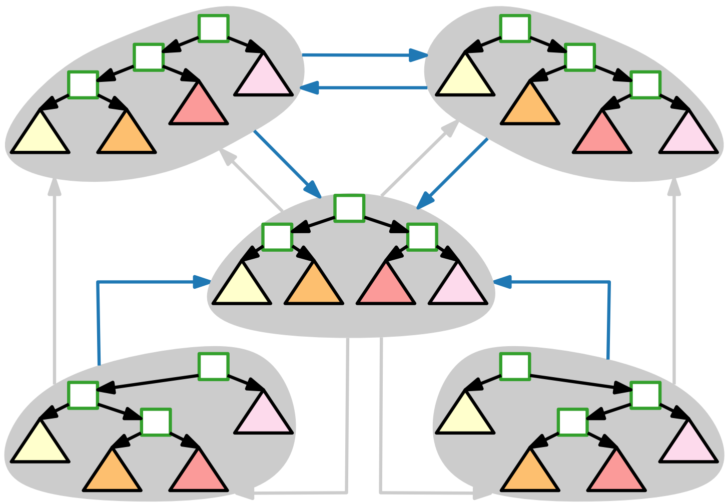
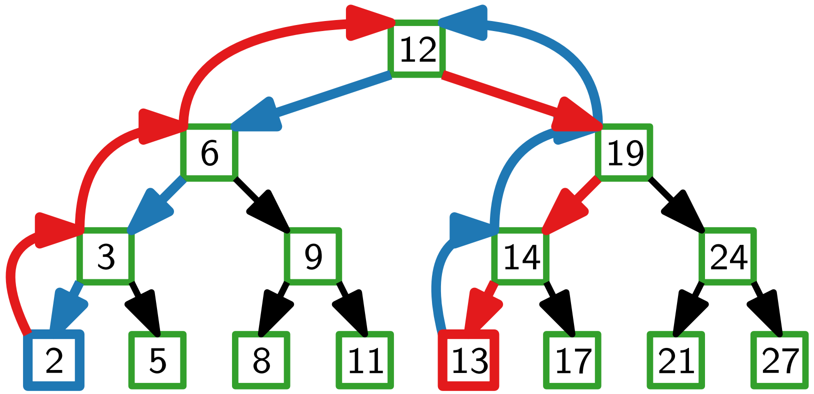
Advanced Algorithms

Optimal Binary Search Trees

Splay Trees



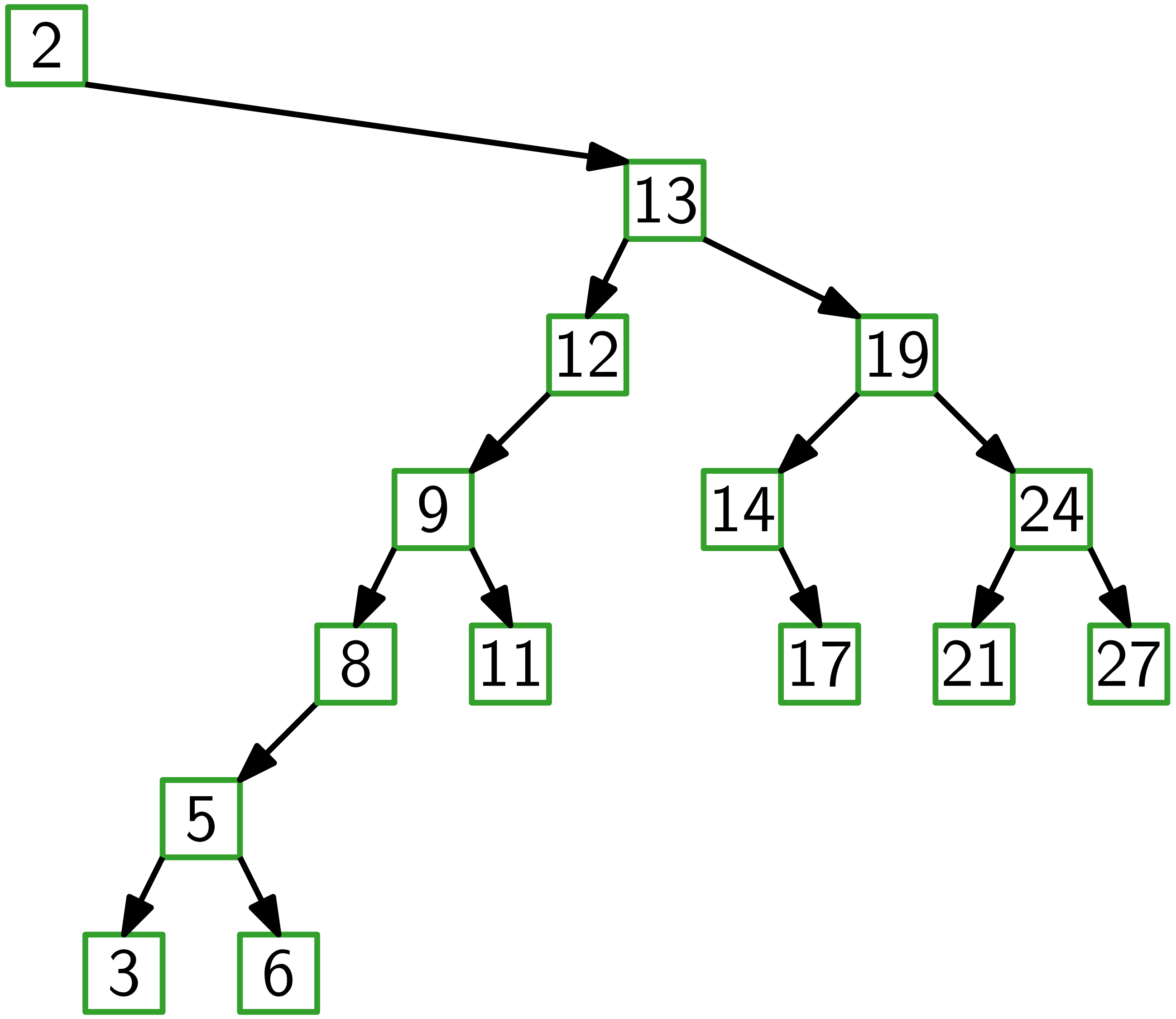


Alexander Wolff

Summer term 2026

How Good is a Binary Search Tree?

Binary search tree (BST):



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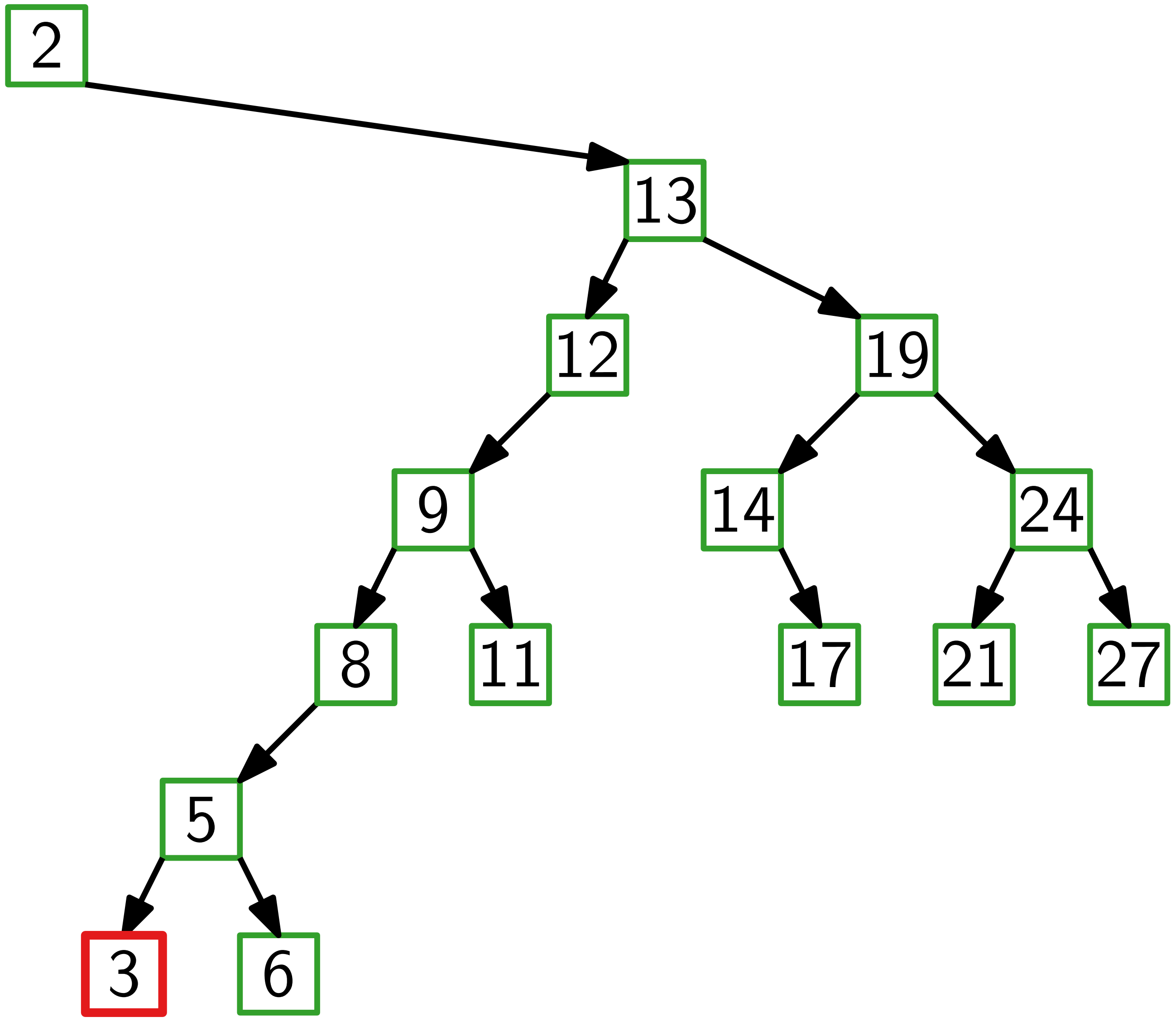

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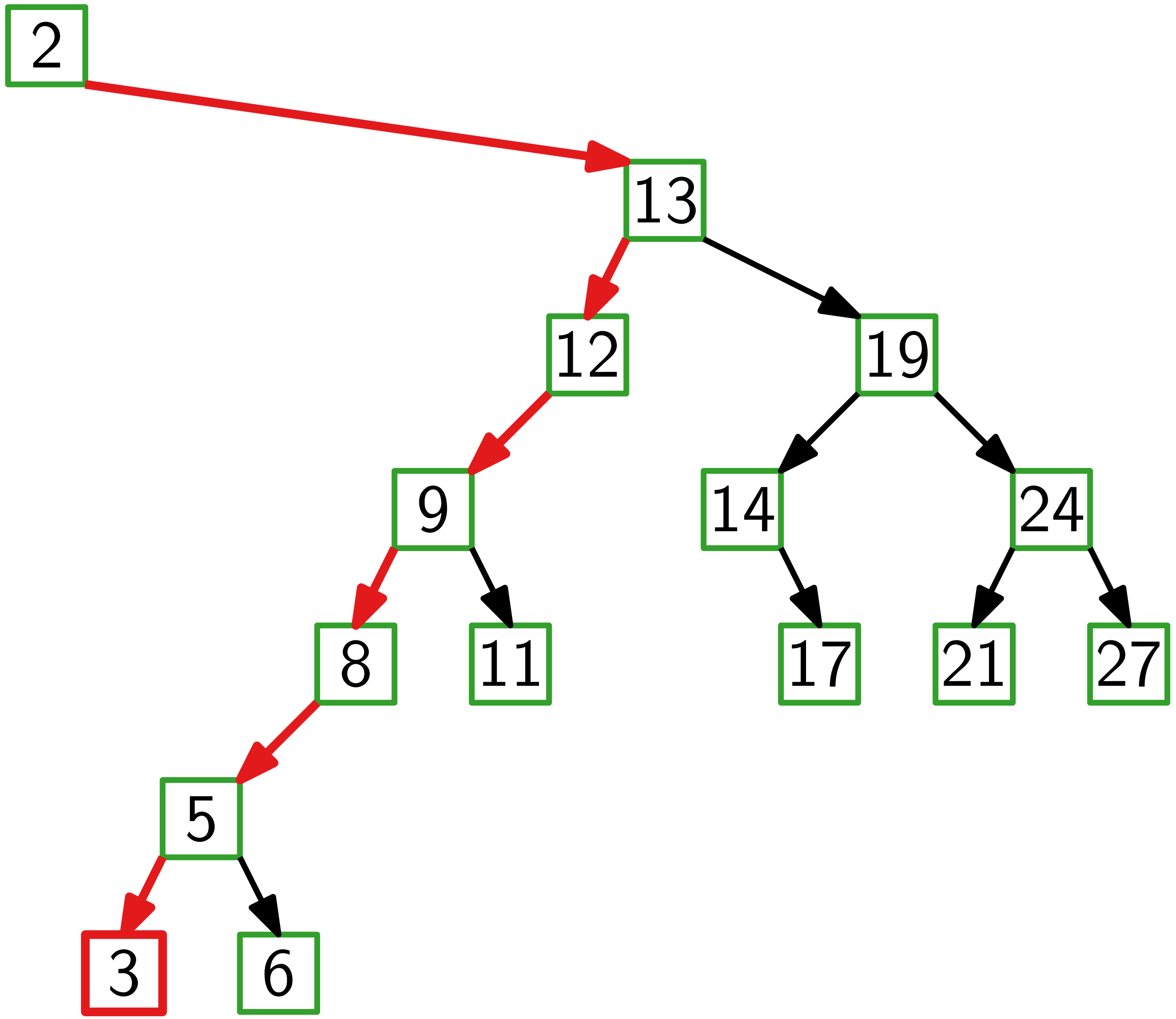

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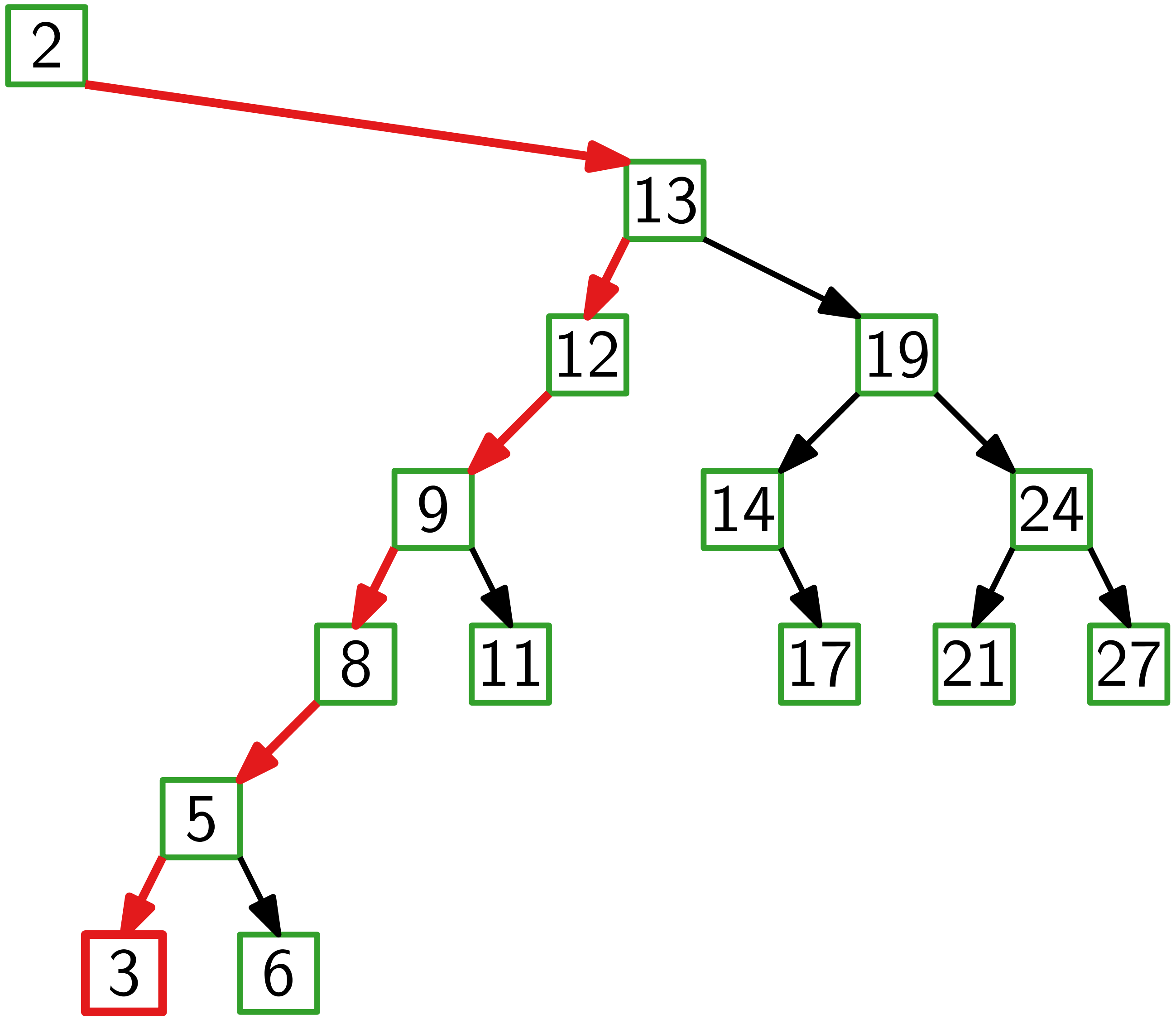

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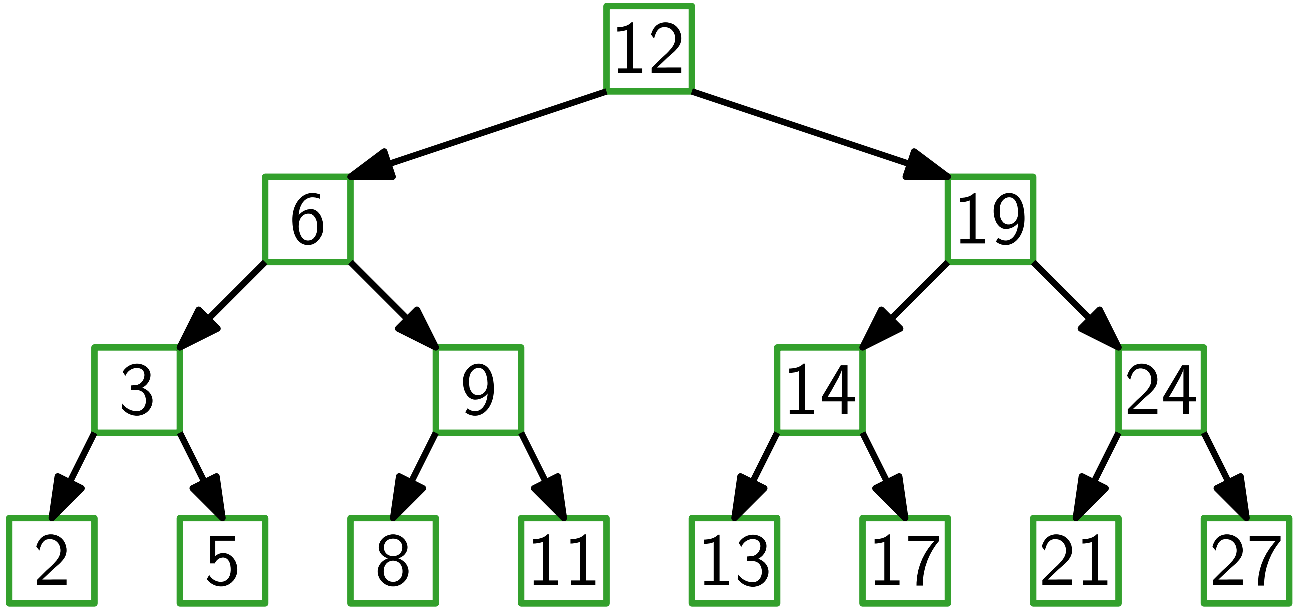

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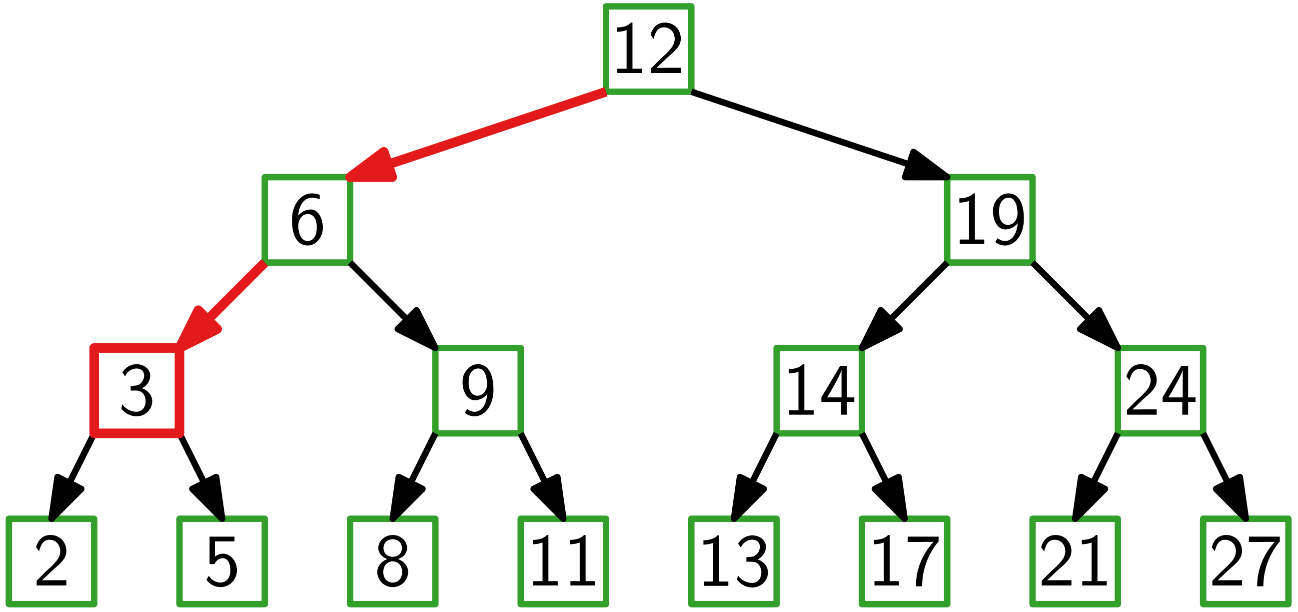

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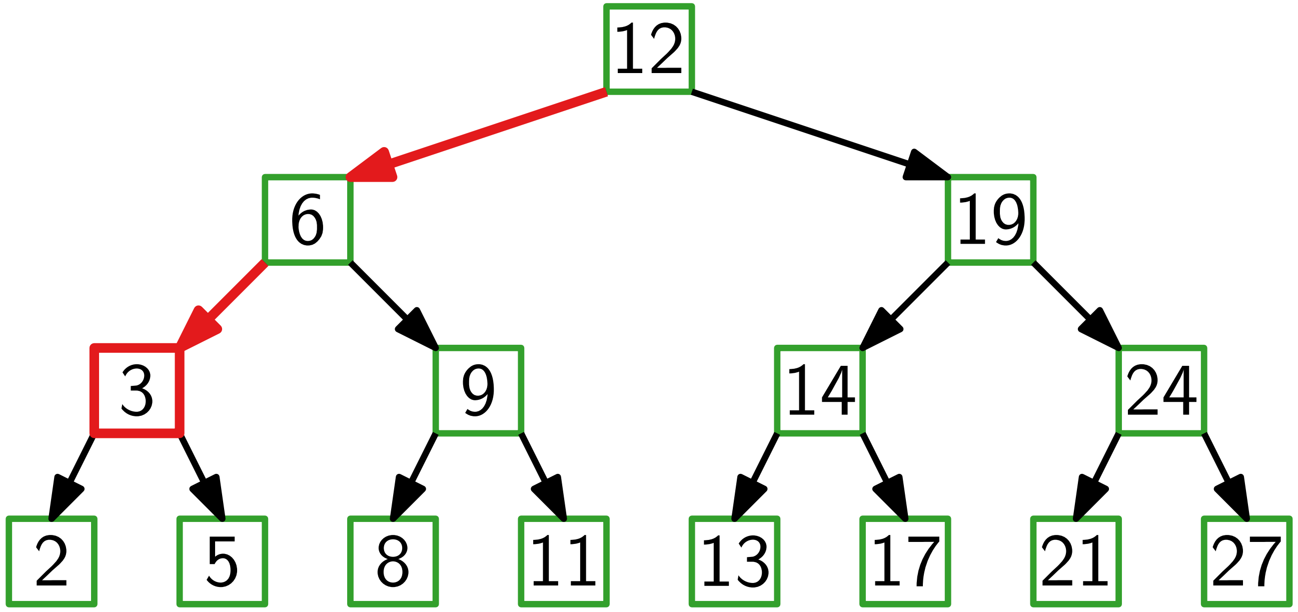

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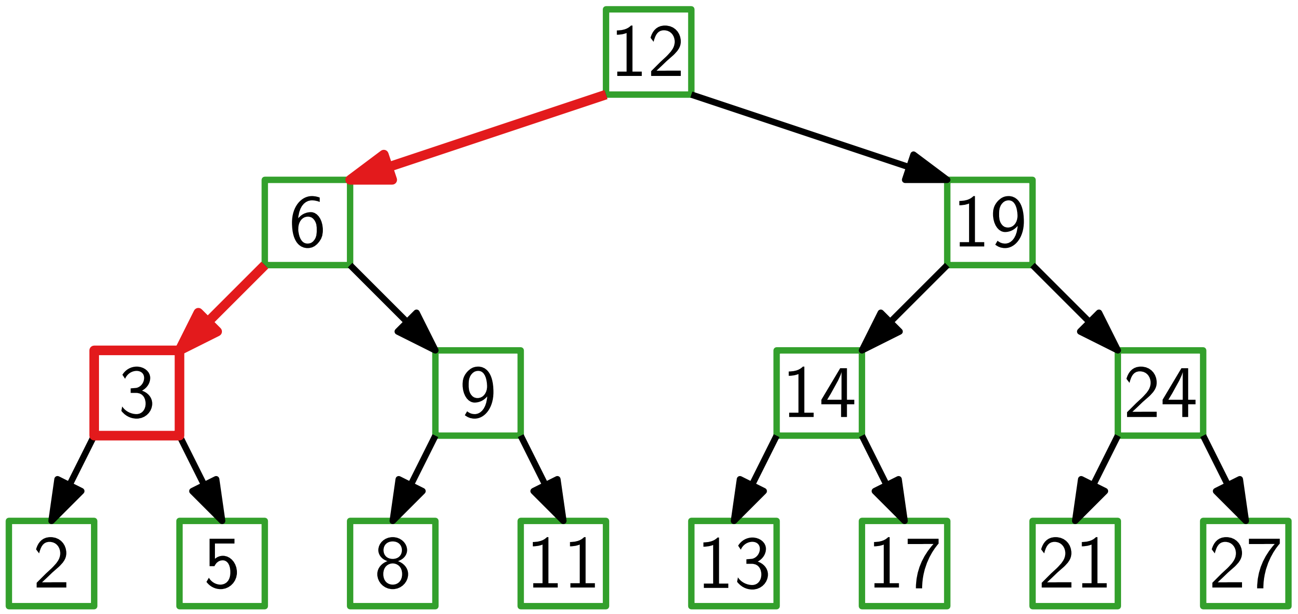

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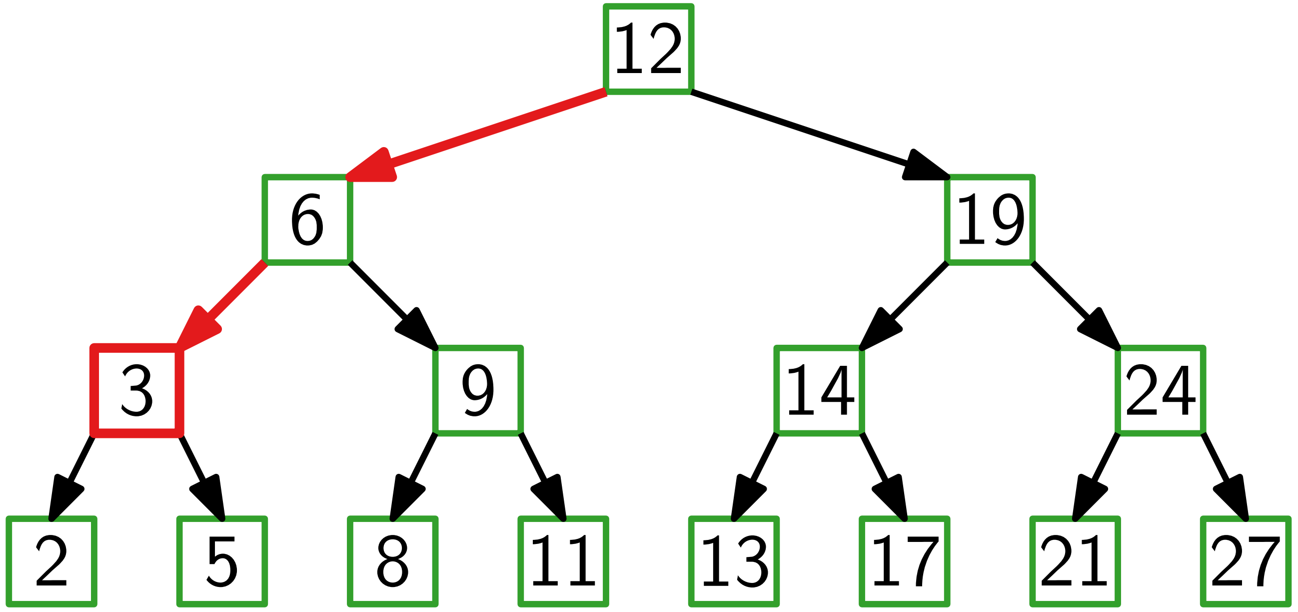

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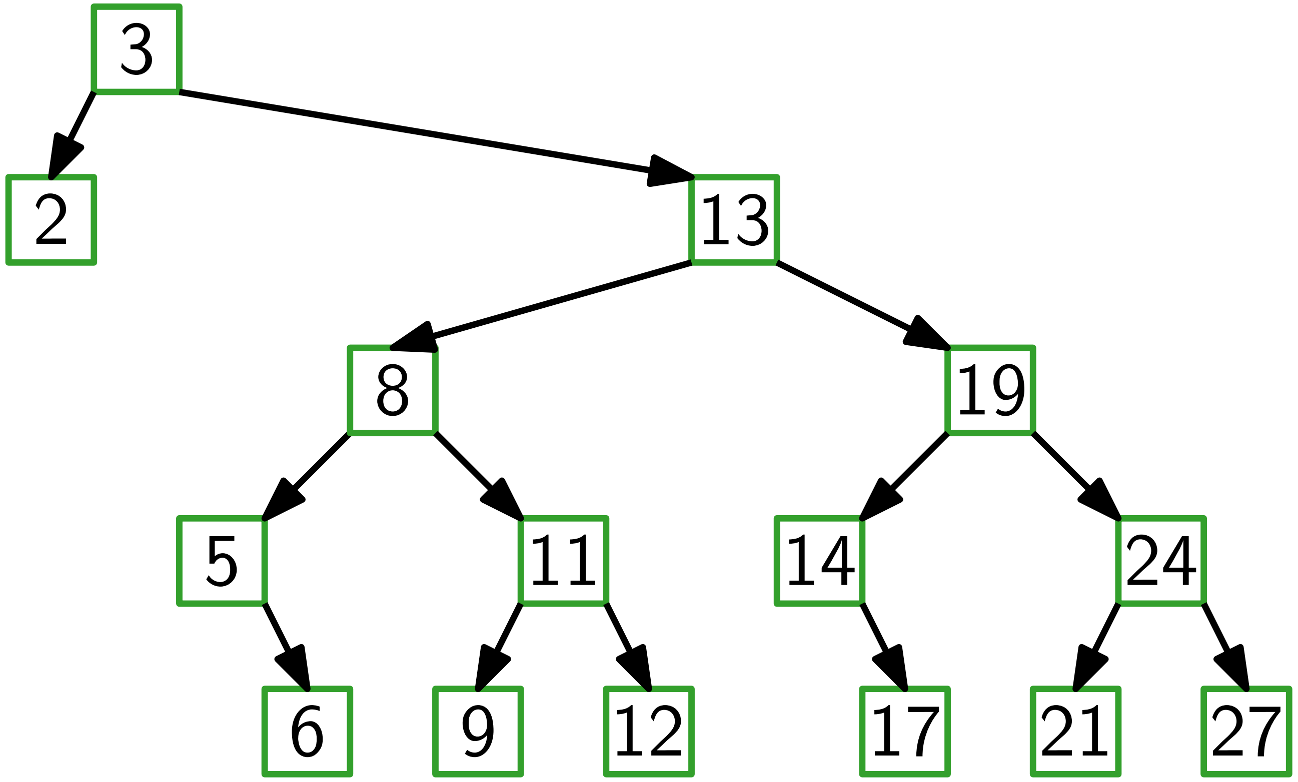

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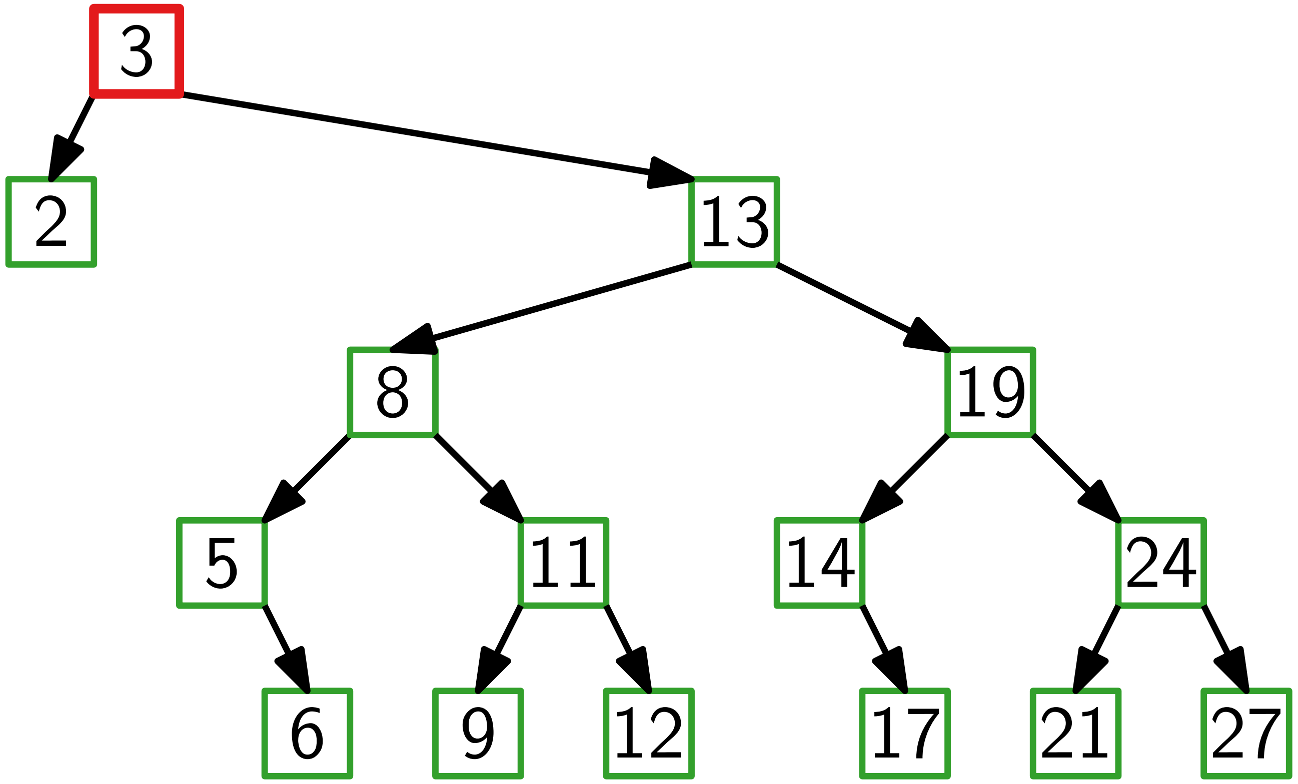

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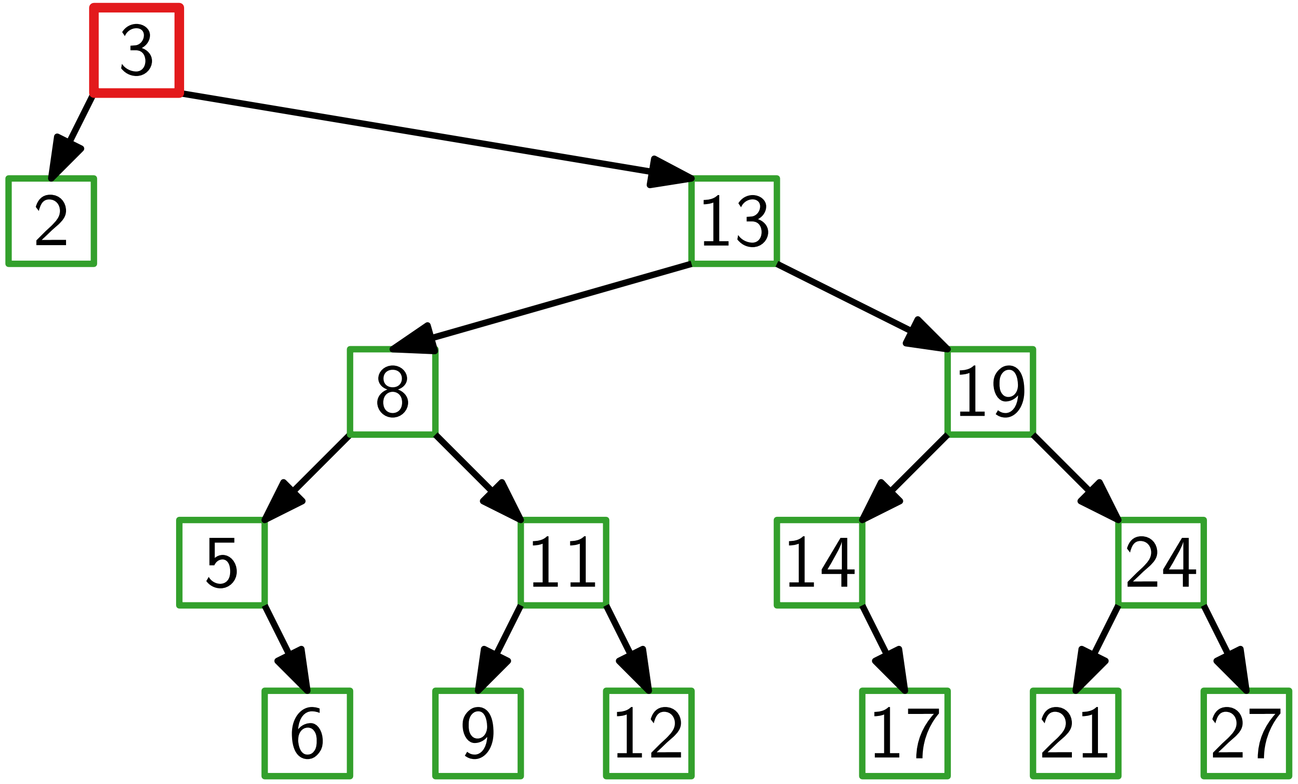

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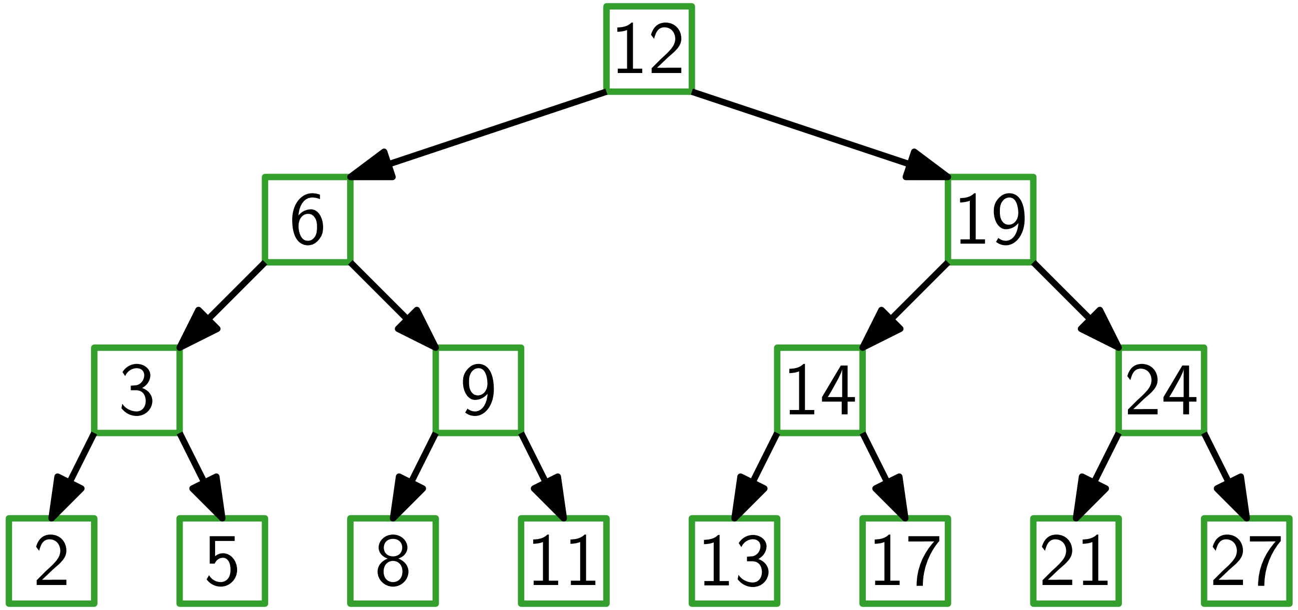

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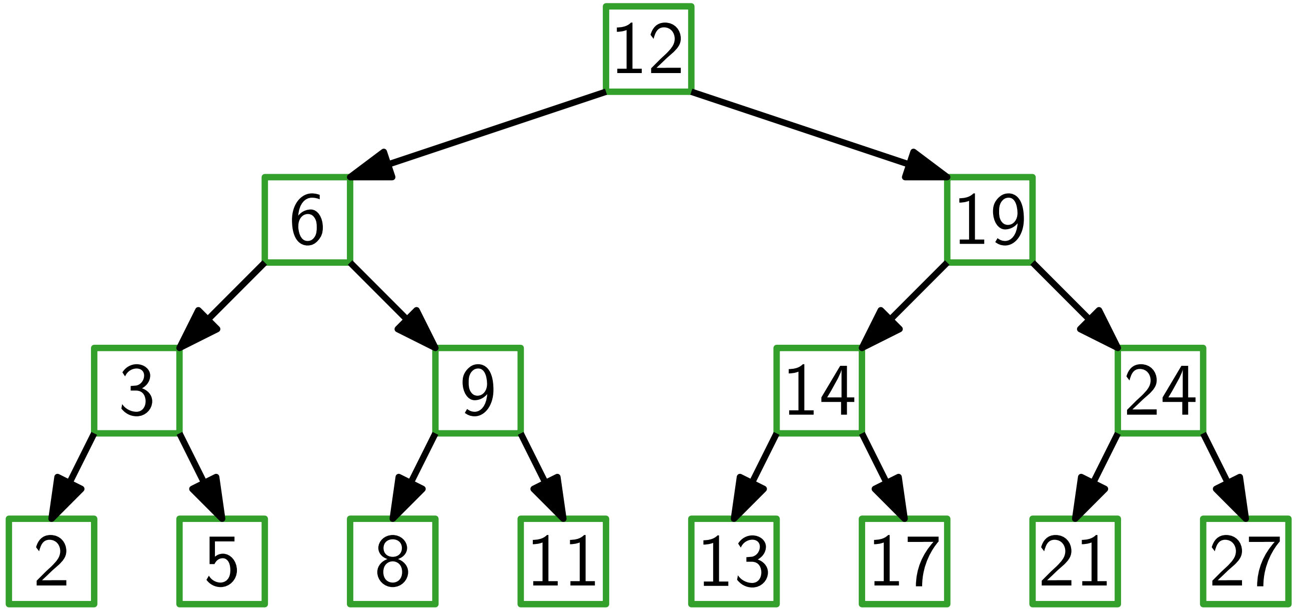

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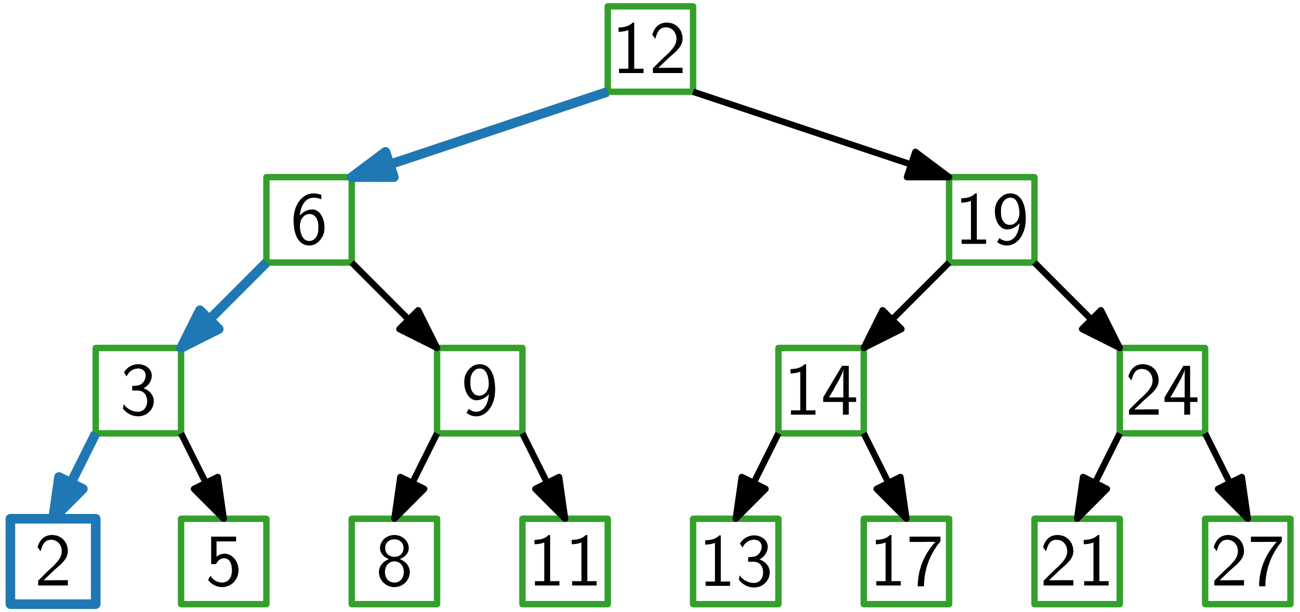

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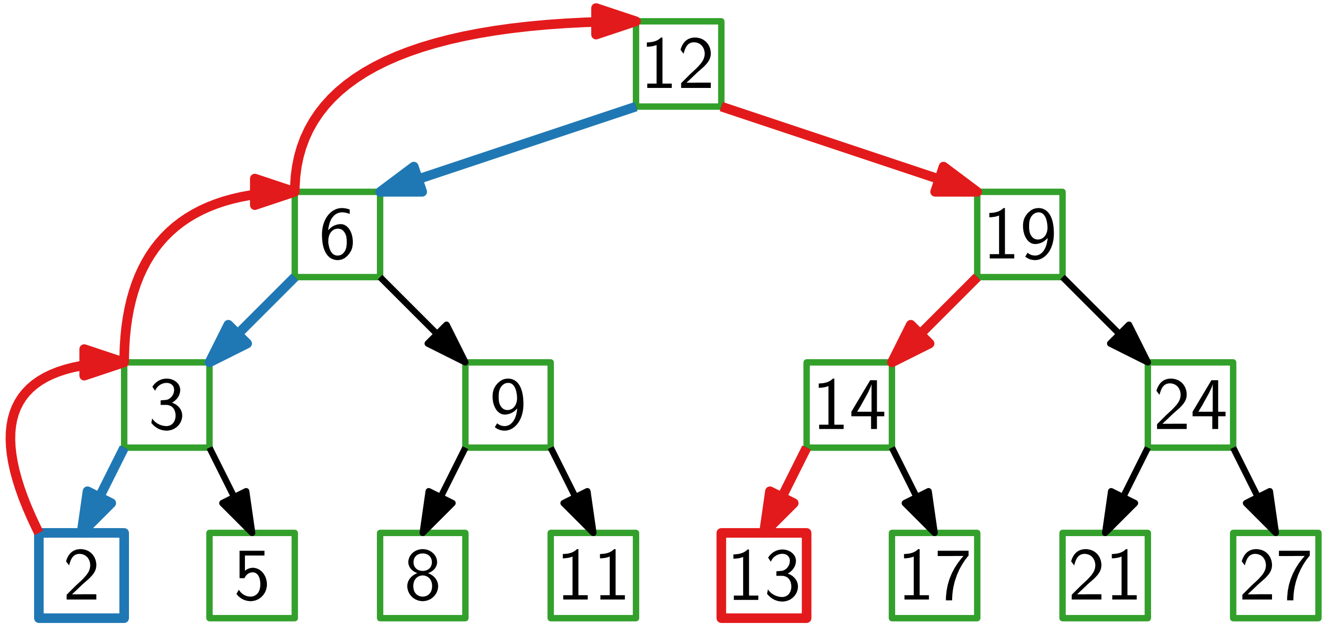

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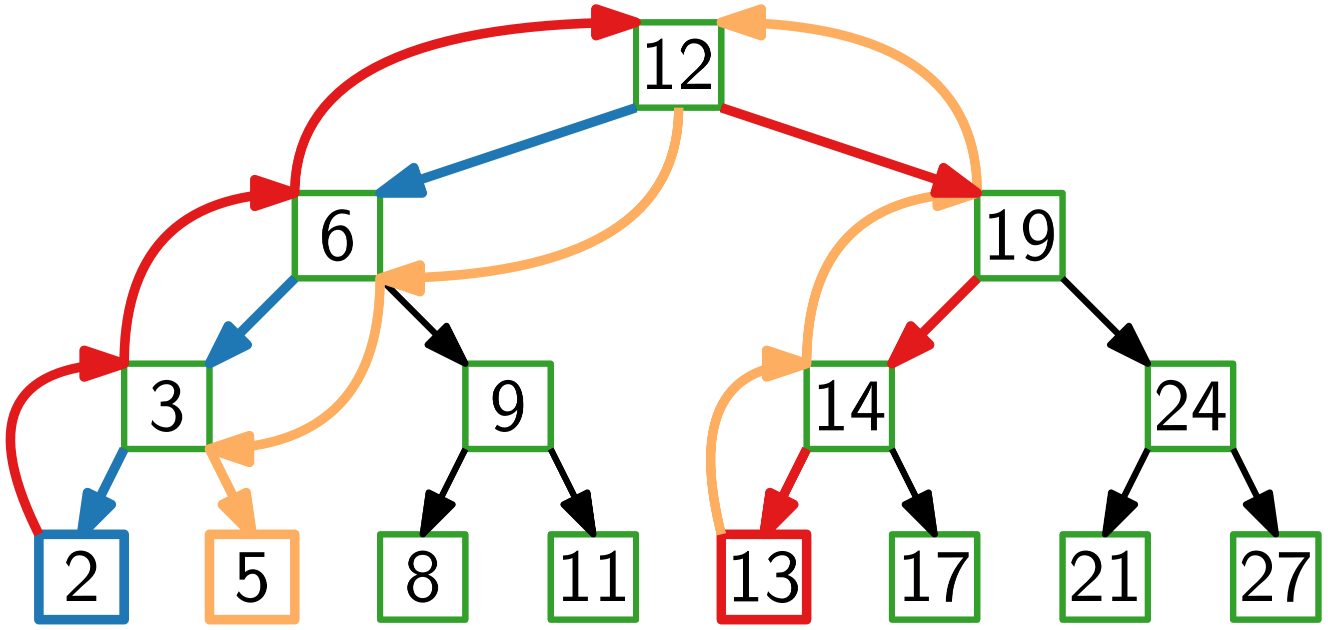

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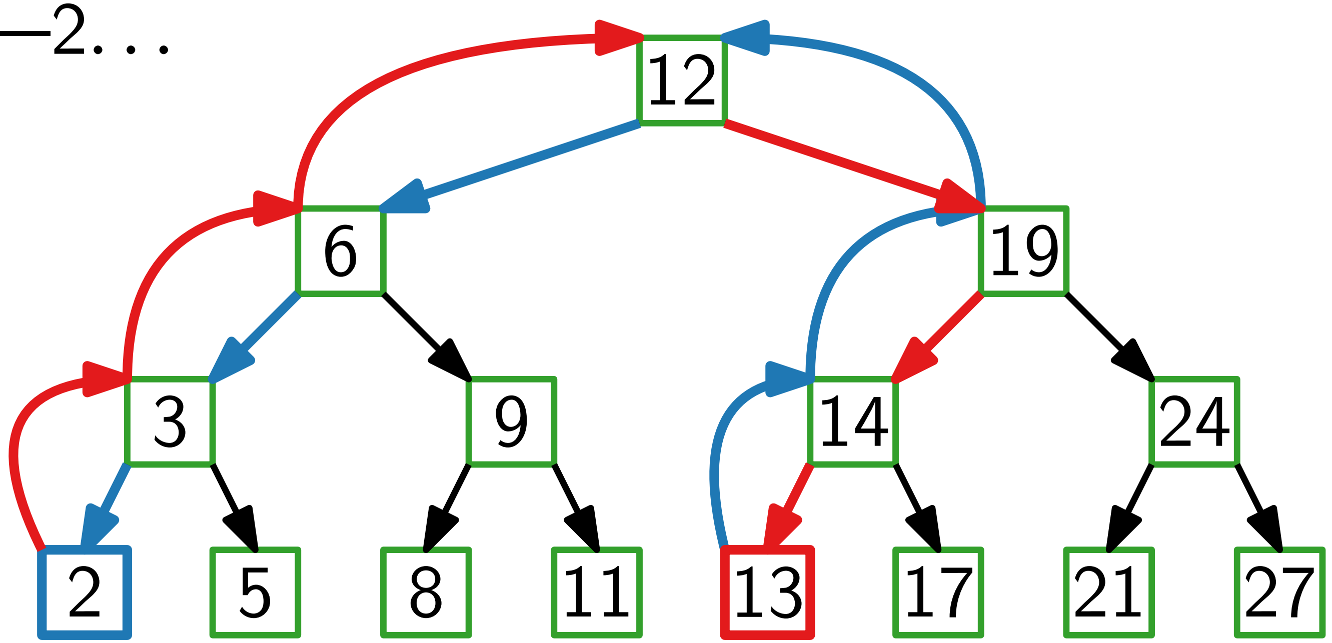

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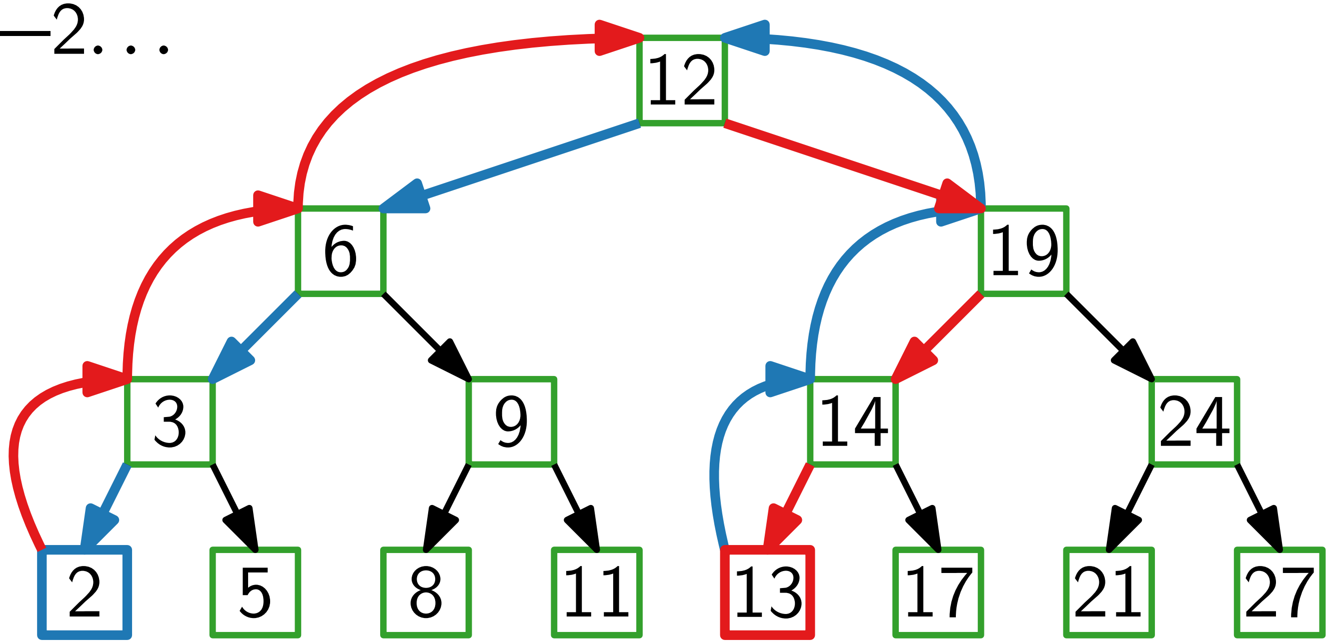

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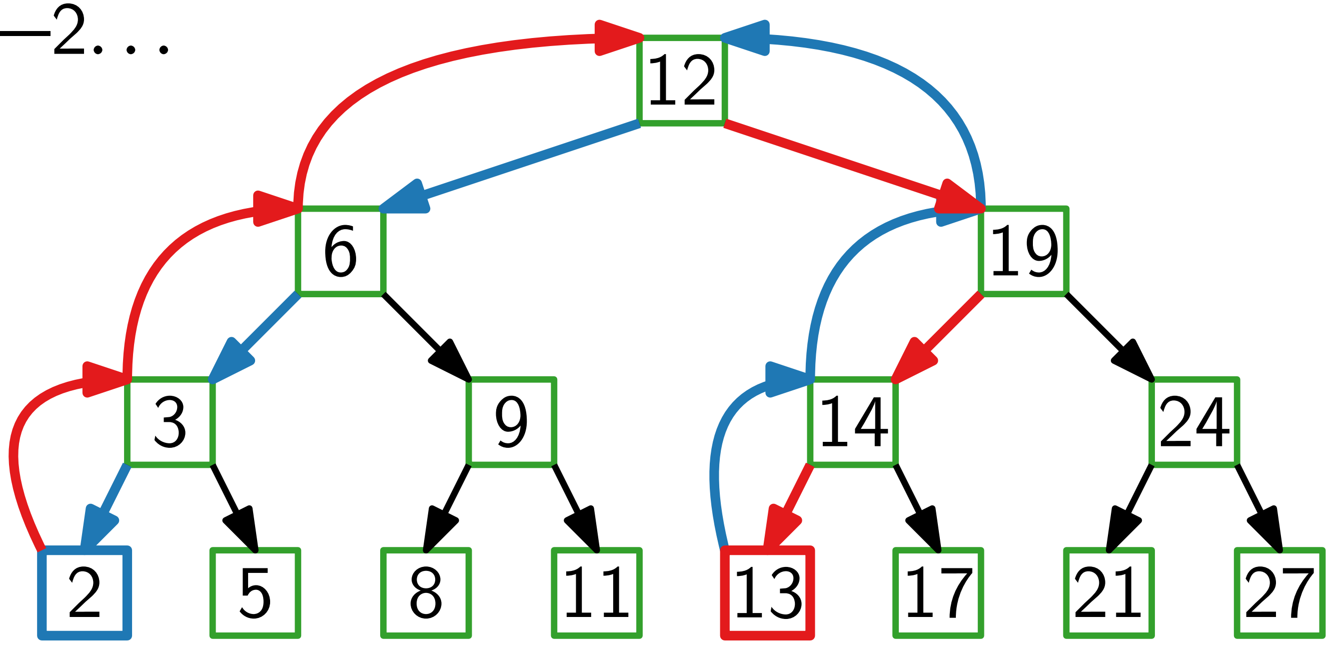

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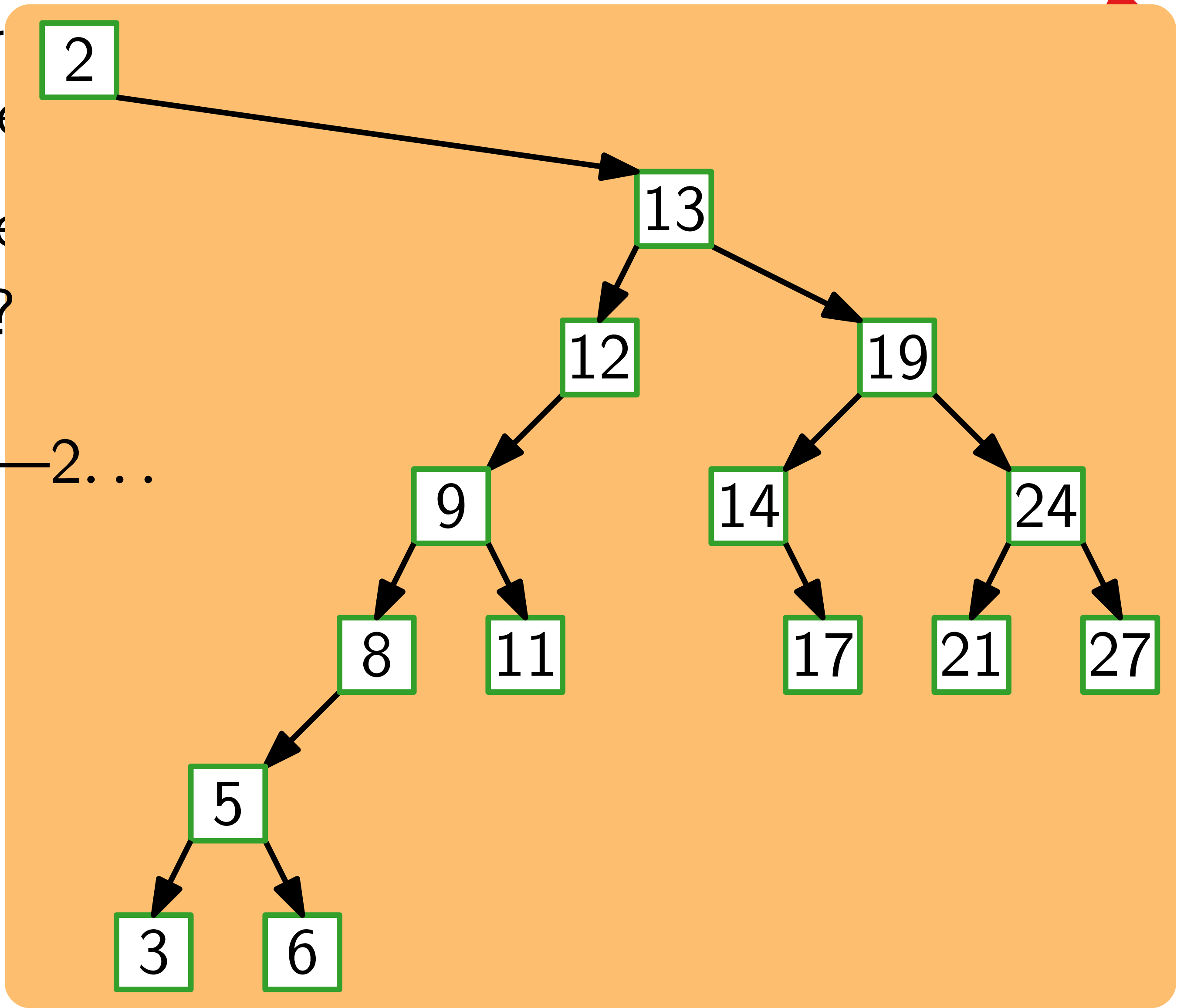

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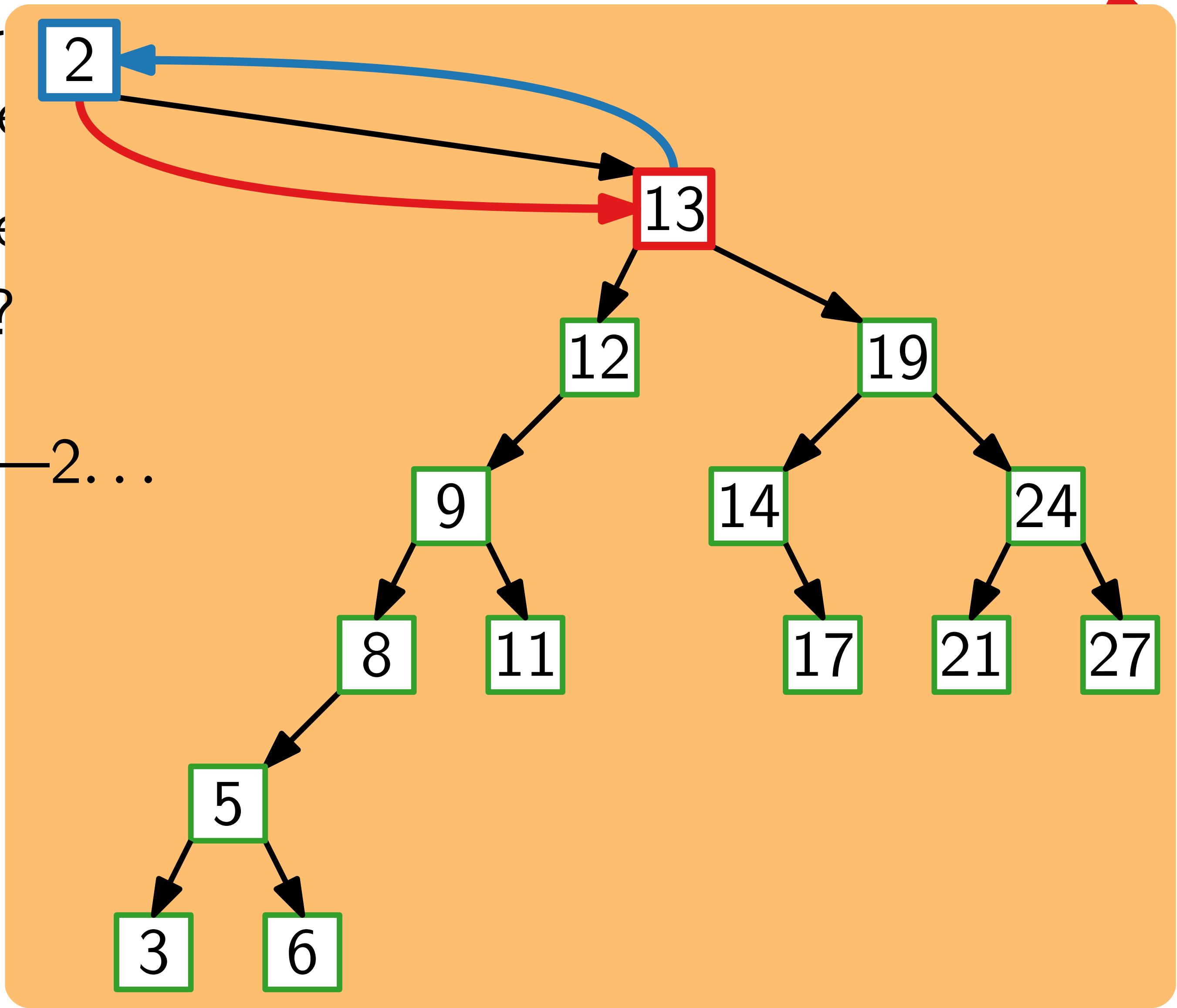

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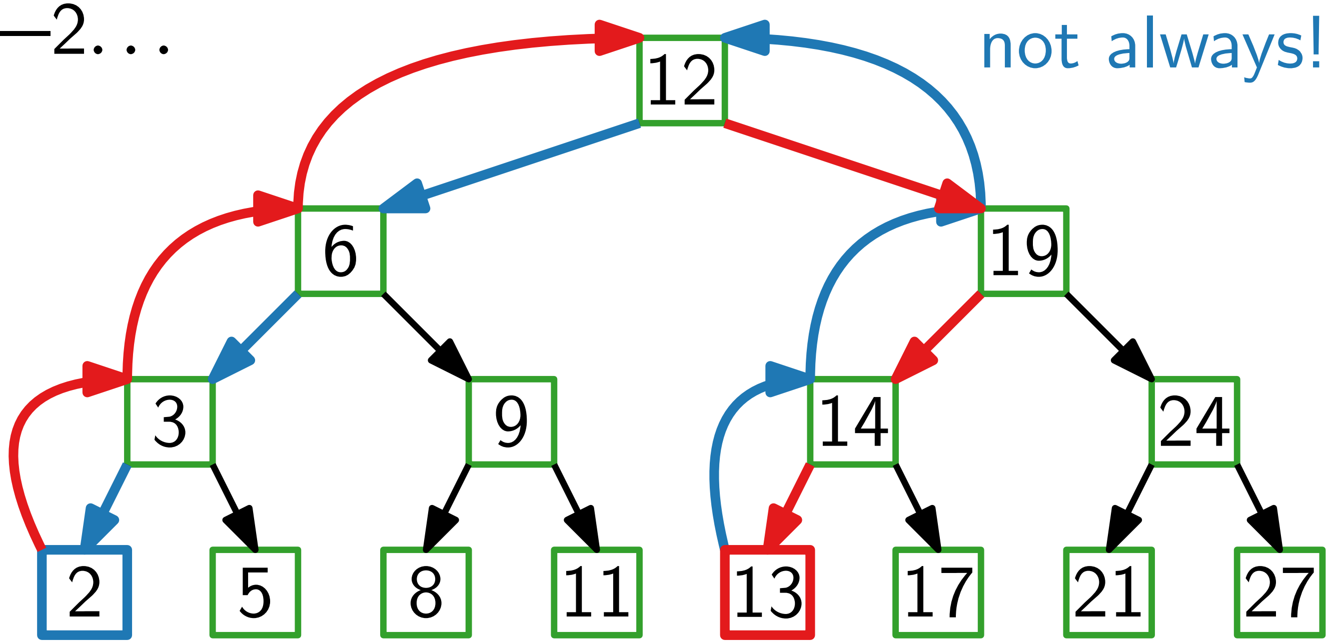

The performance of a BST depends on the model!

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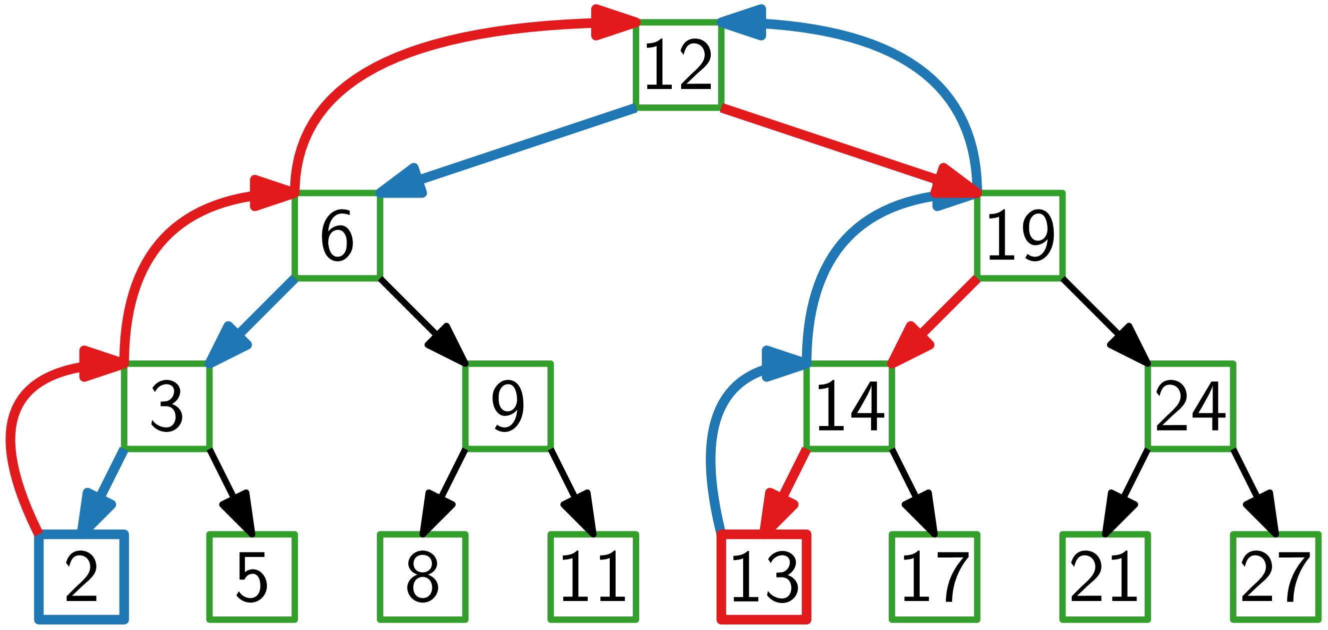

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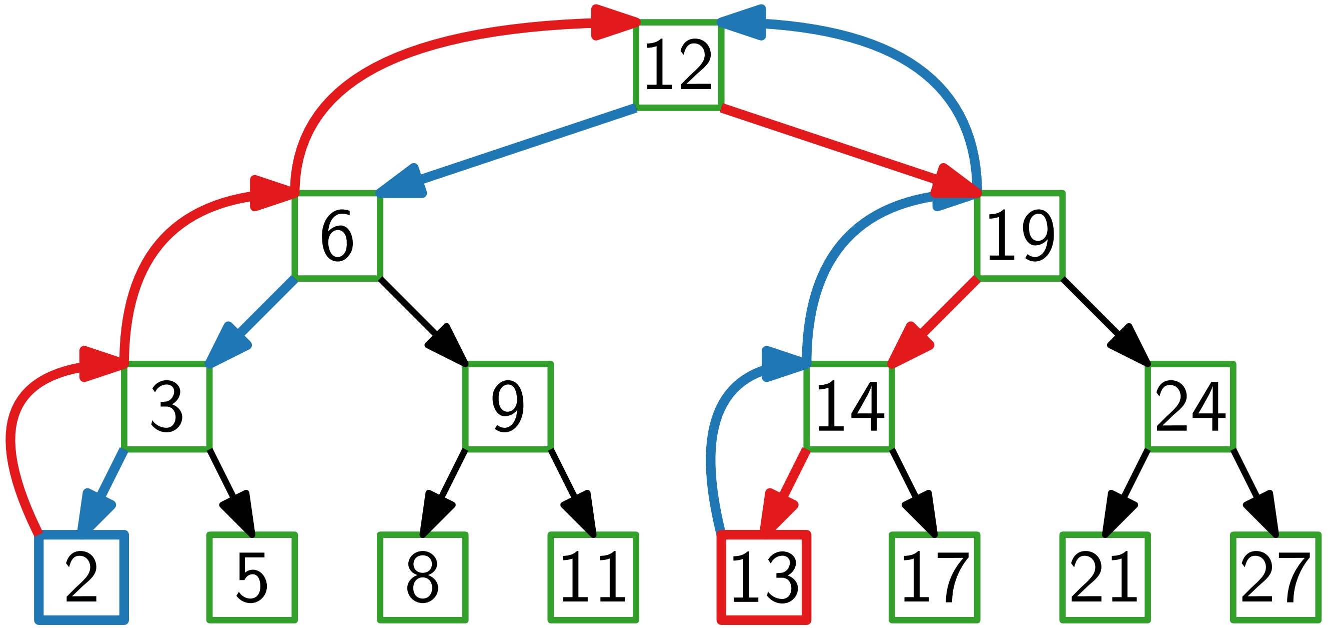

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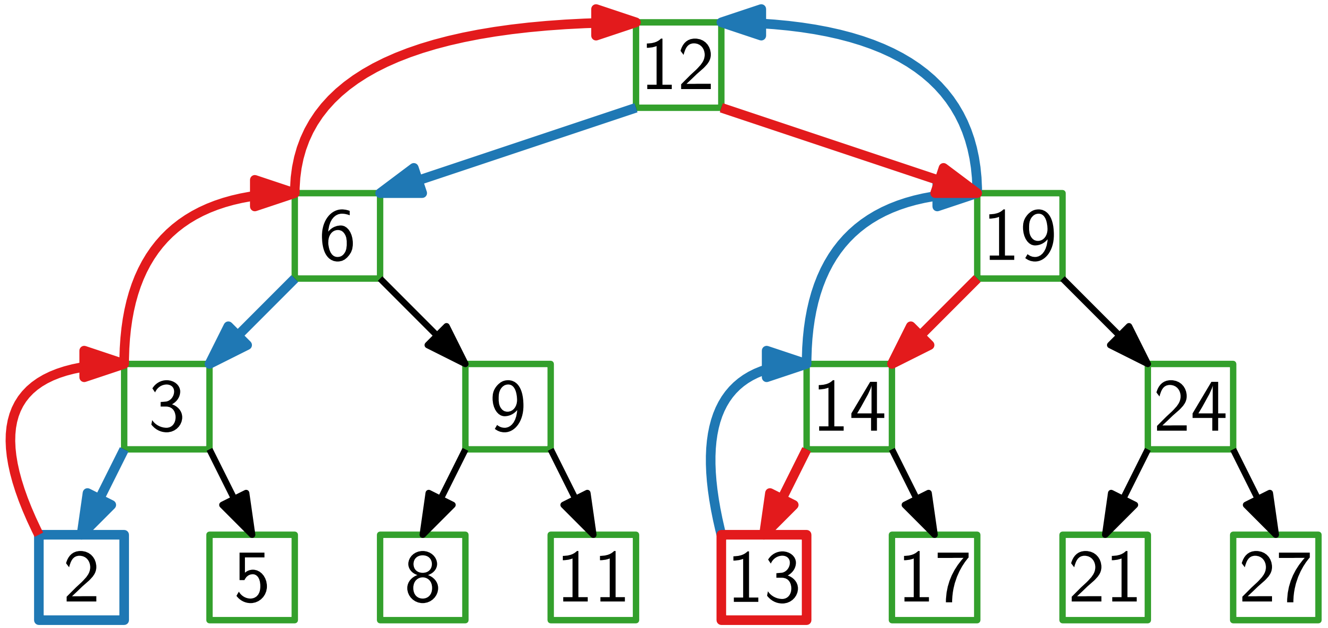

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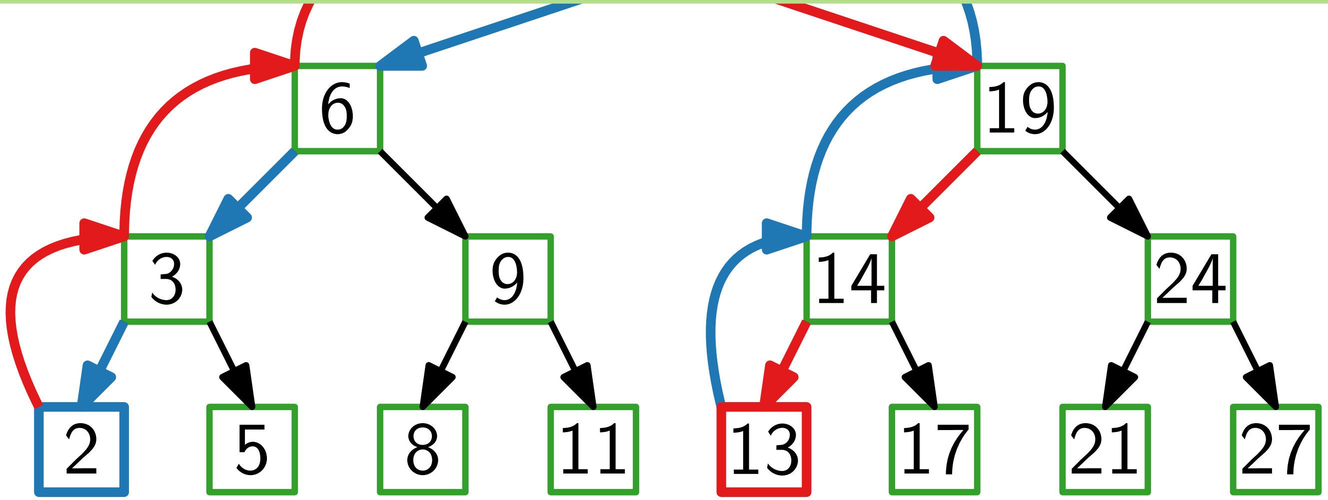

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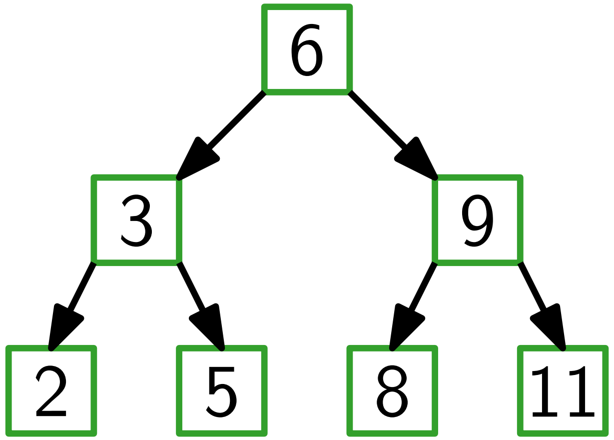

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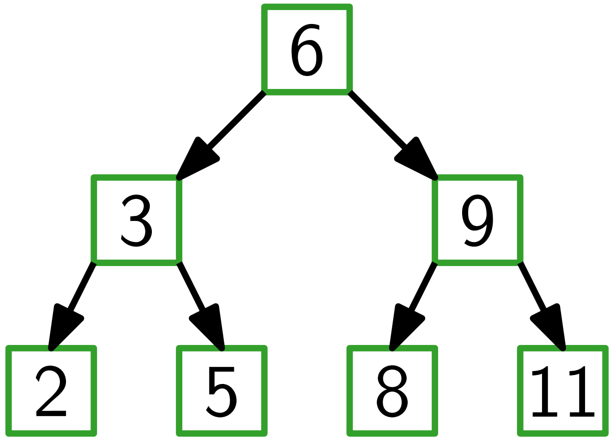
Access Probabilities:

2	3	5	6	8	9	11
2%	20%	30%	8%	20%	15%	5%



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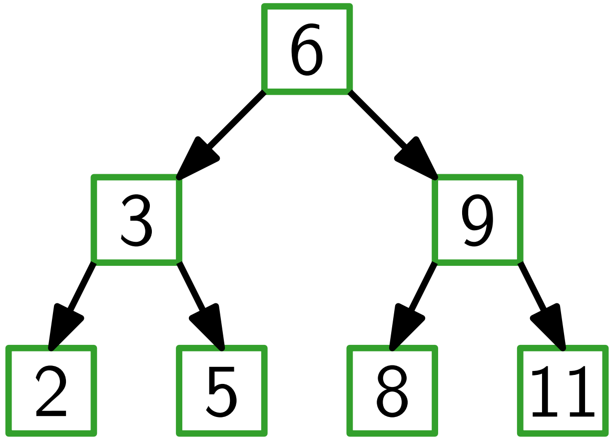

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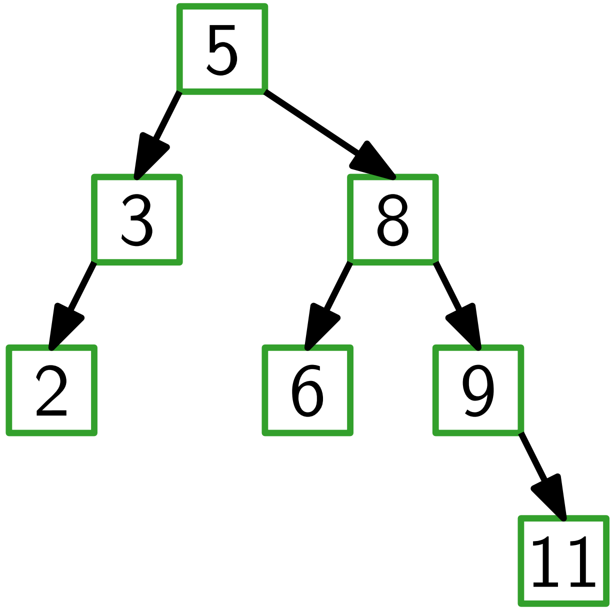

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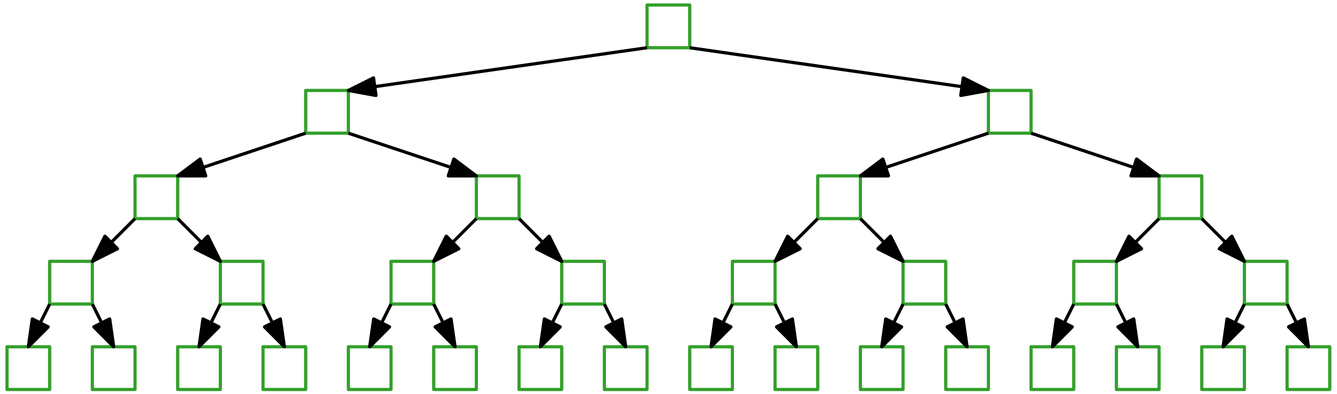

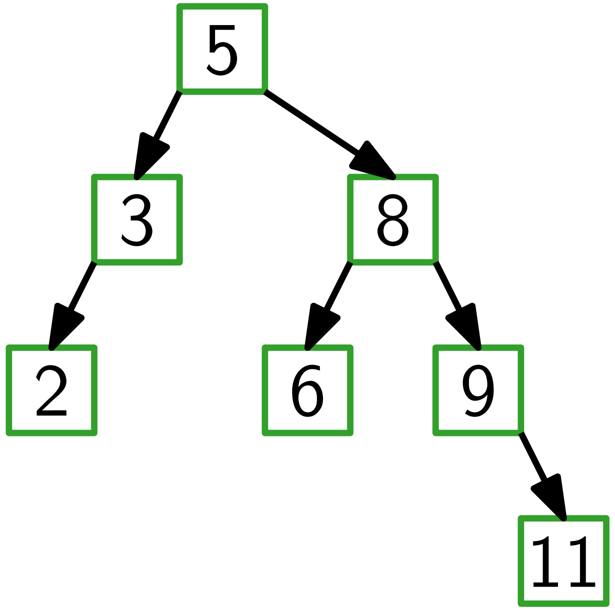

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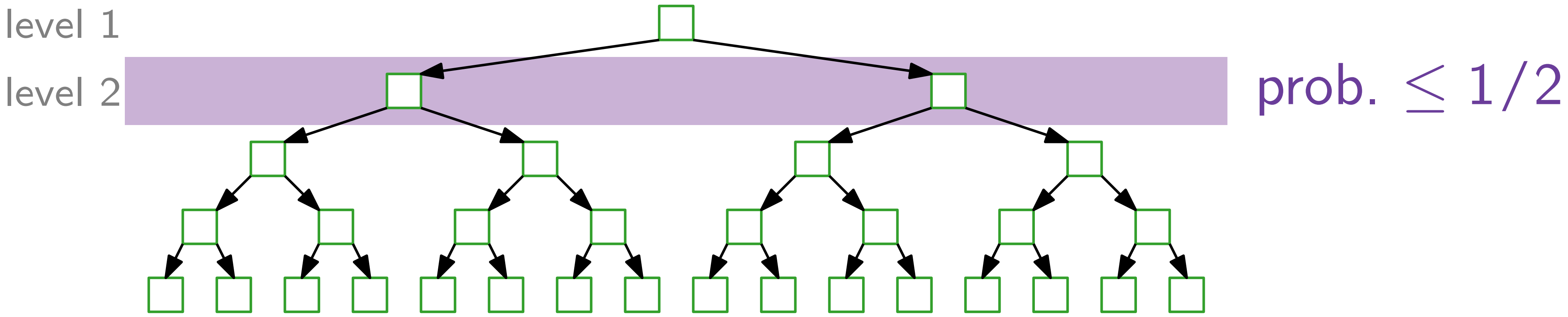

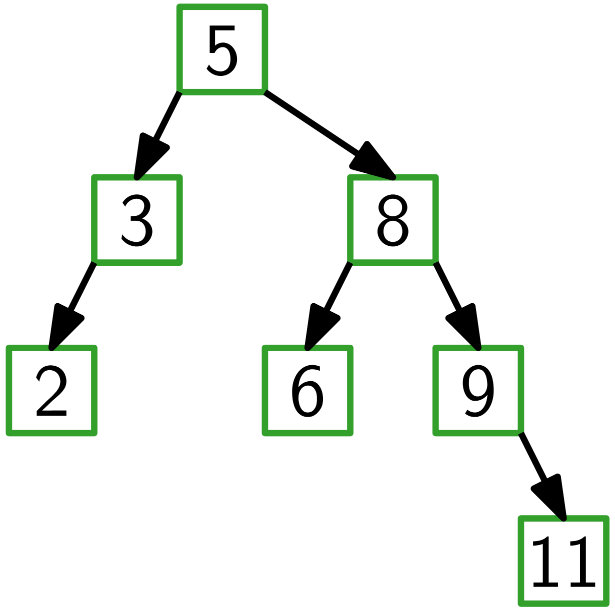

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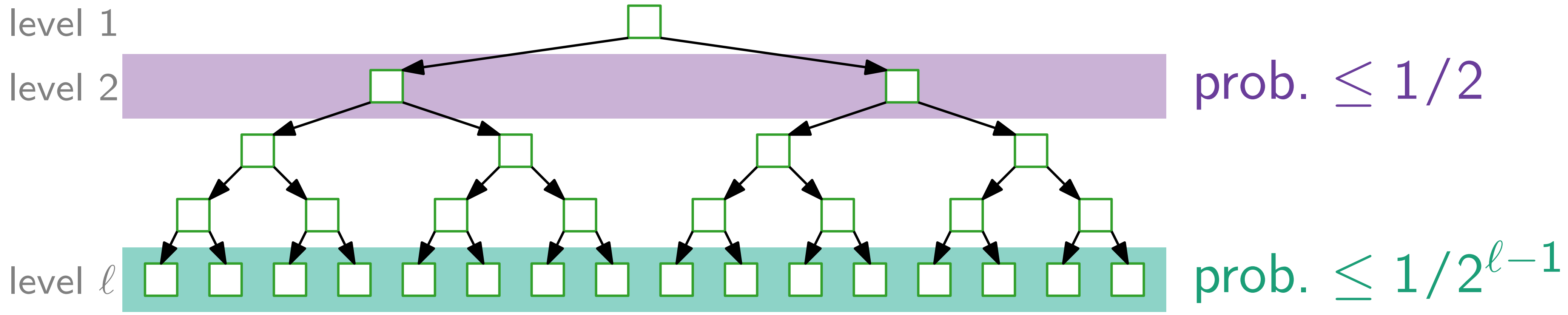

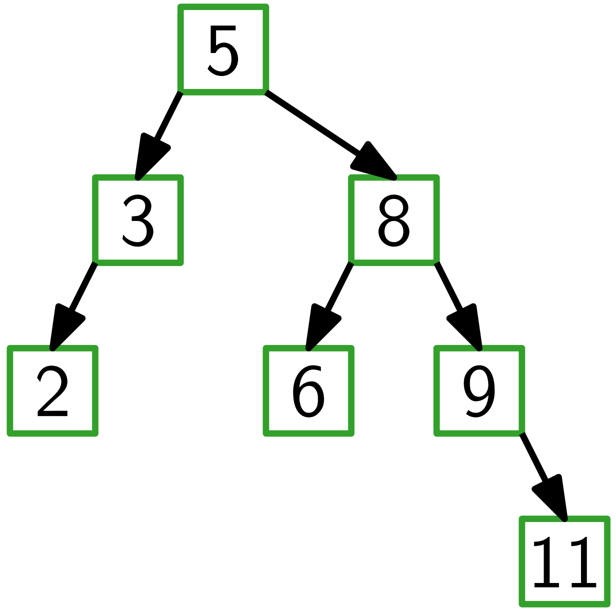

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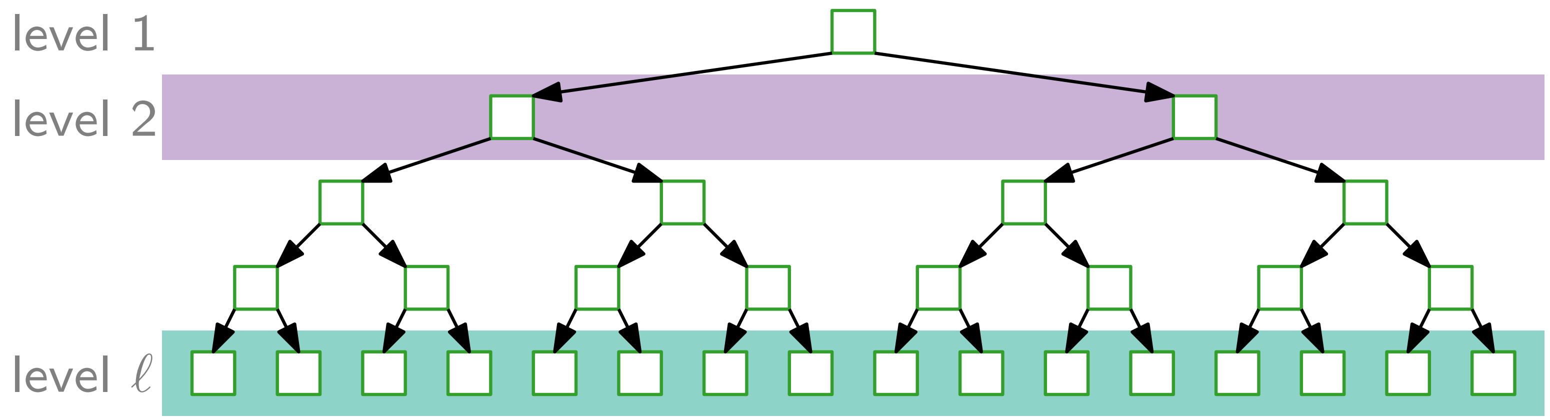

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OPT: prob. $p \Rightarrow$ level

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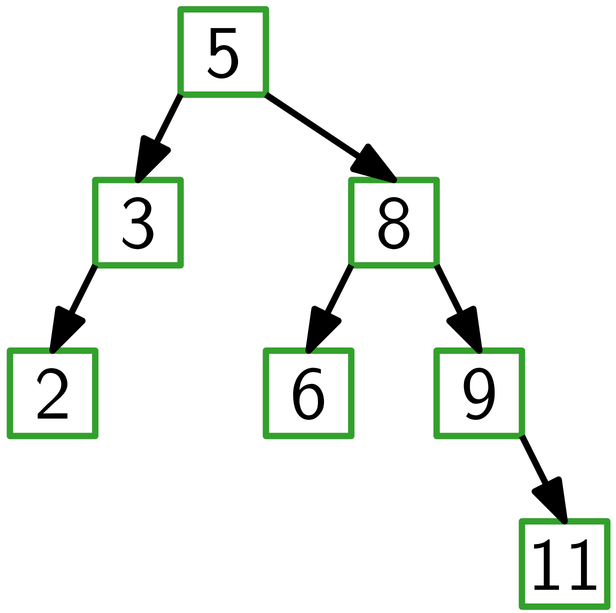

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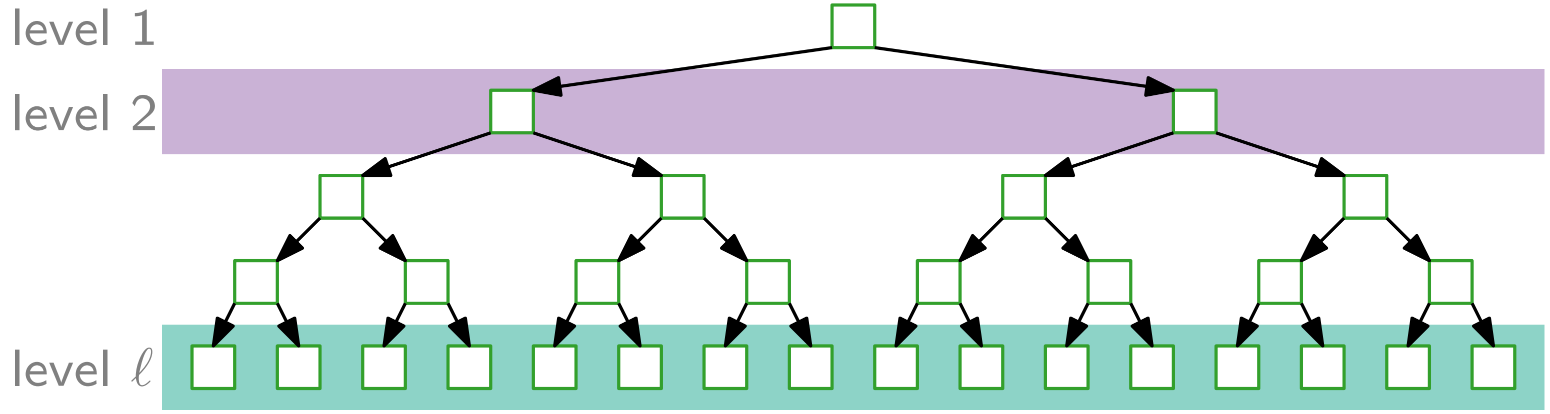

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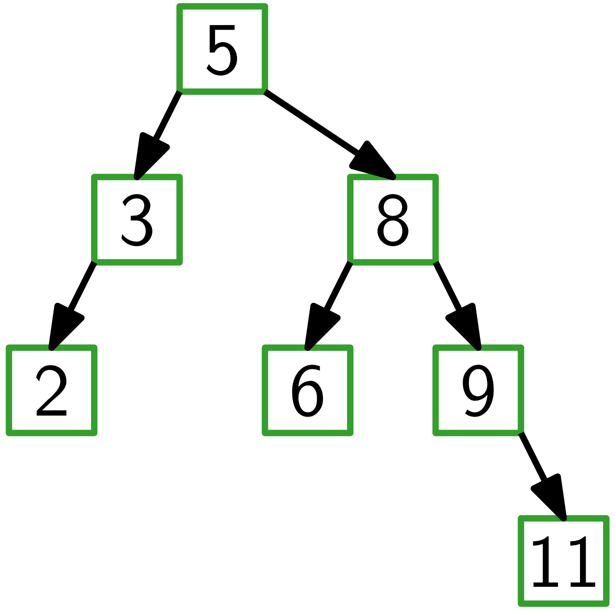

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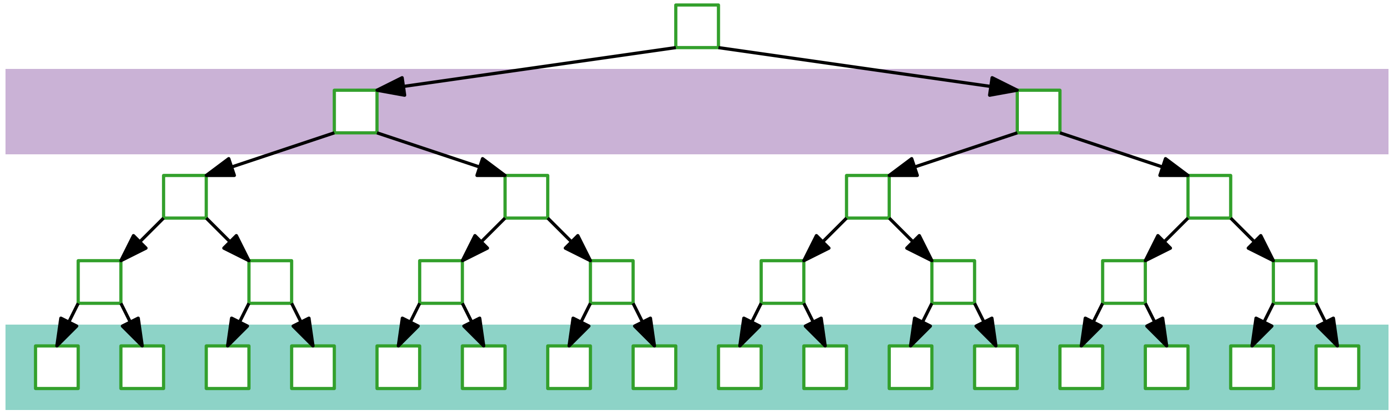
$$i \leq 1 - \log_2 p$$

level 1

level 2

level ℓ



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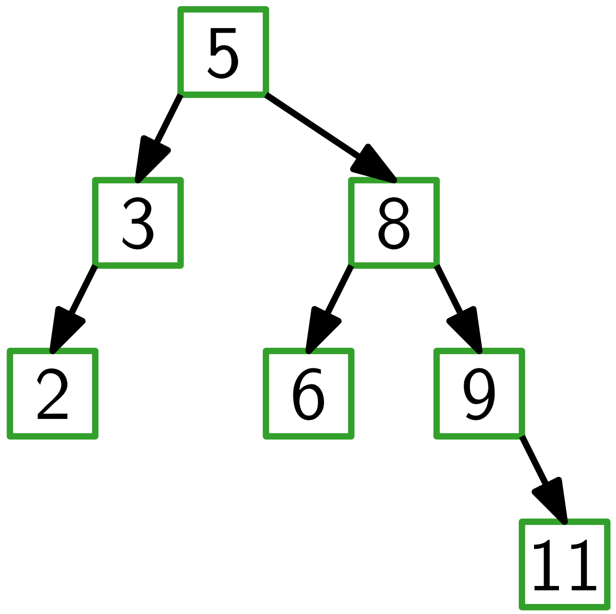

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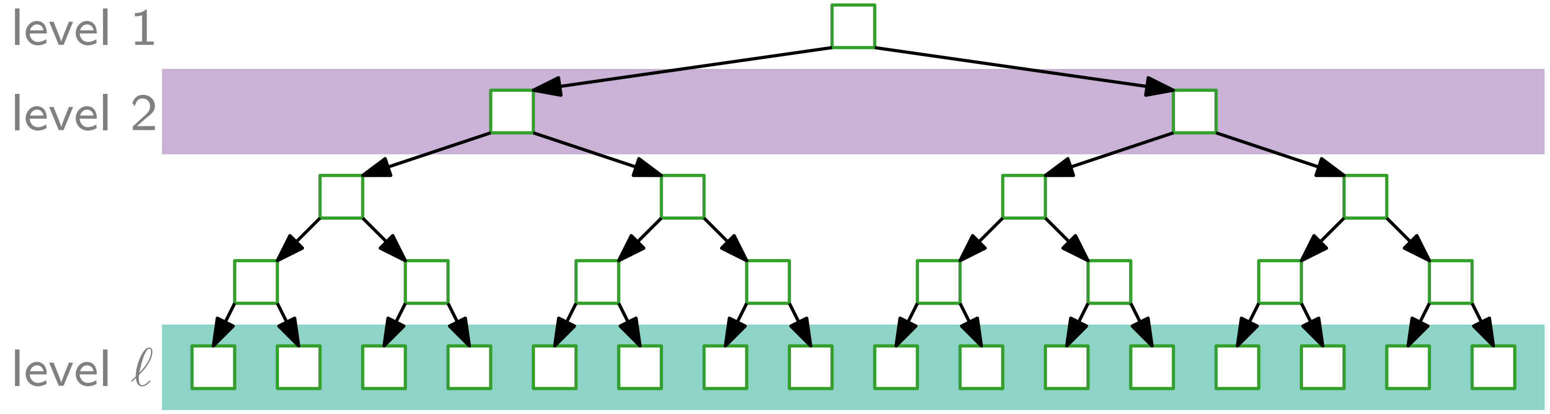

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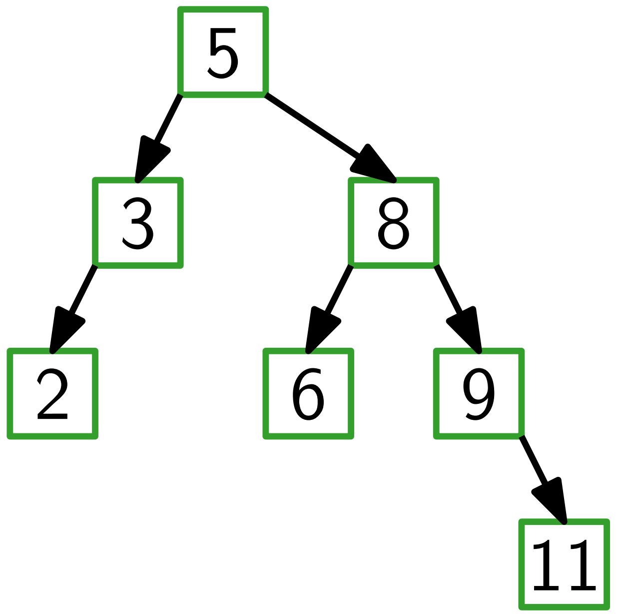

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2%	20%	30%	8%	20%	15%	5%

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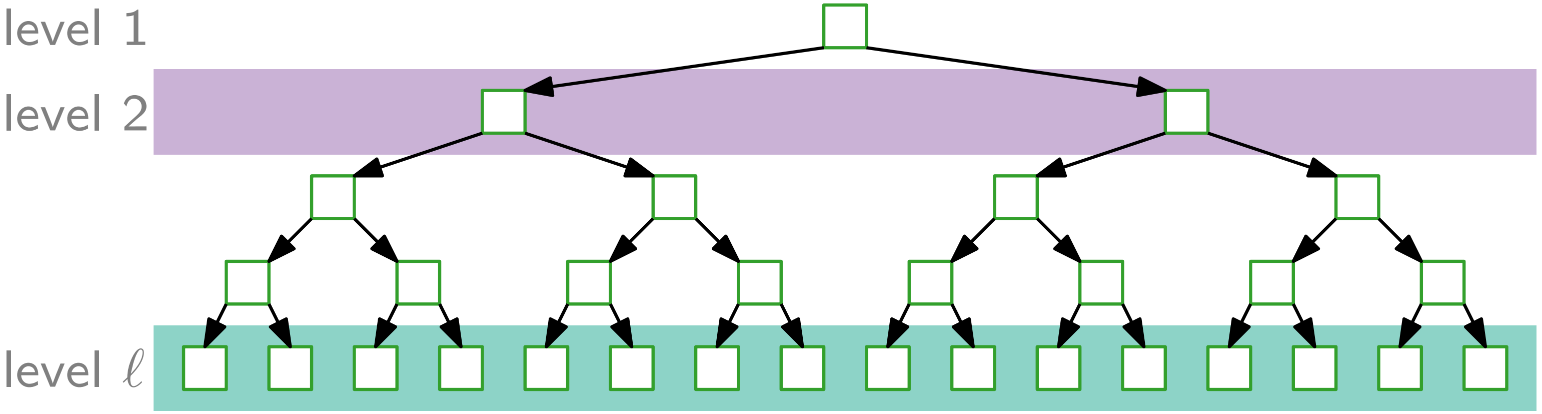

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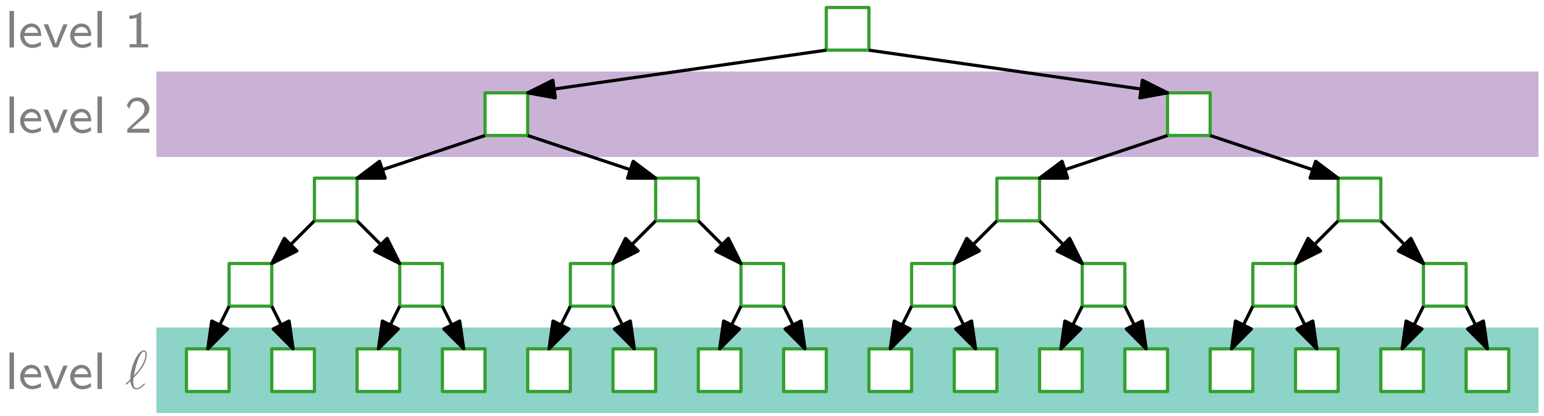

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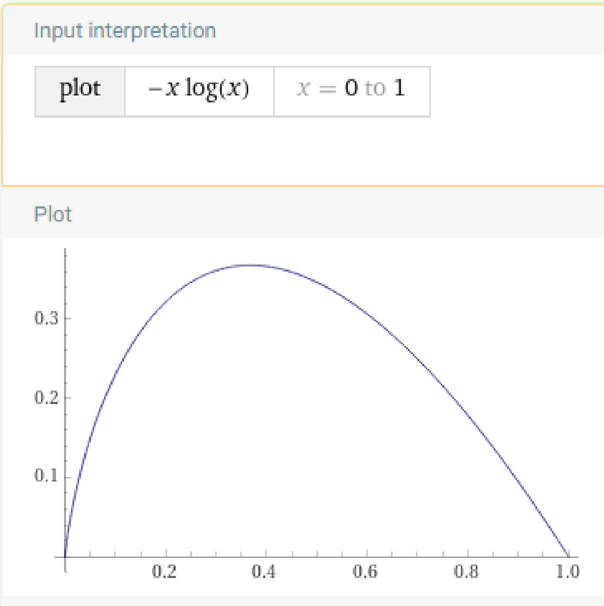

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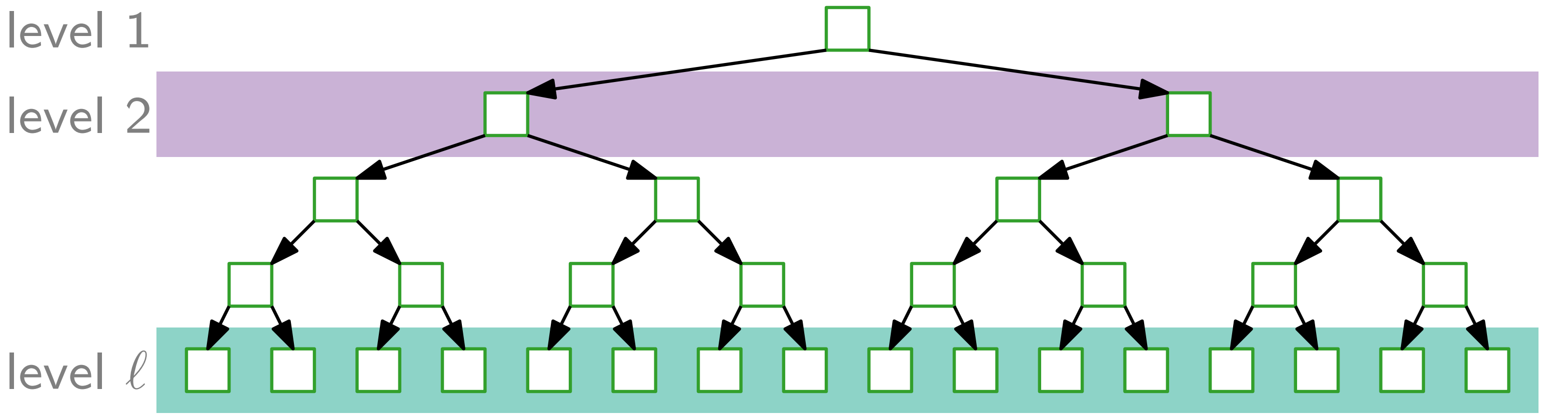

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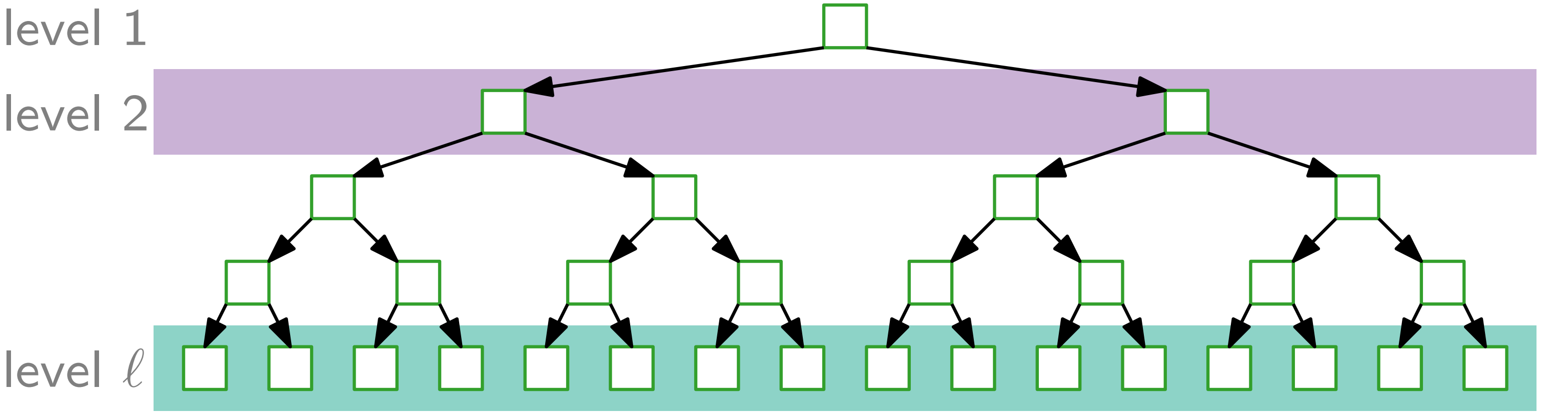

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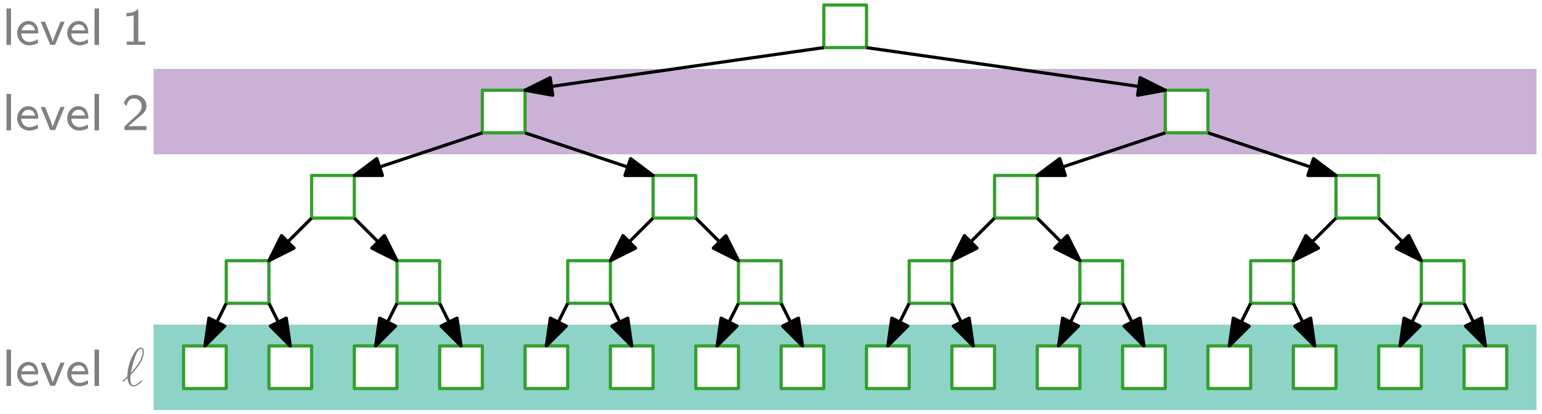

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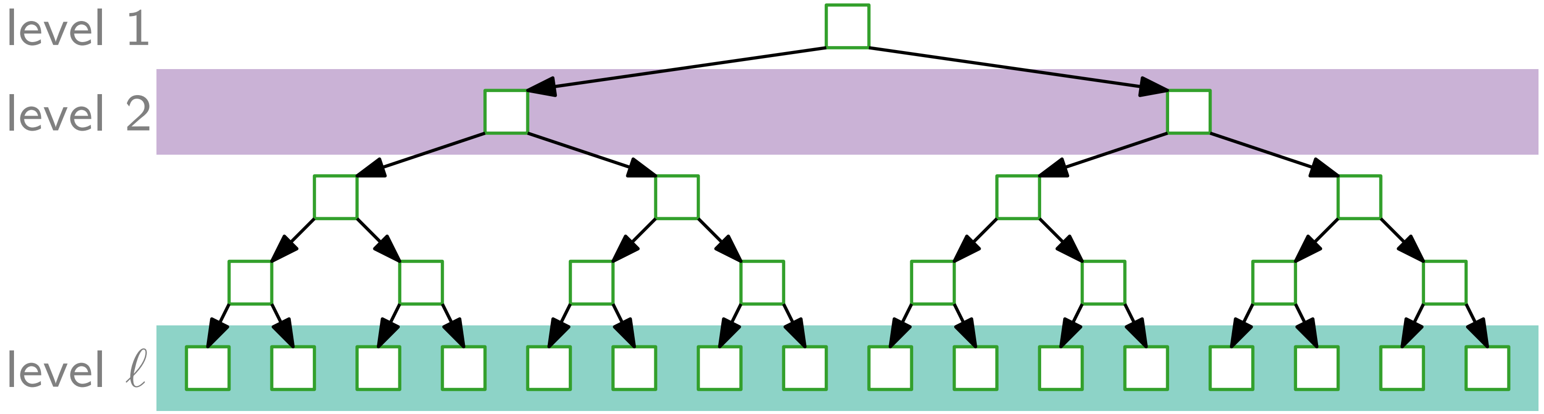

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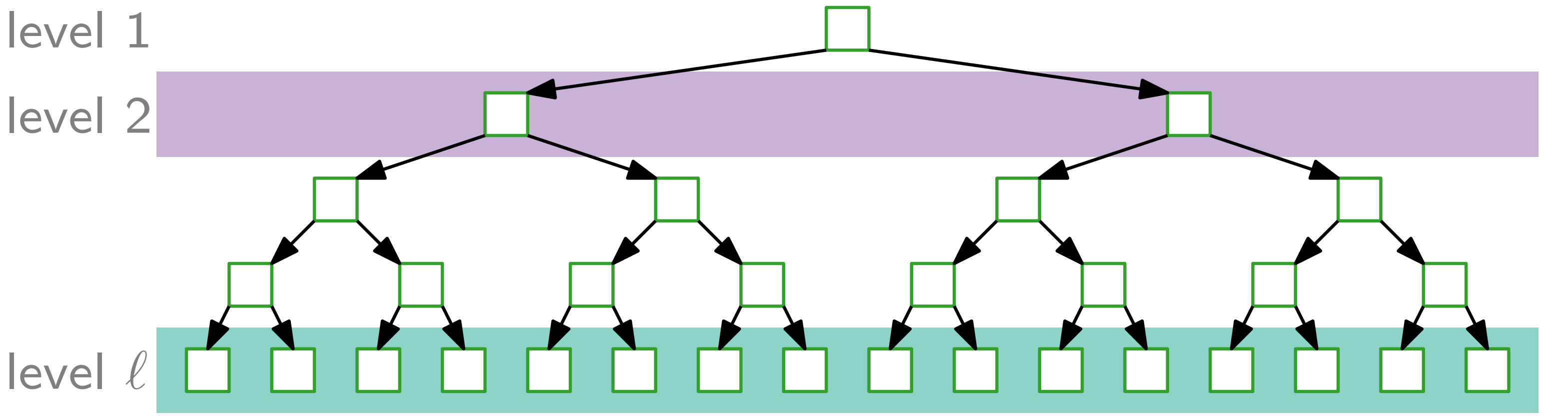

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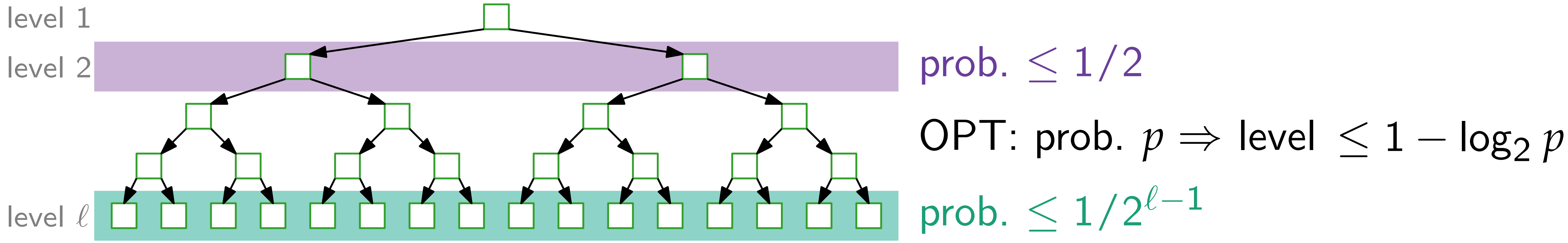

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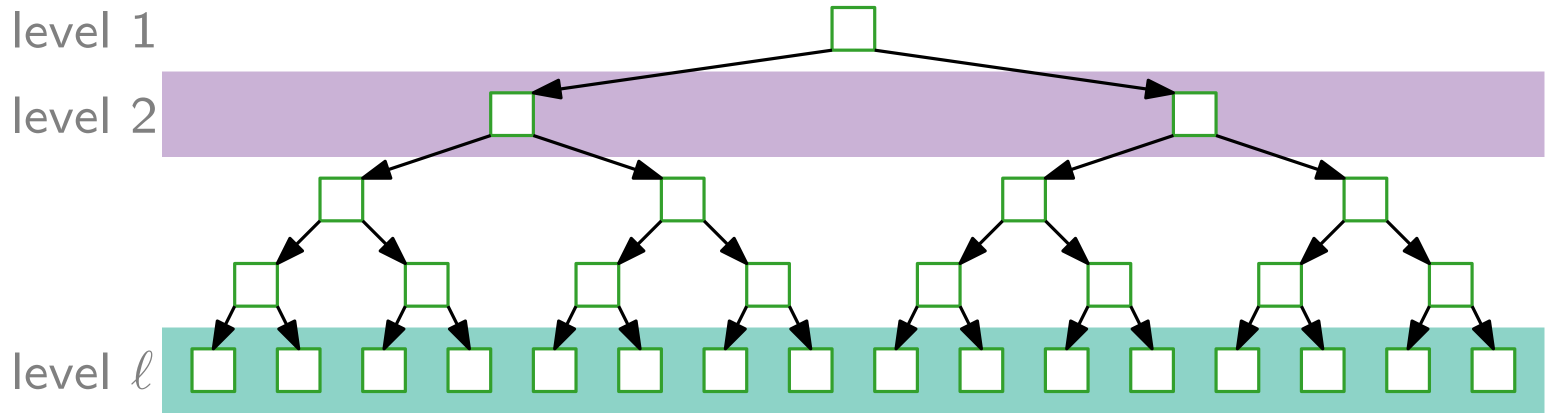

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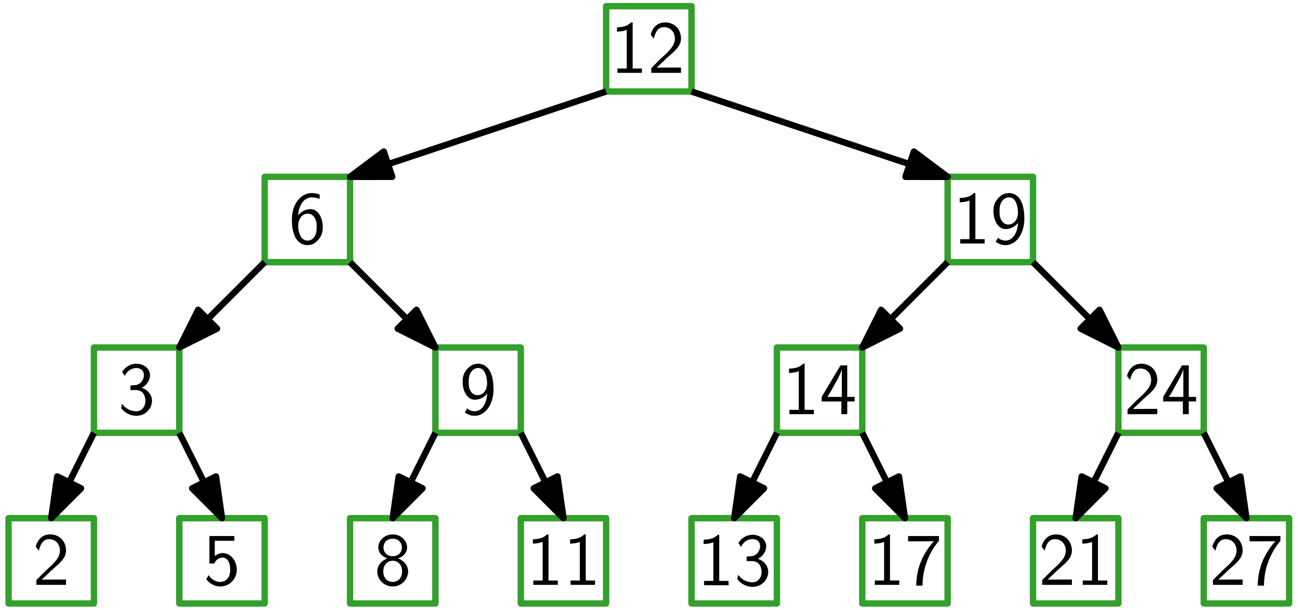

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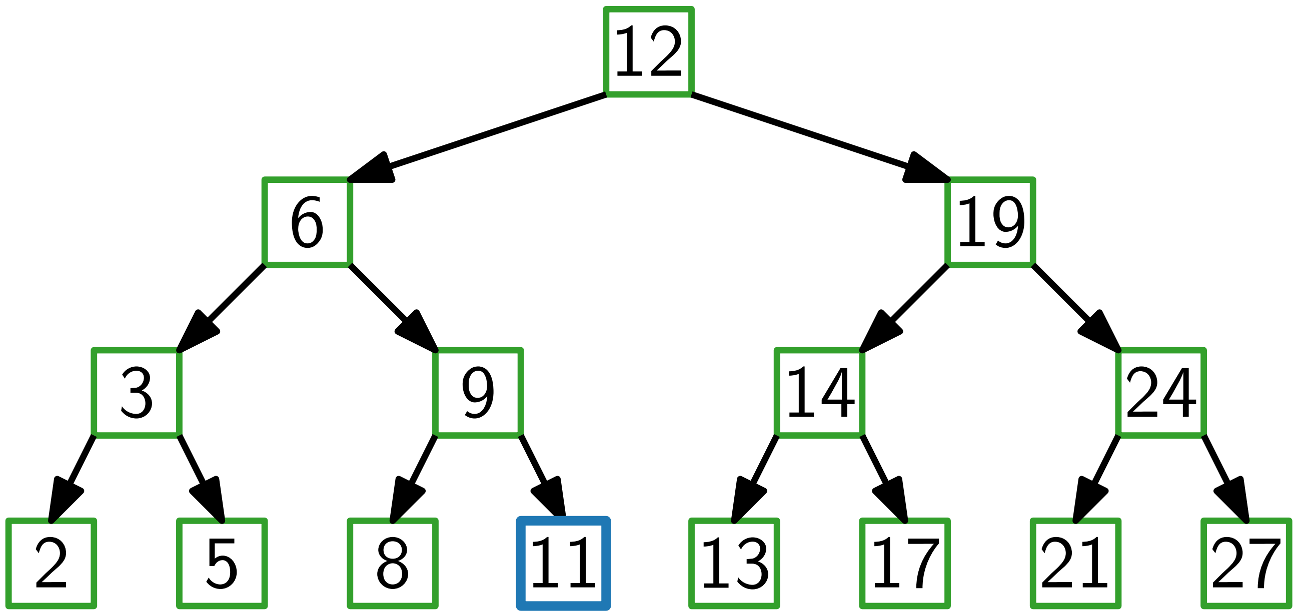

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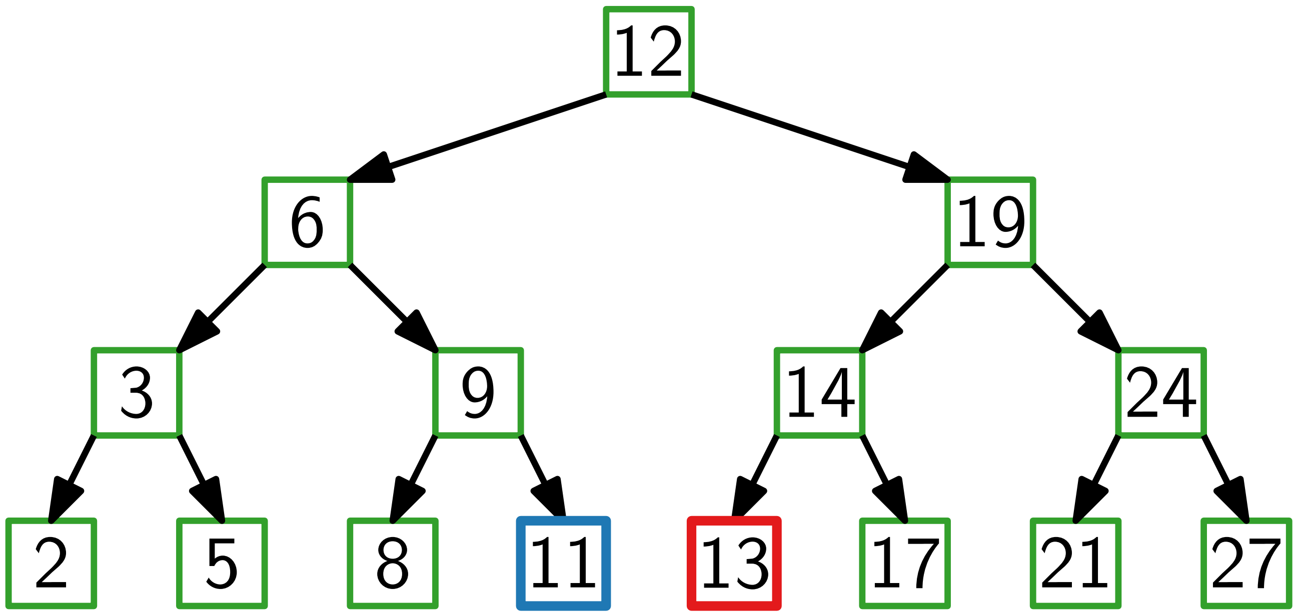

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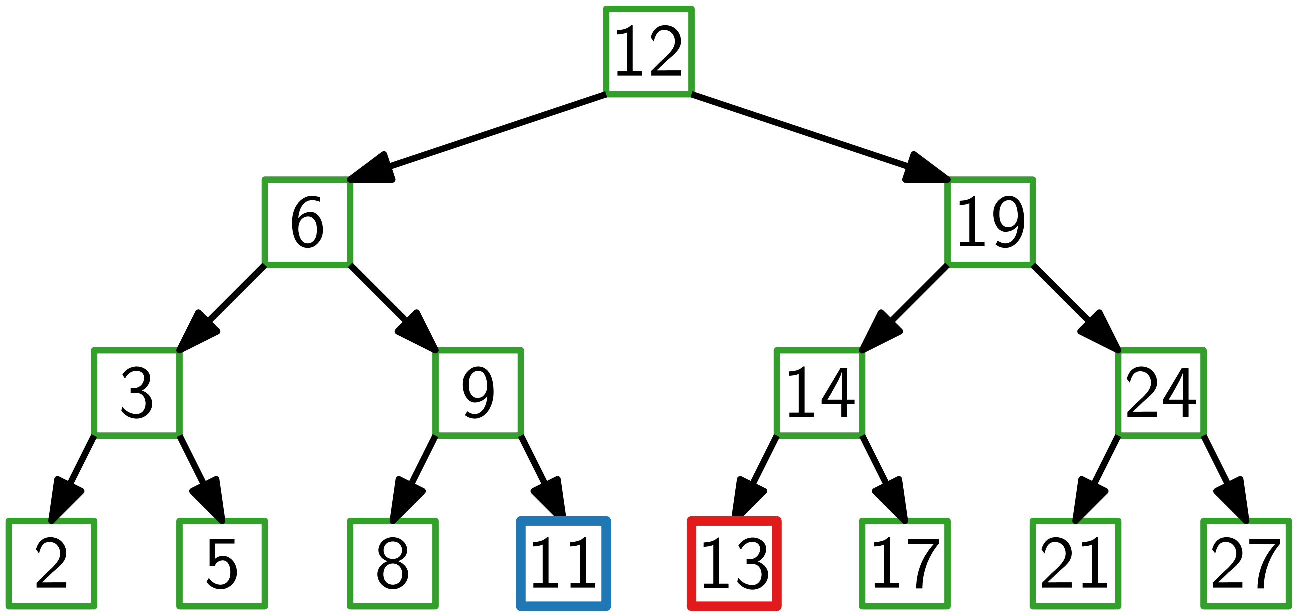

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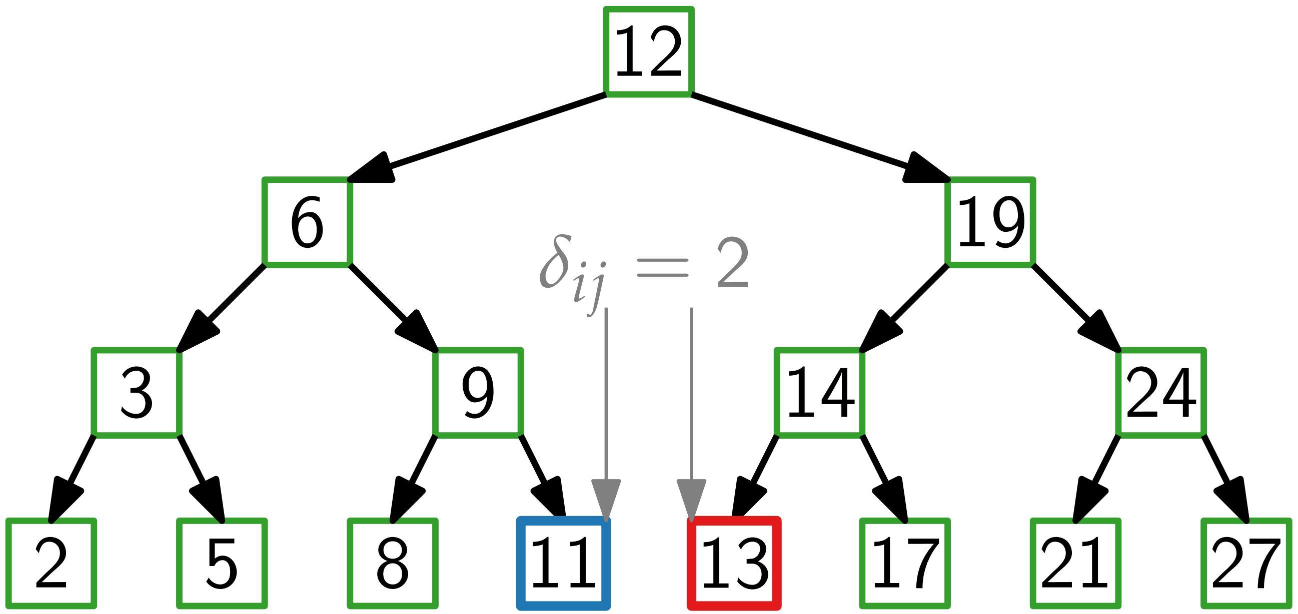

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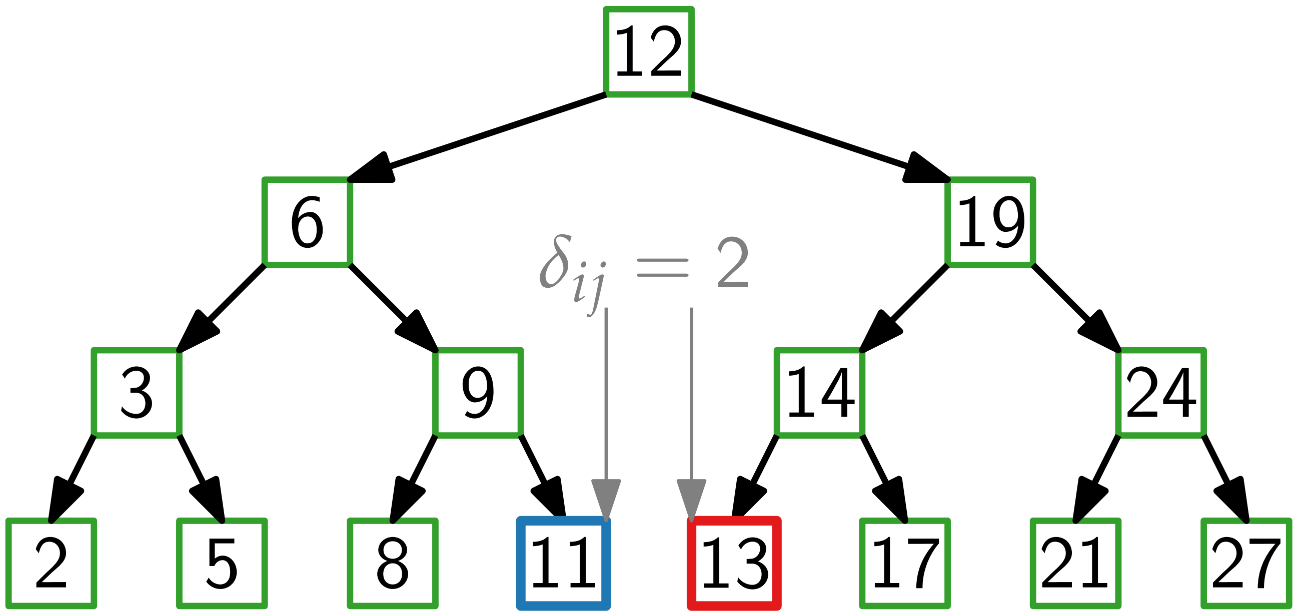

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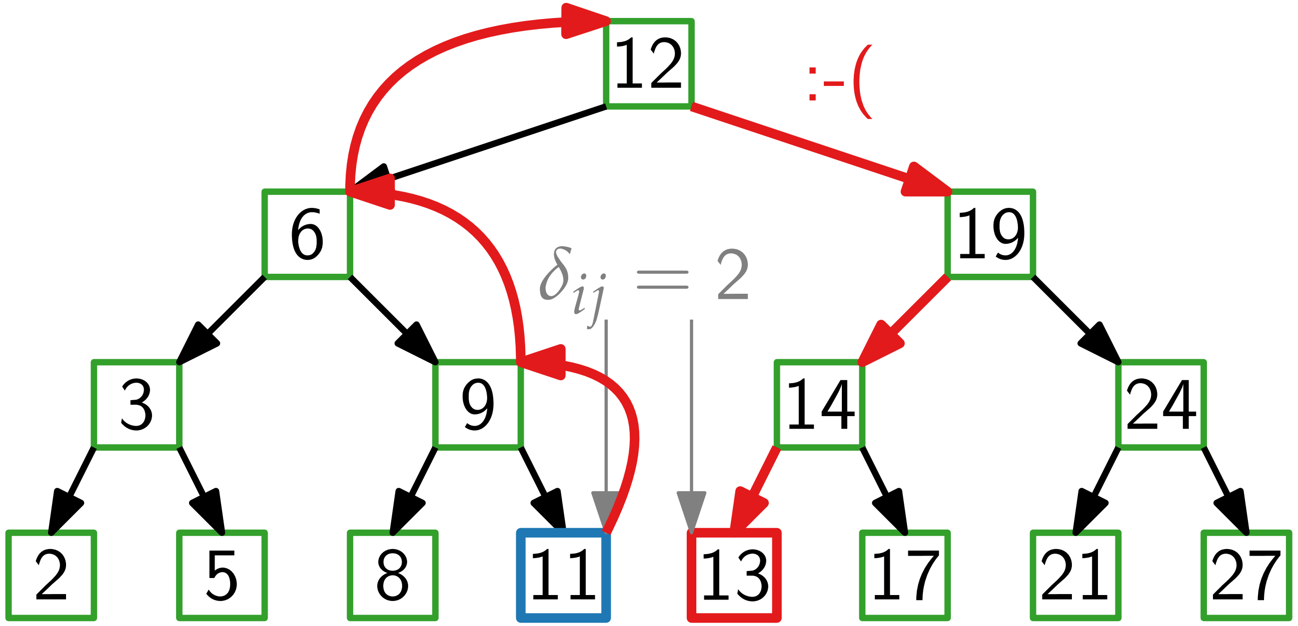

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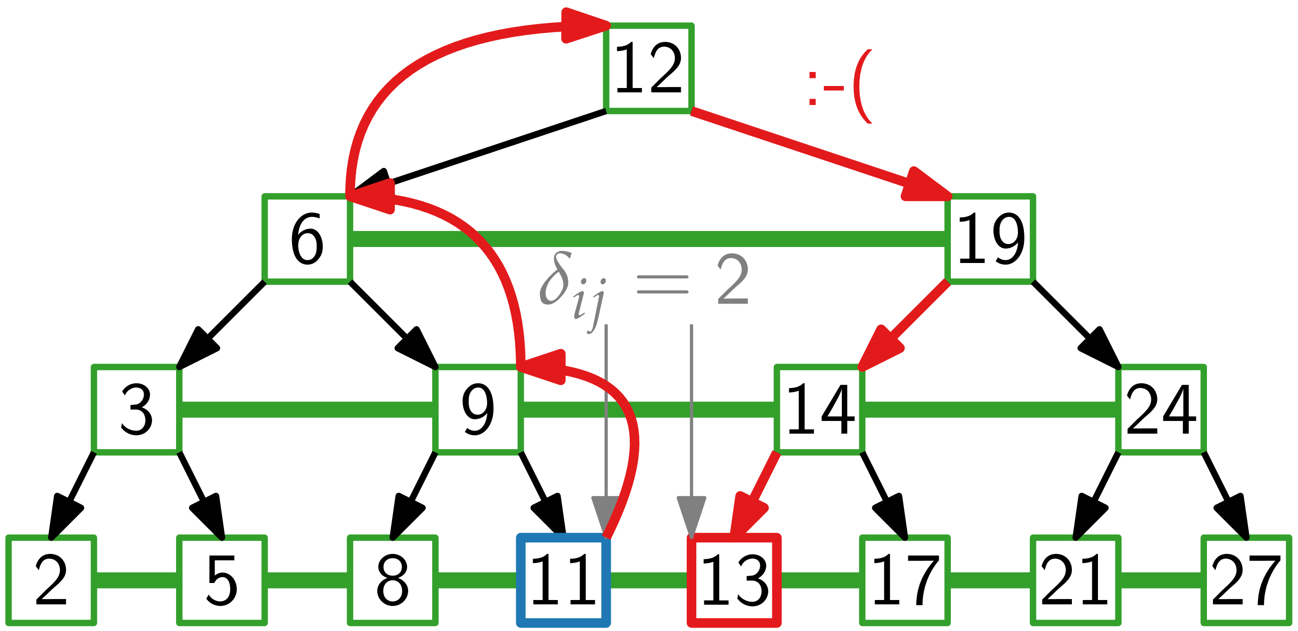

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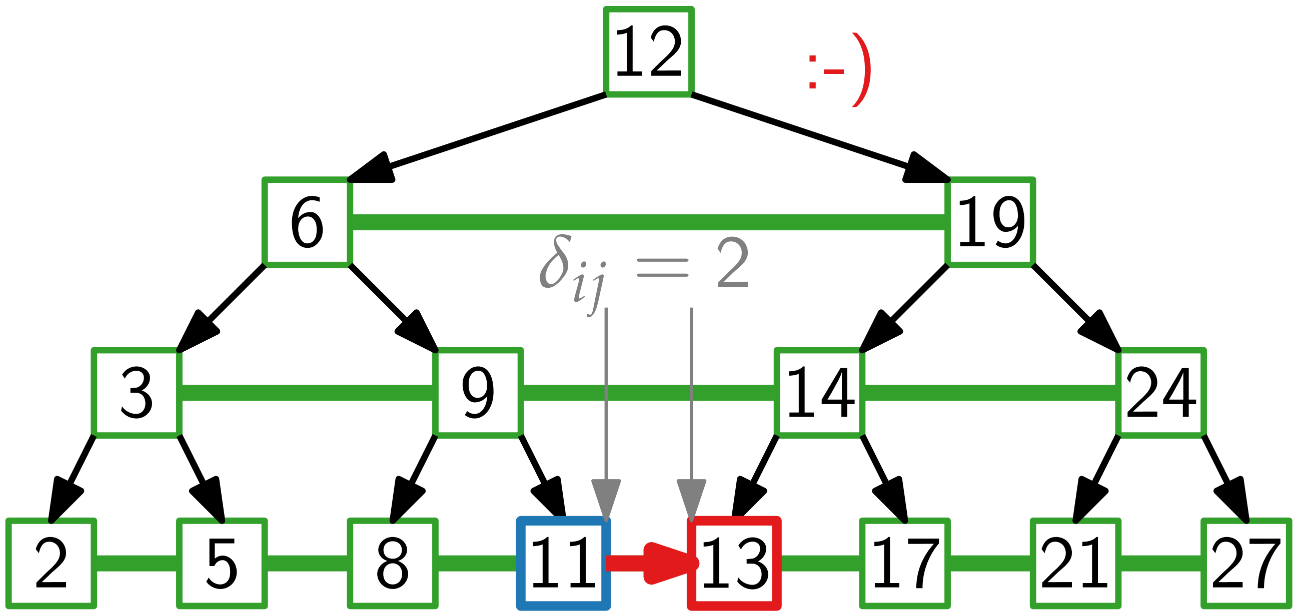

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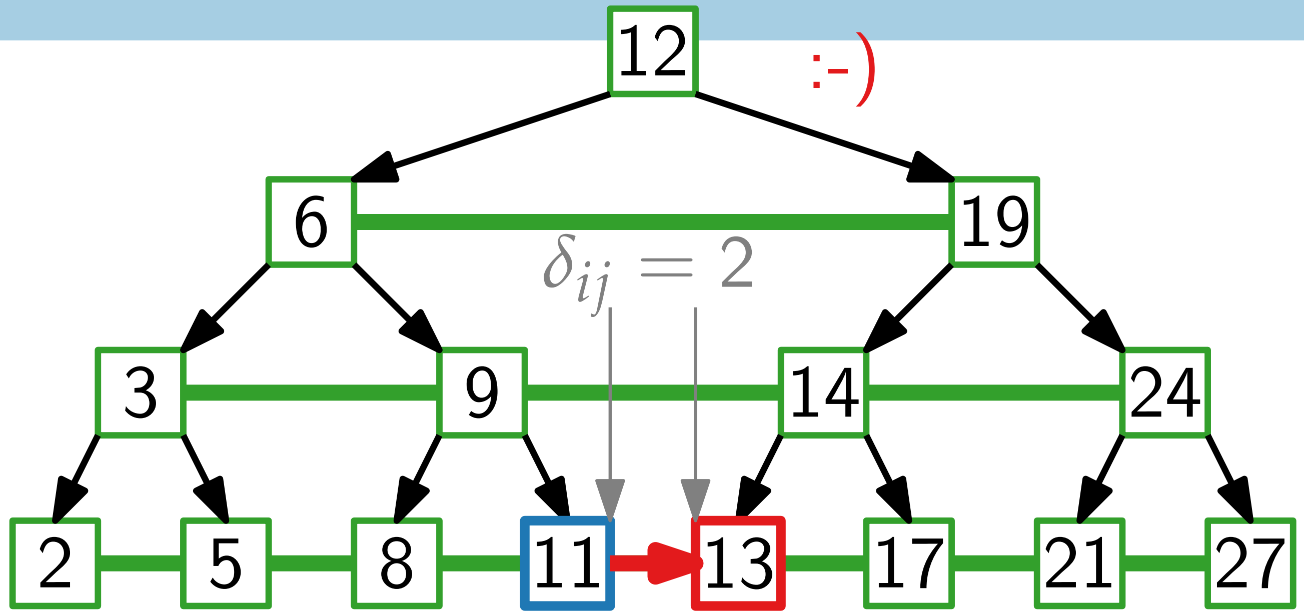
Lemma. A level-linked red–black tree has the dynamic finger property.



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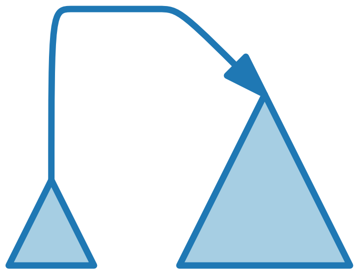

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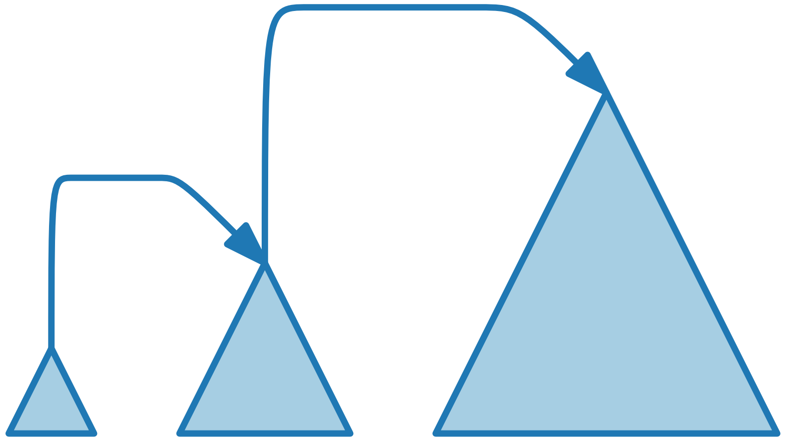

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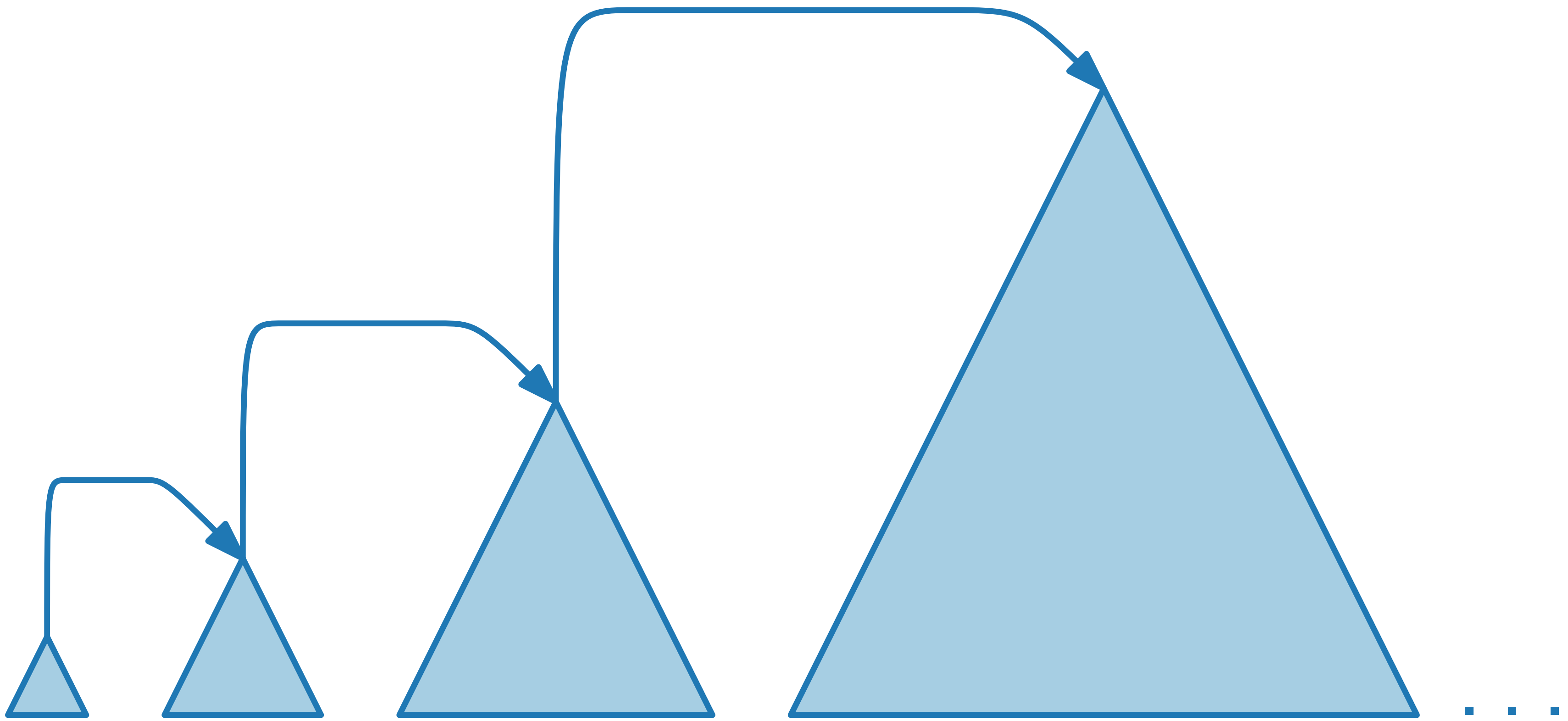

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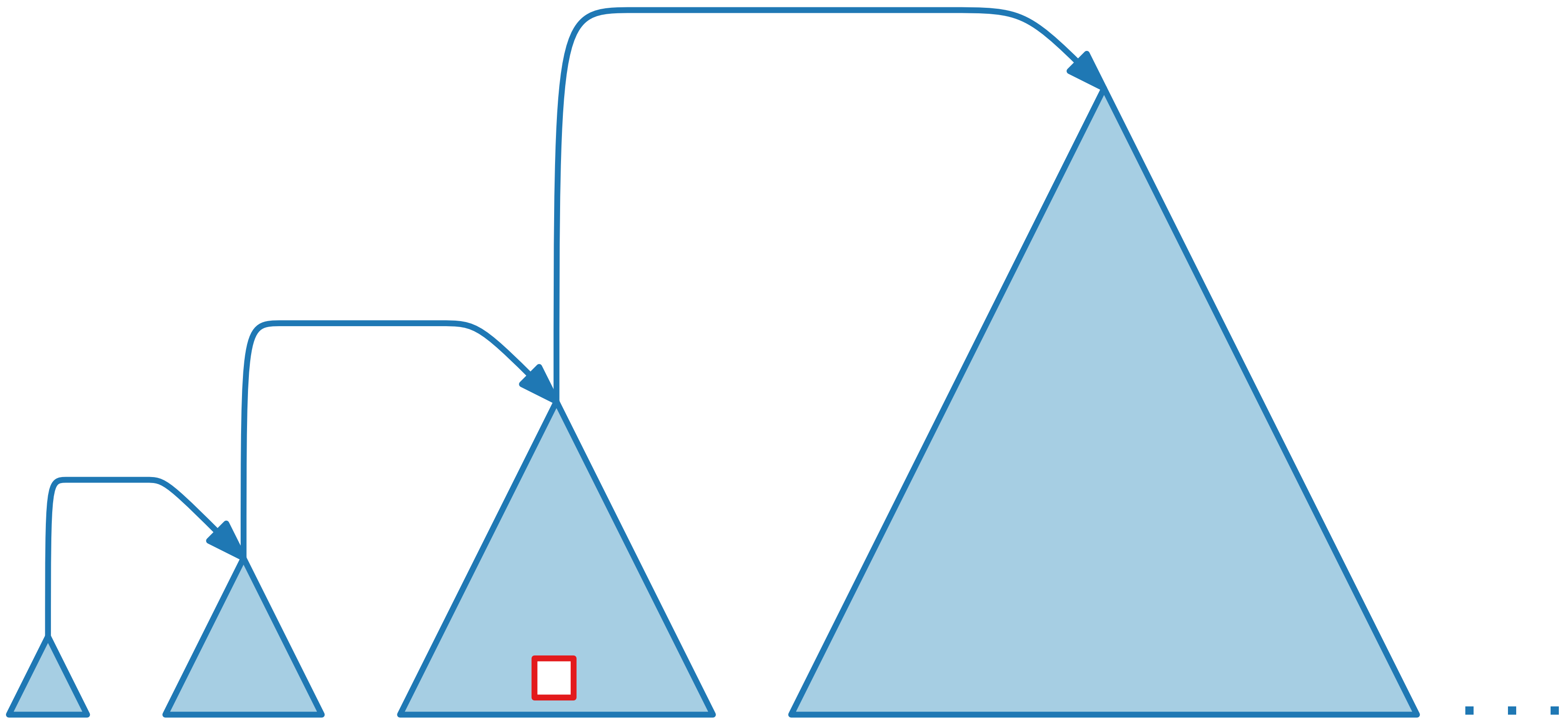

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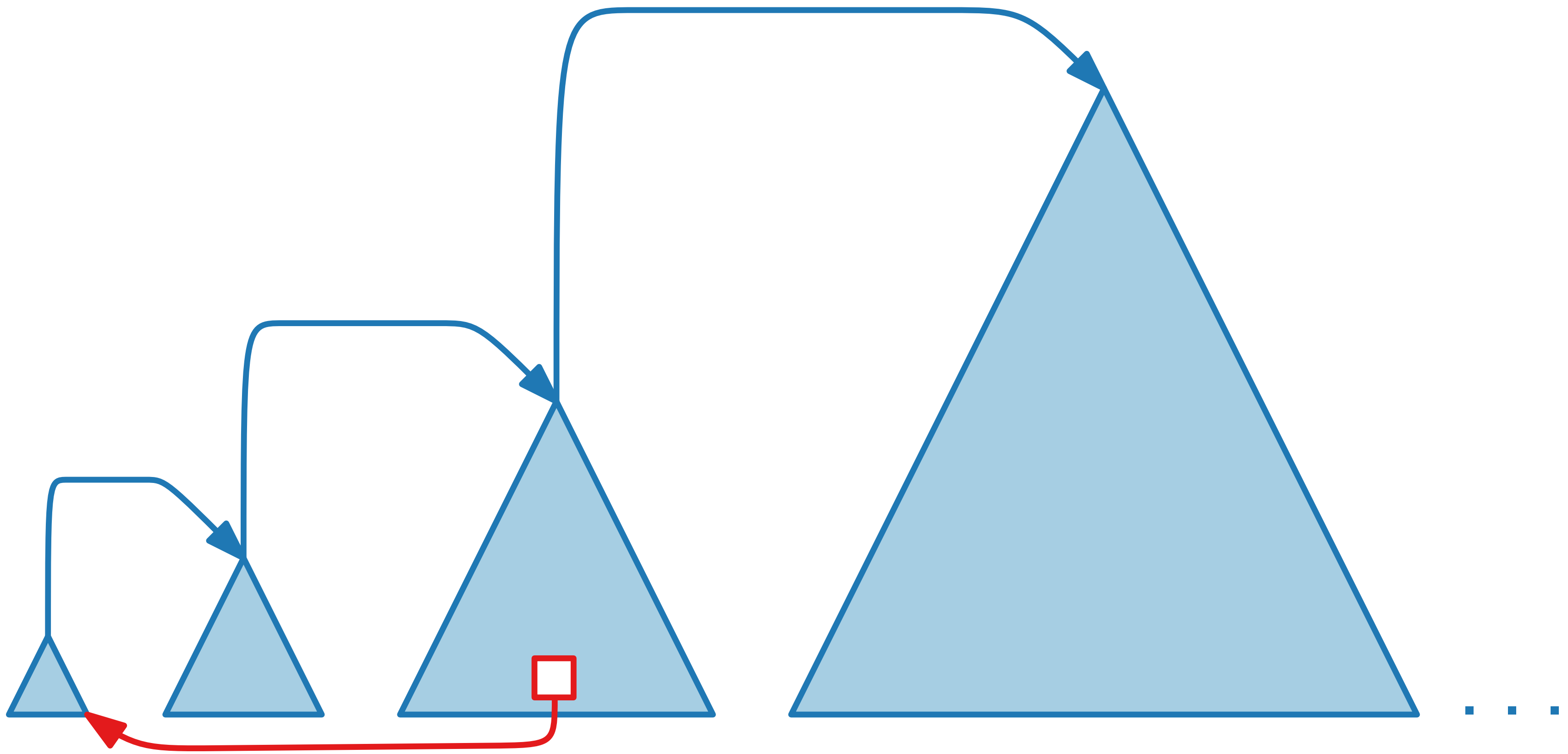

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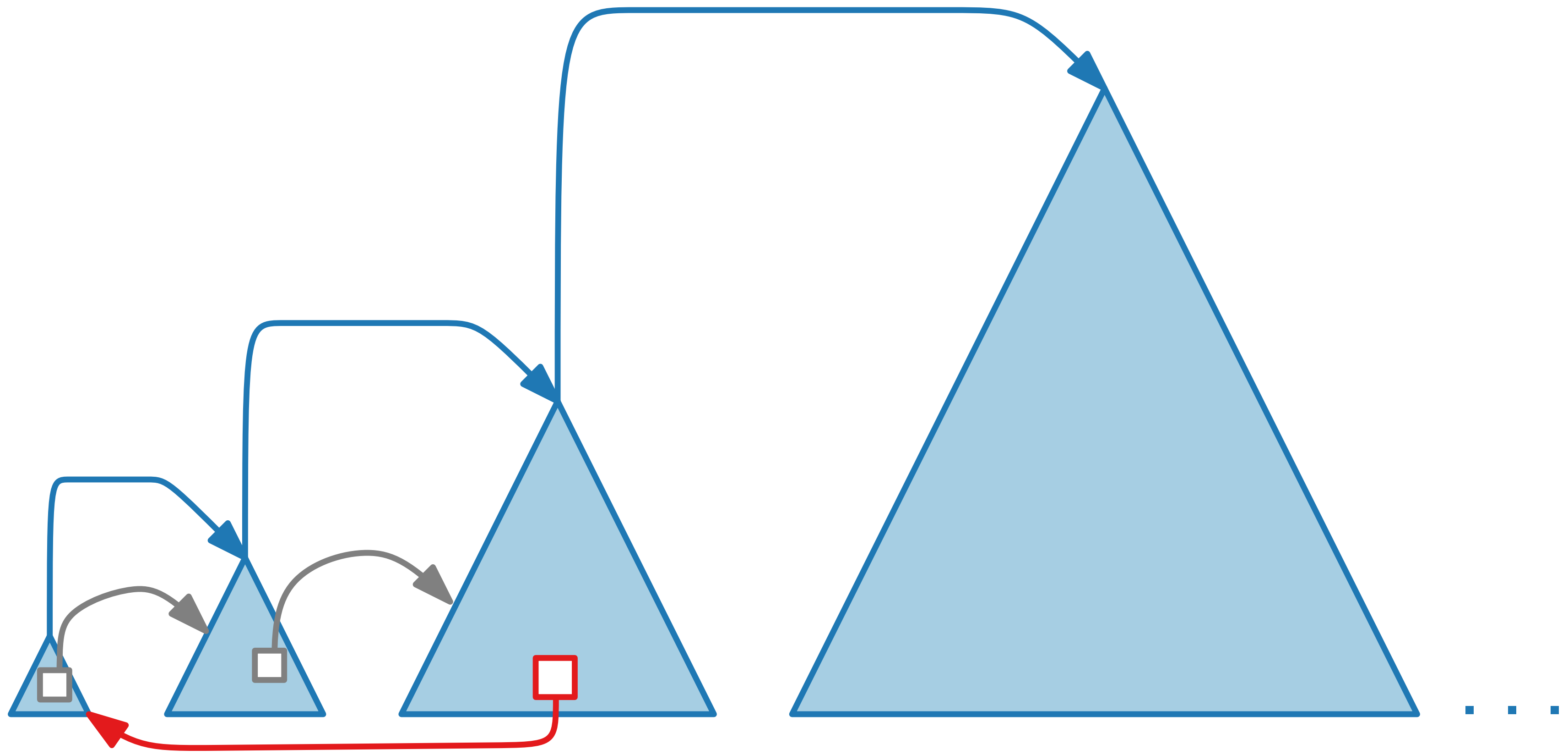

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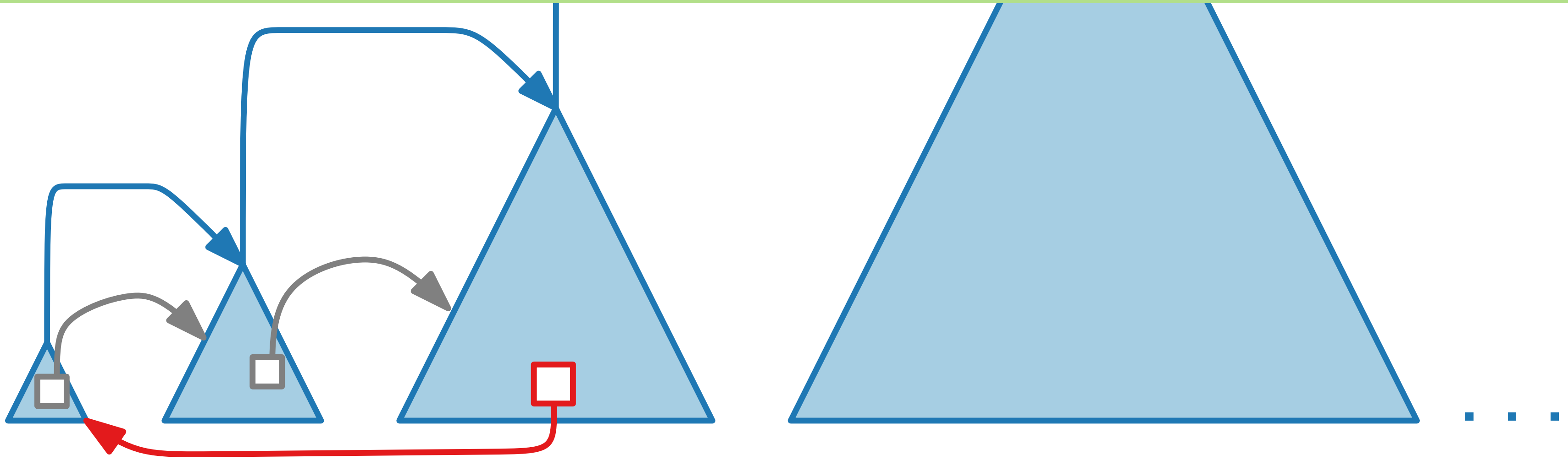

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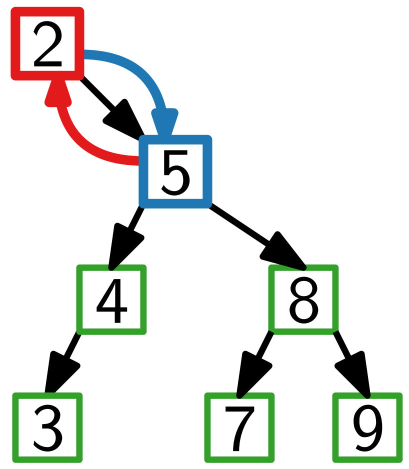

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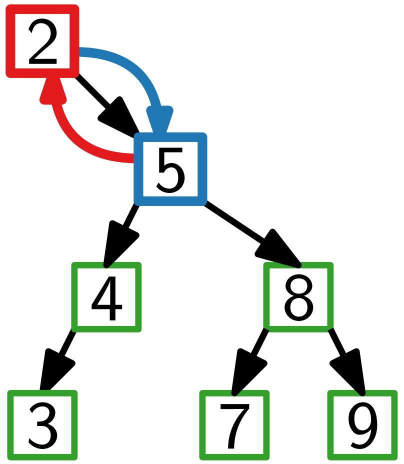

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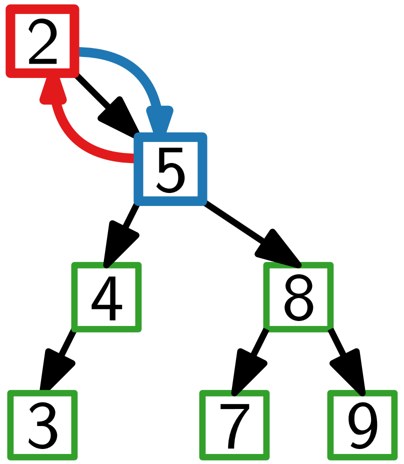

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Definition. A BST is **statically optimal** if queries take (amortized) $O(\mathrm{OPT}_S)$ time for every S.

All These Models . . .

Balanced: Queries take (amortized) $O(\log n)$ time

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Dynamic Finger: Queries take $O(\log \delta_i)$ time (δ_i: rank diff.)

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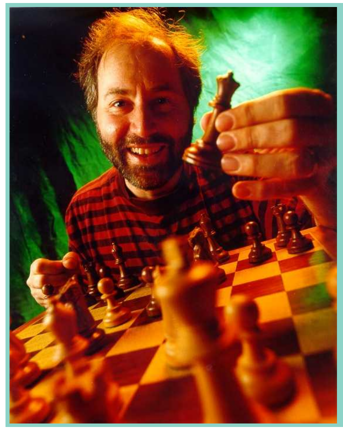

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Robert E. Tarjan

J. ACM 1985

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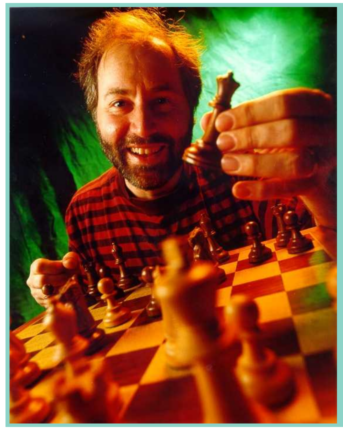

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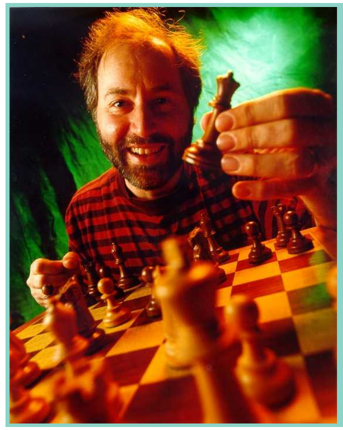

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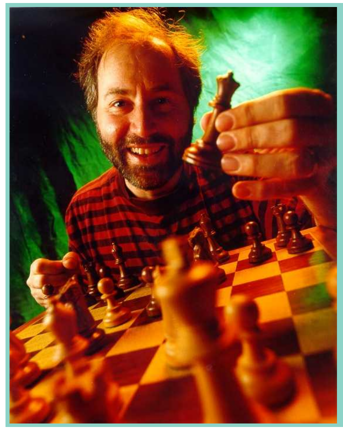

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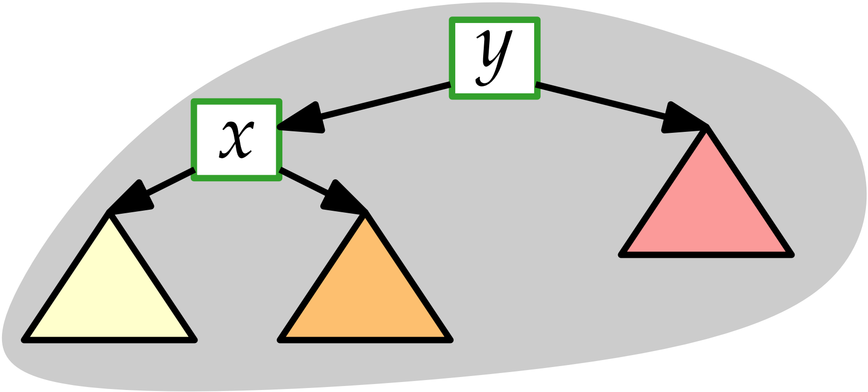

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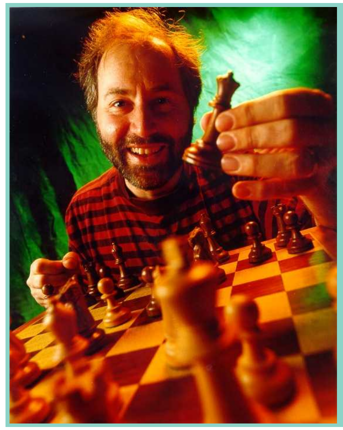

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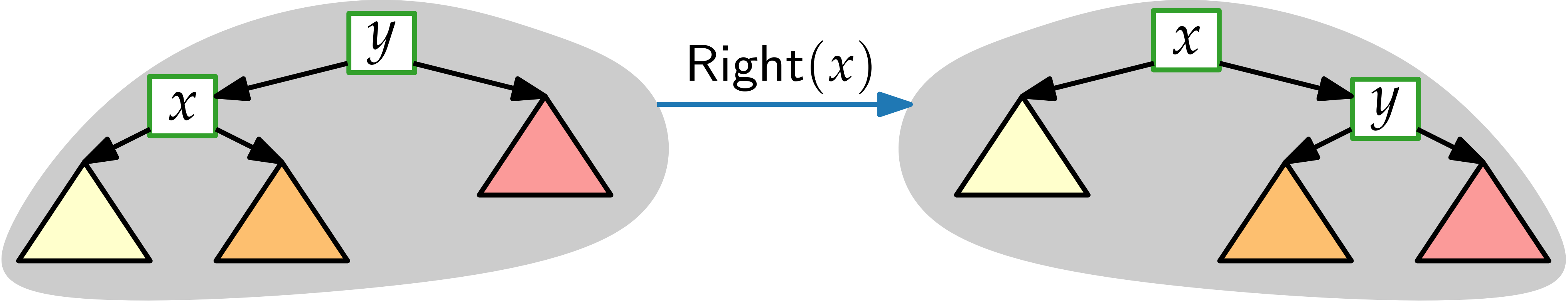

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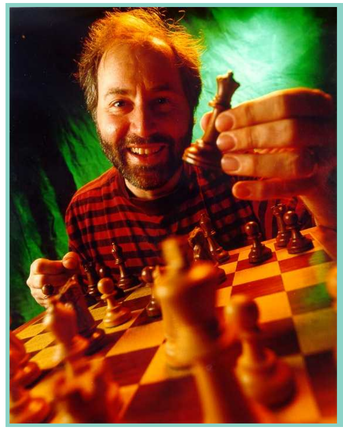

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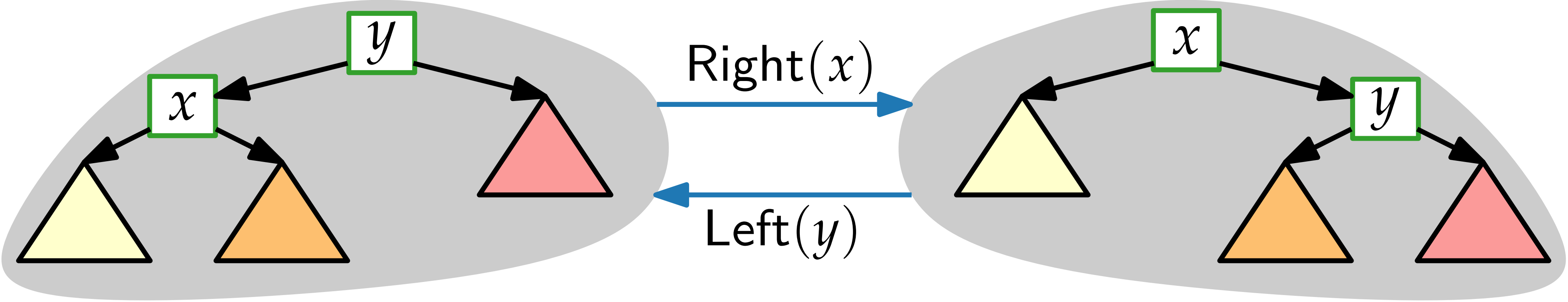

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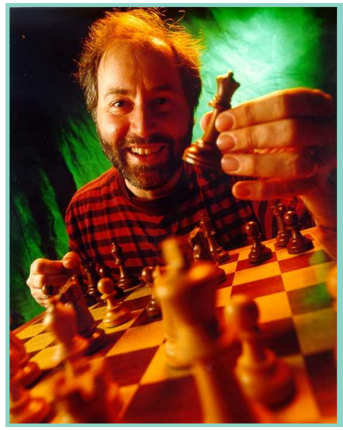

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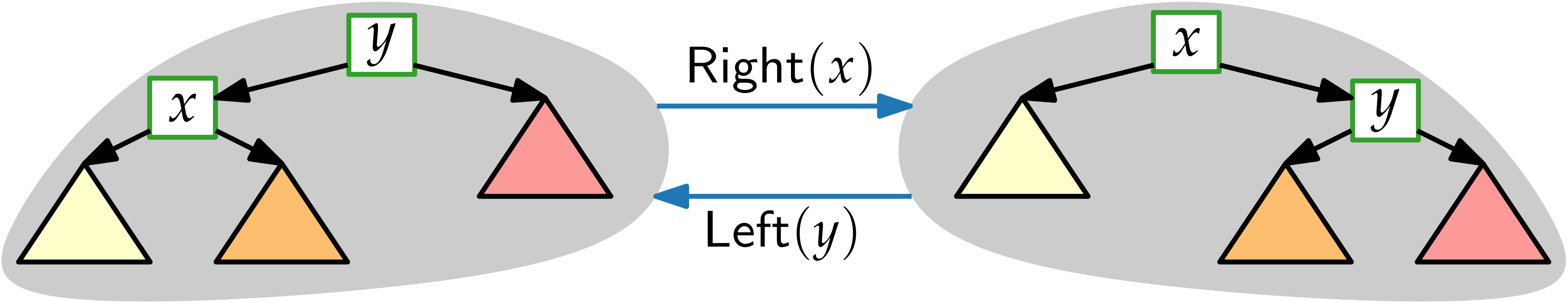

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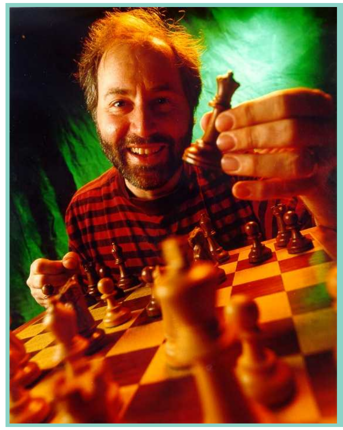

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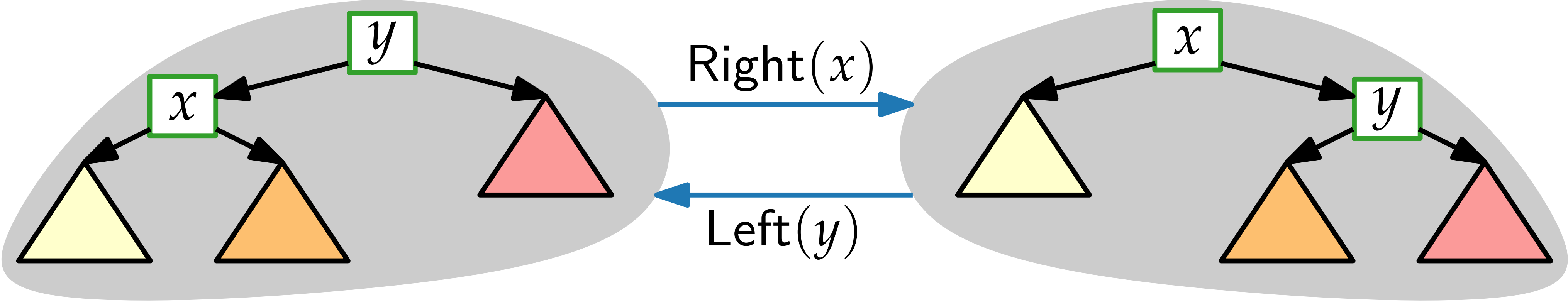

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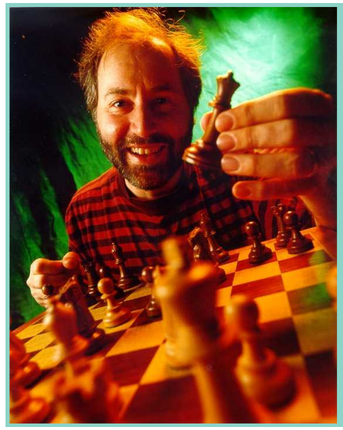

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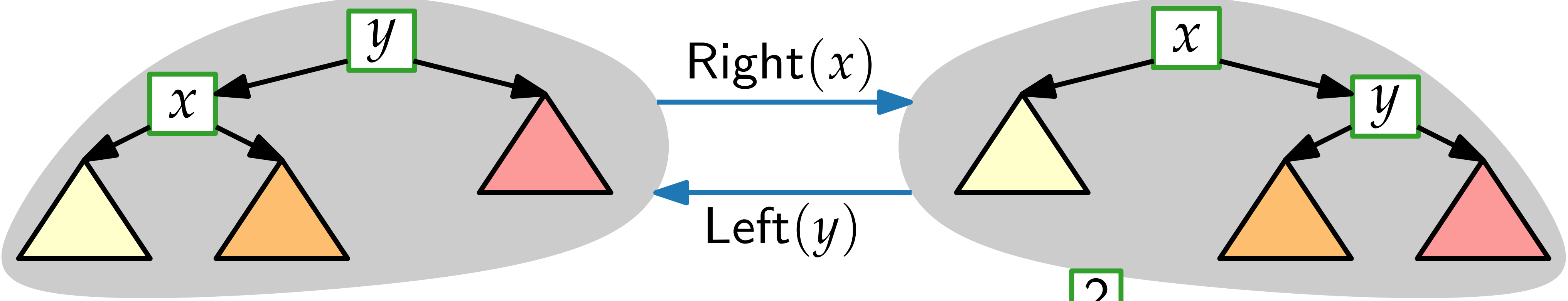

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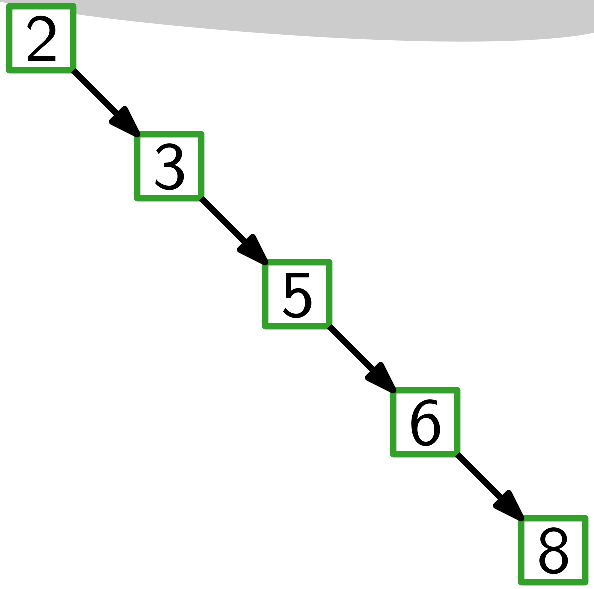

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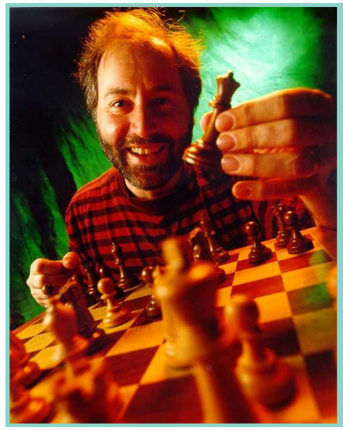

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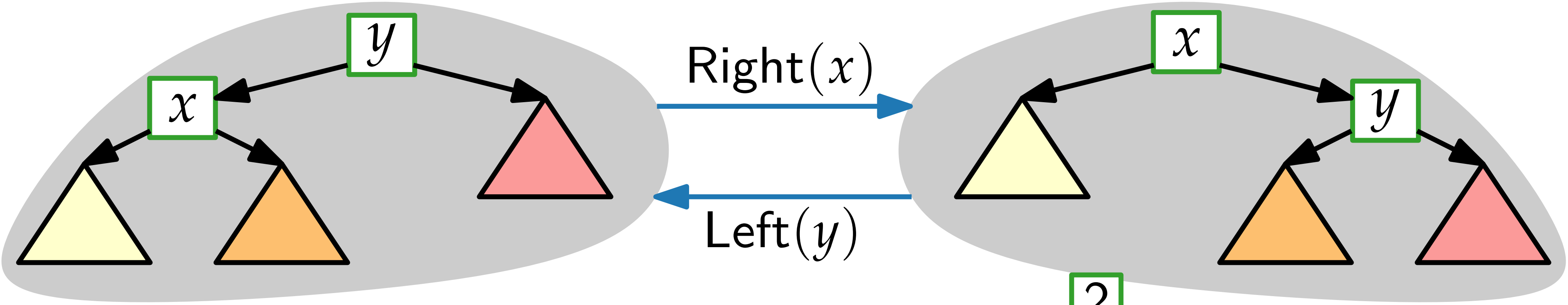

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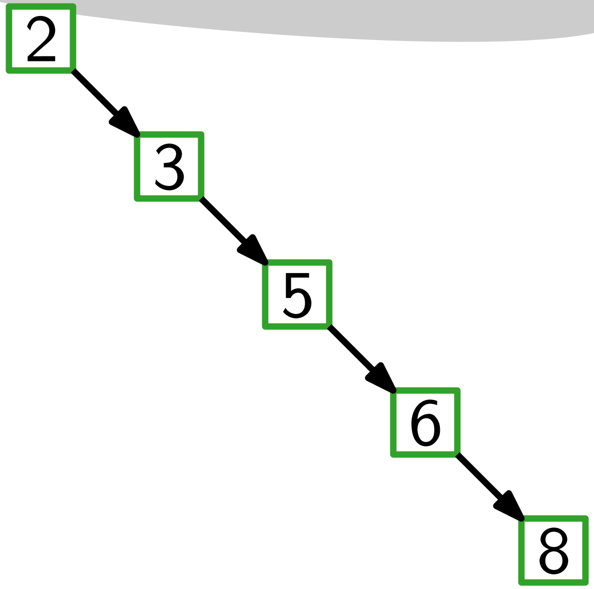

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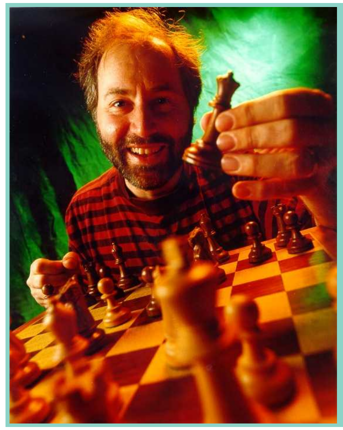

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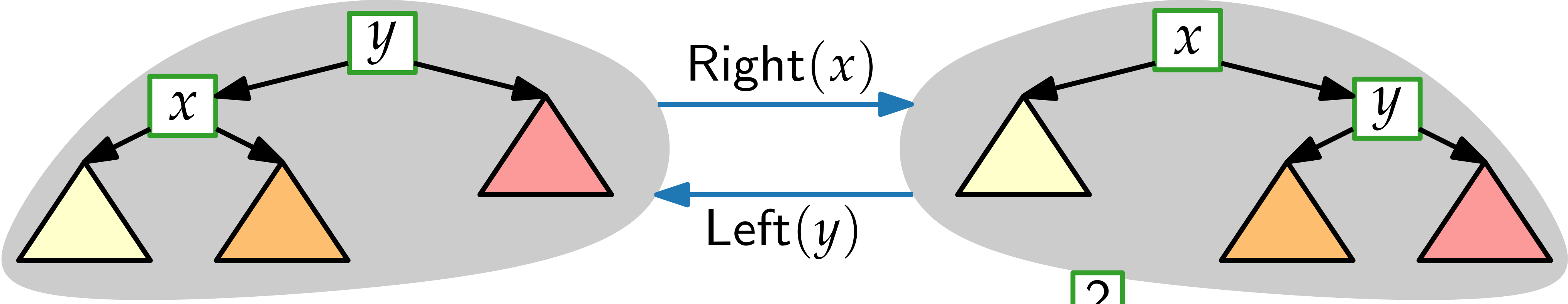

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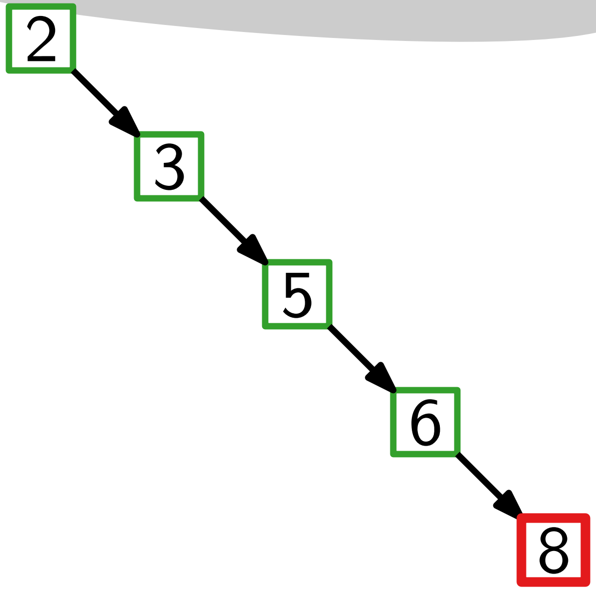

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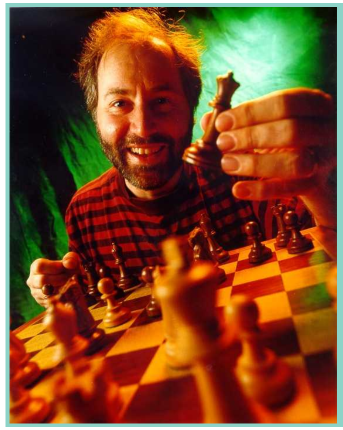

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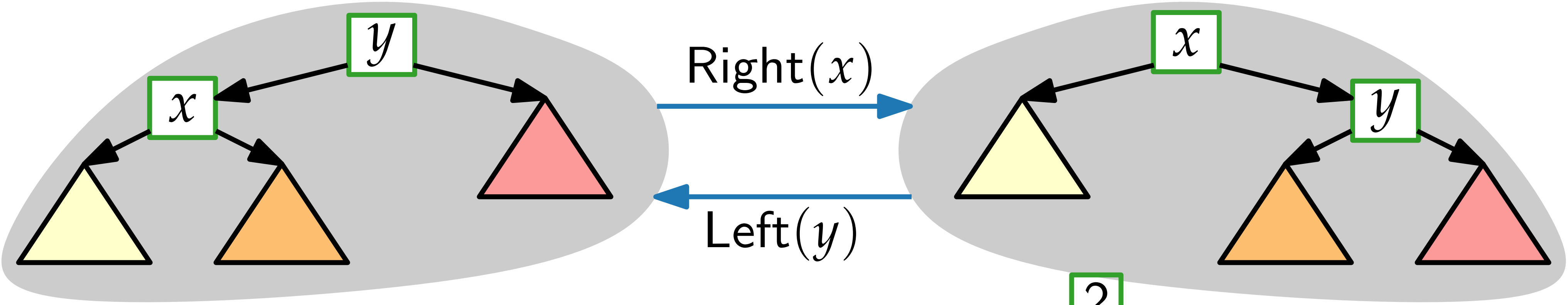

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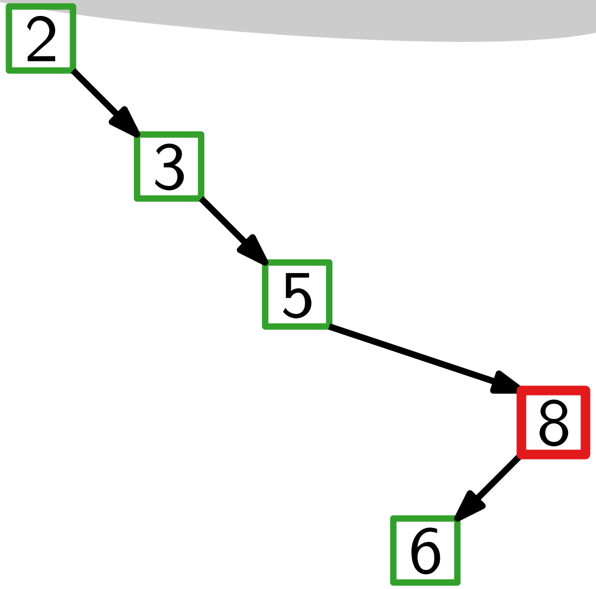

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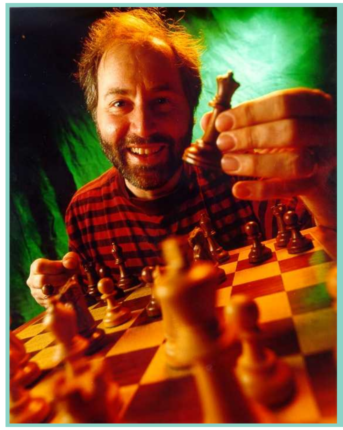

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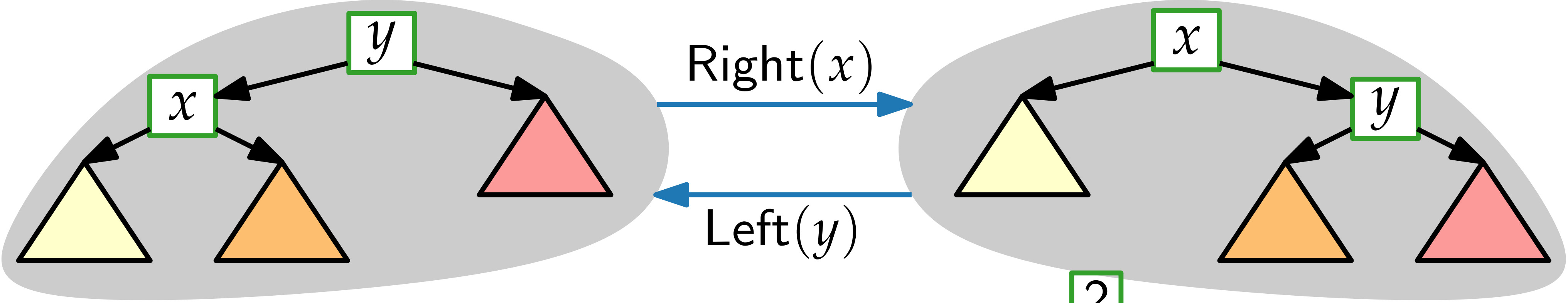

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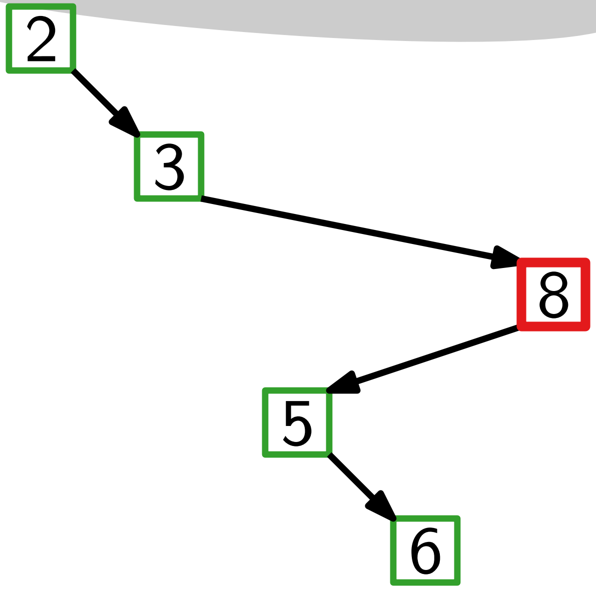

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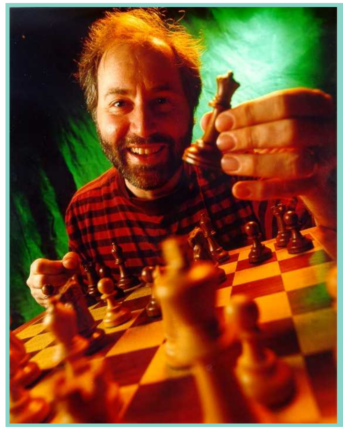

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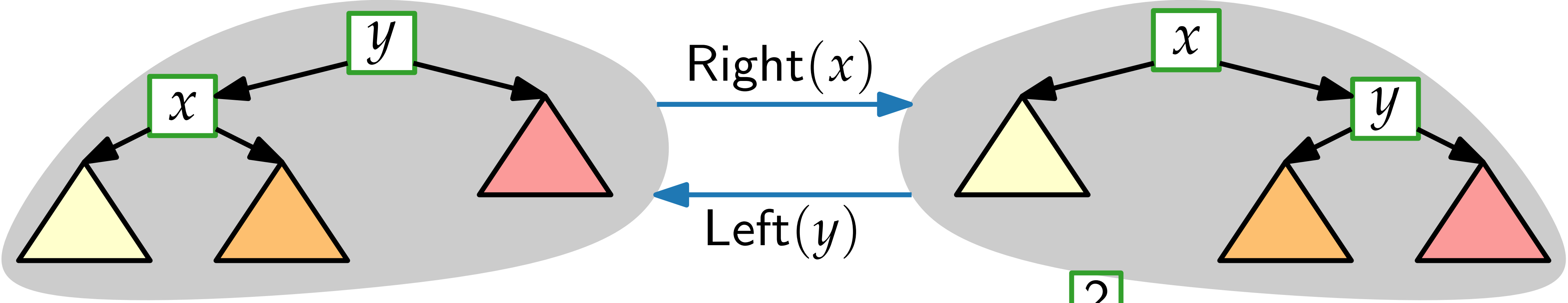

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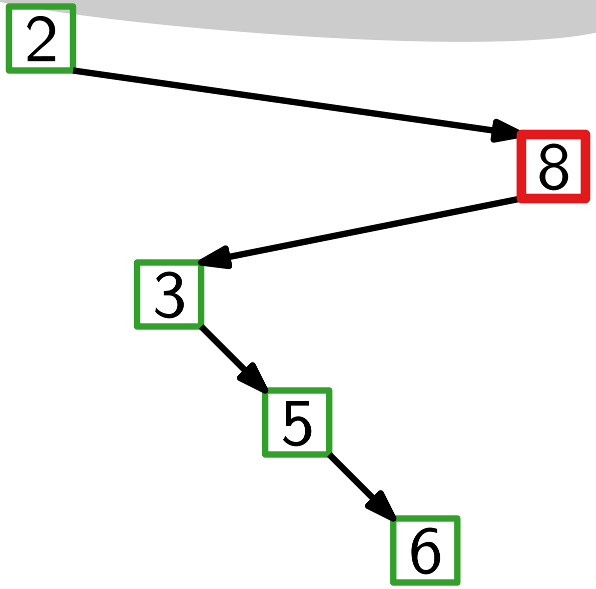

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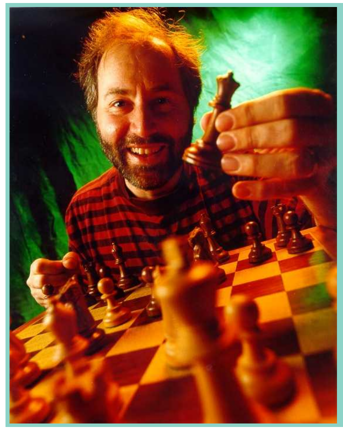

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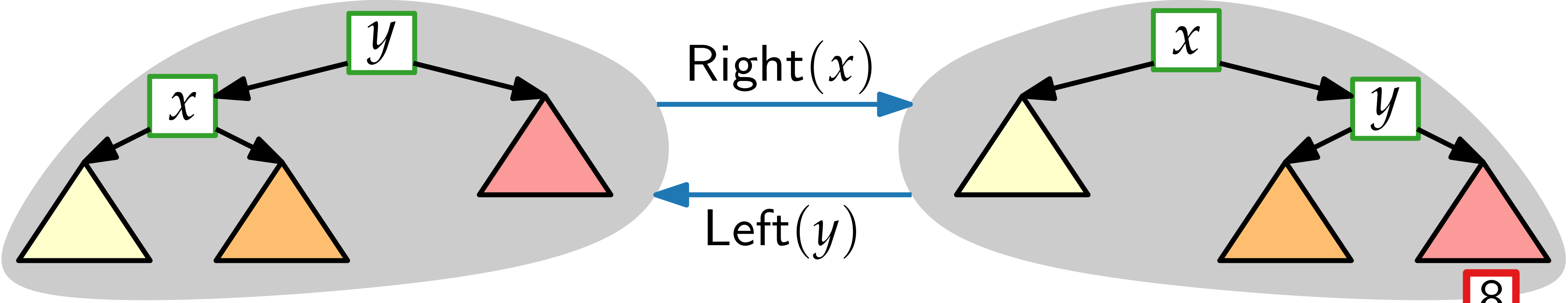

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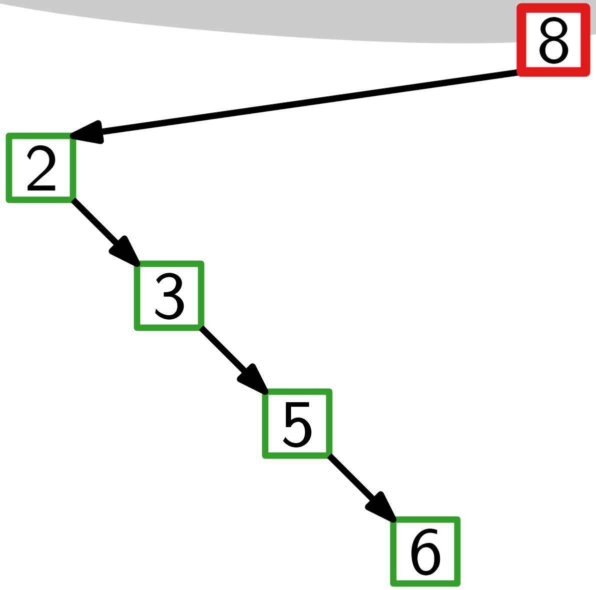

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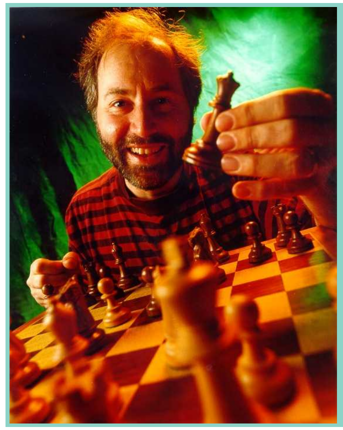

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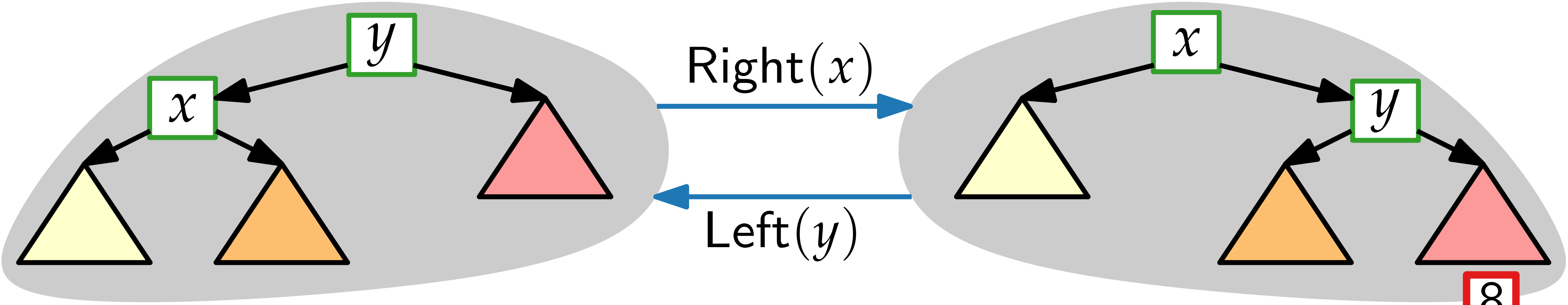

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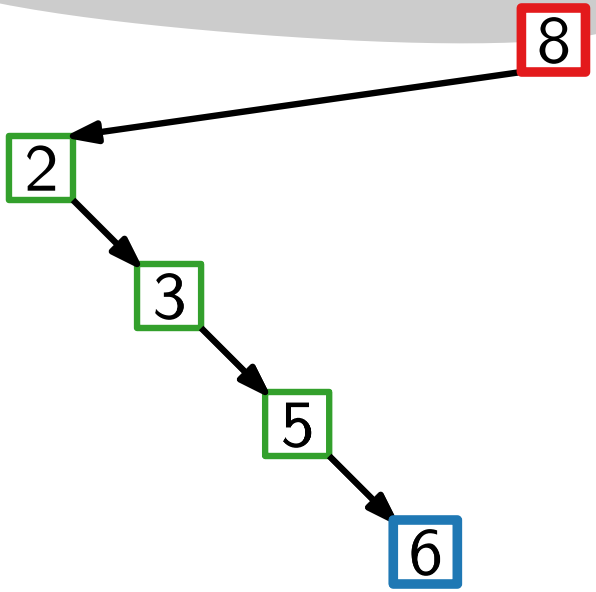

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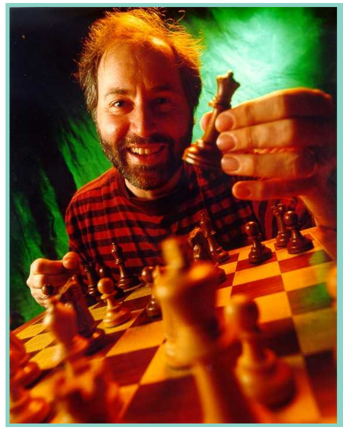

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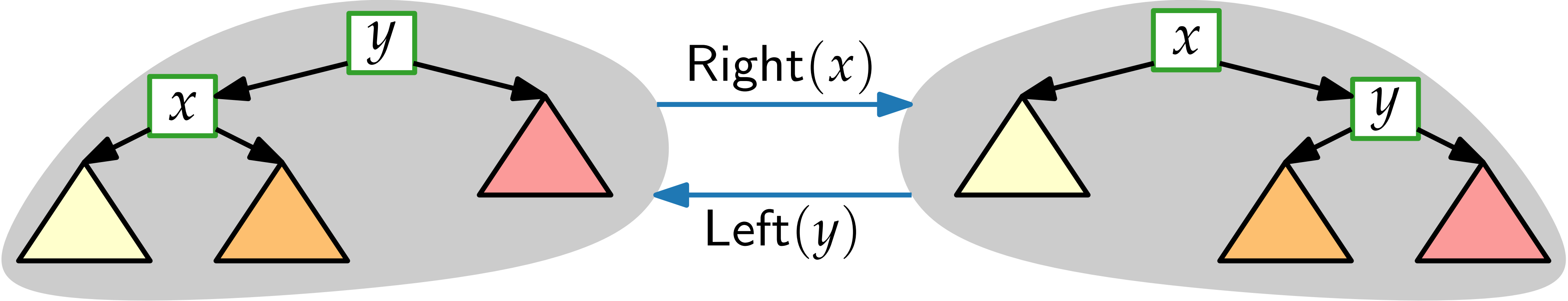

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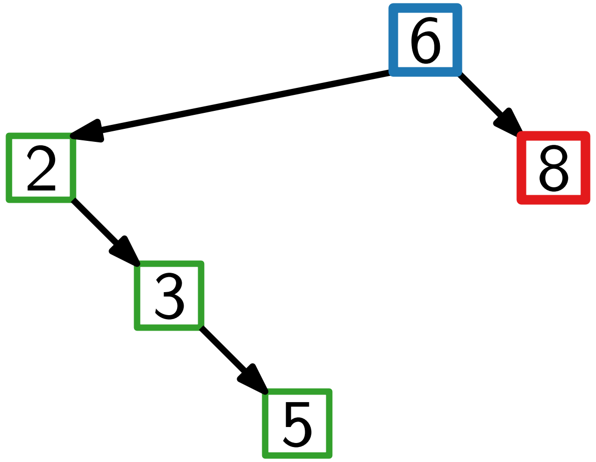

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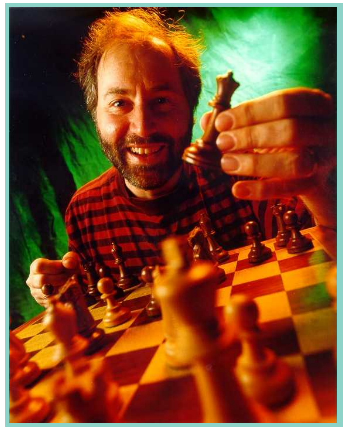

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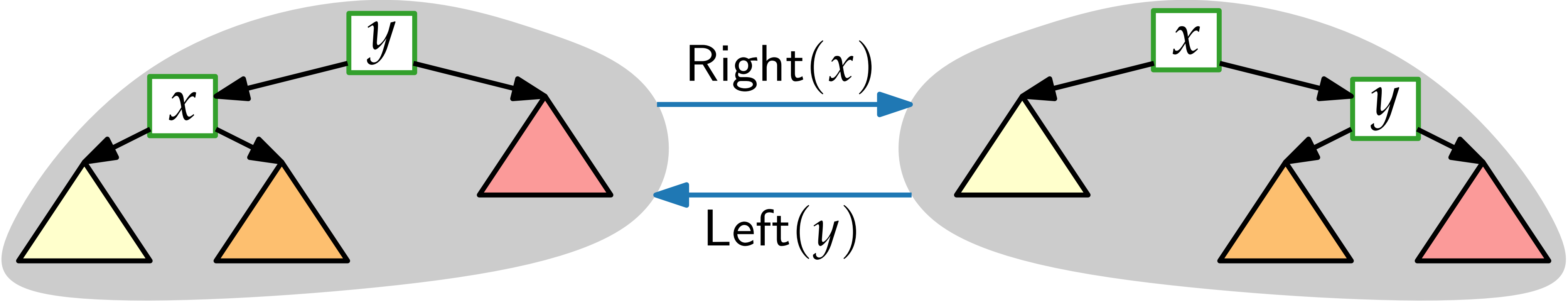

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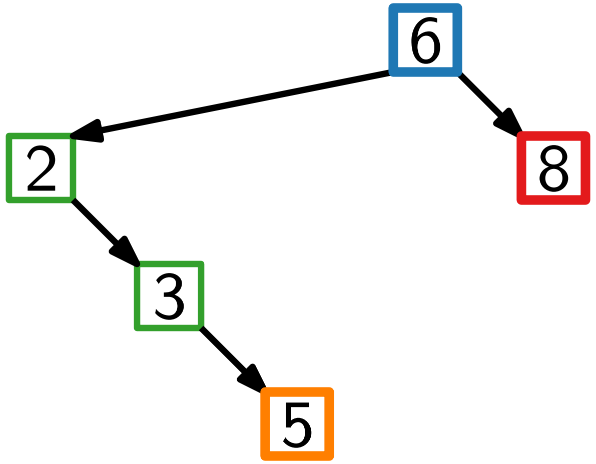

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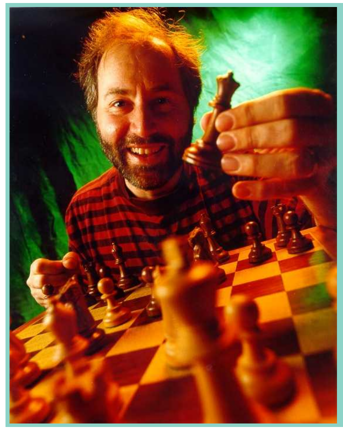

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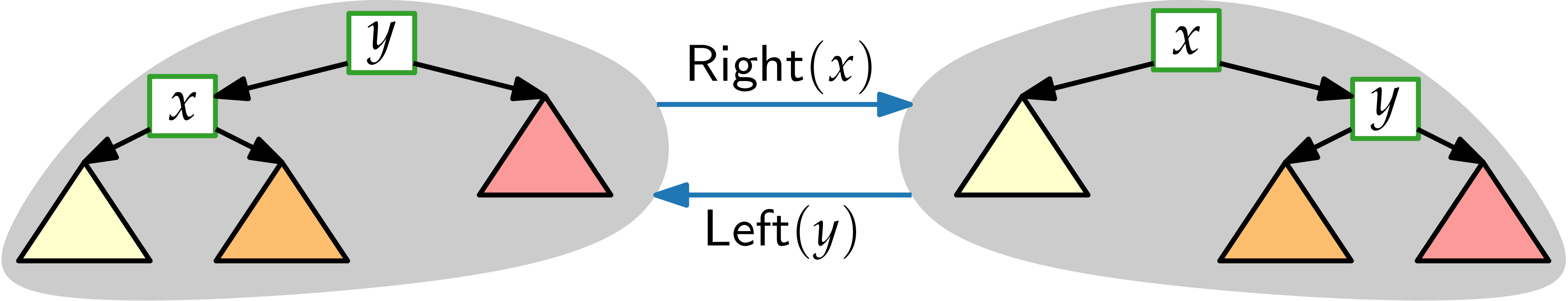

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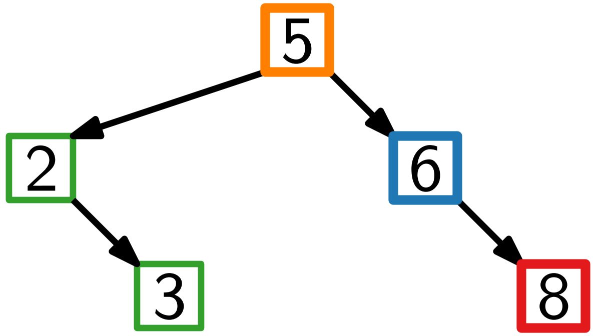

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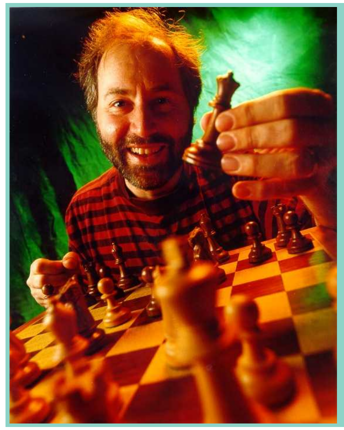

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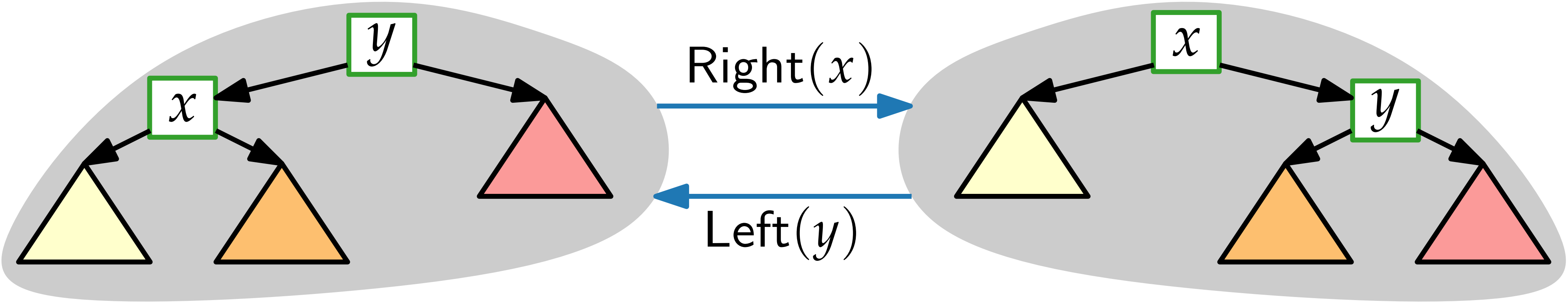

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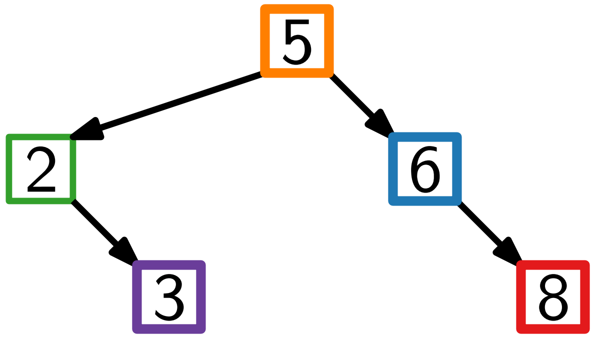

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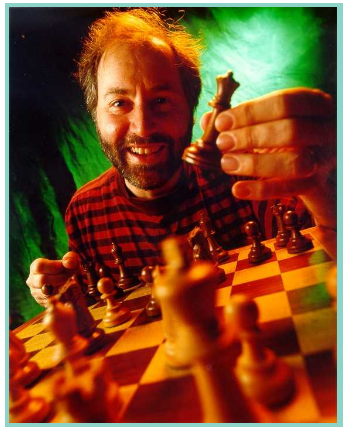

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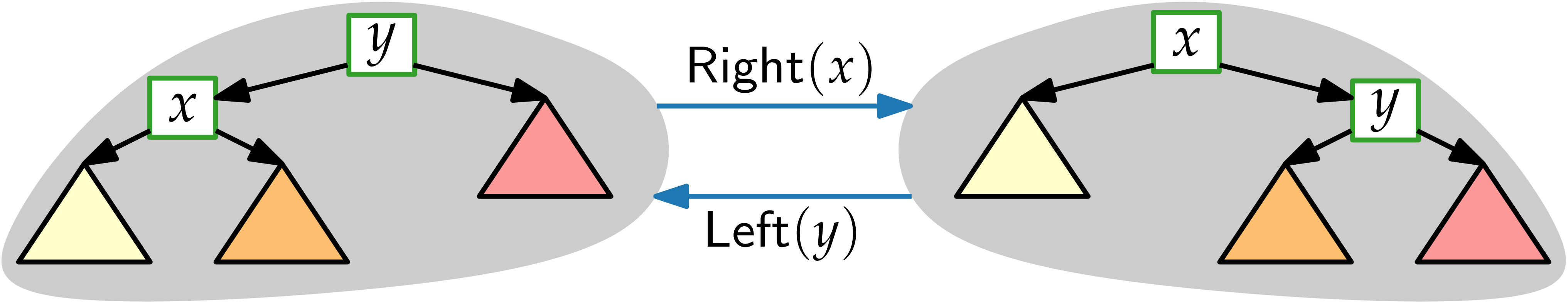

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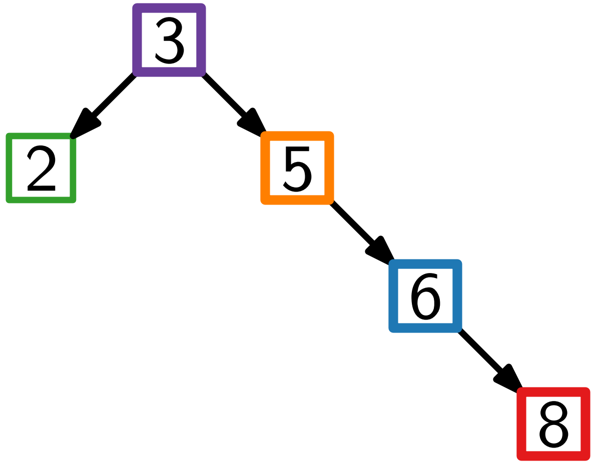

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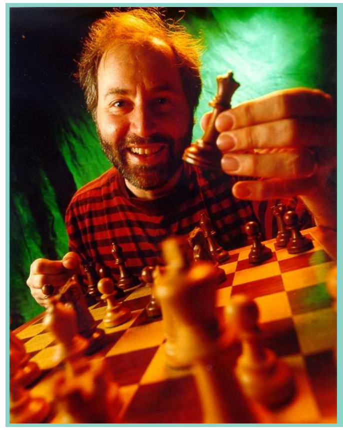

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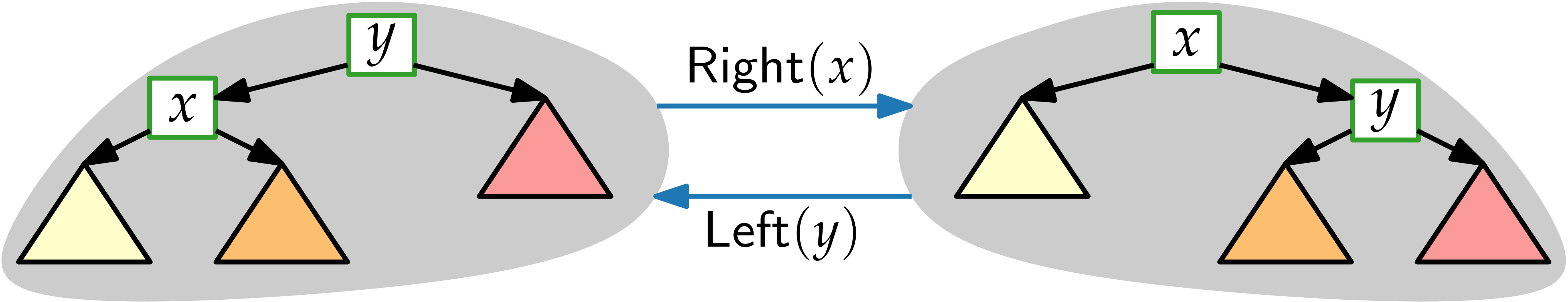

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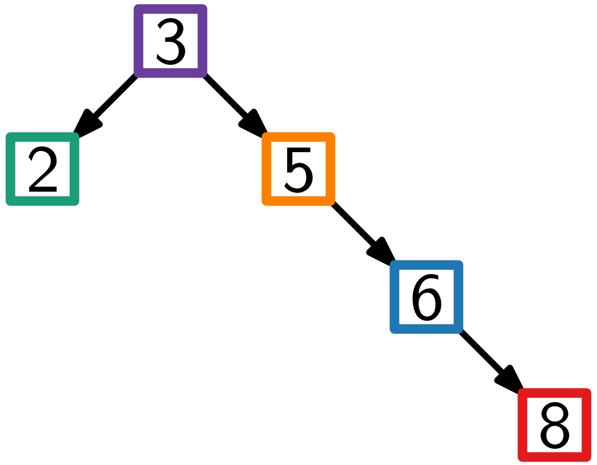

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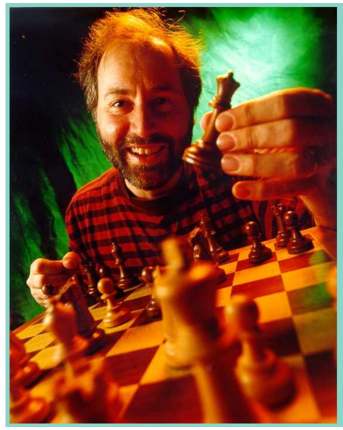

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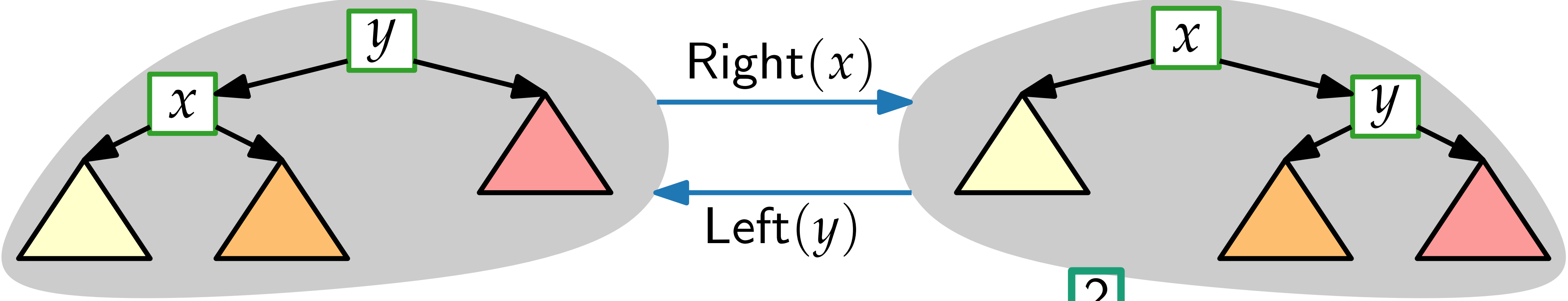

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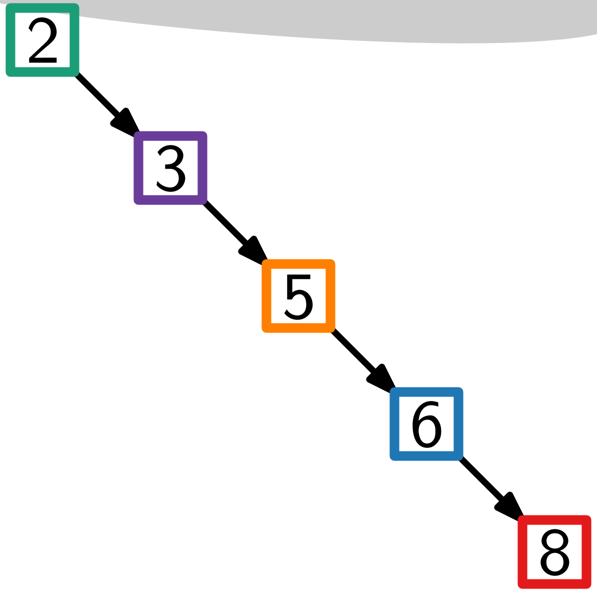

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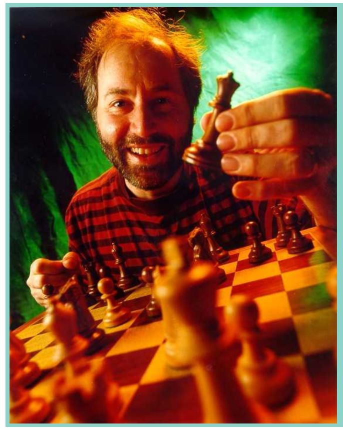

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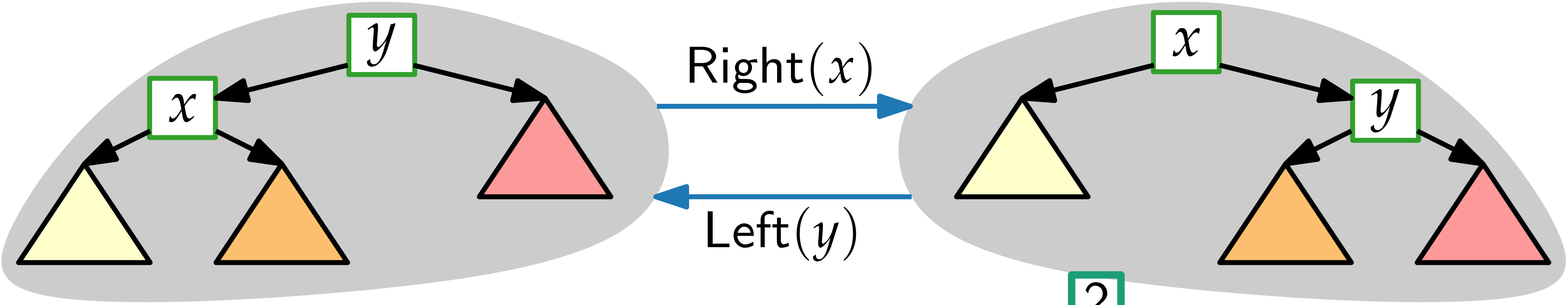

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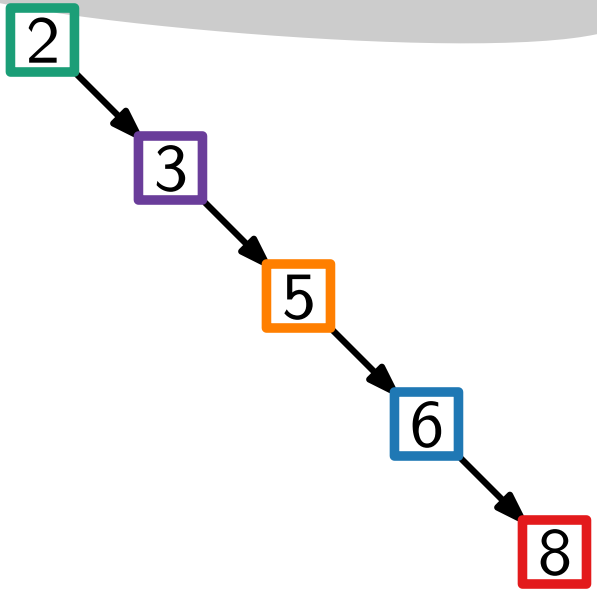
We're back at the start...



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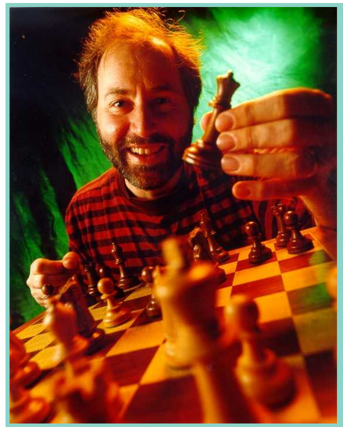

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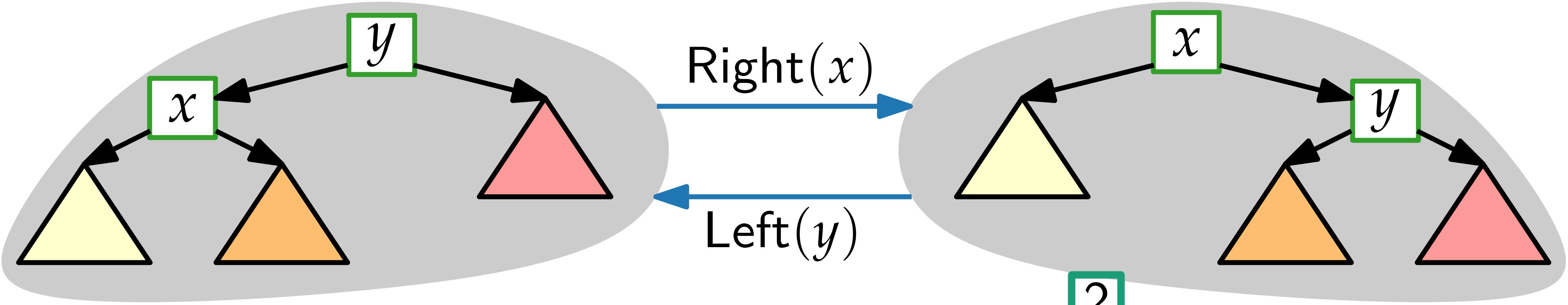
Idea: Whenever we query a key, rotate it to the root.

Known from the lecture *Algorithms and Data Structures* (ADS):



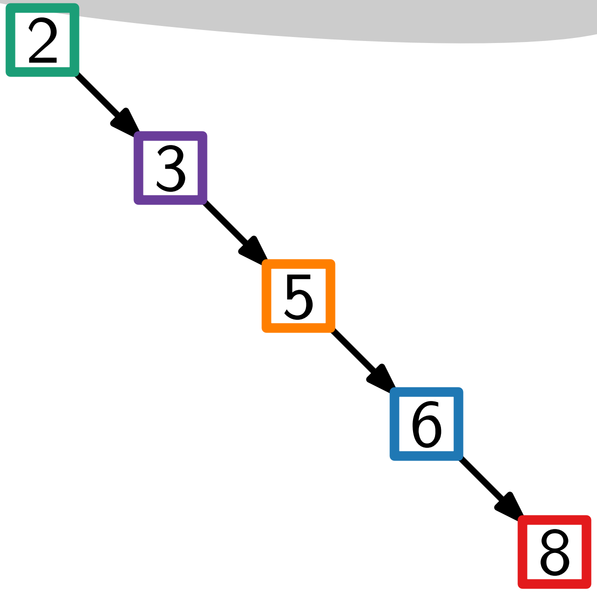
New:

Splay(x): Rotate x to the root

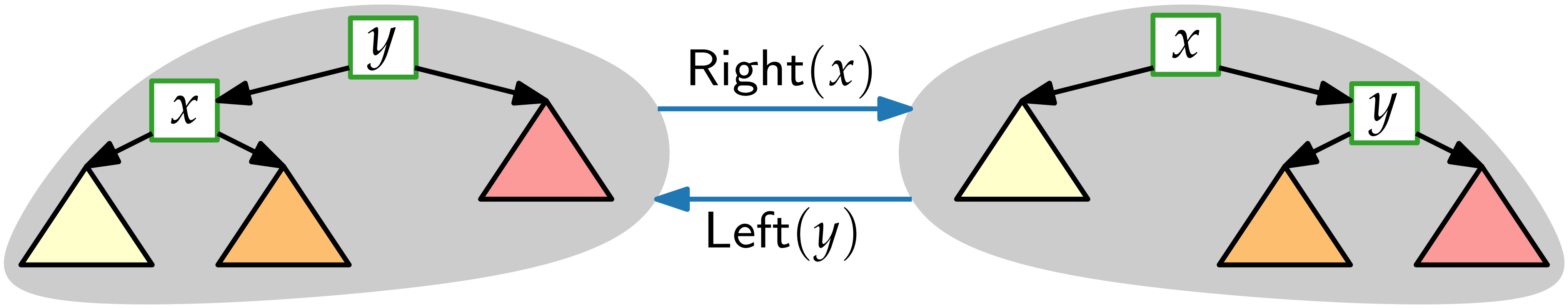
Query(x): Splay(x), then return root

Query(8) Query(6) Query(5)
Query(3)
Query(2)

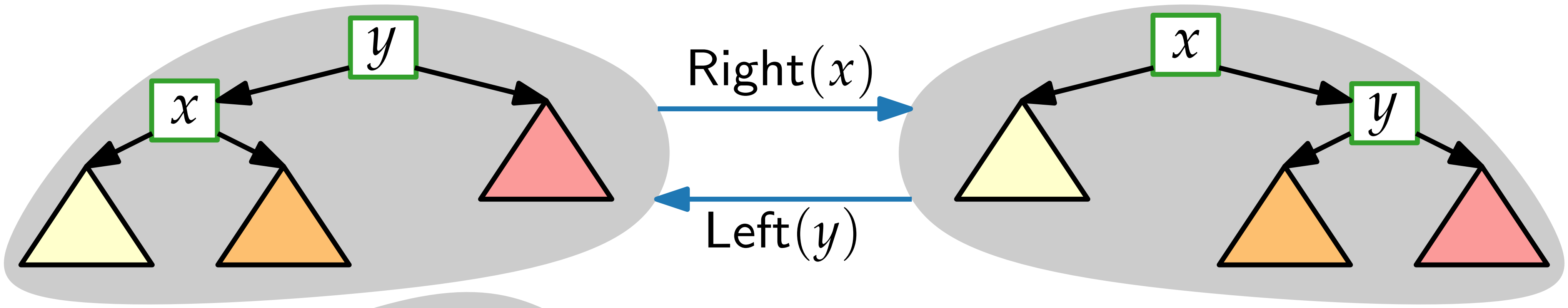
We're back at the start...
and we did $\Theta(n^2)$ rotations

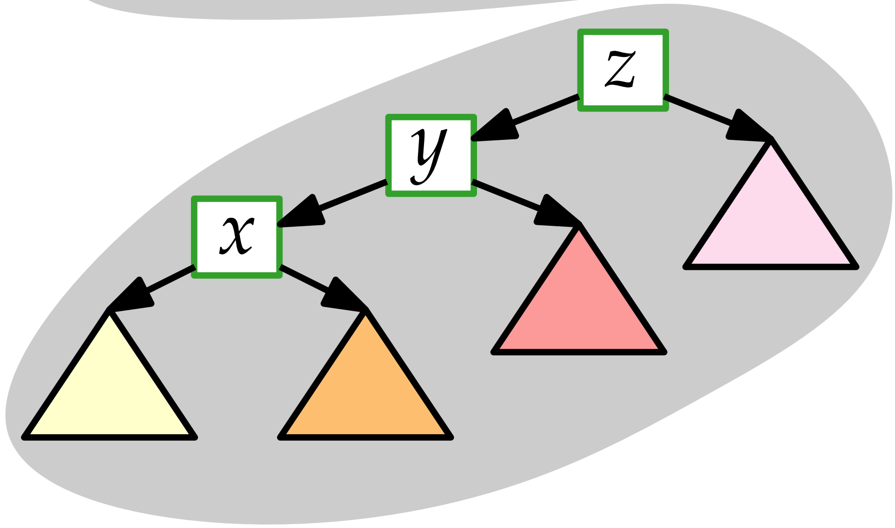


Rotations II

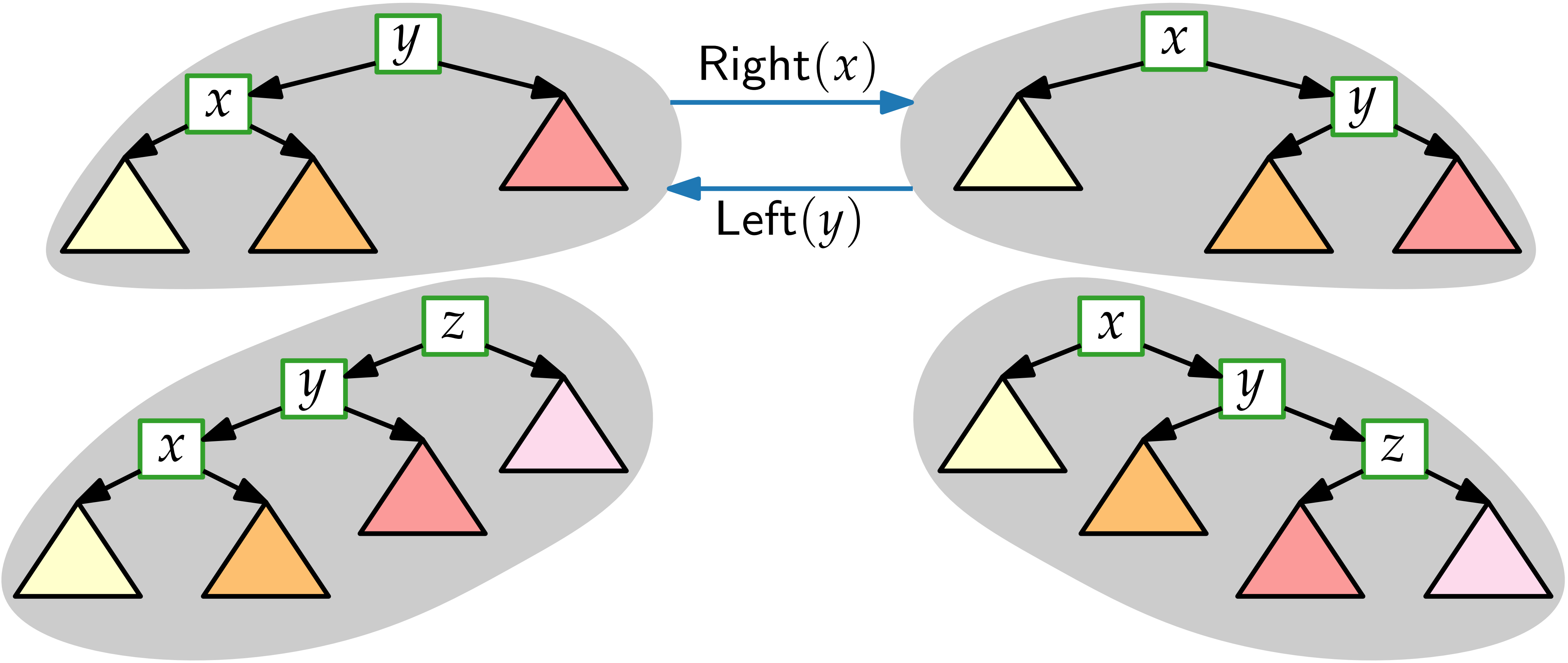


Rotations II

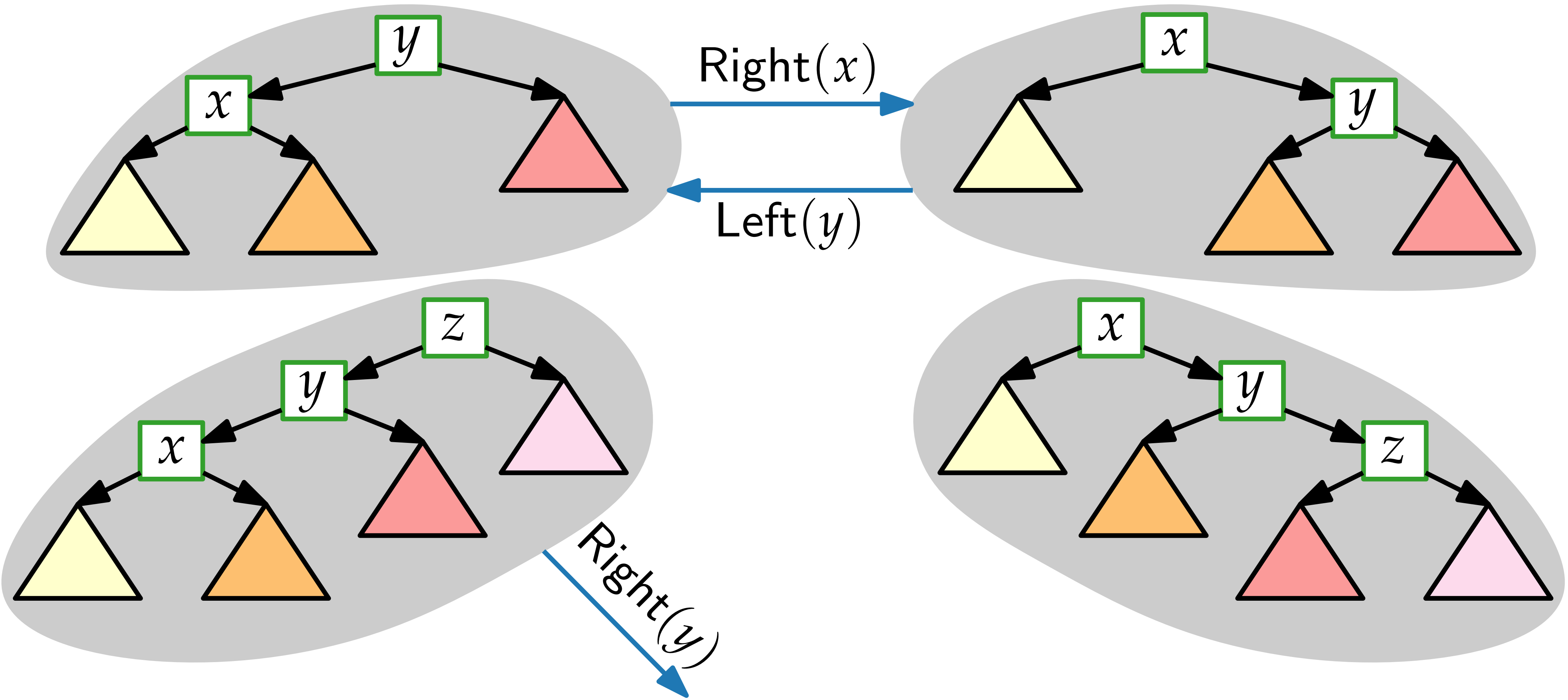




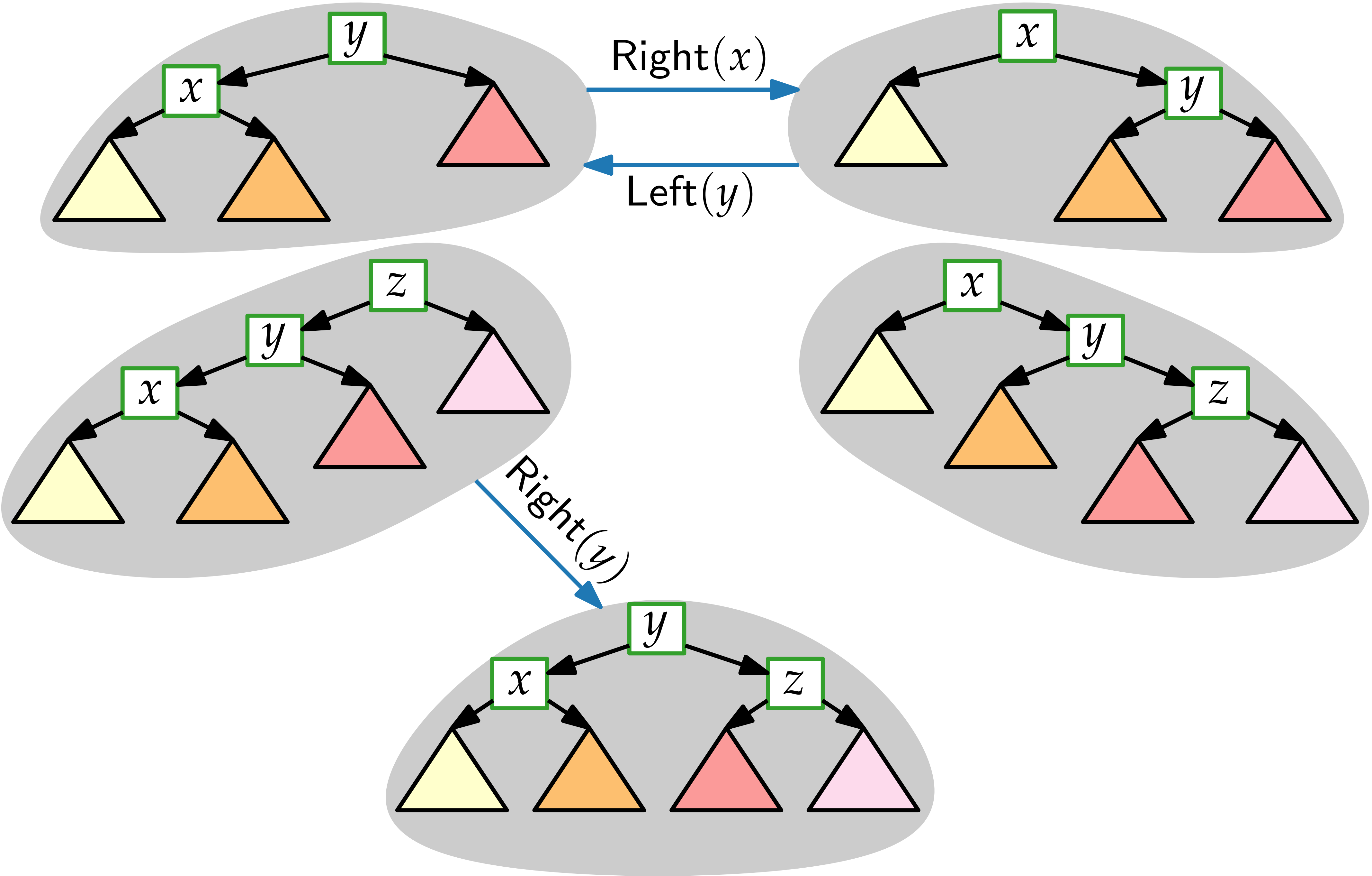
Rotations II



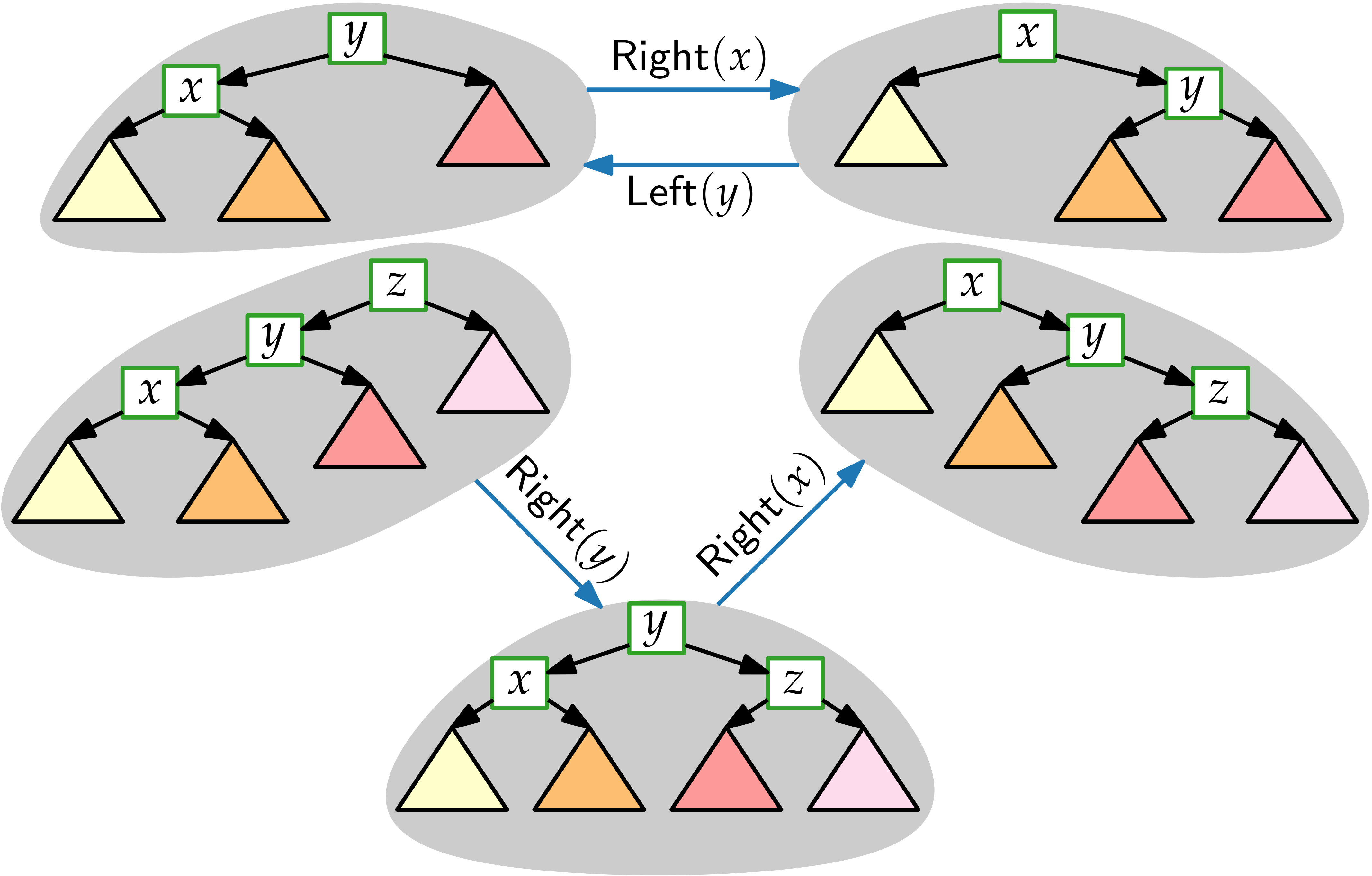
Rotations II



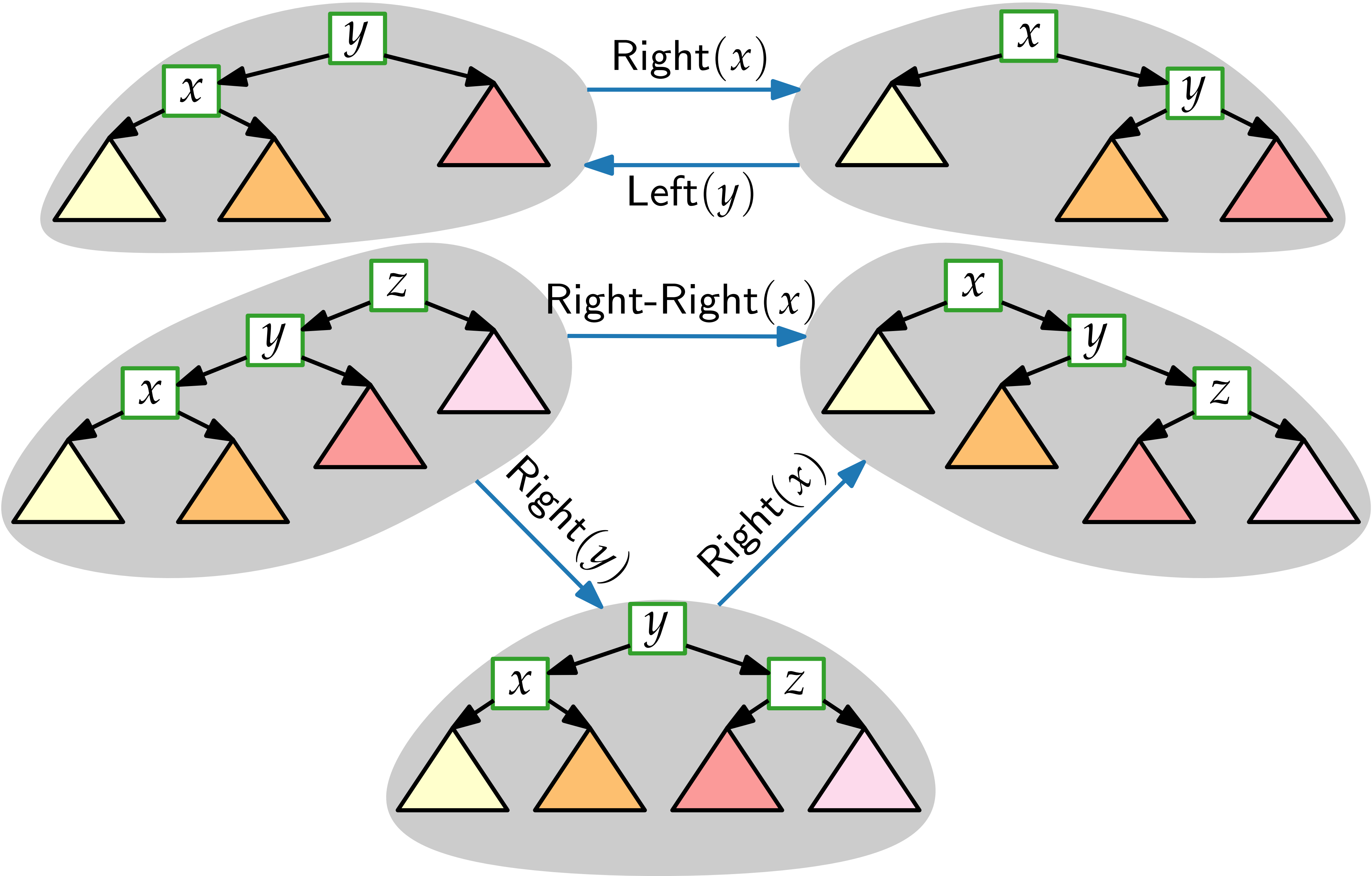
Rotations II



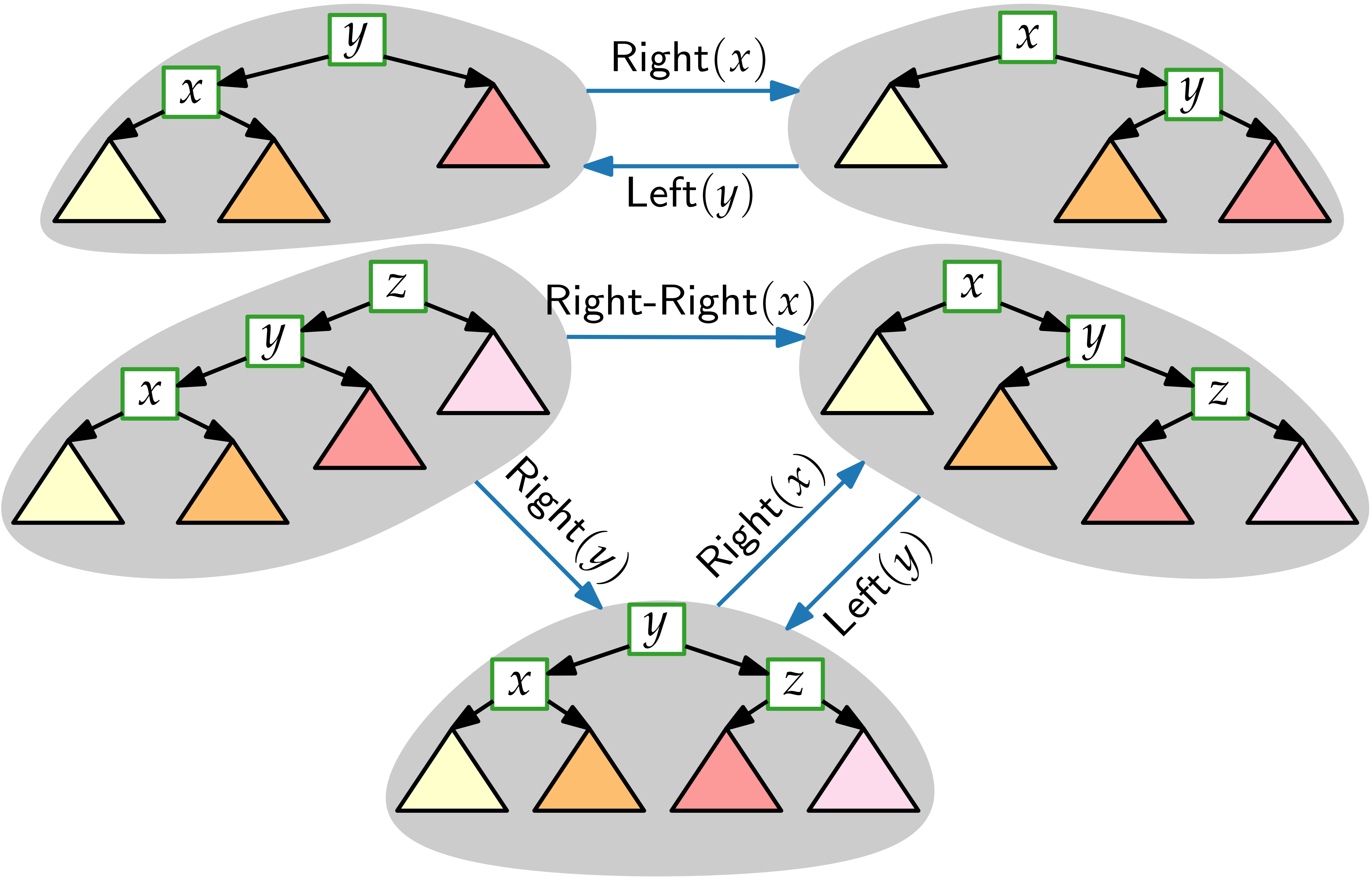
Rotations II



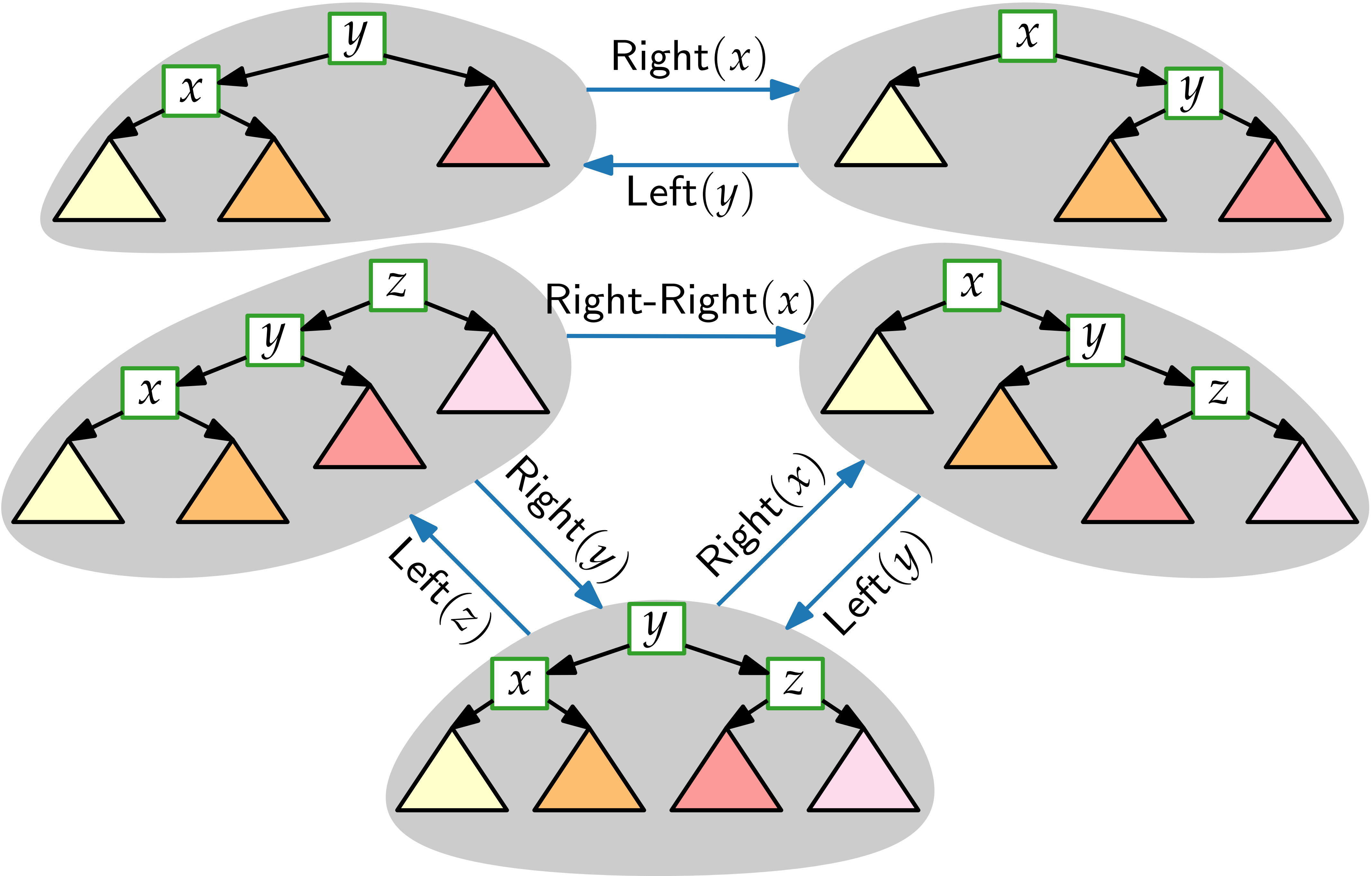
Rotations II



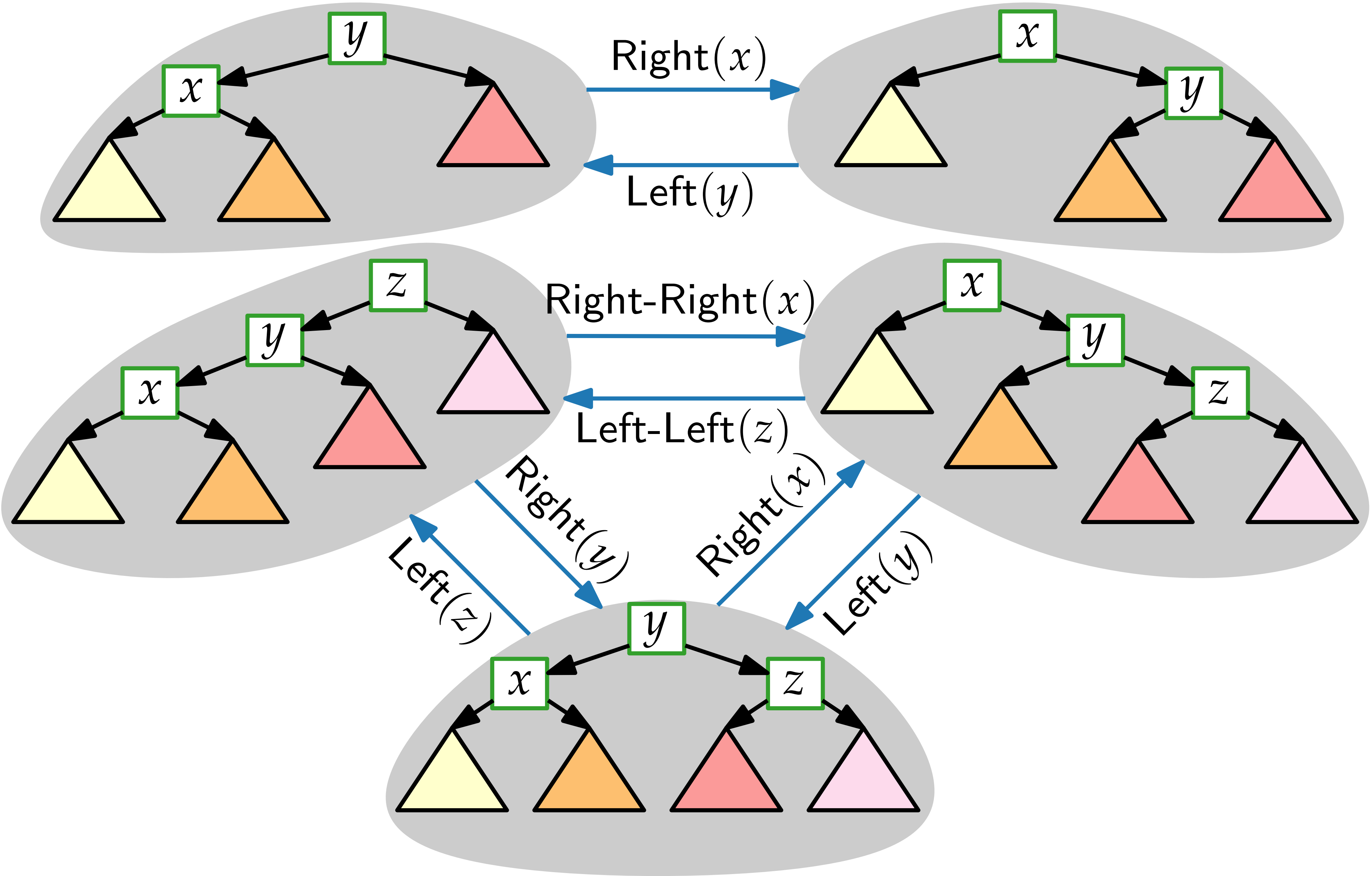
Rotations II



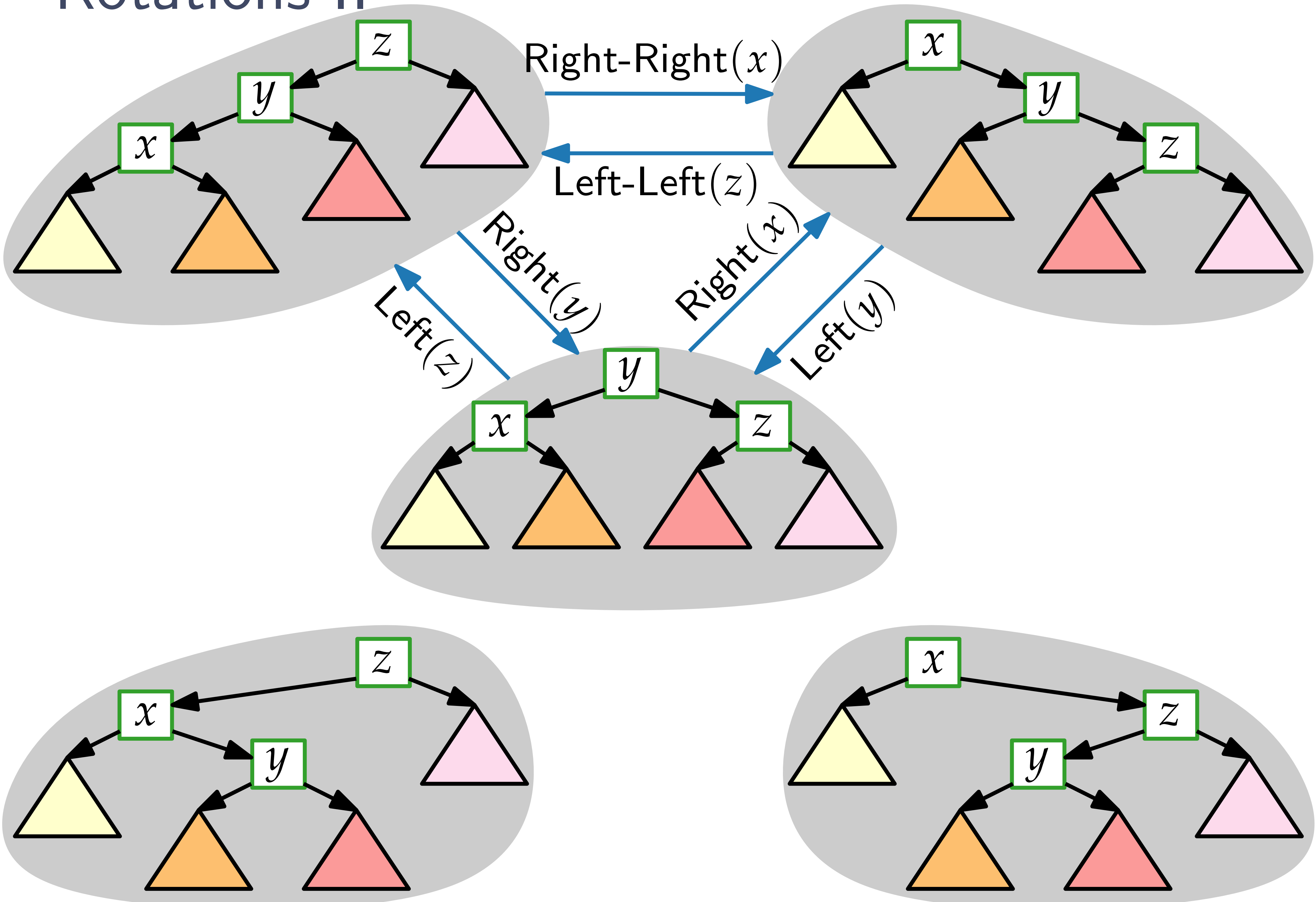
Rotations II



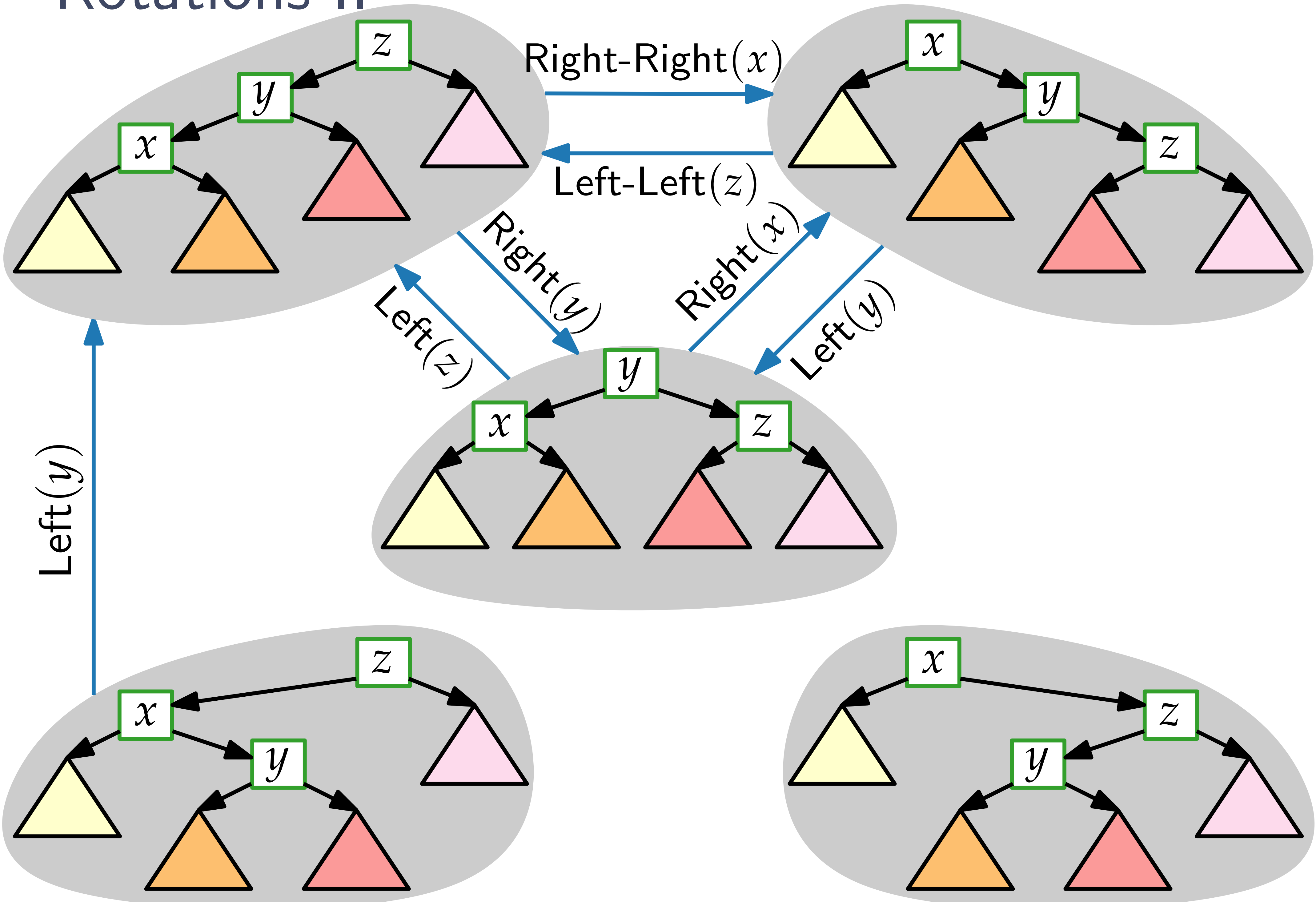
Rotations II



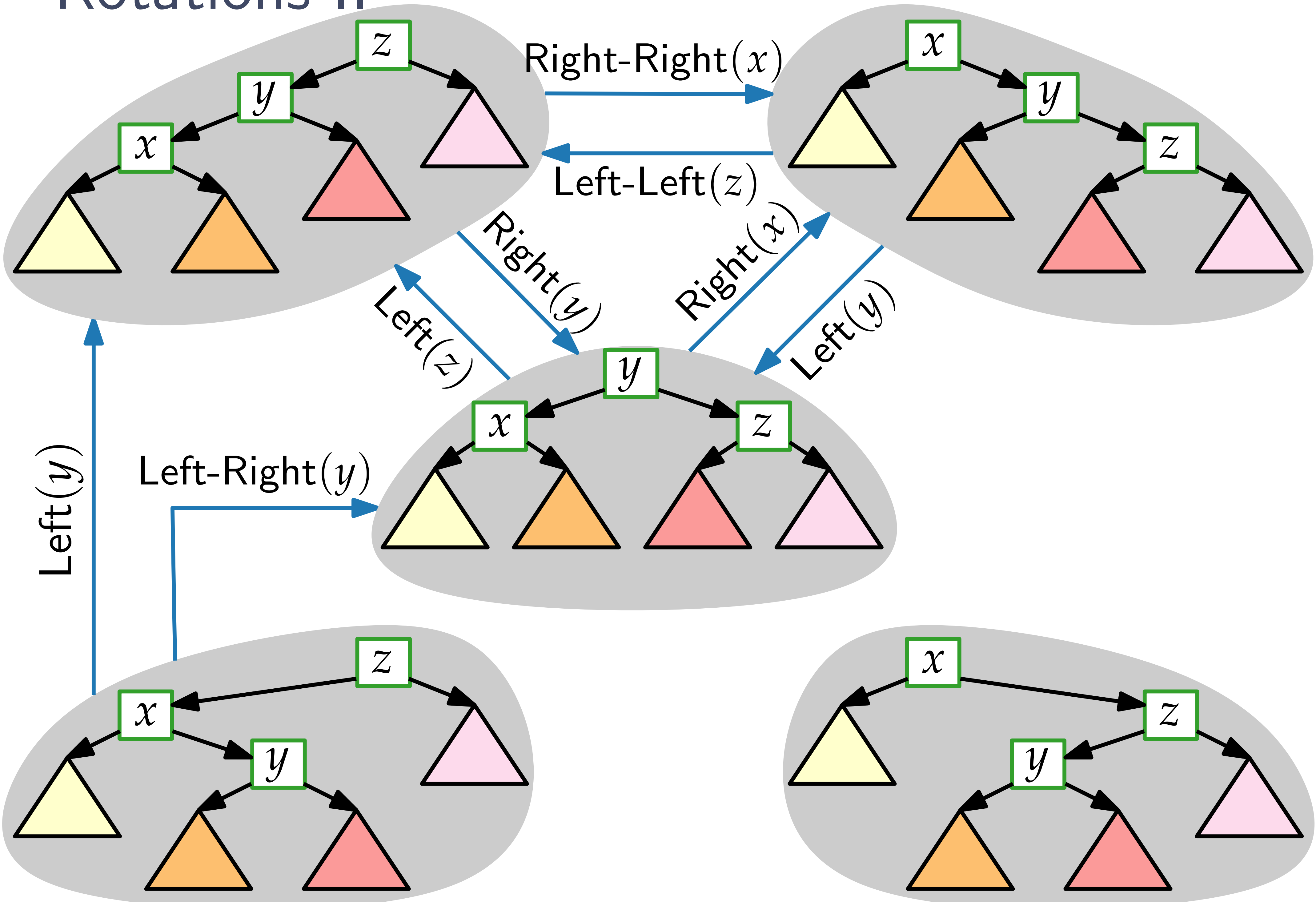
Rotations II



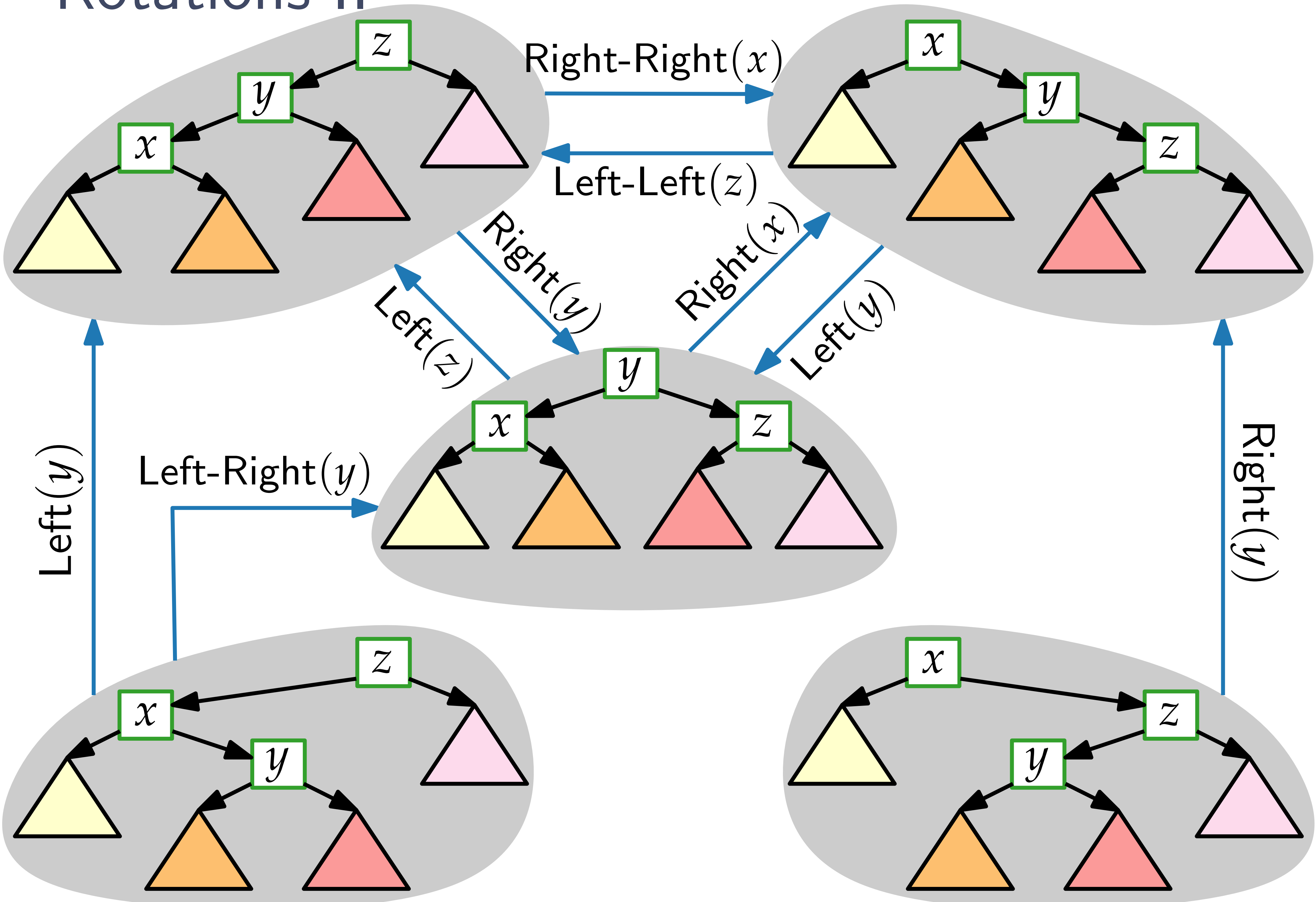
Rotations II



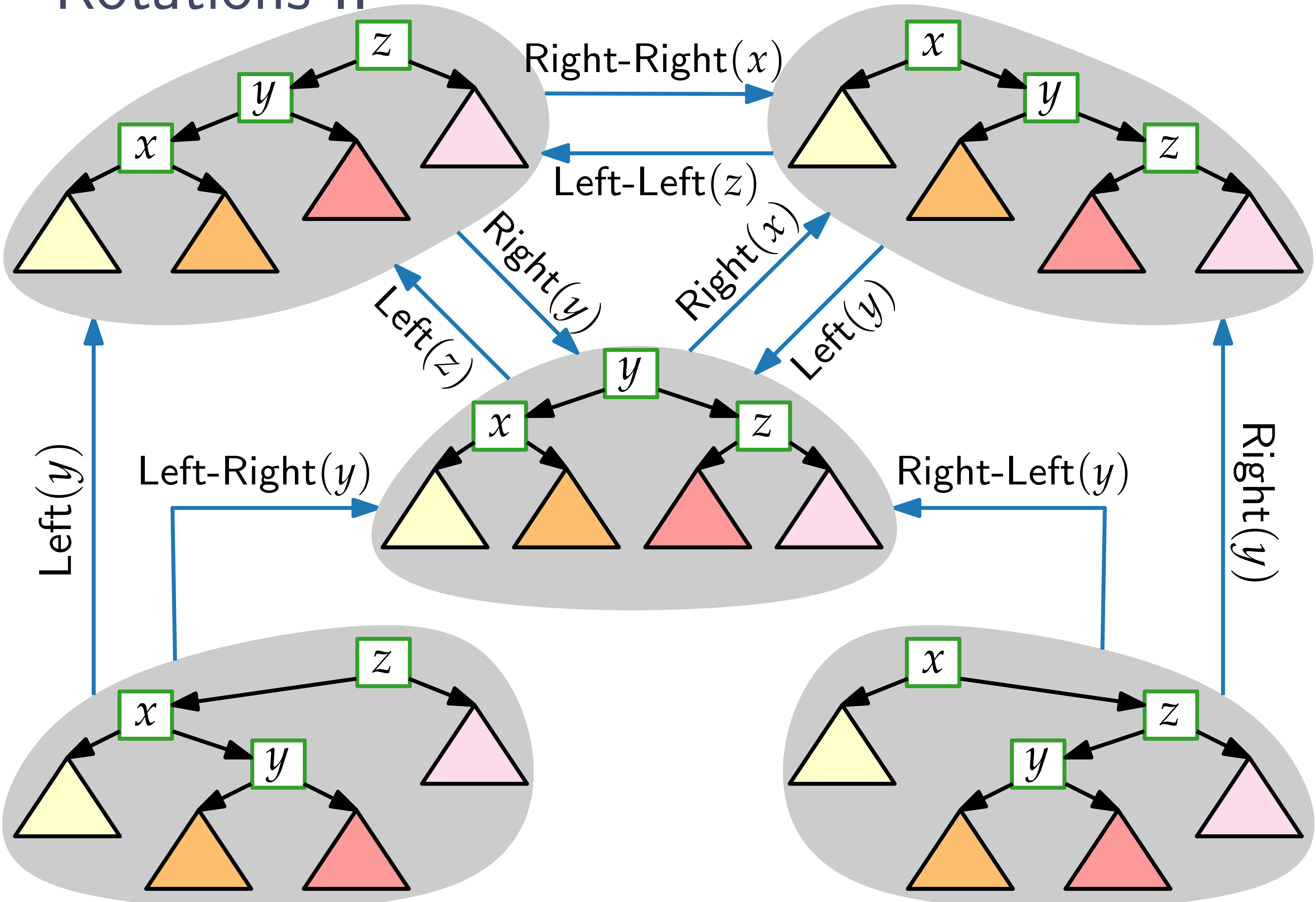
Rotations II



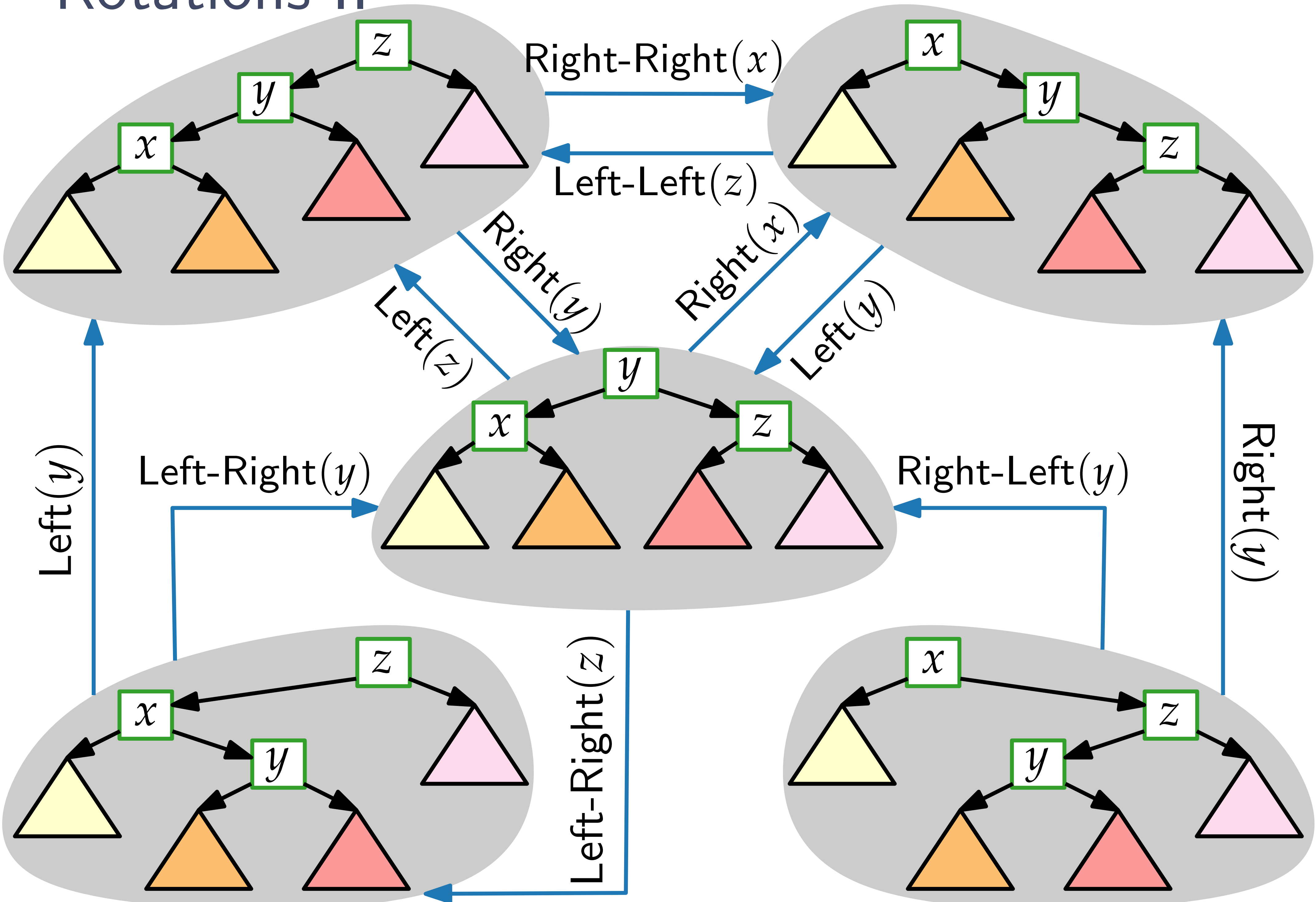
Rotations II



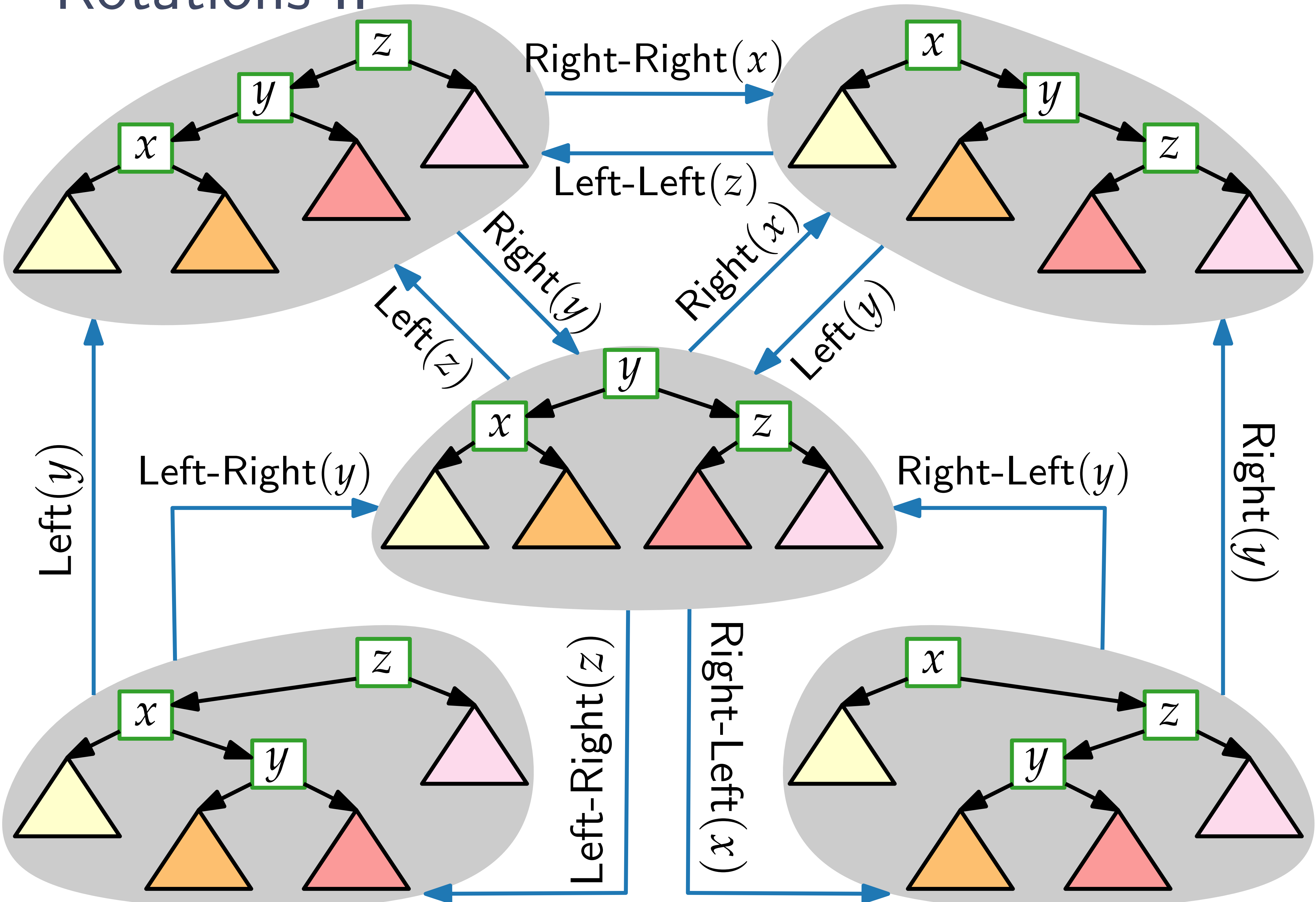
Rotations II



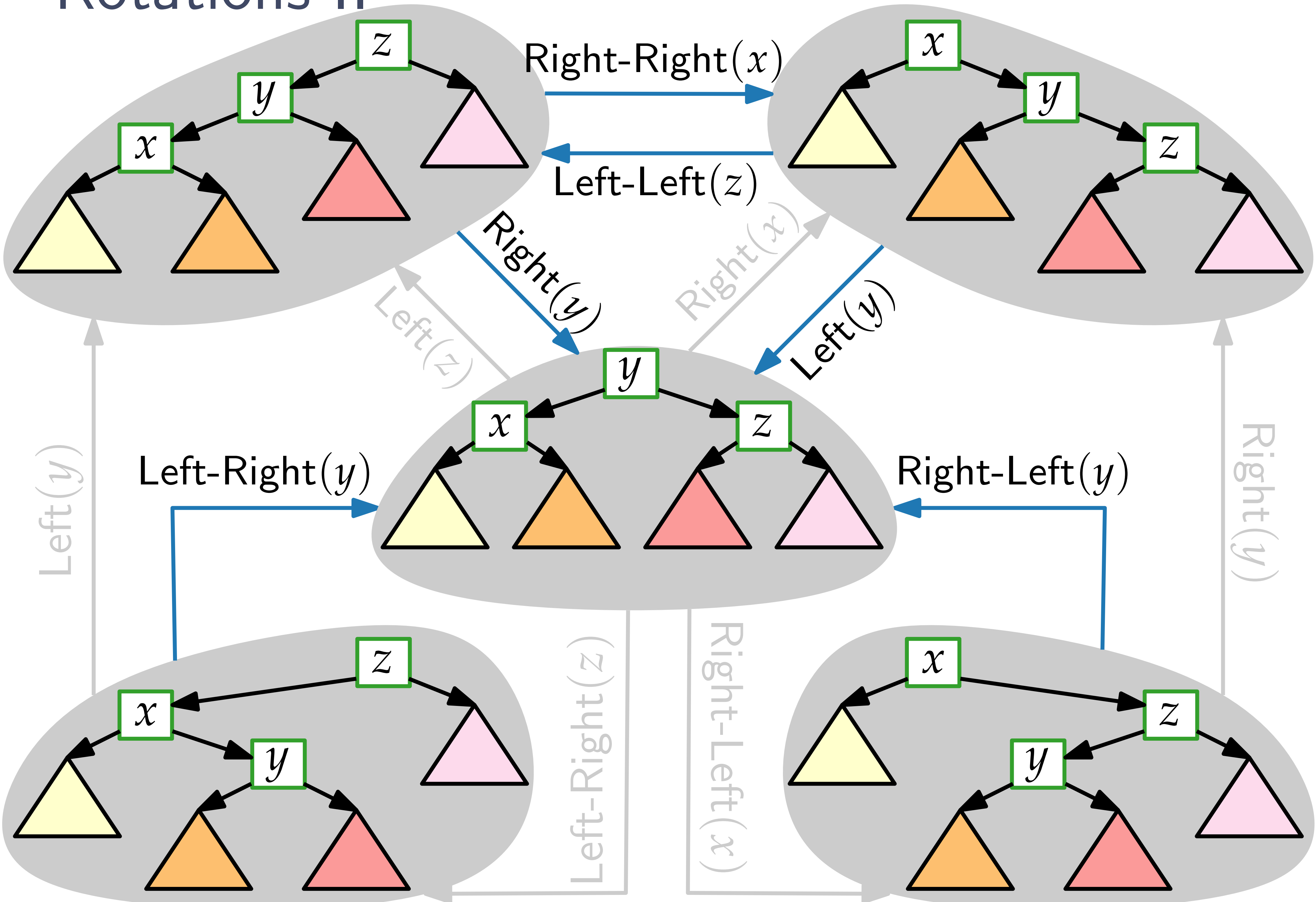
Rotations II



Rotations II



Rotations II



Splay

Algorithm: Splay(x)

Splay

Algorithm: Splay(x)

if $x \neq root$ **then**

Splay

Algorithm: Splay(x)

if $x \neq root$ **then**

  $y =$ parent of x

Splay

Algorithm: Splay(x)

if $x \neq root$ **then**

  $y =$ parent of x

  if $y == root$ **then**

Splay

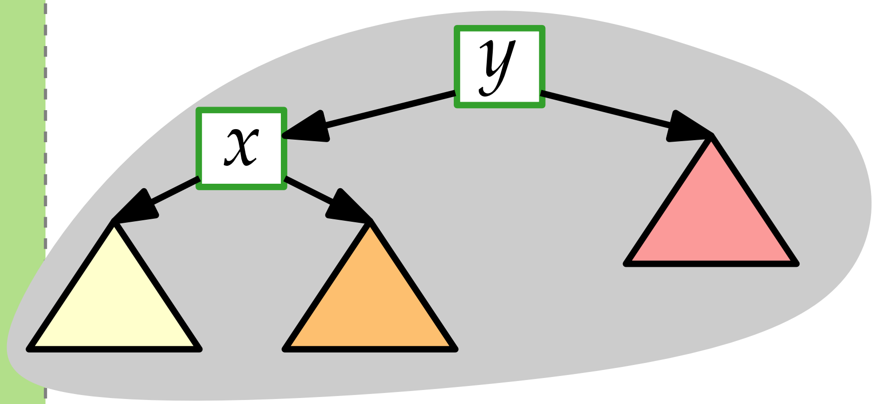
Algorithm: Splay(x)

if $x \neq root$ **then**

  $y =$ parent of x

  if $y == root$ **then**

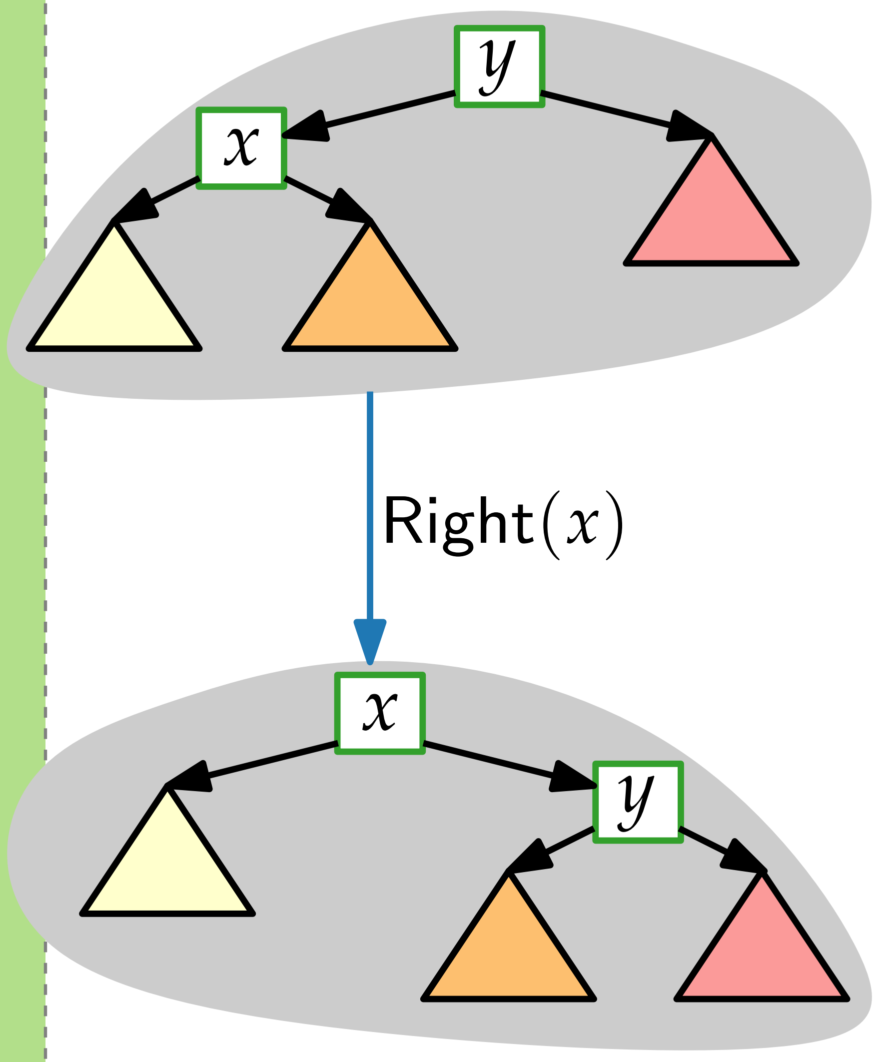
    if $x < y$ **then**



Splay

Algorithm: Splay(x)

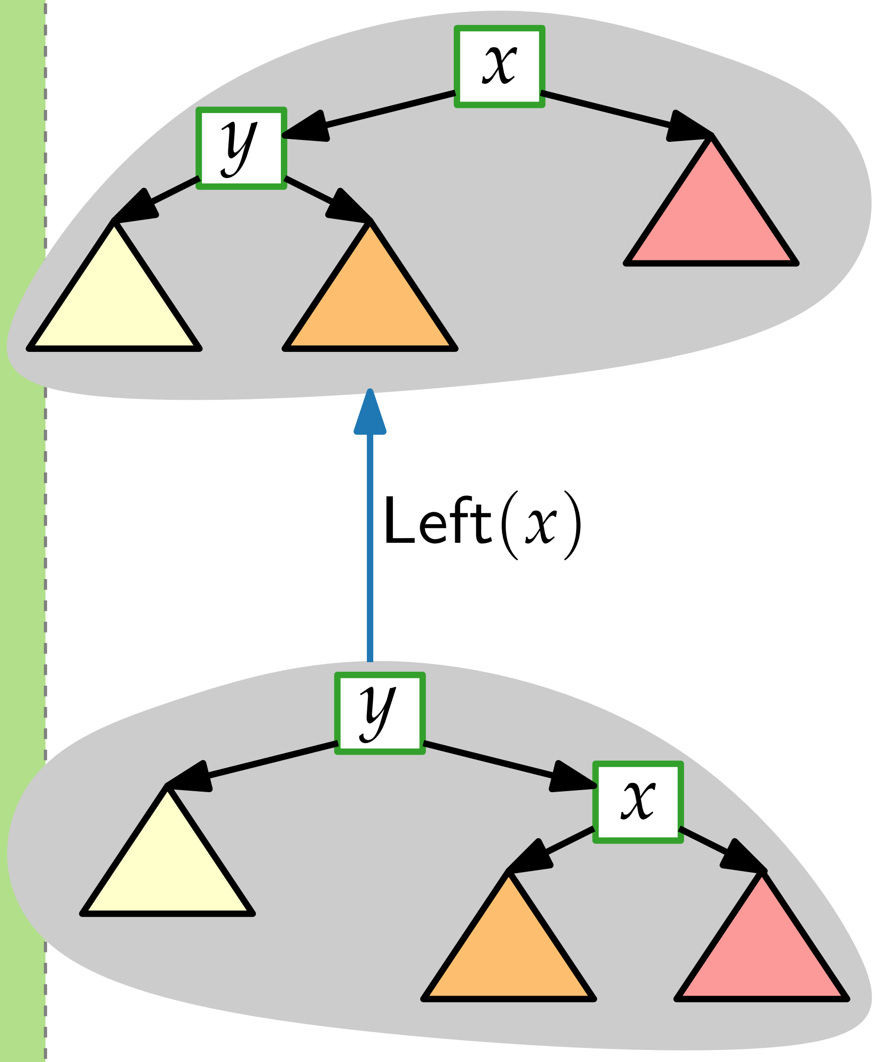
if $x \neq root$ **then**
  $y =$ parent of x
  if $y == root$ **then**
    if $x < y$ **then** Right(x)



Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
  $y =$ parent of x
  if $y == root$ **then**
    if $x < y$ **then** Right(x)
    if $y < x$ **then** Left(x)



Splay

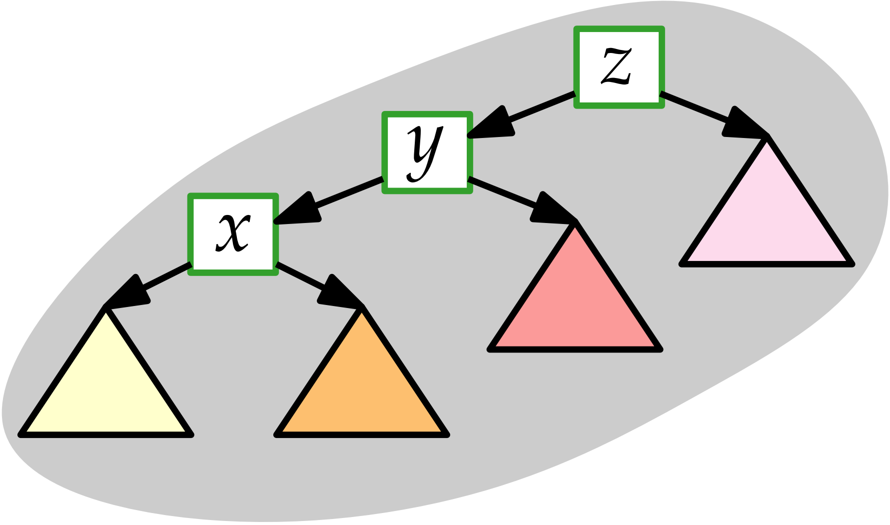
Algorithm: Splay(x)

if $x \neq \mathit{root}$ **then**
    $y =$ parent of x
    if $y == \mathit{root}$ **then**
        if $x < y$ **then** Right(x)
        if $y < x$ **then** Left(x)
    else
        $z =$ parent of y

Splay

Algorithm: Splay(x)

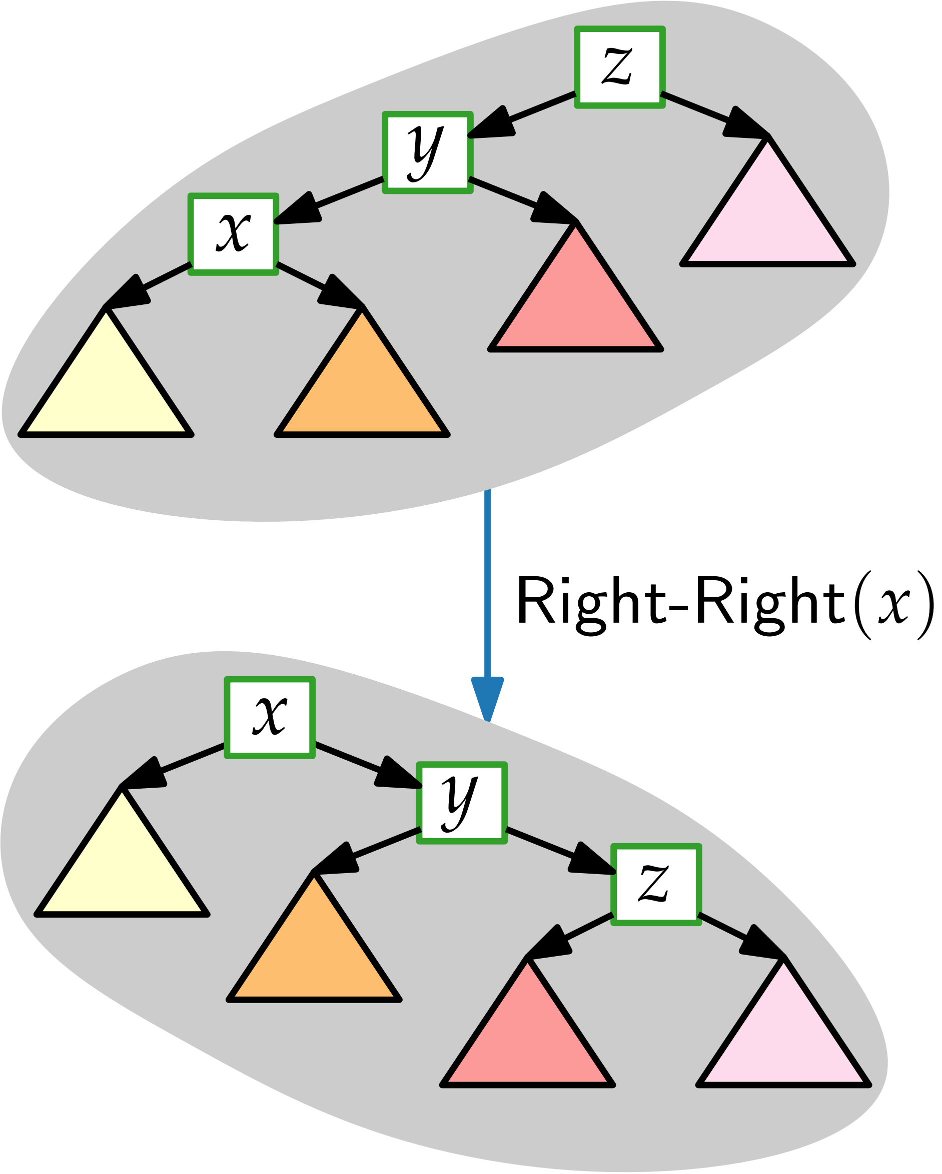
if $x \neq root$ **then**
    y = parent of x
    if $y == root$ **then**
        if $x < y$ **then** Right(x)
        if $y < x$ **then** Left(x)
    else
        z = parent of y
        if $x < y < z$ **then**



Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
- $y =$ parent of x
- **if** $y == root$ **then**
  - **if** $x < y$ **then** Right(x)
  - **if** $y < x$ **then** Left(x)
- **else**
  - $z =$ parent of y
  - **if** $x < y < z$ **then** Right-Right(x)



Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
  $y =$ parent of x
  if $y == root$ **then**
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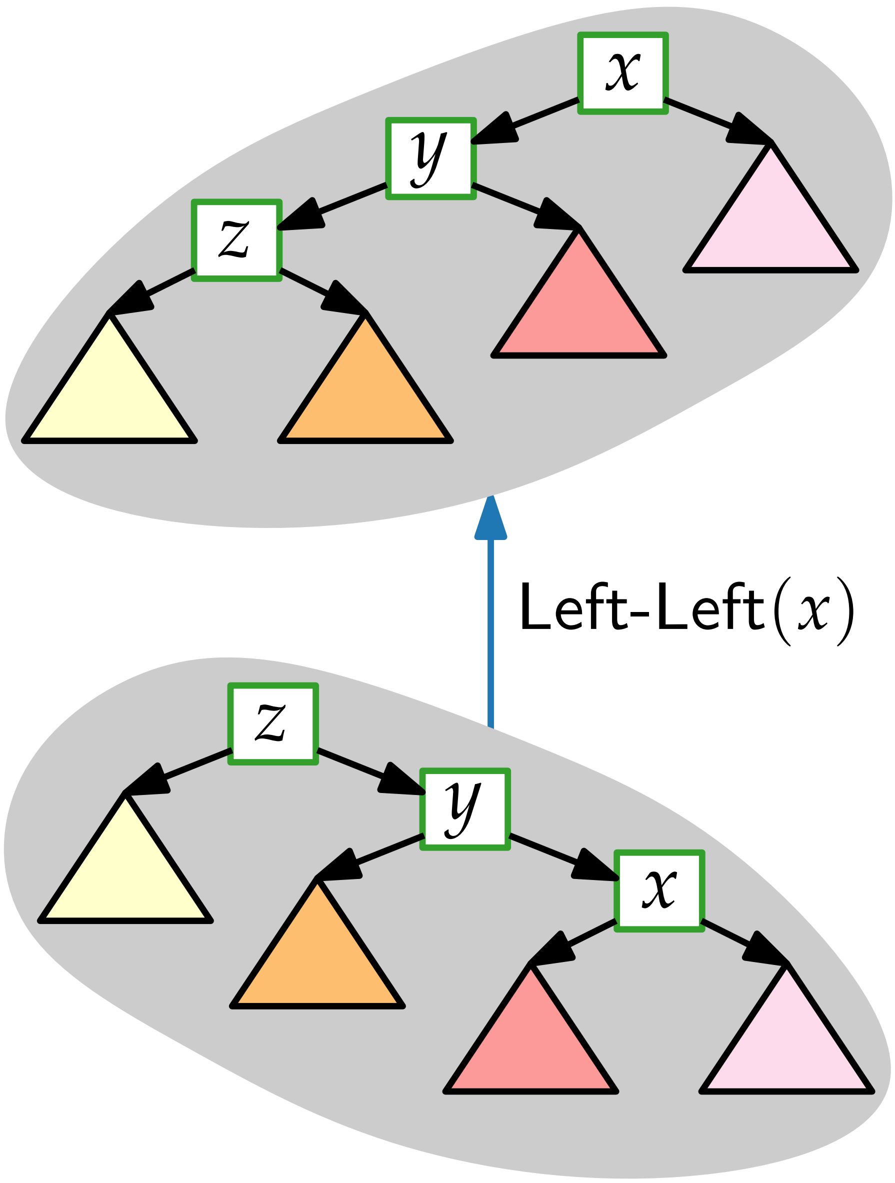
    if $y < x$ **then** Left(x)
  else
    $z =$ parent of y
    if $x < y < z$ **then** Right-Right(x)
    if $z < y < x$ **then** Left-Left(x)



Splay

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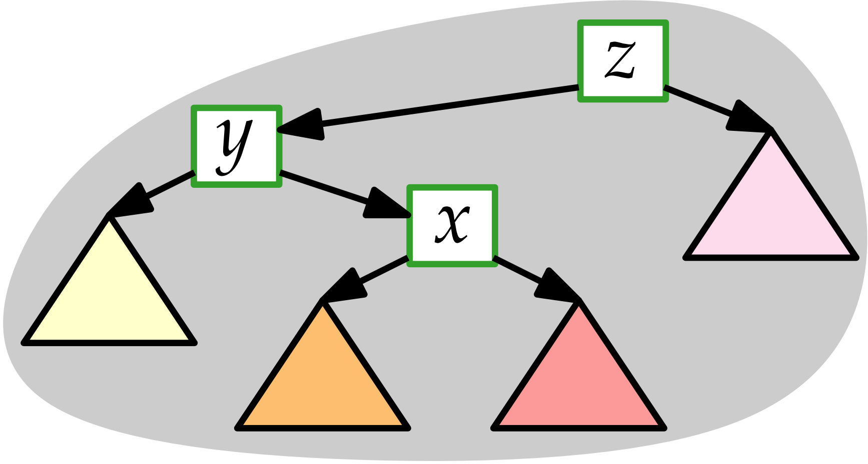
  else
    $z =$ parent of y
    if $x < y < z$ **then** Right-Right(x)
    if $z < y < x$ **then** Left-Left(x)
    if $y < x < z$ **then**



Splay

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if $x \neq root$ **then**
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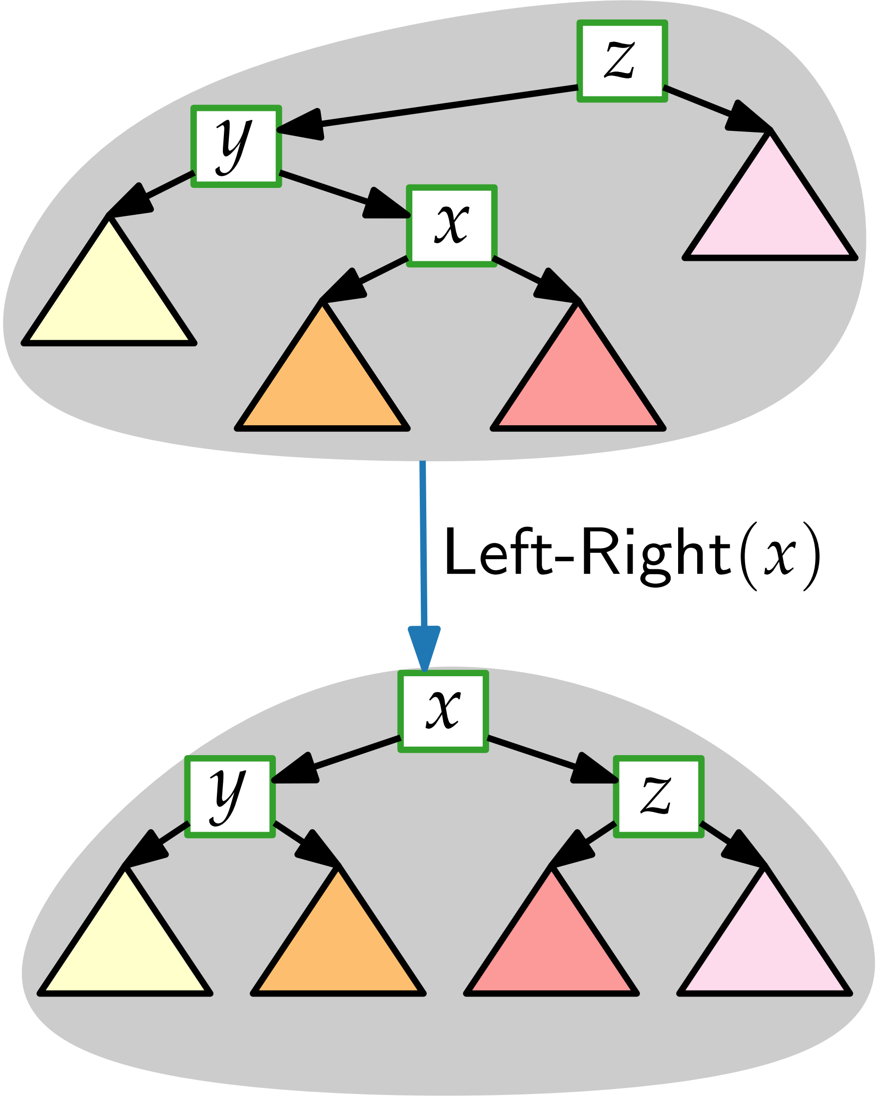
  - **if** $z < y < x$ **then** Left-Left(x)
  - **if** $y < x < z$ **then** Left-Right(x)



Splay

Algorithm: Splay(x)

if $x \neq \mathit{root}$ **then**
- $y =$ parent of x
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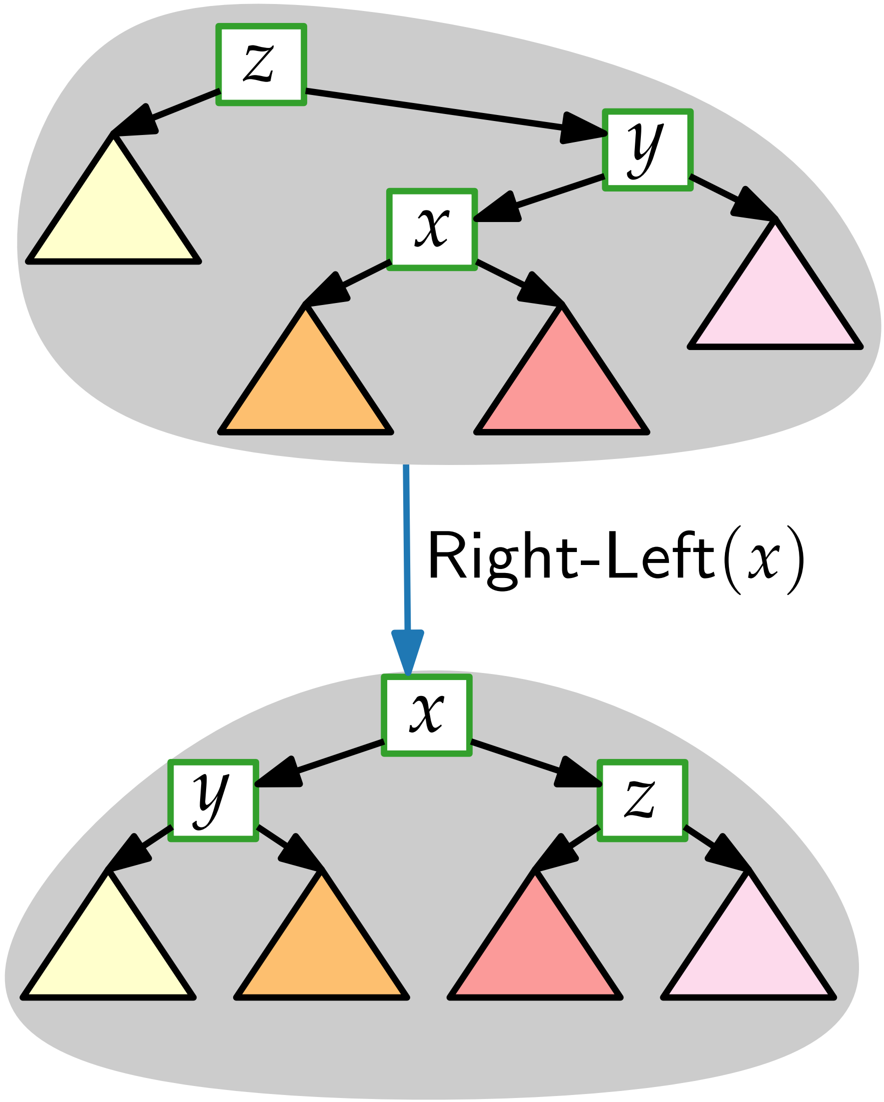
  - **if** $x < y$ **then** Right(x)
  - **if** $y < x$ **then** Left(x)
- **else**
  - $z =$ parent of y
  - **if** $x < y < z$ **then** Right-Right(x)
  - **if** $z < y < x$ **then** Left-Left(x)
  - **if** $y < x < z$ **then** Left-Right(x)
  - **if** $z < x < y$ **then** Right-Left(x)



Splay

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    $z =$ parent of y
    if $x < y < z$ **then** Right-Right(x)
    if $z < y < x$ **then** Left-Left(x)
    if $y < x < z$ **then** Left-Right(x)
    if $z < x < y$ **then** Right-Left(x)
  Splay(x)

Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
- $y =$ parent of x
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  - **if** $x < y$ **then** Right(x)
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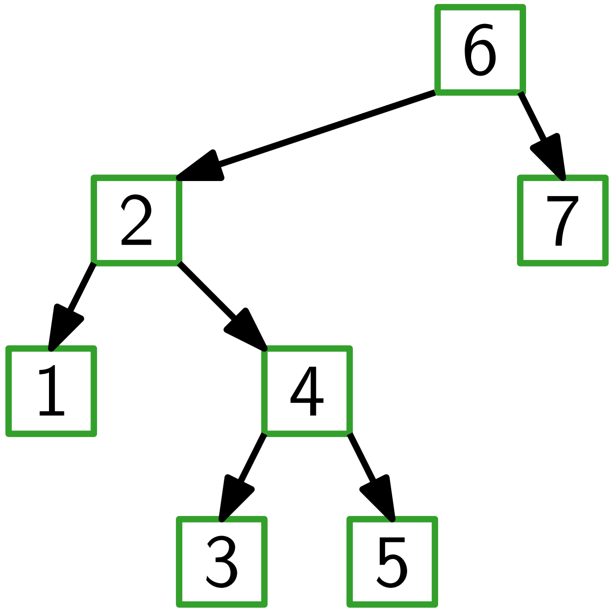
  - **if** $x < y < z$ **then** Right-Right(x)
  - **if** $z < y < x$ **then** Left-Left(x)
  - **if** $y < x < z$ **then** Left-Right(x)
  - **if** $z < x < y$ **then** Right-Left(x)
- Splay(x)

Splay(3):



Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
  $y =$ parent of x
  if $y == root$ **then**
    if $x < y$ **then** Right(x)
    if $y < x$ **then** Left(x)
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    $z =$ parent of y
    if $x < y < z$ **then** Right-Right(x)
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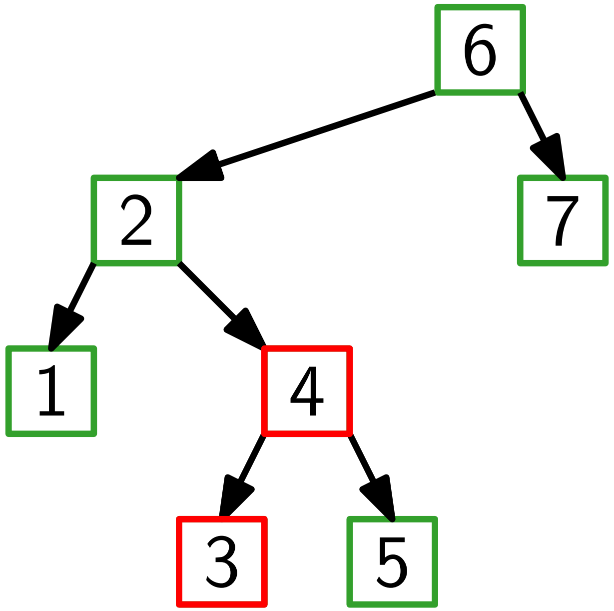
  Splay(x)

Splay(3):



Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
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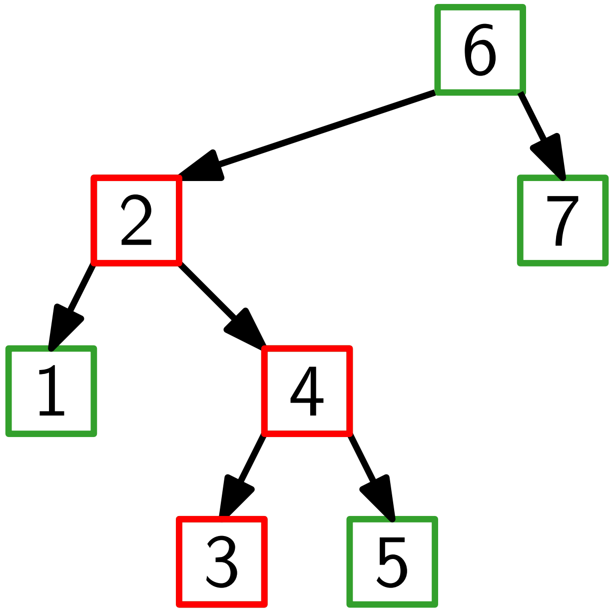
    if $y == root$ **then**
        if $x < y$ **then** Right(x)
        if $y < x$ **then** Left(x)
    else
        z = parent of y
        if $x < y < z$ **then** Right-Right(x)
        if $z < y < x$ **then** Left-Left(x)
        if $y < x < z$ **then** Left-Right(x)
        if $z < x < y$ **then** Right-Left(x)
    Splay(x)

Splay(3):



Splay

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- **else**
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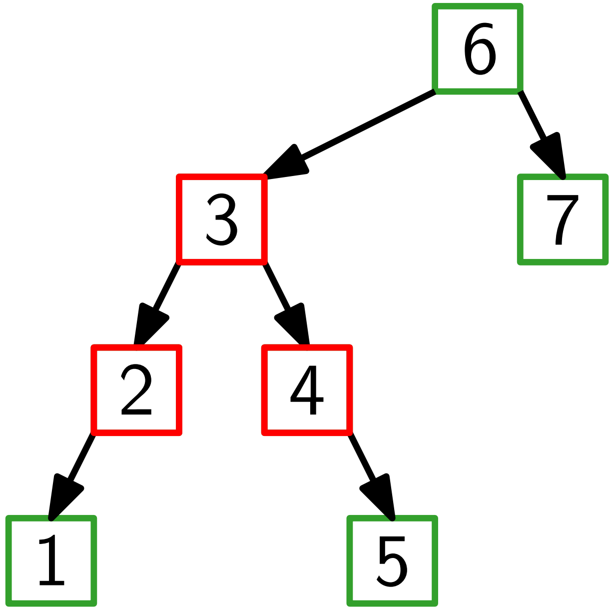
  - **if** $x < y < z$ **then** Right-Right(x)
  - **if** $z < y < x$ **then** Left-Left(x)
  - **if** $y < x < z$ **then** Left-Right(x)
  - **if** $z < x < y$ **then** Right-Left(x)
- Splay(x)

Splay(3):



Splay

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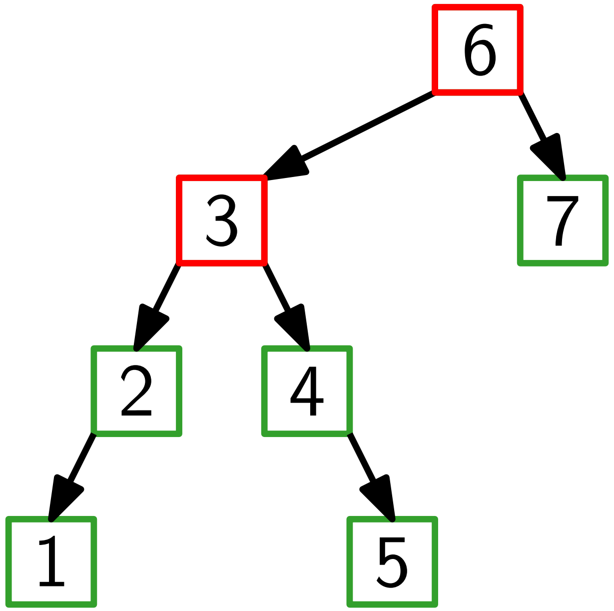
    Splay(x)

Splay(3):



Splay

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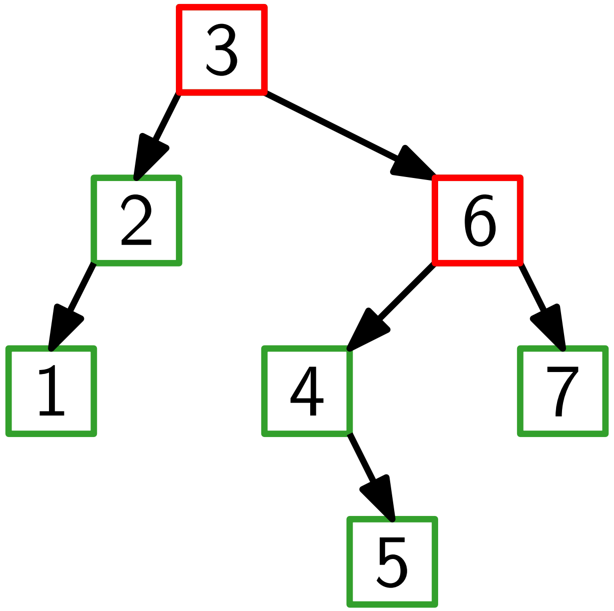
  Splay(x)

Splay(3):



Splay

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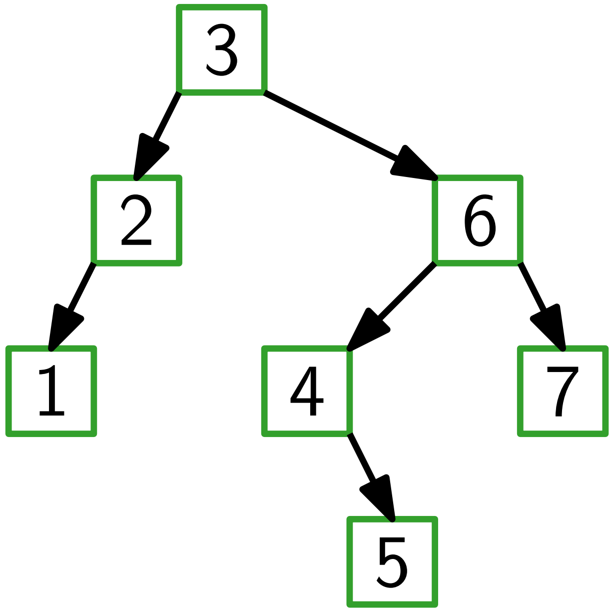
  Splay(x)

Splay(3):



Call Splay(x):

Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
- y = parent of x
- **if** $y == root$ **then**
  - **if** $x < y$ **then** Right(x)
  - **if** $y < x$ **then** Left(x)
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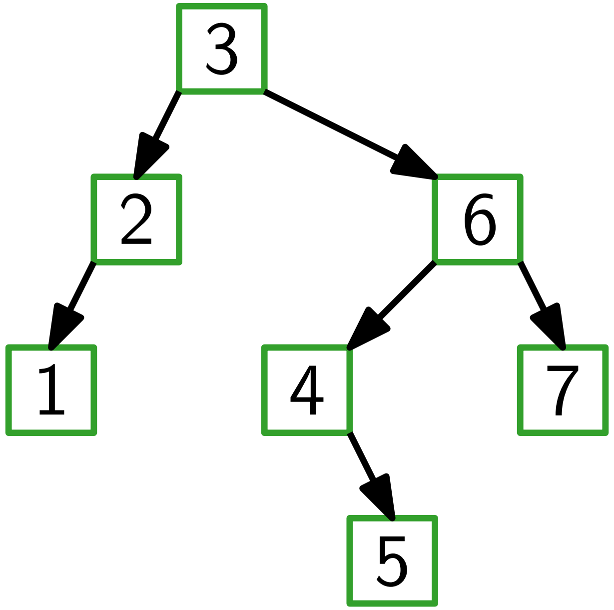
- Splay(x)

Splay(3):



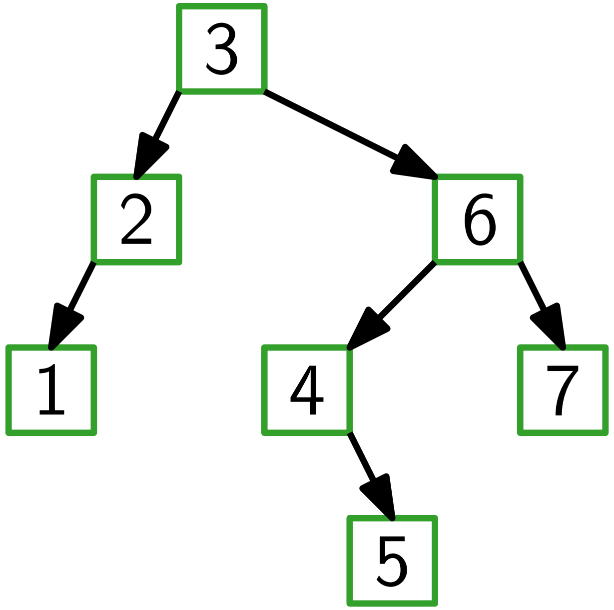
Call Splay(x):
- after Search(x)

Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
  y = parent of x
  if $y == root$ **then**
    if $x < y$ **then** Right(x)
    if $y < x$ **then** Left(x)
  else
    z = parent of y
    if $x < y < z$ **then** Right-Right(x)
    if $z < y < x$ **then** Left-Left(x)
    if $y < x < z$ **then** Left-Right(x)
    if $z < x < y$ **then** Right-Left(x)
  Splay(x)

Splay(3):



Call Splay(x):

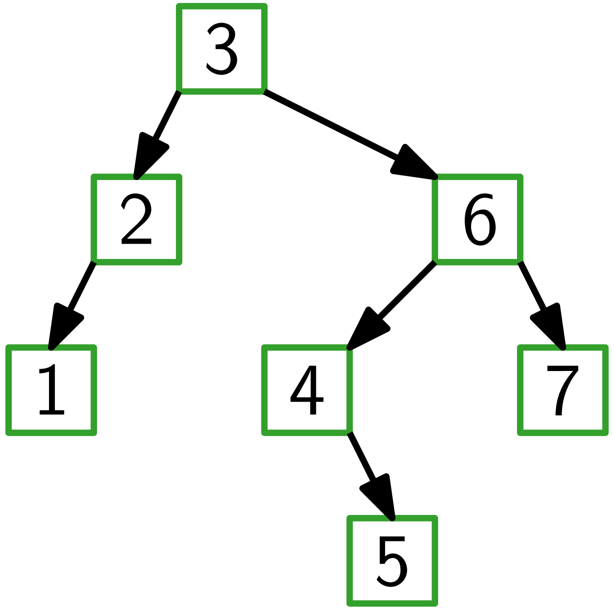
- after Search(x)
- after Insert(x)

Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
- $y =$ parent of x
- **if** $y == root$ **then**
  - **if** $x < y$ **then** Right(x)
  - **if** $y < x$ **then** Left(x)
- **else**
  - $z =$ parent of y
  - **if** $x < y < z$ **then** Right-Right(x)
  - **if** $z < y < x$ **then** Left-Left(x)
  - **if** $y < x < z$ **then** Left-Right(x)
  - **if** $z < x < y$ **then** Right-Left(x)
- Splay(x)

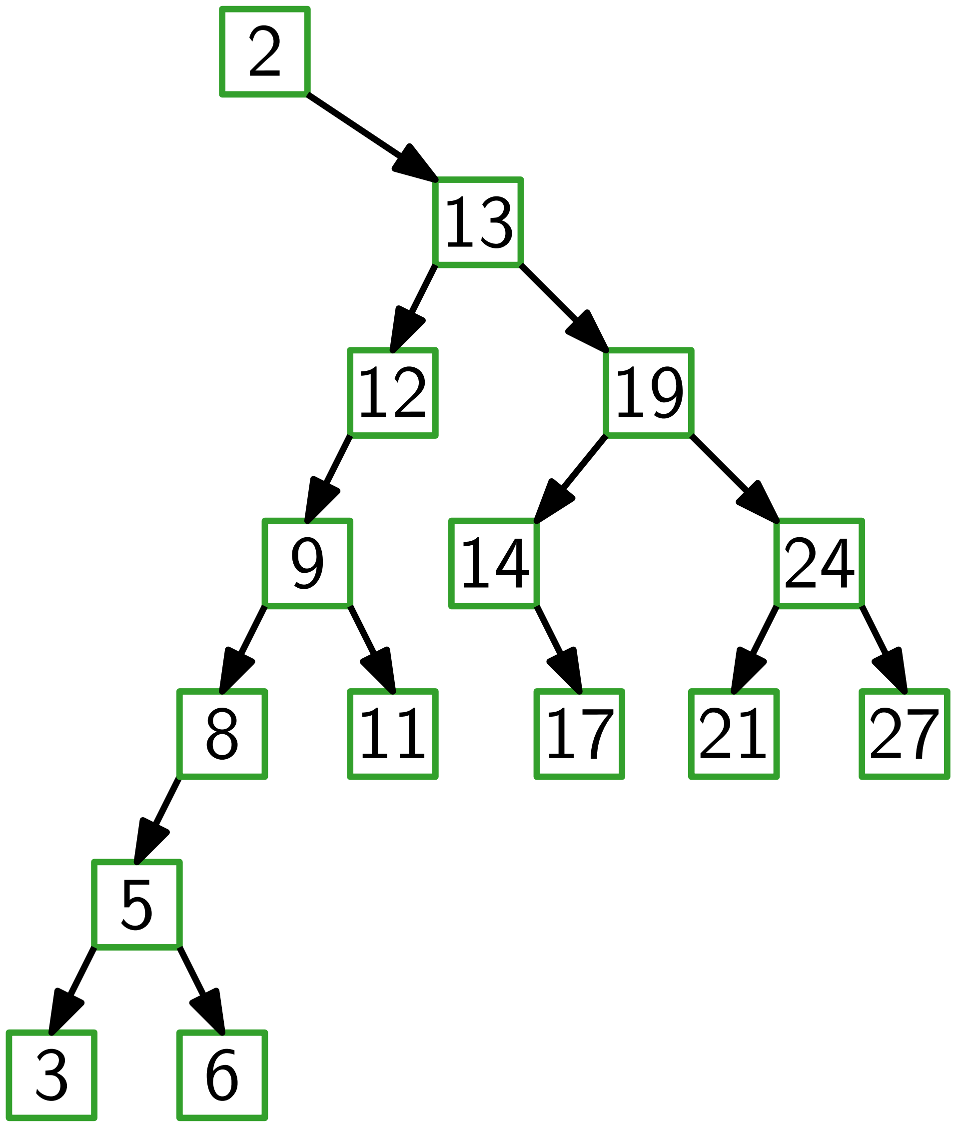
Splay(3):



Call Splay(x):

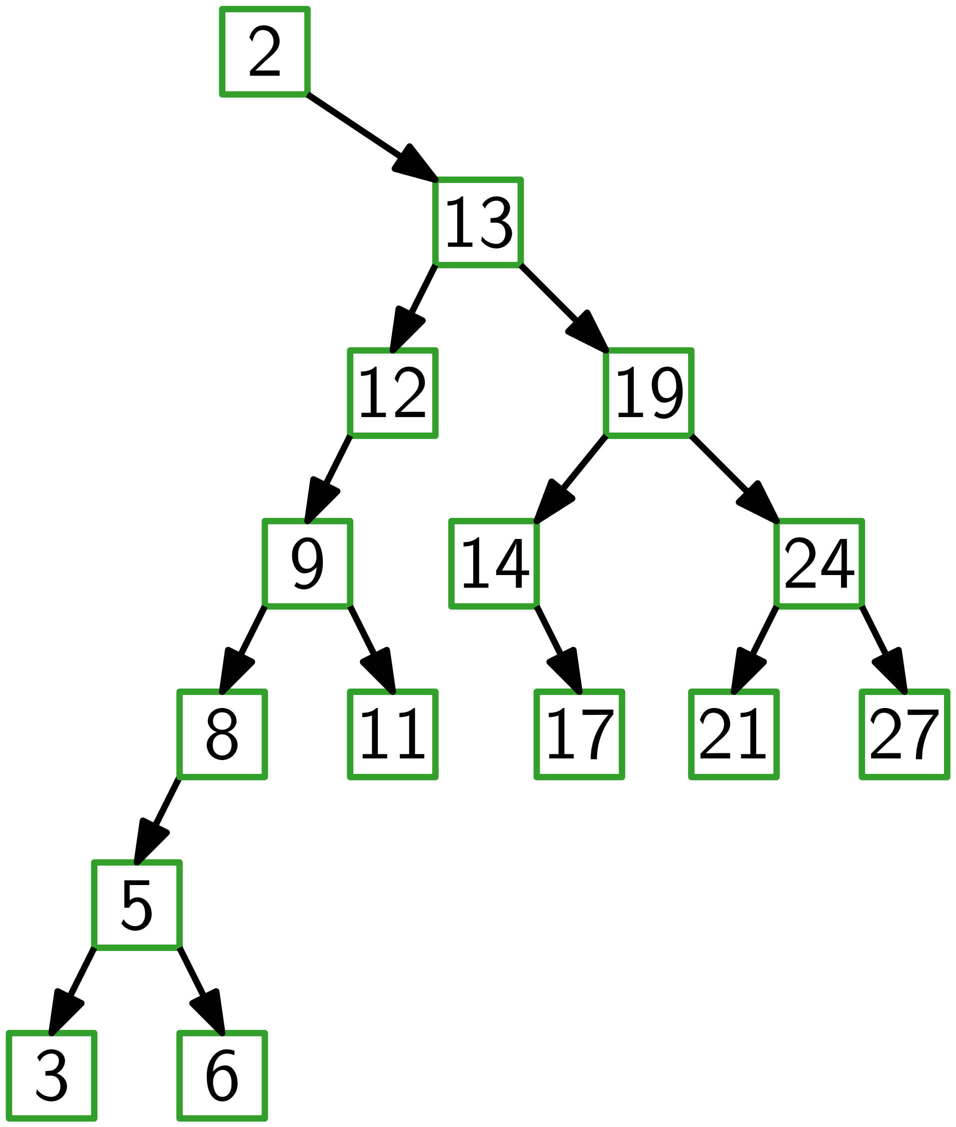
- after Search(x)
- after Insert(x)
- before Delete(x)

Why is Splay Fast?



Why is Splay Fast?

$w(x)$: weight of x (here 1), $W = \sum w(x)$ (here n)



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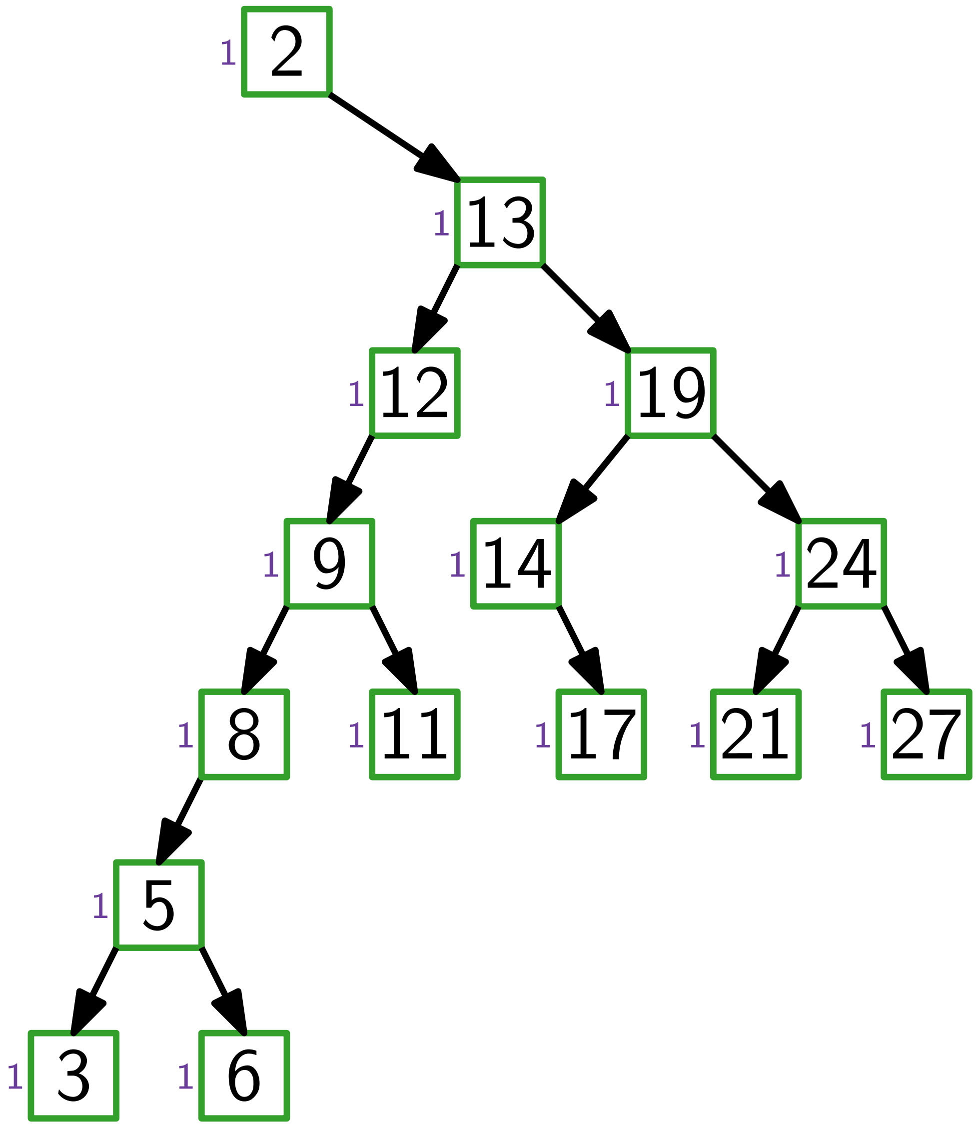

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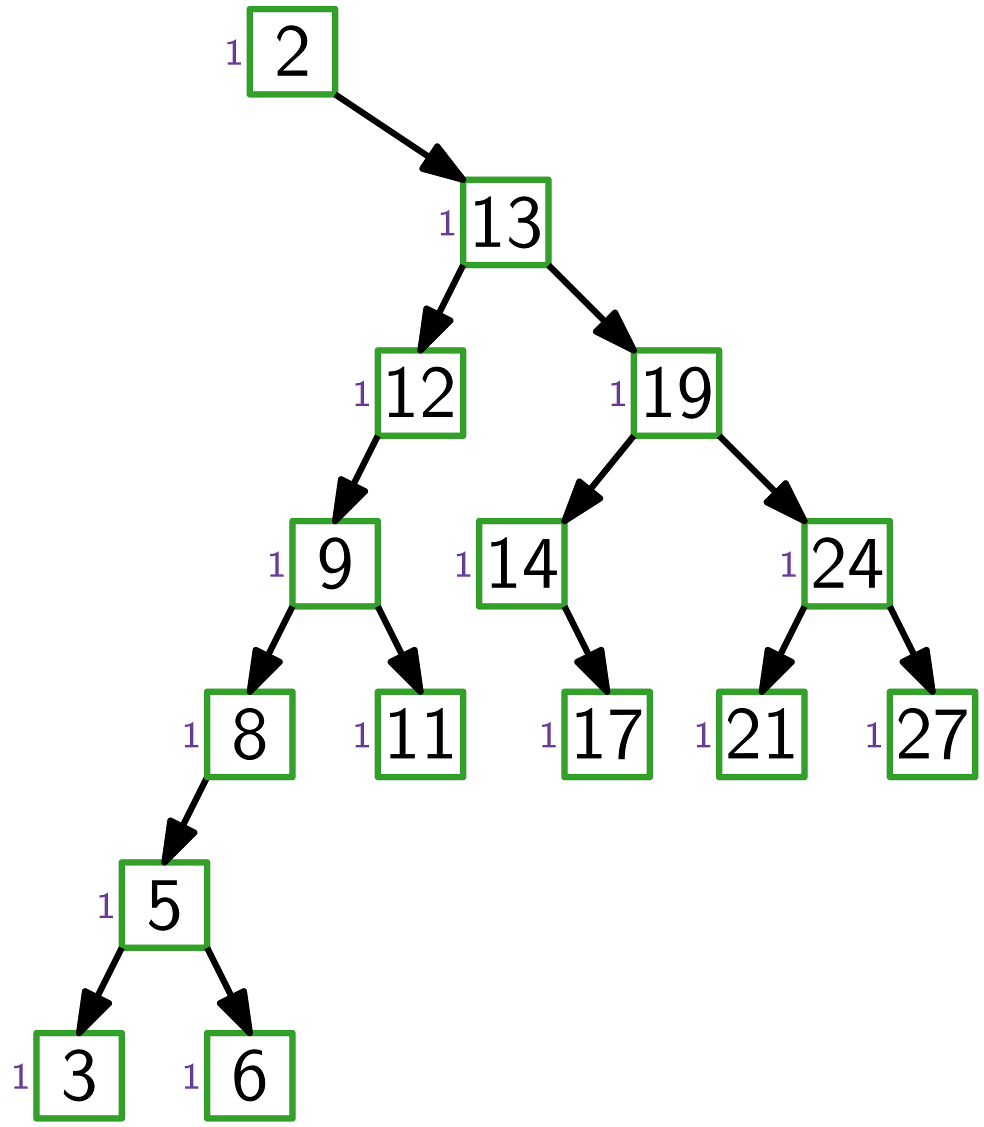

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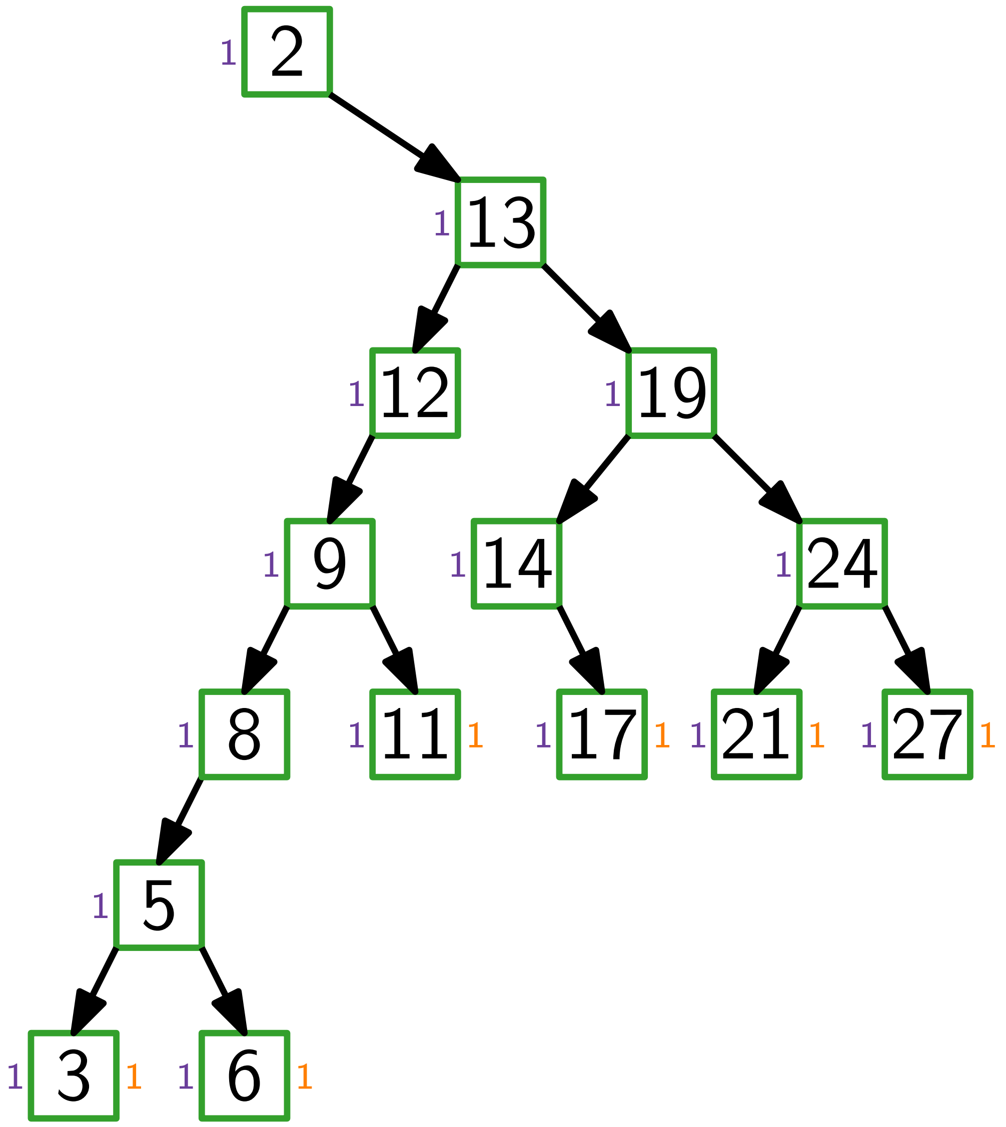

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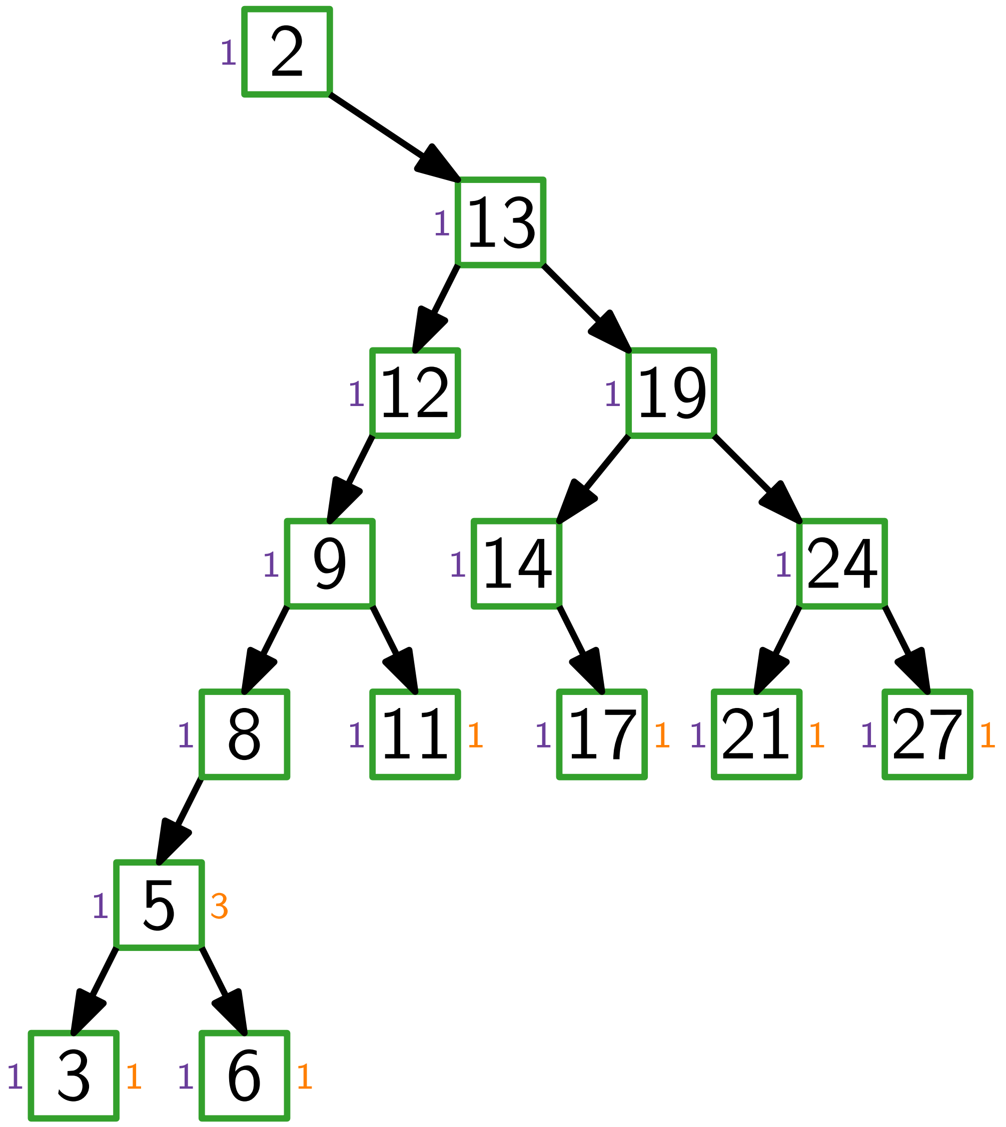

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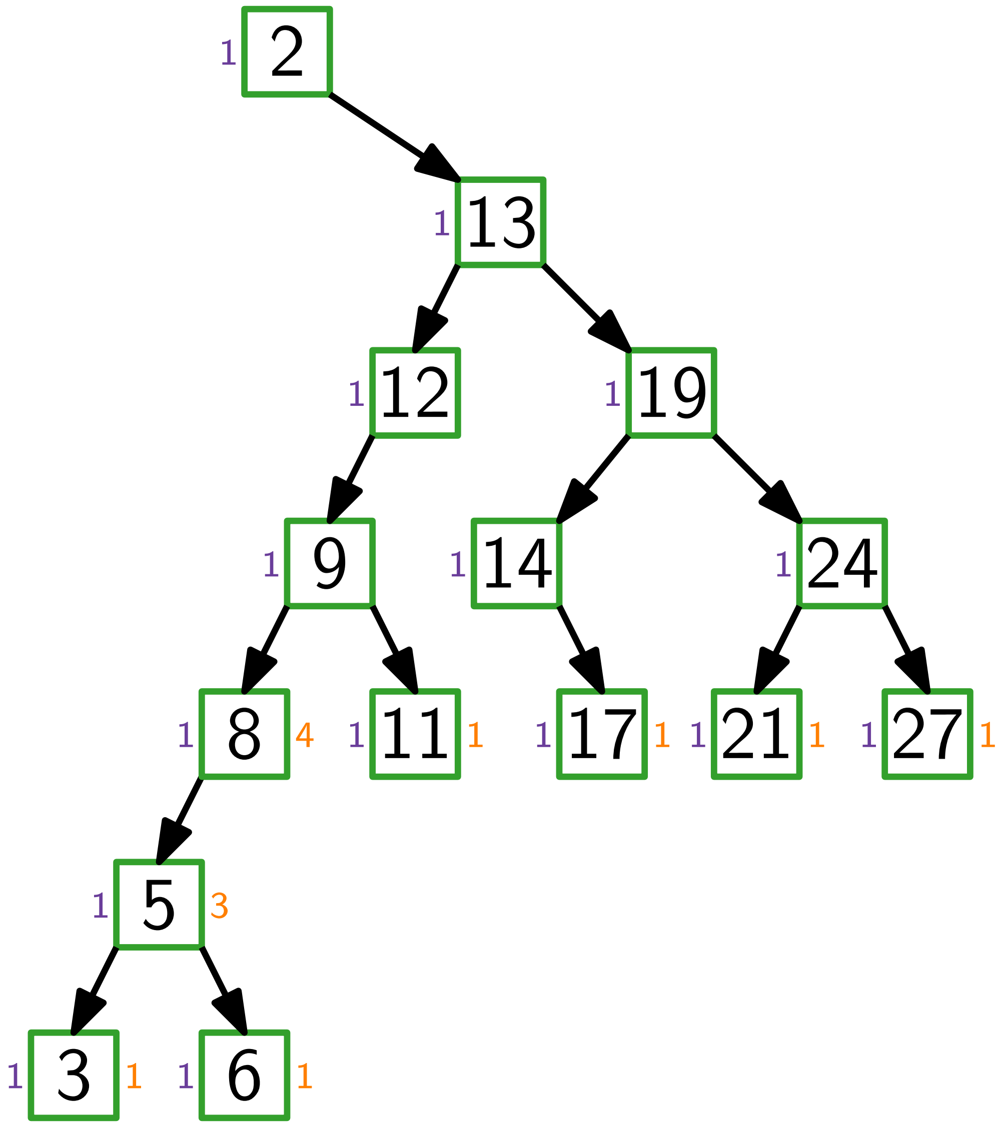

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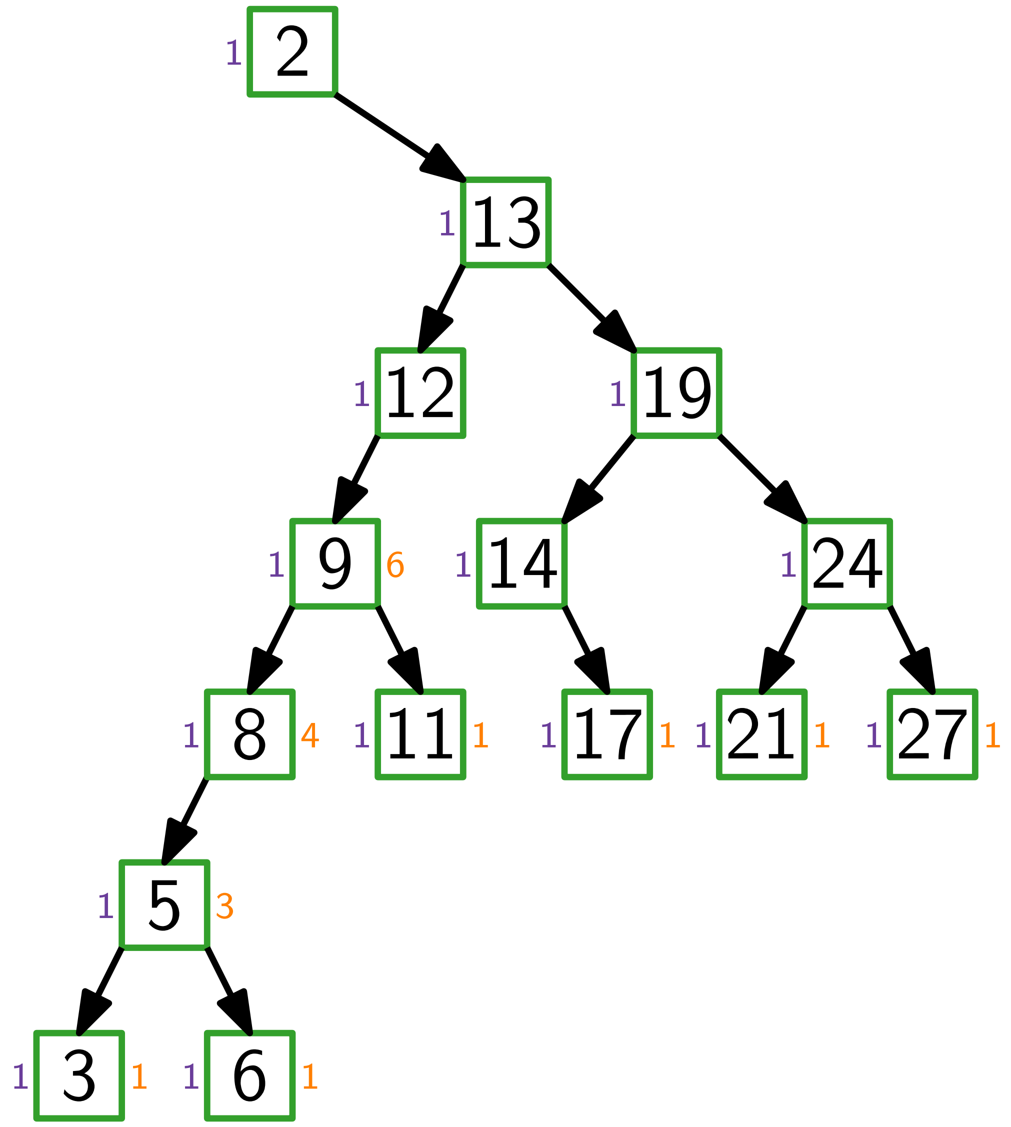

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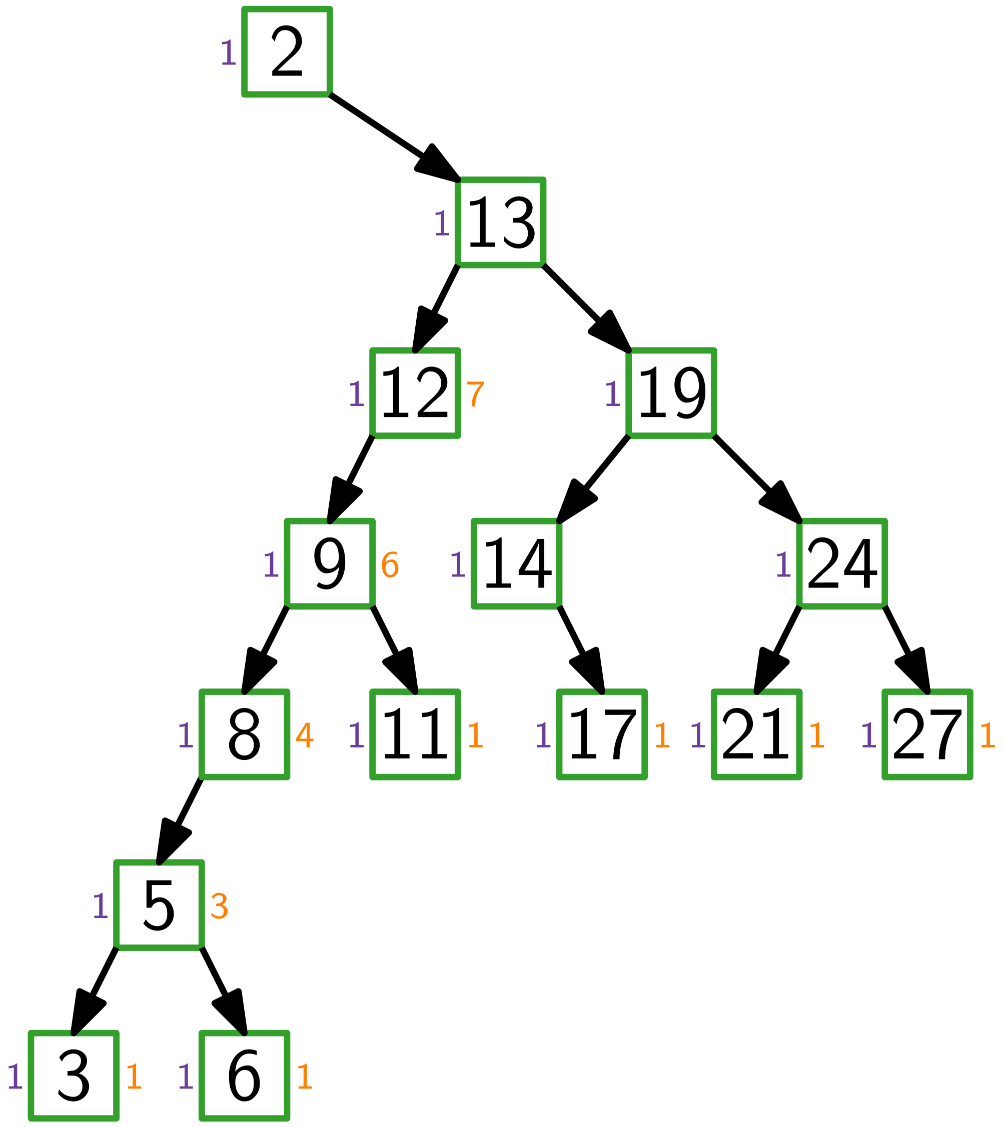

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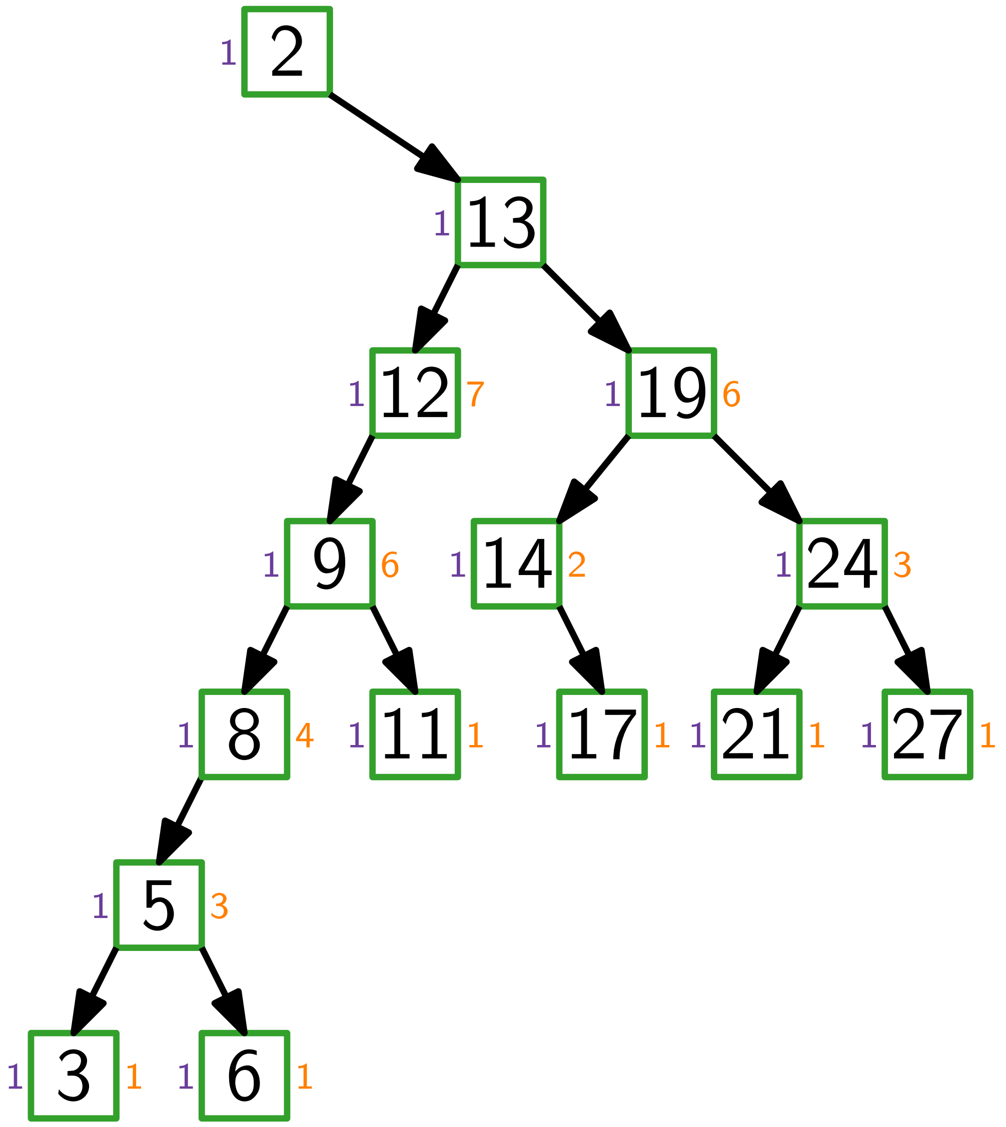

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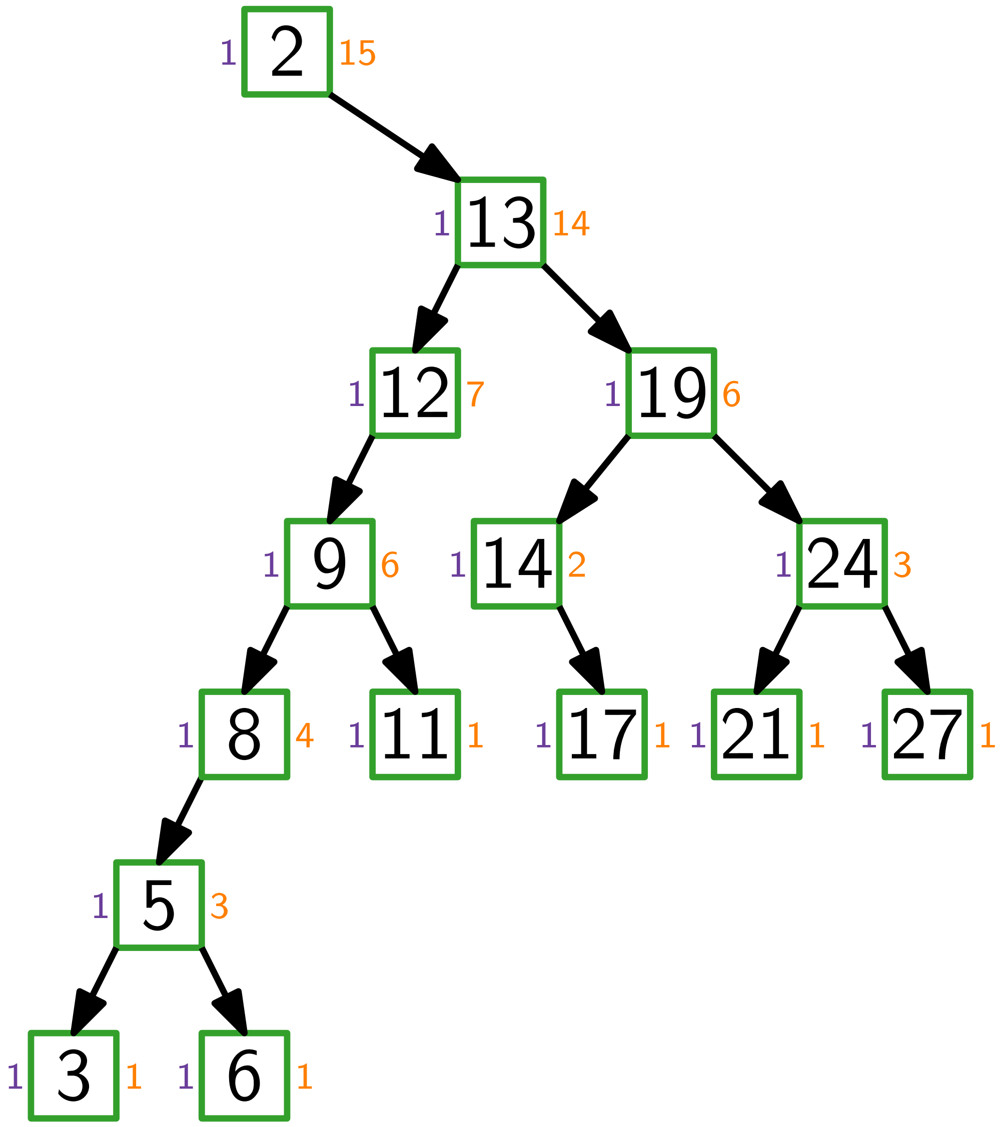

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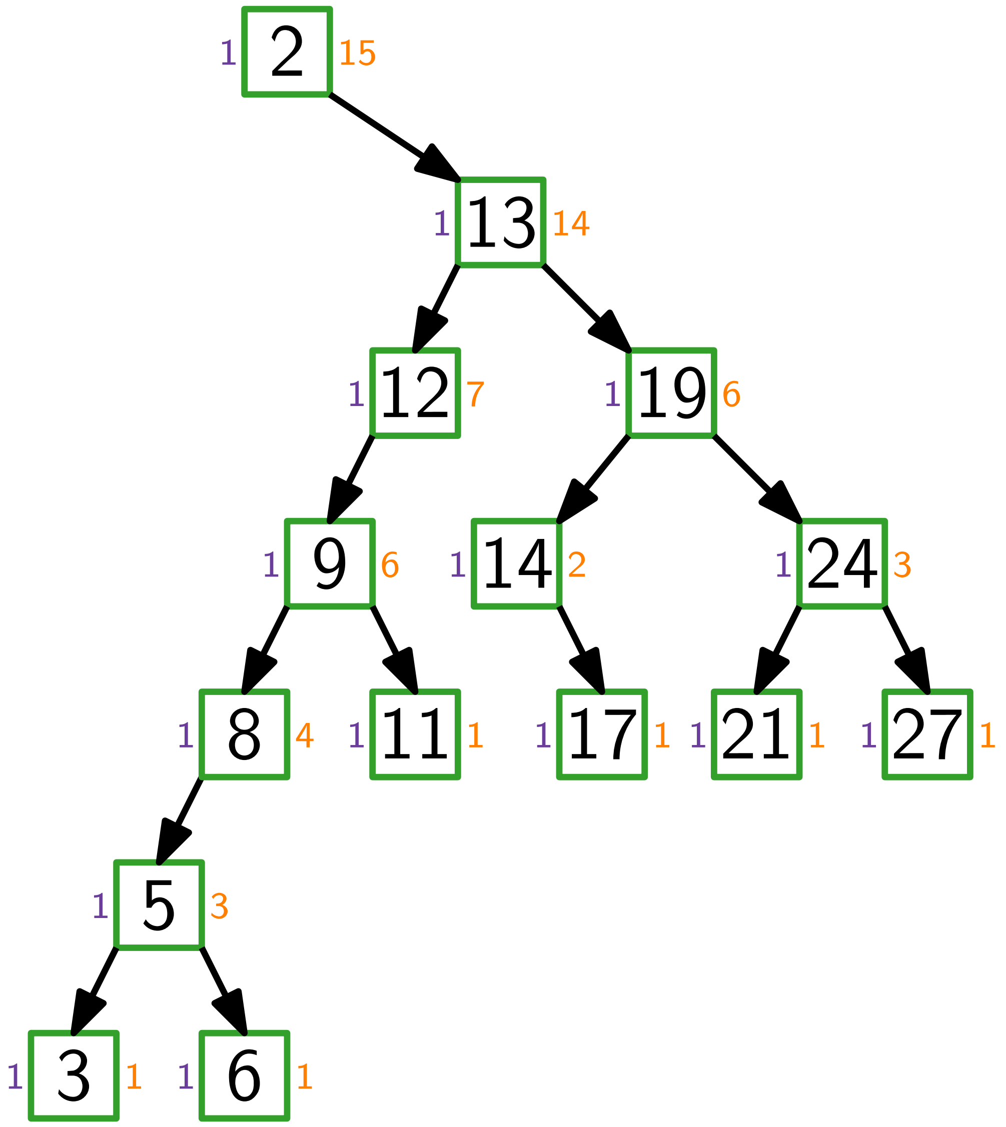

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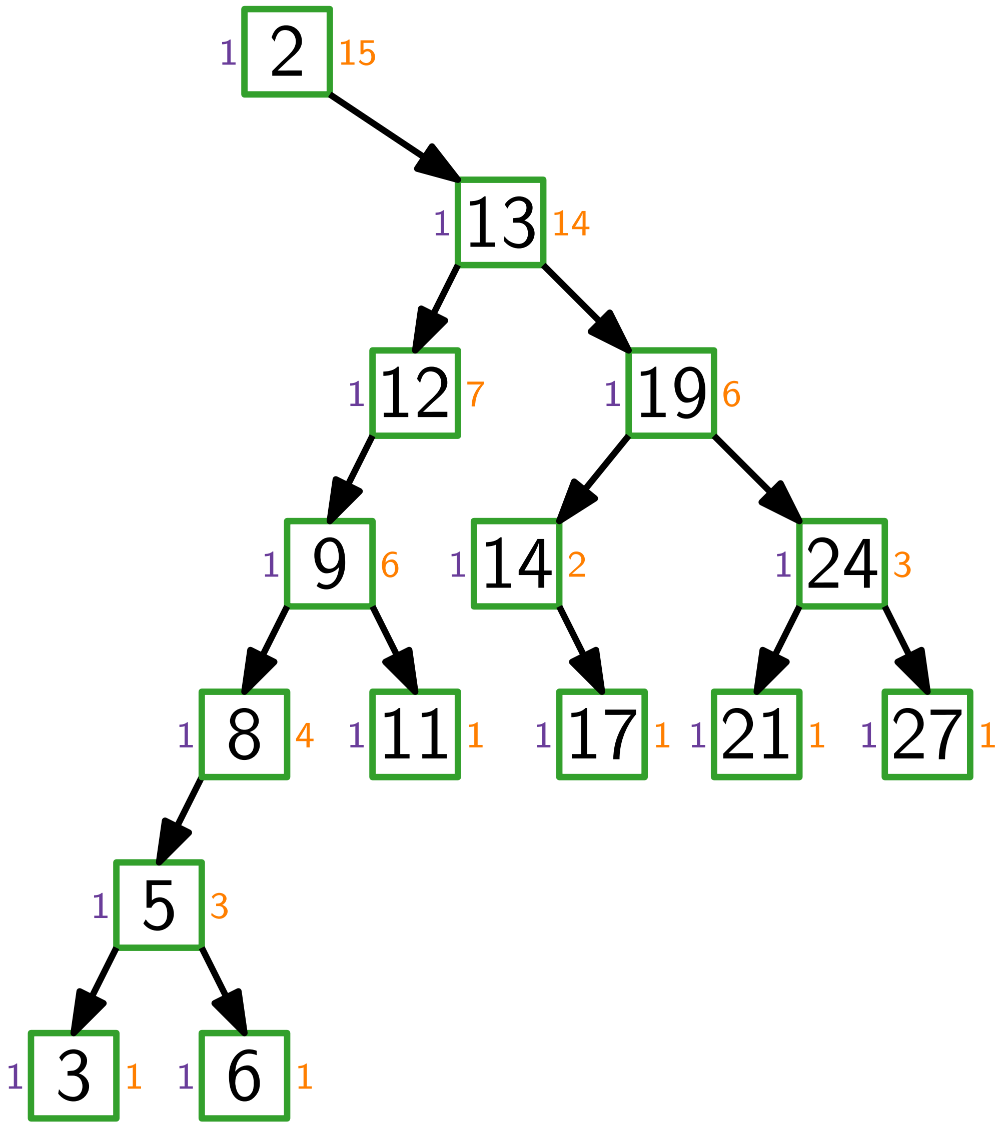

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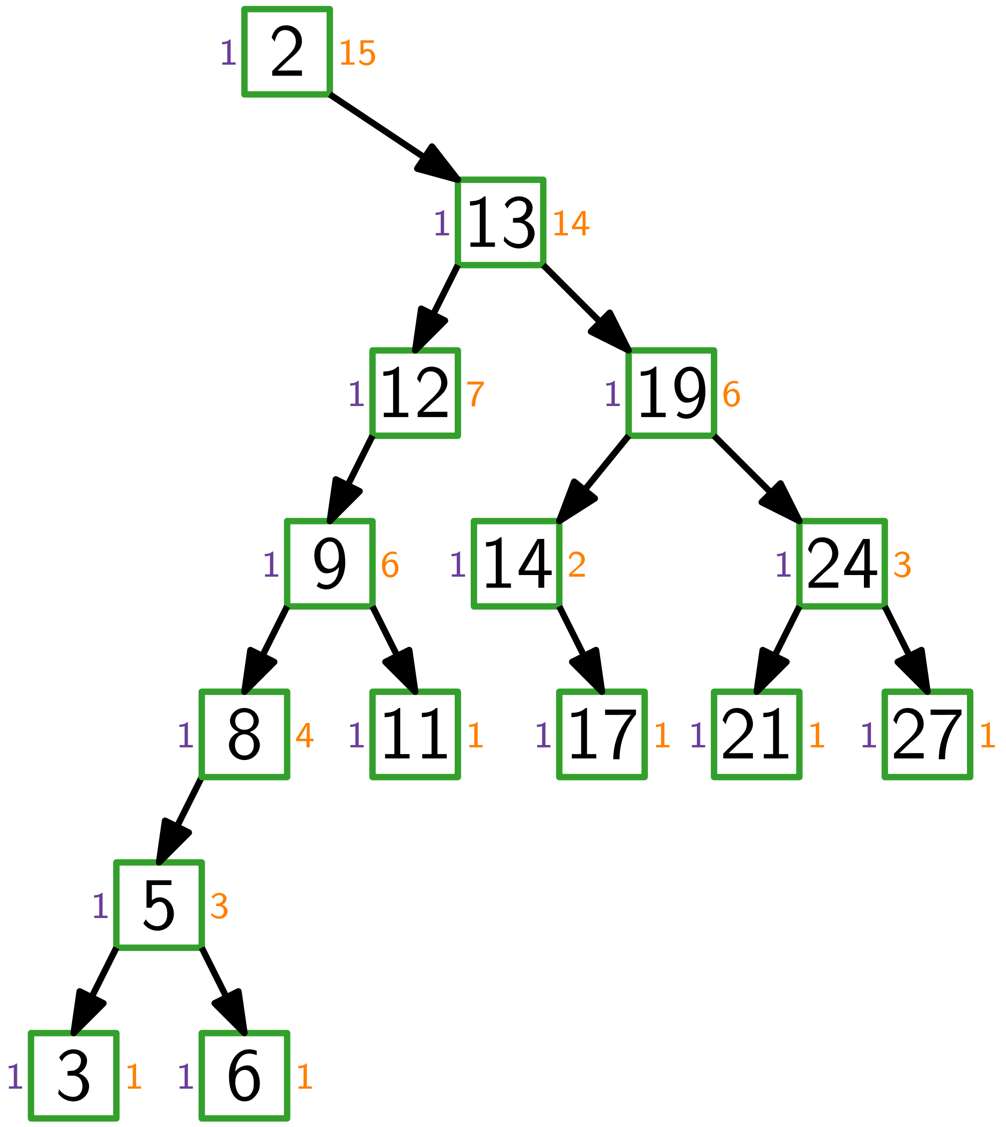

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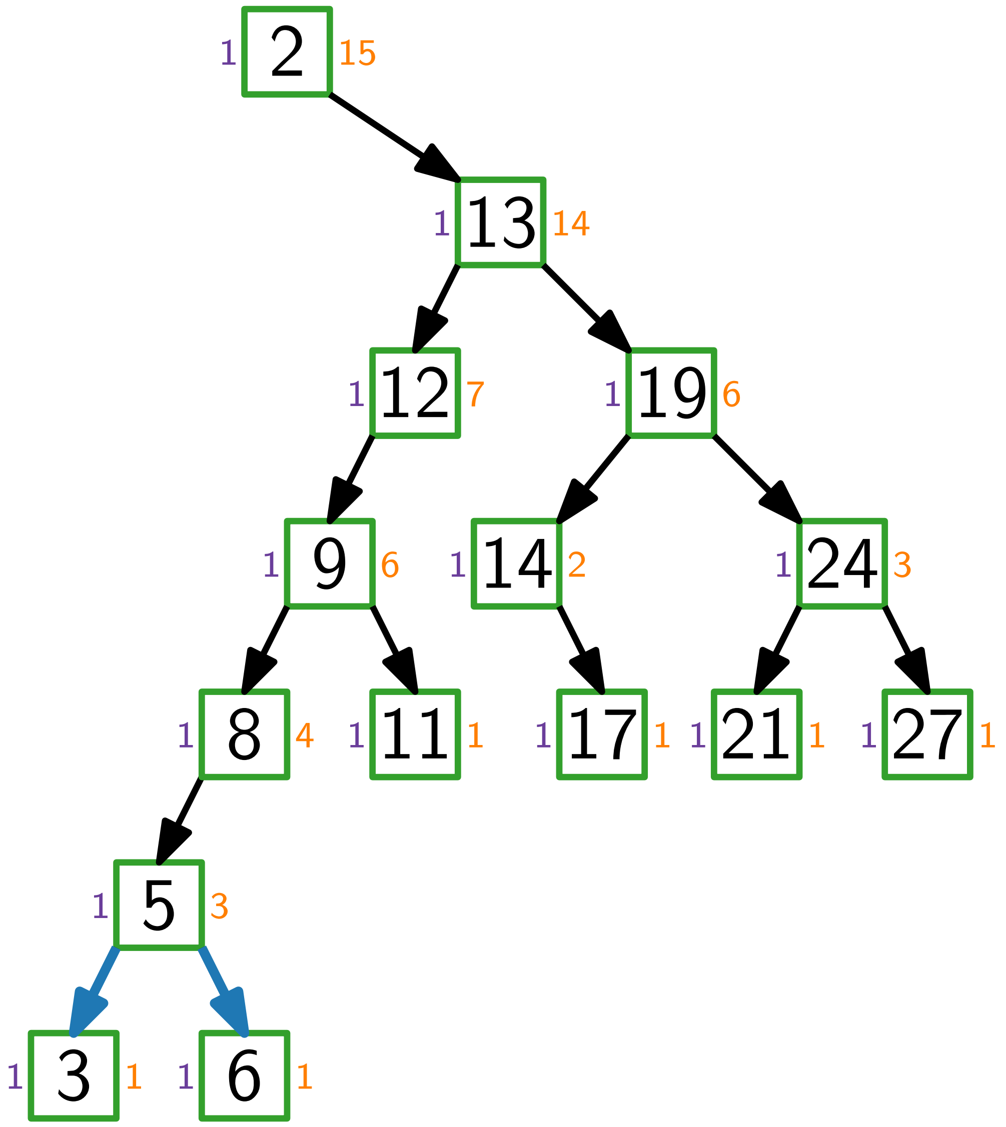

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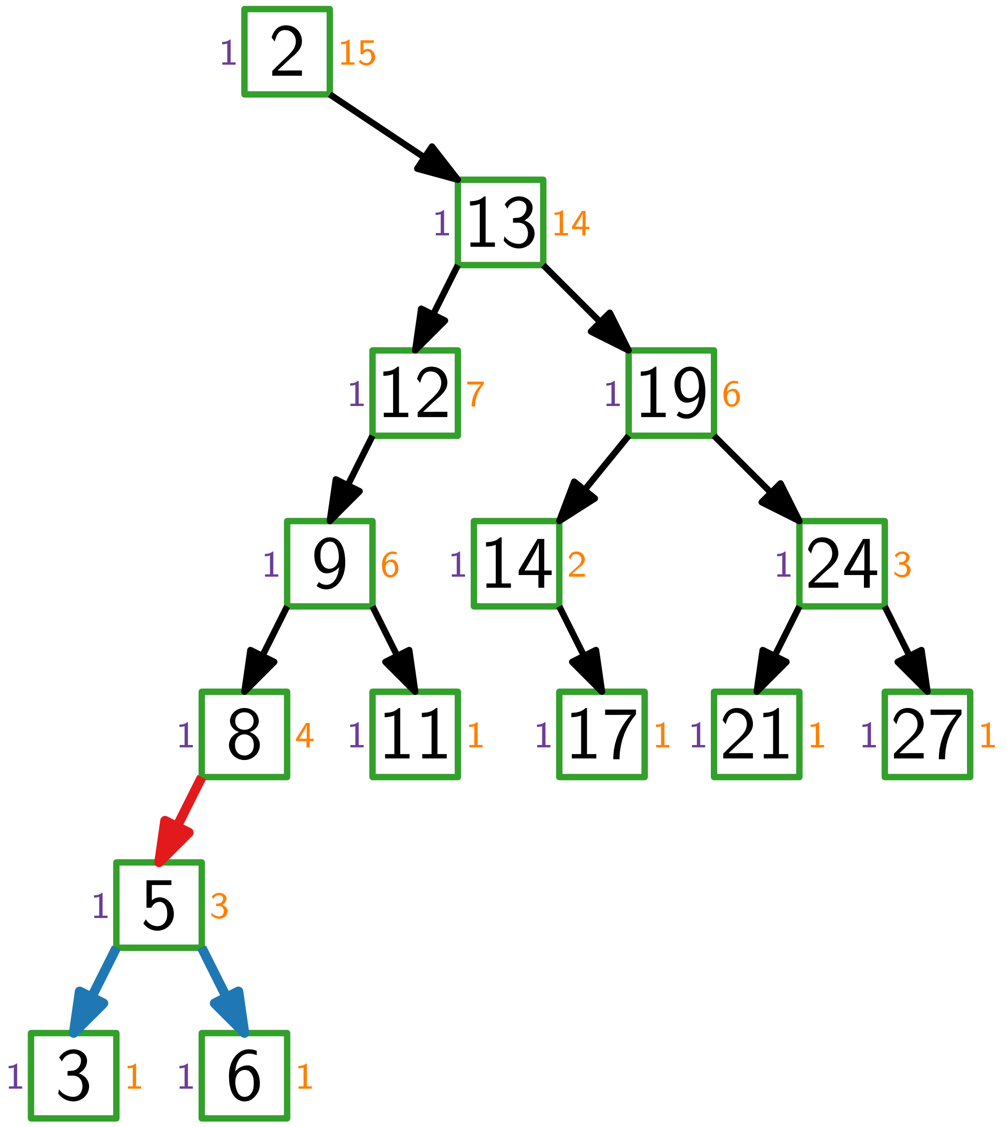

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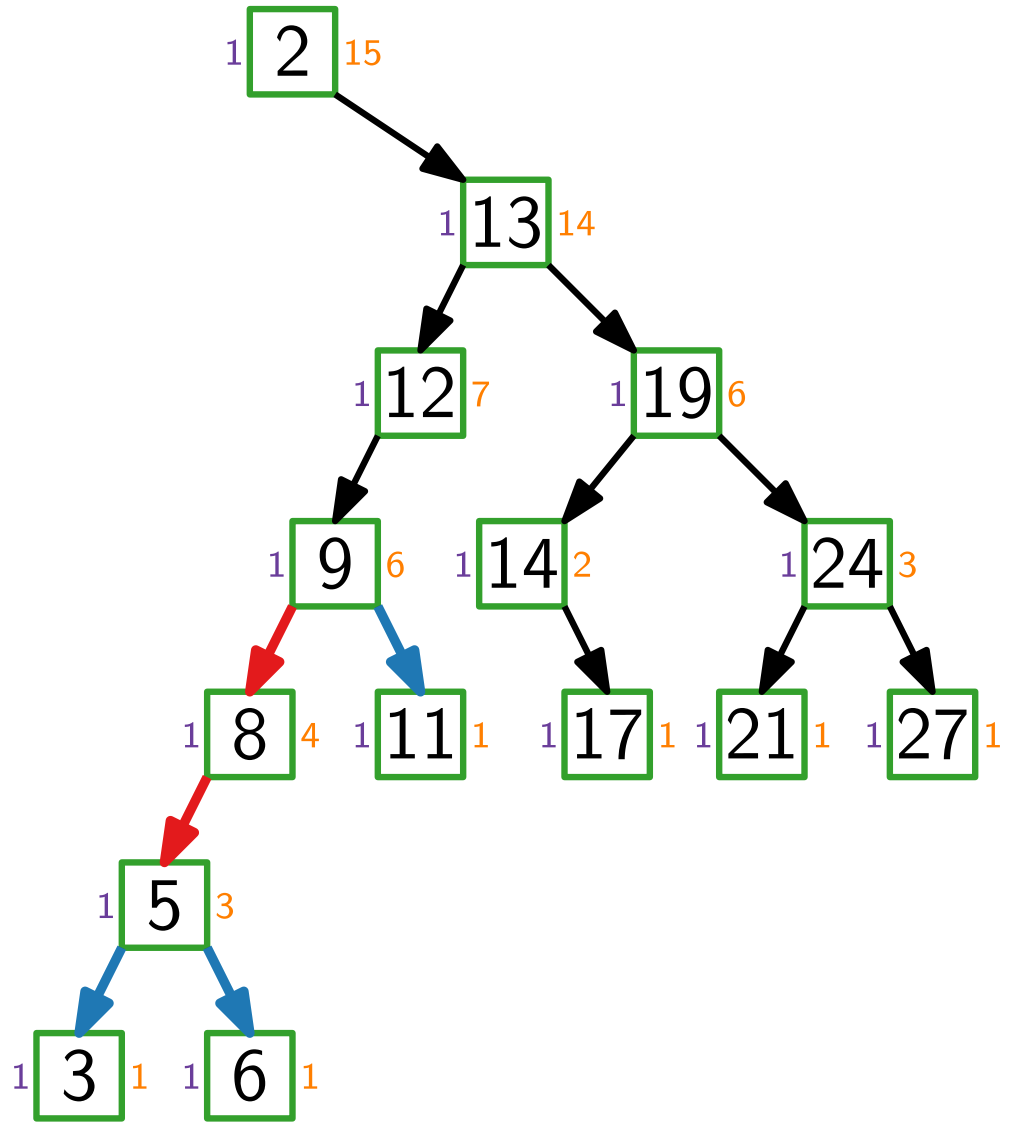

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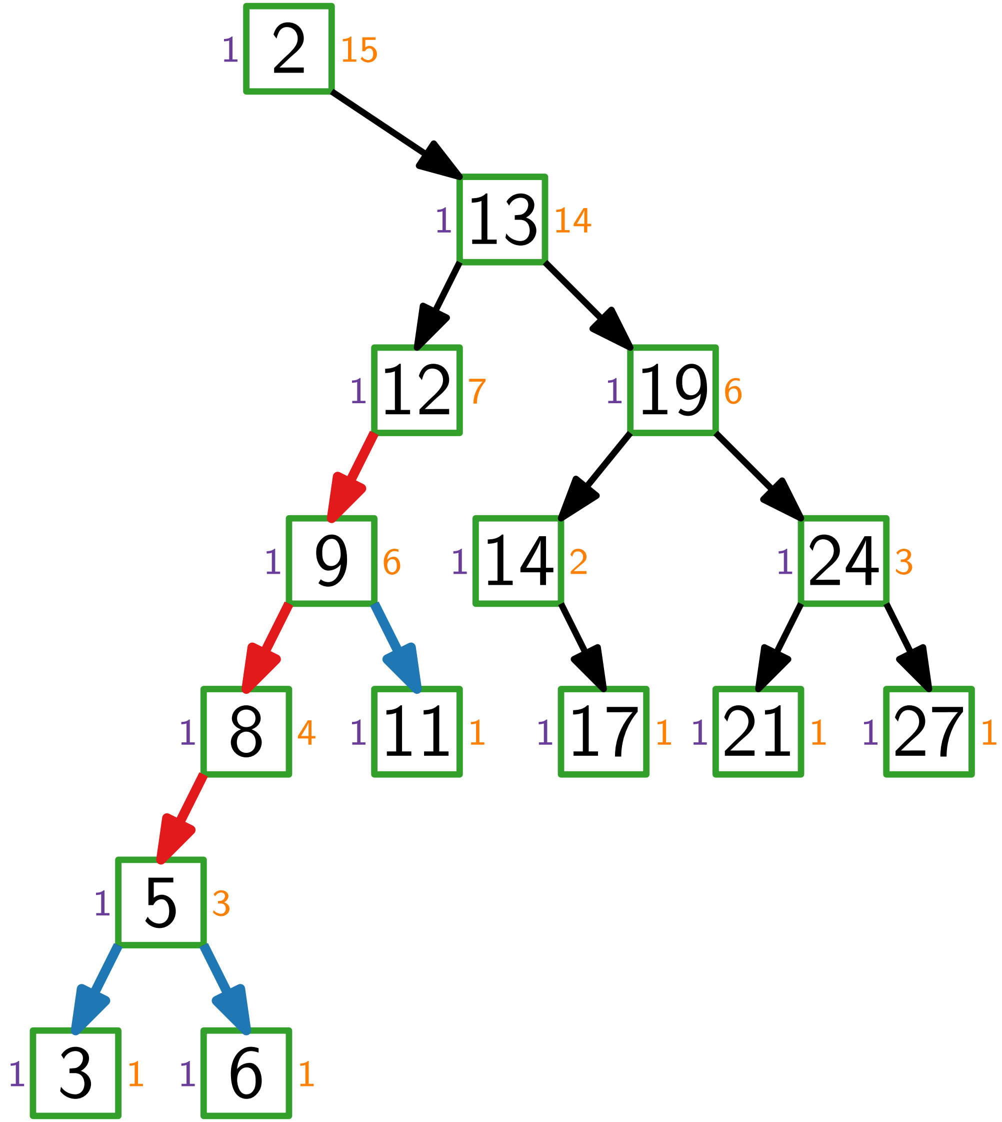

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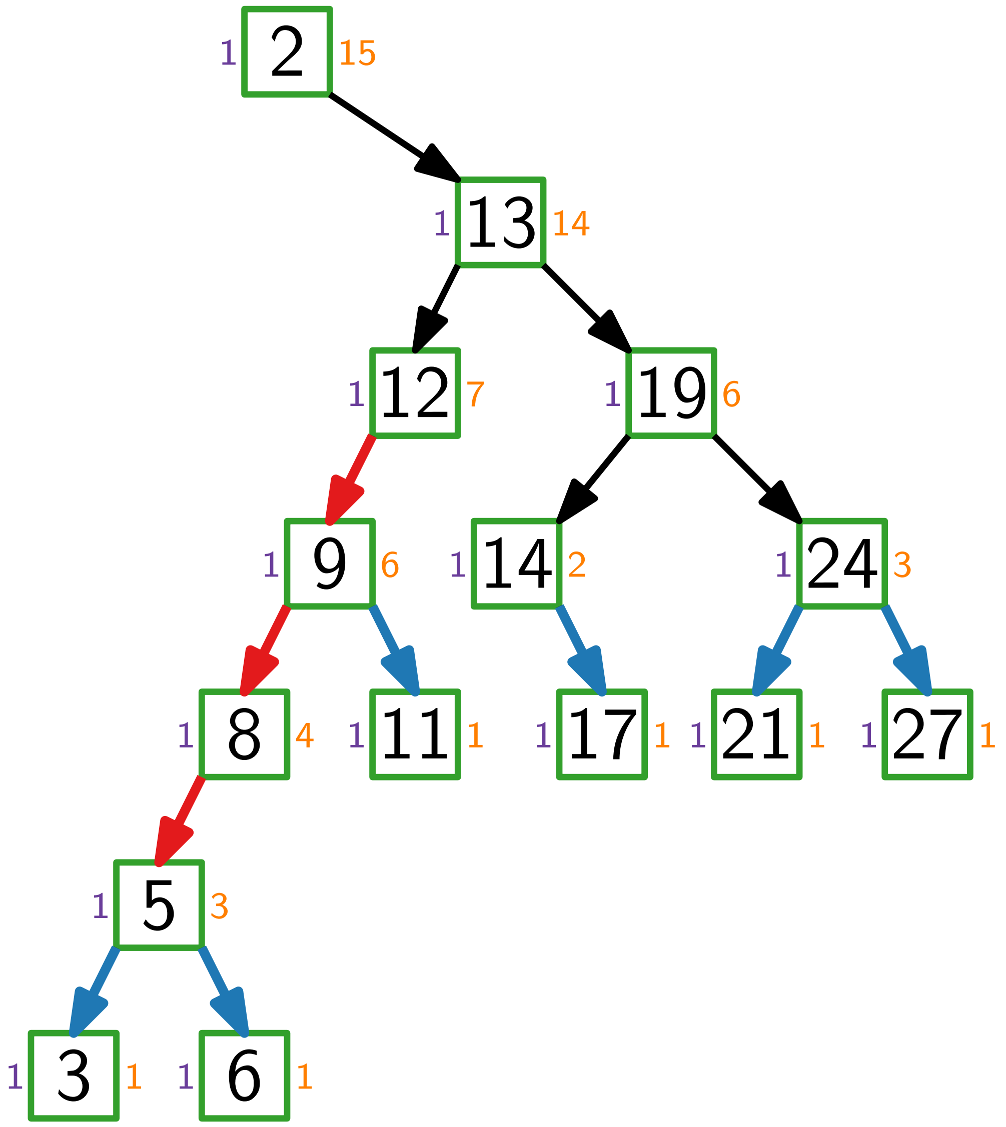

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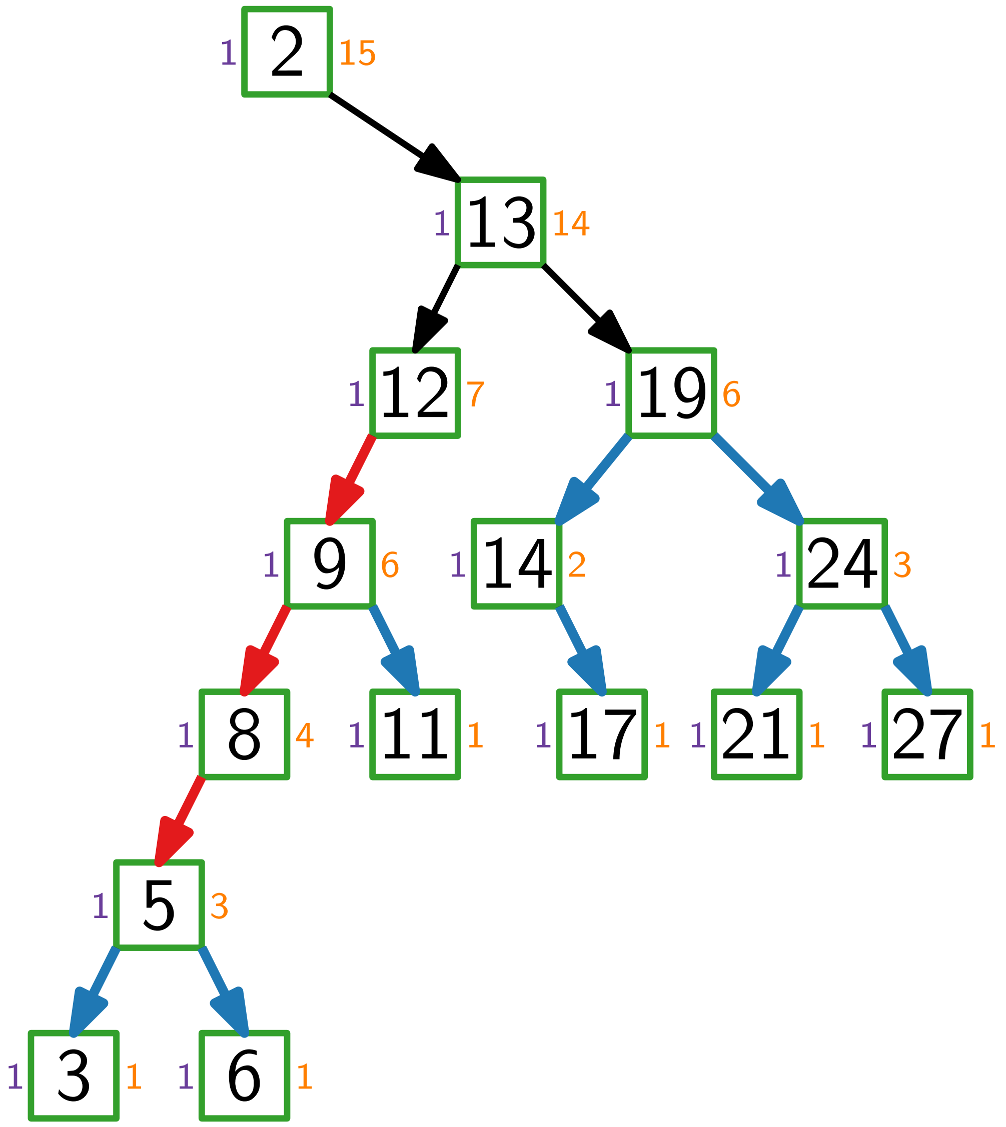

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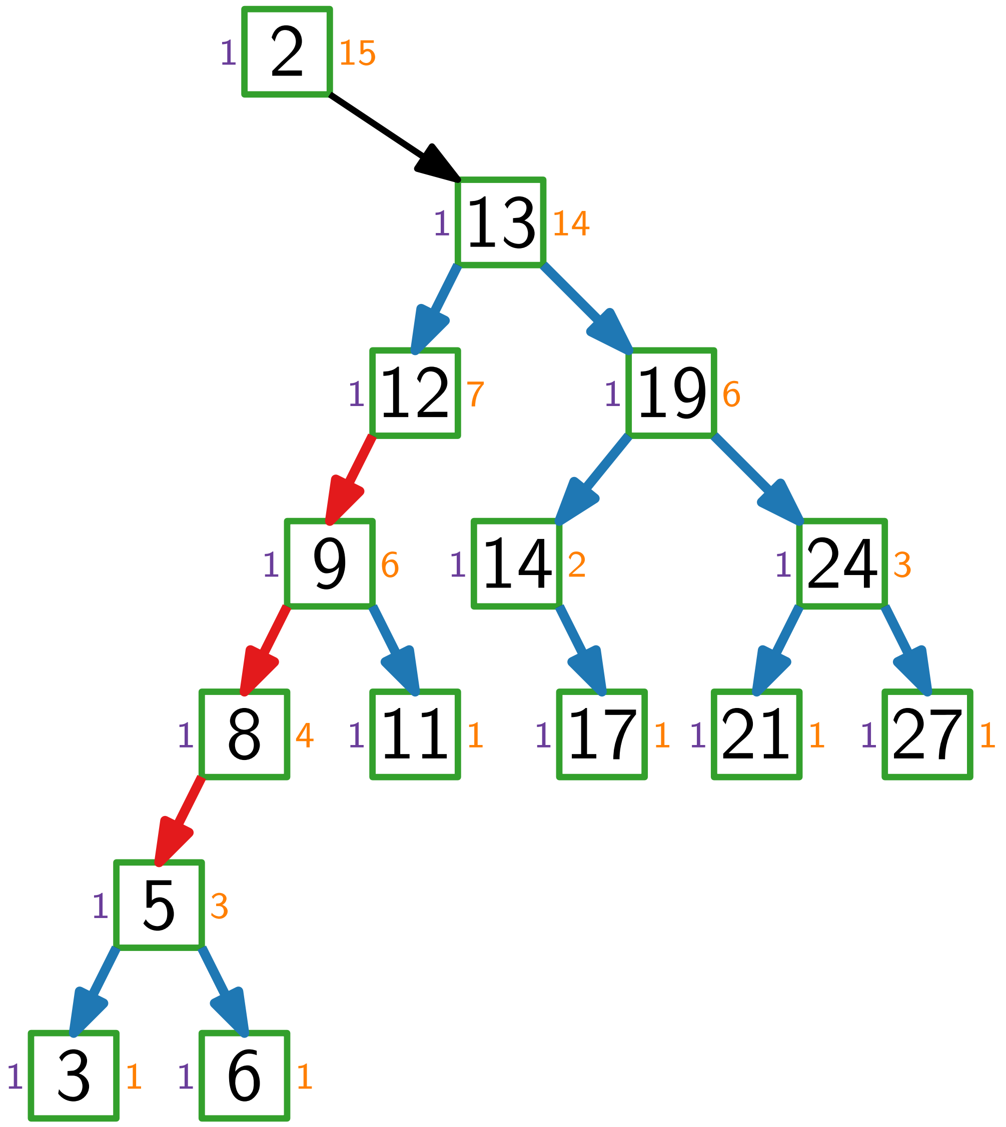

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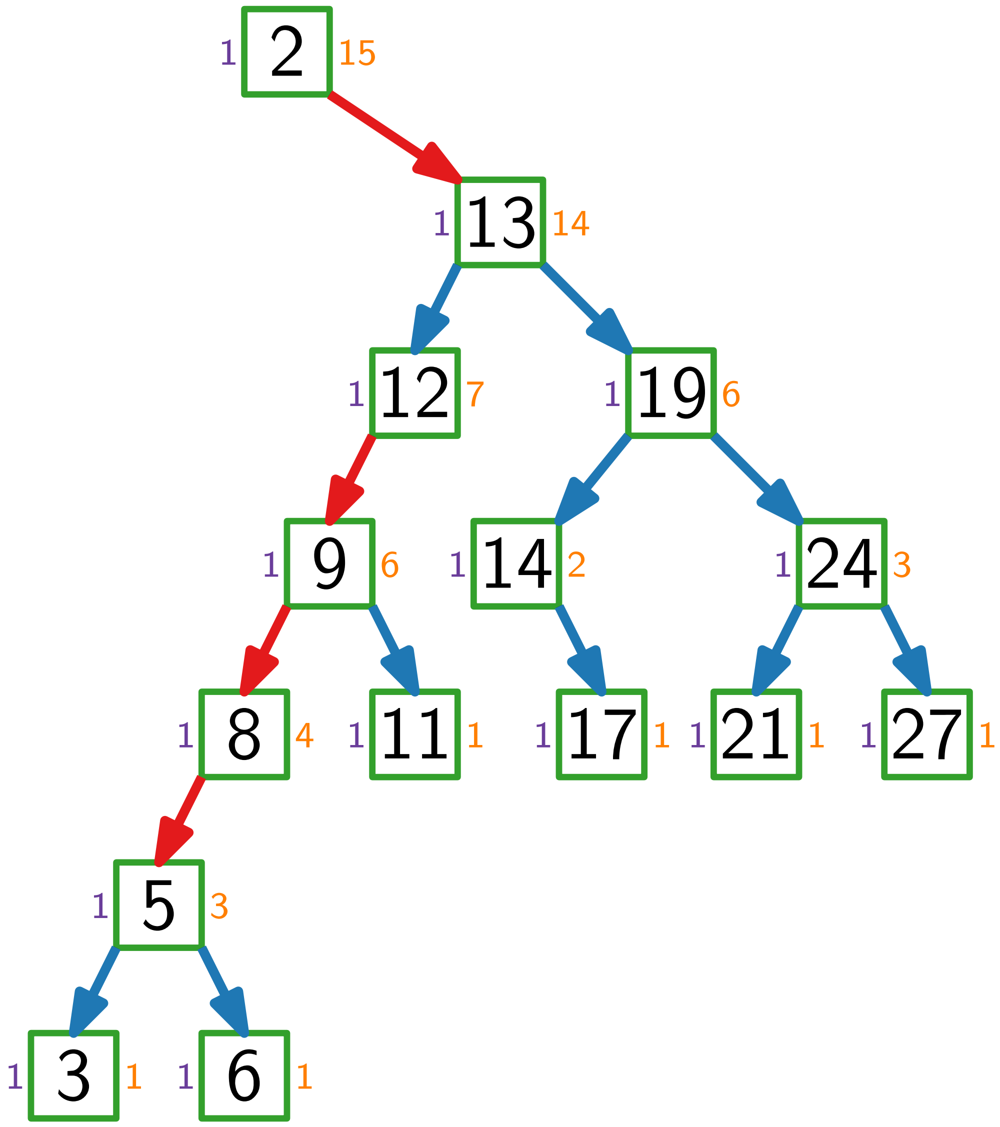

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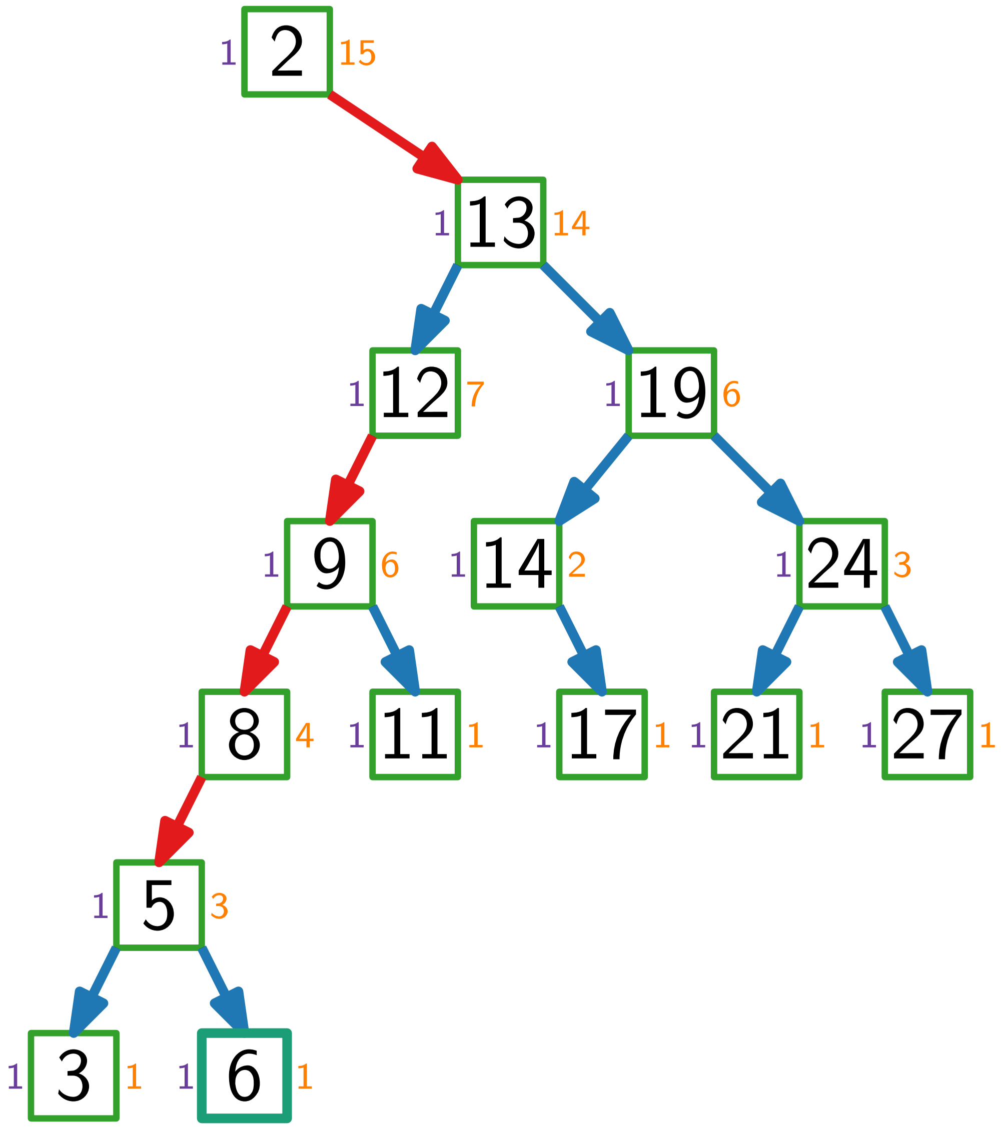

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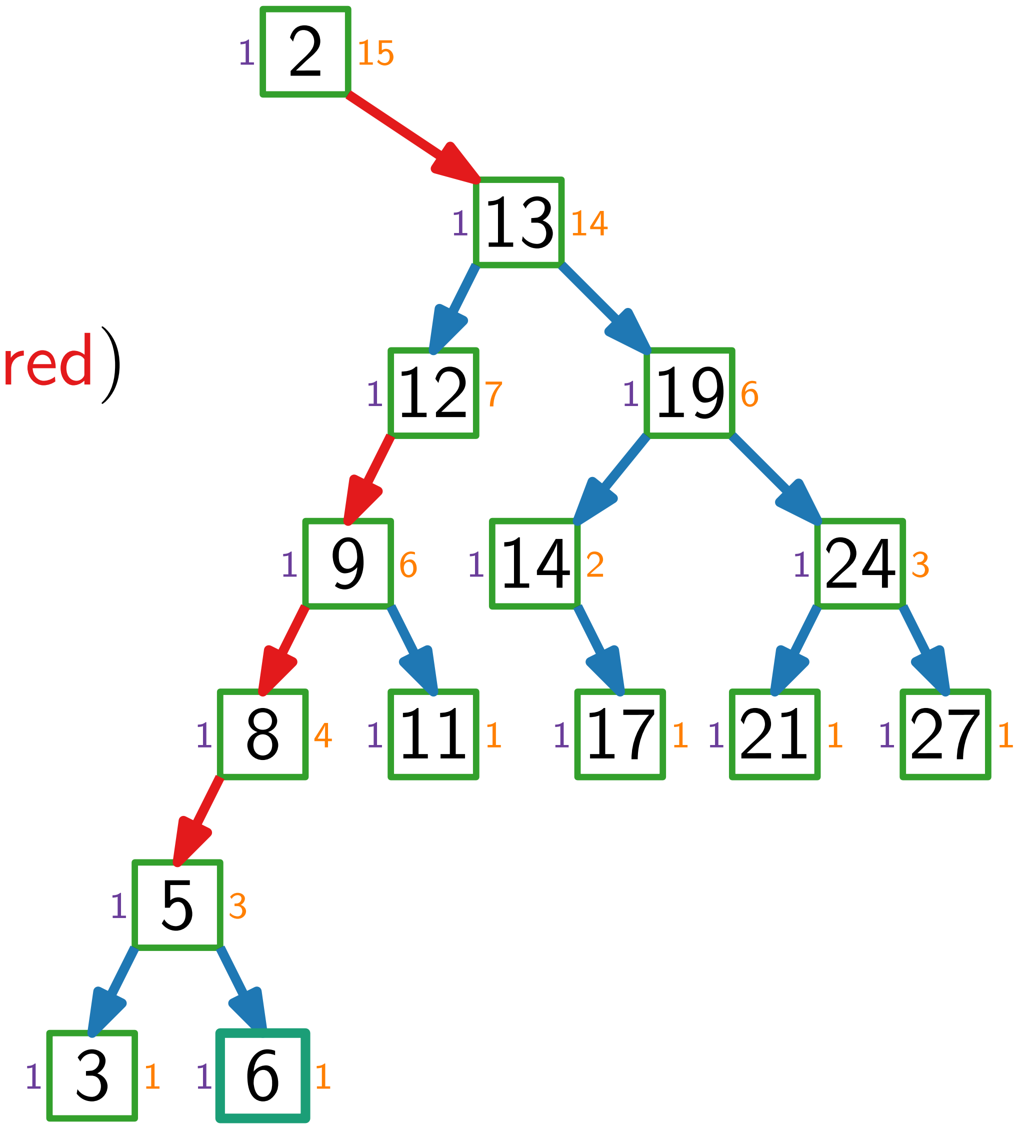

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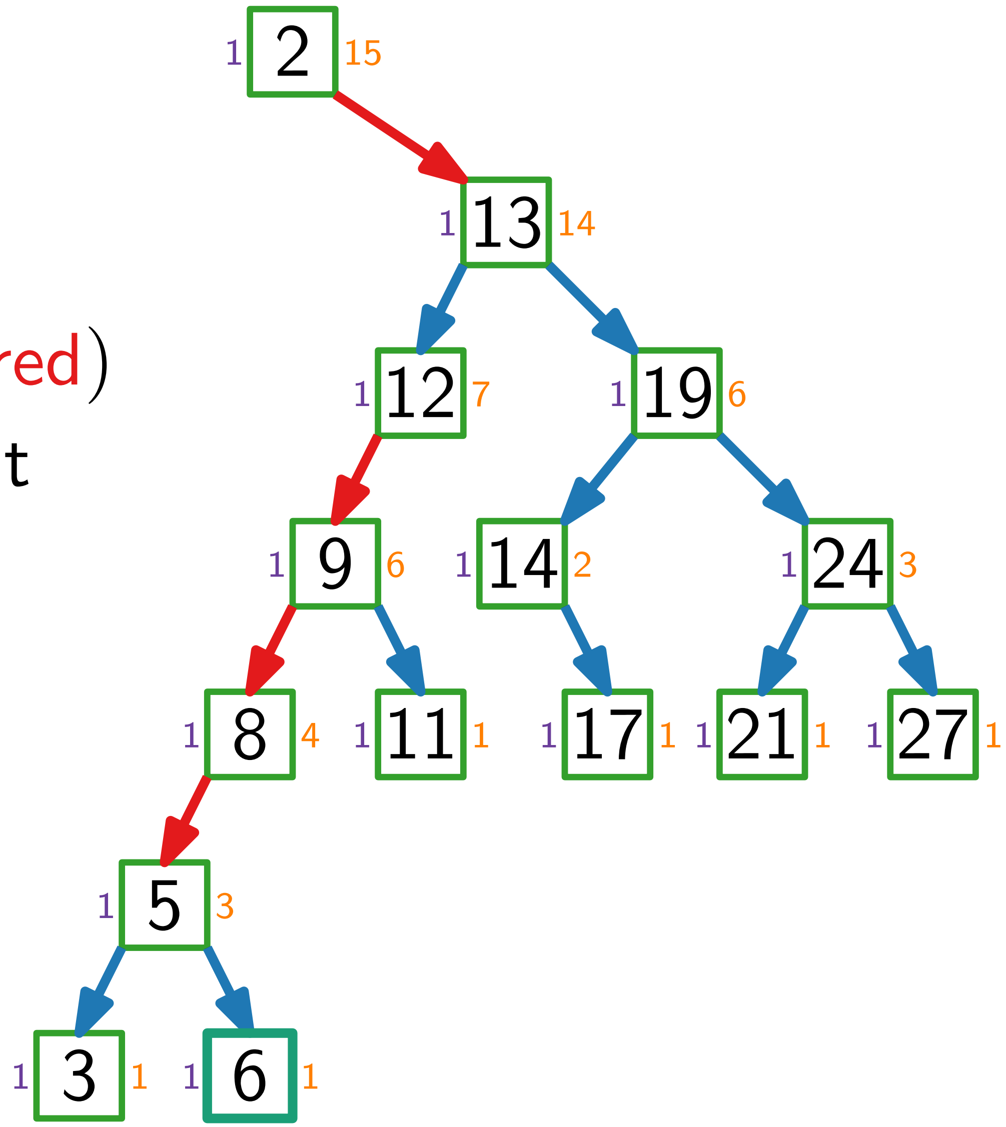

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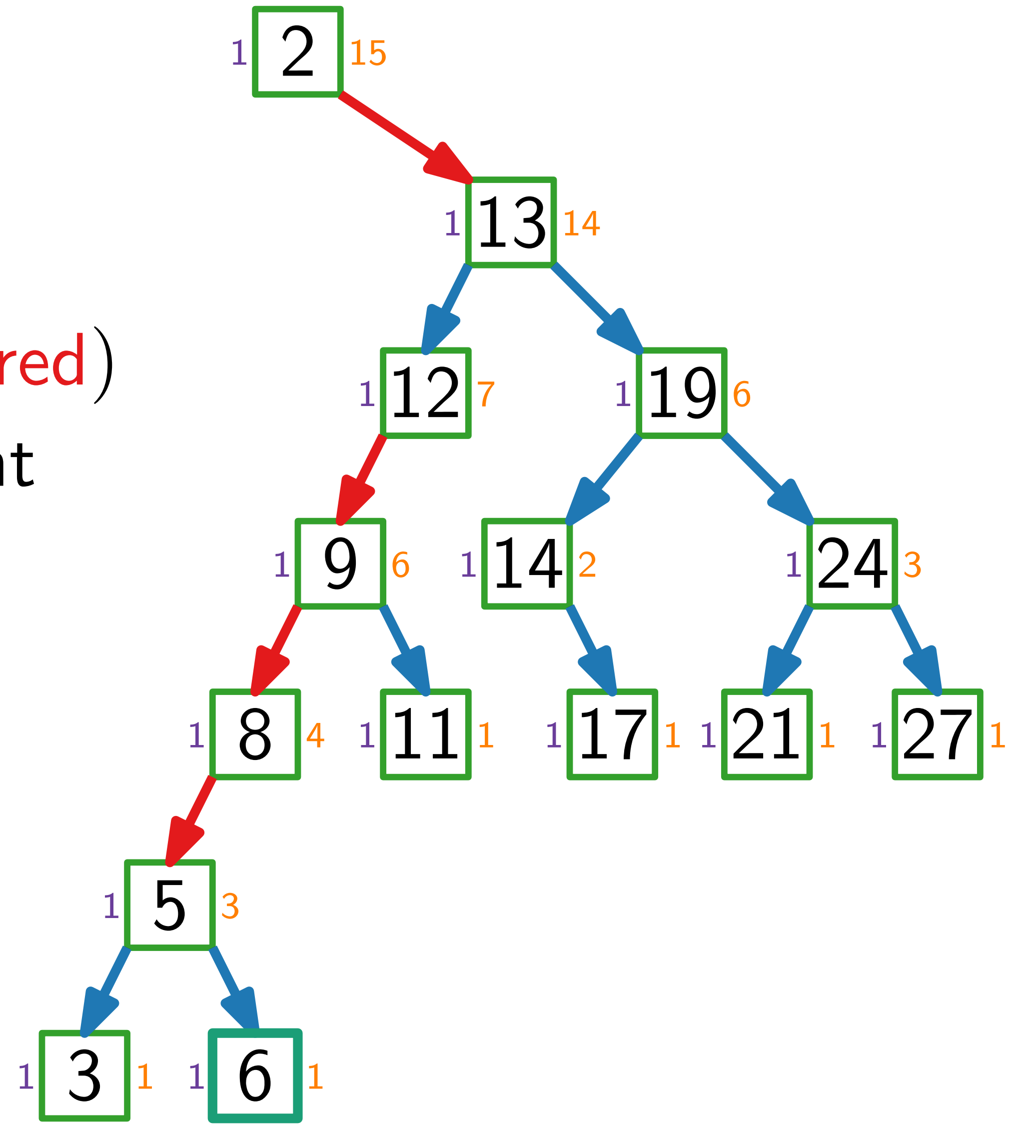

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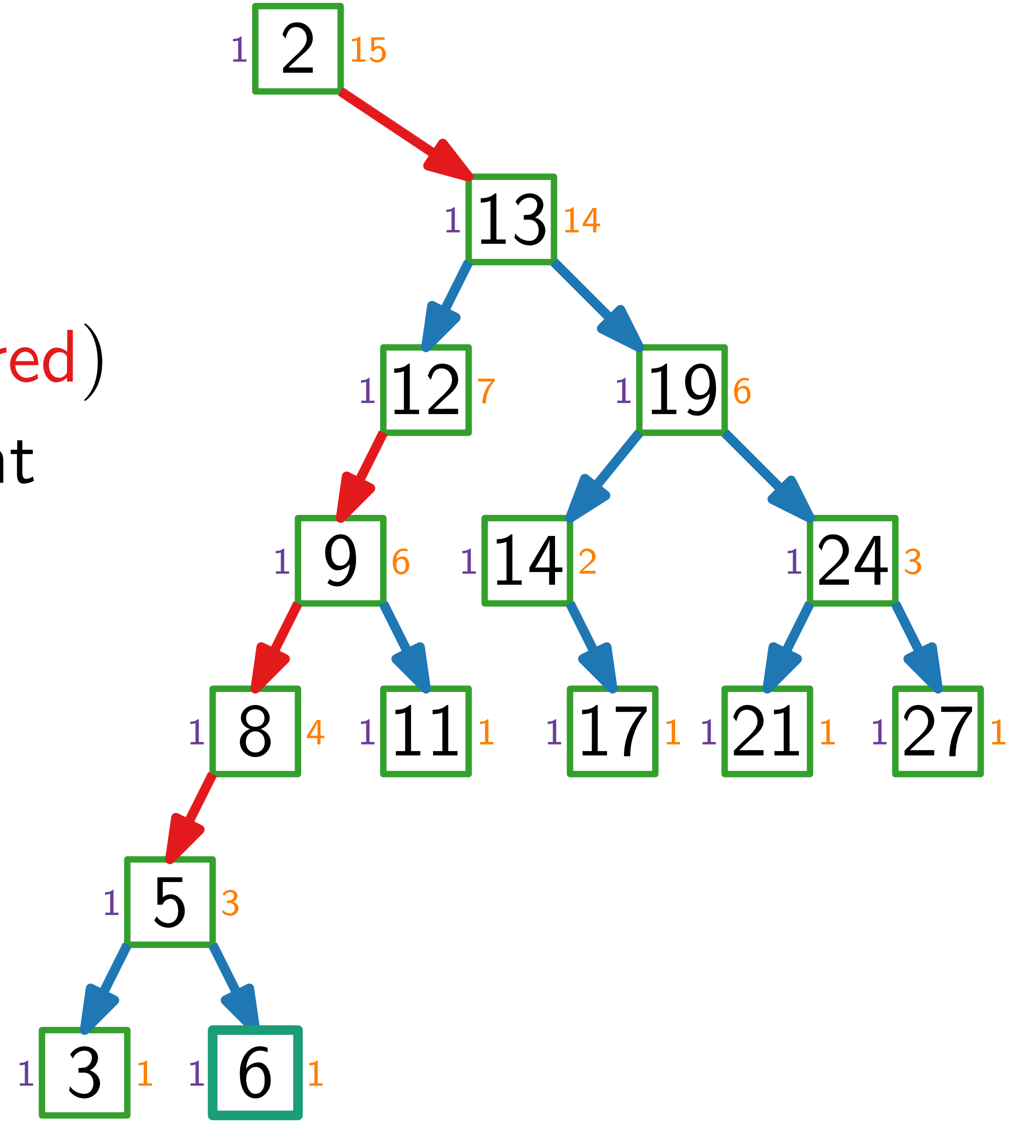

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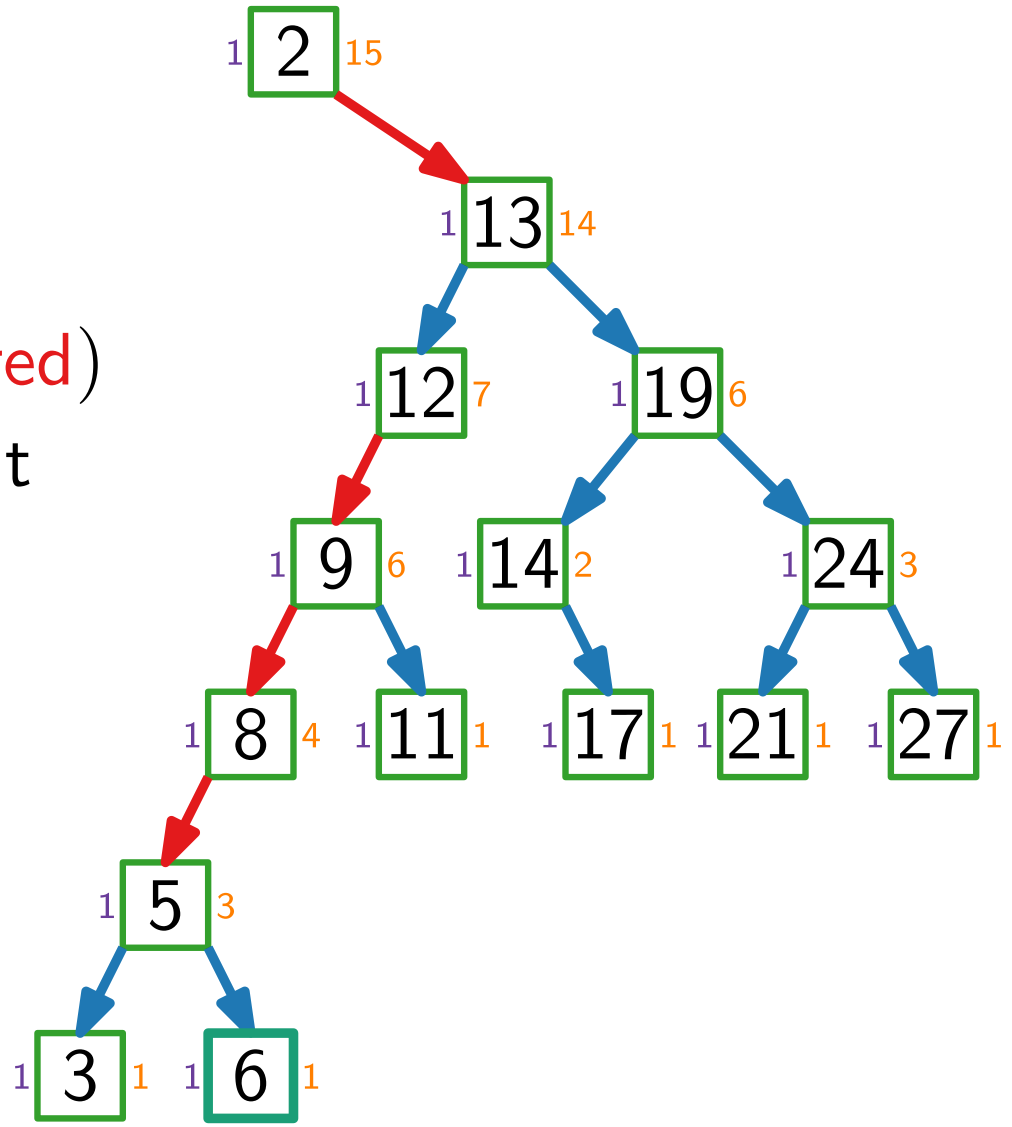

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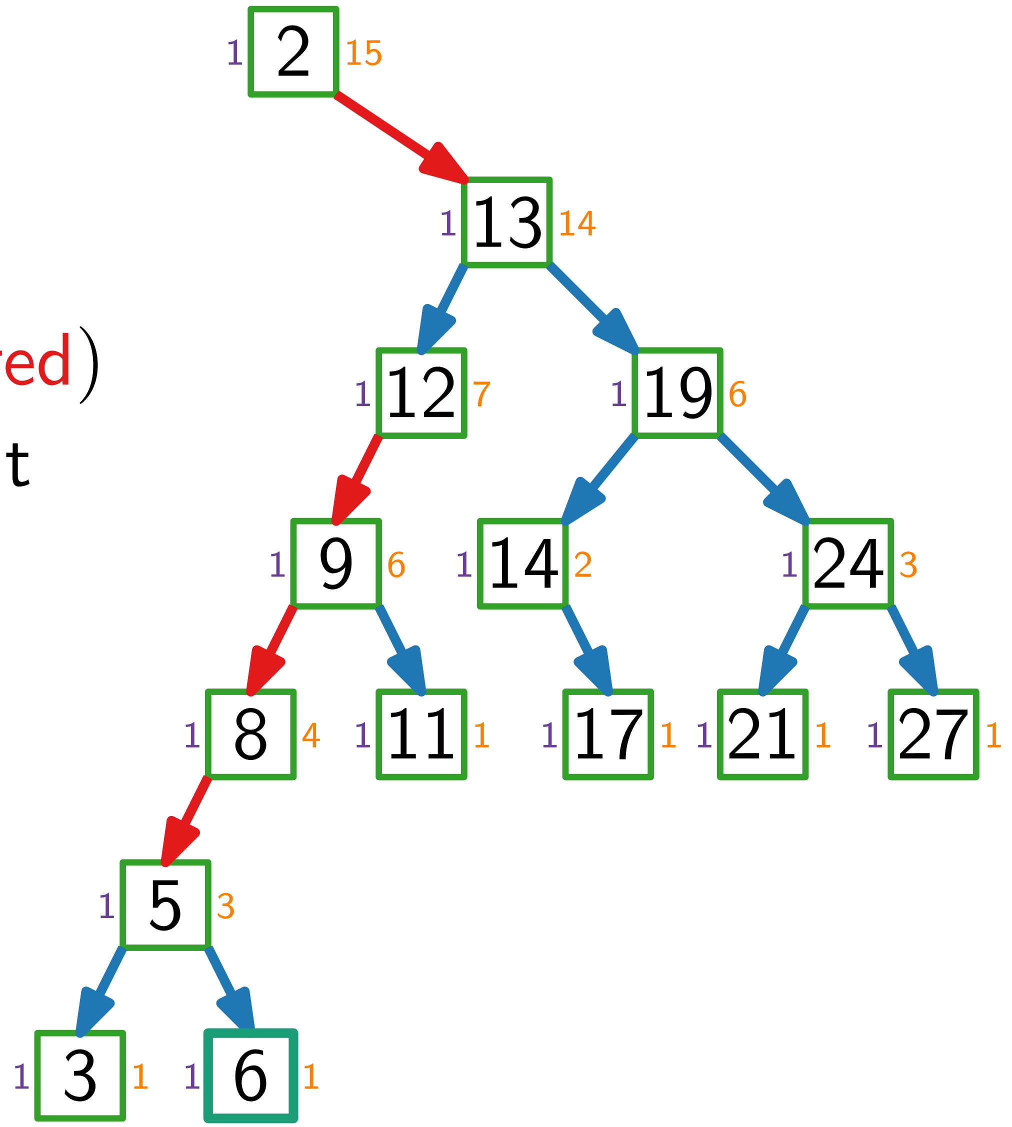

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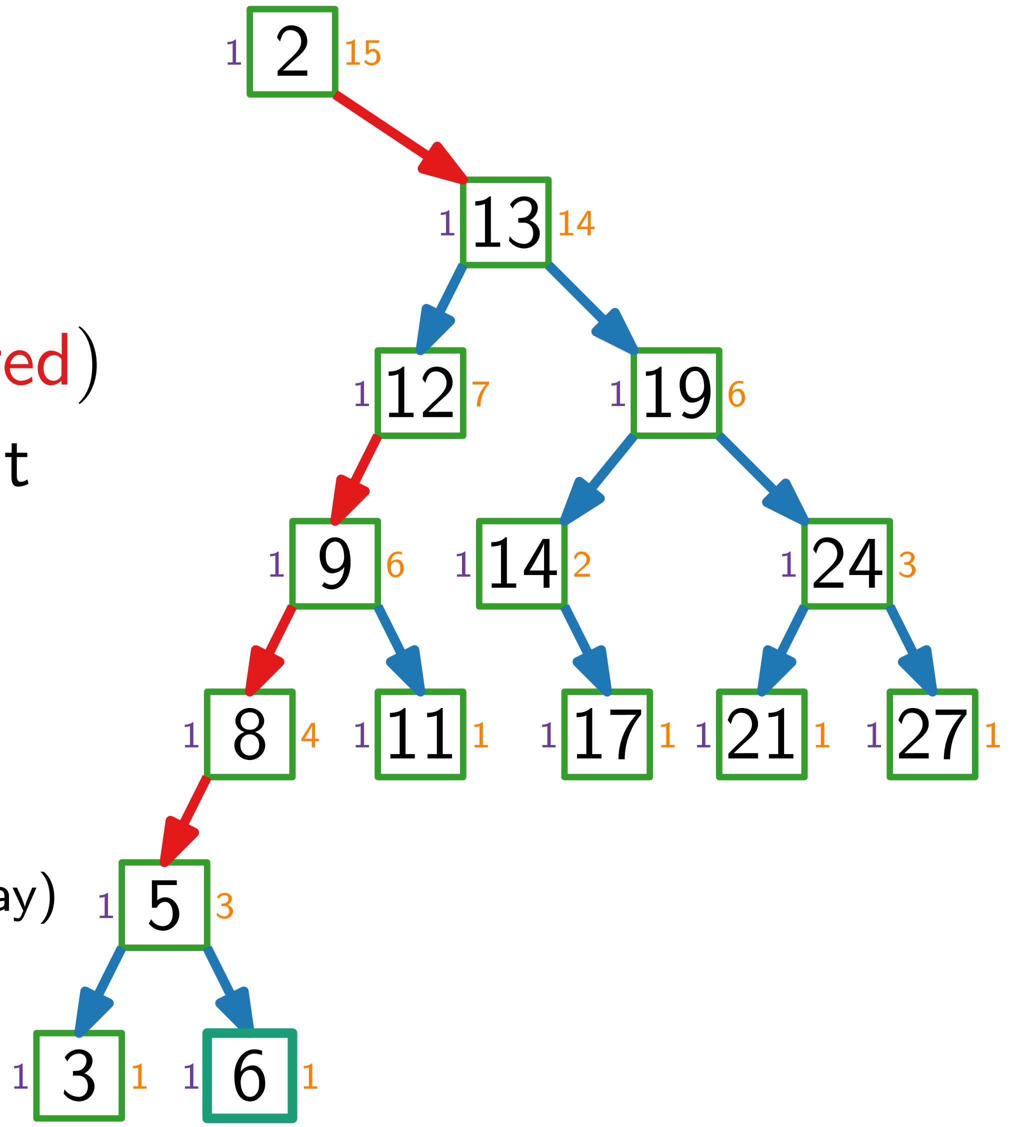
Amortized cost:

(potential before splay)

real cost $+ \Phi_+ - \Phi$

(potential after splay)



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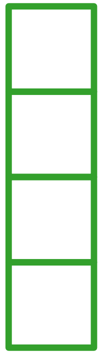

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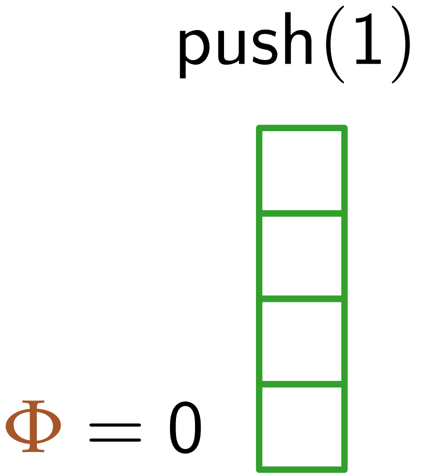

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$\Phi = 1$ $\boxed{1}$

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push(2)

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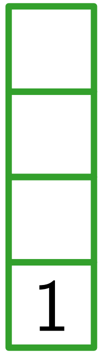

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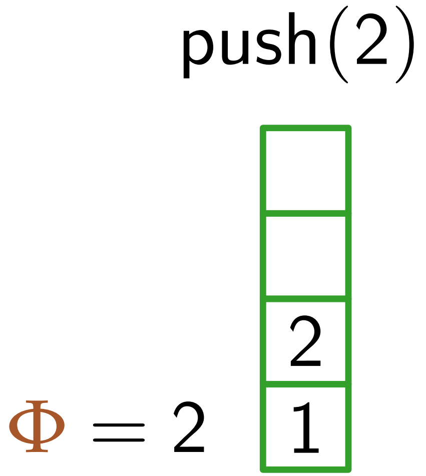

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push(3)

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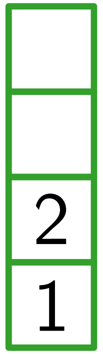
$\Phi = 2$



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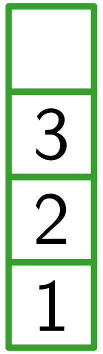

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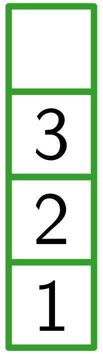

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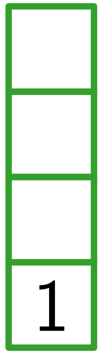

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push:

pop(k):

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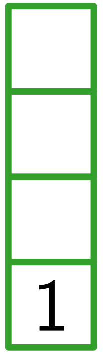

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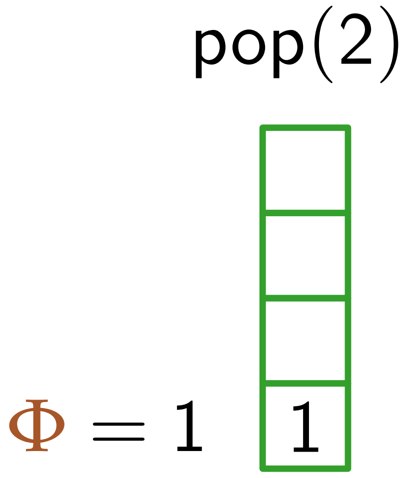

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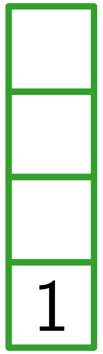

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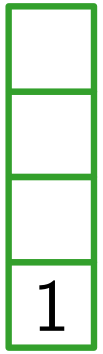

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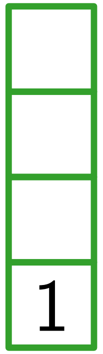

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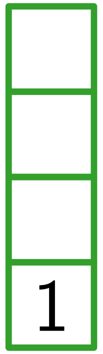

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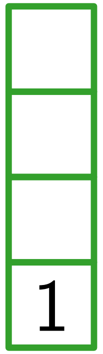

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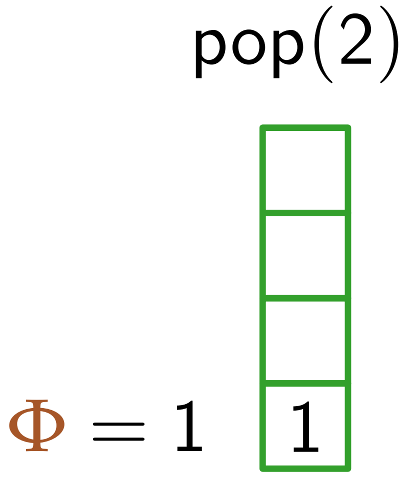

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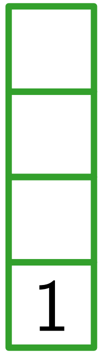

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$w(x)$: weight of x (here 1), $W = \sum w(x)$ (here n)
$s(x)$: sum of all $w(x)$ in subtree of x_i

mark edges:

→ $s(\text{child}) \leq s(\text{parent})/2$ (blue)
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Cost to query x_i: $O(\log W + \#\text{red})$

Idea: blue edges halve the weight
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How can we amortize red edges?

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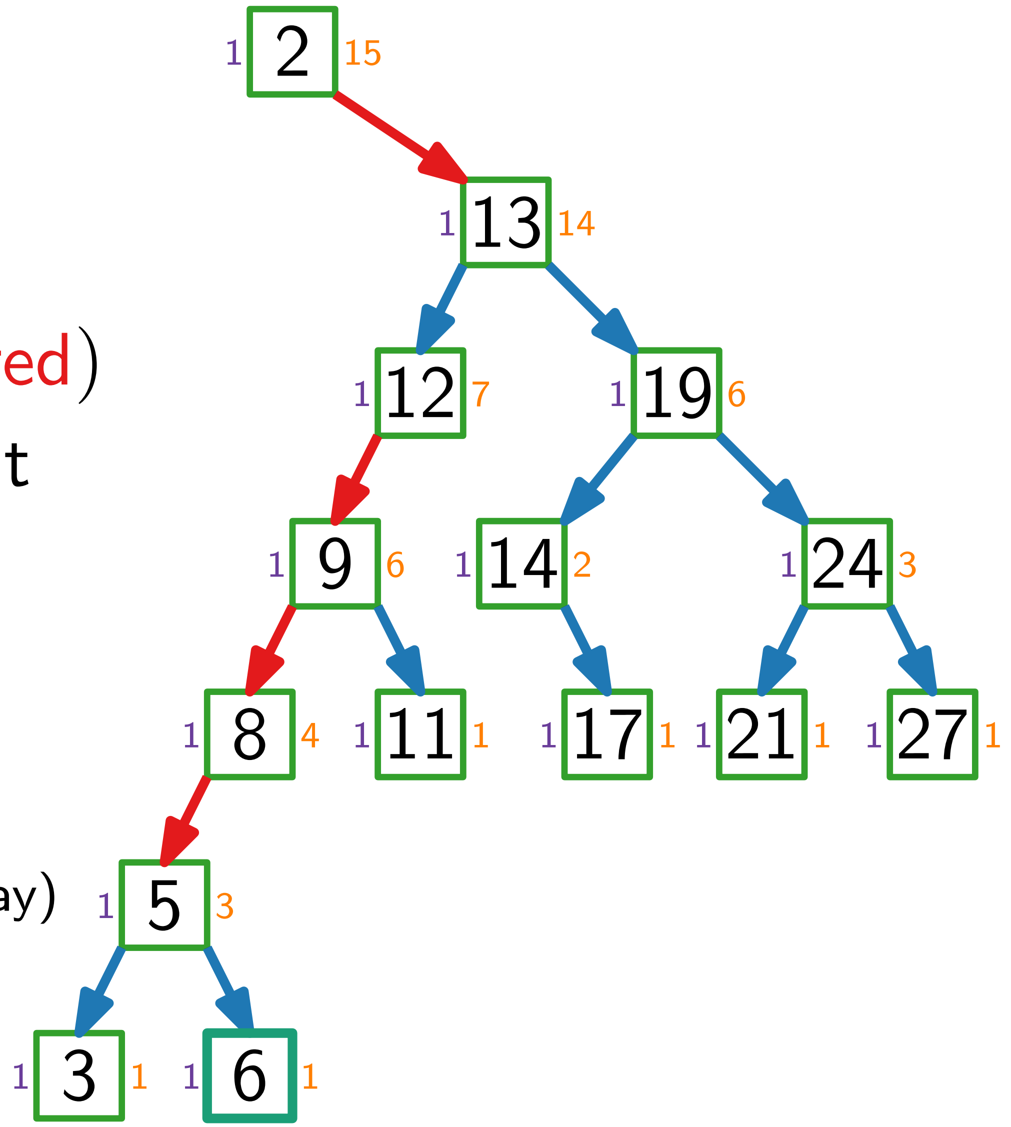

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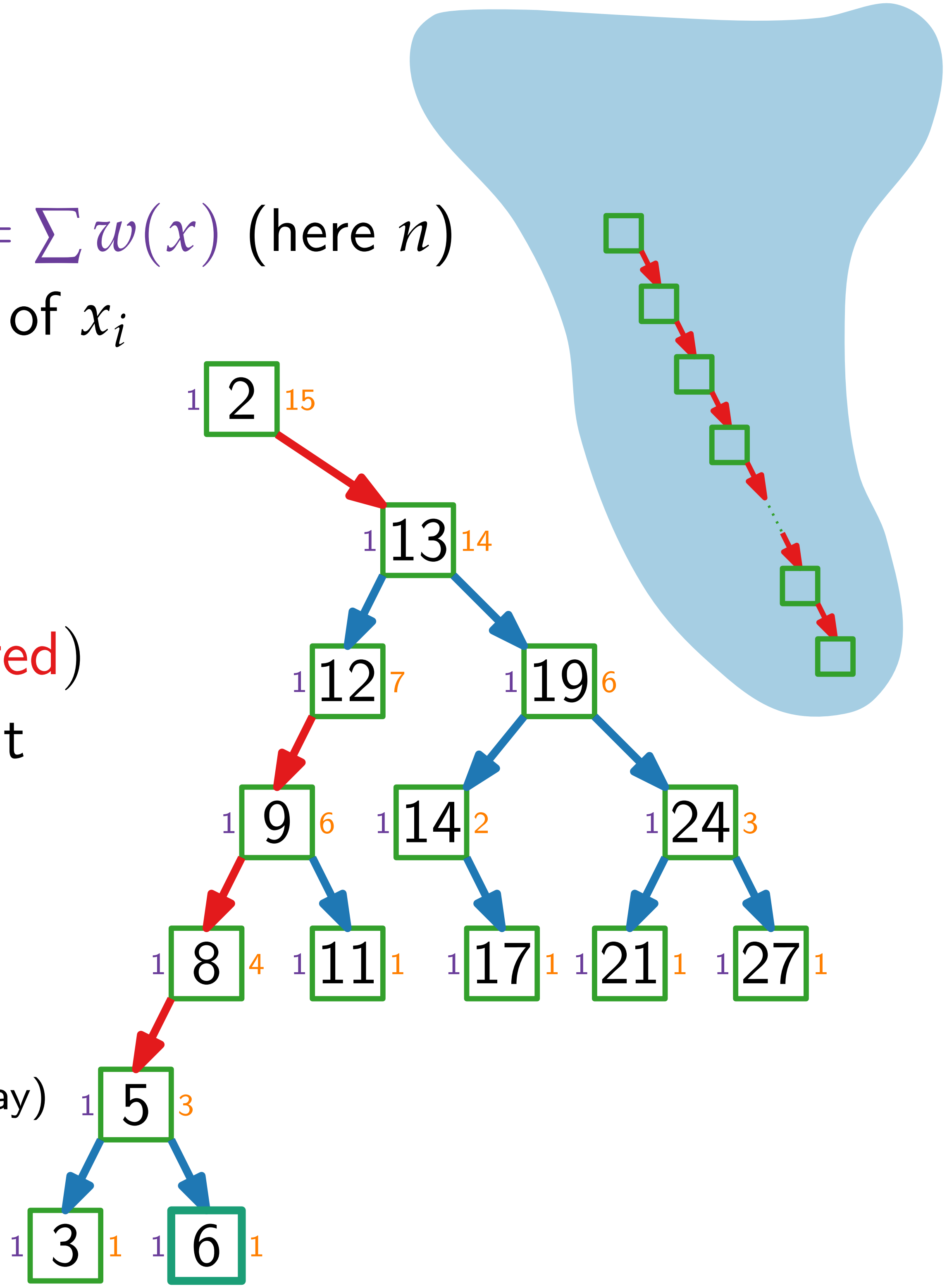

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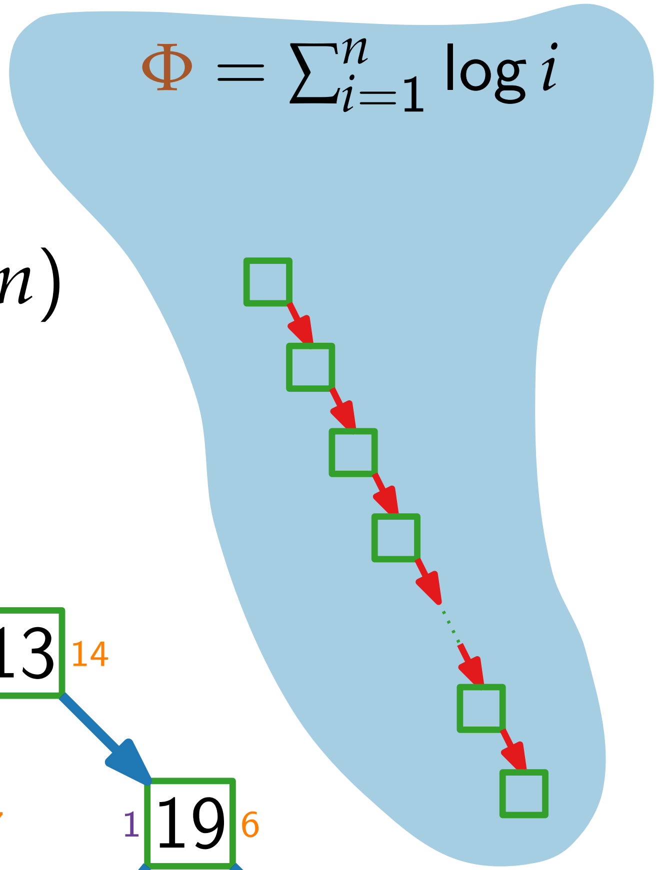

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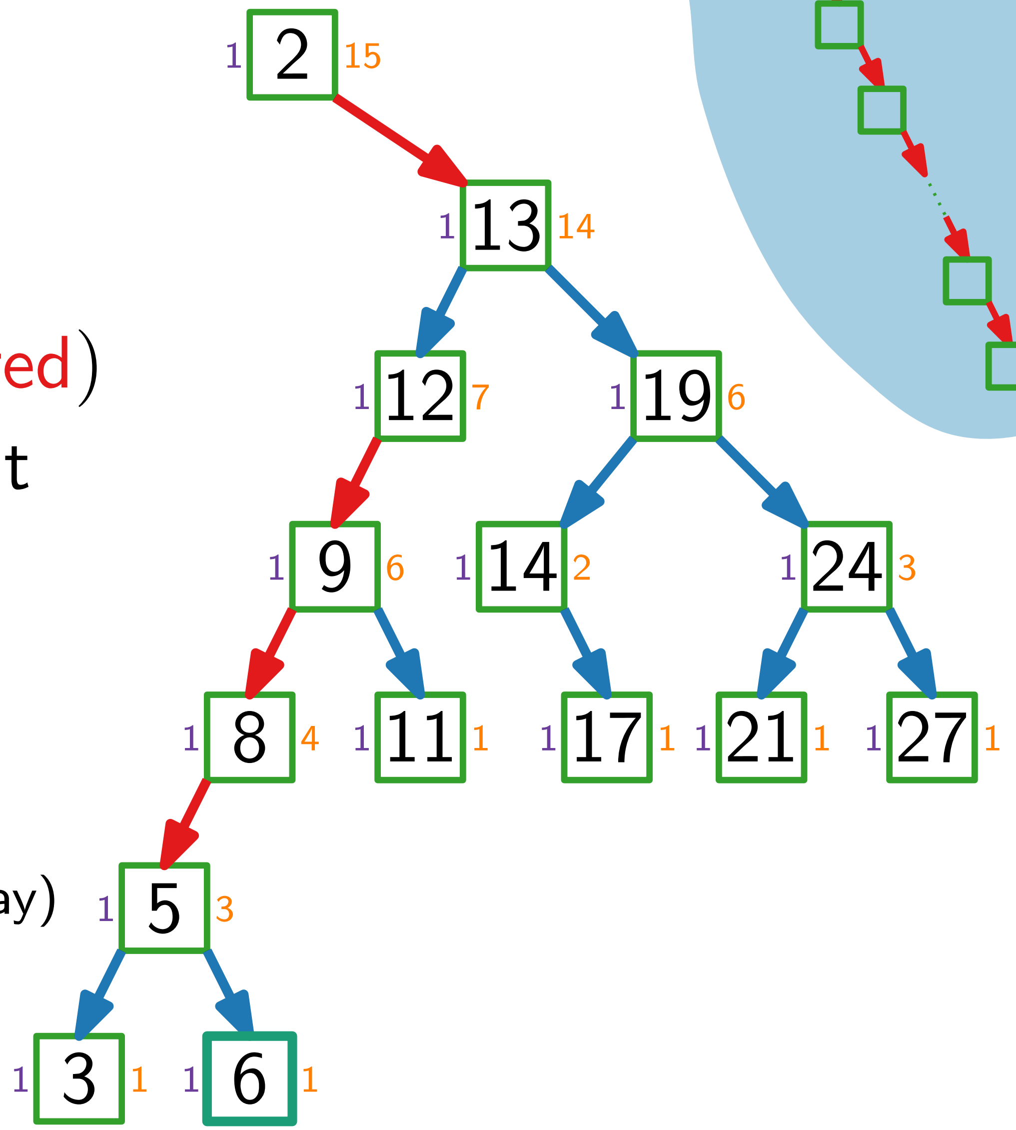

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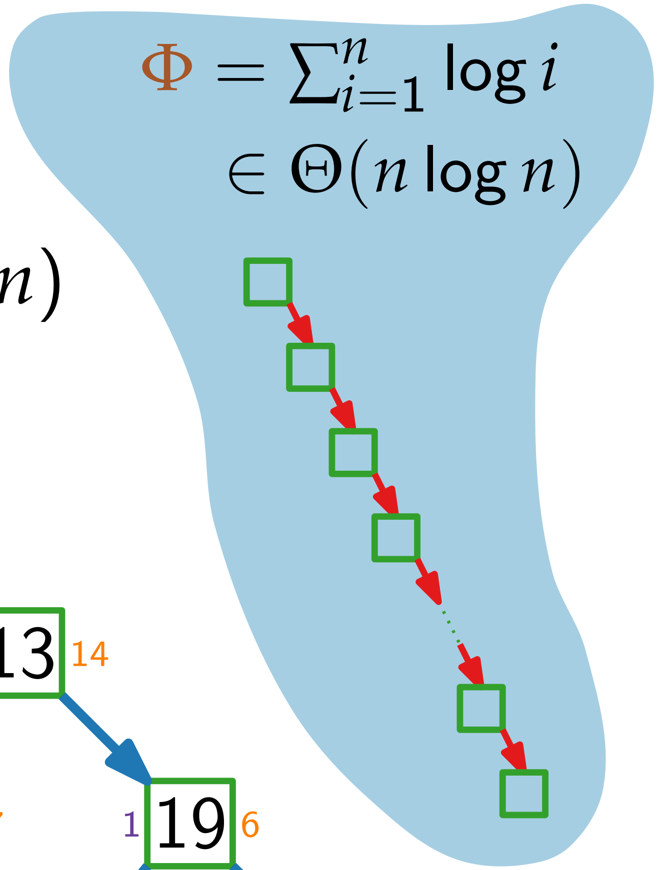

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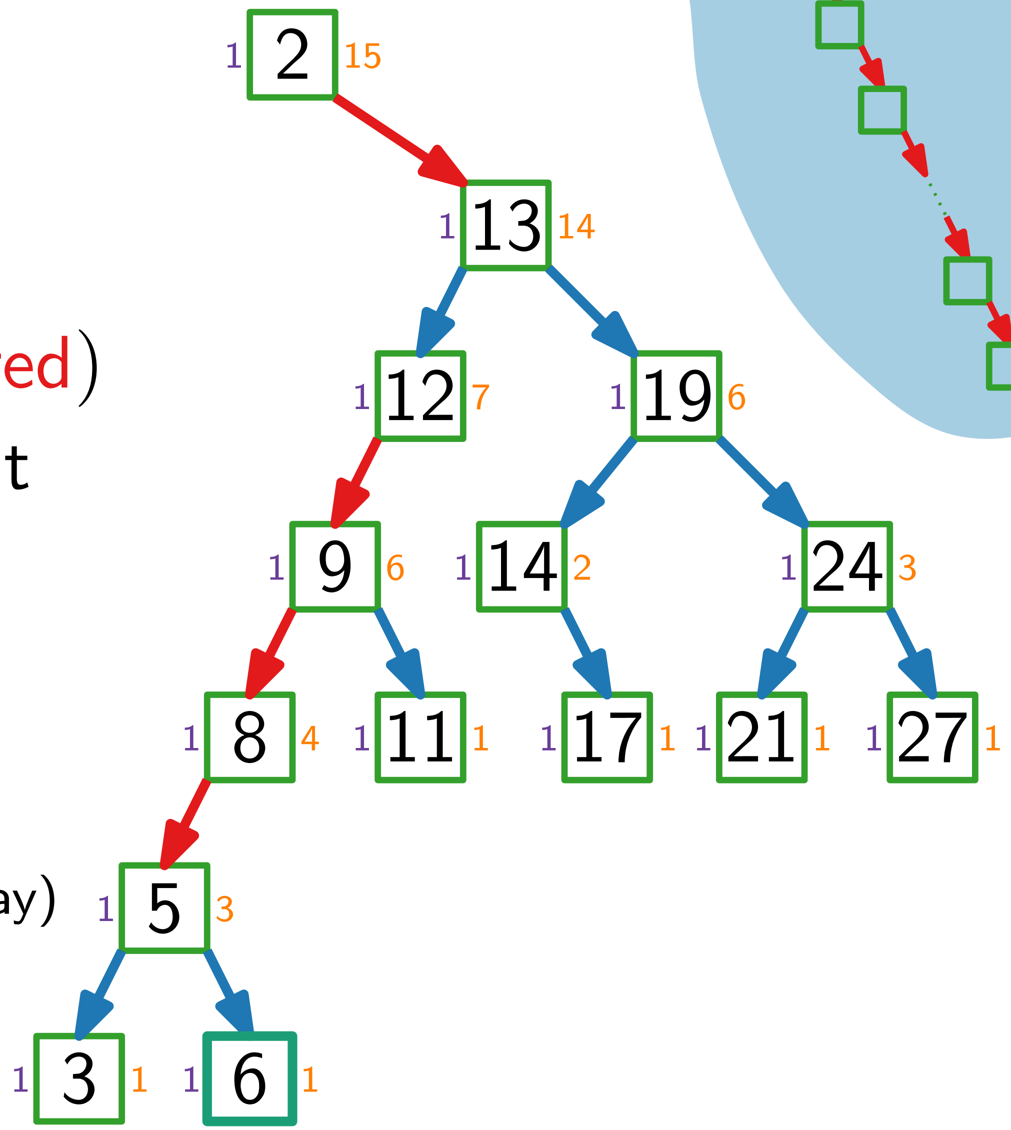

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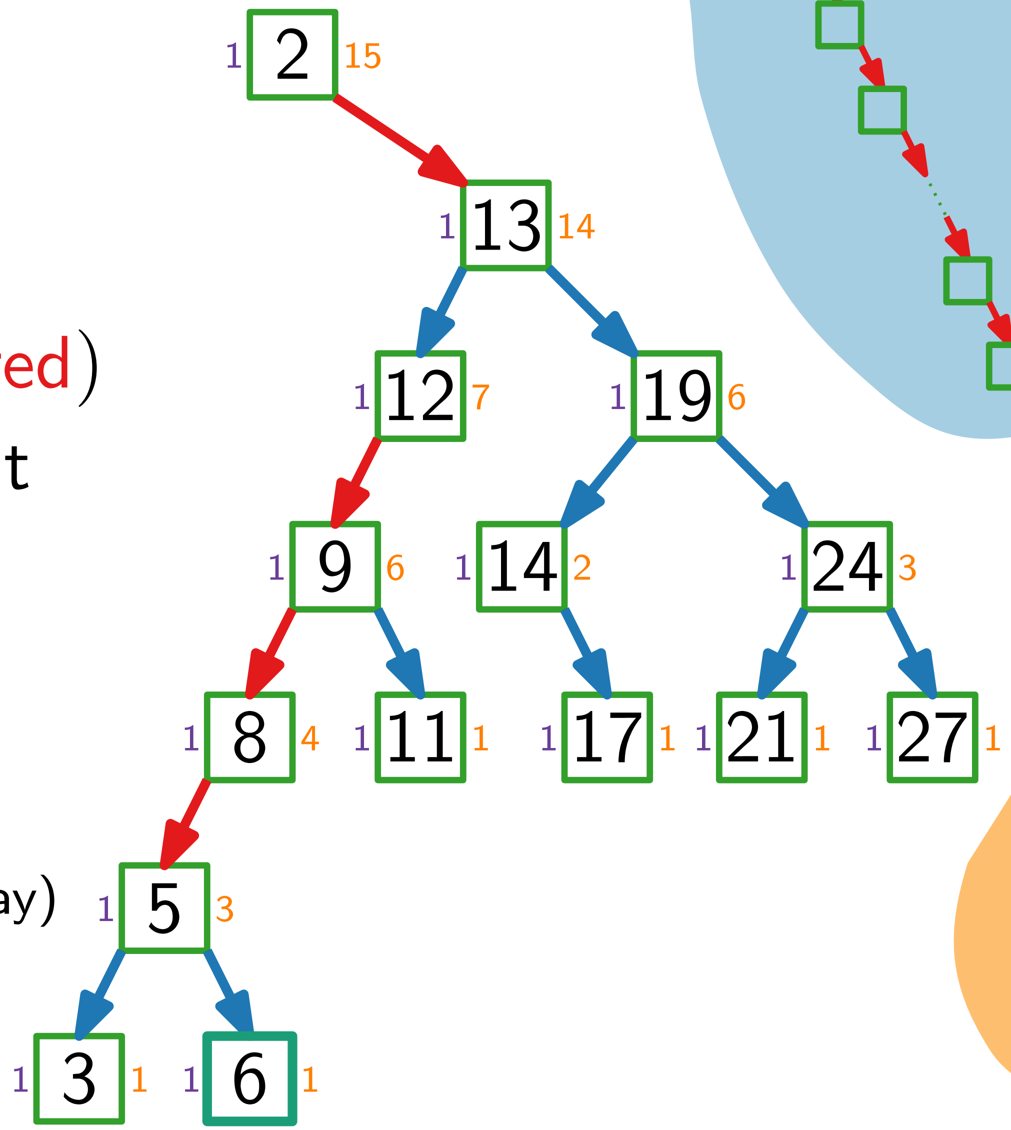
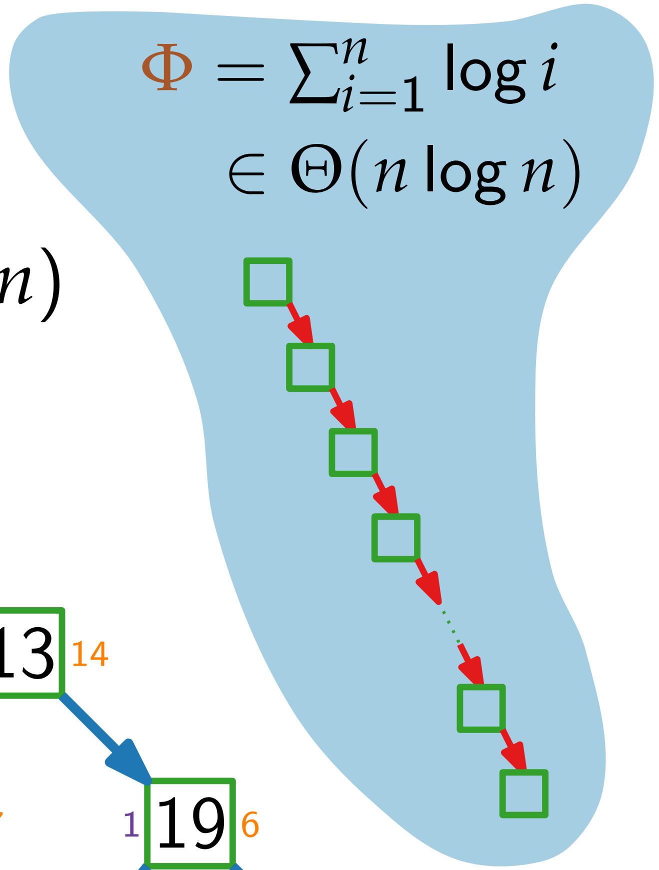
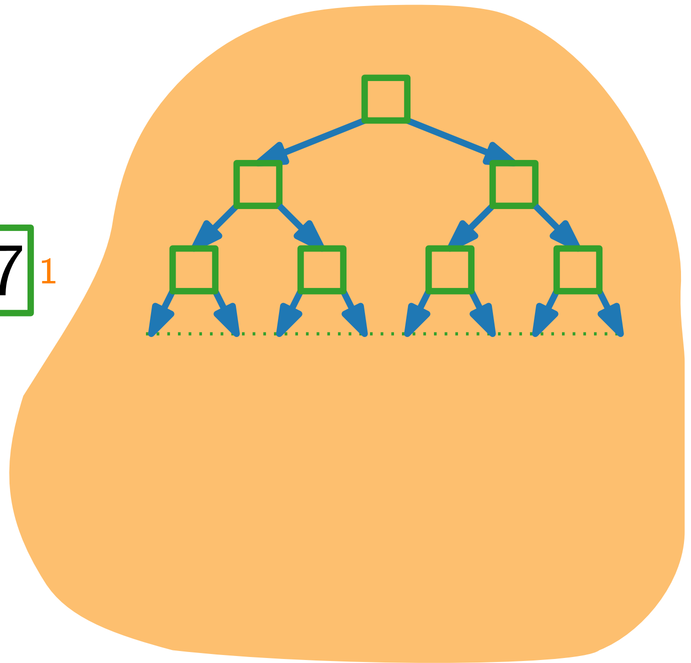


$\Phi = \sum_{i=1}^{n} \log i$

$\in \Theta(n \log n)$





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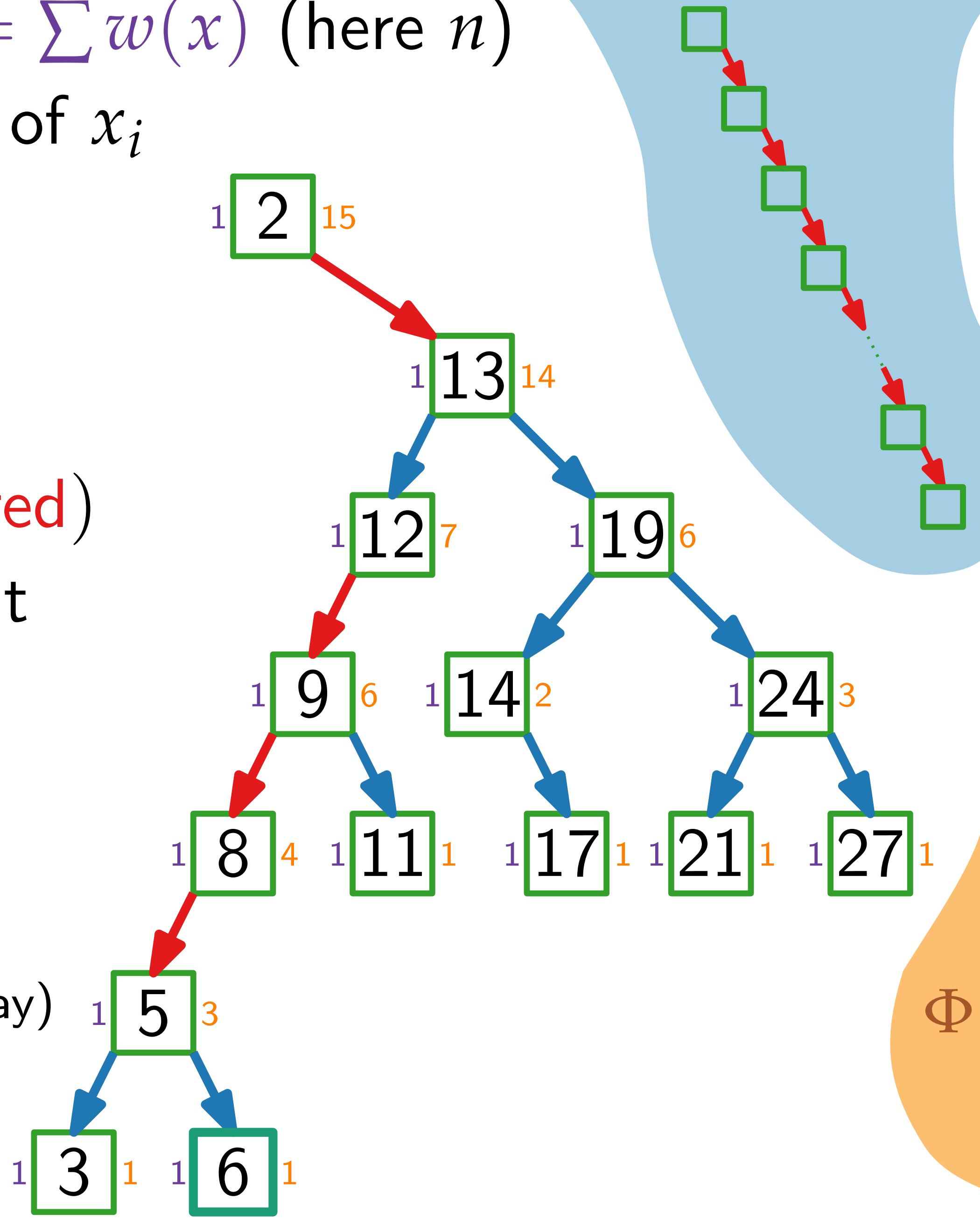

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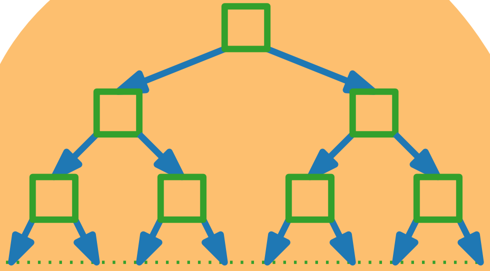

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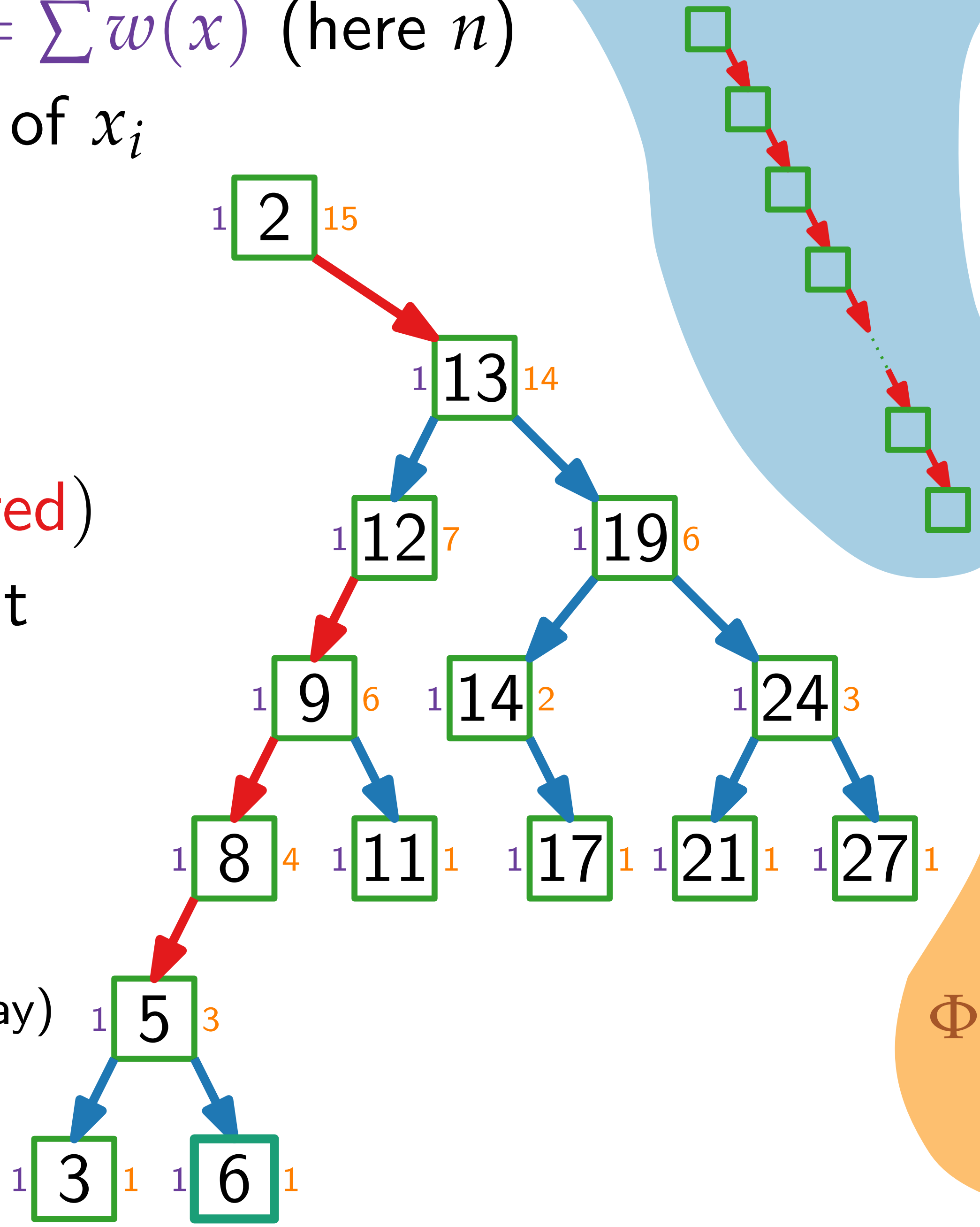

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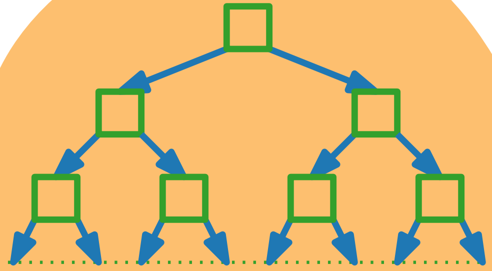

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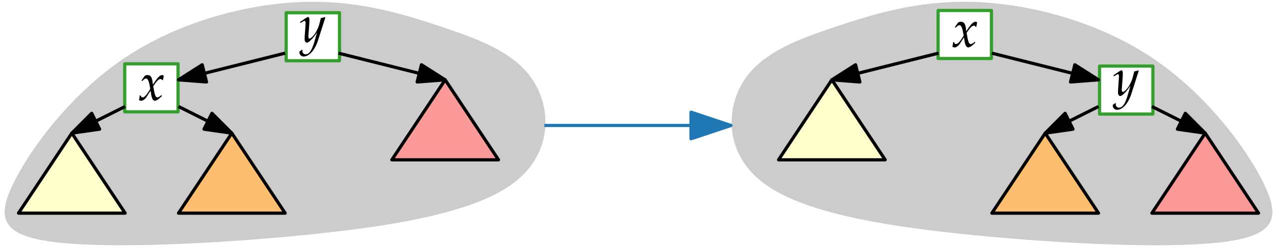

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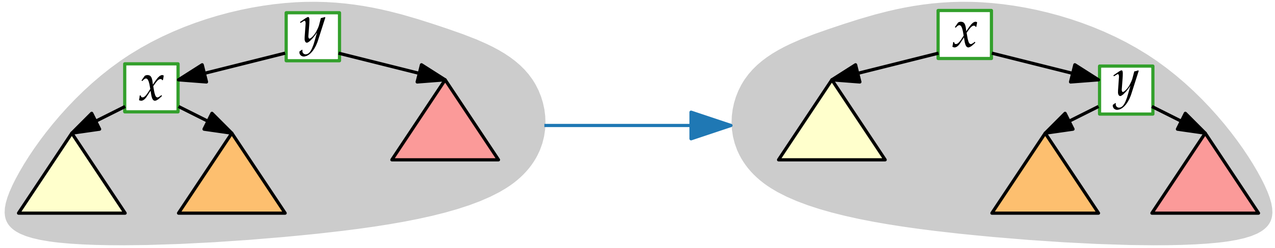

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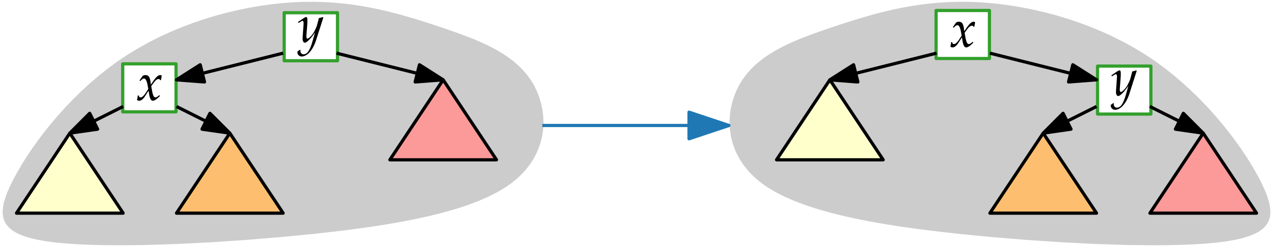

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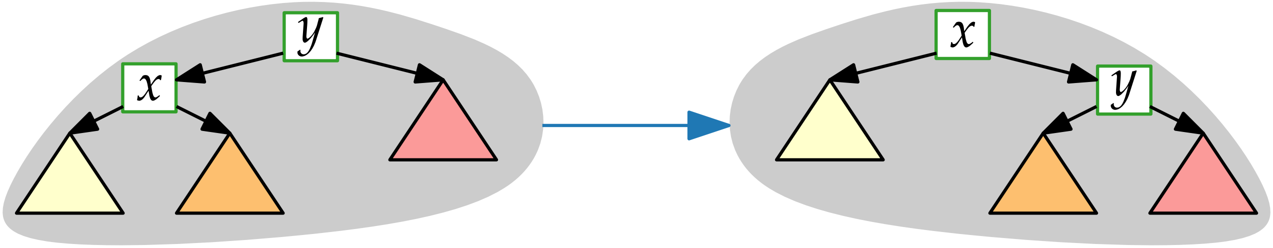

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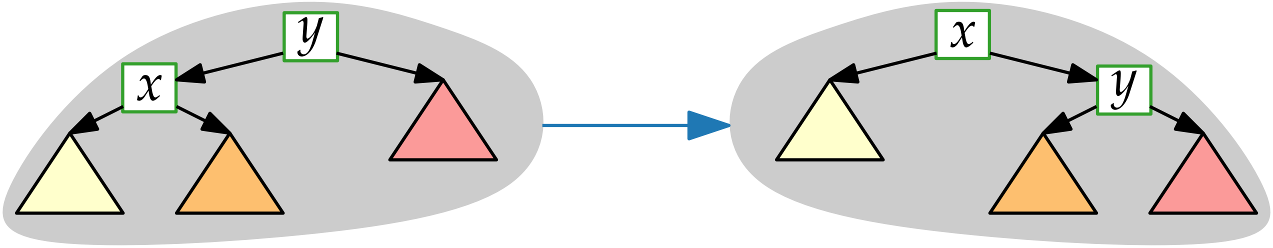

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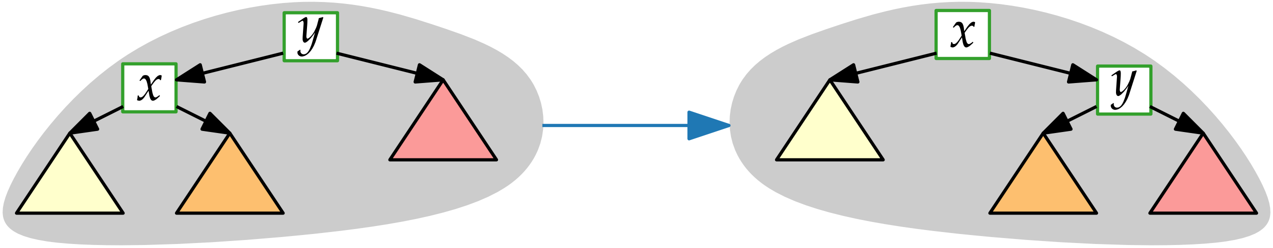

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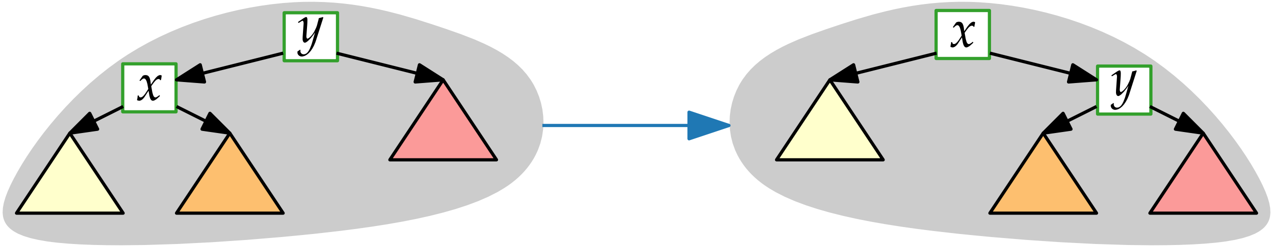

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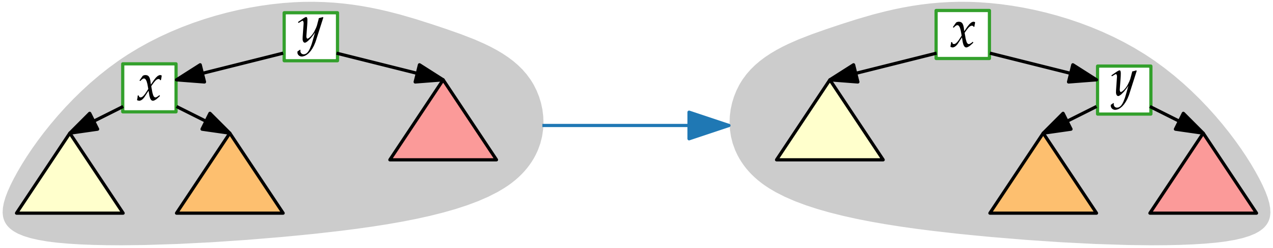

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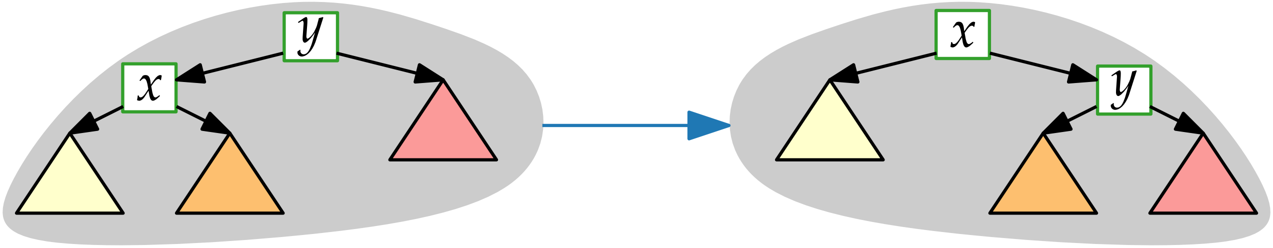

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Left(x) analogue ✓

$\square$

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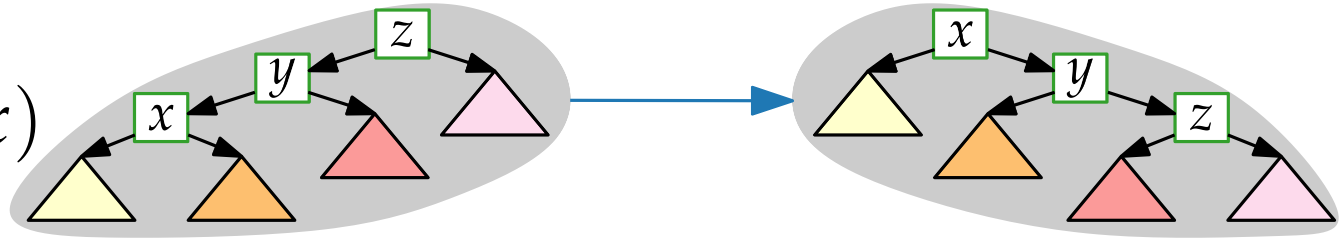
Proof.

Case 1. Right-Right(x)



Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Proof.

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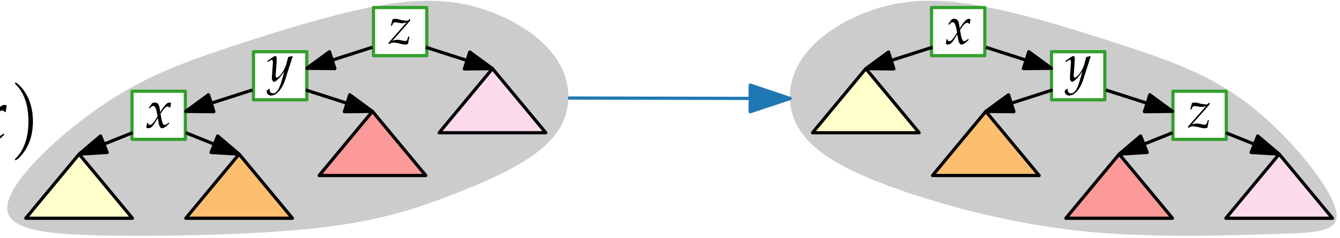

pot. change $= \log s_+(x) + \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) - \log s(z)$

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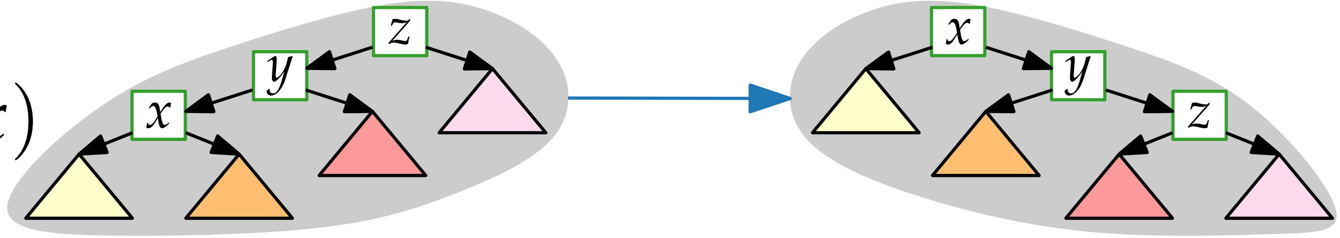

pot. change $= \log s_+(x) + \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) - \log s(z)$

$(s_+(x) = s(z))$

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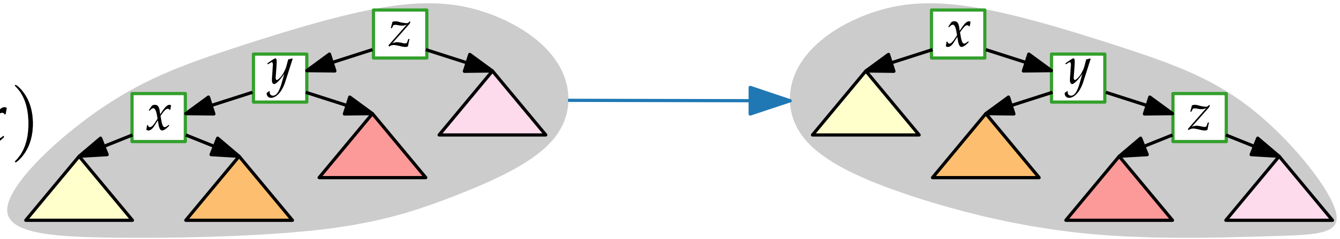

$$\begin{aligned} \text{pot. change} \quad &= \log s_+(x) + \log s_+(y) + \log s_+(z) \\ &\quad - \log s(x) - \log s(y) - \log s(z) \\ (s_+(x) = s(z)) \quad &= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) \end{aligned}$$

Potential after Rotation

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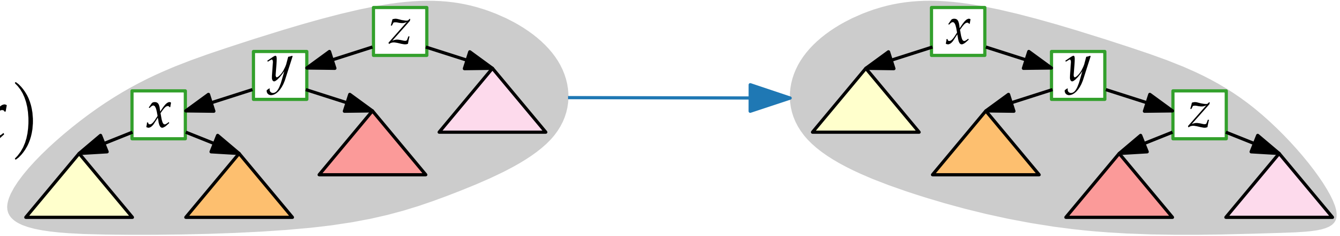

pot. change $= \log s_+(x) + \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) - \log s(z)$

$(s_+(x) = s(z))$ $= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y)$

$(s(x) \leq s(y))$

Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

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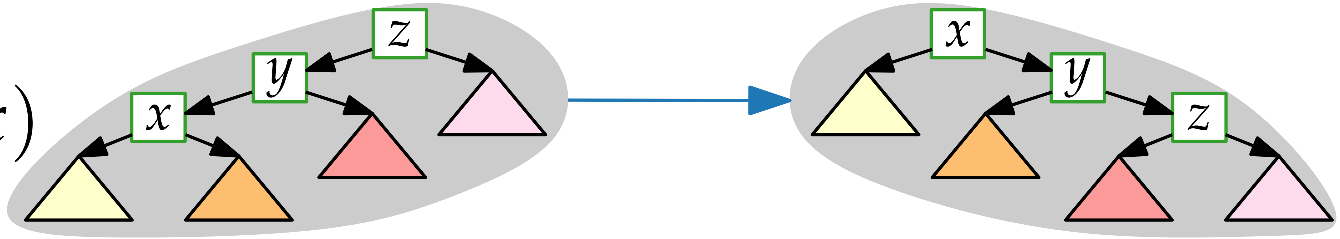

$$
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(s(x) \leq s(y)) \quad &\leq \log s_+(y) + \log s_+(z) - 2\log s(x)
\end{aligned}
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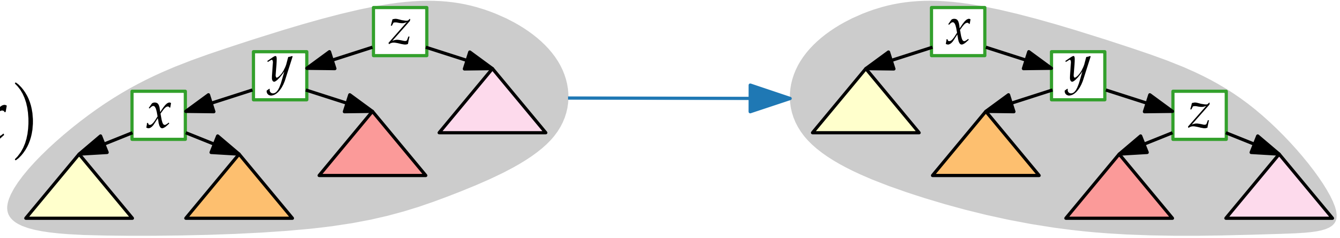

$$
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$(s_+(y) \leq s_+(x))$

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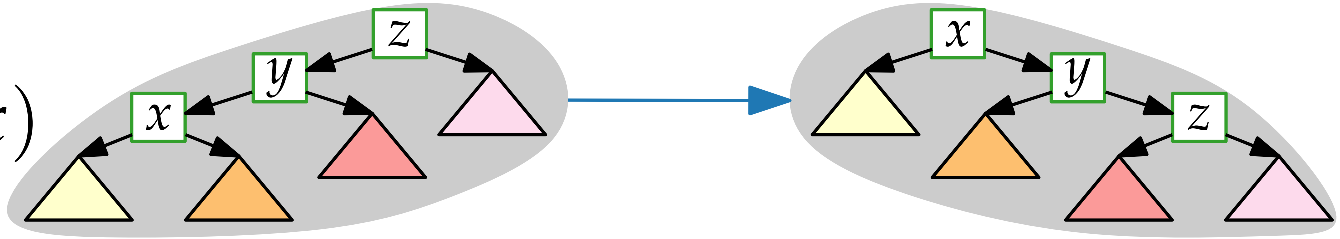

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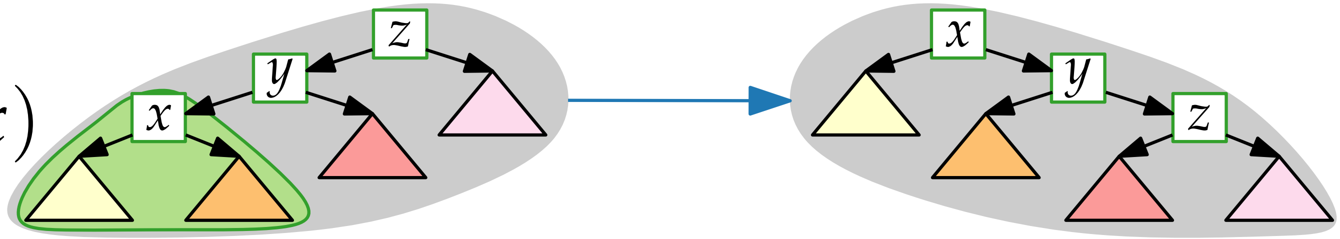

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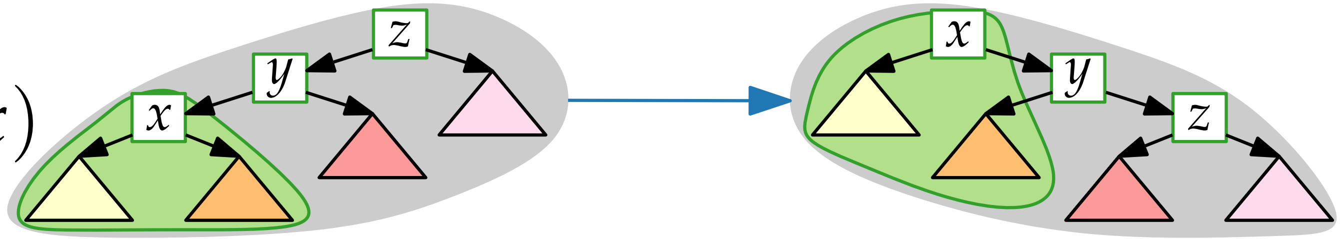

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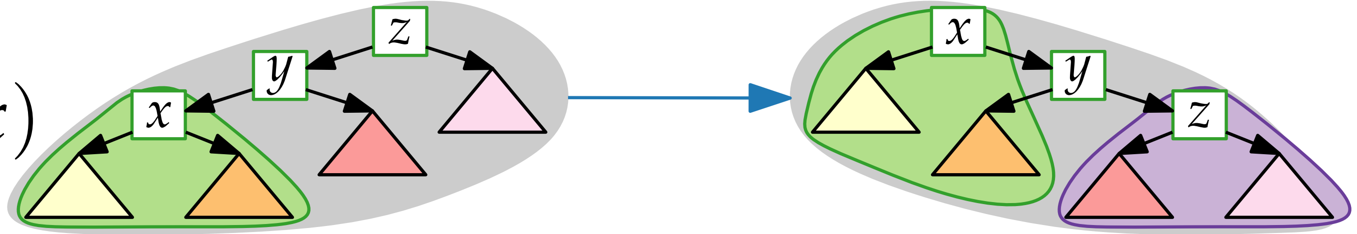

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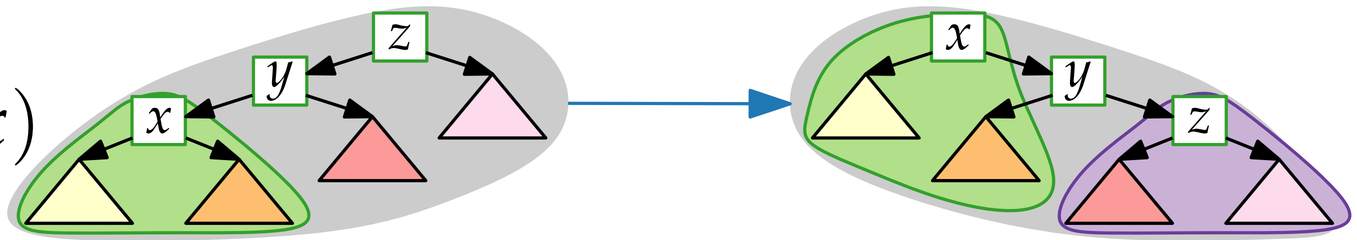

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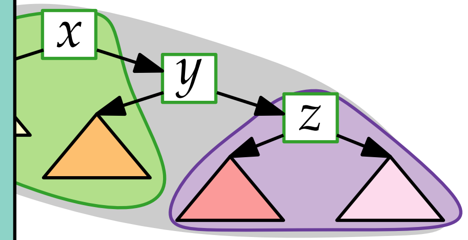

pot. cha

$(s_+(x) =$ … $g\, s(y)$

$(s(x) \leq s$

$(s_+(y) \leq$

Inequality of arithmetic and geometric means (AM-GM):

$$\frac{x_1 + x_2 + \cdots + x_k}{k} \geq \sqrt[k]{x_1 \cdot x_2 \cdot \ldots \cdot x_k}$$

(arithmetic mean) (geometric mean)

for $k = 2$:

$$\frac{x+y}{2} \geq \sqrt{xy} \quad \Rightarrow xy \leq \left(\frac{x+y}{2}\right)^2$$

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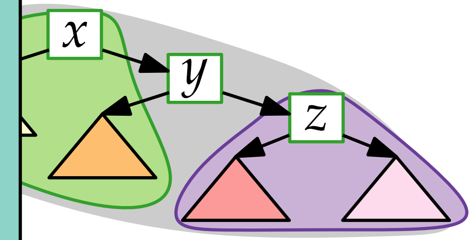

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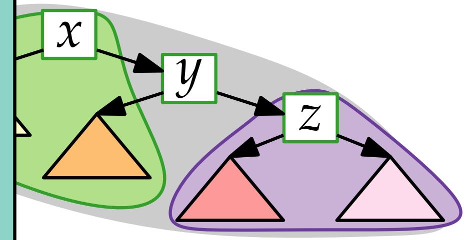

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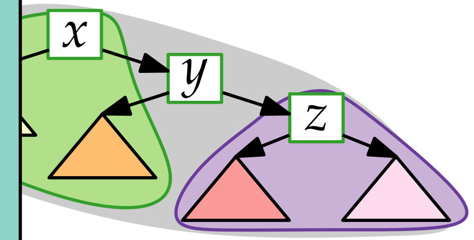

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$$\log s(x) + \log s_+(z) = \log(s(x) s_+(z)) \underset{\text{(AM-GM)}}{\leq} \log(((s(x) + s_+(z))/2)^2)$$

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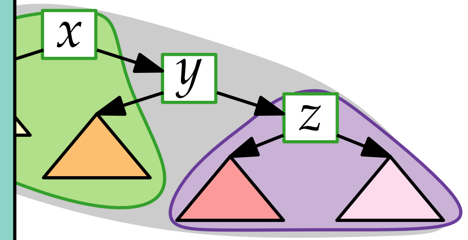

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$\log s(x) + \log s_+(z) = \log(s(x)s_+(z))$

$\underset{\text{(AM-GM)}}{\leq} \log(((s(x) + s_+(z))/2)^2) \underset{(\star)}{\leq} \log((s_+(x)/2)^2)$

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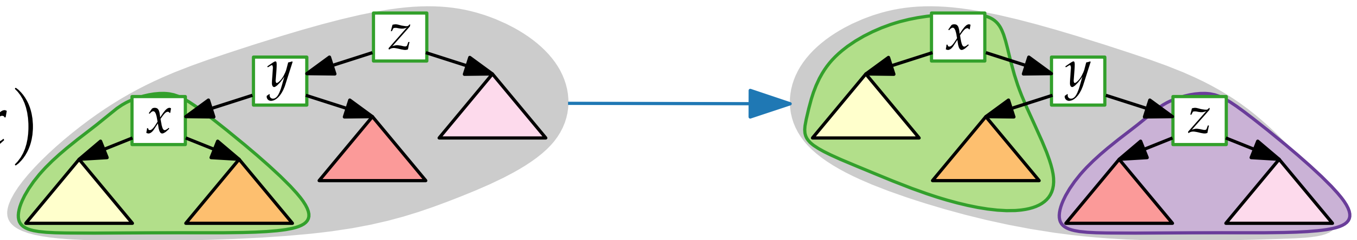

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\text{pot. change} \quad &= \log s_+(x) + \log s_+(y) + \log s_+(z) \\
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(s(x) \leq s(y)) \quad &\leq \log s_+(y) + \log s_+(z) - 2\log s(x) \\
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\end{aligned}$$

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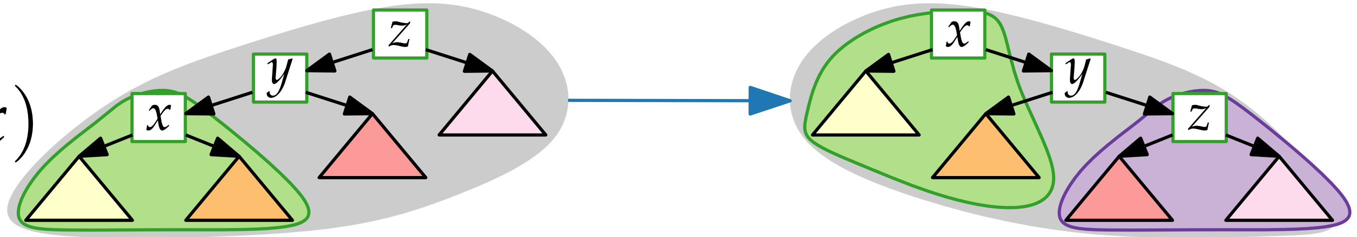

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Potential after Rotation

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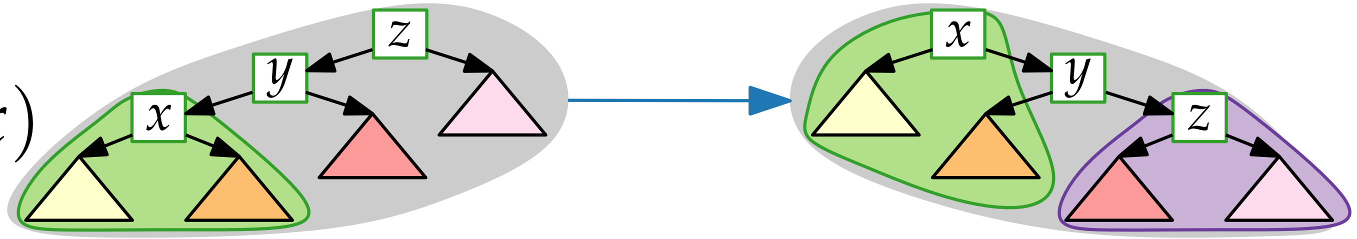

$$
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(s_+(y) \leq s_+(x)) \quad &\leq \log s_+(x) + \log s_+(z) - 2\log s(x) \\
&\leq 3\log s_+(x) - 3\log s(x) - 2
\end{aligned}
$$

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$$
\log s(x) + \log s_+(z) = \log(s(x)s_+(z)) \underset{\text{(AM-GM)}}{\leq} \log(((s(x) + s_+(z))/2)^2) \underset{(\star)}{\leq} \log((s_+(x)/2)^2) = 2\log s_+(x) - 2
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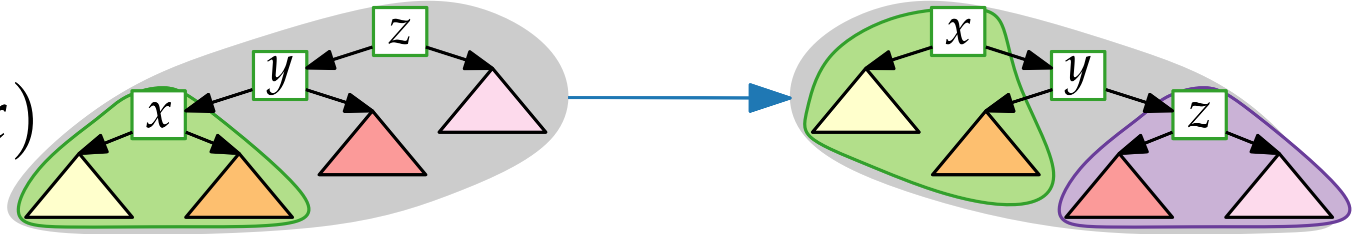

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&\leq 3\log s_+(x) - 3\log s(x) - 2 \quad \checkmark
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$$

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\log s(x) + \log s_+(z) = \log(s(x)s_+(z))
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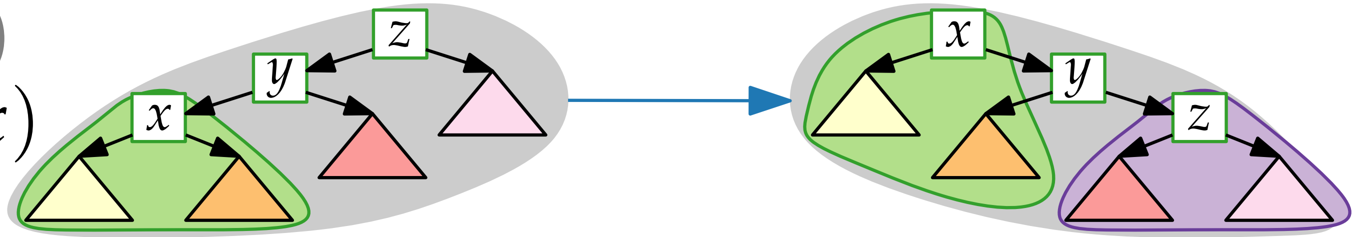
Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Proof. / Left-Left(x)

Case 1. Right-Right(x)



$$
\begin{aligned}
\text{pot. change} \quad & = \log s_+(x) + \log s_+(y) + \log s_+(z) \\
& \quad - \log s(x) - \log s(y) - \log s(z) \\
(s_+(x) = s(z)) \quad & = \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) \\
(s(x) \leq s(y)) \quad & \leq \log s_+(y) + \log s_+(z) - 2\log s(x) \\
(s_+(y) \leq s_+(x)) \quad & \leq \log s_+(x) + \log s_+(z) - 2\log s(x) \\
& \leq 3\log s_+(x) - 3\log s(x) - 2 \quad \checkmark
\end{aligned}
$$

$(\star)$: $s(x) + s_+(z) \leq s_+(x)$

$$\log s(x) + \log s_+(z) = \log(s(x)s_+(z))$$

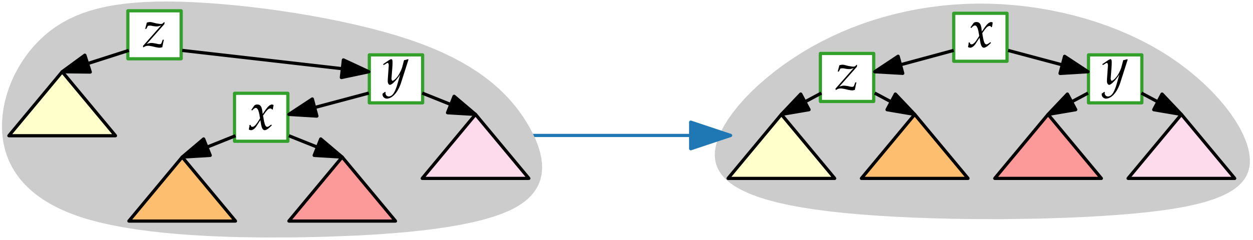
$$\underset{\text{(AM-GM)}}{\leq} \log(((s(x) + s_+(z))/2)^2) \underset{(\star)}{\leq} \log((s_+(x)/2)^2) = 2\log s_+(x) - 2$$

Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Proof.
Case 2. Right-Left(x)



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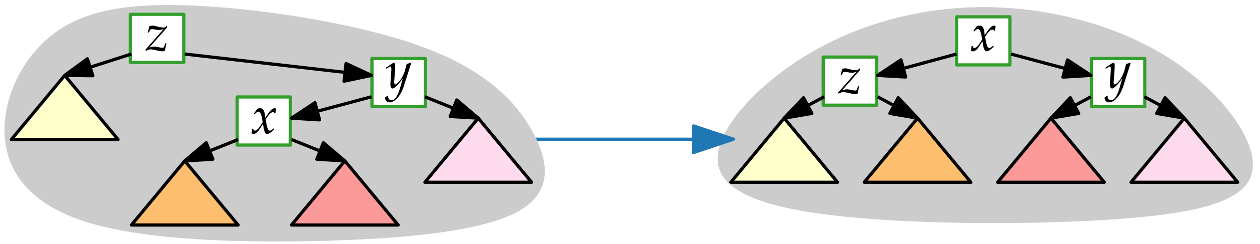

pot. change $= \log s_+(x) + \log s_+(y) + \log s_+(z)$
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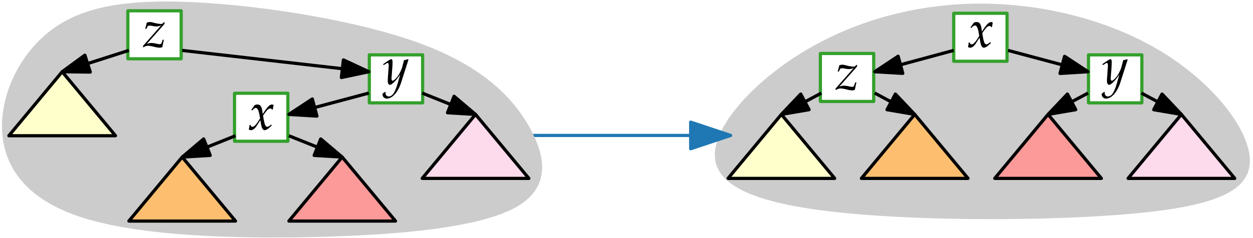

$$\begin{aligned}
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&\quad - \log s(x) - \log s(y) - \log s(z) \\
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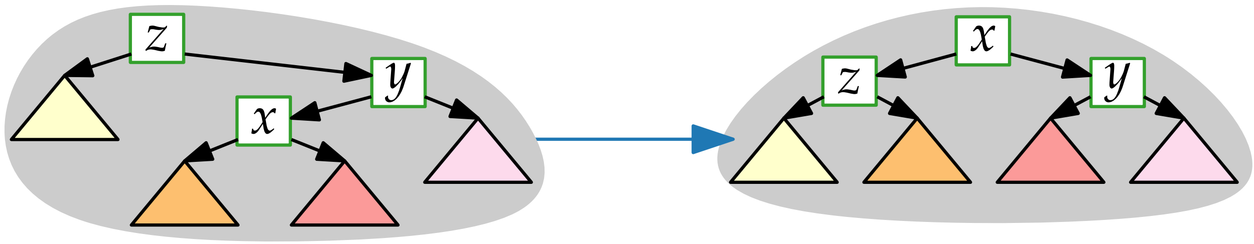

pot. change $= \log s_+(x) + \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) - \log s(z)$

$(s_+(x) = s(z))$ $= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y)$

$(s(x) \leq s(y))$

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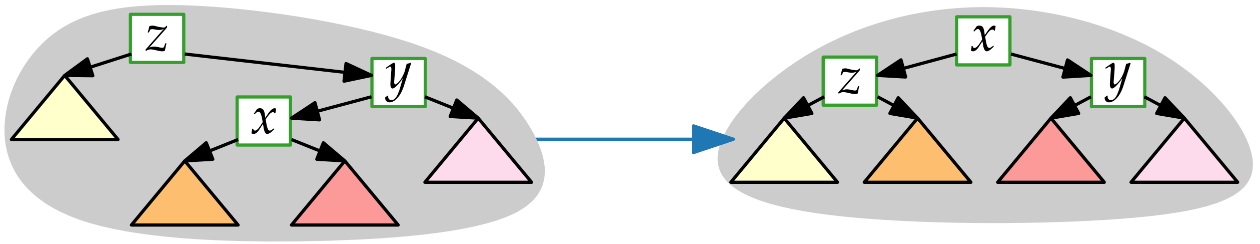

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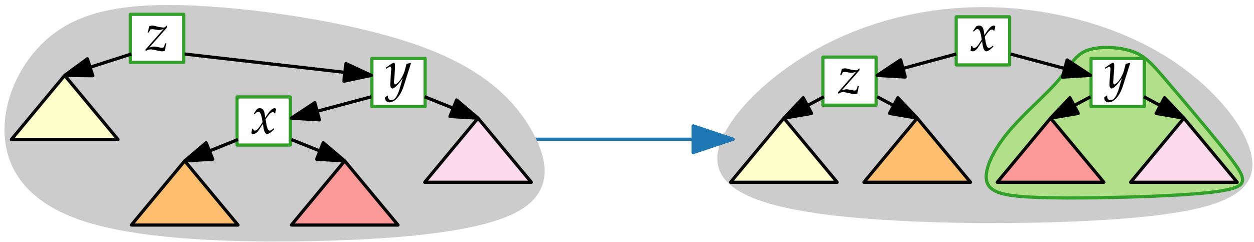

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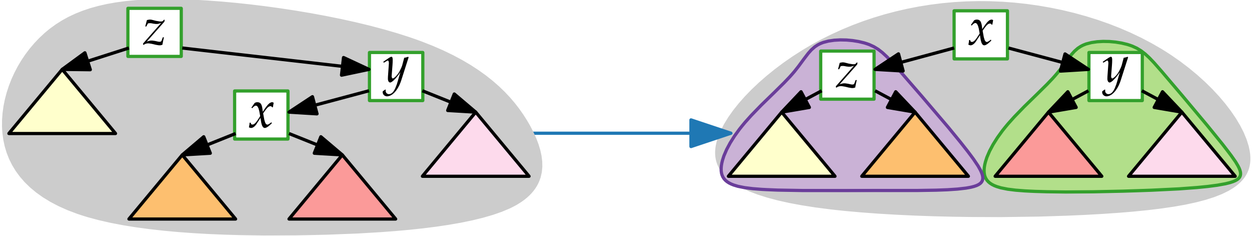

$$
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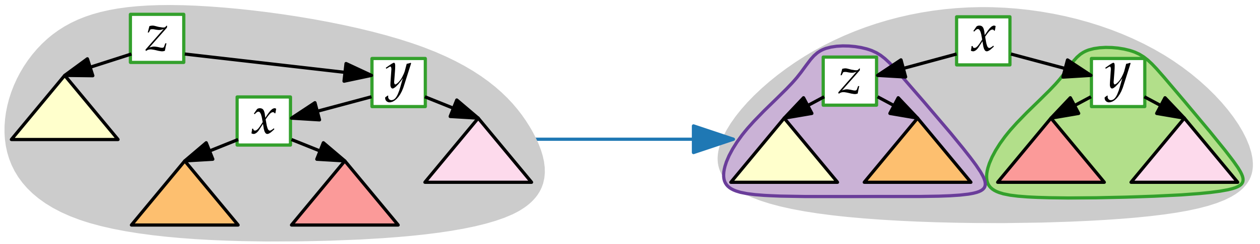

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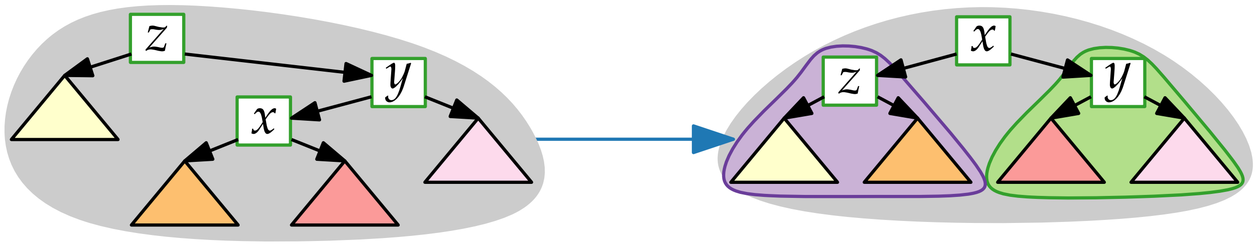

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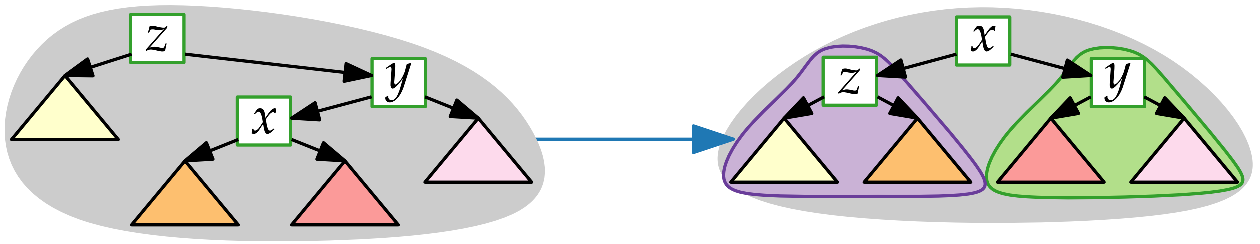

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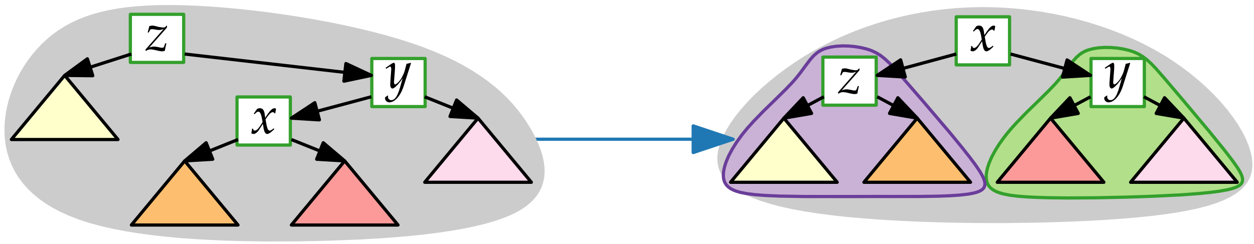

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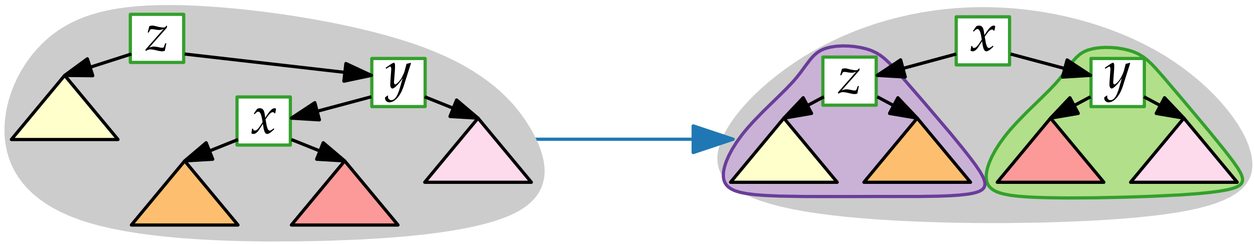

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\end{aligned}$$

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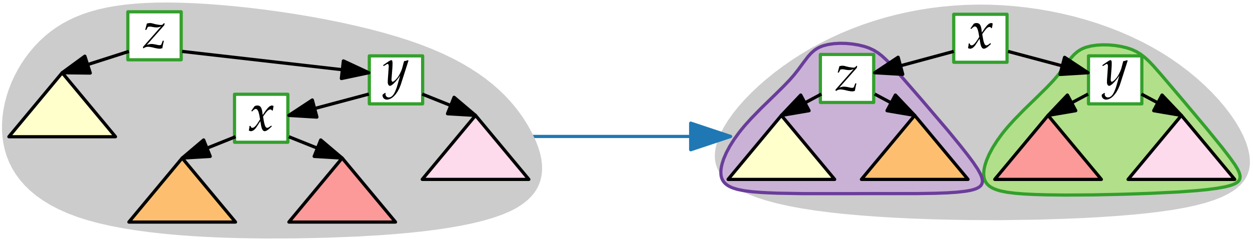

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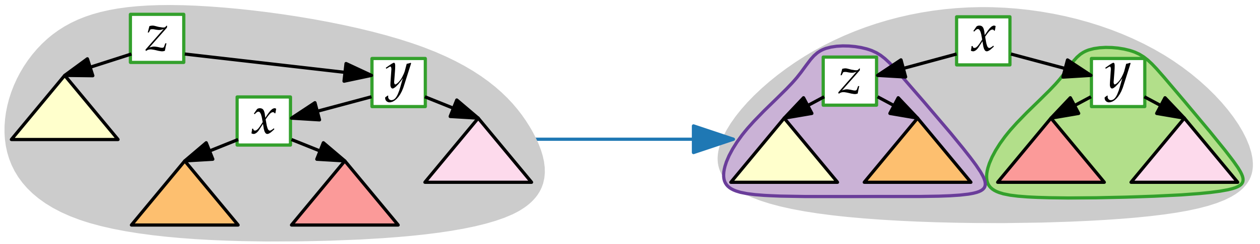

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Proof. / Left-Right(x)

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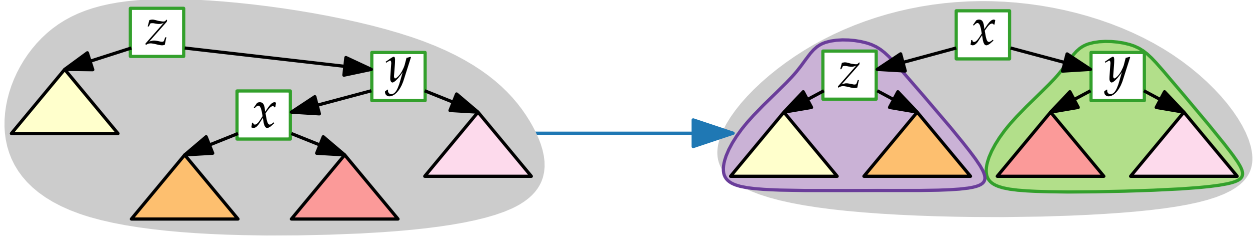

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Access Lemma

Lemma. After a single rotation, the potential increases by $\leq 3\left(\log s_+(x) - \log s(x)\right)$.
After a double rotation, the potential increases by $\leq 3\left(\log s_+(x) - \log s(x)\right) - 2$.

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$\sum_{i=1}^{k}\left(3\left(\log s_i(x) - \log s_{i-1}(x)\right) - 2\right)$

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(id. entries rem.) $= 3\,(\log s_{k+1}(x) - \log s(x)) - 2k$

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root!

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$$
\begin{aligned}
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& +3\,(\log s_{k+1}(x) - \log s_k(x)) \\
\text{(id. entries rem.)} \quad & = 3\,(\log \underbrace{s_{k+1}(x)}_{\text{root!}} - \log s(x)) - 2k \\
& = 3\,(\log W - \log s(x)) - 2k
\end{aligned}
$$

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Lemma. After a single rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x))$.
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Proof. W.l.o.g. k double rotations and 1 single rotation.
Let $s_i(x)$ be $s(x)$ after i single/double rotations.
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All These Models . . .

Balanced: Queries take (amortized) $O(\log n)$ time

Entropy: Queries take expected $O(1 + H)$ time

Dynamic Finger: Queries take $O(\log \delta_i)$ time (δ_i: rank diff.)

Working Set: Queries take $O(\log t)$ time (t: recency)

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Yes!

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All of these properties can be shown by chosing the weight function accordingly.
Note that the actual algorithm is always the same!

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Let S be a sequence of queries.

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$\Rightarrow$ as long as every key is queried at least once, it doesn't change the asymptotic running time.

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Lemma. The (amortized) cost of Splay(x) is $c(\text{Splay}(x)) \leq 1 + 3\log(W/w(x))$.

Definition. A BST has the **entropy property** if queries take expected $O(1 - \sum_{i=1}^{n} p_i \log p_i)$ time.

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$$O(\textstyle\sum_{i=1}^{n} p_i(1 - 3\log p_i)) \;= O(1 - \sum_{i=1}^{n} p_i \log p_i) \qquad \square$$

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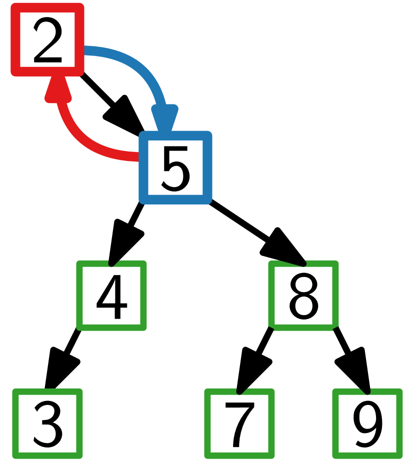

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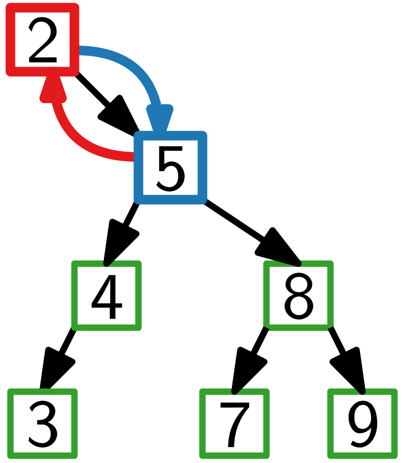

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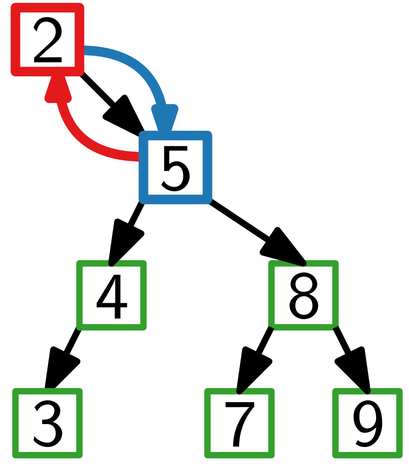

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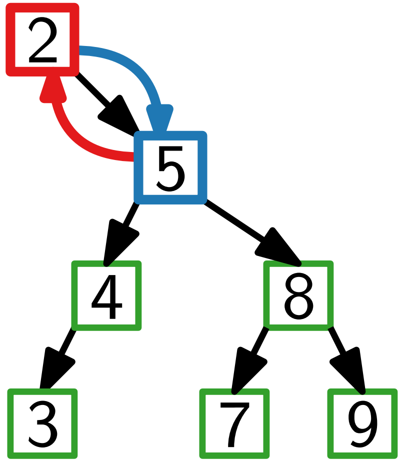

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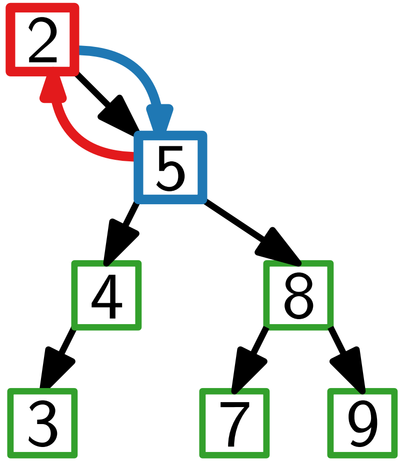

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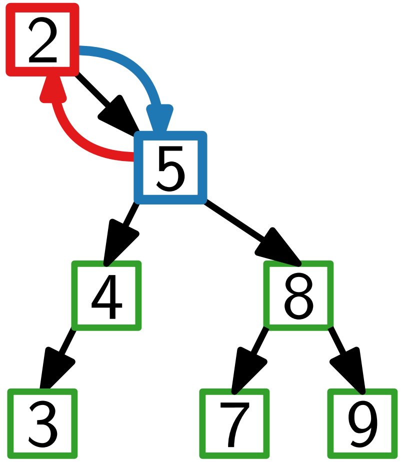

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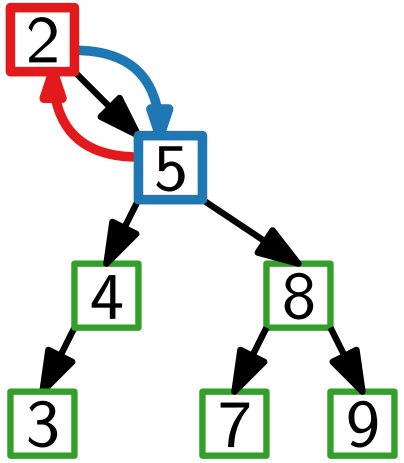

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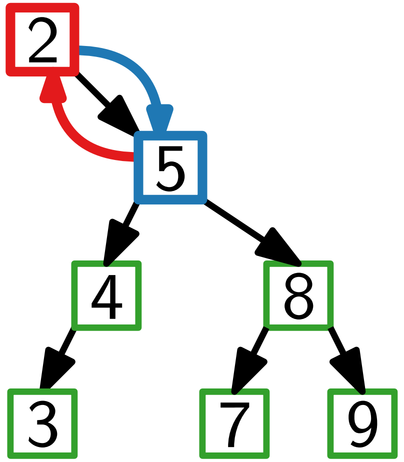

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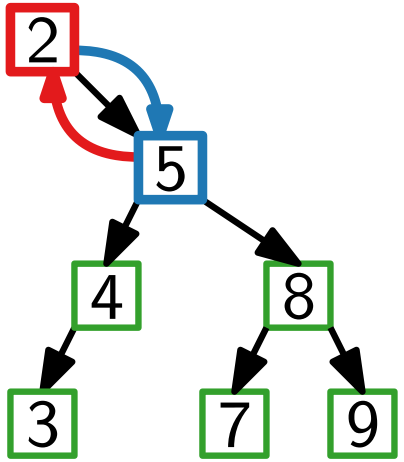

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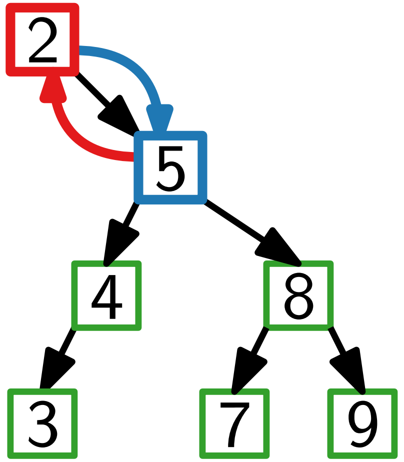

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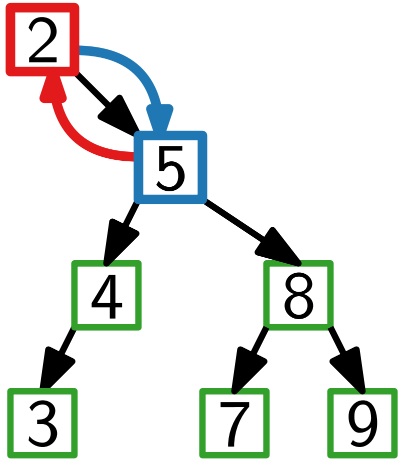

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Conjecture. Splay Trees are dynamically optimal.