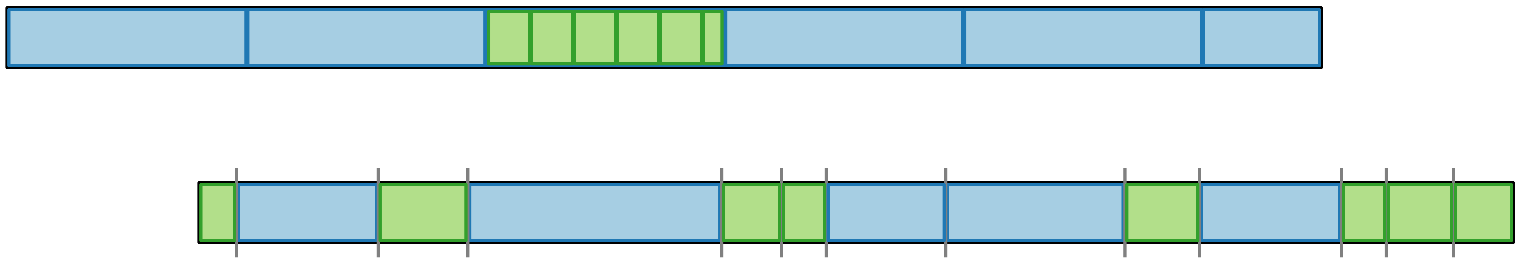
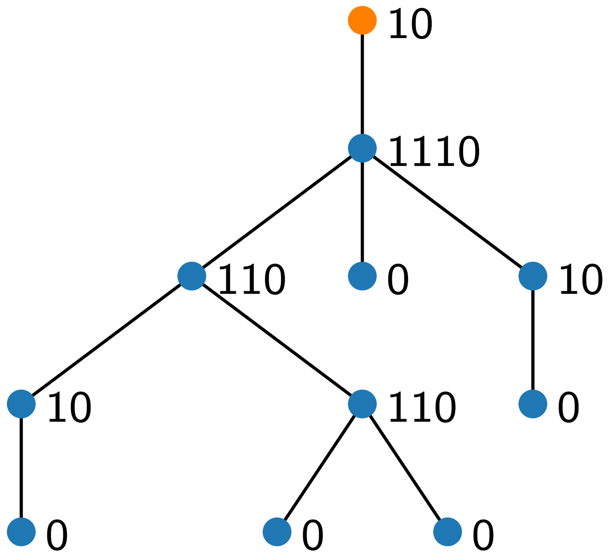
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Advanced Algorithms

Succinct Data Structures

Indexable Dictionaries and Trees





Alexander Wolff

Summer term 2026

Data Structures – Informal Definition

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Remarks.

- We look at data structures as a designer/implementer (and not necessarily as a user).
- To define a data structure and to implement it are two different tasks.

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$\Rightarrow$

- What do we represent?
- How much space is required?
- Dynamic or static?
- Which operations are defined?
- How fast are they?

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Goal.

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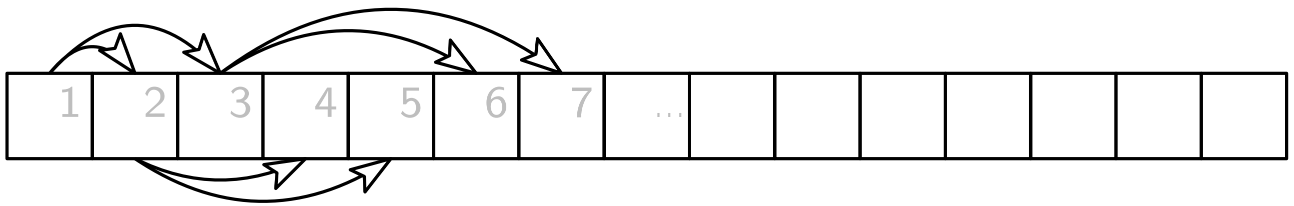

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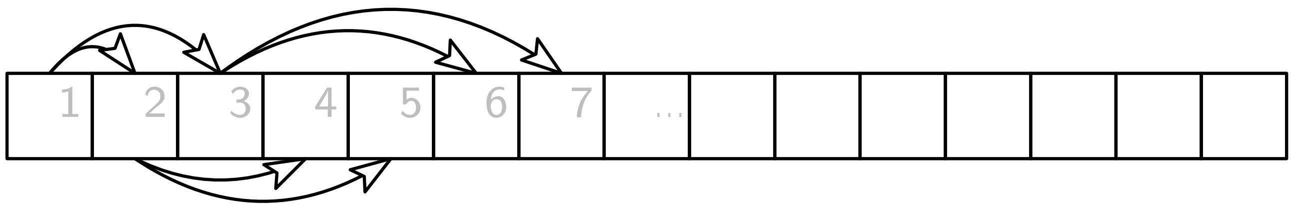

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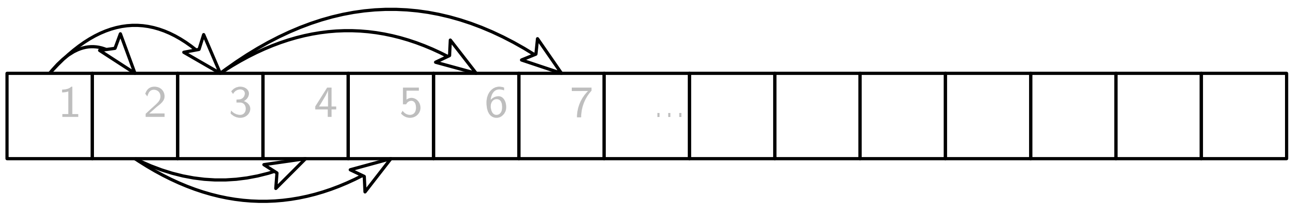

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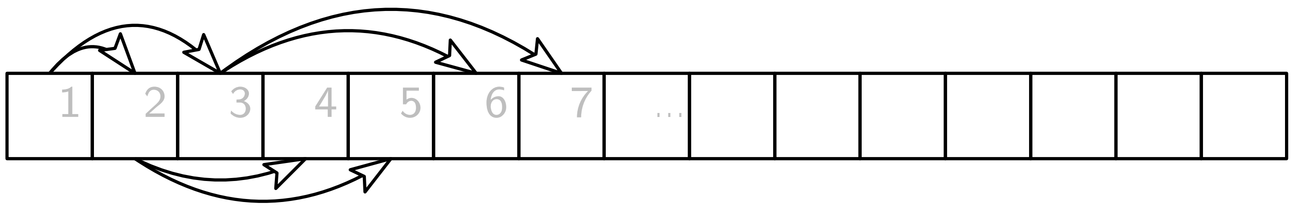

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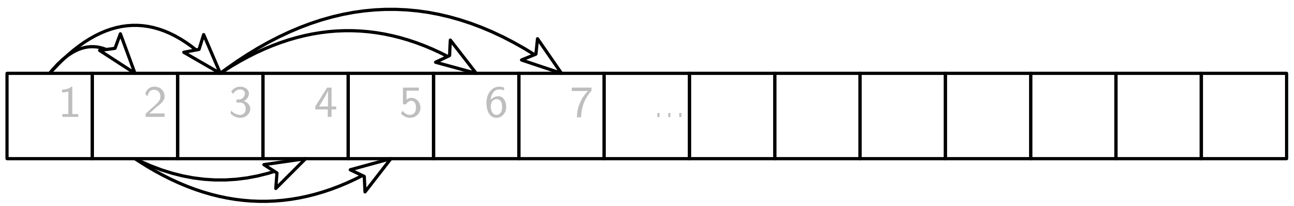

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And unbalanced trees?

Succinct Indexable Dictionary

Represent a subset $S \subseteq \{1, 2, \ldots, n\}$ and support the following operations in $O(1)$ time:

- $\texttt{member}(i)$ returns true iff $i \in S$
- $\texttt{rank}(i) =$ number of elements in S that are at most i
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How many bits of space do we need to distinguish them?

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$\log 2^n = n$ **bits**

our logarithms are all base 2, i.e., $\log_2$

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$$S = \{3, 4, 6, 8, 9, 14\} \text{ where } n = 15$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
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$\texttt{rank}(9) =$

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$\texttt{select}(5) = 9$

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$\texttt{select}(5) = 9$

$\texttt{rank}(9) = 5 = \texttt{rank}(12)$

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$\Rightarrow$ Exercise: Use these methods to answer $\texttt{predecessor}(i)$ and $\texttt{successor}(i)$ in $O(1)$ time.

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b 

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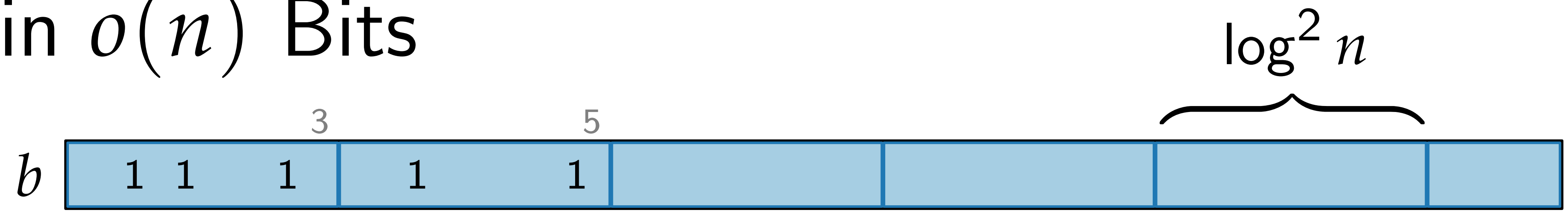

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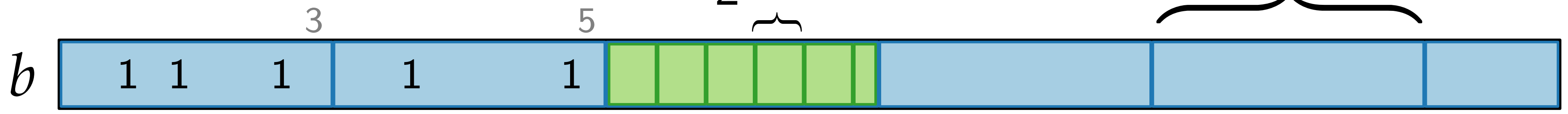

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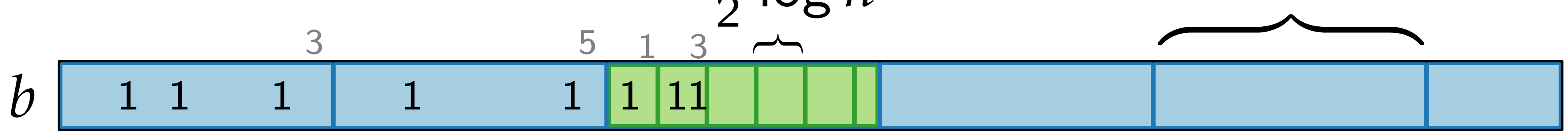

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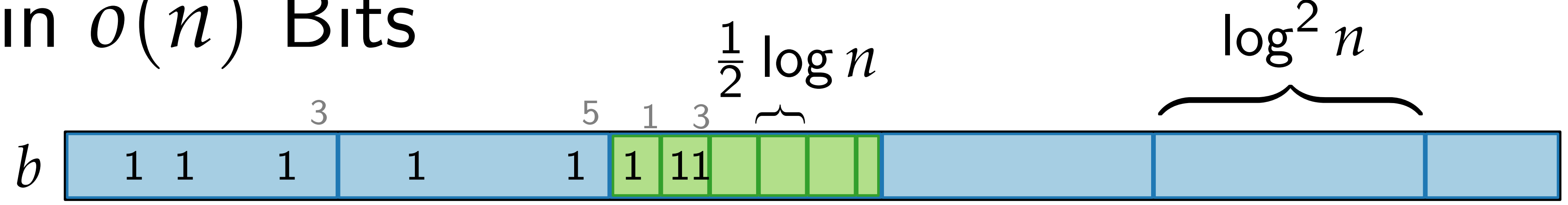

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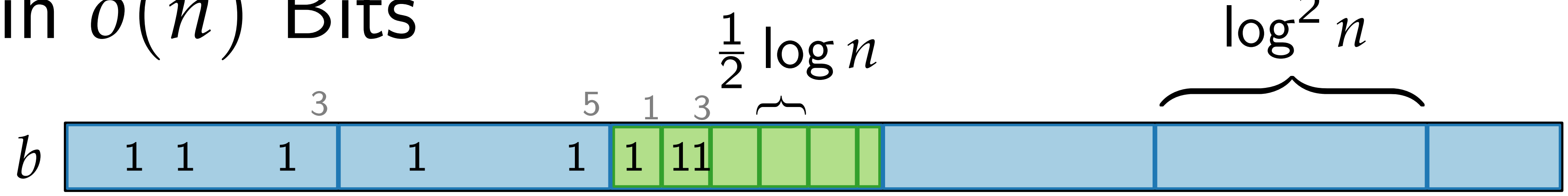

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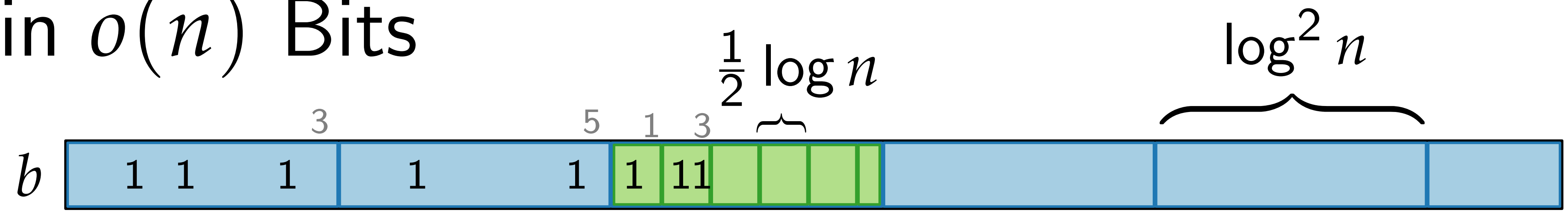

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   $$\Rightarrow O\left(\frac{n}{\log n}\log\log n\right) \subseteq o(n) \text{ bits}$$

3. Use **lookup table** for bitstrings of length $(\frac{1}{2}\log n)$:

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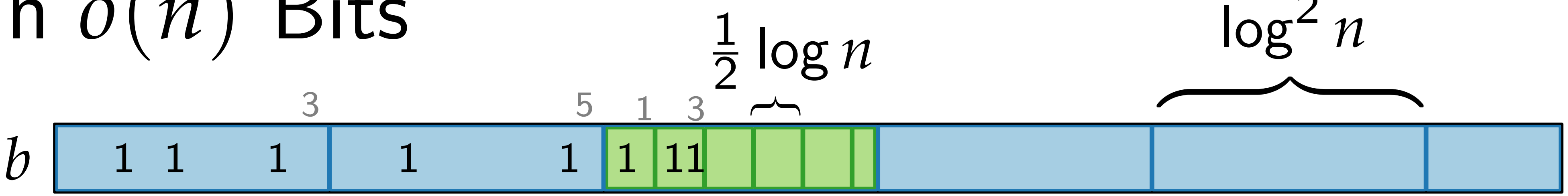

Example: $n = 64 \Rightarrow \frac{1}{2}\log n = 3$

1. Split

   and s $\leq \log n$ bits

   $O(\frac{n}{\log^2 n}\log n) = O(\frac{n}{\log n}) \subseteq o(n)$ bits

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   and each needs $\leq \log\log^2 n = 2\log\log n$ bits

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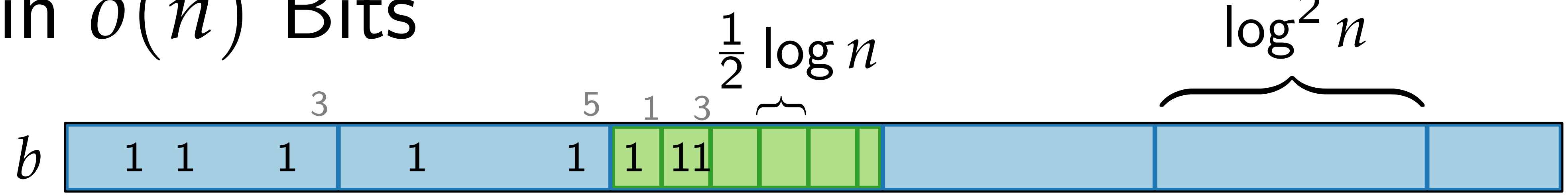
Rank in $o(n)$ Bits

$\log^2 n = (\log n)^2$



1. Split

   and s $\leq \log n$ bits

   $O(\frac{n}{\log^2 n} \log n) = O(\frac{n}{\log n}) \subseteq o(n)$ bits

Example: $n = 64 \Rightarrow \frac{1}{2}\log n = 3$

position $\rightarrow$	1	2	3
bitstring 000	0	0	0
001	0	0	1
010	0	1	1
011	0	1	2
100	1	1	1
101	1	1	2
110	1	2	2
111	1	2	3

2. Split ks

   and each needs $\leq \log \log^2 n = 2 \log \log n$ bits

   $\Rightarrow O(\frac{n}{\log n} \log \log n) \subseteq o(n)$ bits

3. Use **lookup table** for bitstrings of length $(\frac{1}{2}\log n)$:

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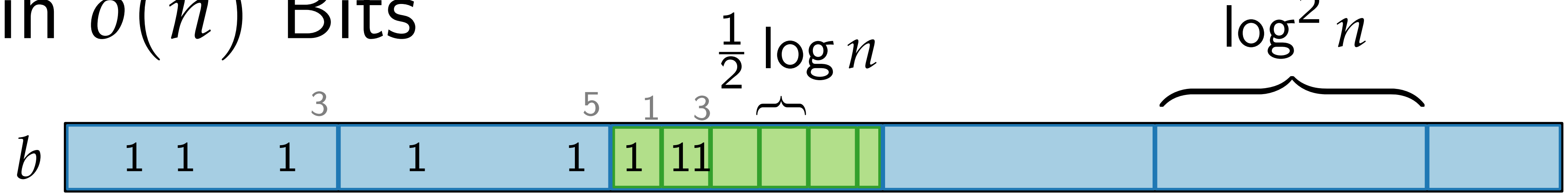

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3. Use **lookup table** for bitstrings of length $(\frac{1}{2}\log n)$: $2^{\frac{1}{2}\log n} =$ distinct bitstrings

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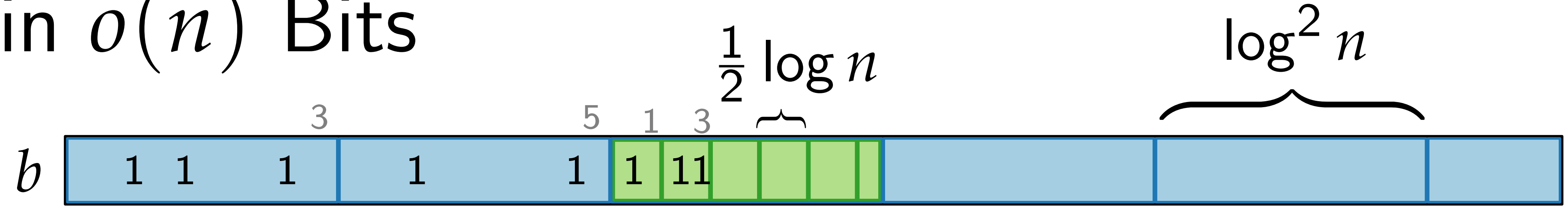

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Rank in $o(n)$ Bits

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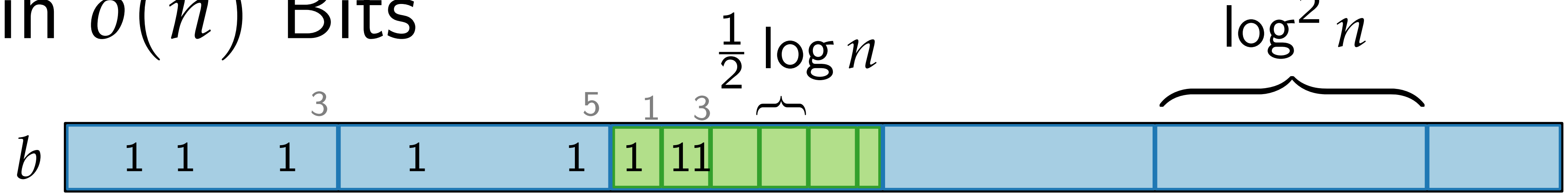

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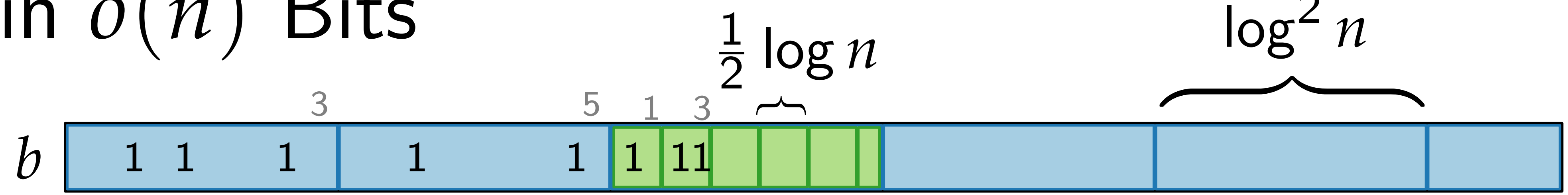
   $\Rightarrow O(\underbrace{\sqrt{n}}_{\#\text{ rows}}\underbrace{\log n}_{\#\text{ columns}}\underbrace{\log\log n}_{\text{rel. rank}}) \subseteq o(n)$ bits

Rank in $o(n)$ Bits

$\log^2 n = (\log n)^2$



1. Split into $(\log^2 n)$-bit **chunks**

   and store cumulative rank: each needs $\leq \log n$ bits

   $$\Rightarrow O\left(\frac{n}{\log^2 n}\log n\right) = O\left(\frac{n}{\log n}\right) \subseteq o(n) \text{ bits}$$

2. Split **chunks** into $(\frac{1}{2}\log n)$-bit **subchunks**

   and store cumulative rank within **chunk**: each needs $\leq \log\log^2 n = 2\log\log n$ bits

   $$\Rightarrow O\left(\frac{n}{\log n}\log\log n\right) \subseteq o(n) \text{ bits}$$

3. Use **lookup table** for bitstrings of length $(\frac{1}{2}\log n)$: $2^{\frac{1}{2}\log n} = \sqrt{n}$ distinct bitstrings

   $$\Rightarrow O(\sqrt{n}\log n \log\log n) \subseteq o(n) \text{ bits}$$

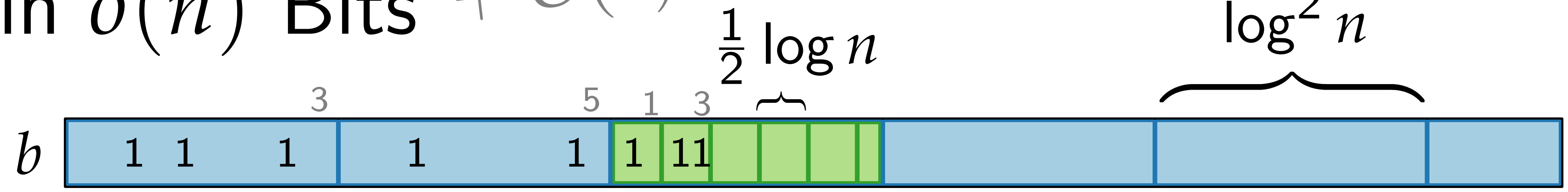
4. $\mathtt{rank}(i) = \mathtt{rank}$ of **chunk**

   $+$ relative $\mathtt{rank}$ of **subchunk** within **chunk**

   $+$ relative $\mathtt{rank}$ of element i within **subchunk**

Rank in $o(n)$ Bits $+\ O(1)$ Time

$\log^2 n = (\log n)^2$



1. Split into $(\log^2 n)$-bit **chunks**
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$\Rightarrow O(1)$ time

(assume read/write numbers in $O(1)$ time)

Select in $o(n)$ Bits

b

Select in $o(n)$ Bits

1. Store indices of every $(\log n \log\log n)$-th 1 in array

Select in $o(n)$ Bits

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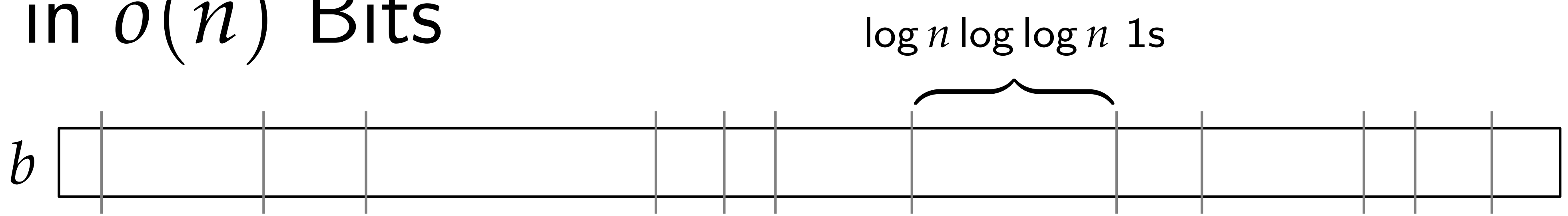
Select in $o(n)$ Bits

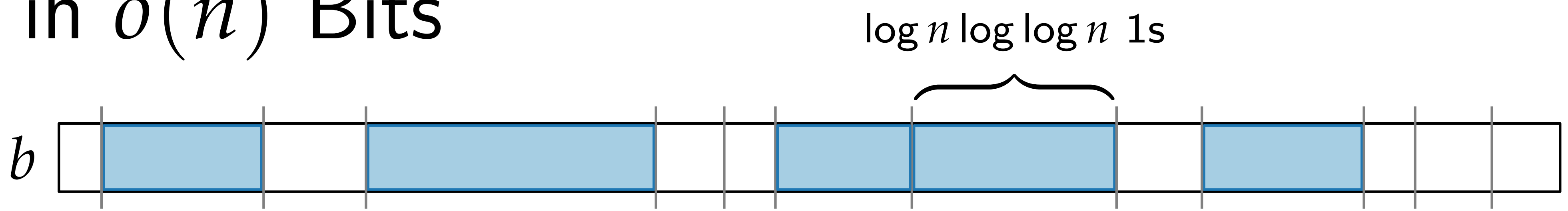


1. Store indices of every $(\log n \log \log n)$-th 1 in array

$$\Rightarrow O\left(\frac{n}{\log n \log \log n} \log n\right) = O\left(\frac{n}{\log \log n}\right) \subseteq o(n) \text{ bits}$$

2. Within group of $(\log n \log \log n)$ 1s whose length is, say, r bits:

Select in $o(n)$ Bits



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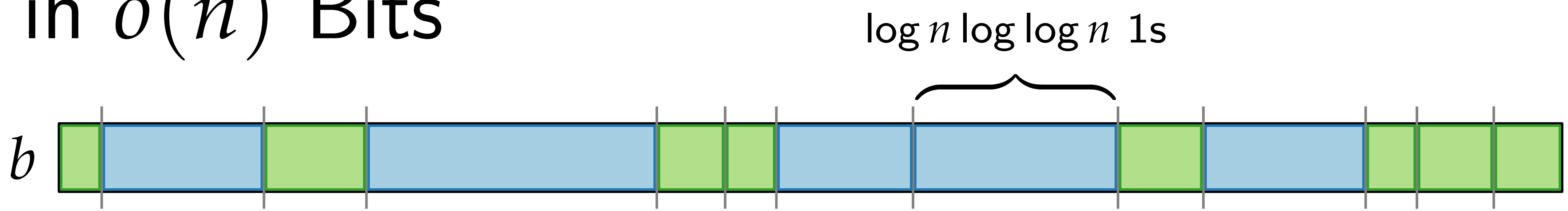
   if $r \geq (\log n \log\log n)^2$
   then store indices of all 1s in the group in an array

$$\Rightarrow O\Big(\underbrace{\frac{n}{(\log n \log\log n)^2}}_{\#\text{ groups}} \underbrace{(\log n \log\log n)}_{\#\text{ 1s}} \underbrace{\log n}_{\text{index}}\Big) \subseteq O\left(\frac{n}{\log\log n}\right) \text{ bits}$$

Select in $o(n)$ Bits



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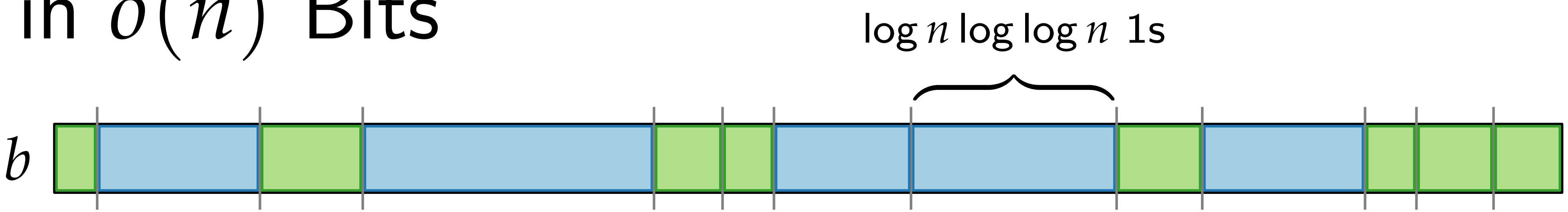
   else problem is reduced to bitstrings of length $r < (\log n \log\log n)^2$

Select in $o(n)$ Bits



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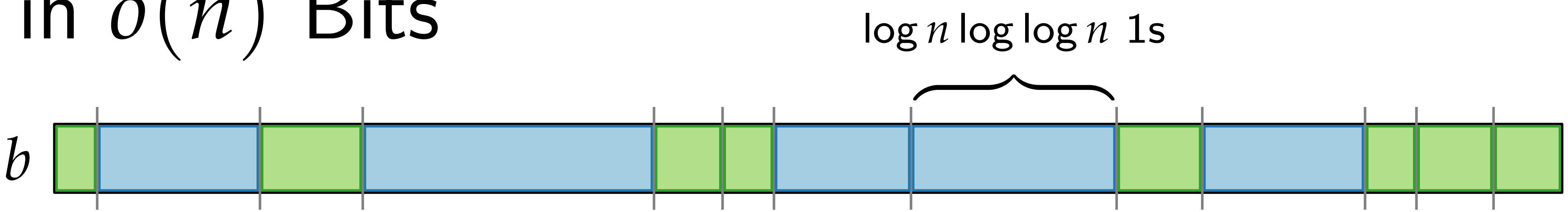
3. Repeat 1. and 2. on reduced bitstrings

Select in $o(n)$ Bits



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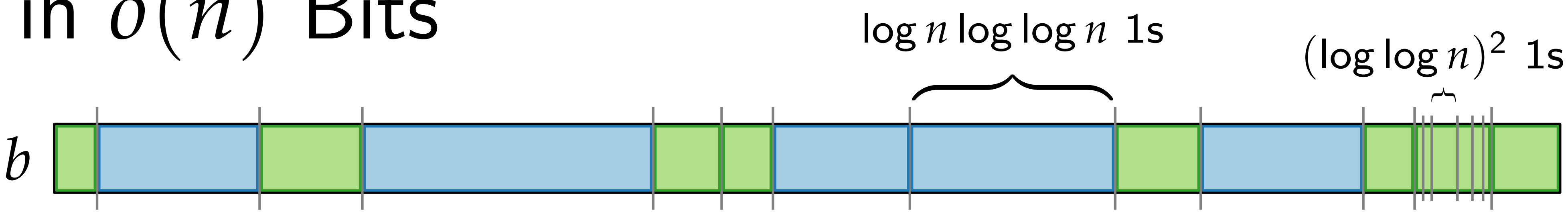
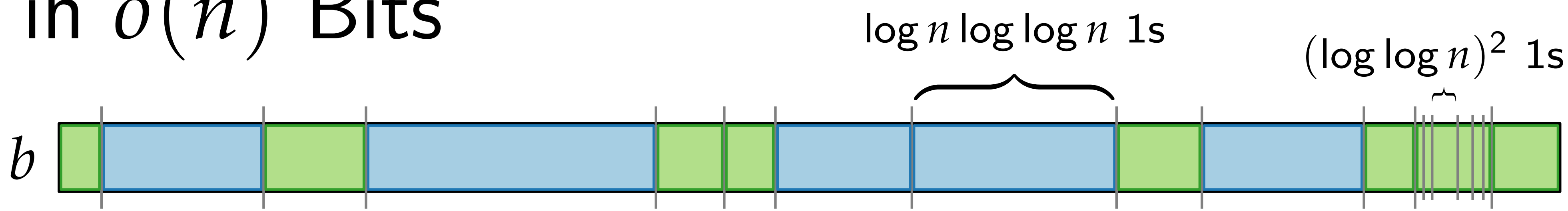


3. Repeat 1. and 2. on reduced bitstrings ($r < (\log n \log\log n)^2$):

   1' Store relative indices of every $(\log\log n)^2$-th 1 in array

Select in $o(n)$ Bits



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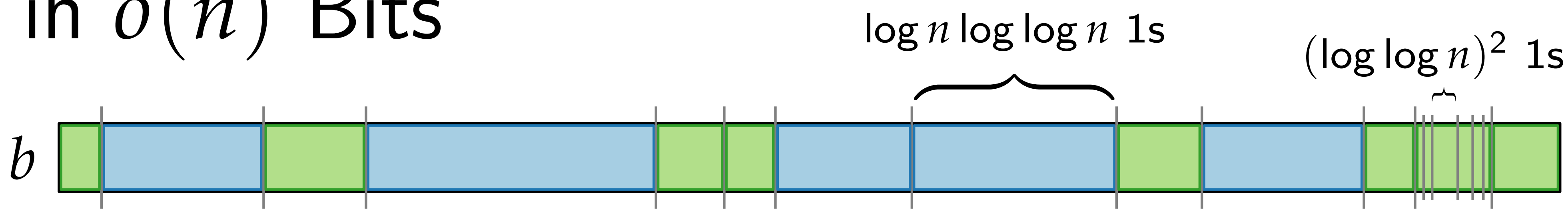
Select in $o(n)$ Bits



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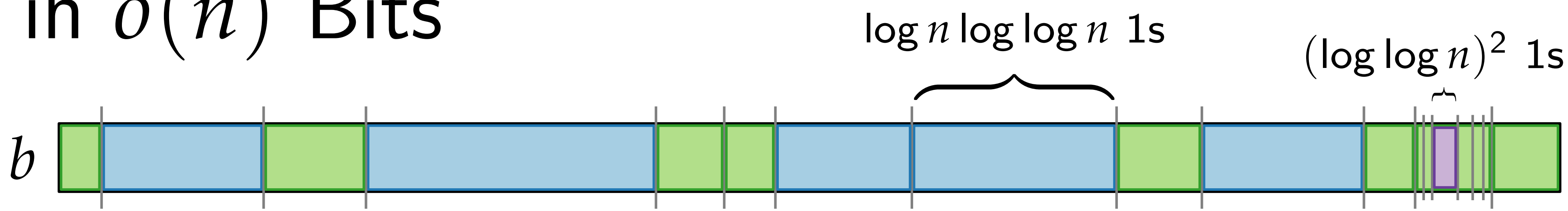
Select in $o(n)$ Bits



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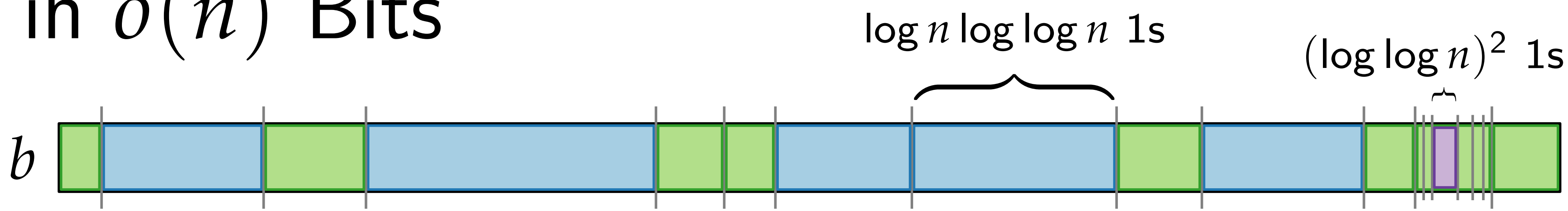
then store relative index of every 1 in the subgroup in an array.

Select in $o(n)$ Bits



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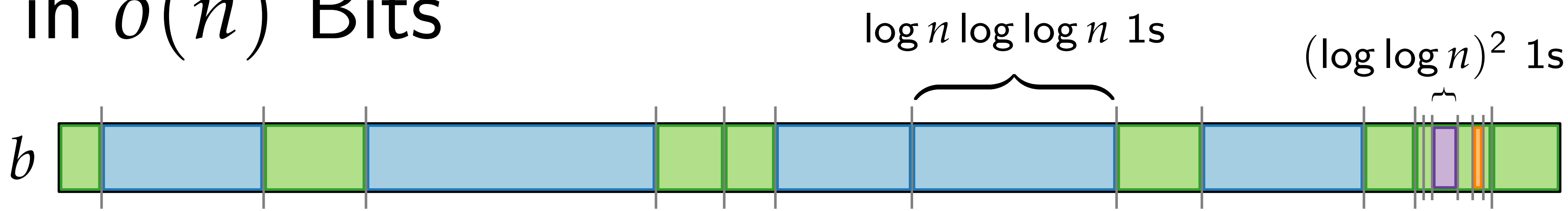
Select in $o(n)$ Bits



3. Repeat 1. and 2. on reduced bitstrings ($r < (\log n \log\log n)^2$):

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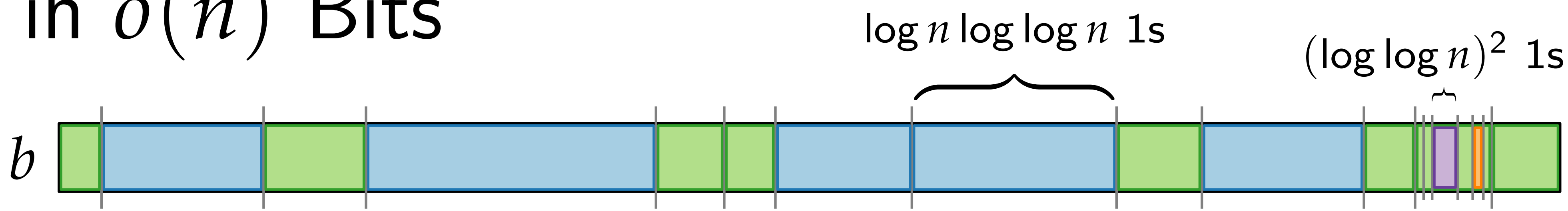

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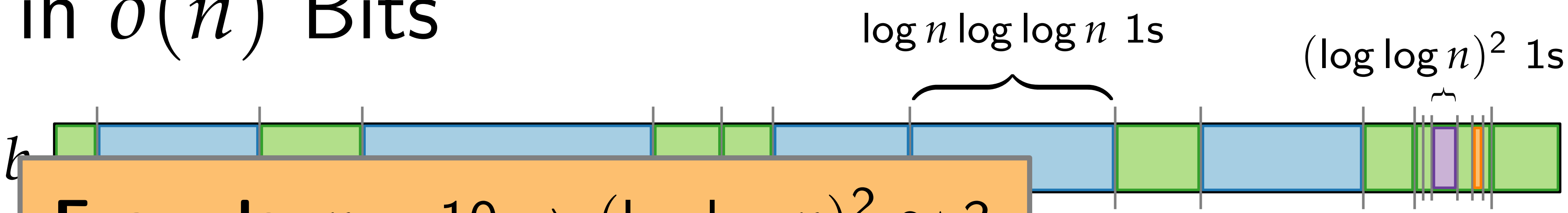
4. Use **lookup table** for bitstrings of length $r' \leq (\log\log n)^4$:

Select in $o(n)$ Bits



Example: $n = 10 \Rightarrow (\log\log n)^2 \approx 3$

$\Rightarrow r' < (\log\log n)^4 \approx 9$

bitstring / select →	1	2	3
00000111	6	7	8
00001011	5	7	8
00001101	5	6	8
⋮			
11001000	1	2	5
11010000	1	2	4
11100000	1	2	3

3. Repea ... $n \log\log n)^2$):

1' Store ... 1 in array

... $n)^2$ $\log\log n) = O(\frac{n}{\log\log n})$ bits

2' With ... gth is, say, r' bits:

if r' ...

then ... group in an array.

... $n)^2 \log\log n) = O(\frac{n}{\log\log n})$ bits

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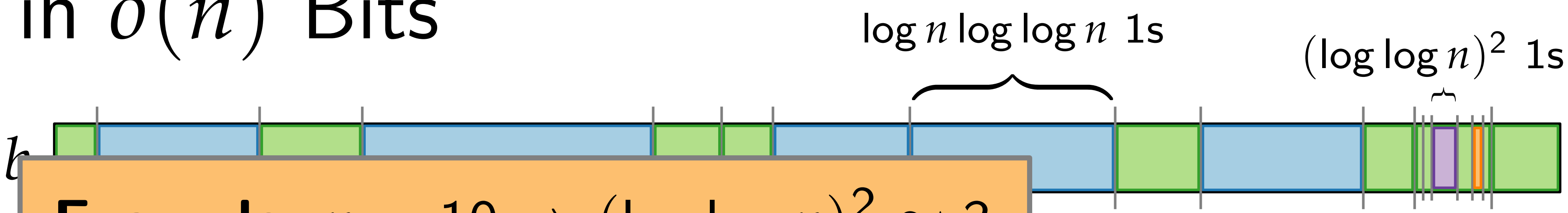
Select in $o(n)$ Bits



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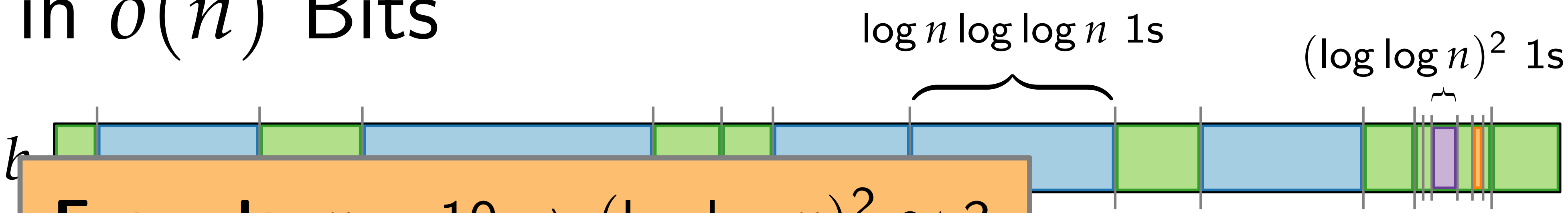
else problem is reduced to bitstrings of length $r' < (\log\log n)^4$

4. Use **lookup table** for bitstrings of length $r' \leq (\log\log n)^4$:

$\underbrace{2^{(\log\log n)^4}}_{\#\text{ rows}} \in O(2^{\frac{1}{2}\log n}) = O(\sqrt{n})$; $\underbrace{(\log\log n)^2}_{\#\text{ columns}} \in O(\log n)$

Select in $o(n)$ Bits



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bitstring	00000111	6	7	8
	00001011	5	7	8
	00001101	5	6	8
	$\vdots$			
	11001000	1	2	5
	11010000	1	2	4
	11100000	1	2	3

3. Repea … $n \log\log n)^2$):

1' Store … 1 in array

… $n)^2 \log\log n) = O(\frac{n}{\log\log n})$ bits

2' With … gth is, say, r' bits:

if r'

then … group in an array.

… $n)^2 \log\log n) = O(\frac{n}{\log\log n})$ bits

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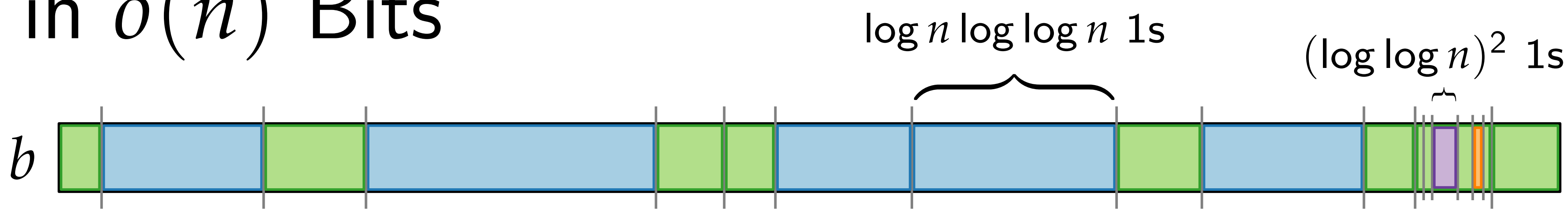
$\underbrace{2^{(\log\log n)^4}}_{\#\text{ rows}} \in O(2^{\frac{1}{2}\log n}) = O(\sqrt{n})$; $\underbrace{(\log\log n)^2}_{\#\text{ columns}} \in O(\log n)$ $\Rightarrow O(\underbrace{\sqrt{n}}_{\#\text{ rows}}\,\underbrace{\log n}_{\#\text{ columns}}\,\underbrace{\log\log n}_{\text{rel. index}}) = o(n)$ bits

Select in $o(n)$ Bits

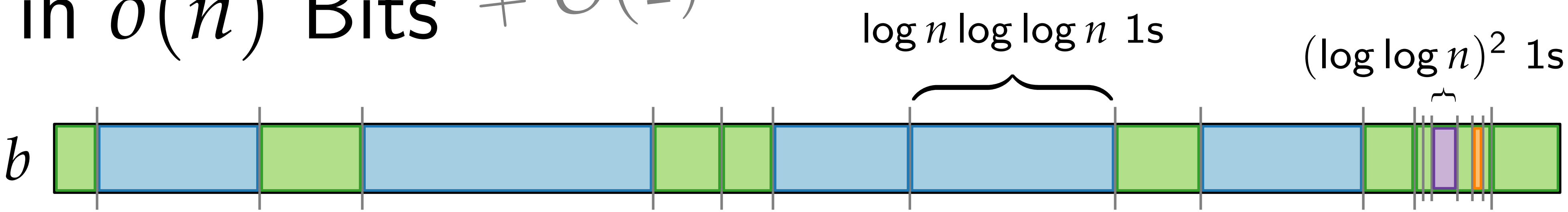


4. $\texttt{select}(j) = \texttt{select}$ J-th group where $J = \lfloor j/(\log n \log\log n) \rfloor$

   $+$ directly $\texttt{select}$ $(j-J)$-th 1
   or $\texttt{select}$ J'-th subgroup where $J' = \lfloor (j-J)/(\log\log n)^2 \rfloor$

   $+$ directly $\texttt{select}$ $(j-J-J')$-th 1
   or $\texttt{select}$ it in the lookup table

Select in $o(n)$ Bits + $O(1)$ Time



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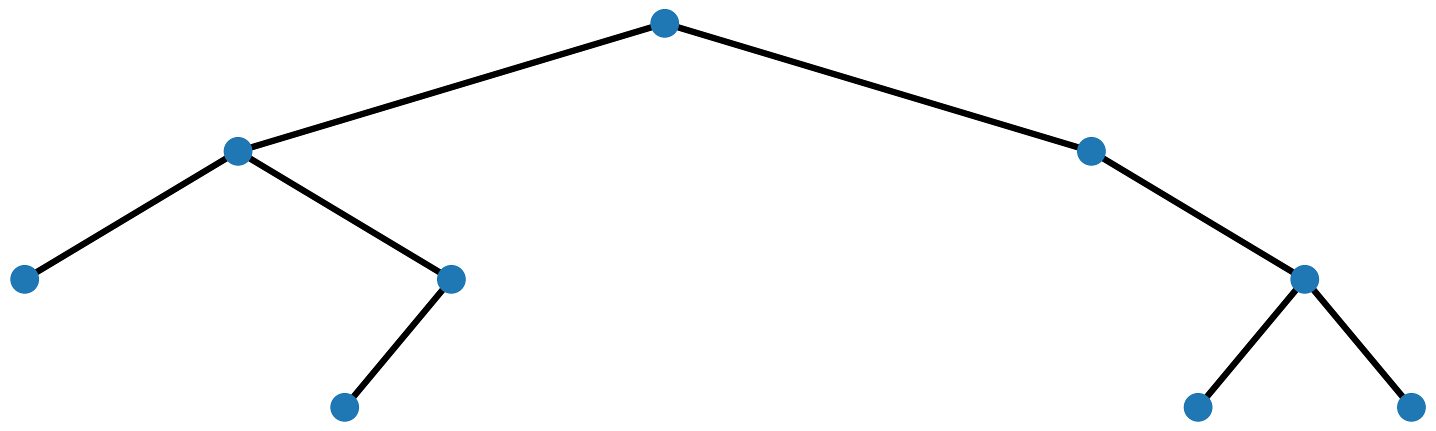
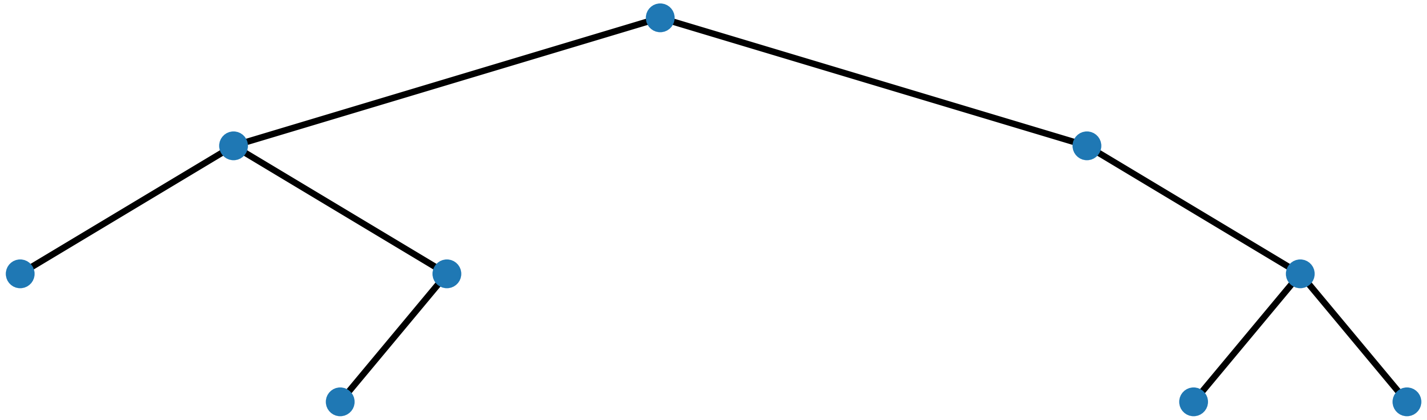
Succinct Representation of Binary Trees



Number of binary trees on n vertices: $C_n =$

Succinct Representation of Binary Trees

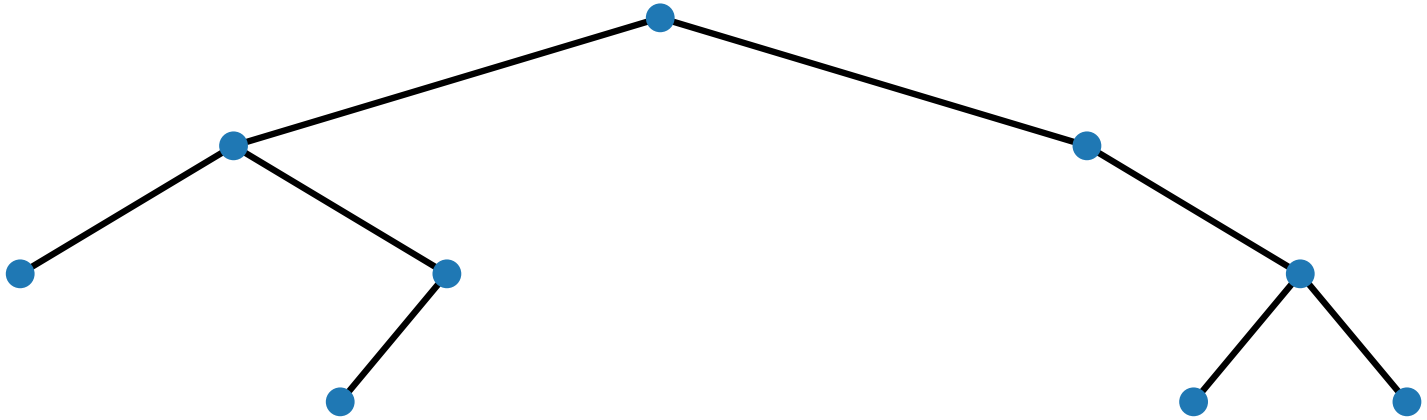


Number of binary trees on n vertices: $C_n =$

$n = 0$: "empty tree"

1 possibility

Succinct Representation of Binary Trees



Number of binary trees on n vertices: $C_n =$

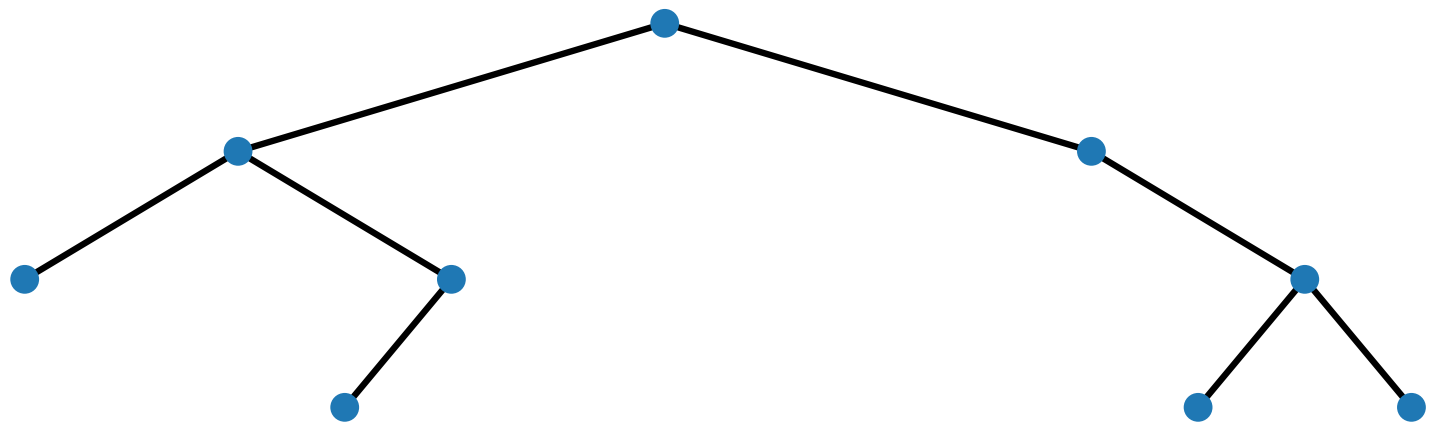
$n = 0$: "empty tree"

1 possibility

$n = 1$: 

1 possibility

Succinct Representation of Binary Trees



Number of binary trees on n vertices: $C_n =$

$n = 0$: “empty tree”

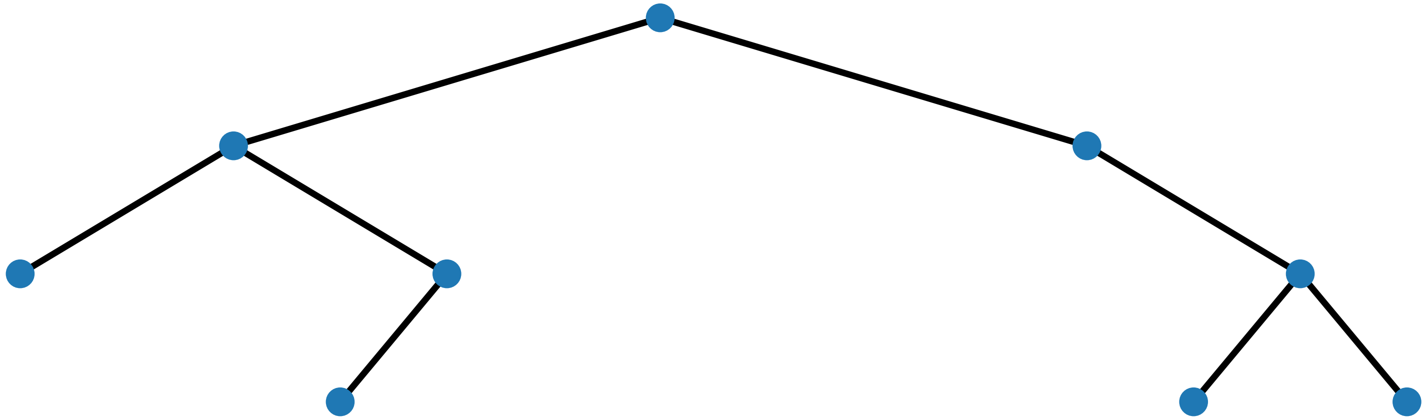
1 possibility

$n = 1$:

1 possibility

$n = 2$: start with root

Succinct Representation of Binary Trees



Number of binary trees on n vertices: $C_n =$

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1 possibility

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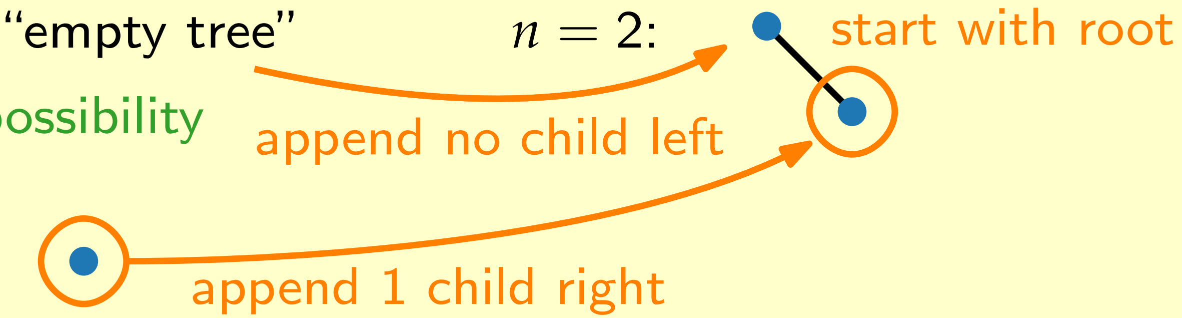
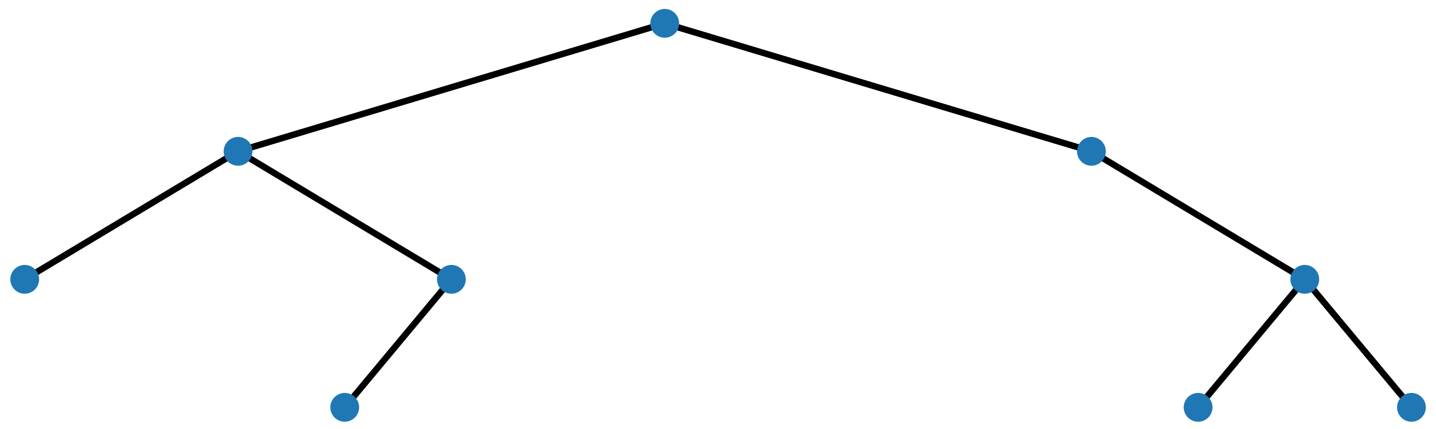
1 possibility

$n = 2$:



Succinct Representation of Binary Trees



Number of binary trees on n vertices: $C_n =$

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1 possibility

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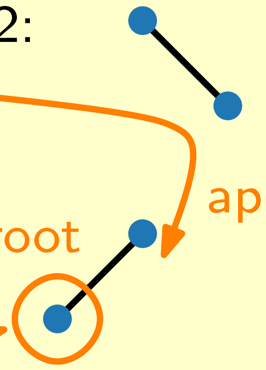
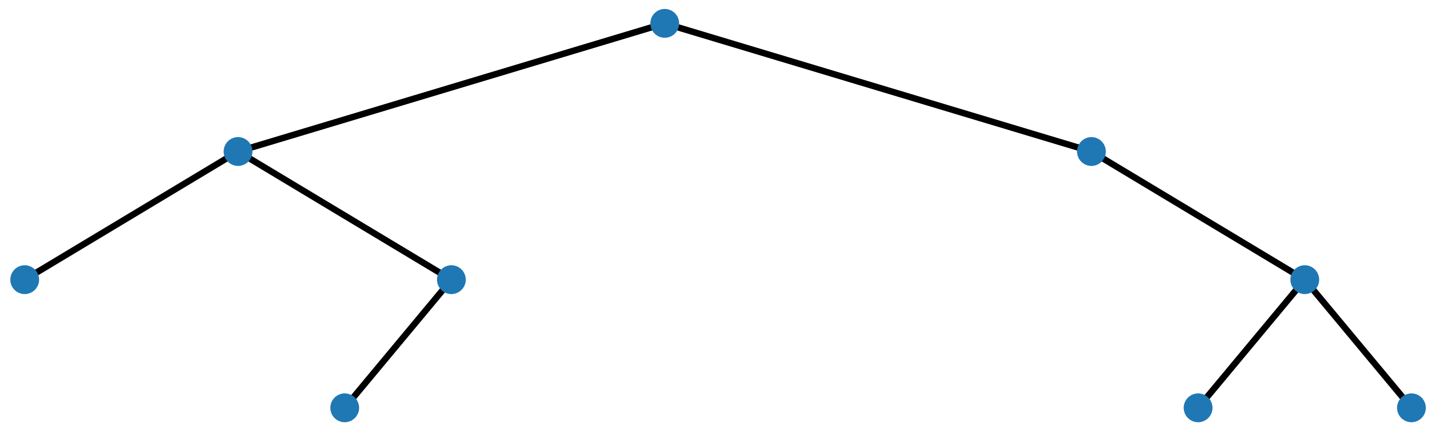
1 possibility

$n = 2$:



Succinct Representation of Binary Trees



Number of binary trees on n vertices: $C_n =$

$n = 0$: "empty tree"

1 possibility

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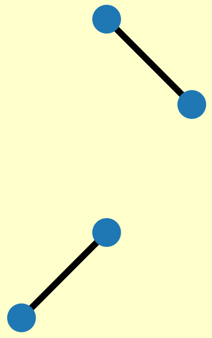
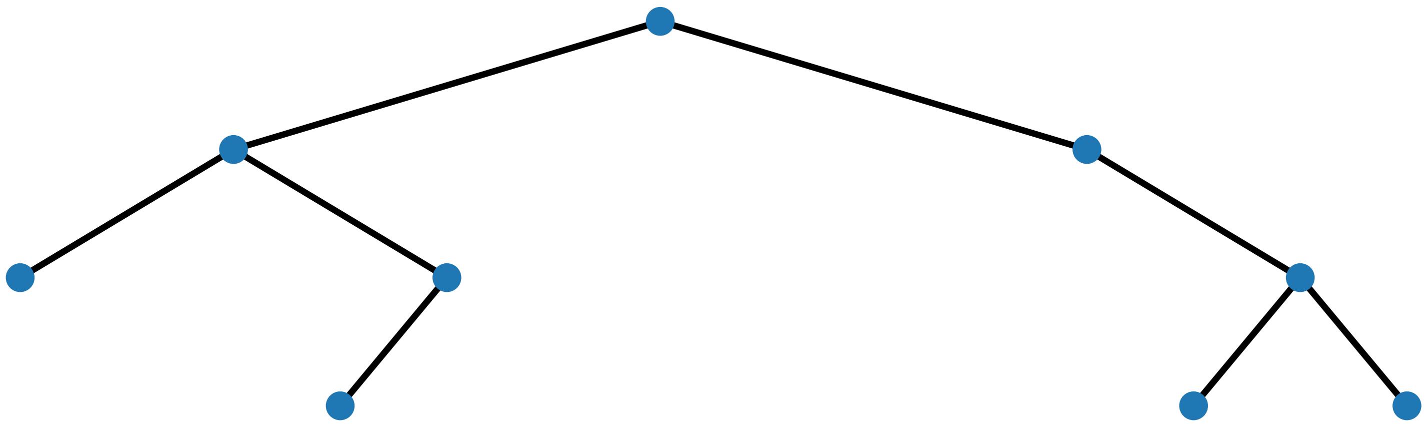
1 possibility

$n = 2$: 

2 possibilities

Succinct Representation of Binary Trees



Number of binary trees on n vertices: $C_n =$

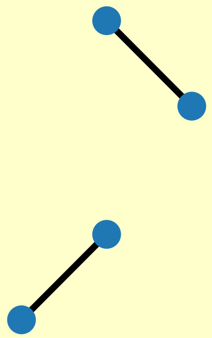
$n = 0$: “empty tree”

1 possibility

$n = 1$:

1 possibility

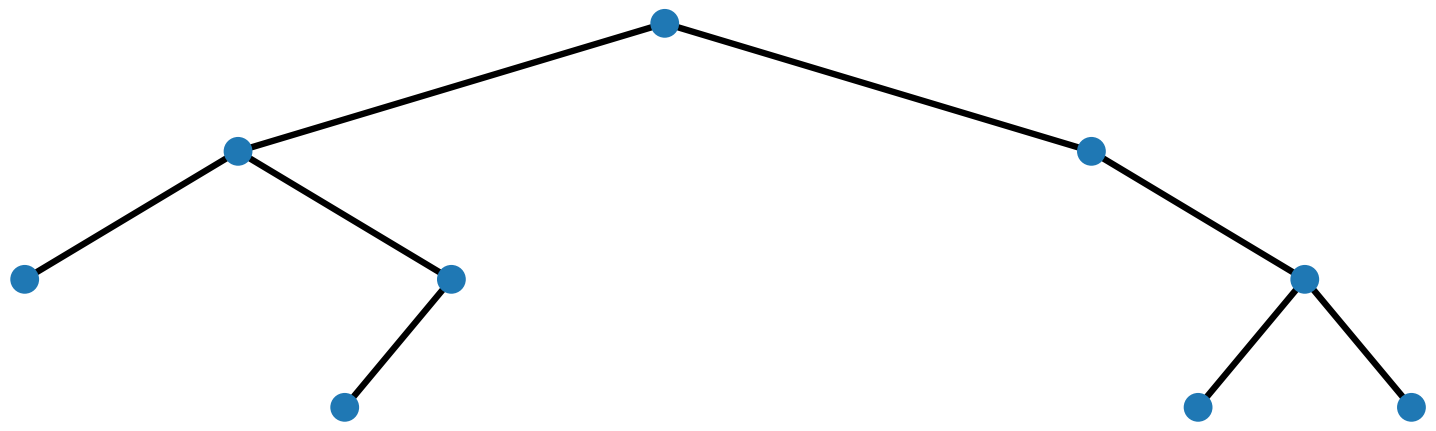
$n = 2$:



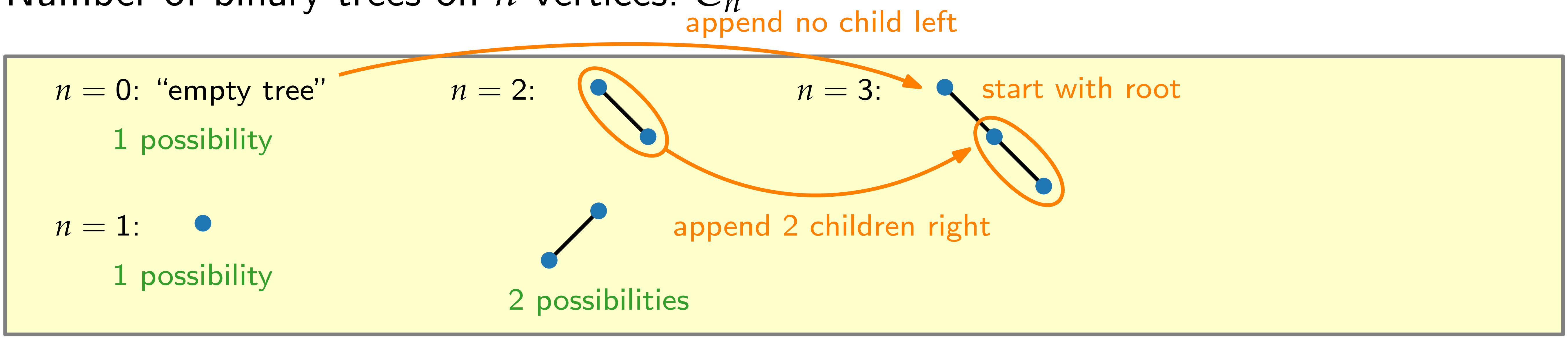
2 possibilities

$n = 3$: start with root

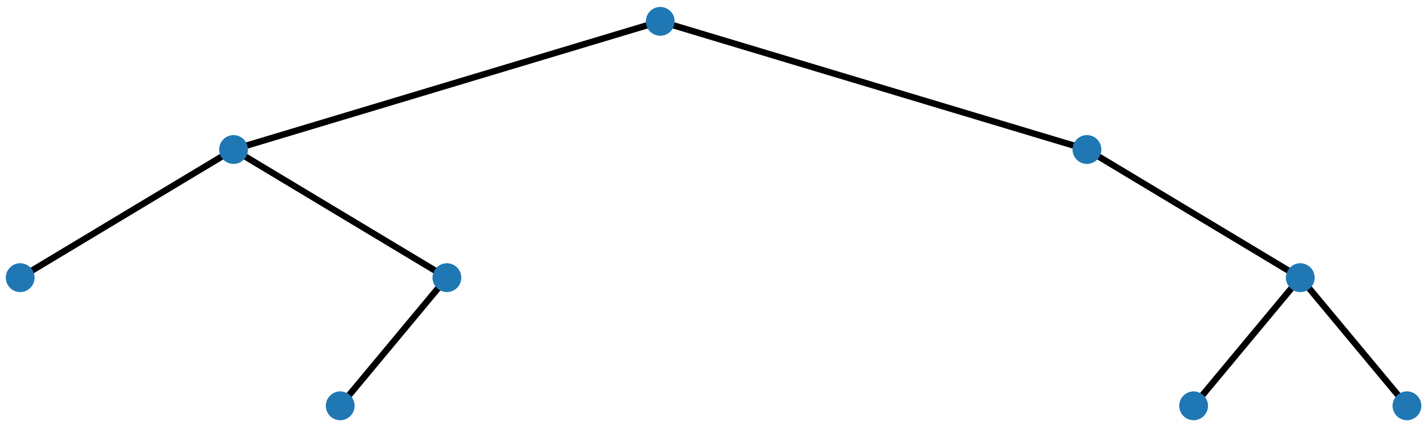
Succinct Representation of Binary Trees



Number of binary trees on n vertices: $C_n =$



Succinct Representation of Binary Trees



Number of binary trees on n vertices: $C_n =$

append no child left

$n = 0$: “empty tree”

1 possibility

$n = 1$:

1 possibility

$n = 2$:

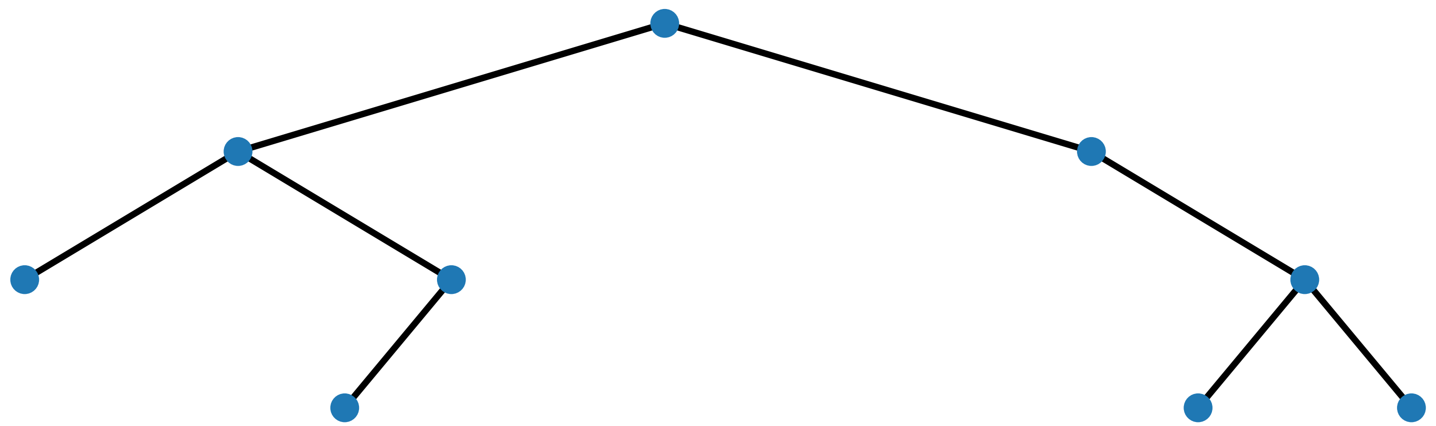
2 possibilities

append 2 children right

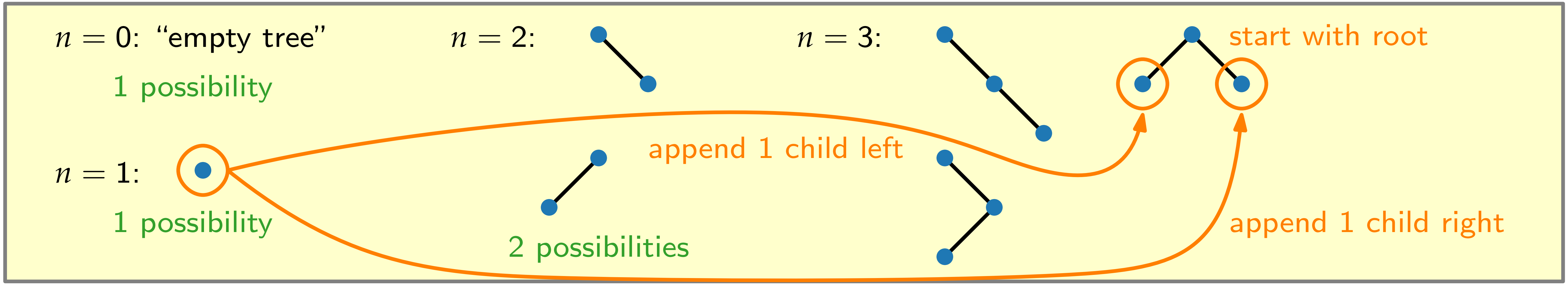
$n = 3$:

start with root

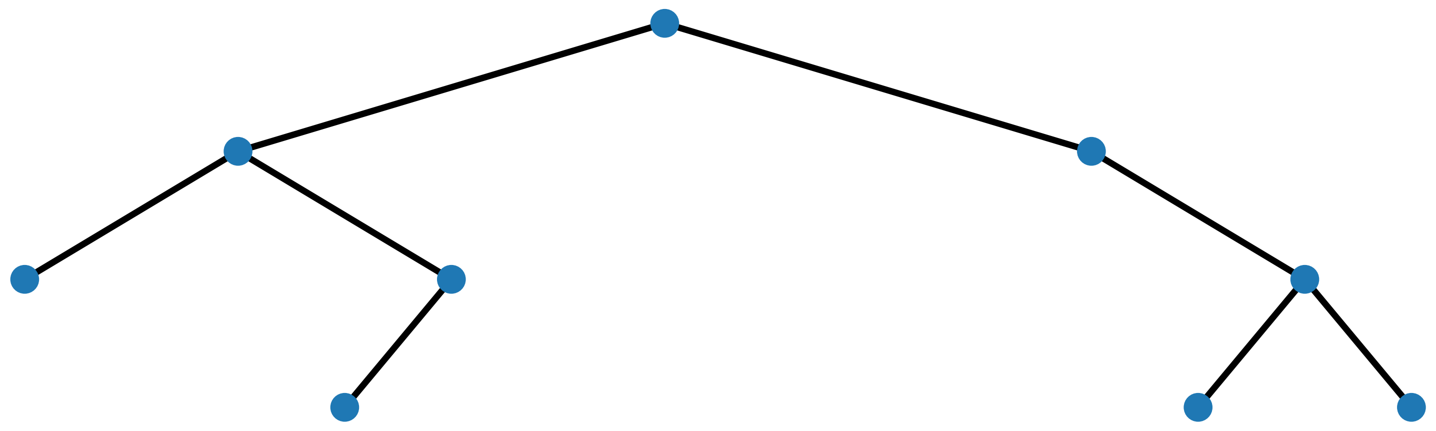
Succinct Representation of Binary Trees



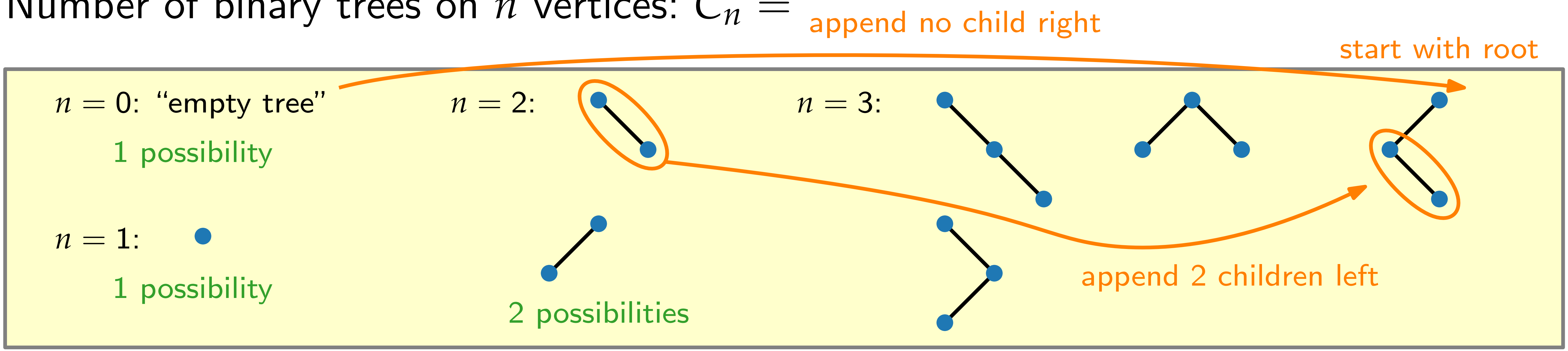
Number of binary trees on n vertices: $C_n =$



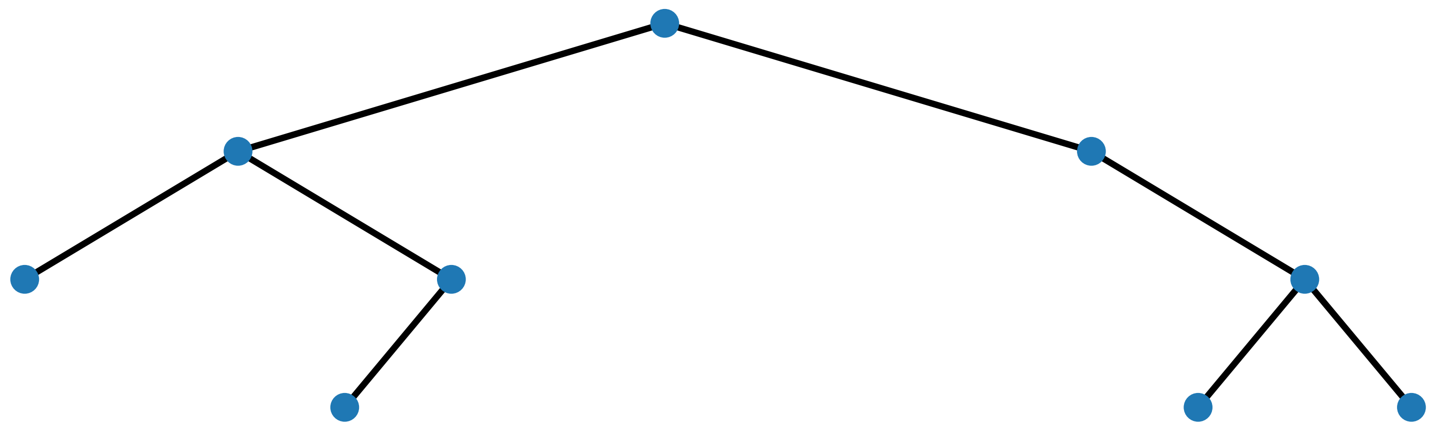
Succinct Representation of Binary Trees



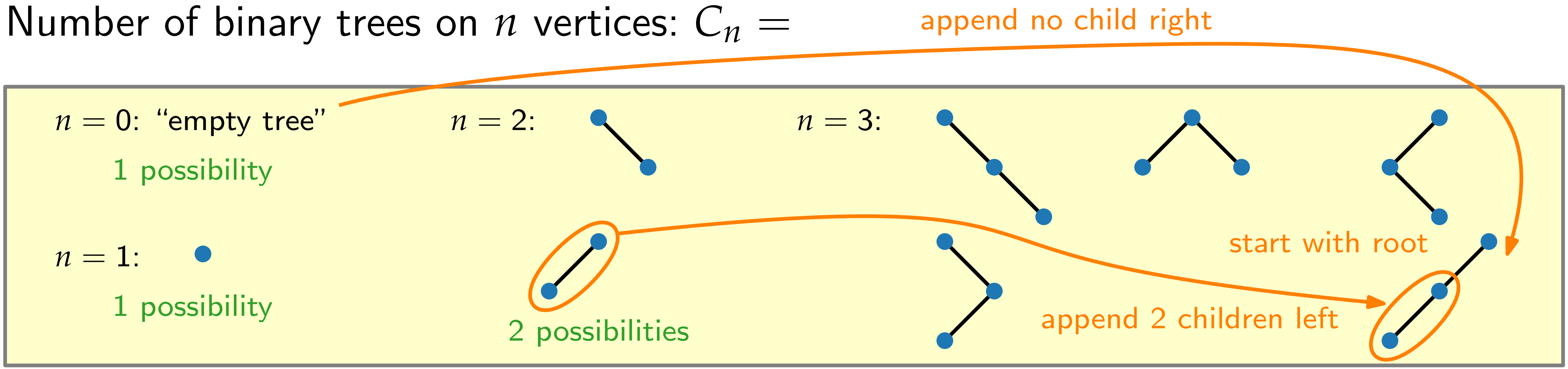
Number of binary trees on n vertices: $C_n =$



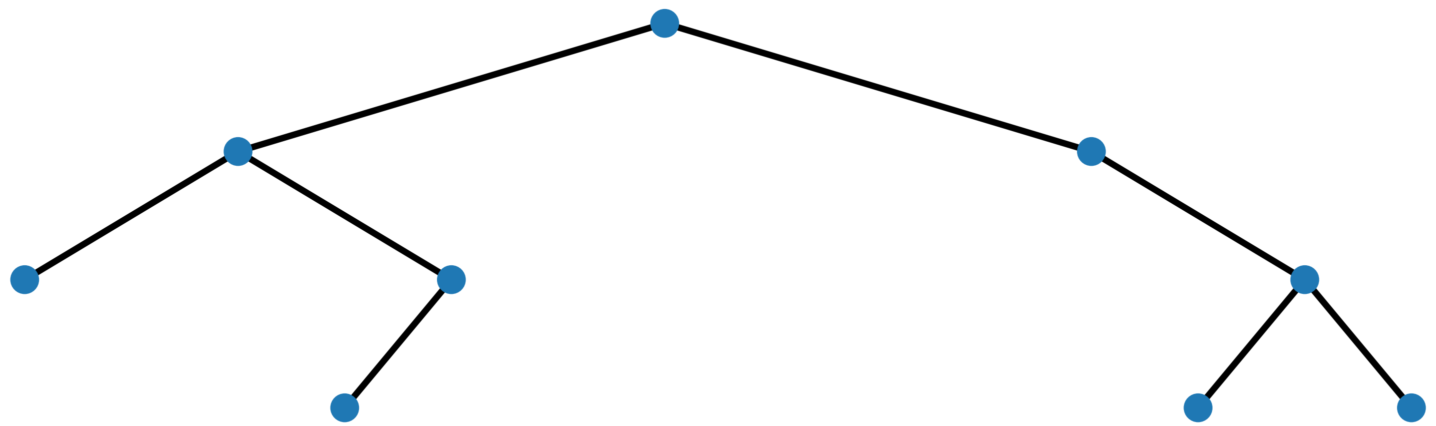
Succinct Representation of Binary Trees



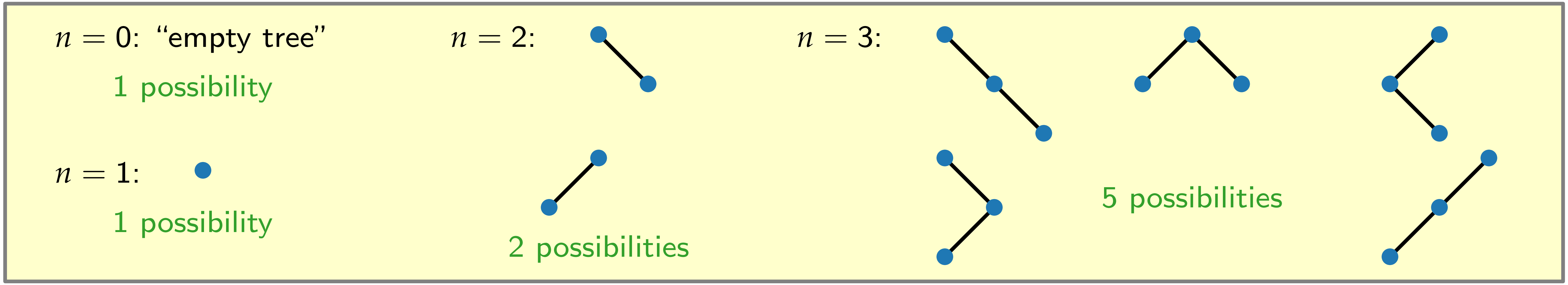
Number of binary trees on n vertices: $C_n =$



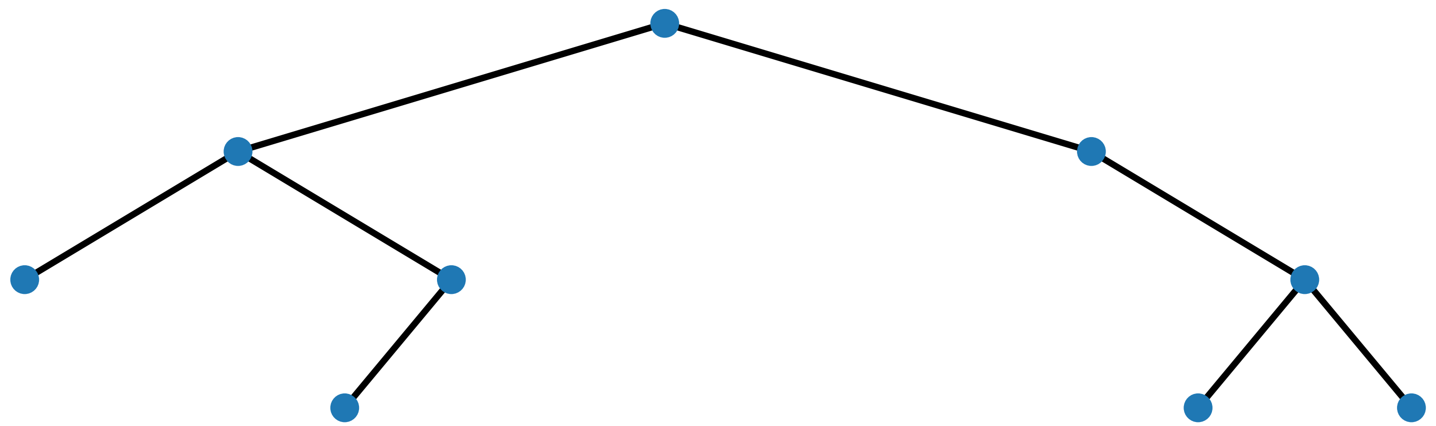
Succinct Representation of Binary Trees



Number of binary trees on n vertices: $C_n =$



Succinct Representation of Binary Trees



Number of binary trees on n vertices: $C_n = \sum_{i=0}^{n-1}$

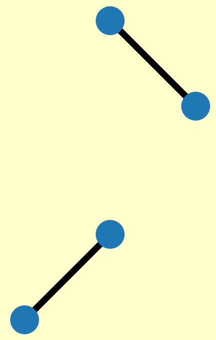
$n = 0$: “empty tree”

1 possibility

$n = 1$:

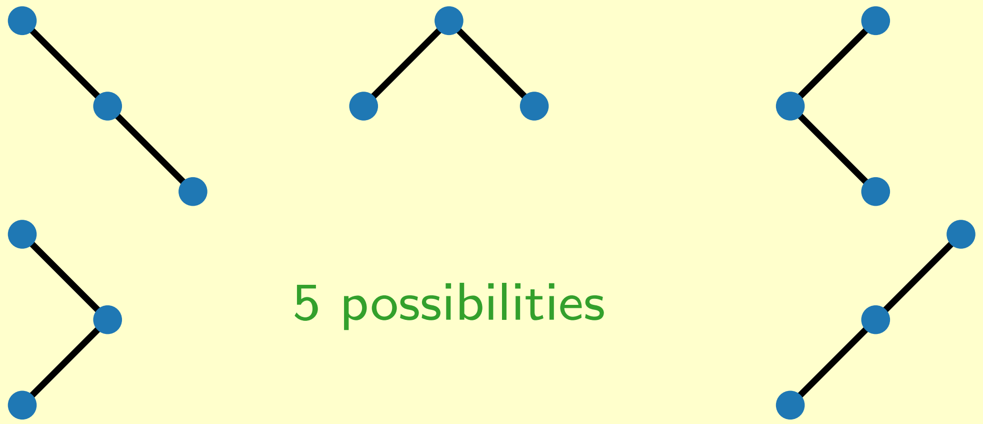
1 possibility

$n = 2$:



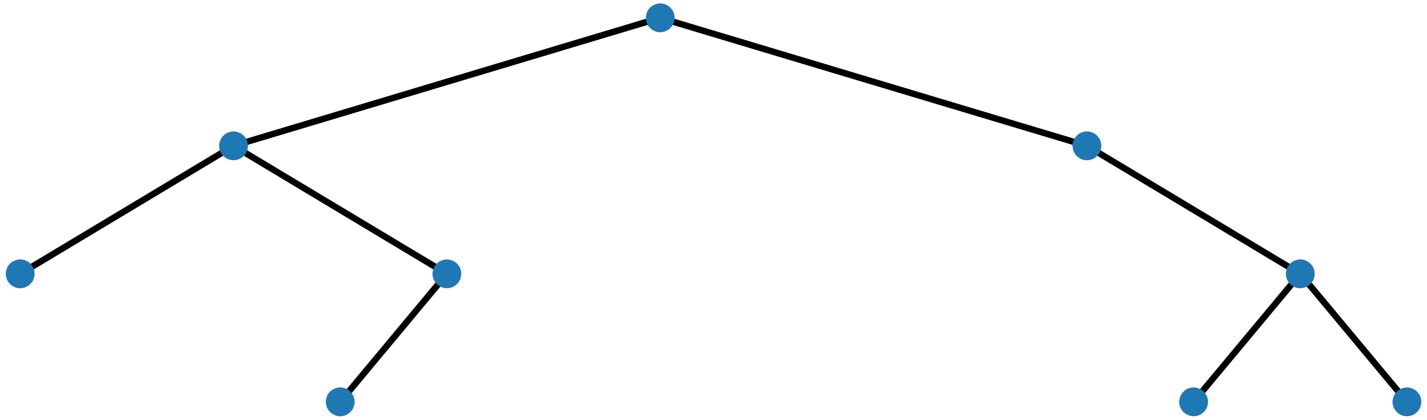
2 possibilities

$n = 3$:



5 possibilities

Succinct Representation of Binary Trees



Number of binary trees on n vertices: $C_n = \sum_{i=0}^{n-1} C_i \cdot C_{n-1-i} =$

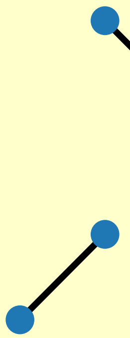
$n = 0$: "empty tree"

1 possibility

$n = 1$: 

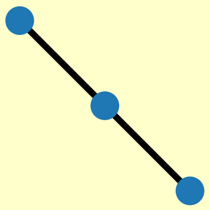
1 possibility

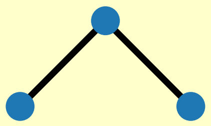
$n = 2$:

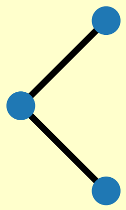


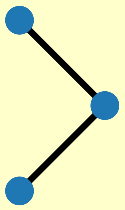
2 possibilities

$n = 3$:

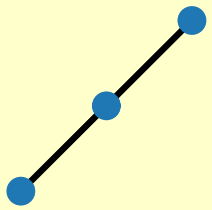




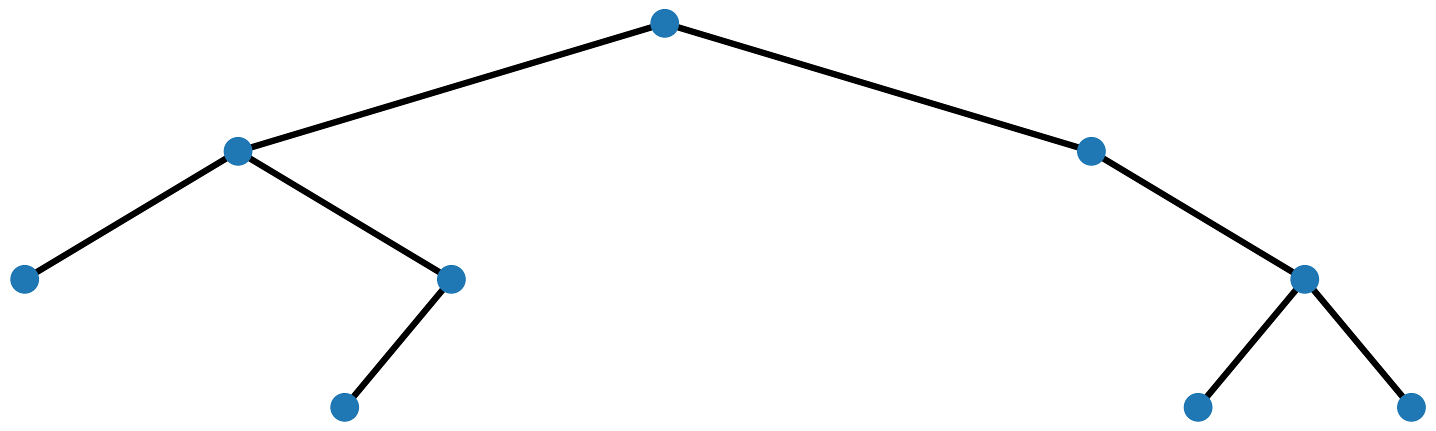




5 possibilities



Succinct Representation of Binary Trees



C_n is the n-th *Catalan number* and $C_0 = 1$

Number of binary trees on n vertices: $C_n = \sum_{i=0}^{n-1} C_i \cdot C_{n-1-i} =$

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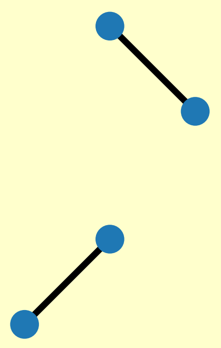
1 possibility

$n = 1$:

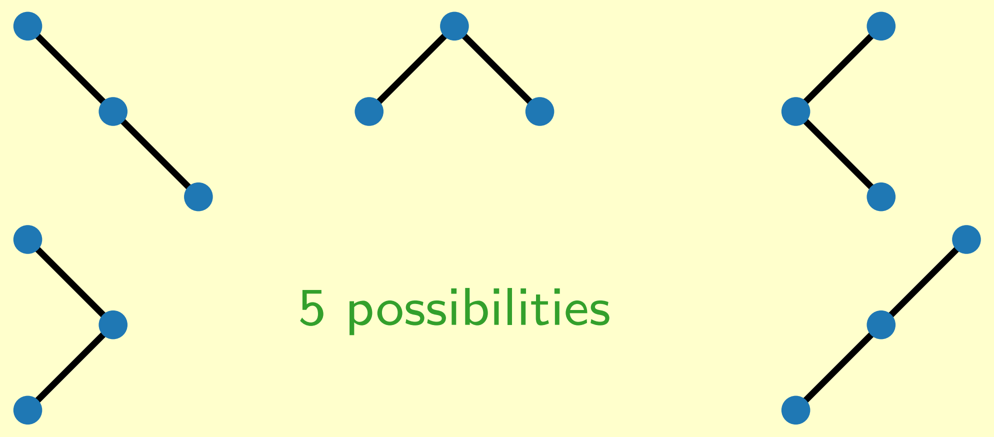
1 possibility

$n = 2$:



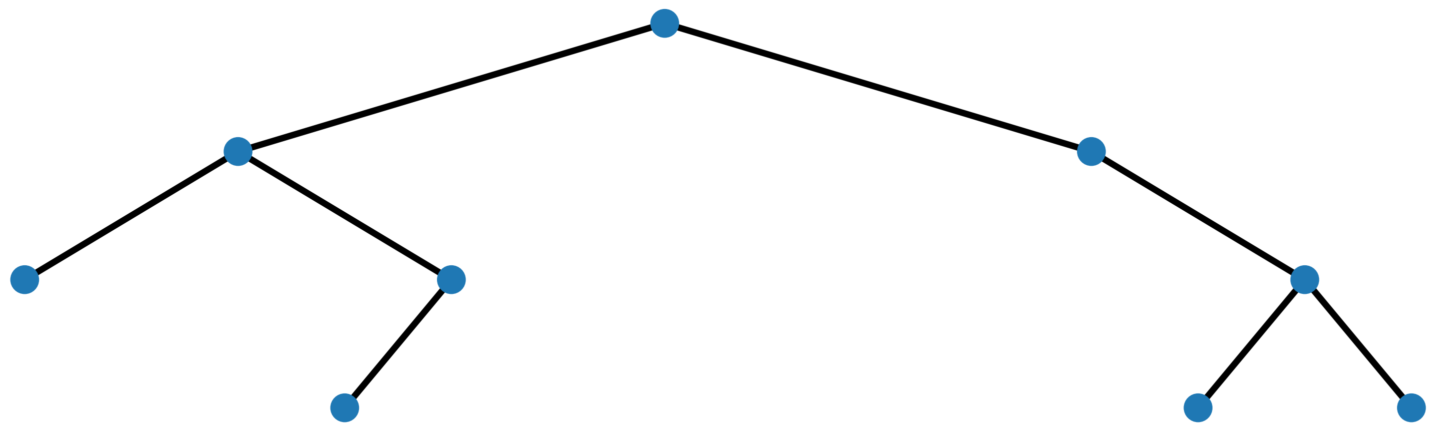
2 possibilities

$n = 3$:



5 possibilities

Succinct Representation of Binary Trees



C_n is the n-th *Catalan number* and $C_0 = 1$

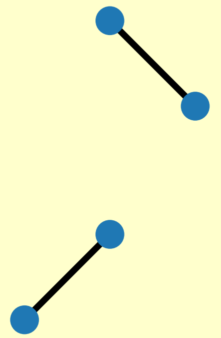
Number of binary trees on n vertices: $C_n = \sum_{i=0}^{n-1} C_i \cdot C_{n-1-i} = \frac{(2n)!}{(n+1)!\,n!}$

$n = 0$: "empty tree"

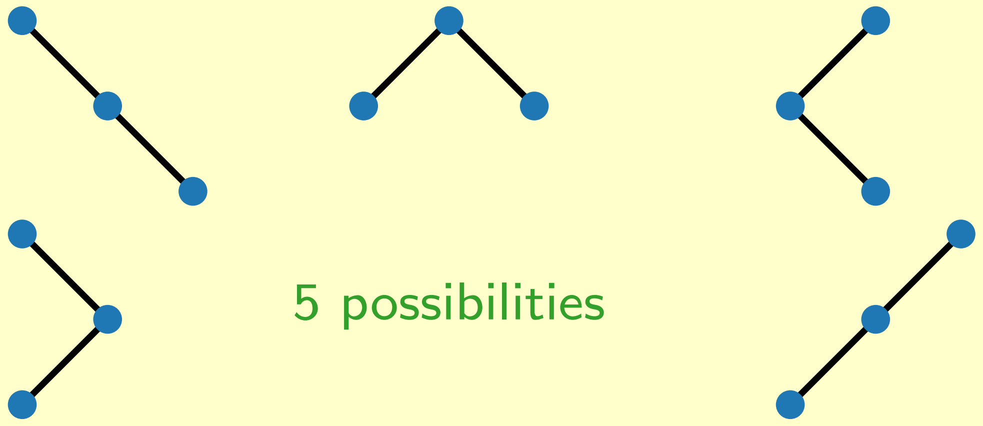
1 possibility

$n = 1$: 

1 possibility

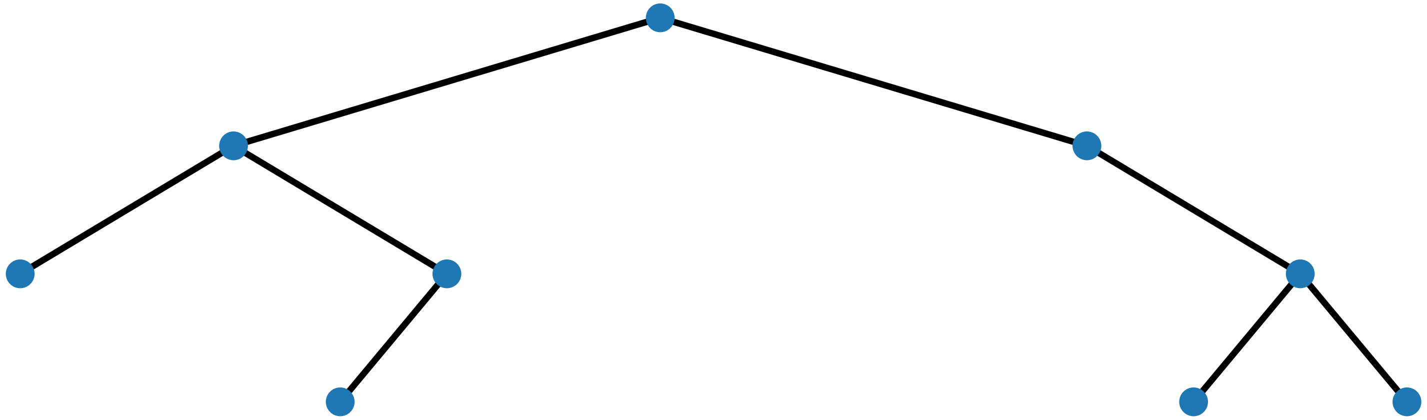
$n = 2$: 

2 possibilities

$n = 3$: 

5 possibilities

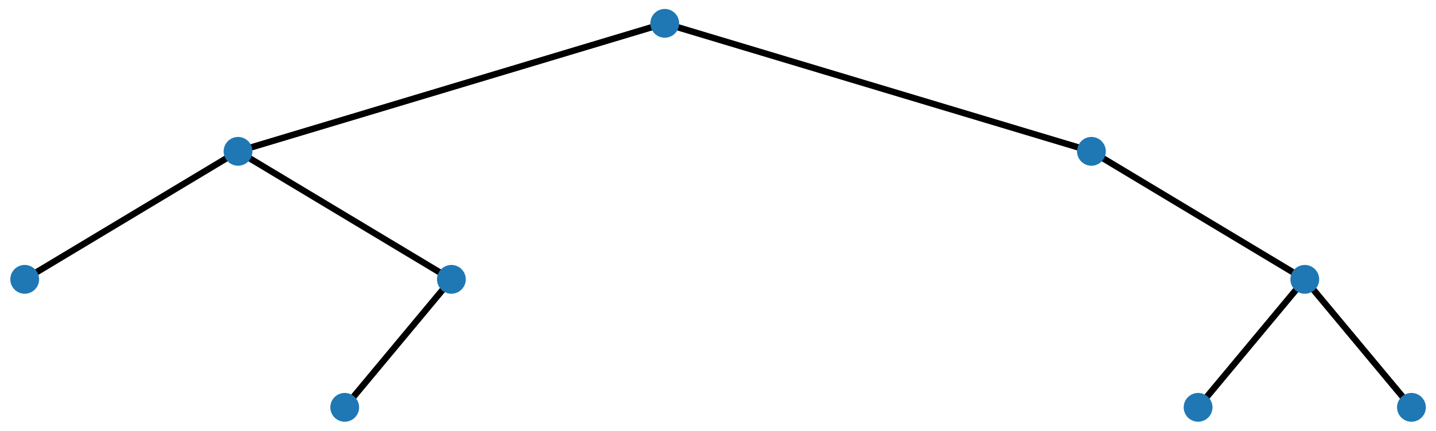
Succinct Representation of Binary Trees



Number of binary trees on n vertices: $C_n = \sum_{i=0}^{n-1} C_i \cdot C_{n-1-i} = \frac{(2n)!}{(n+1)!\,n!}$

$$\log C_n = 2n + o(n) \text{ (by Stirling's approximation)}$$

Succinct Representation of Binary Trees



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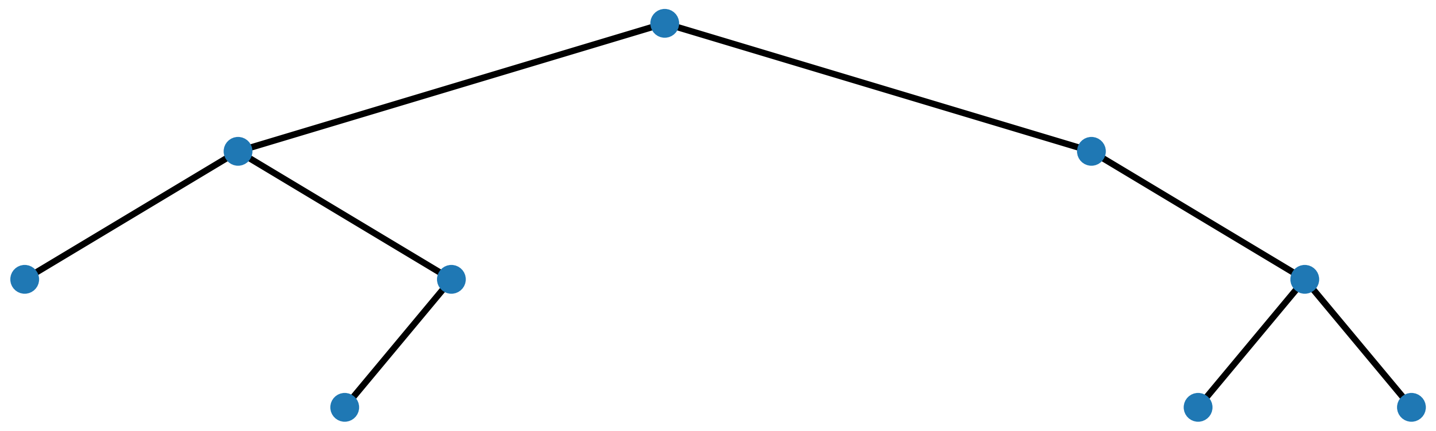
$$\log C_n = 2n + o(n) \text{ (by Stirling's approximation)}$$

$\Rightarrow$ We can use $2n + o(n)$ bits to represent binary trees.

Succinct Representation of Binary Trees



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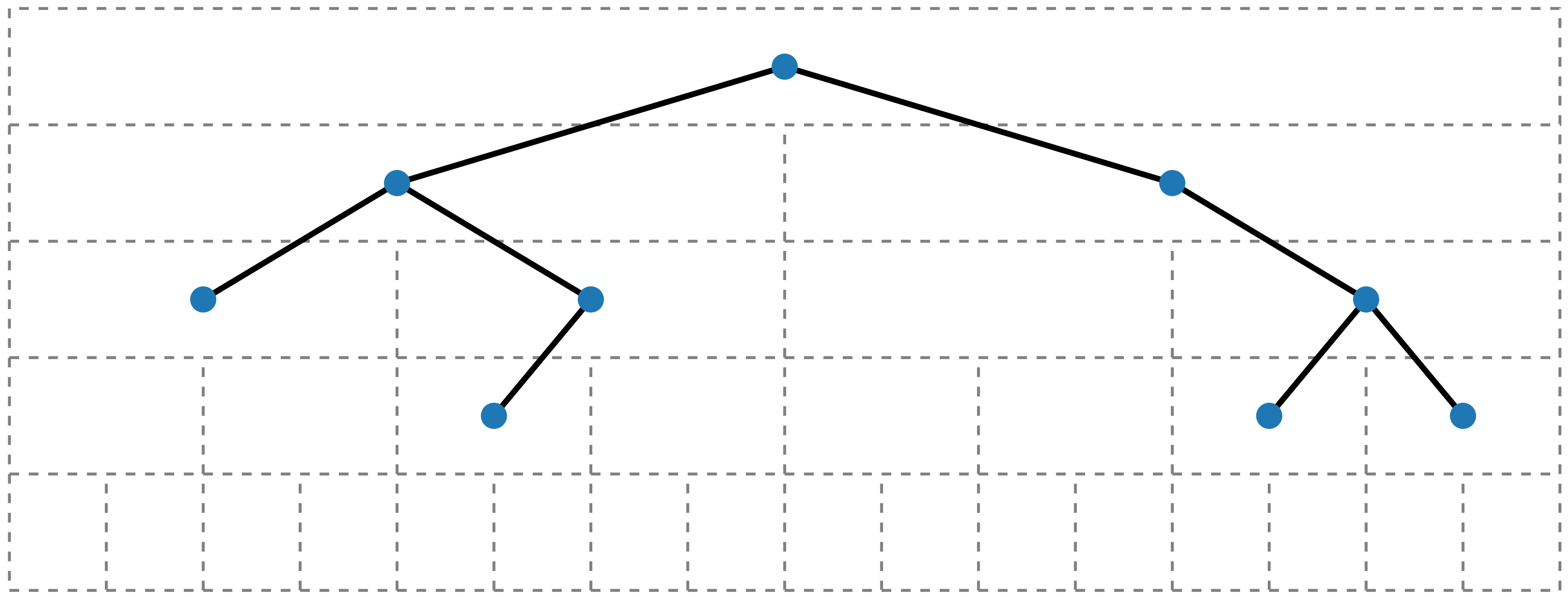
$$\log C_n = 2n + o(n) \text{ (by Stirling's approximation)}$$

$\Rightarrow$ We can use $2n + o(n)$ bits to represent binary trees.

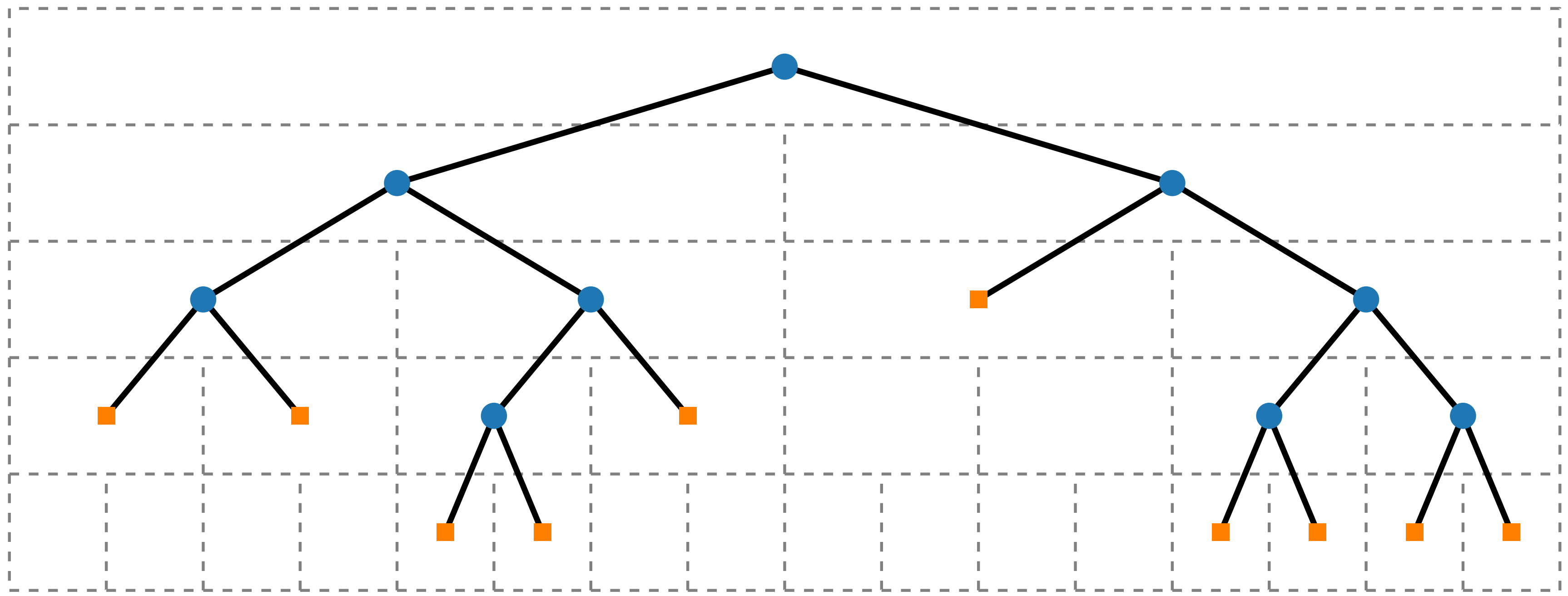
Difficulty is when a binary tree is not full.

Succinct Representation of Binary Trees



Idea.

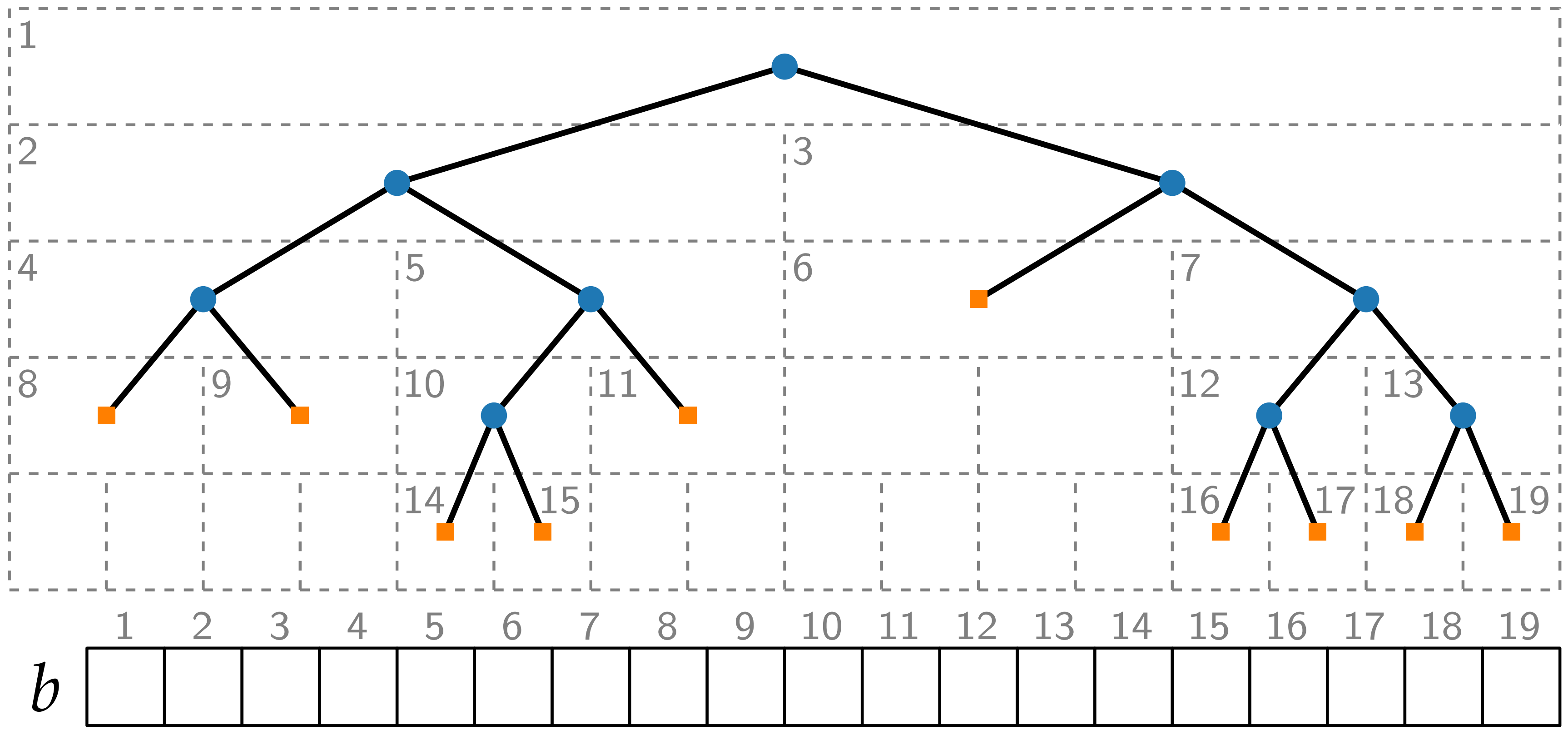
Succinct Representation of Binary Trees



Idea.

- Add external nodes to have out-degree 2 or 0 at every node

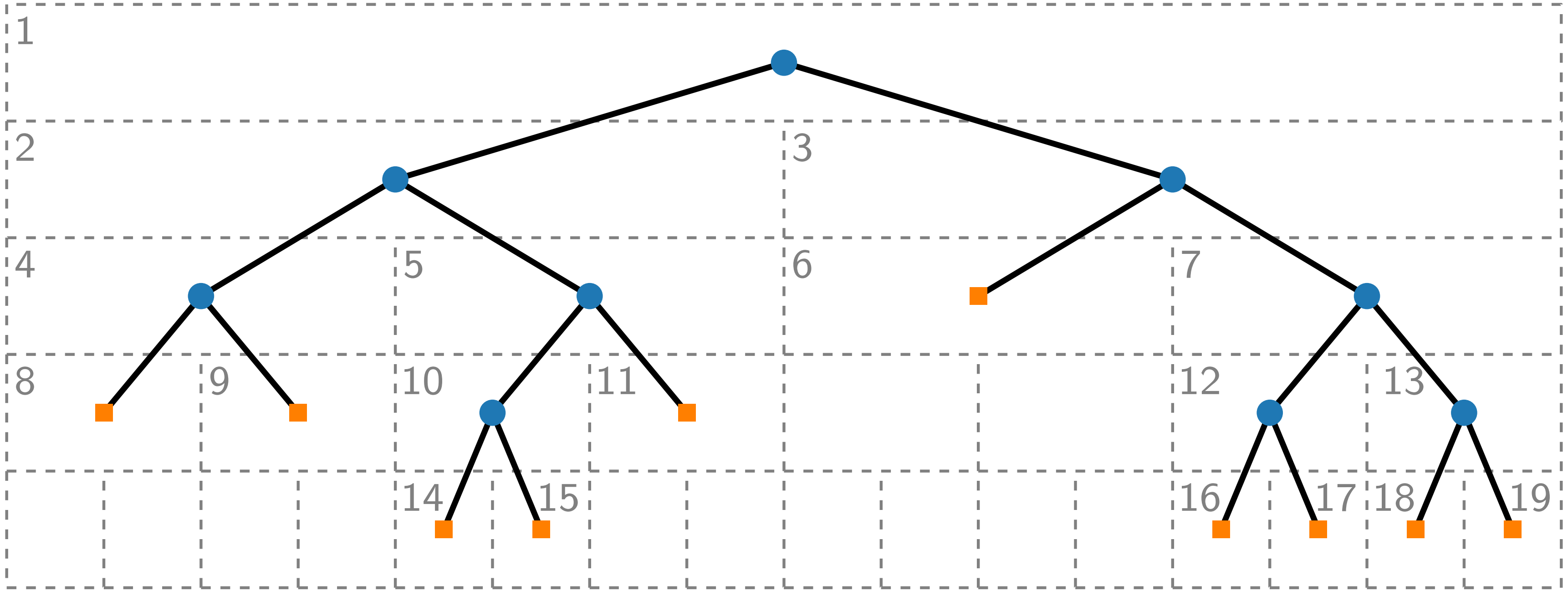
Succinct Representation of Binary Trees



Idea.

- Add external nodes to have out-degree 2 or 0 at every node

Succinct Representation of Binary Trees

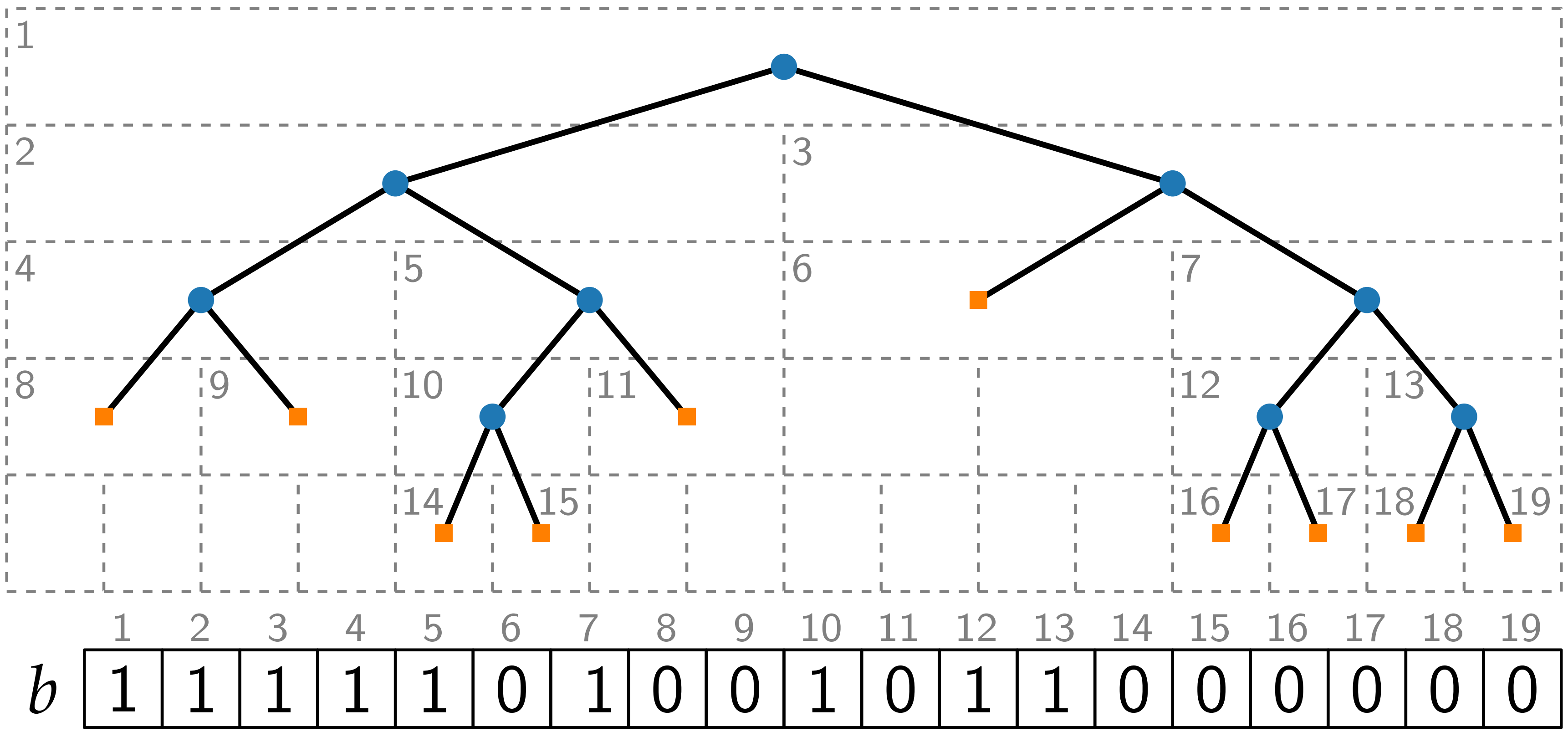


	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
b	1	1	1	1	1	0	1	0	0	1	0	1	1	0	0	0	0	0	0

Idea.

- Add external nodes to have out-degree 2 or 0 at every node
- Read internal nodes as 1
- Read external nodes as 0

Succinct Representation of Binary Trees



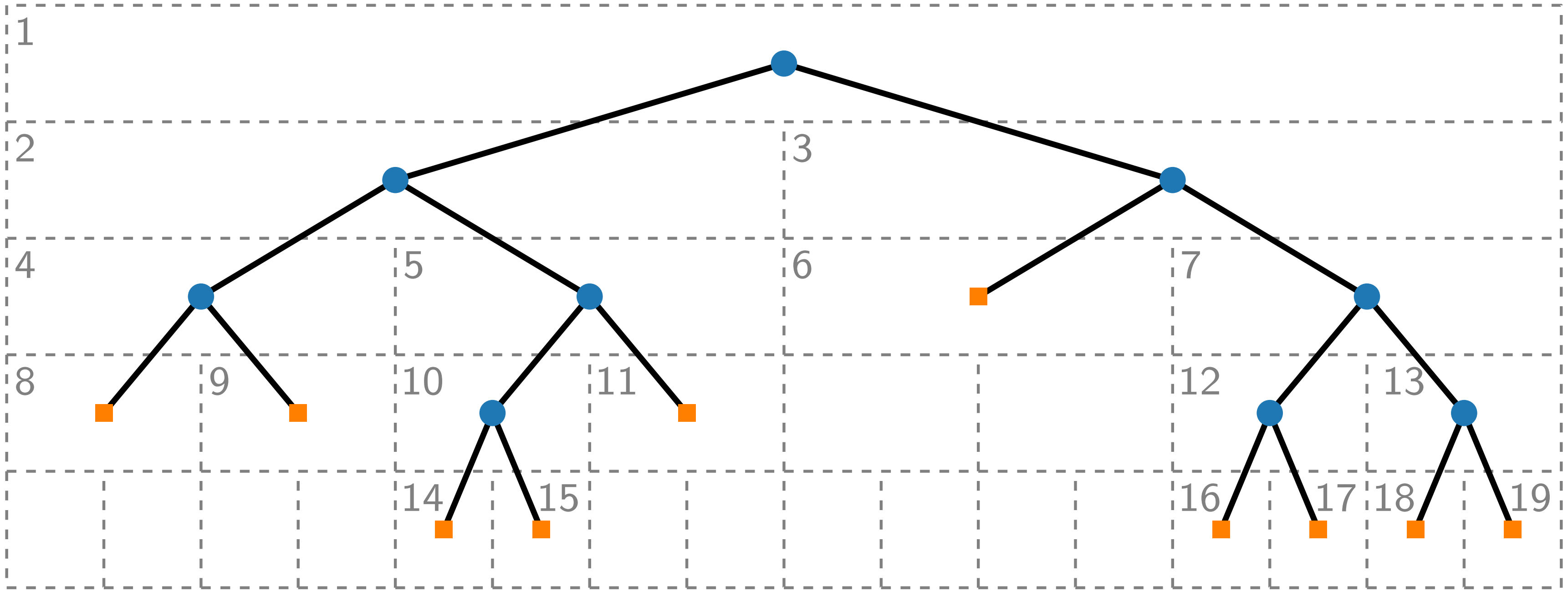
Idea.

- Add external nodes to have out-degree 2 or 0 at every node
- Read internal nodes as 1
- Read external nodes as 0

Size.

- $2n + 1$ bits for b
- $o(n)$ for `rank` and `select`

Succinct Representation of Binary Trees



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
b	1	1	1	1	1	0	1	0	0	1	0	1	1	0	0	0	0	0	0

Size.

- $2n + 1$ bits for b
- $o(n)$ for `rank` and `select`

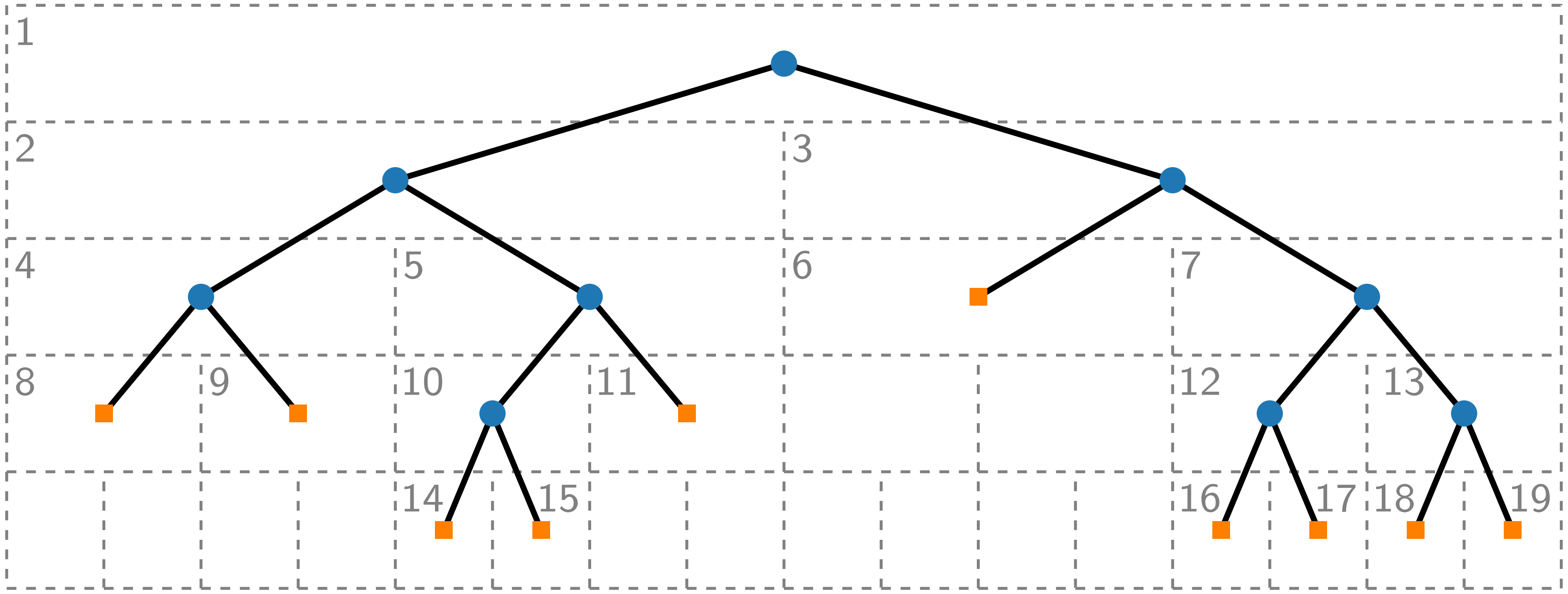
Idea.

- Add external nodes to have out-degree 2 or 0 at every node
- Read internal nodes as 1
- Read external nodes as 0

Operations.

- $\texttt{parent}(i) =$?
- $\texttt{leftChild}(i) =$?
- $\texttt{rightChild}(i) =$?

Succinct Representation of Binary Trees



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
b	1	1	1	1	1	0	1	0	0	1	0	1	1	0	0	0	0	0	0

Size.

- $2n + 1$ bits for b
- $o(n)$ for `rank` and `select`

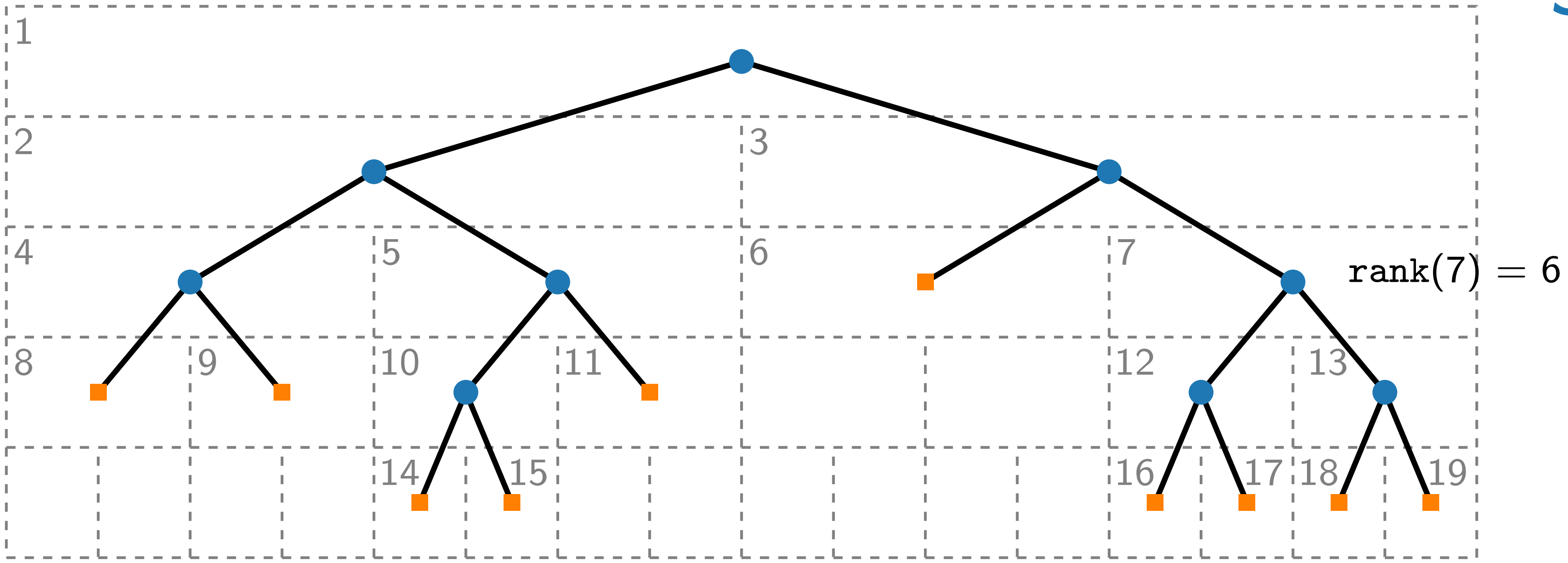
Idea.

- Add external nodes to have out-degree 2 or 0 at every node
- Read internal nodes as 1
- Read external nodes as 0
- Use `rank` and `select`

Operations.

- $\texttt{parent}(i) = \ ?$
- $\texttt{leftChild}(i) = \ ?$
- $\texttt{rightChild}(i) = \ ?$

Succinct Representation of Binary Trees



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
b	1	1	1	1	1	0	1	0	0	1	0	1	1	0	0	0	0	0	0

Size.

- $2n+1$ bits for b
- $o(n)$ for `rank` and `select`

Idea.

- Add external nodes to have out-degree 2 or 0 at every node
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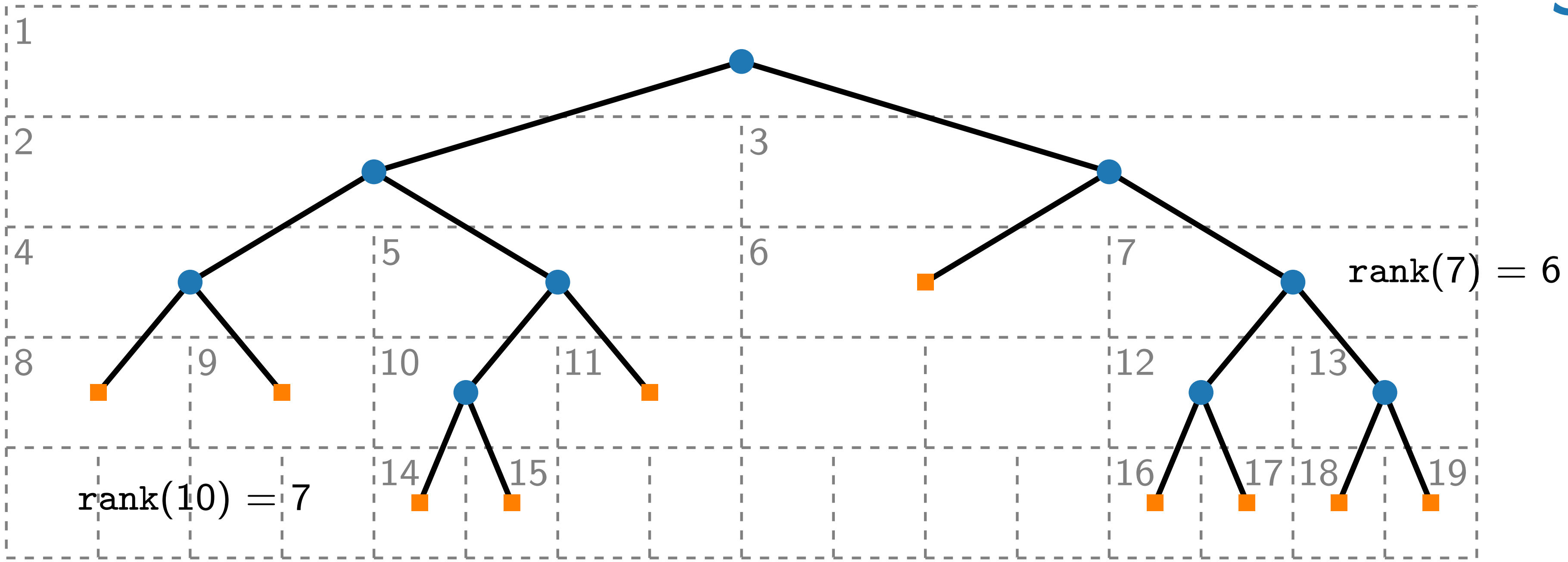
Operations.

- $\texttt{parent}(i) =$?
- $\texttt{leftChild}(i) =$?
- $\texttt{rightChild}(i) =$?

Succinct Representation of Binary Trees



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
b	1	1	1	1	1	0	1	0	0	1	0	1	1	0	0	0	0	0	0

Size.

- $2n+1$ bits for b
- $o(n)$ for `rank` and `select`

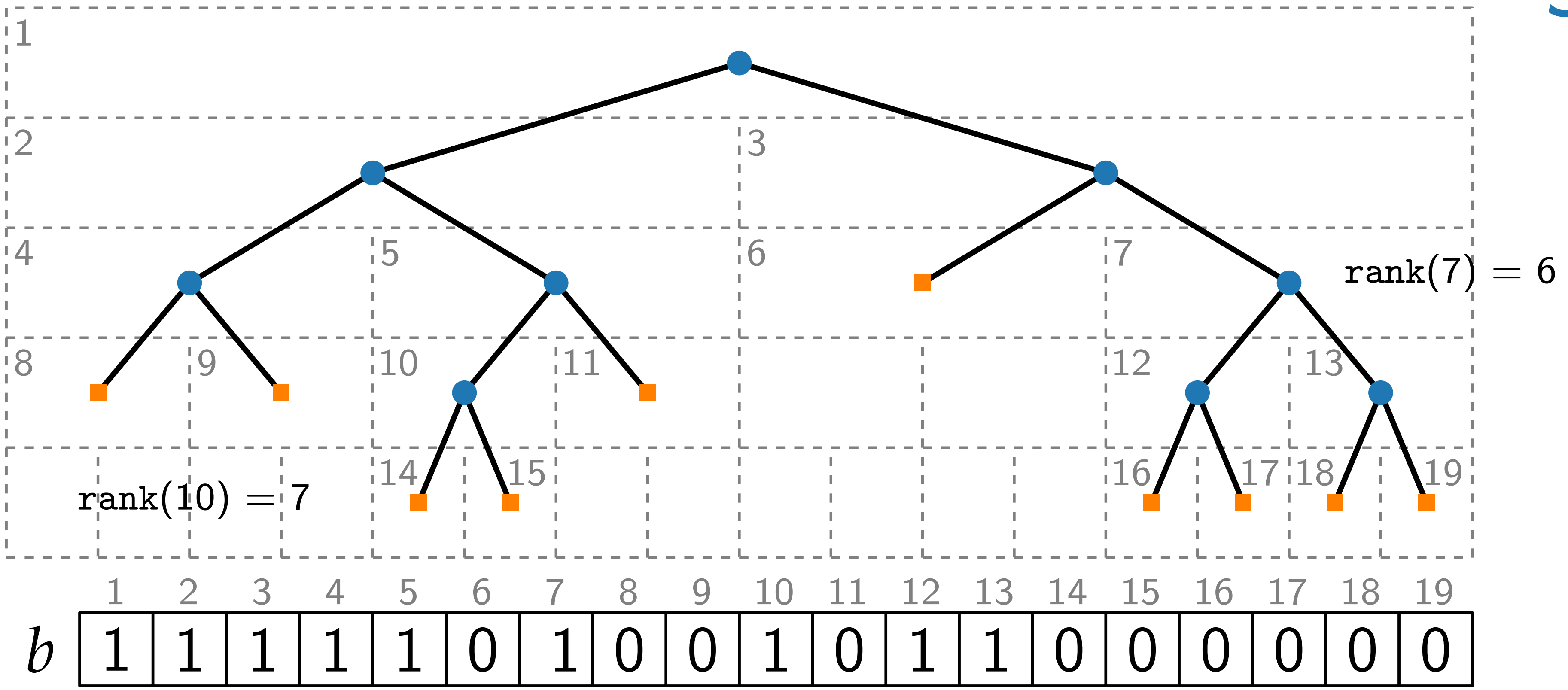
Idea.

- Add external nodes to have out-degree 2 or 0 at every node
- Read internal nodes as 1
- Read external nodes as 0
- Use `rank` and `select`

Operations.

- $\texttt{parent}(i) = \ ?$
- $\texttt{leftChild}(i) = \ ?$
- $\texttt{rightChild}(i) = \ ?$

Succinct Representation of Binary Trees



Size.

- $2n+1$ bits for b
- $o(n)$ for rank and select

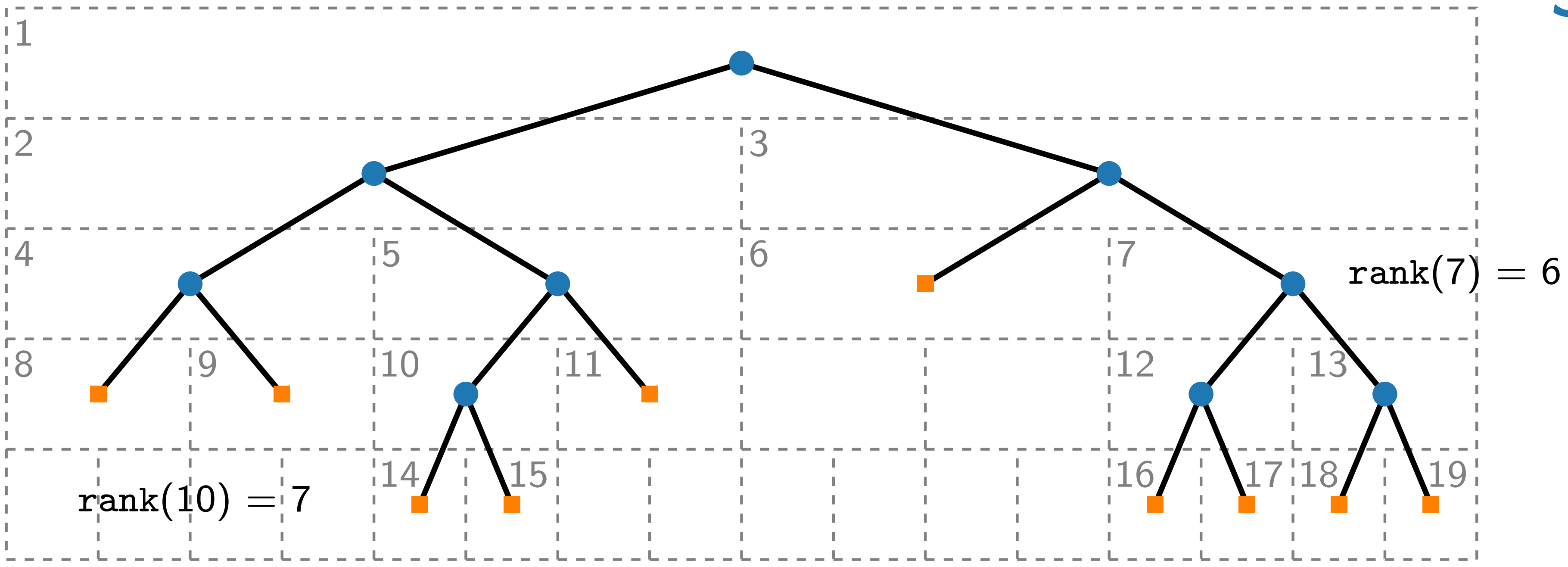
Idea.

- Add external nodes to have out-degree 2 or 0 at every node
- Read internal nodes as 1
- Read external nodes as 0
- Use rank and select

Operations.

- $\mathtt{parent}(i) = \ \ ?$
- $\mathtt{leftChild}(i) = 2\,\mathtt{rank}(i)$
- $\mathtt{rightChild}(i) = 2\,\mathtt{rank}(i) + 1$

Succinct Representation of Binary Trees



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
b	1	1	1	1	1	0	1	0	0	1	0	1	1	0	0	0	0	0	0

Size.

- $2n + 1$ bits for b
- $o(n)$ for `rank` and `select`

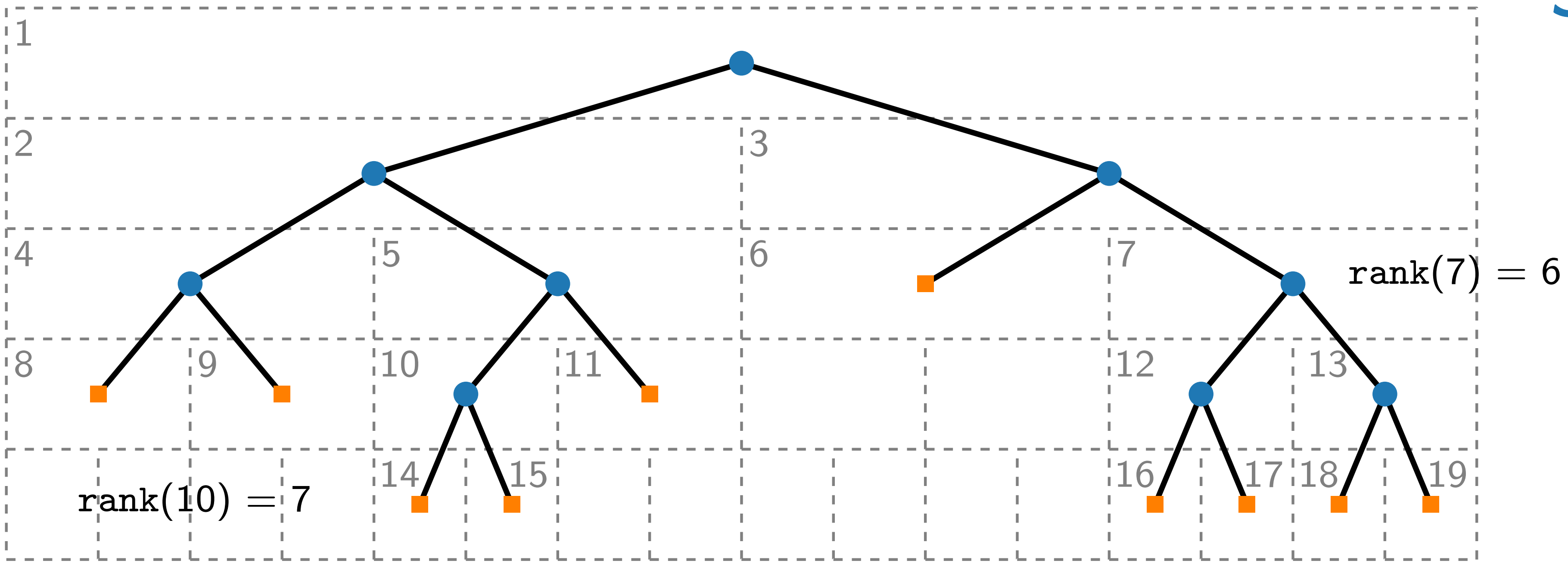
Idea.

- Add external nodes to have out-degree 2 or 0 at every node
- Read internal nodes as 1
- Read external nodes as 0
- Use `rank` and `select`

Operations.

- $\texttt{parent}(i) = \texttt{select}(\lfloor \frac{i}{2} \rfloor)$
- $\texttt{leftChild}(i) = 2\,\texttt{rank}(i)$
- $\texttt{rightChild}(i) = 2\,\texttt{rank}(i) + 1$

Succinct Representation of Binary Trees



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
b	1	1	1	1	1	0	1	0	0	1	0	1	1	0	0	0	0	0	0

Size.

- $2n + 1$ bits for b
- $o(n)$ for `rank` and `select`

Idea.

- Add external nodes to have out-degree 2 or 0 at every node
- Read internal nodes as 1
- Read external nodes as 0
- Use `rank` and `select`

Operations.

- $\texttt{parent}(i) = \texttt{select}(\lfloor \frac{i}{2} \rfloor)$
- $\texttt{leftChild}(i) = 2\,\texttt{rank}(i)$
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Proof is exercise.

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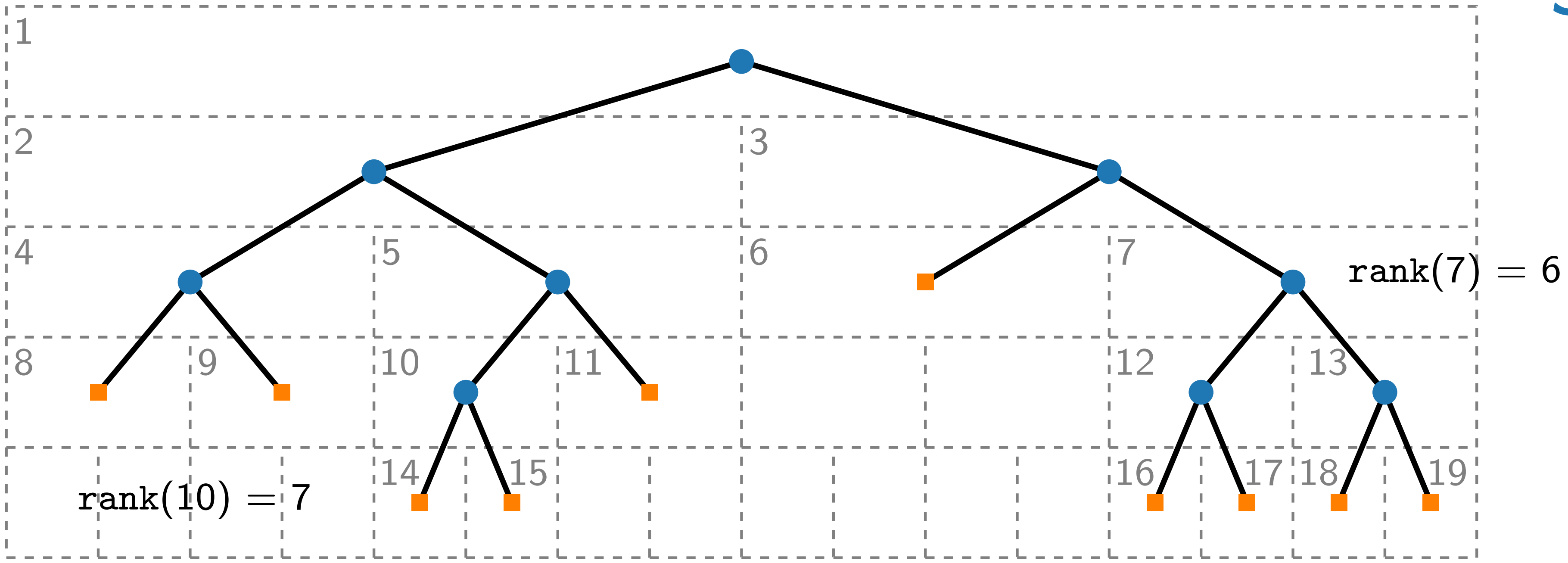

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
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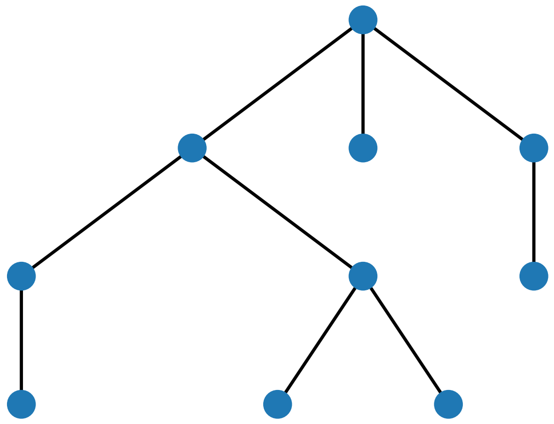
- $\texttt{rightChild}(i) = 2\,\texttt{rank}(i) + 1$
- $\texttt{rank}(i)$ is index for array storing actual values

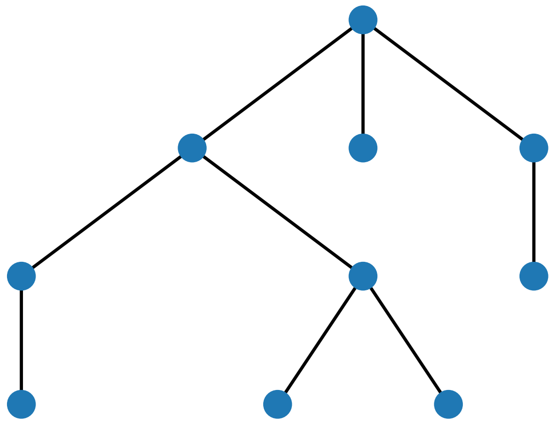
Proof is exercise.

Succinct Representation of Trees – LOUDS



Succinct Representation of Trees – LOUDS

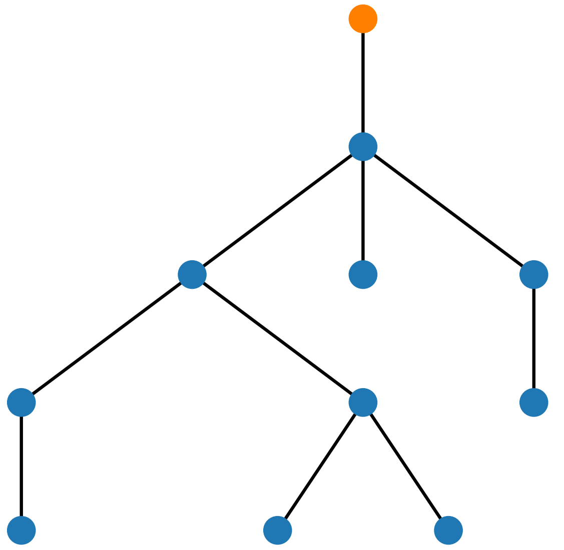
[Level Order Unary Degree Sequence]



Succinct Representation of Trees – LOUDS

[Level Order Unary Degree Sequence]

- add extra root with out-degree 1



Succinct Representation of Trees – LOUDS

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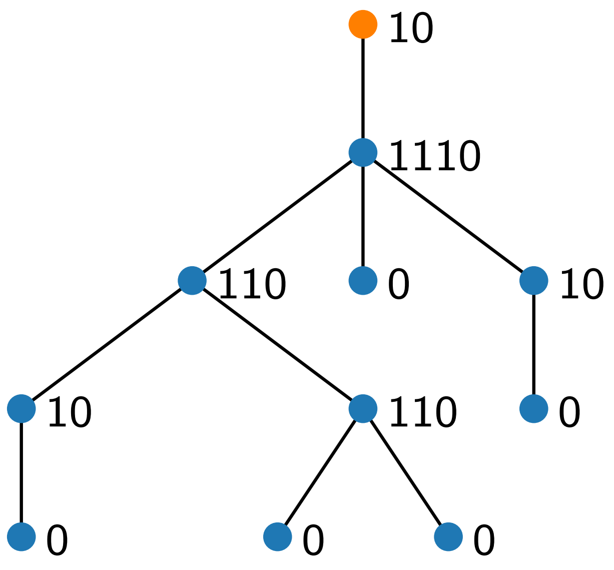
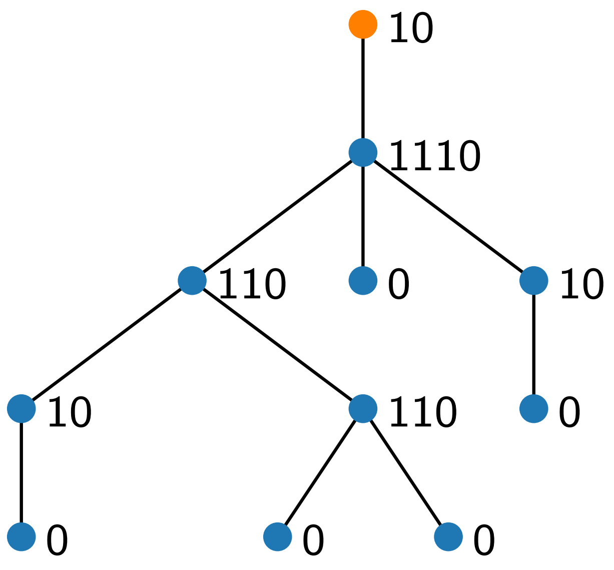


- add extra root with out-degree 1
- unary encoding of out-degree terminated by a 0

Succinct Representation of Trees – LOUDS

[Level Order Unary Degree Sequence]



- add extra root with out-degree 1
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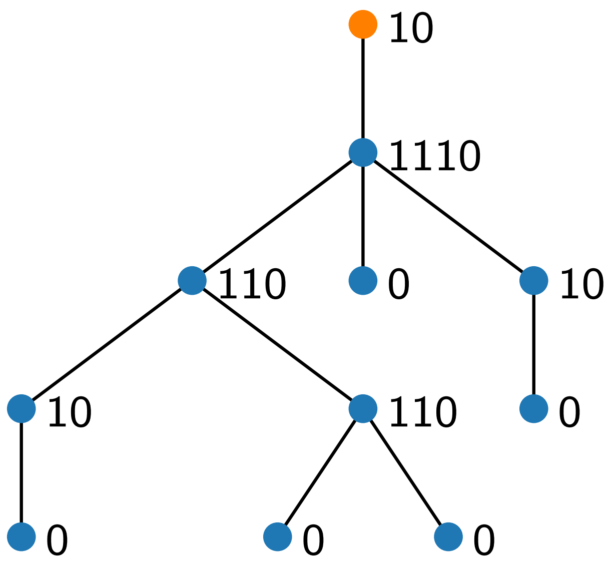

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Size.

- each vertex (except root) is represented twice, namely with a 1 and with a 0
- $o(n)$ bits for `rank` and `select`

Succinct Representation of Trees – LOUDS

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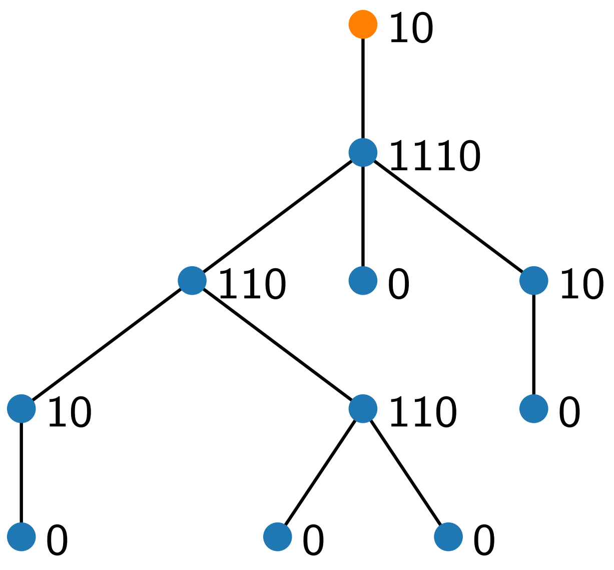

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$\Rightarrow 2n + o(n)$ bits

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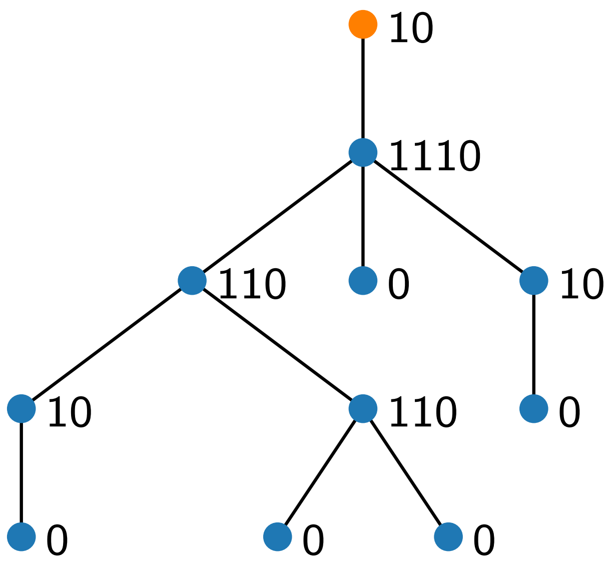

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1	0	1	1	1	0	1	1	0	0	1	0	1	0	1	1	0	0	0	0	0

Operations.

- Let i be index of 1 in LOUDS sequence. This 1 represents a node (e.g. first 1 represents the root).
- $\texttt{rank}(i)$ is index for array storing actual values of the nodes.

Succinct Representation of Trees – LOUDS

[Level Order Unary Degree Sequence]

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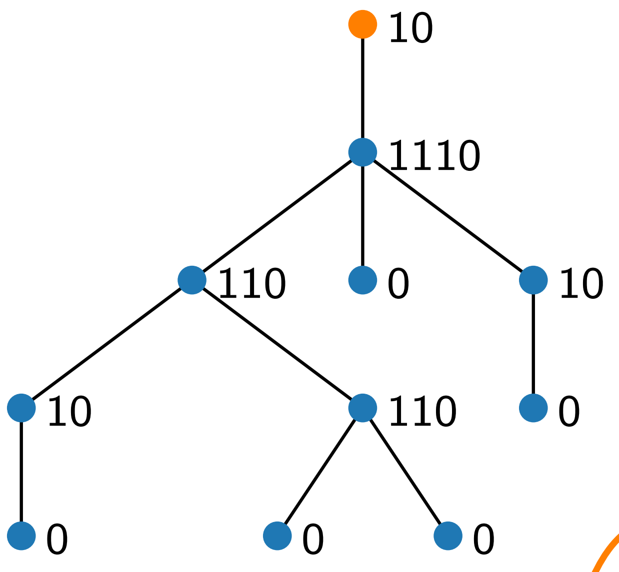

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	0	1	1	1	0	1	1	0	0	1	0	1	0	1	1	0	0	0	0	0

execute $\texttt{select}(j)$ on the 0s instead of the 1s

execute $\texttt{rank}(i)$ on the 1s (as before)

- $\texttt{firstChild}(i) = \texttt{select}_0(\texttt{rank}_1(i)) + 1$

Succinct Representation of Trees – LOUDS

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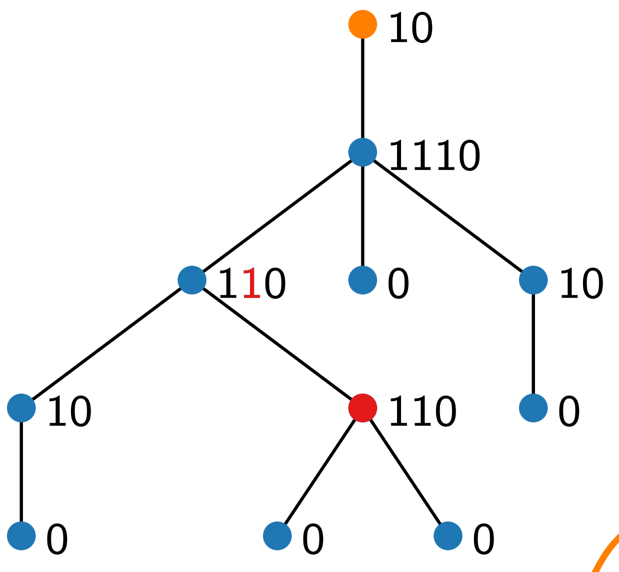

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  $\texttt{firstChild}(8) = \texttt{select}_0(\texttt{rank}_1(8)) + 1$

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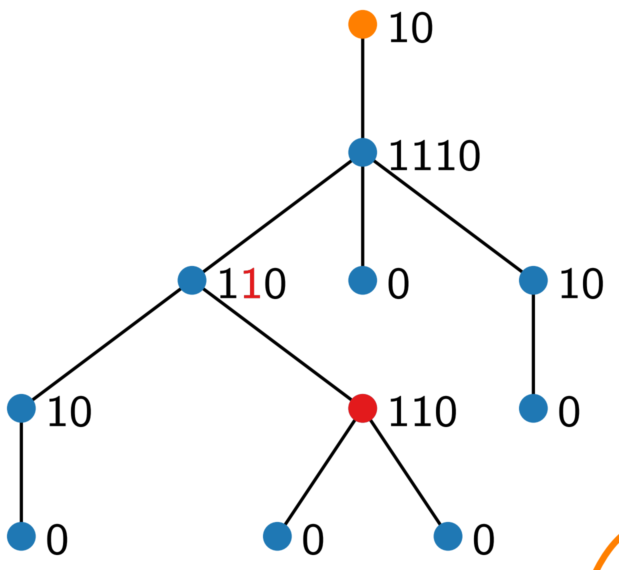

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
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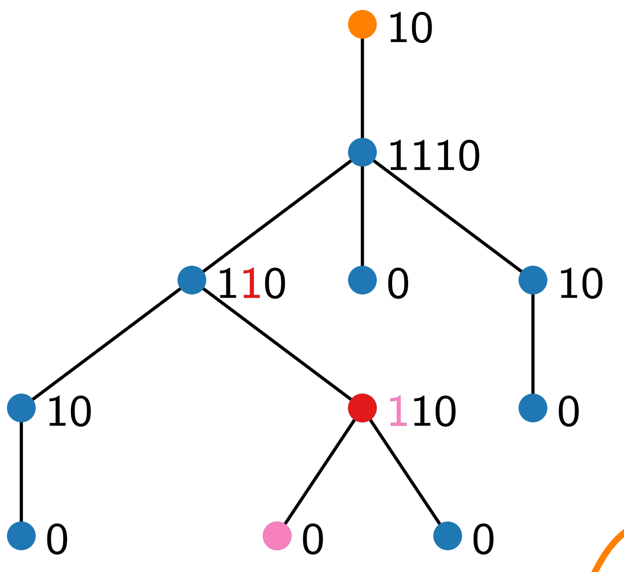
$\texttt{firstChild}(8) = \texttt{select}_0(\texttt{rank}_1(8)) + 1$
$= \texttt{select}_0(6) + 1$

Succinct Representation of Trees – LOUDS

[Level Order Unary Degree Sequence]



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$\texttt{firstChild}(8) = \texttt{select}_0(\texttt{rank}_1(8)) + 1$
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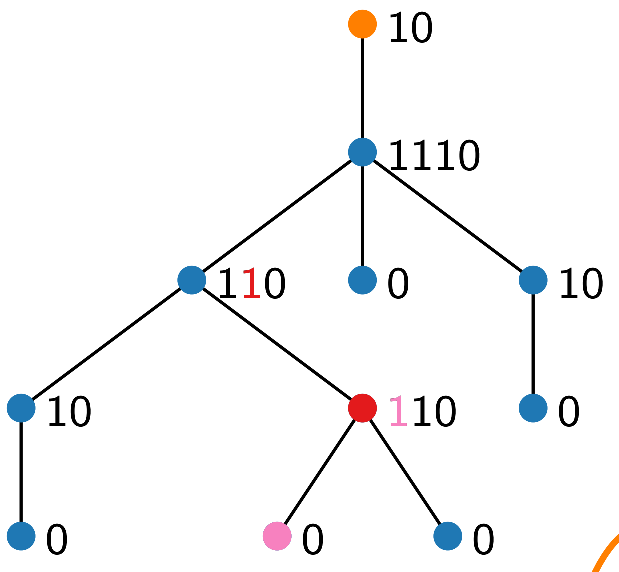
Succinct Representation of Trees – LOUDS

[Level Order Unary Degree Sequence]



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- $\texttt{nextSibling}(i) = i + 1$

Succinct Representation of Trees – LOUDS

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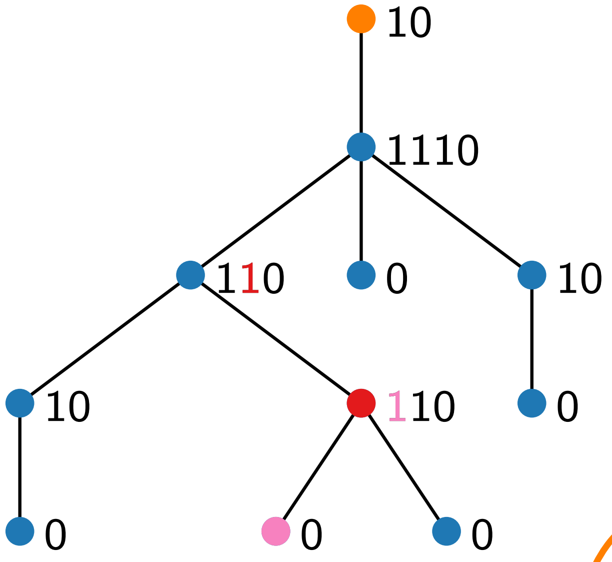

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Exercise: $\texttt{child}(i, j)$ with validity check

Succinct Representation of Trees – LOUDS

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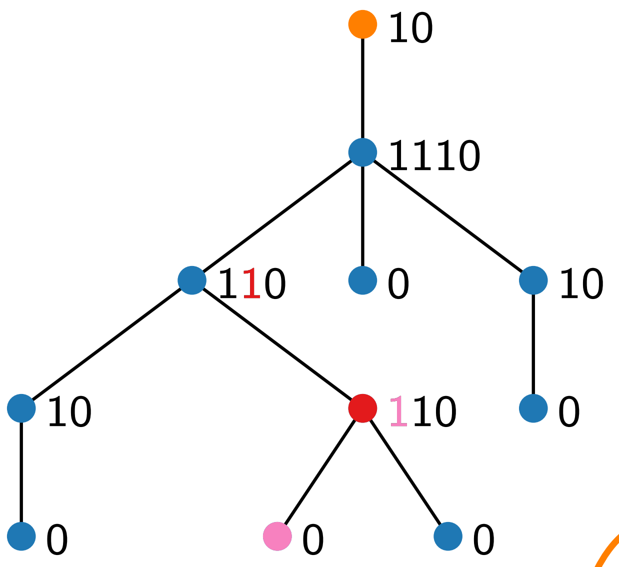

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- $\mathtt{firstChild}(i) = \mathtt{select}_0(\mathtt{rank}_1(i)) + 1$
- $\mathtt{parent}(i) = \mathtt{select}_1(\mathtt{rank}_0(i))$

$\mathtt{firstChild}(8) = \mathtt{select}_0(\mathtt{rank}_1(8)) + 1$
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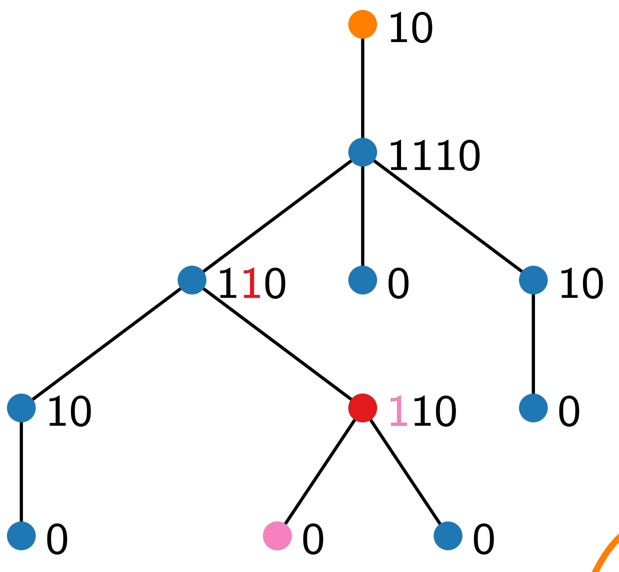

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  $= \texttt{select}_0(6) + 1 = 14 + 1 = 15$

- $\texttt{parent}(i) = \texttt{select}_1(\texttt{rank}_0(i))$

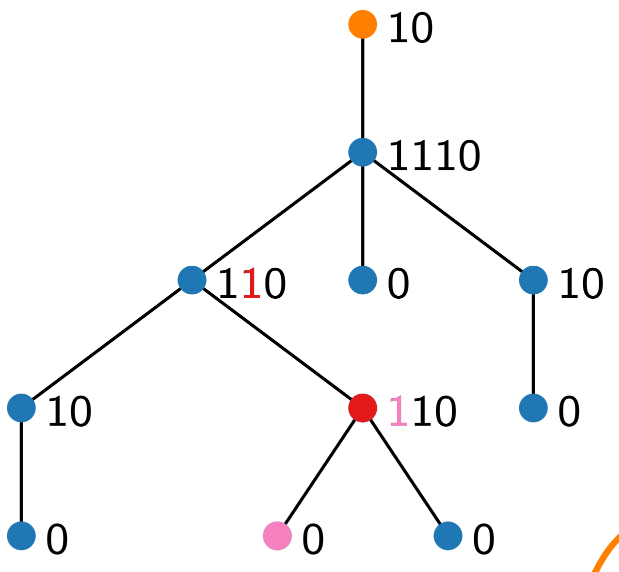
  $\texttt{parent}(8) = \texttt{select}_1(\texttt{rank}_0(8))$

- $\texttt{nextSibling}(i) = i + 1$

Exercise: $\texttt{child}(i, j)$ with validity check

Succinct Representation of Trees – LOUDS

[Level Order Unary Degree Sequence]



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- $\mathtt{parent}(i) = \mathtt{select}_1(\mathtt{rank}_0(i))$

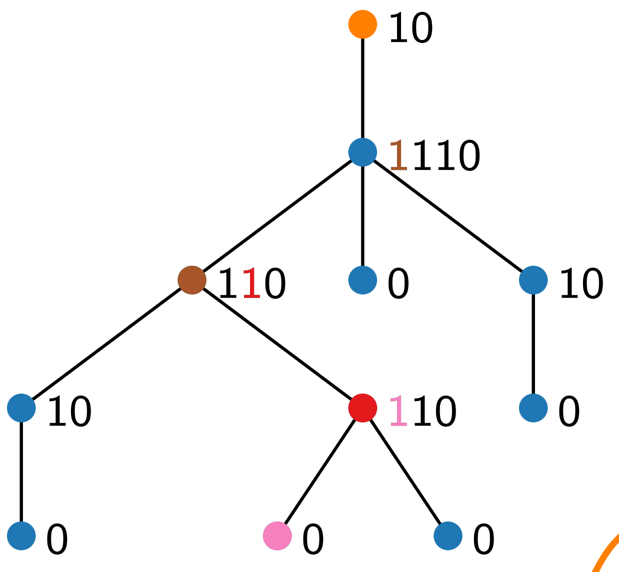
  $\mathtt{parent}(8) = \mathtt{select}_1(\mathtt{rank}_0(8))$
  $= \mathtt{select}_1(2)$

- $\mathtt{nextSibling}(i) = i + 1$

Exercise: $\mathtt{child}(i, j)$
with validity check

Succinct Representation of Trees – LOUDS

[Level Order Unary Degree Sequence]



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1	0	1	1	1	0	1	1	0	0	1	0	1	0	1	1	0	0	0	0	0

execute select(j) on the 0s instead of the 1s

execute rank(i) on the 1s (as before)

- $\texttt{firstChild}(i) = \texttt{select}_0(\texttt{rank}_1(i)) + 1$

  $\texttt{firstChild}(8) = \texttt{select}_0(\texttt{rank}_1(8)) + 1$
  $= \texttt{select}_0(6) + 1 = 14 + 1 = 15$

- $\texttt{parent}(i) = \texttt{select}_1(\texttt{rank}_0(i))$

  $\texttt{parent}(8) = \texttt{select}_1(\texttt{rank}_0(8))$
  $= \texttt{select}_1(2) = 3$

- $\texttt{nextSibling}(i) = i + 1$

Exercise: $\texttt{child}(i, j)$ with validity check

Discussion

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  - space efficient
  - support fast operations

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  - support fast operations

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    that is, insertions & deletions
  - number of operations is limited,
  - complex $\rightarrow$ harder to implement,
  - the $o(n)$ and $O(1)$ terms hide constants that may dominate before any asymptotic advantage over the "best" compact data structures becomes apparent.

    $\rightarrow$ primarily a theoretical result (also does not consider hardware architecture)

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- `rank` and `select` form the basis for many succinct representations (e.g., for specific types of trees or strings).

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    $\rightarrow$ primarily a theoretical result (also does not consider hardware architecture)
- `rank` and `select` form the basis for many succinct representations (e.g., for specific types of trees or strings).
- There are implementations of succinct data structures being used in practice for large data sets in information retrieval, language model representation, bioinformatics, etc.

Literature

Main reference:

- Lecture 17 of Advanced Data Structures (MIT, Fall'17) by Erik Demaine
- [Jacobson 1989] "Space efficient Static Trees and Graphs"

Recommendations:

- Lecture 18 of Demaine's course on compact & succinct arrays & trees