 =$ suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1 =$ suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2 =$ suffixes with index $i \equiv 2(3)$

Each $S_i \in \mathcal{S}_0$ can be written as (t_i, S_{i+1}) s.t. $S_{i+1} \in \mathcal{S}_1$.

$\mathcal{S}(T) =$ suffixes of $T =$

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Step 2: Sorting $\mathcal{S}_0$

Shortened notation: $T = t_0 t_1 \dots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for $x \bmod y = z$.

	0	1	2	3	4	5	6	7	8	9	10	11
$T=$	y	a	b	b	a	d	a	b	b	a	d	o

$\mathcal{S}_0$ = suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1$ = suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2$ = suffixes with index $i \equiv 2(3)$

Each $S_i \in \mathcal{S}_0$ can be written as (t_i, S_{i+1}) s.t. $S_{i+1} \in \mathcal{S}_1$.

Observation. Let $S_i, S_j \in \mathcal{S}_0$. Then $S_i < S_j$ if and only if

- $t_i < t_j$; or
- $t_i = t_j$ and $S_{i+1} < S_{j+1}$.

$\mathcal{S}(T)$ = suffixes of T =

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Step 2: Sorting $\mathcal{S}_0$

Shortened notation: $T = t_0 t_1 \ldots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for x mod $y = z$.

	0	1	2	3	4	5	6	7	8	9	10	11
$T=$	y	a	b	b	a	d	a	b	b	a	d	o

$\mathcal{S}_0$ = suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1$ = suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2$ = suffixes with index $i \equiv 2(3)$

Each $S_i \in \mathcal{S}_0$ can be written as (t_i, S_{i+1}) s.t. $S_{i+1} \in \mathcal{S}_1$.

Observation. Let $S_i, S_j \in \mathcal{S}_0$. Then $S_i < S_j$ if and only if

- $t_i < t_j$; or
- $t_i = t_j$ and $S_{i+1} < S_{j+1}$.

$\Rightarrow$ $\mathcal{S}_o$ can be sorted by sorting all tuples (t_i, S_{i+1}) with $i \equiv 0(3)$. This can be done via RadixSort in $\mathcal{O}(n)$ time since the ordering of the entries in $\mathcal{S}_1$ is already implicit in A_{12}.

$\mathcal{S}(T)$ = suffixes of T =

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Construction of Suffix Arrays – Overview

Shortened notation: $T = t_0 t_1 \ldots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for $x \bmod y = z$.

	0	1	2	3	4	5	6	7	8	9	10	11
$T=$	y	a	b	b	a	d	a	b	b	a	d	o

$\mathcal{S}_0 =$ suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1 =$ suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2 =$ suffixes with index $i \equiv 2(3)$

```
ConstructSuffixArray(string $T$)
    if $n \in \mathcal{O}(1)$ then
        construct $A$ in $\mathcal{O}(1)$ time.
    else
        sort $\mathcal{S}_1 \cup \mathcal{S}_2$ into an array $A_{12}$
        use $A_{12}$ to sort $\mathcal{S}_0$ into an array $A_0$
        merge $A_{12}$ with $A_0$
```

$\mathcal{S}(T) =$ suffixes of $T =$

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

For simplicity, we assume $n \equiv 0(3)$.

Step 3: Merging A_{12} and A_0

Shortened notation: $T = t_0 t_1 \ldots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for x mod $y = z$.

	0	1	2	3	4	5	6	7	8	9	10	11
$T=$	y	a	b	b	a	d	a	b	b	a	d	o

$\mathcal{S}_0$ = suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1$ = suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2$ = suffixes with index $i \equiv 2(3)$

$\mathcal{S}(T)$ = suffixes of T =

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Step 3: Merging A_{12} and A_0

Shortened notation: $T = t_0 t_1 \dots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for $x \bmod y = z$.

	0	1	2	3	4	5	6	7	8	9	10	11
$T=$	y	a	b	b	a	d	a	b	b	a	d	o

$\mathcal{S}_0$ = suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1$ = suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2$ = suffixes with index $i \equiv 2(3)$

Each $S_i \in \mathcal{S}_0$ can be written as (t_i, S_{i+1}) s.t. $S_{i+1} \in \mathcal{S}_1$ and as (t_i, t_{i+1}, S_{i+2}) s.t. $S_{i+2} \in \mathcal{S}_2$.

$\mathcal{S}(T)$ = suffixes of T =

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Step 3: Merging A_{12} and A_0

Shortened notation: $T = t_0 t_1 \dots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for x mod $y = z$.

	0	1	2	3	4	5	6	7	8	9	10	11
$T=$	y	a	b	b	a	d	a	b	b	a	d	o

$\mathcal{S}_0$ = suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1$ = suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2$ = suffixes with index $i \equiv 2(3)$

Each $S_i \in \mathcal{S}_0$ can be written as (t_i, S_{i+1}) s.t. $S_{i+1} \in \mathcal{S}_1$ and as (t_i, t_{i+1}, S_{i+2}) s.t. $S_{i+2} \in \mathcal{S}_2$.

Observation. Let $S_i \in \mathcal{S}_0$.

- Let $S_j \in \mathcal{S}_1$. Then $S_i < S_j$ if and only if
  - $t_i < t_j$; or
  - $t_i = t_j$ and $S_{i+1} < S_{j+1}$ where $S_{j+1} \in \mathcal{S}_2$.
- Let $S_j \in \mathcal{S}_2$. Then $S_i < S_j$ if and only if
  - $t_i < t_j$; or
  - $t_i = t_j$ and $t_{i+1} < t_{j+1}$; or
  - $t_i t_{i+1} = t_j t_{j+1}$ and $S_{i+2} < S_{j+2}$ where $S_{j+2} \in \mathcal{S}_1$.

$\mathcal{S}(T)$ = suffixes of T =

S_0	y a b b a d a b b a d o
S_1	a b b a d a b b a d o
S_2	b b a d a b b a d o
S_3	b a d a b b a d o
S_4	a d a b b a d o
S_5	d a b b a d o
S_6	a b b a d o
S_7	b b a d o
S_8	b a d o
S_9	a d o
S_{10}	d o
S_{11}	o

Step 3: Merging A_{12} and A_0

Shortened notation: $T = t_0 t_1 \dots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for x mod $y = z$.

	0	1	2	3	4	5	6	7	8	9	10	11
$T=$	y	a	b	b	a	d	a	b	b	a	d	o

$\mathcal{S}_0$ = suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1$ = suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2$ = suffixes with index $i \equiv 2(3)$

Each $S_i \in \mathcal{S}_0$ can be written as (t_i, S_{i+1}) s.t. $S_{i+1} \in \mathcal{S}_1$ and as (t_i, t_{i+1}, S_{i+2}) s.t. $S_{i+2} \in \mathcal{S}_2$.

Observation. Let $S_i \in \mathcal{S}_0$.

- Let $S_j \in \mathcal{S}_1$. Then $S_i < S_j$ if and only if
  - $t_i < t_j$; or
  - $t_i = t_j$ and $S_{i+1} < S_{j+1}$ where $S_{j+1} \in \mathcal{S}_2$.
- Let $S_j \in \mathcal{S}_2$. Then $S_i < S_j$ if and only if
  - $t_i < t_j$; or
  - $t_i = t_j$ and $t_{i+1} < t_{j+1}$; or
  - $t_i t_{i+1} = t_j t_{j+1}$ and $S_{i+2} < S_{j+2}$ where $S_{j+2} \in \mathcal{S}_1$.

Since the ordering of $\mathcal{S}_1 \cup \mathcal{S}_2$ is already implicit in A_{12}, we can perform these comparisons in $\mathcal{O}(1)$ time.

Step 3: Merging A_{12} and A_0

Shortened notation: $T = t_0 t_1 \dots t_{n-1}$ and $x \equiv z(y)$ is a shorthand for $x \bmod y = z$.

	0	1	2	3	4	5	6	7	8	9	10	11
$T=$	y	a	b	b	a	d	a	b	b	a	d	o

$\mathcal{S}_0$ = suffixes with index $i \equiv 0(3)$

$\mathcal{S}_1$ = suffixes with index $i \equiv 1(3)$

$\mathcal{S}_2$ = suffixes with index $i \equiv 2(3)$

Each $S_i \in \mathcal{S}_0$ can be written as (t_i, S_{i+1}) s.t. $S_{i+1} \in \mathcal{S}_1$ and as (t_i, t_{i+1}, S_{i+2}) s.t. $S_{i+2} \in \mathcal{S}_2$.

Observation. Let $S_i \in \mathcal{S}_0$.

- Let $S_j \in \mathcal{S}_1$. Then $S_i < S_j$ if and only if
  - $t_i < t_j$; or
  - $t_i = t_j$ and $S_{i+1} < S_{j+1}$ where $S_{j+1} \in \mathcal{S}_2$.
- Let $S_j \in \mathcal{S}_2$. Then $S_i < S_j$ if and only if
  - $t_i < t_j$; or
  - $t_i = t_j$ and $t_{i+1} < t_{j+1}$; or
  - $t_i t_{i+1} = t_j t_{j+1}$ and $S_{i+2} < S_{j+2}$ where $S_{j+2} \in \mathcal{S}_1$.

Since the ordering of $\mathcal{S}_1 \cup \mathcal{S}_2$ is already implicit in A_{12}, we can perform these comparisons in $\mathcal{O}(1)$ time.

$\Rightarrow$ A_{12} and A_0 can be merged as in MergeSort to obtain A.

Construction of Suffix Arrays – Summary

```
ConstructSuffixArray(string $T$)
    if $n \in \mathcal{O}(1)$ then
        construct $A$ in $\mathcal{O}(1)$ time.
    else
        sort $\mathcal{S}_1 \cup \mathcal{S}_2$ into an array $A_{12}$
        use $A_{12}$ to sort $\mathcal{S}_0$ into an array $A_0$
        merge $A_{12}$ with $A_0$
```

Runtime of each step:

Construction of Suffix Arrays – Summary

```
ConstructSuffixArray(string $T$)
    if $n \in \mathcal{O}(1)$ then
        construct $A$ in $\mathcal{O}(1)$ time.
    else
        sort $\mathcal{S}_1 \cup \mathcal{S}_2$ into an array $A_{12}$
        use $A_{12}$ to sort $\mathcal{S}_0$ into an array $A_0$
        merge $A_{12}$ with $A_0$
```

Runtime of each step:

$$\mathcal{O}(n) + Z\left(\tfrac{2}{3}n\right)$$

Construction of Suffix Arrays – Summary

```
ConstructSuffixArray(string $T$)
    if $n \in \mathcal{O}(1)$ then
        construct $A$ in $\mathcal{O}(1)$ time.
    else
        sort $\mathcal{S}_1 \cup \mathcal{S}_2$ into an array $A_{12}$
        use $A_{12}$ to sort $\mathcal{S}_0$ into an array $A_0$
        merge $A_{12}$ with $A_0$
```

Runtime of each step:

$\mathcal{O}(n) + Z\left(\frac{2}{3}n\right)$

$\mathcal{O}(n)$

Construction of Suffix Arrays – Summary

```
ConstructSuffixArray(string $T$)
    if $n \in \mathcal{O}(1)$ then
        construct $A$ in $\mathcal{O}(1)$ time.
    else
        sort $\mathcal{S}_1 \cup \mathcal{S}_2$ into an array $A_{12}$
        use $A_{12}$ to sort $\mathcal{S}_0$ into an array $A_0$
        merge $A_{12}$ with $A_0$
```

Runtime of each step:

$\mathcal{O}(n) + Z\left(\frac{2}{3}n\right)$

$\mathcal{O}(n)$

$\mathcal{O}(n)$

Construction of Suffix Arrays – Summary

```
ConstructSuffixArray(string $T$)
    if $n \in \mathcal{O}(1)$ then
        construct $A$ in $\mathcal{O}(1)$ time.
    else
        sort $\mathcal{S}_1 \cup \mathcal{S}_2$ into an array $A_{12}$
        use $A_{12}$ to sort $\mathcal{S}_0$ into an array $A_0$
        merge $A_{12}$ with $A_0$
```

Runtime of each step:

$\mathcal{O}(n) + Z\left(\frac{2}{3}n\right)$

$\mathcal{O}(n)$

$\mathcal{O}(n)$

Total running time:

Construction of Suffix Arrays – Summary

```
ConstructSuffixArray(string $T$)
    if $n \in \mathcal{O}(1)$ then
        construct $A$ in $\mathcal{O}(1)$ time.
    else
        sort $\mathcal{S}_1 \cup \mathcal{S}_2$ into an array $A_{12}$
        use $A_{12}$ to sort $\mathcal{S}_0$ into an array $A_0$
        merge $A_{12}$ with $A_0$
```

Runtime of each step:

$\mathcal{O}(n) + Z\left(\frac{2}{3}n\right)$

$\mathcal{O}(n)$

$\mathcal{O}(n)$

Total running time:

$$Z(n) = \begin{cases} \mathcal{O}(1), & \text{if } n = \mathcal{O}(1) \\ \mathcal{O}(n) + Z\left(\frac{2}{3}n\right), & \text{otherwise} \end{cases}$$

Construction of Suffix Arrays – Summary

```
ConstructSuffixArray(string $T$)
    if $n \in \mathcal{O}(1)$ then
        construct $A$ in $\mathcal{O}(1)$ time.
    else
        sort $\mathcal{S}_1 \cup \mathcal{S}_2$ into an array $A_{12}$
        use $A_{12}$ to sort $\mathcal{S}_0$ into an array $A_0$
        merge $A_{12}$ with $A_0$
```

Runtime of each step:

$\mathcal{O}(n) + Z(\frac{2}{3}n)$

$\mathcal{O}(n)$

$\mathcal{O}(n)$

Total running time:

$$Z(n) = \begin{cases} \mathcal{O}(1), & \text{if } n = \mathcal{O}(1) \\ \mathcal{O}(n) + Z(\frac{2}{3}n), & \text{otherwise} \end{cases}$$

$$\overset{\text{Master Theorem}}{\Rightarrow} Z(n) \in \mathcal{O}(n)$$

Summary and Discussion

Let T be a string over an alphabet Σ where $n = |T|$.

Lemma. A suffix array for T can be used to compute an LCP ("longest common prefix") array and a suffix tree of T in $\mathcal{O}(n)$ time. [without proof]

Summary and Discussion

Let T be a string over an alphabet Σ where $n = |T|$.

Lemma. A suffix array for T can be used to compute an LCP ("longest common prefix") array and a suffix tree of T in $\mathcal{O}(n)$ time. [without proof]

Theorem. A suffix tree for T can computed in $\mathcal{O}(n)$ time and space. It can be used to answer String Matching queries of length m in $\mathcal{O}(m \log |\Sigma| + k)$ time.

Summary and Discussion

Let T be a string over an alphabet Σ where $n = |T|$.

Lemma. A suffix array for T can be used to compute an LCP ("longest common prefix") array and a suffix tree of T in $\mathcal{O}(n)$ time. [without proof]

Theorem. A suffix tree for T can computed in $\mathcal{O}(n)$ time and space. It can be used to answer String Matching queries of length m in $\mathcal{O}(m \log |\Sigma| + k)$ time.

Theorem. A suffix array for T can computed in $\mathcal{O}(n)$ time and space. It can be used to answer String Matching queries of length m in $\mathcal{O}(m \log n + k)$ time.

Summary and Discussion

Let T be a string over an alphabet Σ where $n = |T|$.

Lemma. A suffix array for T can be used to compute an LCP ("longest common prefix") array and a suffix tree of T in $\mathcal{O}(n)$ time. [without proof]

Theorem. A suffix tree for T can computed in $\mathcal{O}(n)$ time and space. It can be used to answer String Matching queries of length m in $\mathcal{O}(m \log |\Sigma| + k)$ time.

Theorem. A suffix array for T can computed in $\mathcal{O}(n)$ time and space. It can be used to answer String Matching queries of length m in $\mathcal{O}(m \log n + k)$ time.

Remark. The suffix array is a simpler and more compact alternative to the suffix tree.

Summary and Discussion

Let T be a string over an alphabet Σ where $n = |T|$.

Lemma. A suffix array for T can be used to compute an LCP ("longest common prefix") array and a suffix tree of T in $\mathcal{O}(n)$ time. [without proof]

Theorem. A suffix tree for T can computed in $\mathcal{O}(n)$ time and space. It can be used to answer $\textsc{String Matching}$ queries of length m in $\mathcal{O}(m \log |\Sigma| + k)$ time.

Theorem. A suffix array for T can computed in $\mathcal{O}(n)$ time and space. It can be used to answer $\textsc{String Matching}$ queries of length m in $\mathcal{O}(m \log n + k)$ time.

Remark. The suffix array is a simpler and more compact alternative to the suffix tree.

The suffix tree (and the suffix array + LCP array) have several additional applications:

- Finding the longest repeated substring.
- Finding the longest common substring of two strings.
- ...

Literature and References

The content of this presentation is based on Dorothea Wagner’s slides for a lecture on “String-Matching: Suffixbäume“ as part of the course “Algorithmen II“ held at KIT WS 13/14. Most figures and examples were taken from these slides.

Literature:

- Simple Linear Work Suffix Array Construction. Kärkkäinen and Sanders, ICALP’03
- Optimal suffix tree construction with large alphabets. Farach, FOCS’97
- Algorithms on Strings, Trees and Sequences: Computer Science and Computational Biology. Gusfield, 1999, Cambridge University Press