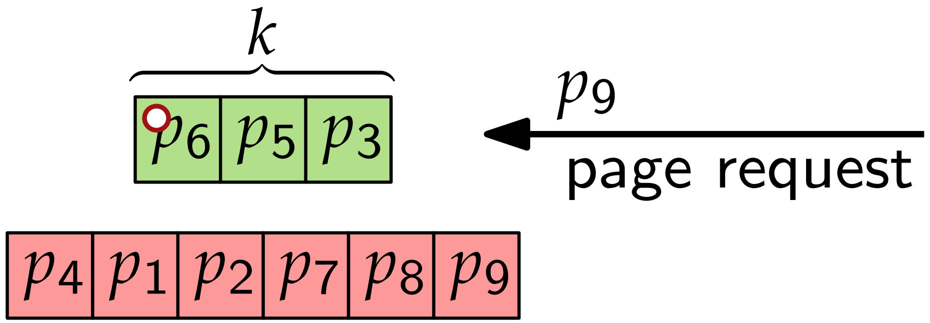
Advanced Algorithms

Online Algorithms

Ski-Rental Problem and Paging



Alexander Wolff

Summer term 2026

Introduction

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Ski-Rental Problem

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- Is it worth **buying** new skis?
- Or should we rather **rent** them?

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- But what if there is not always enough snow? Or snow but "bad" weather?
- Is it worth **buying** new skis?
- Or should we rather **rent** them?
- We don't know the weather (much) in advance.

Ski-Rental Problem – Definition

Behavior.

- Every day when there is “good” weather, you go skiing.
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Task.

- Not knowing T, devise a strategy if and when to buy skis.

Ski-Rental Problem – Strategies I and II

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Buying costs M
T good days

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- Imagine this was the only good day the whole winter.

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- So Strategy I is M times worse than the optimal strategy.

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Strategy II: never buy, **always rent**

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competitive ratio

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costs for deterministic startegy

costs for optimal startegy

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Strategy IV: throw a coin; **HEADS:** buy on the $\mathbf{M}$-th good day
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- Case $T = \alpha M$: $\frac{E[c_{\text{StrategyIV}}]}{c_{\text{OPT}}} = \frac{\frac{1}{2}\cdot\alpha M+\frac{1}{2}\cdot((1+\alpha)M-1)}{\alpha M} = 1 + \frac{1}{2\alpha} - \frac{1}{2\alpha M} \overset{M\rightsquigarrow\infty}{=} 1 + \frac{1}{2\alpha} = 2$

not better than the deterministic Strategy III

Ski-Rental Problem – Strategy IV

Renting costs 1 per day
Buying costs M
T good days

Can we get below this bound using randomization? – Let's try!

Strategy IV: throw a coin; **HEADS:** buy on the $\mathbf{M}$-th good day
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$\Rightarrow$ Strategy IV (with $\alpha = \frac{\sqrt{5}-1}{2} \approx 0.62$) is 1.81-competitive, randomized, and better than any deterministic strategy.

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- With a more sophisticated probability distribution for the point in time when we buy skis, we can get an expected competitive ratio of even $\frac{e}{e-1} \approx 1.58$.

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Given (offline/online):

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$$\overbrace{\boxed{p_1}\boxed{p_5}\boxed{p_8}}^{k}$$

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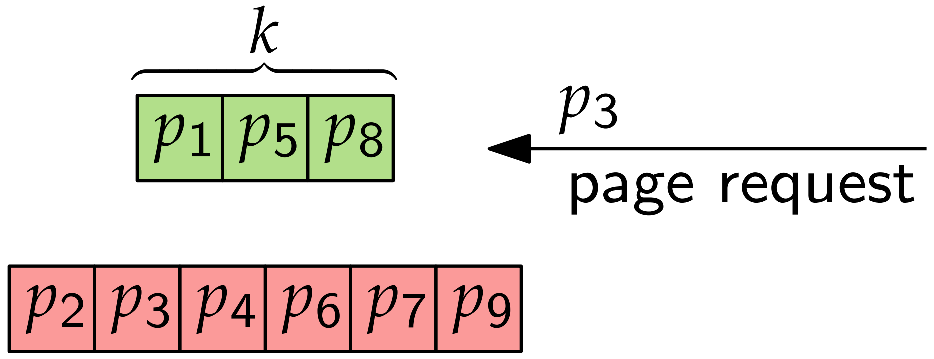

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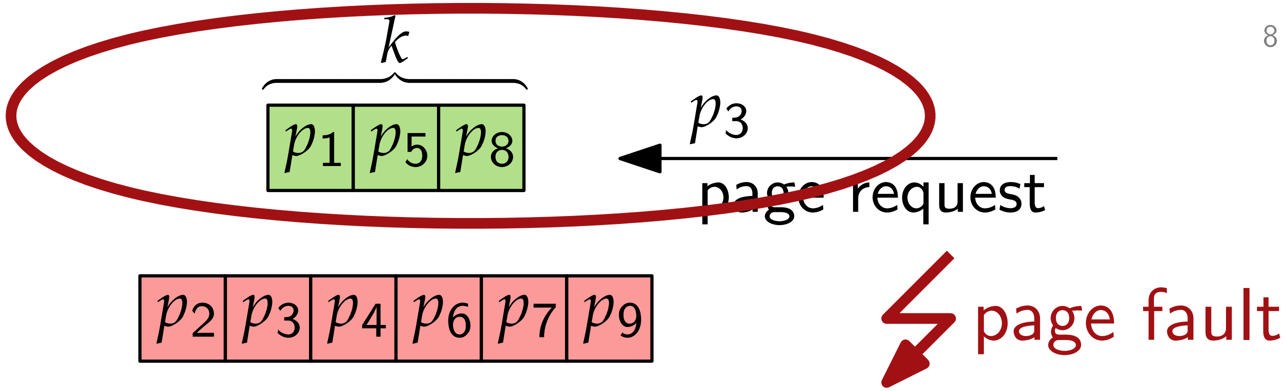

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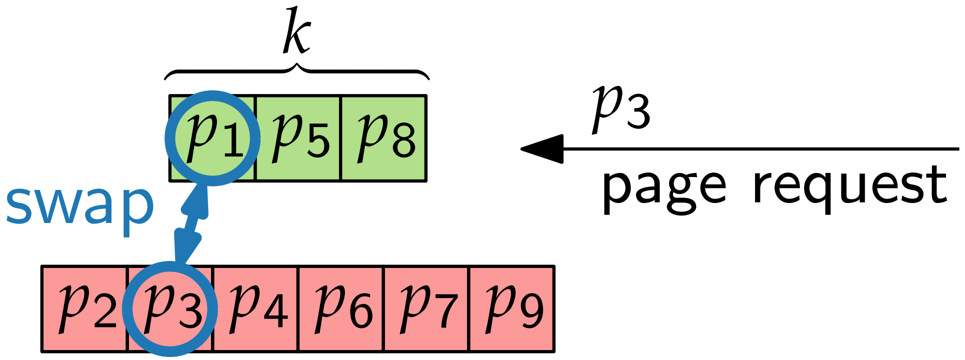

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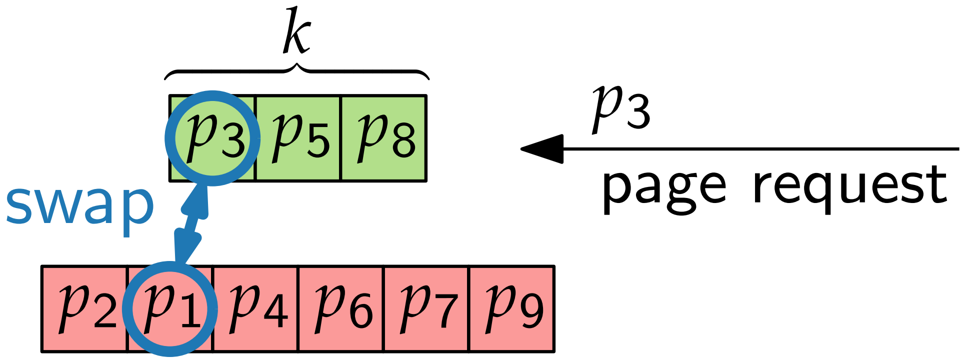

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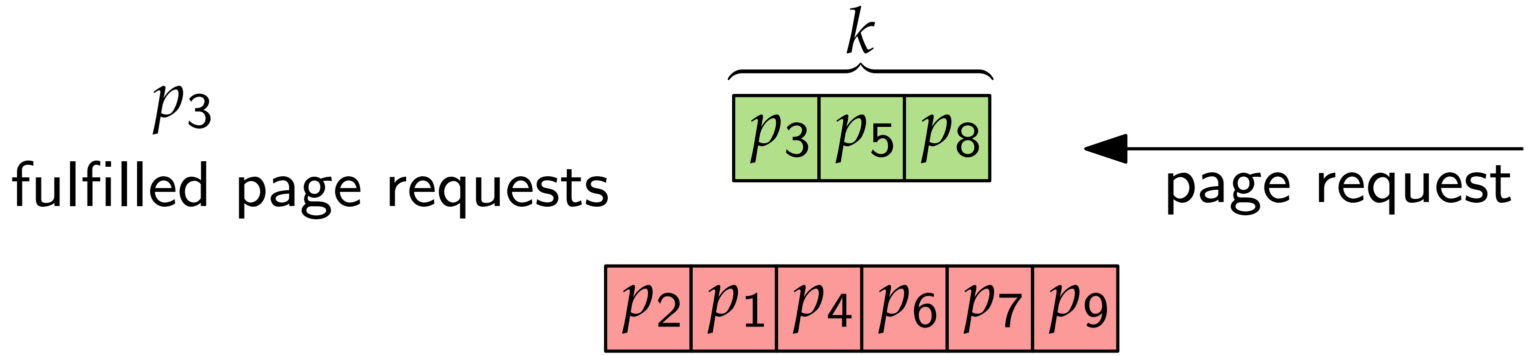

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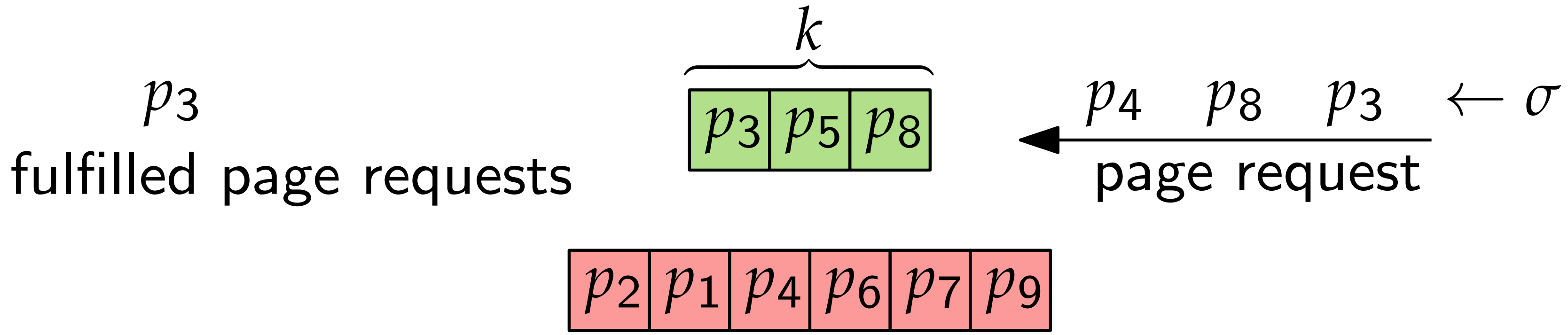

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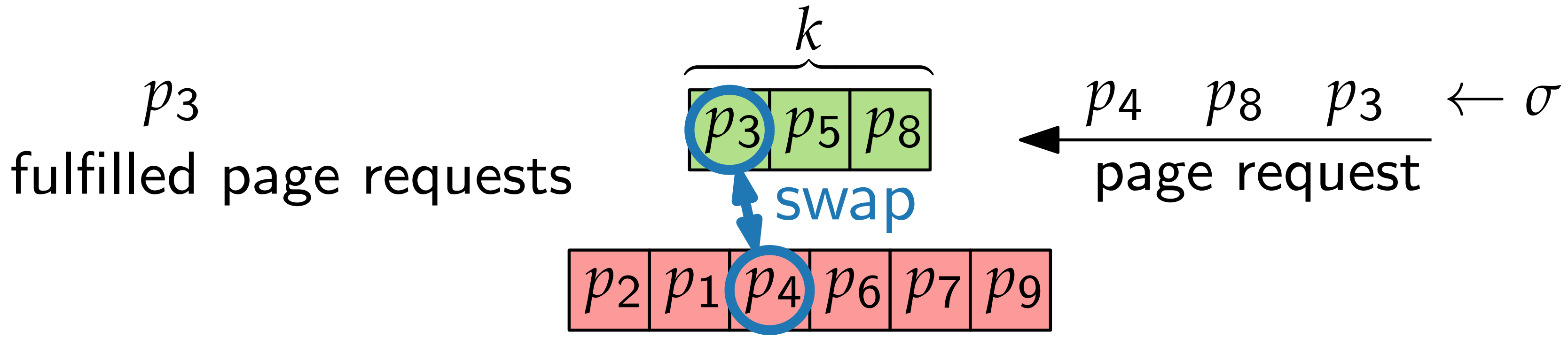

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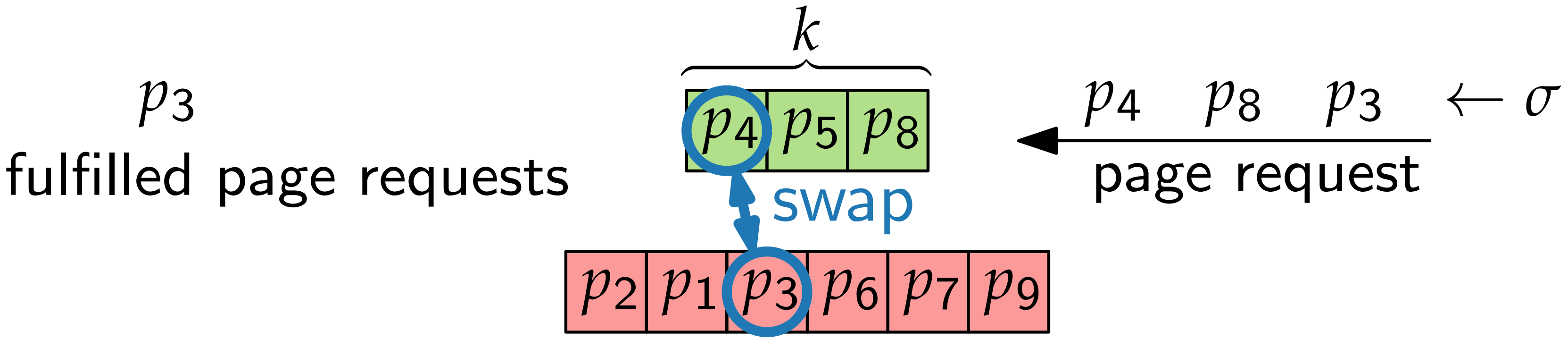

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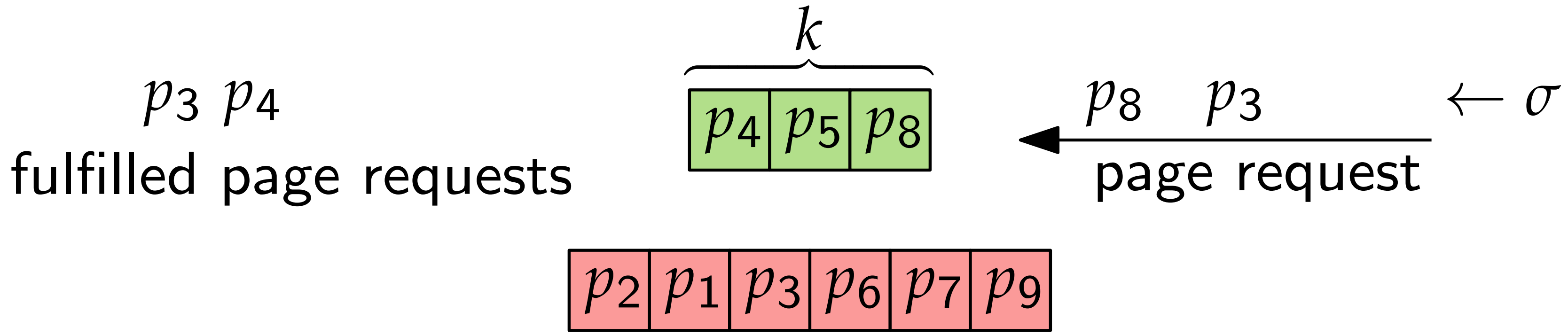

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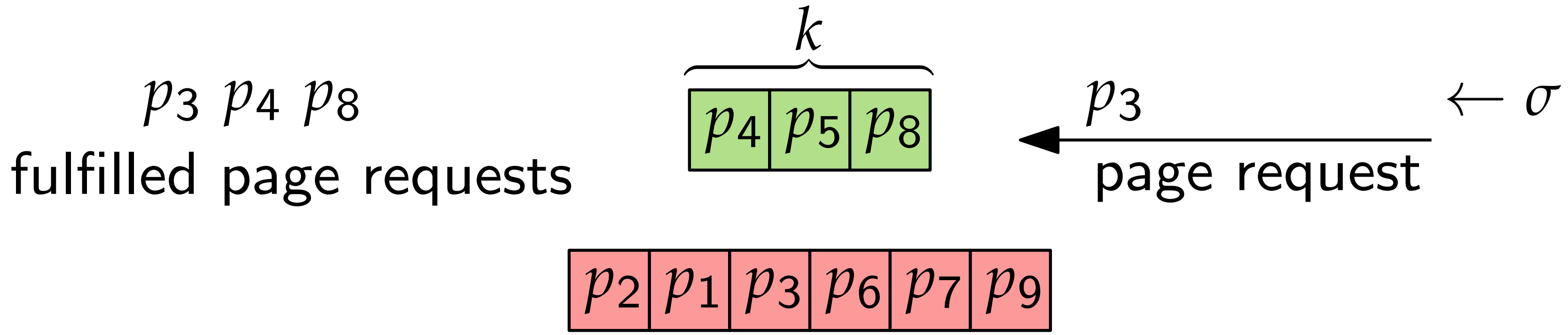

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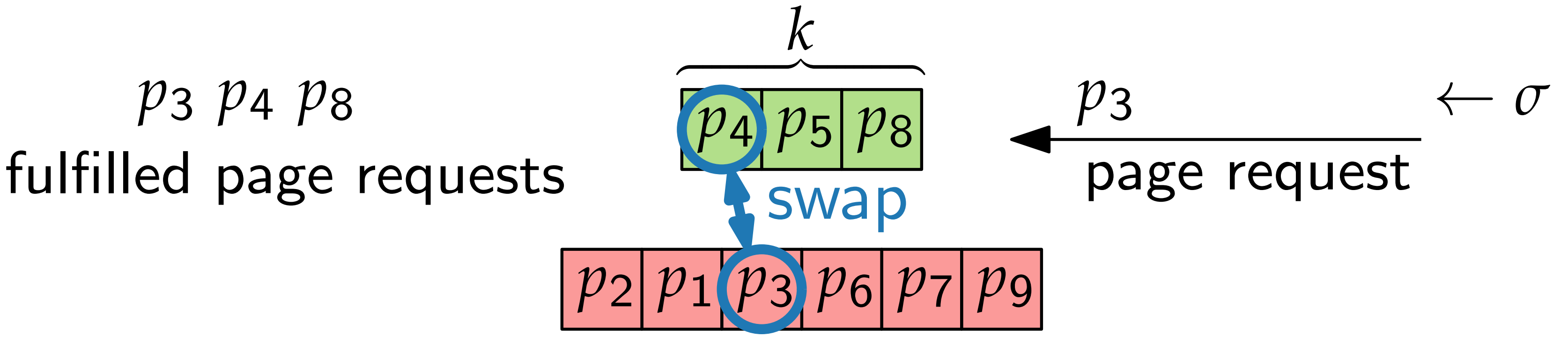

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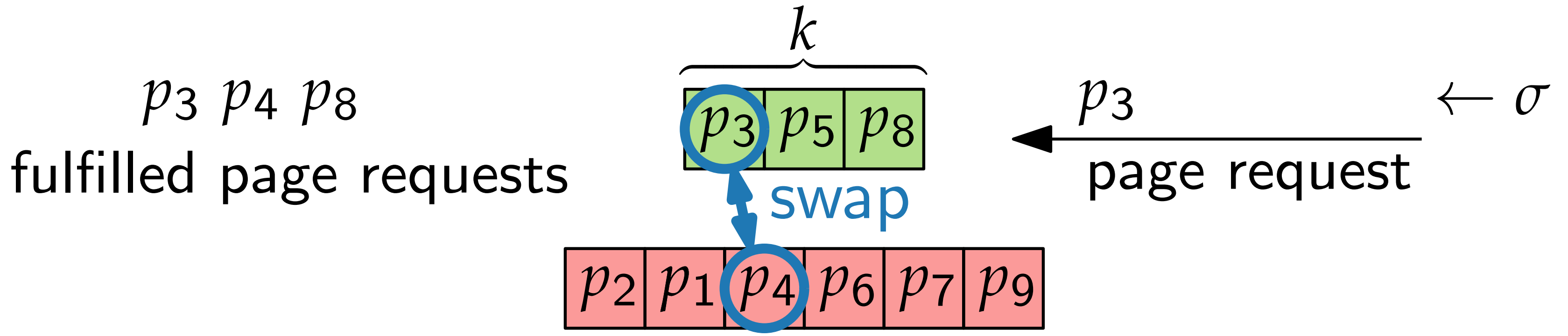

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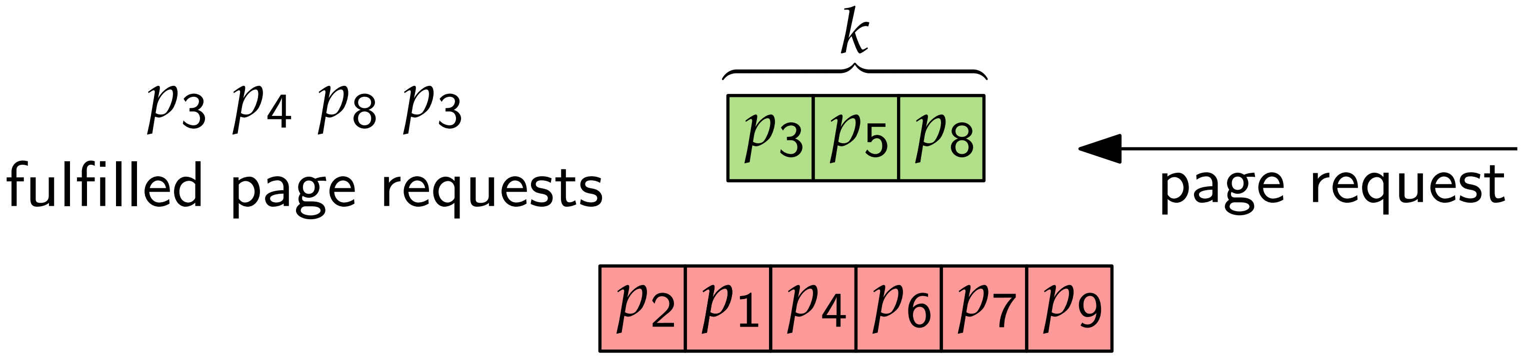

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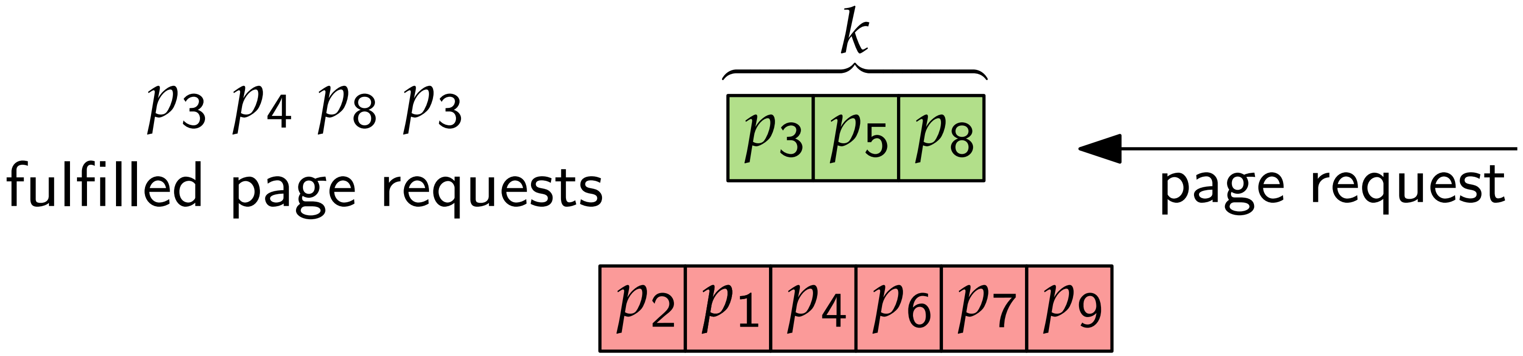

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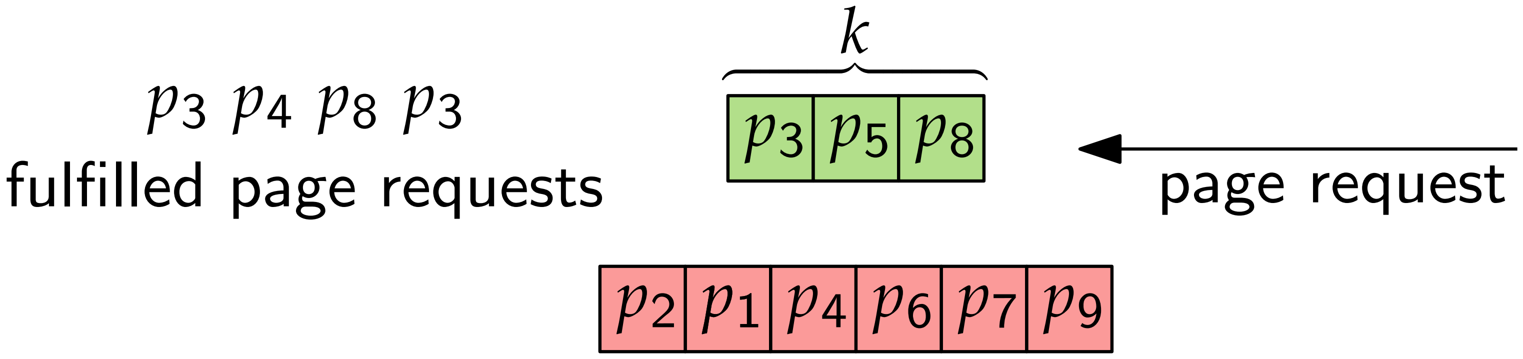

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Objective value:

- Minimize the number of page faults while fulfilling σ.

Paging: Det. Strategies

- On a page fault, a paging algorithm chooses which page to evict from the cache.

Paging: Det. Strategies

$p_4\ p_8\ p_8\ p_5\ p_4$

fulfilled page requests

$\overbrace{\boxed{p_4}\boxed{p_5}\boxed{p_8}}^{k}$

p_3 $\leftarrow \sigma$

page request

$\boxed{p_1}\boxed{p_2}\boxed{p_3}\boxed{p_6}\boxed{p_7}\boxed{p_9}$

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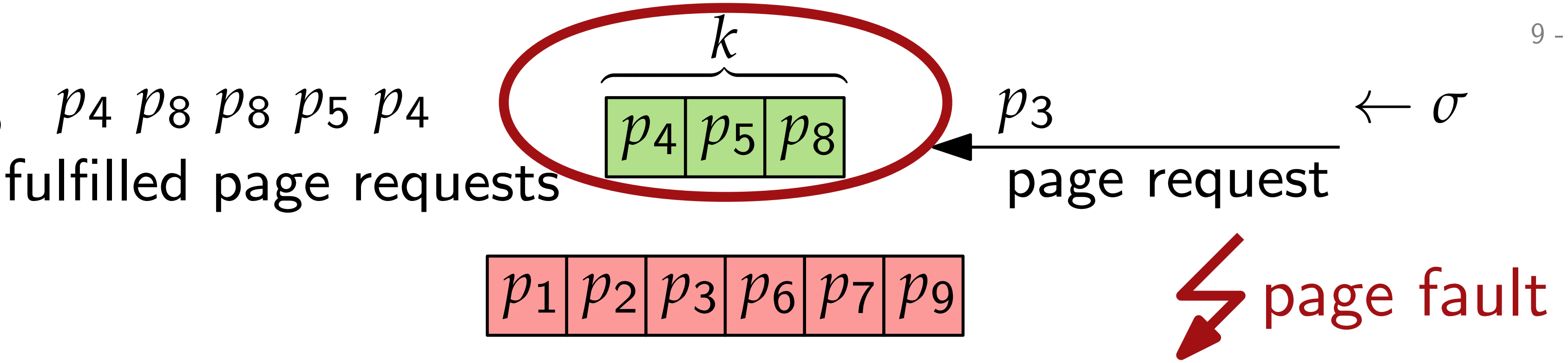

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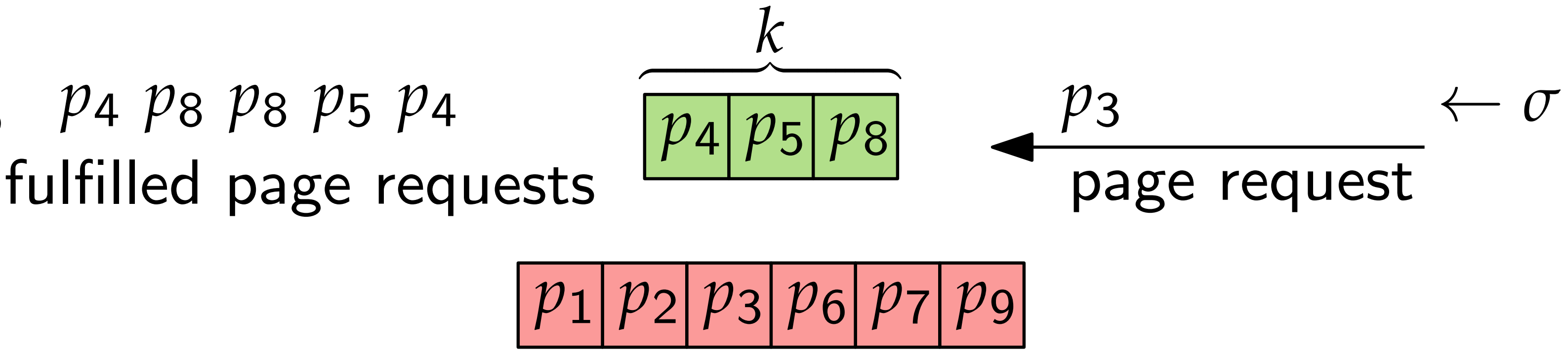

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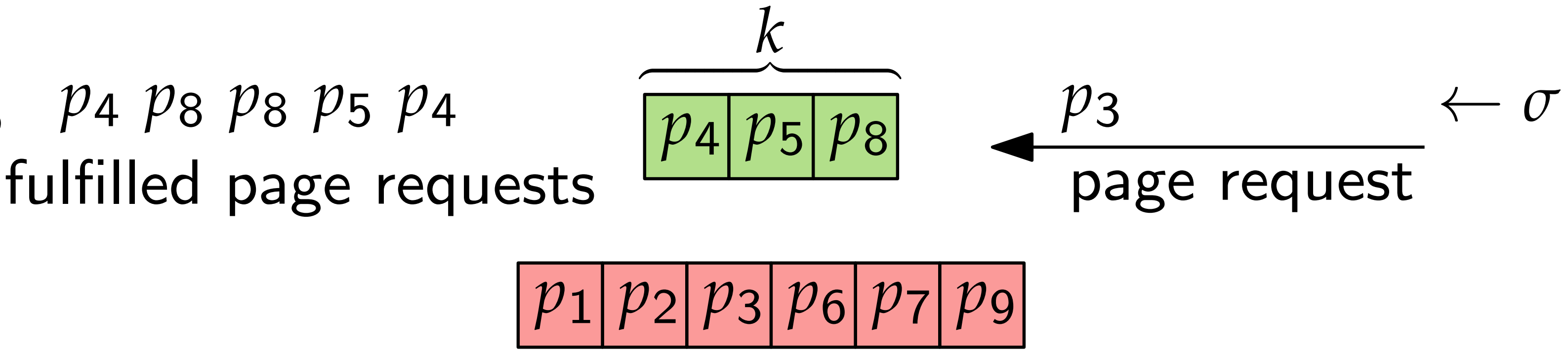

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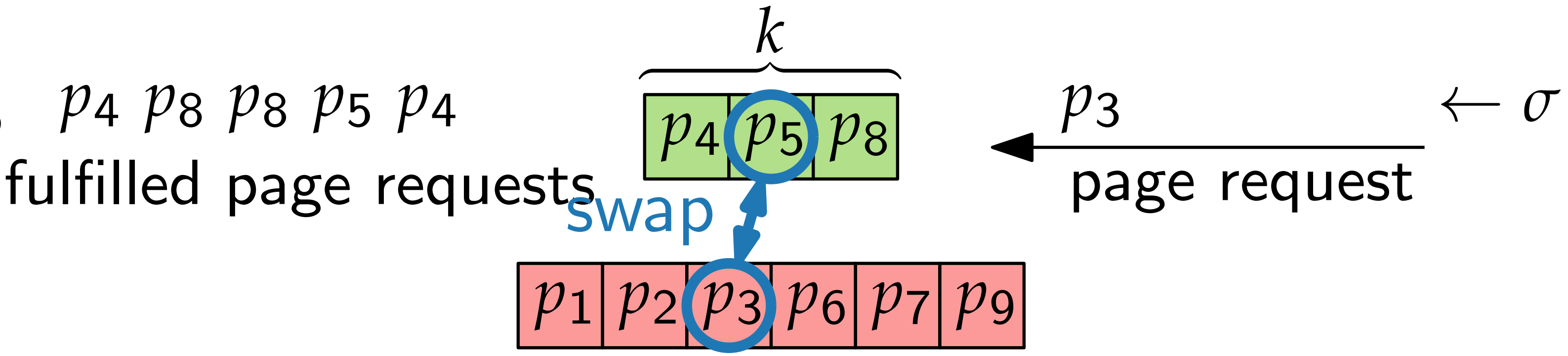

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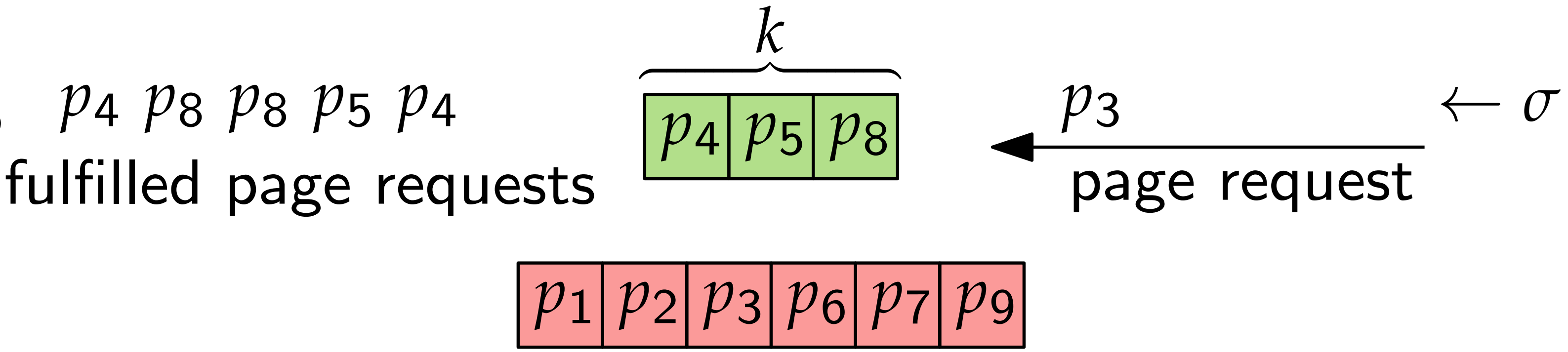

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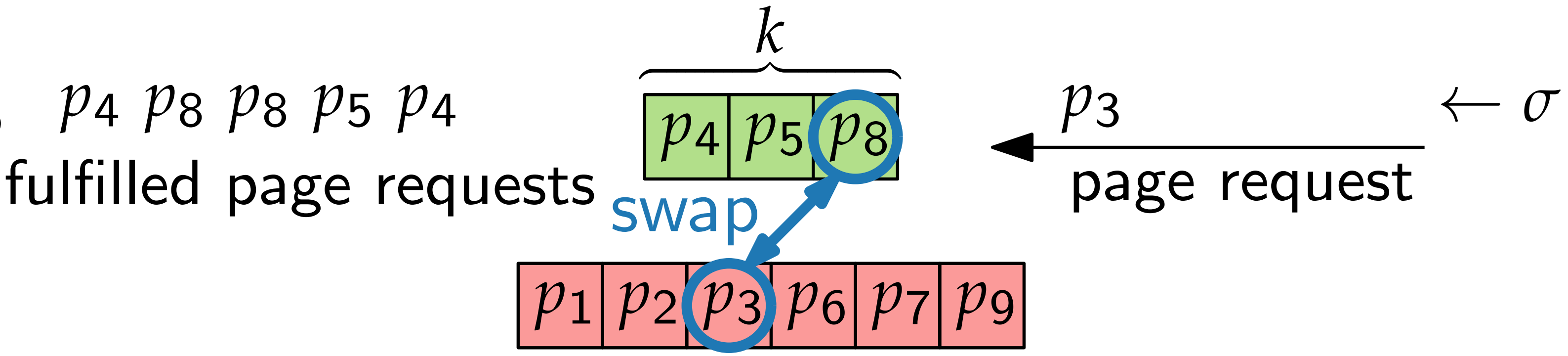

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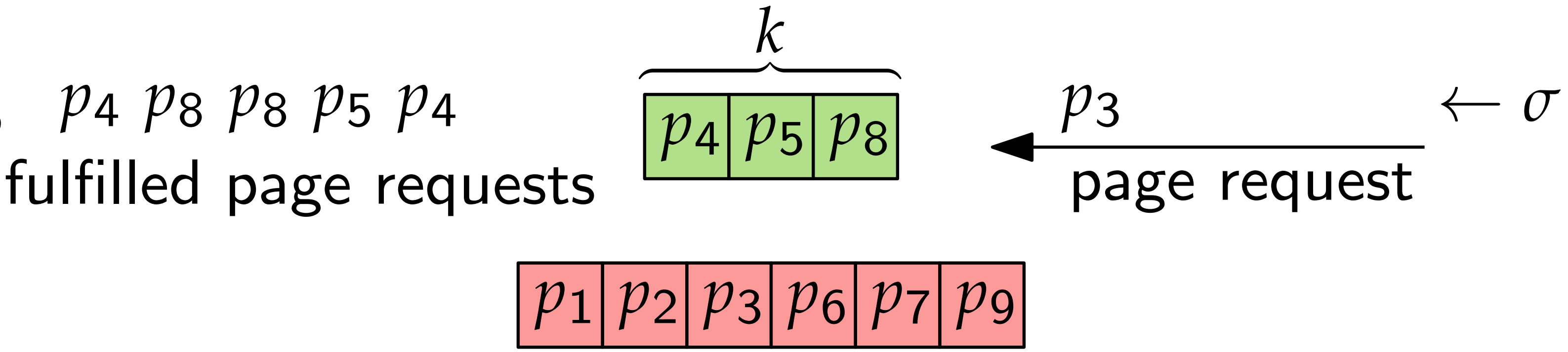

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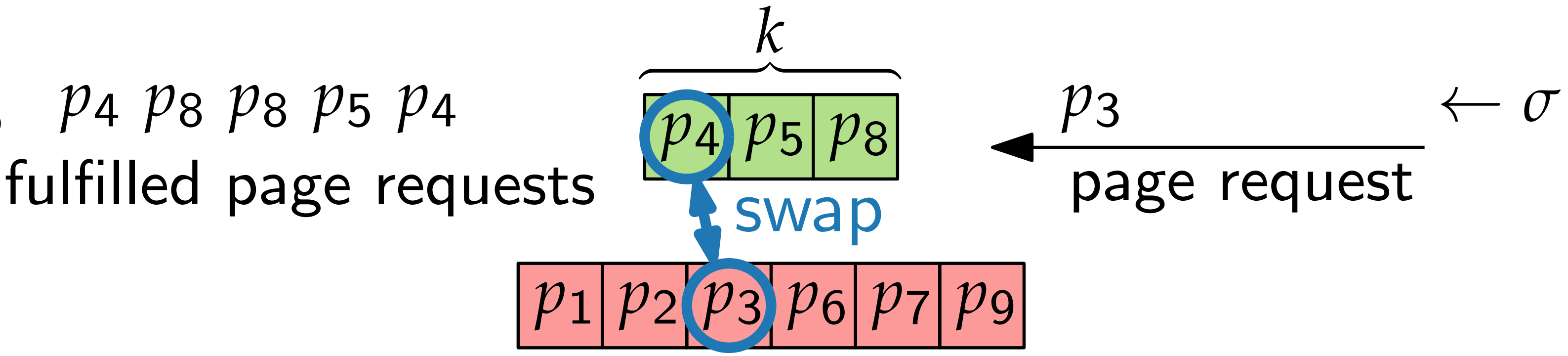
Paging: Det. Strategies



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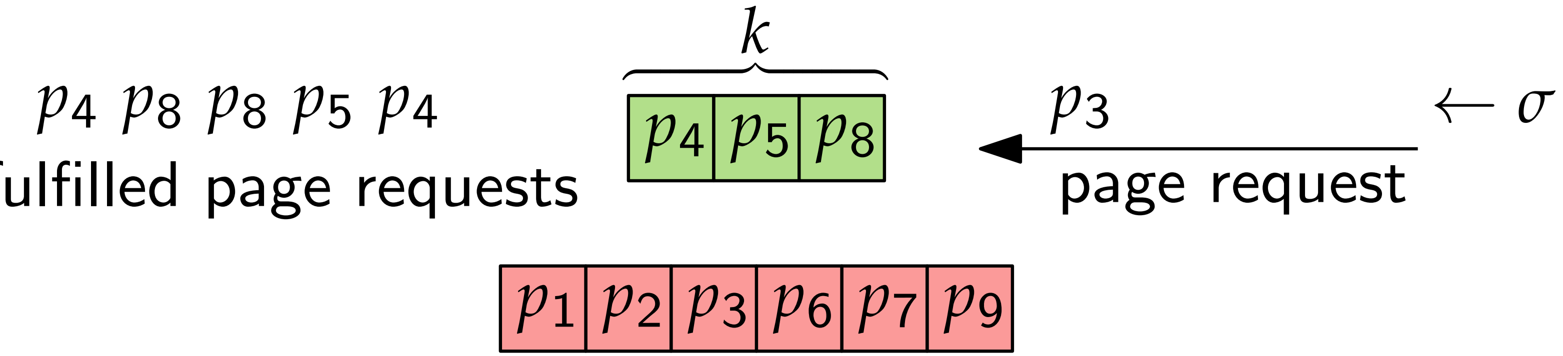

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Paging: Det. Strategies

$p_4\ p_8\ p_8\ p_5\ p_4$ fulfilled page requests $\overbrace{\boxed{p_4}\boxed{p_5}\boxed{p_8}}^{k}$ $\xleftarrow[\text{page request}]{p_3}$ $\leftarrow \sigma$

$\boxed{p_1}\boxed{p_2}\boxed{p_3}\boxed{p_6}\boxed{p_7}\boxed{p_9}$

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$\Rightarrow$ The competitive ratio cannot be better than $\dfrac{|\sigma^\star|}{\left\lceil \frac{|\sigma^\star|}{k} \right\rceil} \overset{|\sigma^\star| \rightsquigarrow \infty}{=} k$.

$\square$

Paging: Randomized Strategies

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$$\overbrace{\boxed{p_1}\boxed{p_2}\boxed{p_3}}^{k}$$

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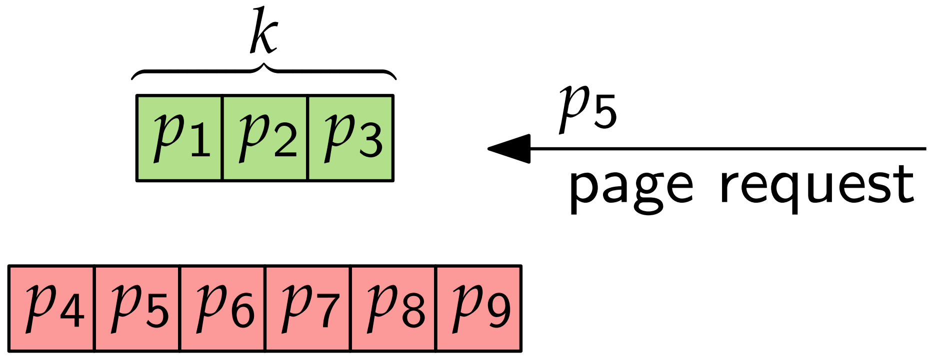

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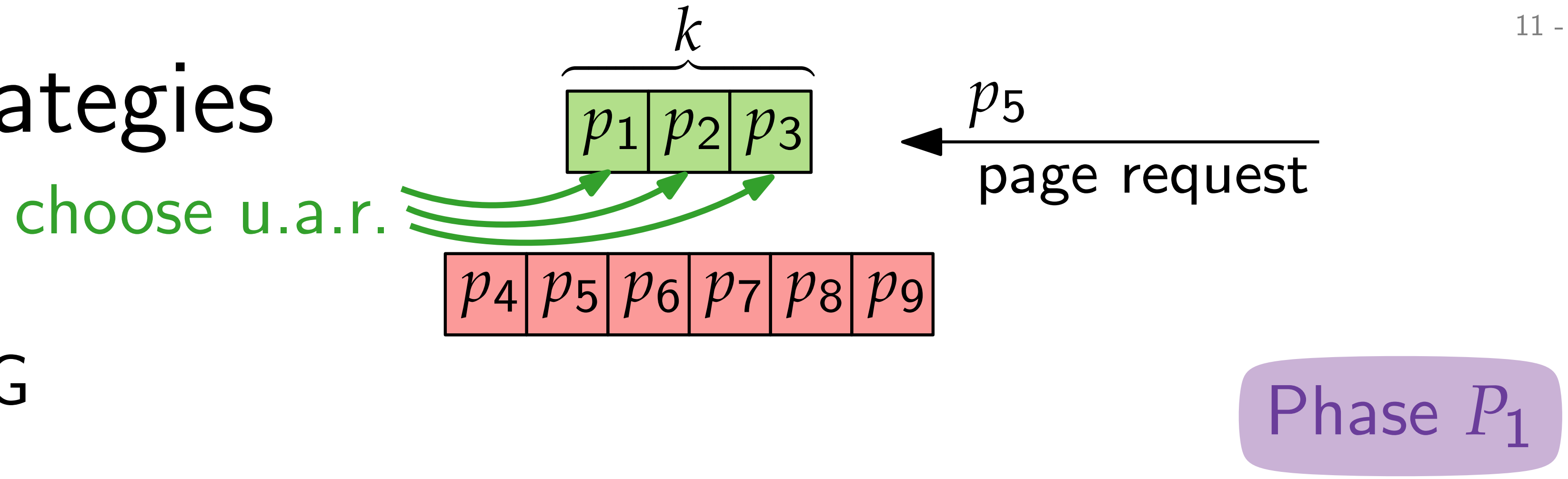

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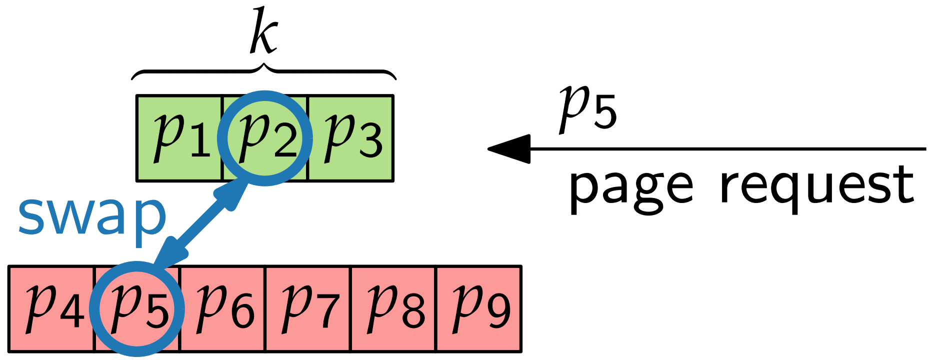

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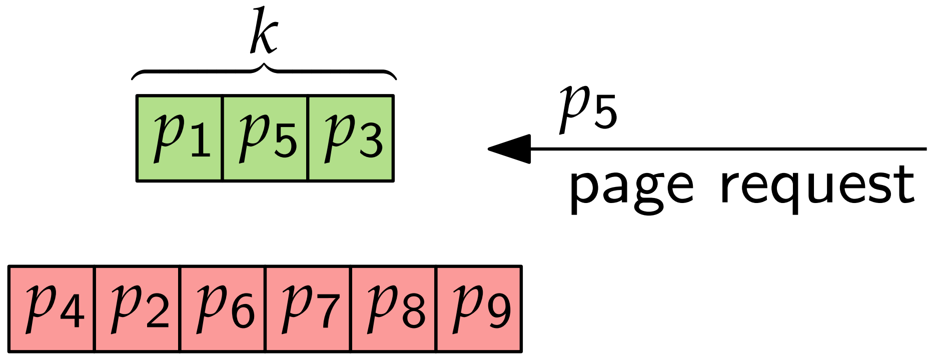

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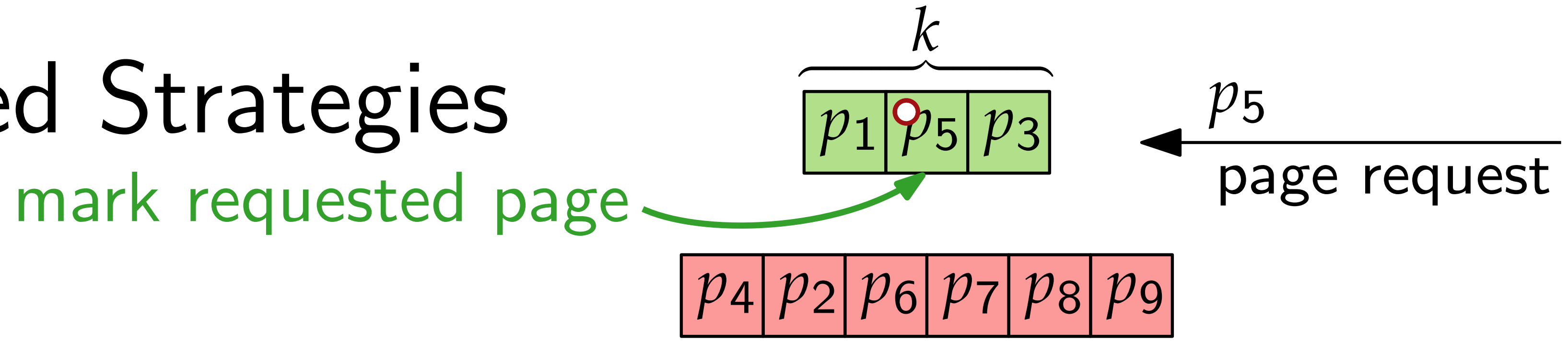

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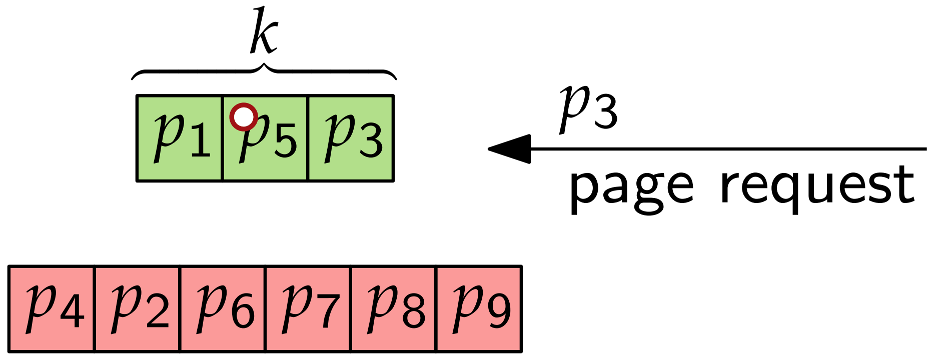

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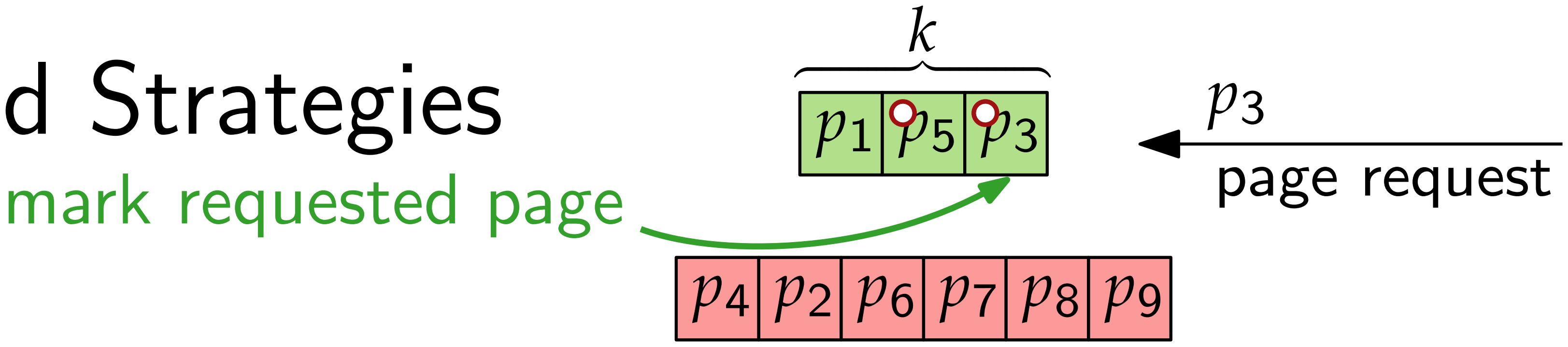

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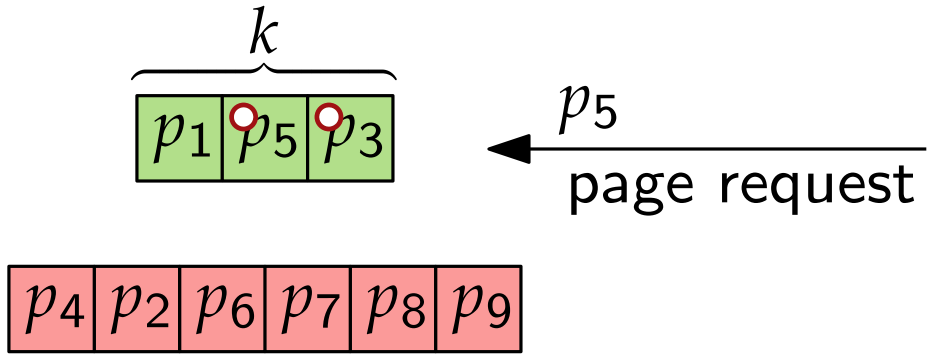

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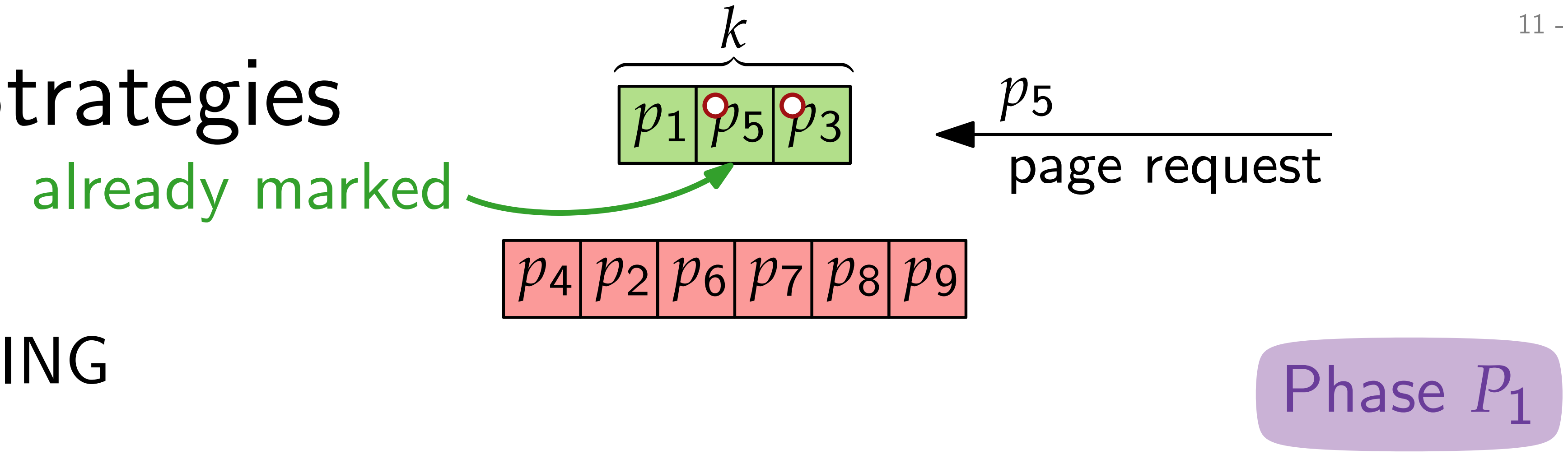

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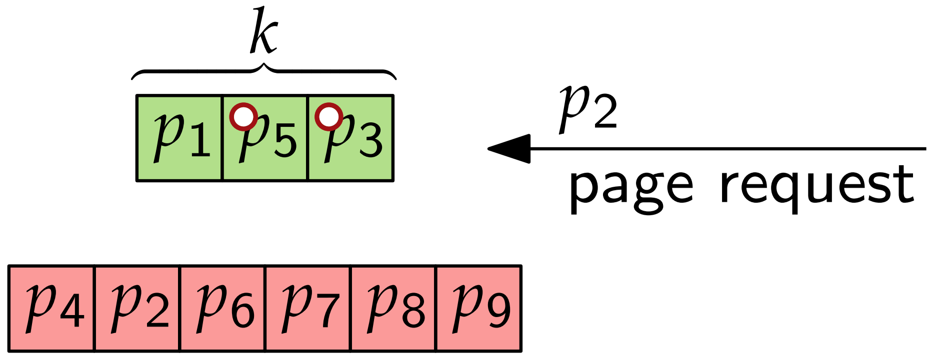

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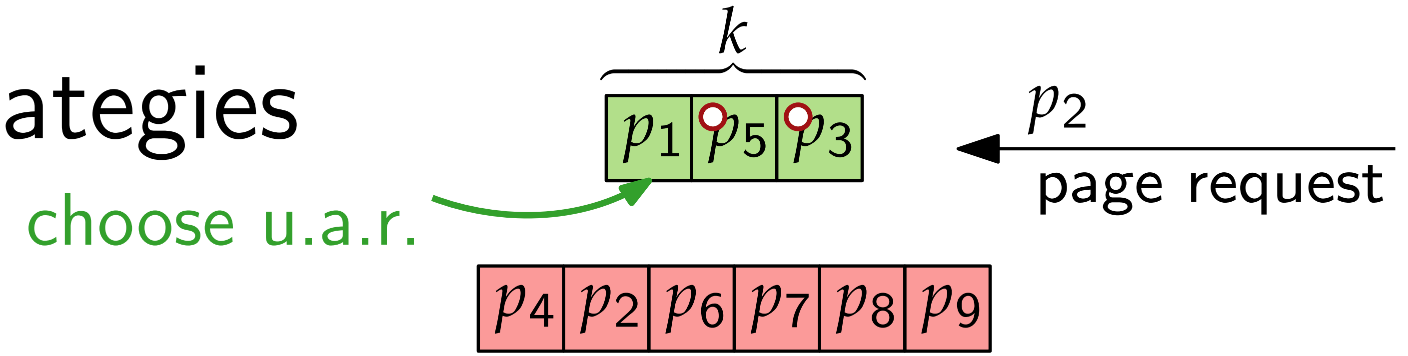

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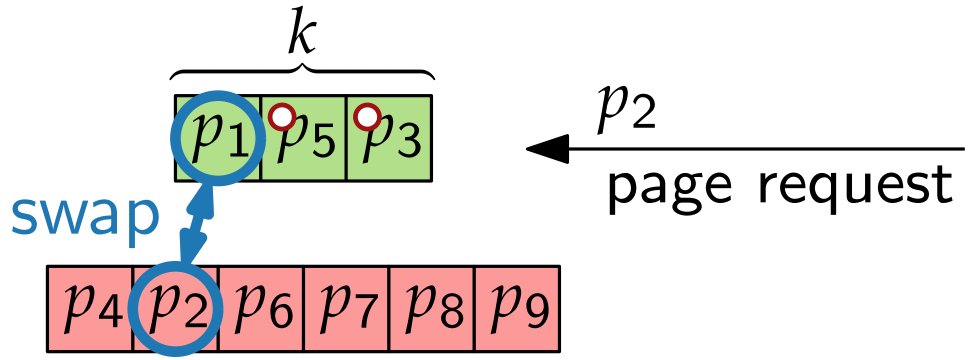

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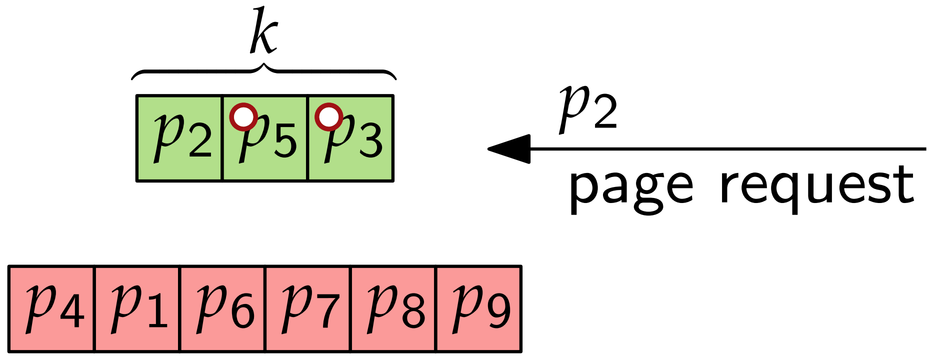

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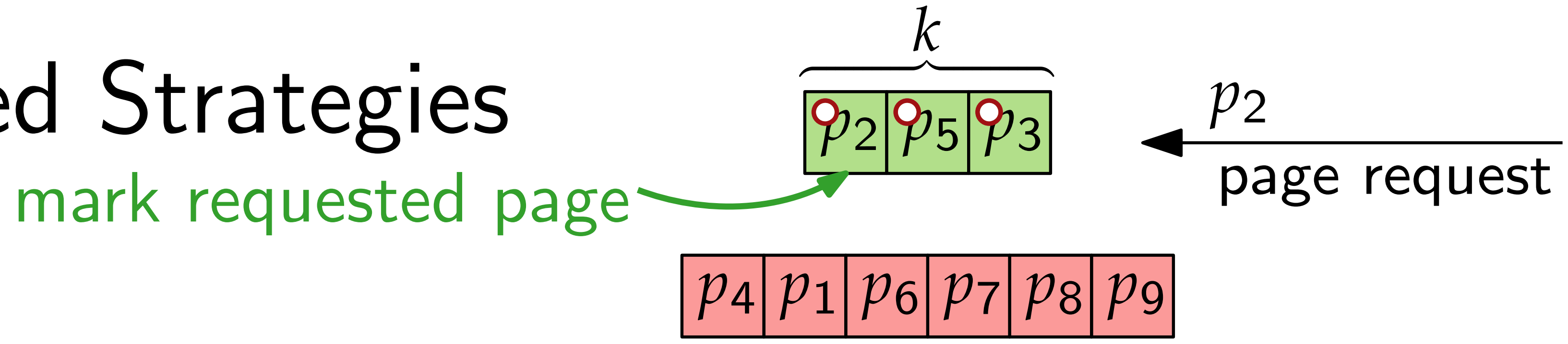

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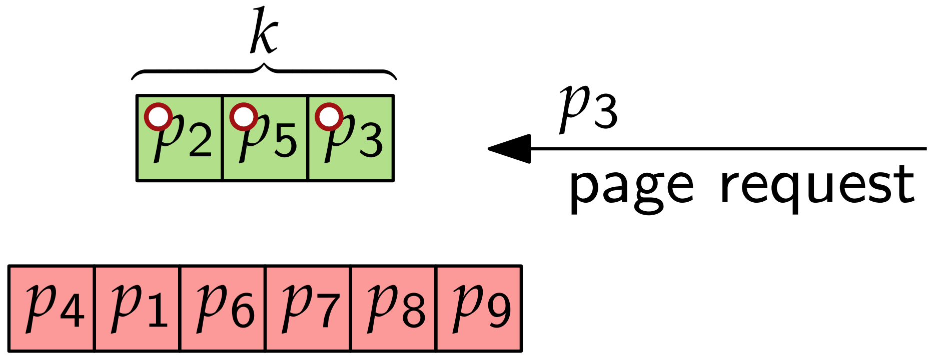

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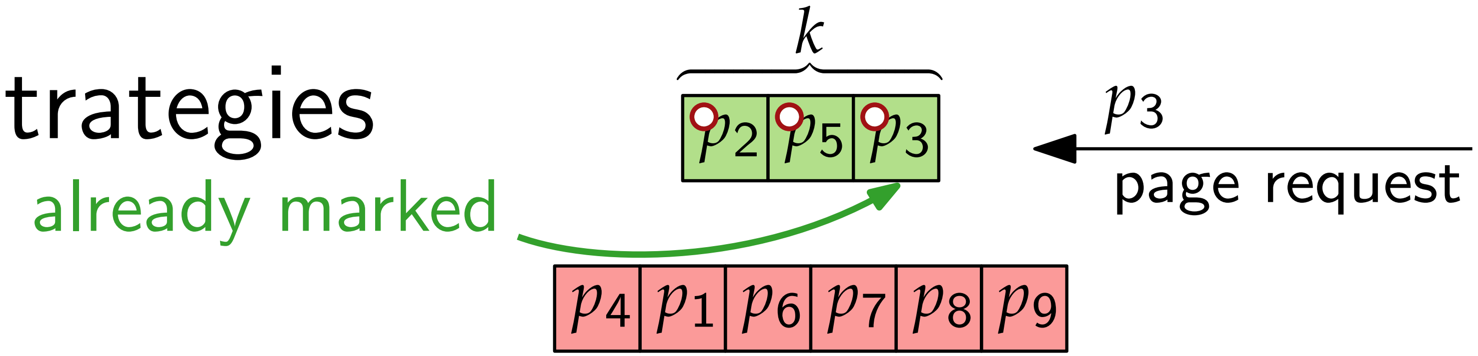

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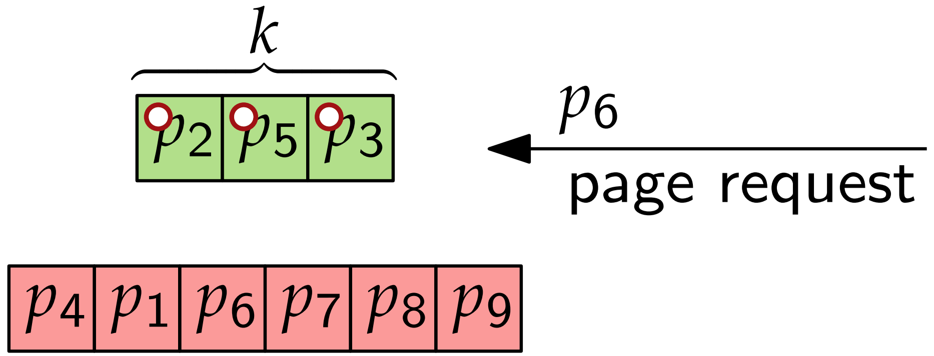

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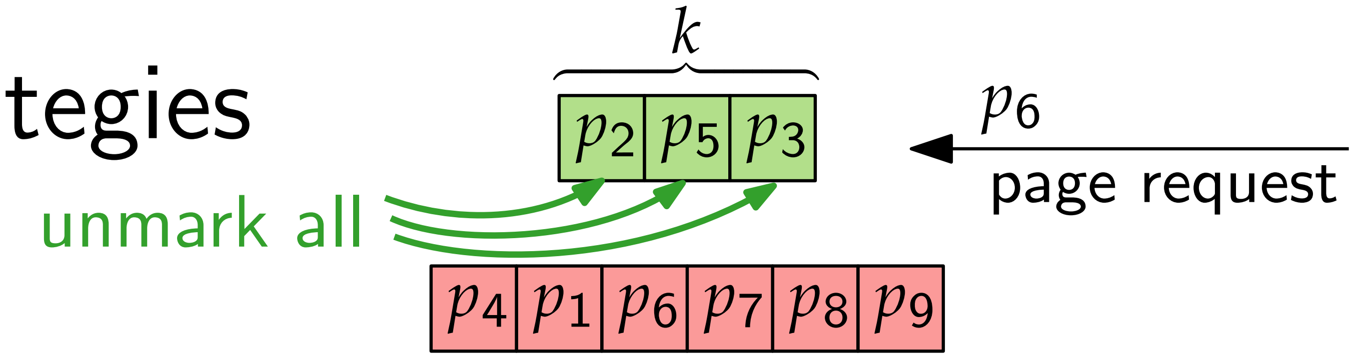

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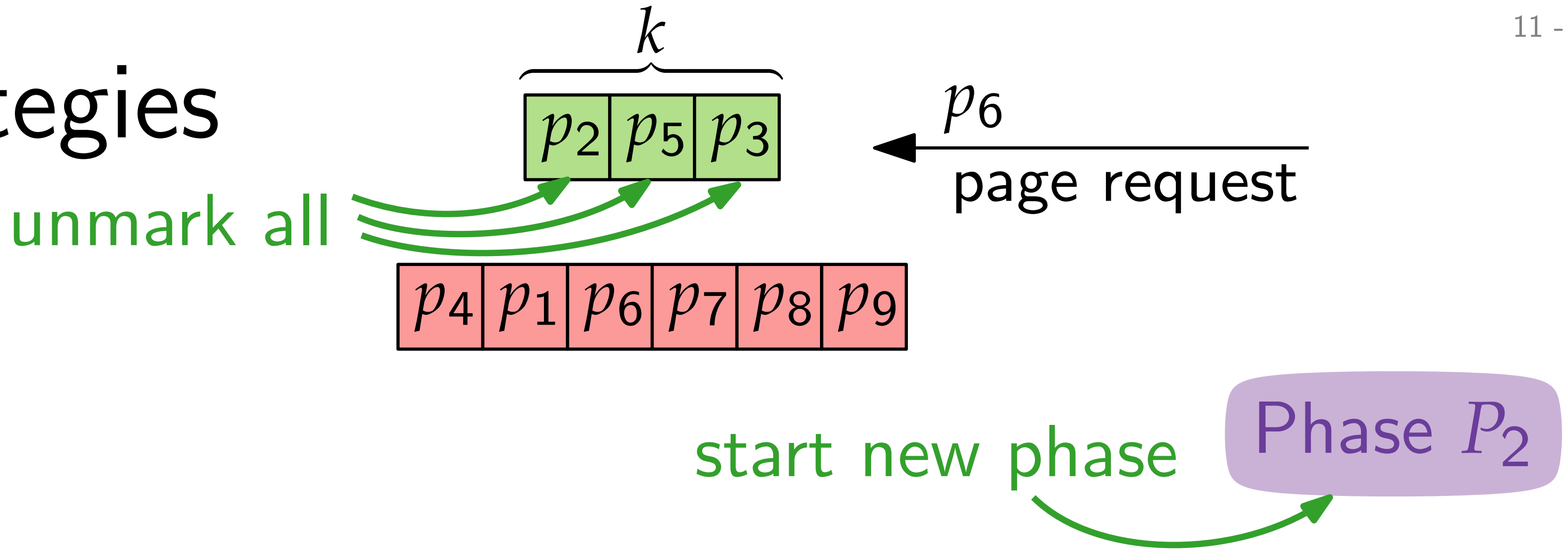

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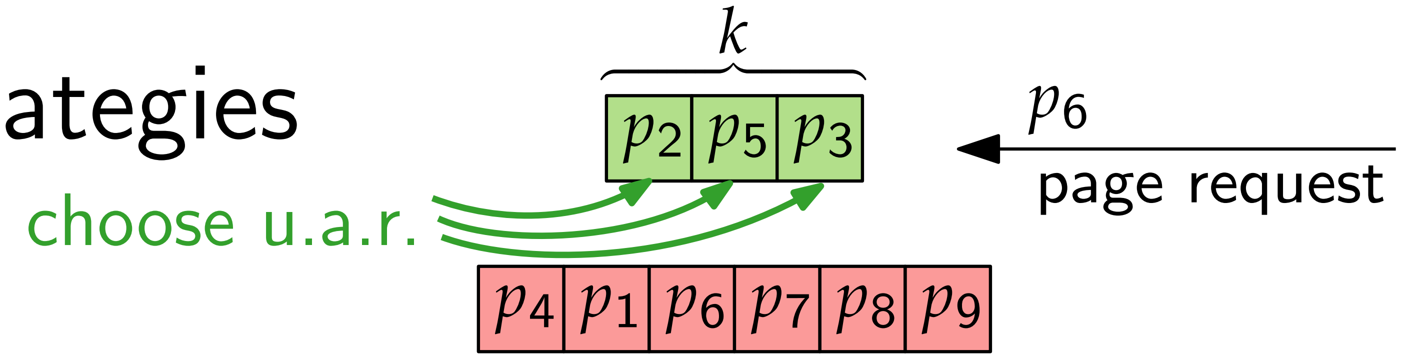

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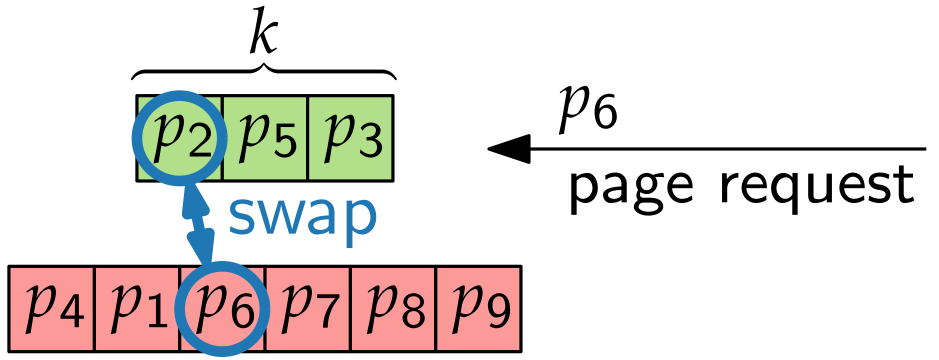

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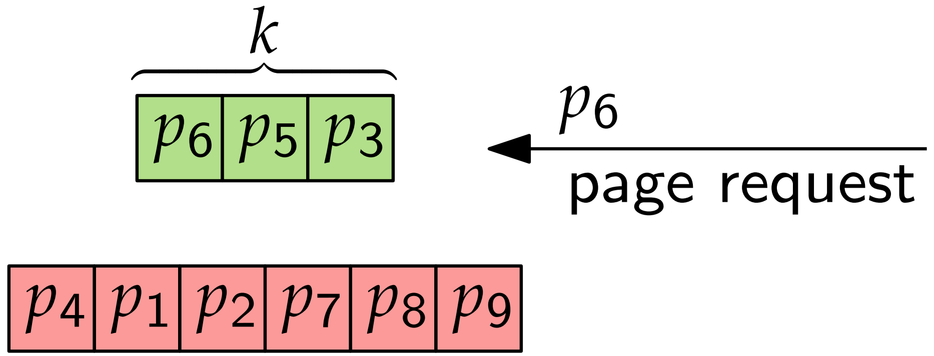
Paging: Randomized Strategies



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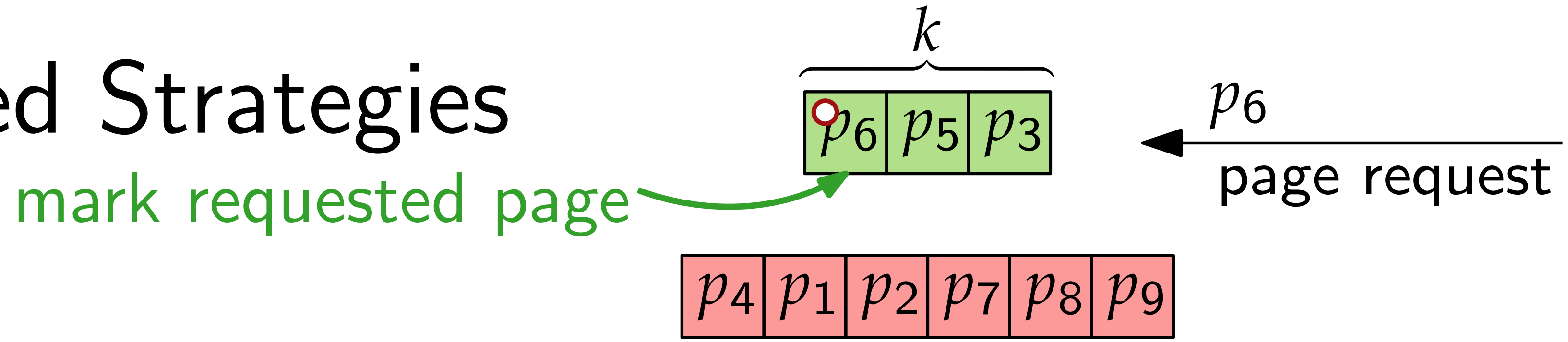

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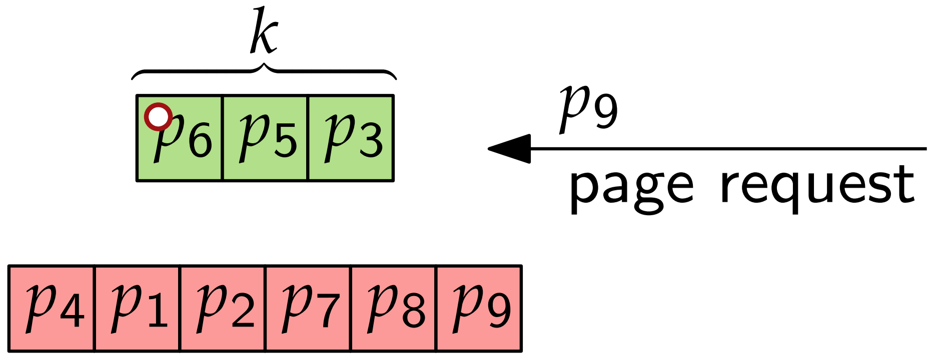

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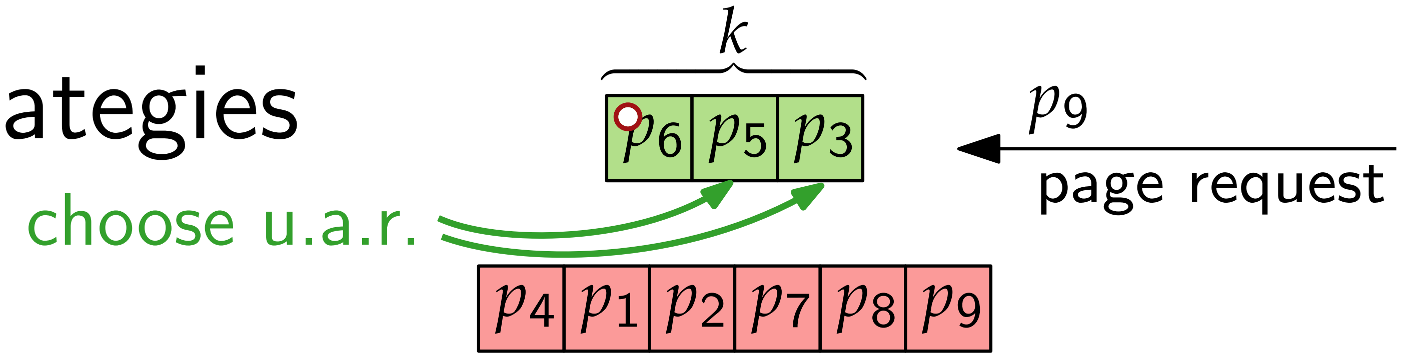

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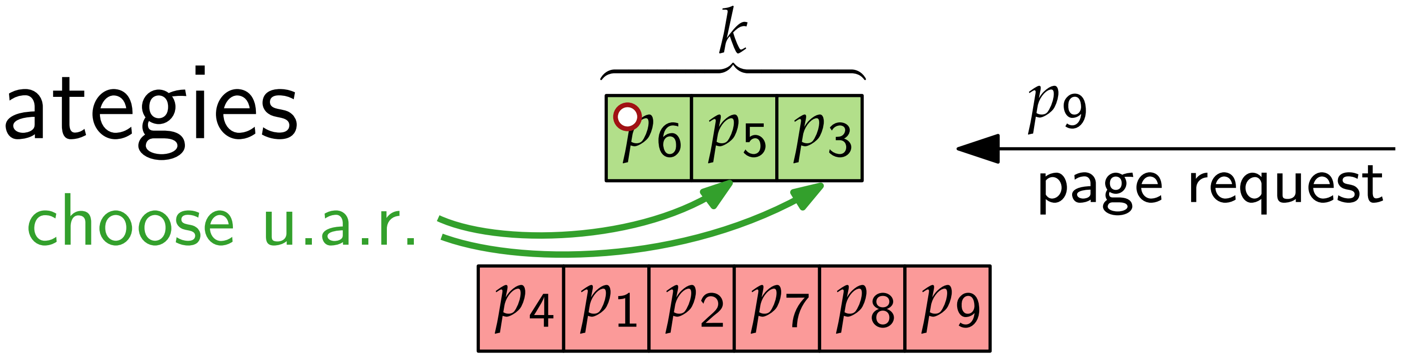

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Theorem 3. MARKING is $2H_k$-competitive.

Remark.

$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$ is the k-th harmonic number. For $k \geq 2$: $H_k < 1 + \ln k$.

Paging – Rand. Strategy – Analysis

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Proof.

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Each page in $S_{\text{MIN}} \setminus S_{\text{MARK}}$ at the end of P_i has caused MIN to produce a fault since all k pages in S_{MARK} have been requested by σ during P_i.

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  Over all phases, all $\frac{d_{\text{begin}}}{2}$ and $\frac{d_{\text{end}}}{2}$ cancel out, except the first $\frac{d_{\text{begin}}}{2}$ and the last $\frac{d_{\text{end}}}{2}$.
- Since the first $d_{\text{begin}} = 0$, MIN has at least $\frac{c}{2}$ faults per phase.

Paging – Rand. Strategy – Analysis

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Proof.

- For the clean pages, MARKING has c faults.

Paging – Rand. Strategy – Analysis

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Paging – Rand. Strategy – Analysis

Reminder. No deterministic strategy is better than k-competitive.

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$\Rightarrow$ **Exponential improvement!**

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- Online algorithms operate in a setting different from that of classical algorithms. However, this setting of incomplete information is very natural and occurs often in real-world applications. Can you think of further examples?

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- We can also transform a classical problem with incomplete information into an online problem. E.g.: Matching problem for ride sharing.

Discussion

- Online algorithms operate in a setting different from that of classical algorithms. However, this setting of incomplete information is very natural and occurs often in real-world applications. Can you think of further examples?

- We can also transform a classical problem with incomplete information into an online problem. E.g.: Matching problem for ride sharing.

- Randomization can help to improve our behavior on worst-case instances. You may also think of: we are less predictable for an adversary.

Literature

Main source:

- Sabine Storandt's lecture script "Randomized Algorithms" (2016/2017)

Original papers:

- [Belady '66] "A Study of Replacement Algorithms for Virtual-Storage Computer."
- [Sleator, Tarjan '85] "Amortized Efficiency of List Update and Paging Rules."
- [Fiat, Karp, Luby, McGeoch, Sleator, Young '91] "Competitive Paging Algorithms."