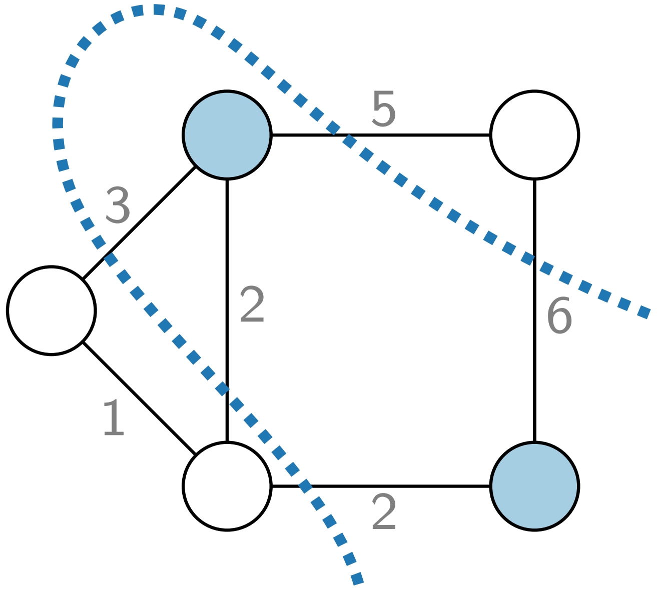
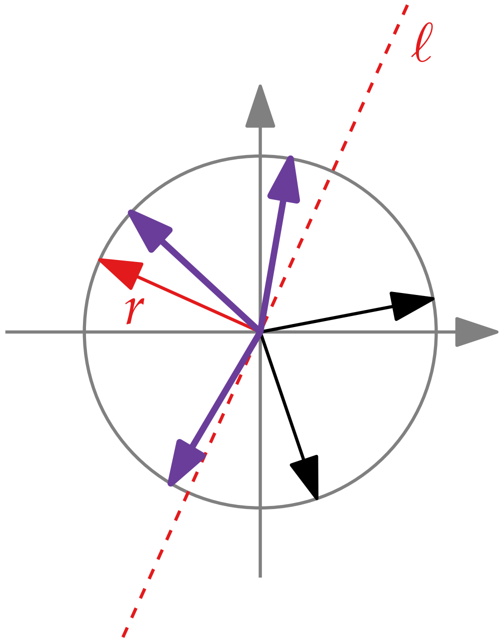
Advanced Algorithms

QP-Relaxation

for MaxCut



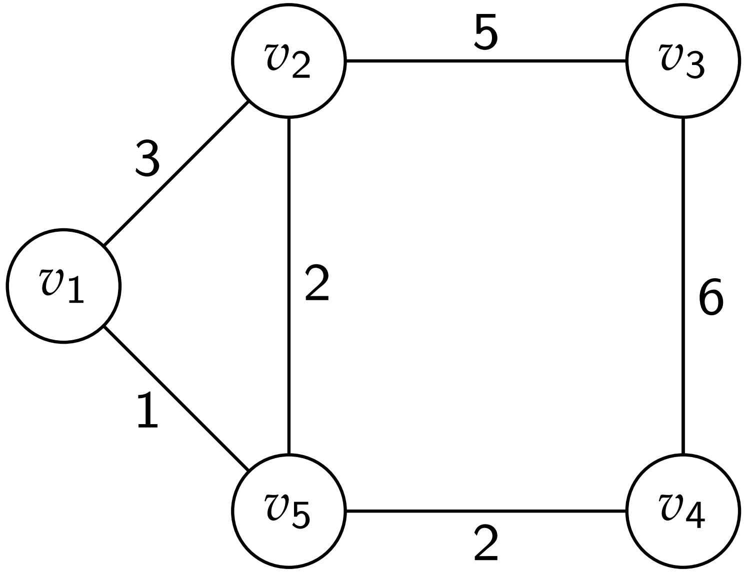


Alexander Wolff

Summer term 2026

Cut

- Let G be a graph with integral edge weights $w\colon E(G) \to \mathbb{N}$.



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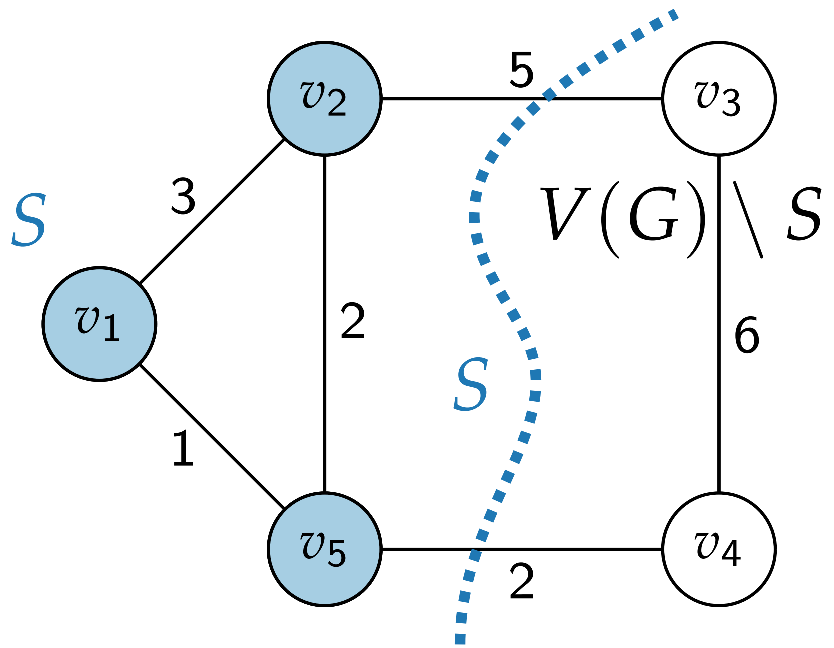

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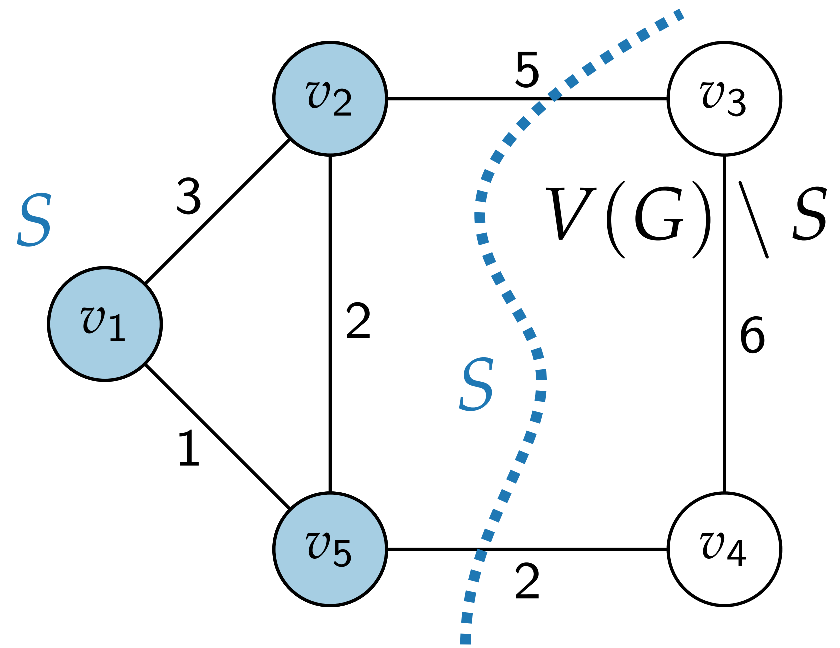
$$w(S) = \sum_{\substack{uv \in E,\\ u \in S, v \notin S}} w(uv)$$



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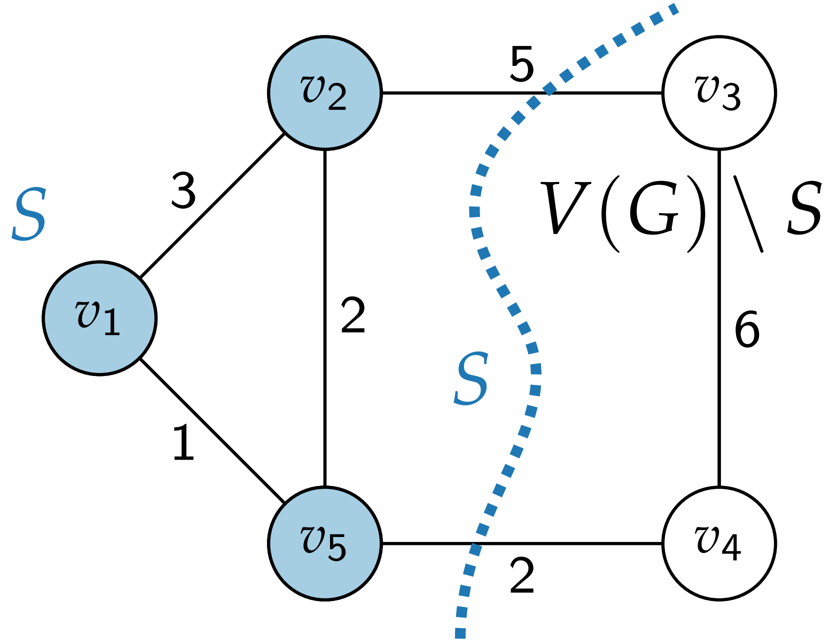

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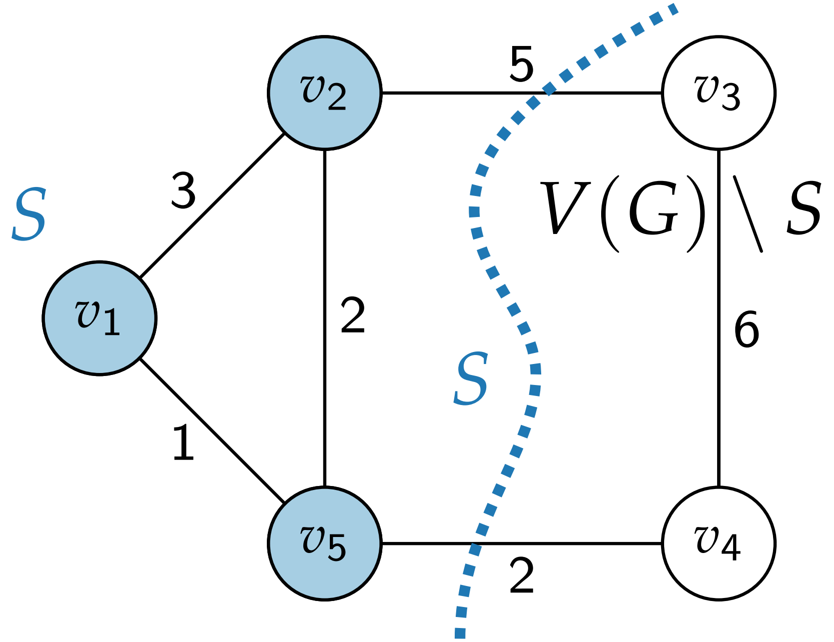

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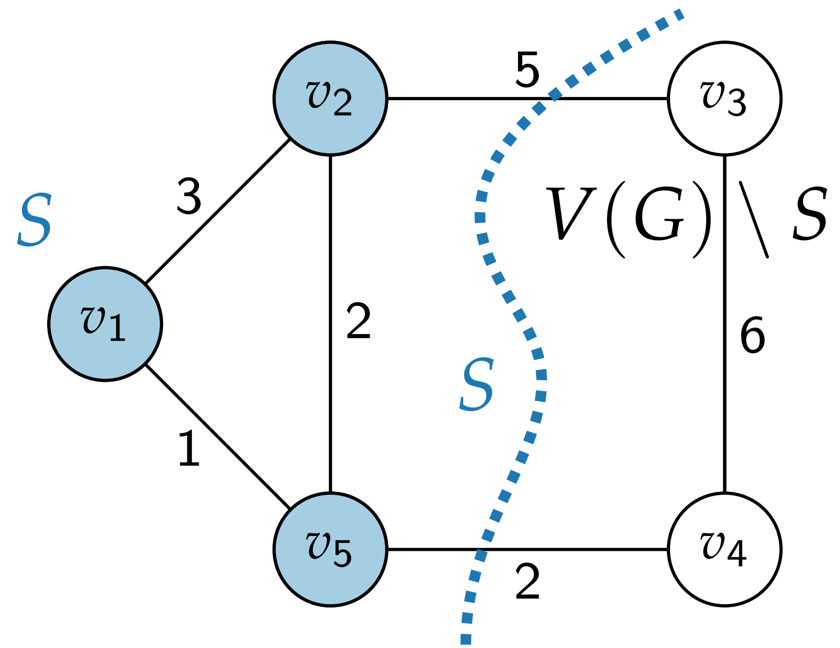
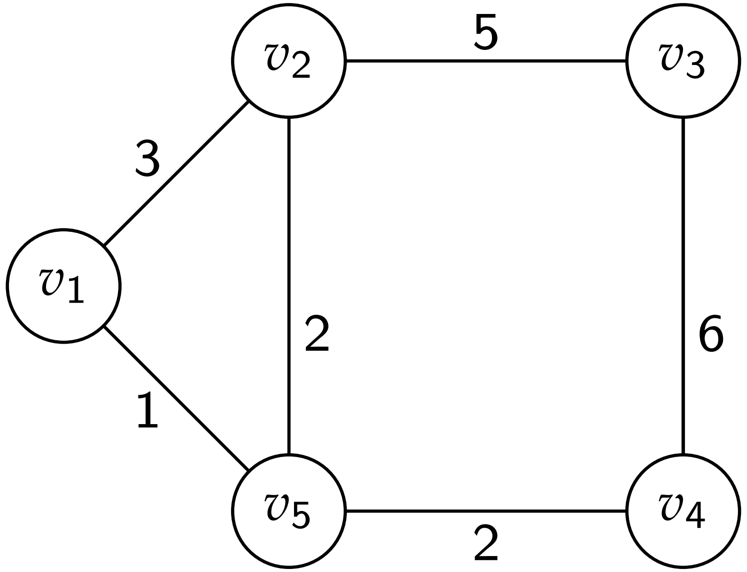


$$w(\{v_1, v_2, v_5\}) = w(\{v_3, v_4\})$$
$$= w(v_2v_3) + w(v_4v_5) = 7$$

The **MinCut** Problem

Input. Graph G, edge weights $w\colon E(G) \to \mathbb{N}$.

Output. Cut $(S, V(G) \setminus S)$ of G of **minimum** weight.



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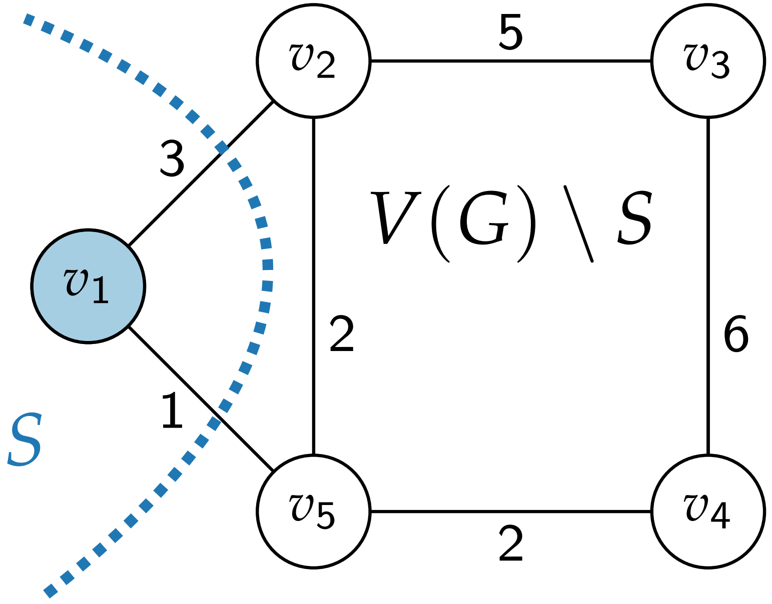

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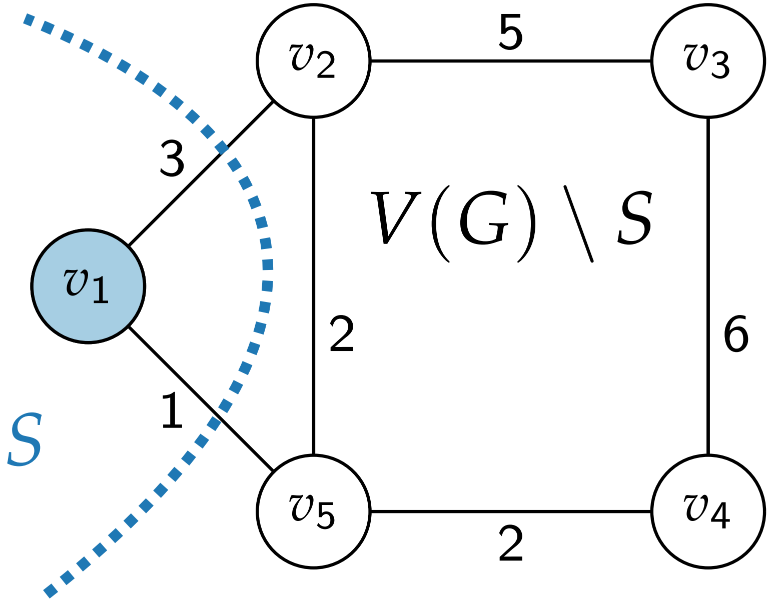

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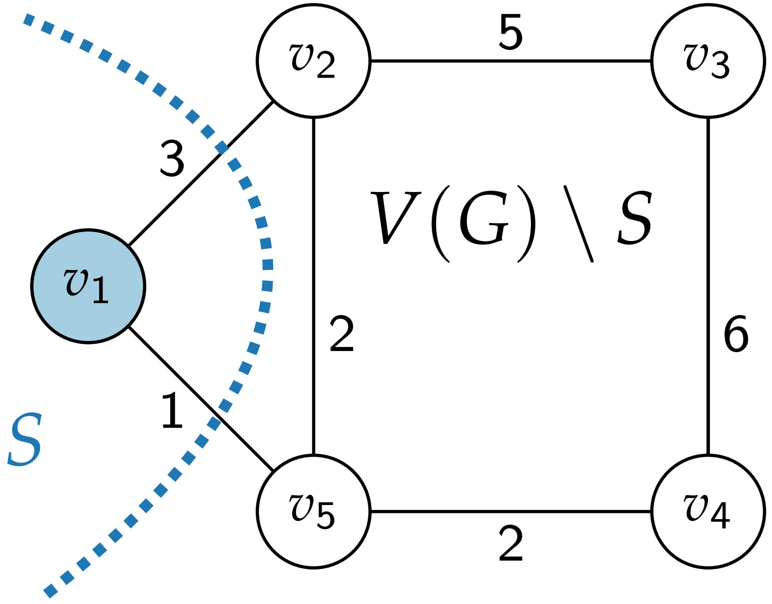
- Can be solved optimally in polynomial time, e.g., by the Stoer–Wagner algorithm.



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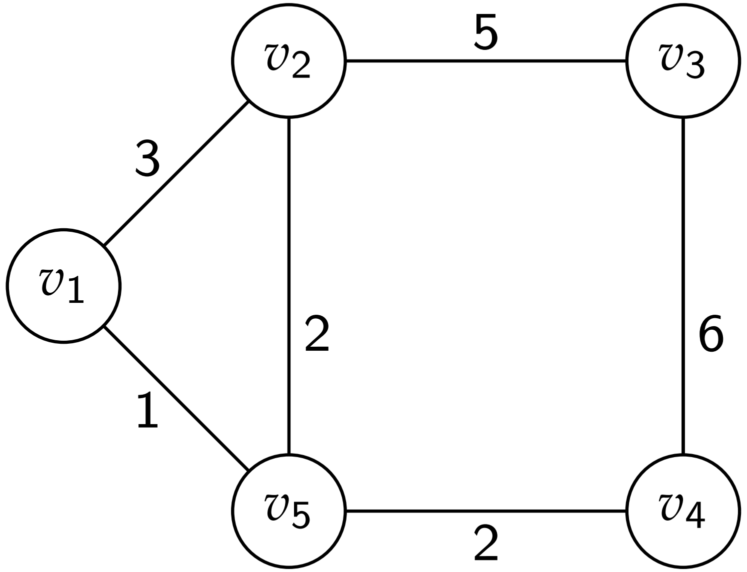

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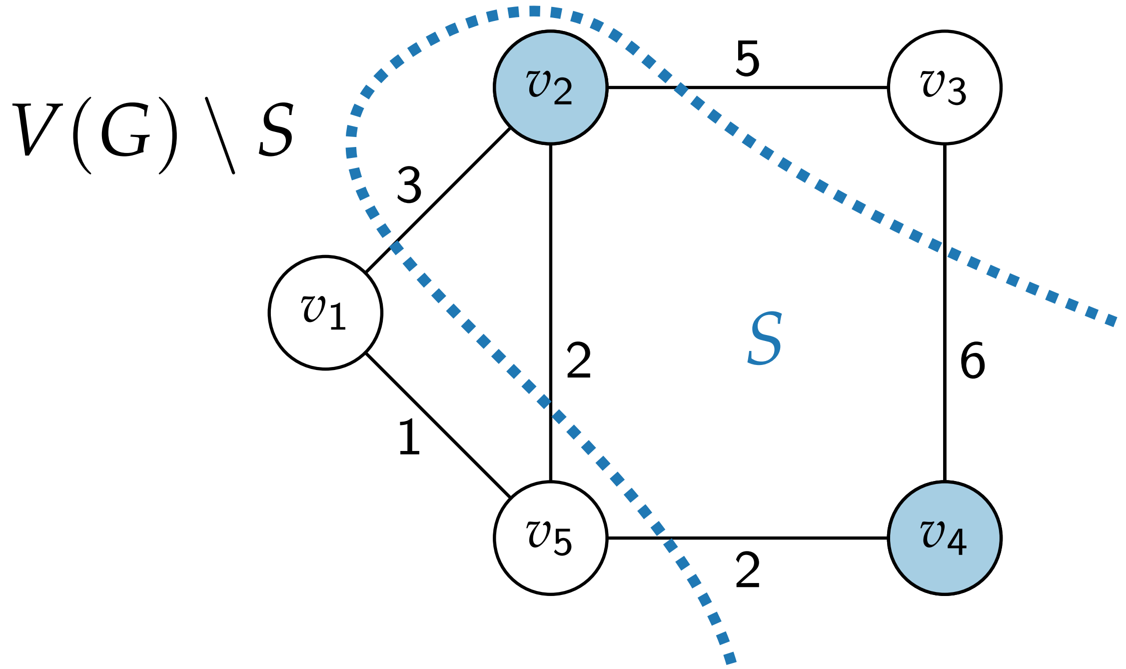

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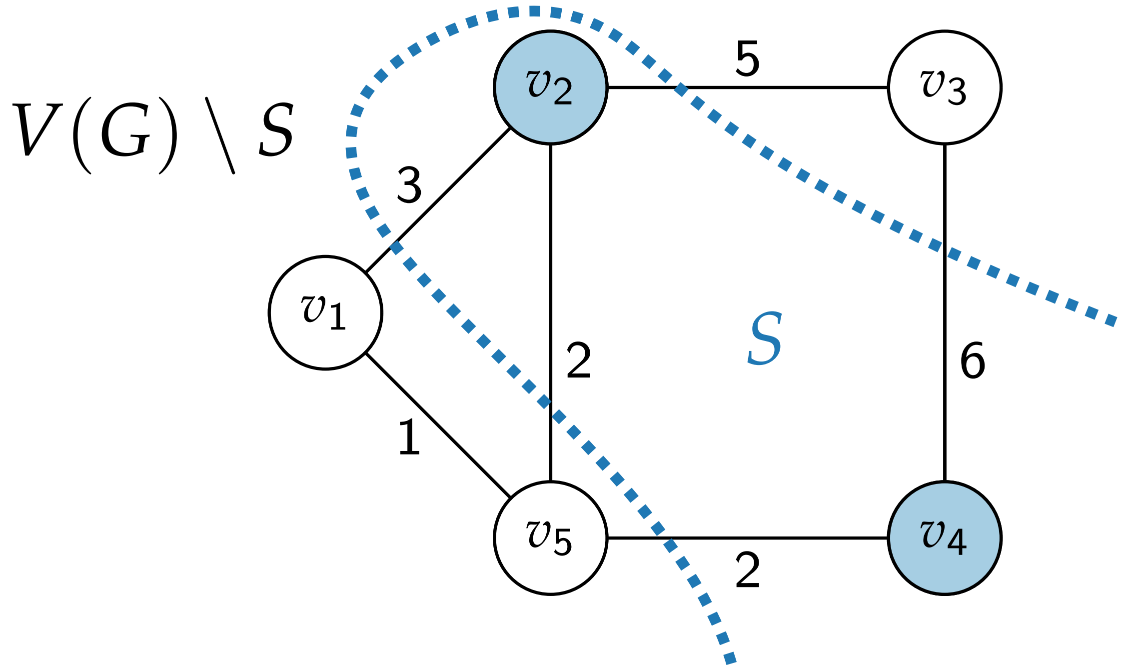

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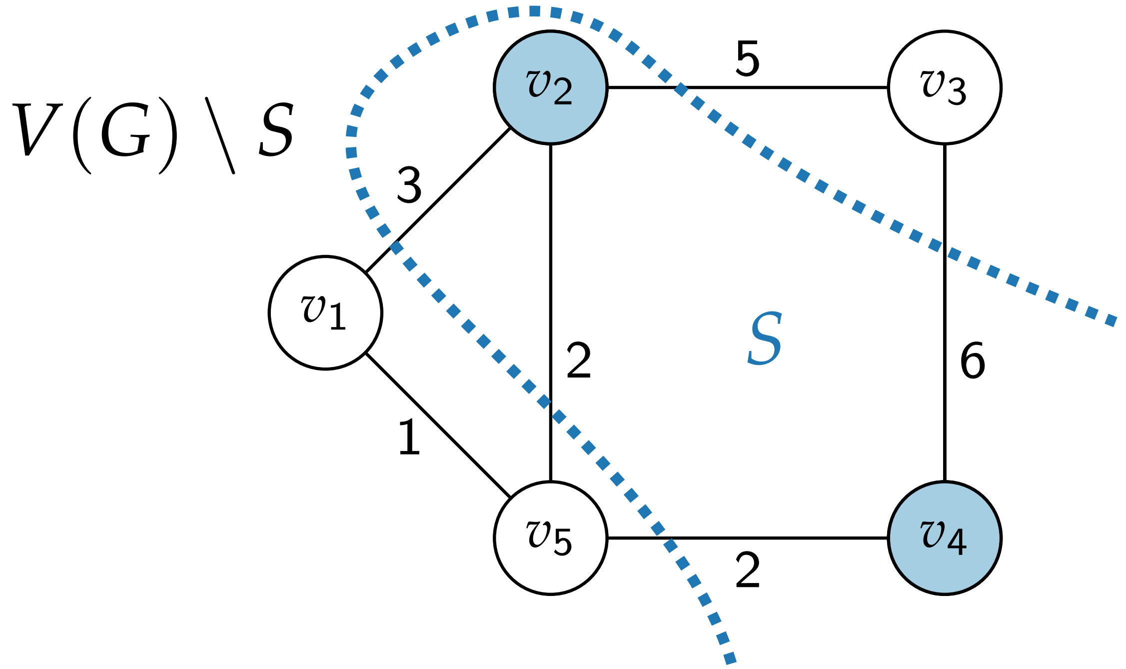
- NP-complete to find a cut of maximum weight.



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Randomized Approximation for (Unweighted) MaxCut

$\textsc{CoinFlipMaxCut}(G, w \equiv 1)$

$S \leftarrow \emptyset$
foreach $v \in V(G)$ **do**
  if coin flip shows $\textsc{Heads}$ **then**
    $S \leftarrow S \cup \{v\}$
return $w(S)$ and S

Randomized Approximation for (Unweighted) MaxCut

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- Can be "de-randomized". Exercise.

LP-Relaxation

Integer Linear Program

$$
\begin{array}{llll}
\textbf{maximize} & c^\intercal x & & \\
\textbf{subject to} & Ax & \leq & b \\
 & x & \geq & 0 \\
 & x & \in & \mathbb{Z}^n
\end{array}
$$

LP-Relaxation

Integer Linear Program

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LP-Relaxation

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Solve in polynomial time

Solution for LP

$x^\star$

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e.g. rounding

Assignment for ILP

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Solve in
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Solution for LP

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e.g. rounding

Assignment for ILP

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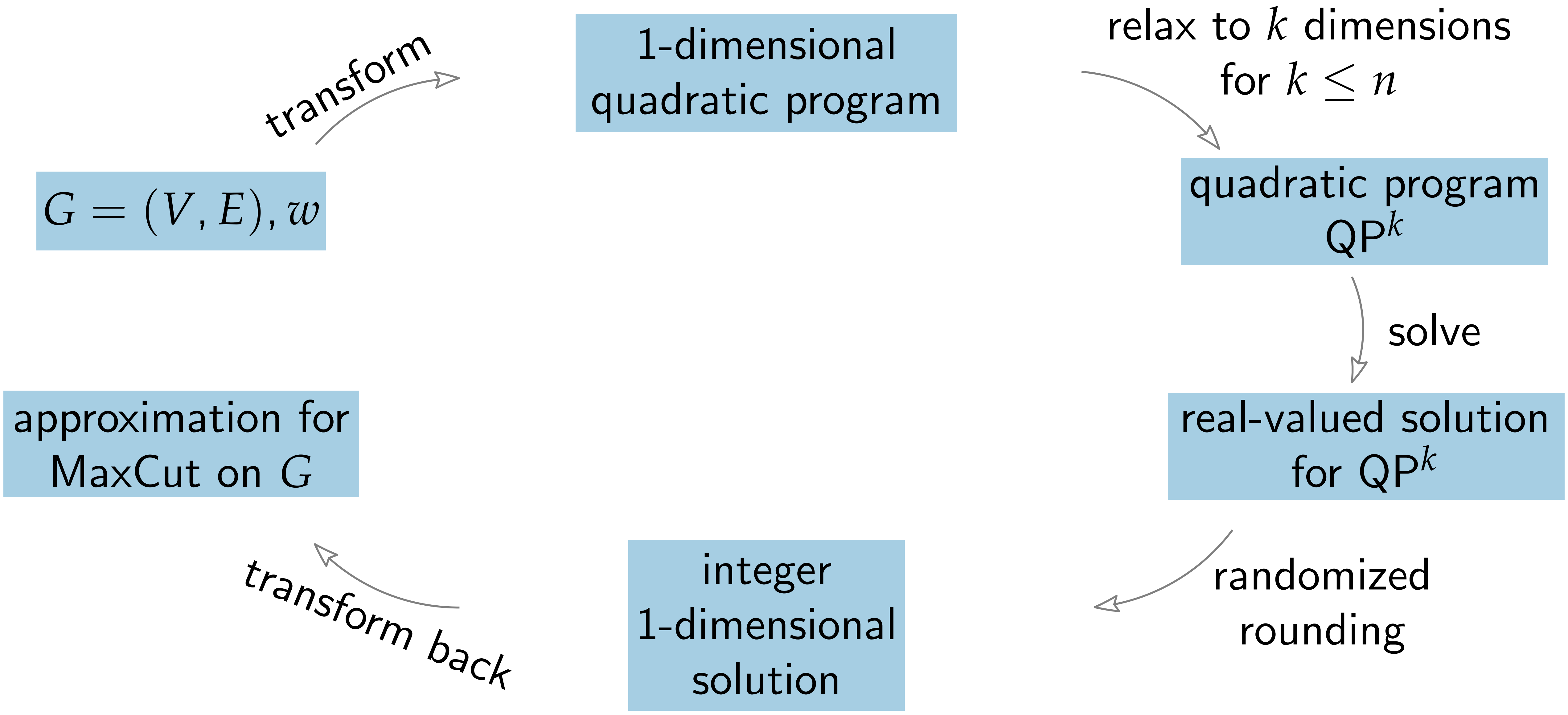
Solution,
approximation,
or bound

Goemans–Williamson Algorithm for MaxCut



Goemans–Williamson Algorithm for MaxCut

$G = (V, E), w$

transform

1-dimensional quadratic program

relax to k dimensions for $k \leq n$

quadratic program QP^k

solve

real-valued solution for QP^k

randomized rounding

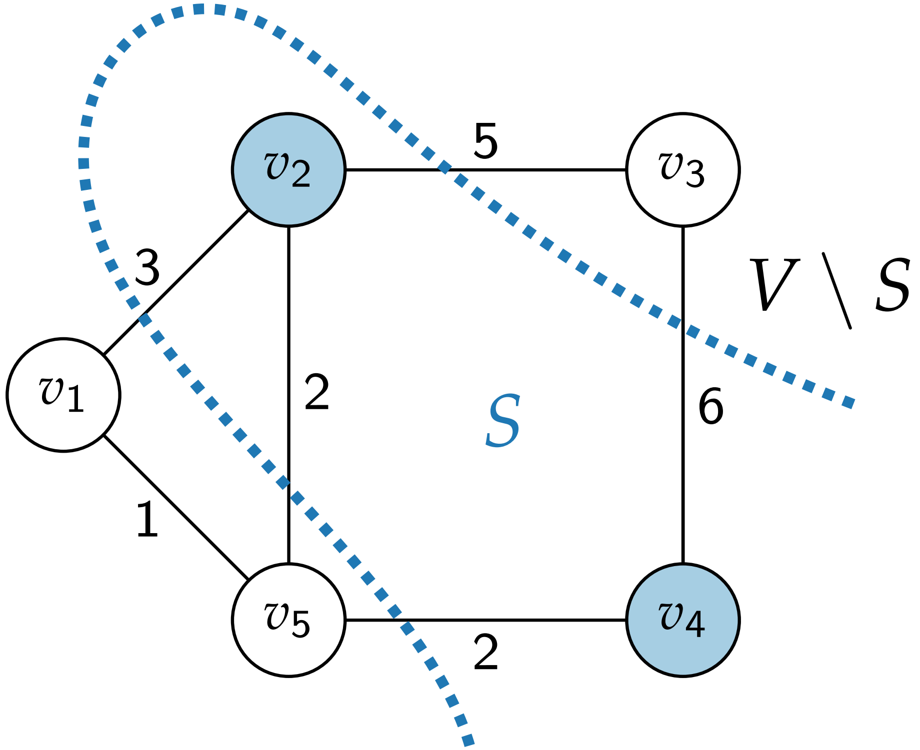
integer 1-dimensional solution

transform back

approximation for MaxCut on G

QP(G, w)

Idea.



QP(G, w)

maximize

subject to

$\mathrm{QP}(G, w)$

Idea.

- Indicator variable for each vertex v_i:
  $x_i \in \{1, -1\}$

QP(G, w)

maximize

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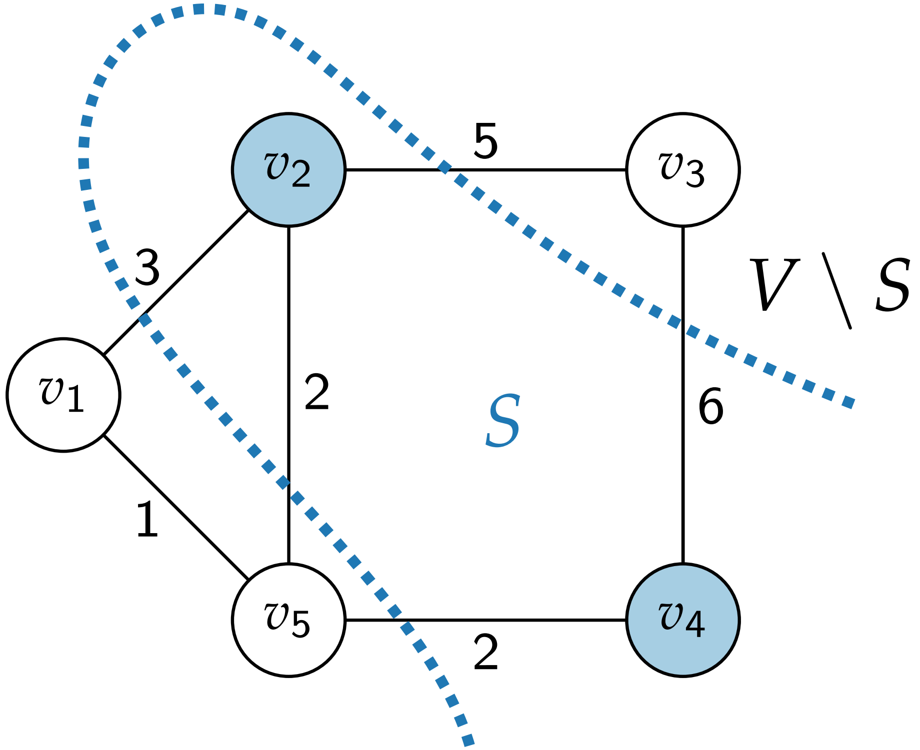

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QP(G, w)

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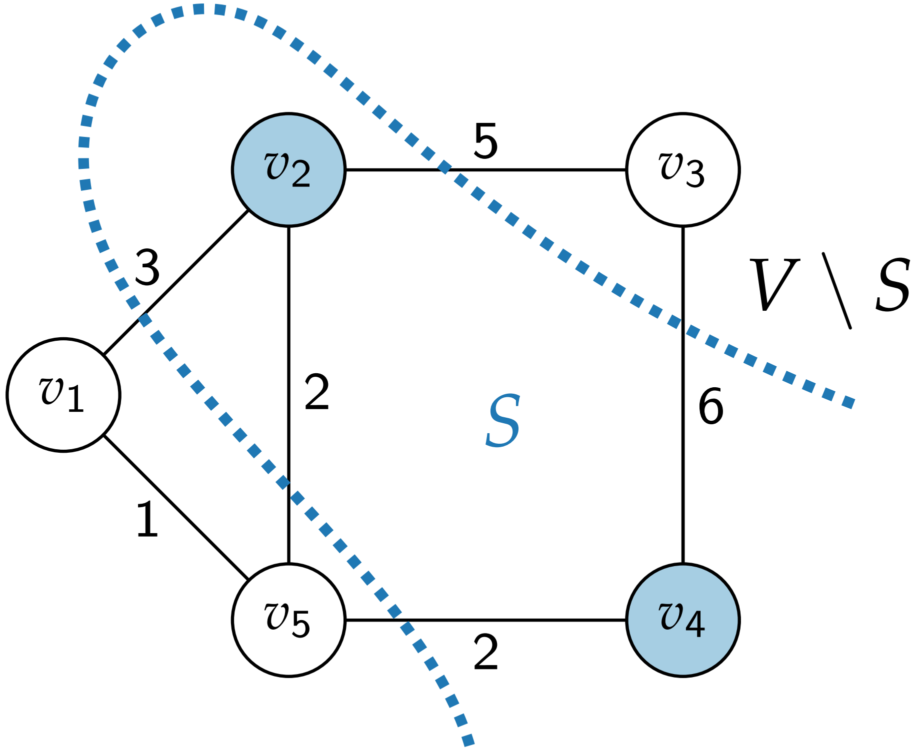
subject to $x_i^2 = 1$



QP(G, w)

Idea.

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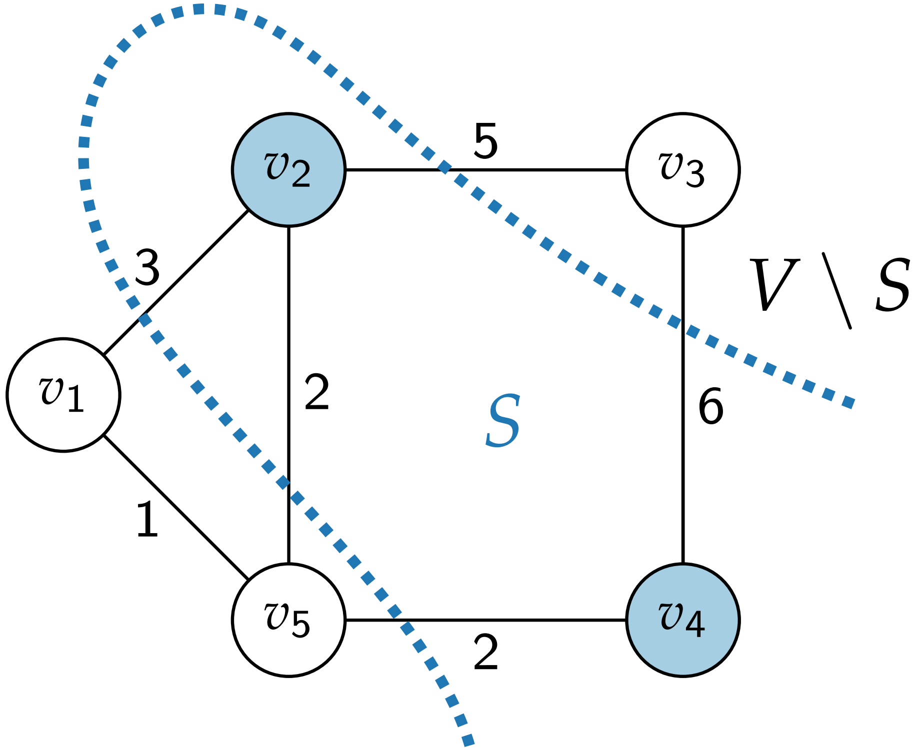
- $x_i \cdot x_j = \begin{cases} 1 & \text{if } i, j \text{ on the same side} \\ -1 & \text{otherwise} \end{cases}$



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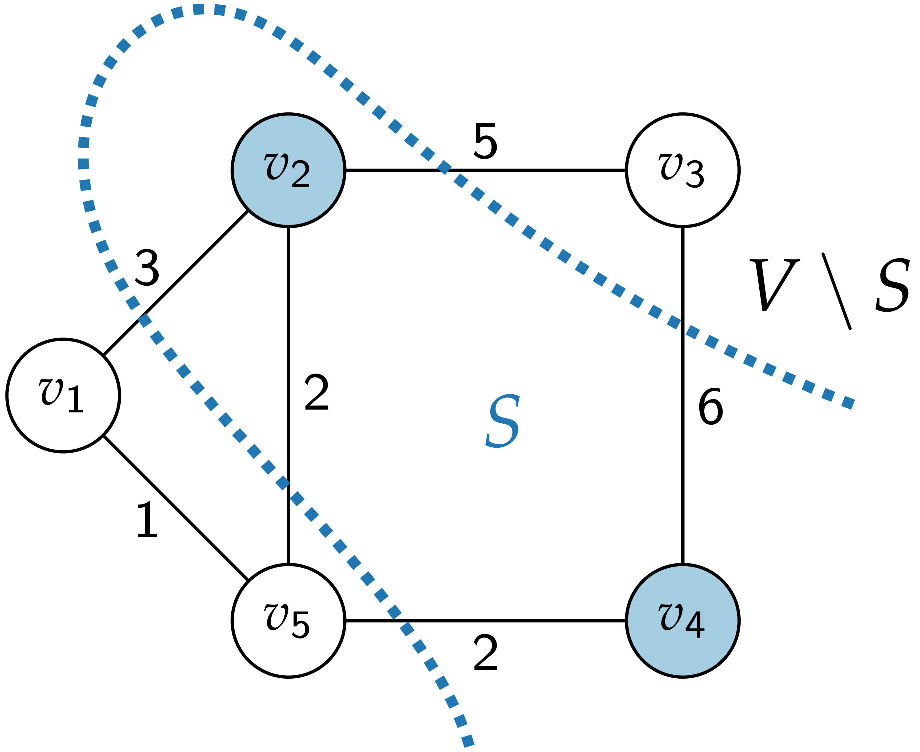

QP(G, w)

maximize $(1 - x_i x_j)$

subject to $x_i^2 = 1$

QP(G, w)

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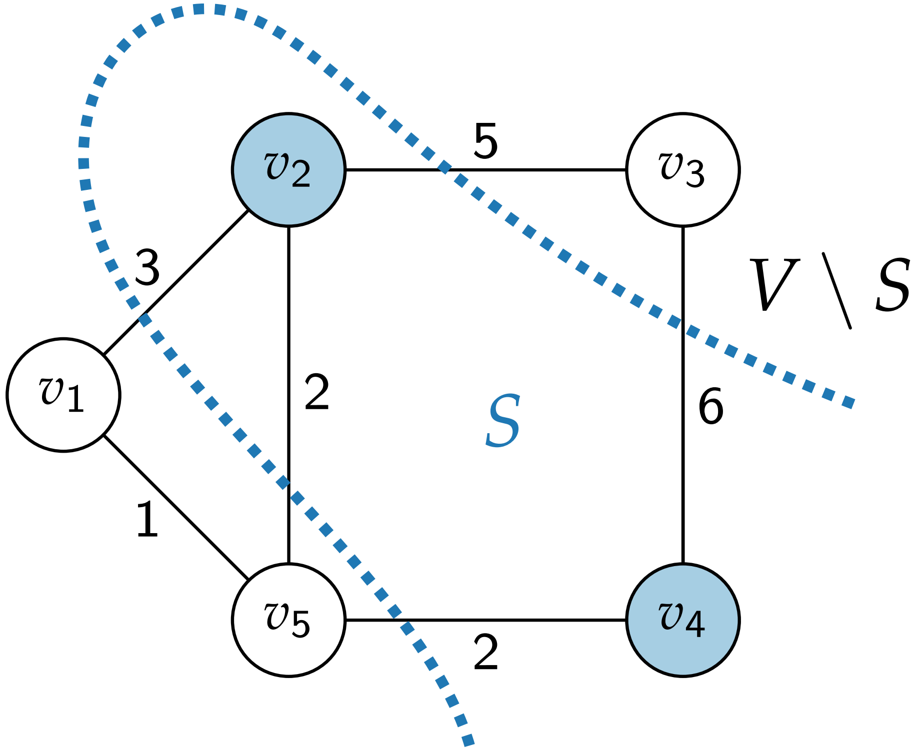
QP(G, w)

maximize $\quad w_{ij}(1 - x_i x_j)$

subject to $\quad x_i^2 = 1$



- Weight matrix w_{ij}

	1	2	3	4	5
1		3			1
2	3		5		2
3		5		6	
4			6		2
5	1	2		2	

$\mathrm{QP}(G, w)$

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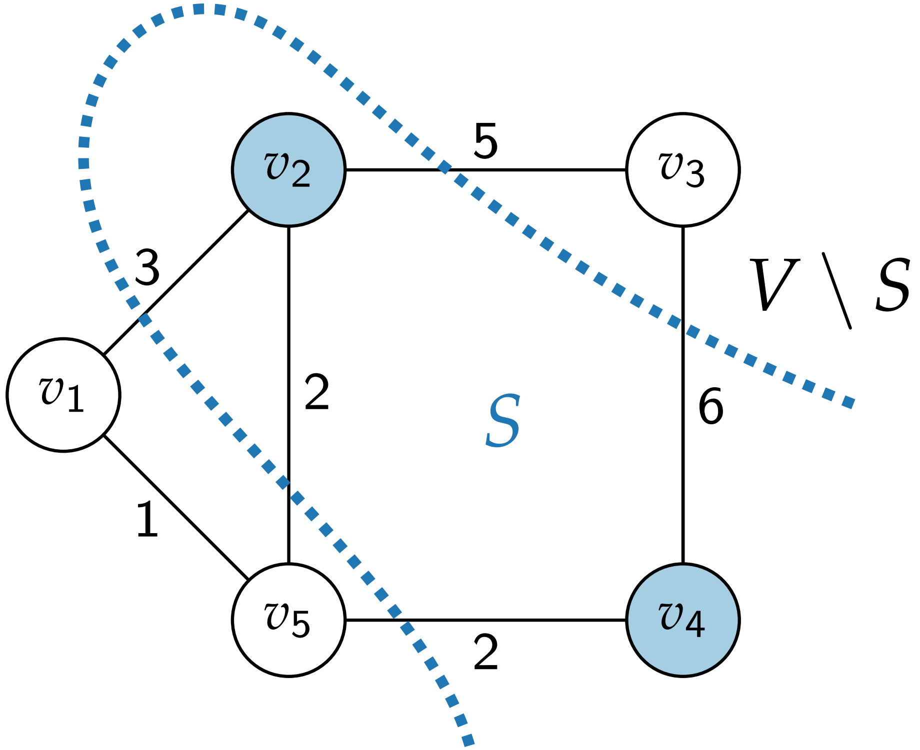
QP(G, w)

maximize $\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij}(1 - x_i x_j)$

subject to $x_i^2 = 1$



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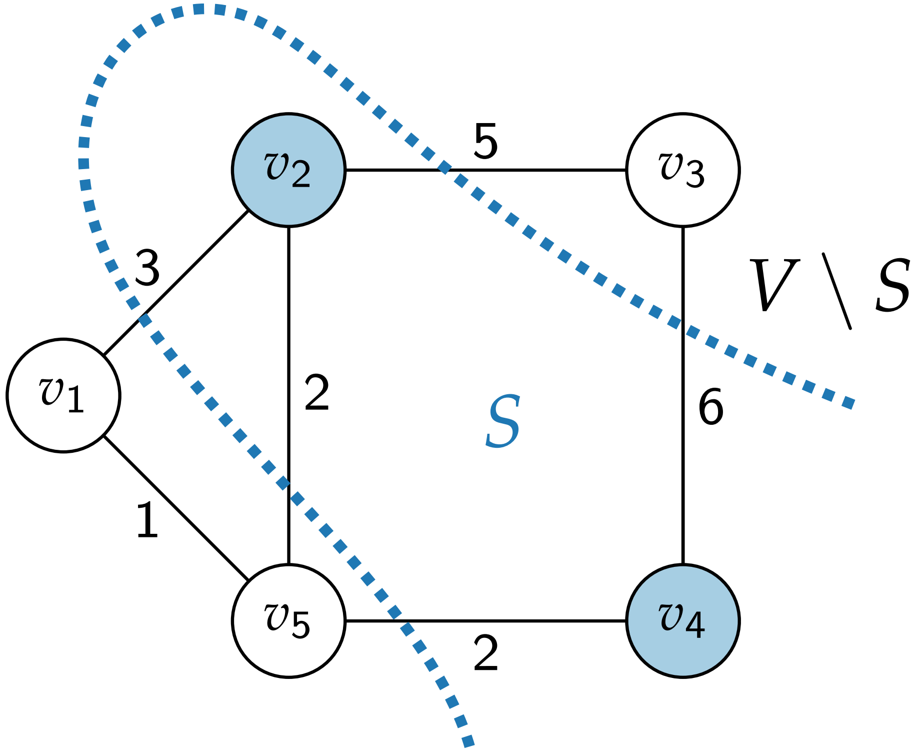

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- Solution

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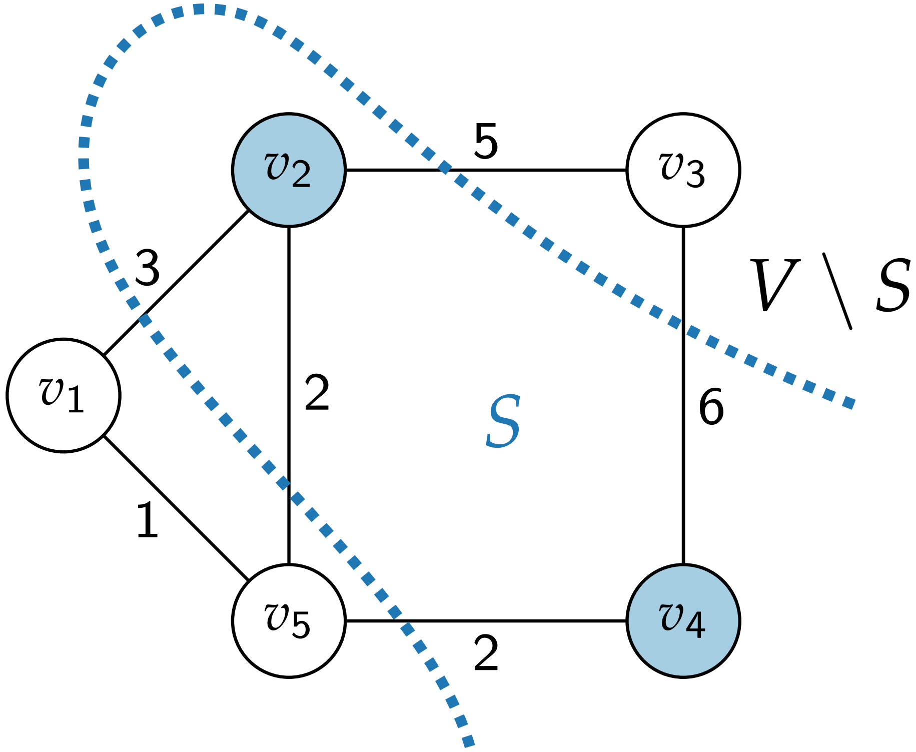

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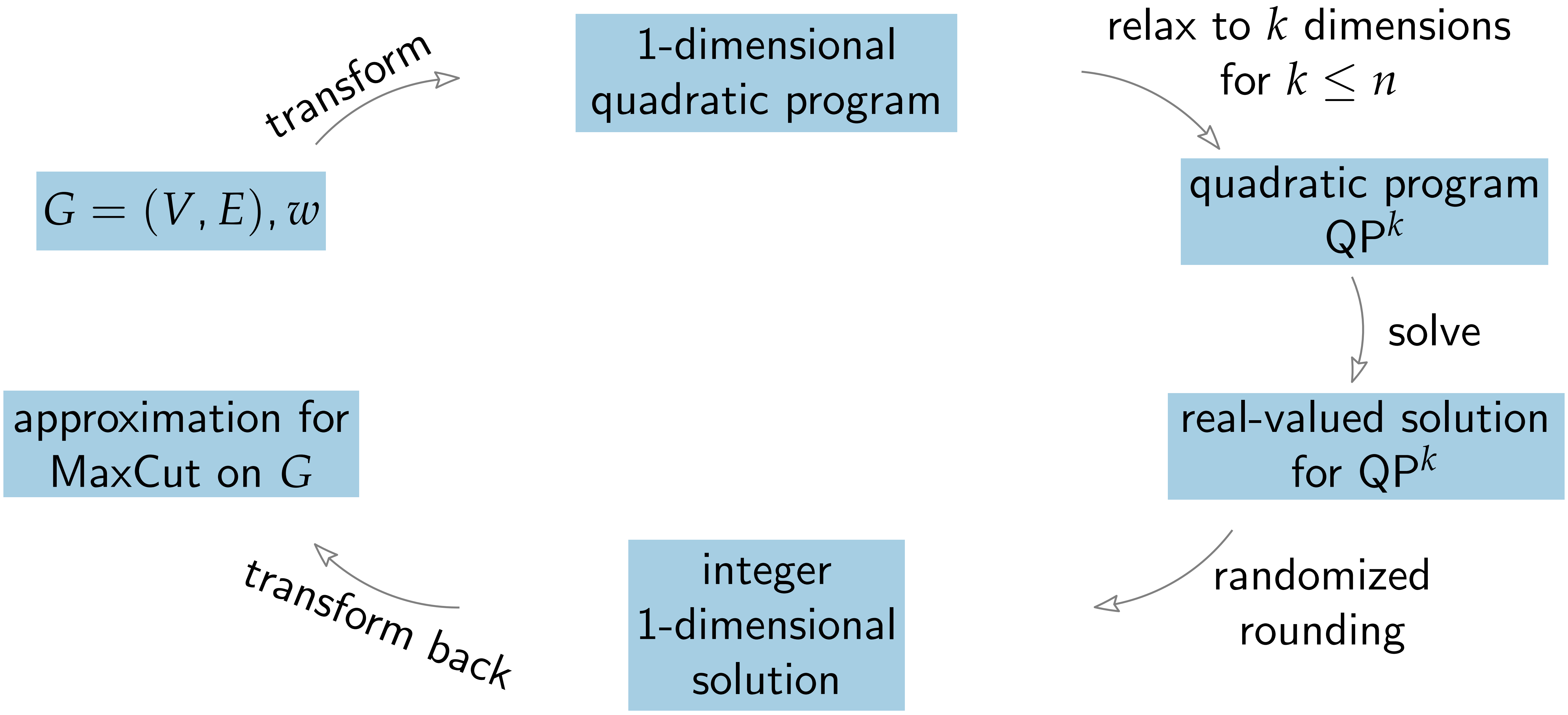
Note.

- Solving QP(G, w) is NP-hard.
- Otherwise MaxCut would not be NP-hard.

Goemans–Williamson Algorithm for MaxCut



Goemans–Williamson Algorithm for MaxCut

$G = (V, E), w$

transform

1-dimensional quadratic program

relax to k dimensions for $k \leq n$

quadratic program QP^k

solve

real-valued solution for QP^k

randomized rounding

integer 1-dimensional solution

transform back

approximation for MaxCut on G

- Here explained for $k = 2$,
- but unknown if QP^2 can be solved optimally in polynomial time.
- QP^n can be solved in poly. time.

Relaxation of QP(G, w)

QP$^2(G, w)$

maximize $\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij}(1 - x^i \cdot x^j)$

subject to

$$x^i \cdot x^i = 1$$
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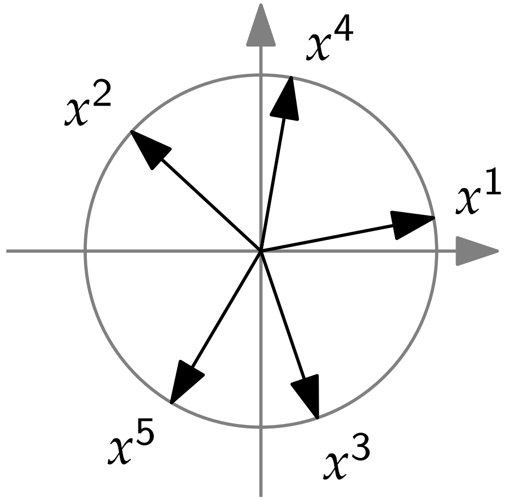

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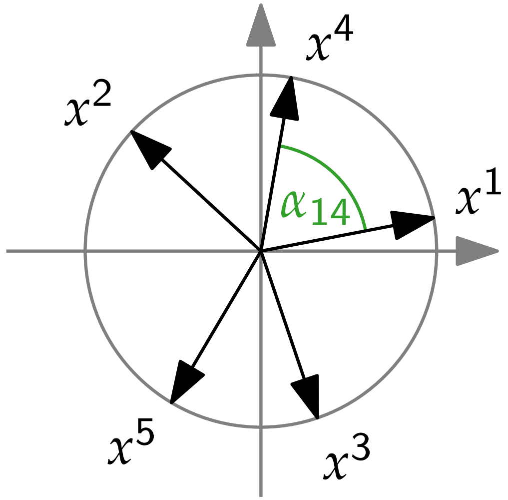

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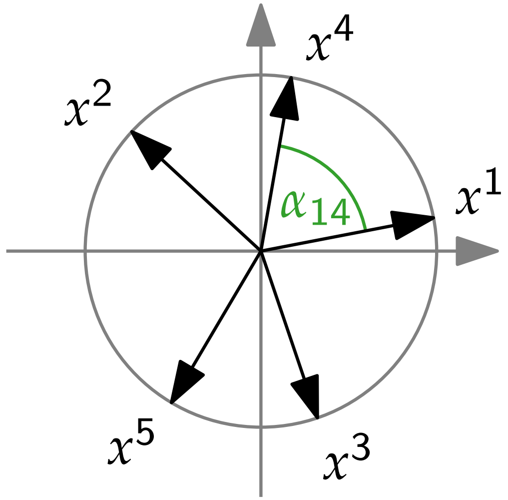

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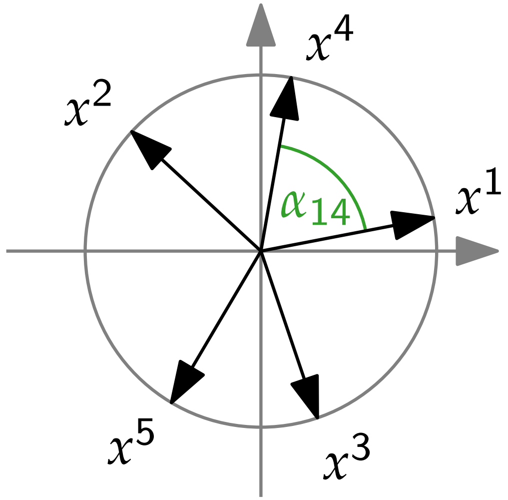

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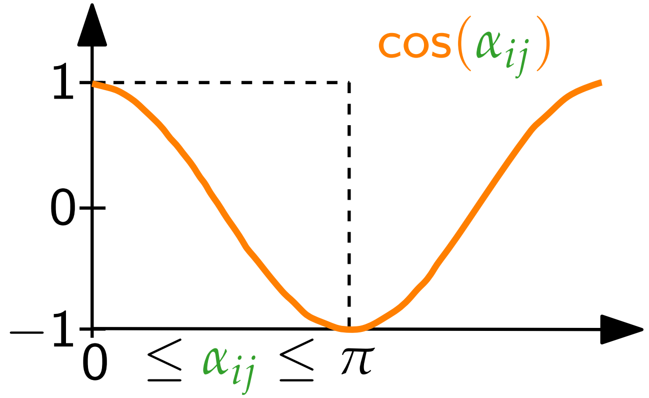

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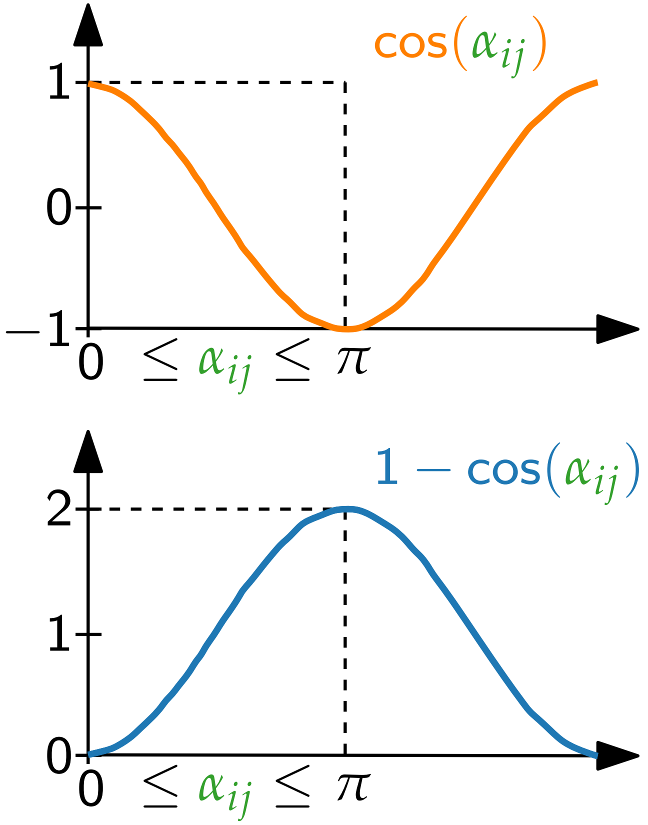

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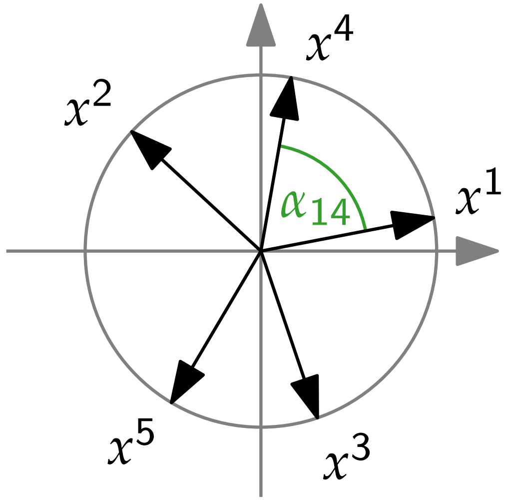

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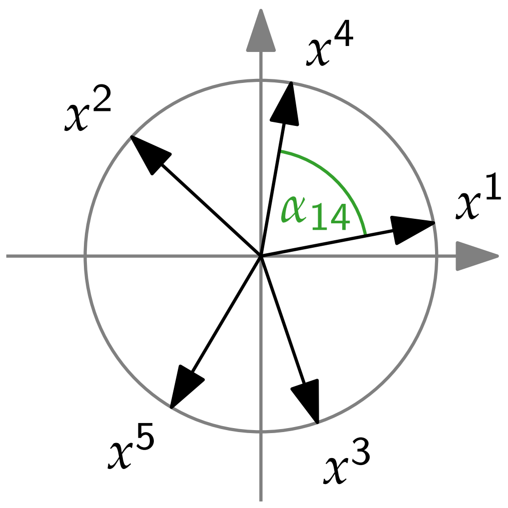
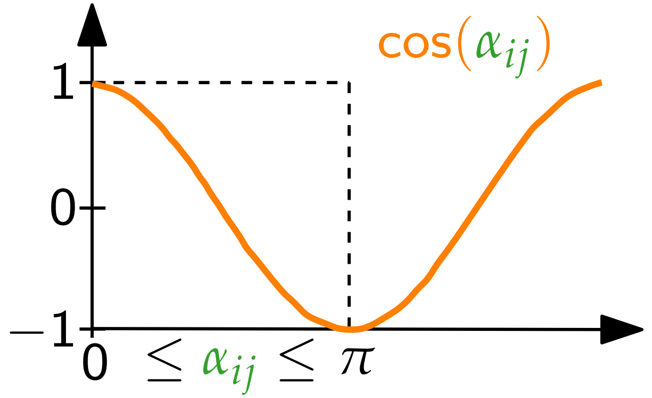
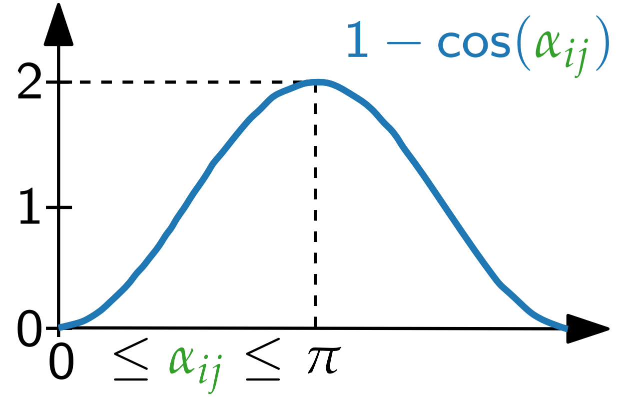


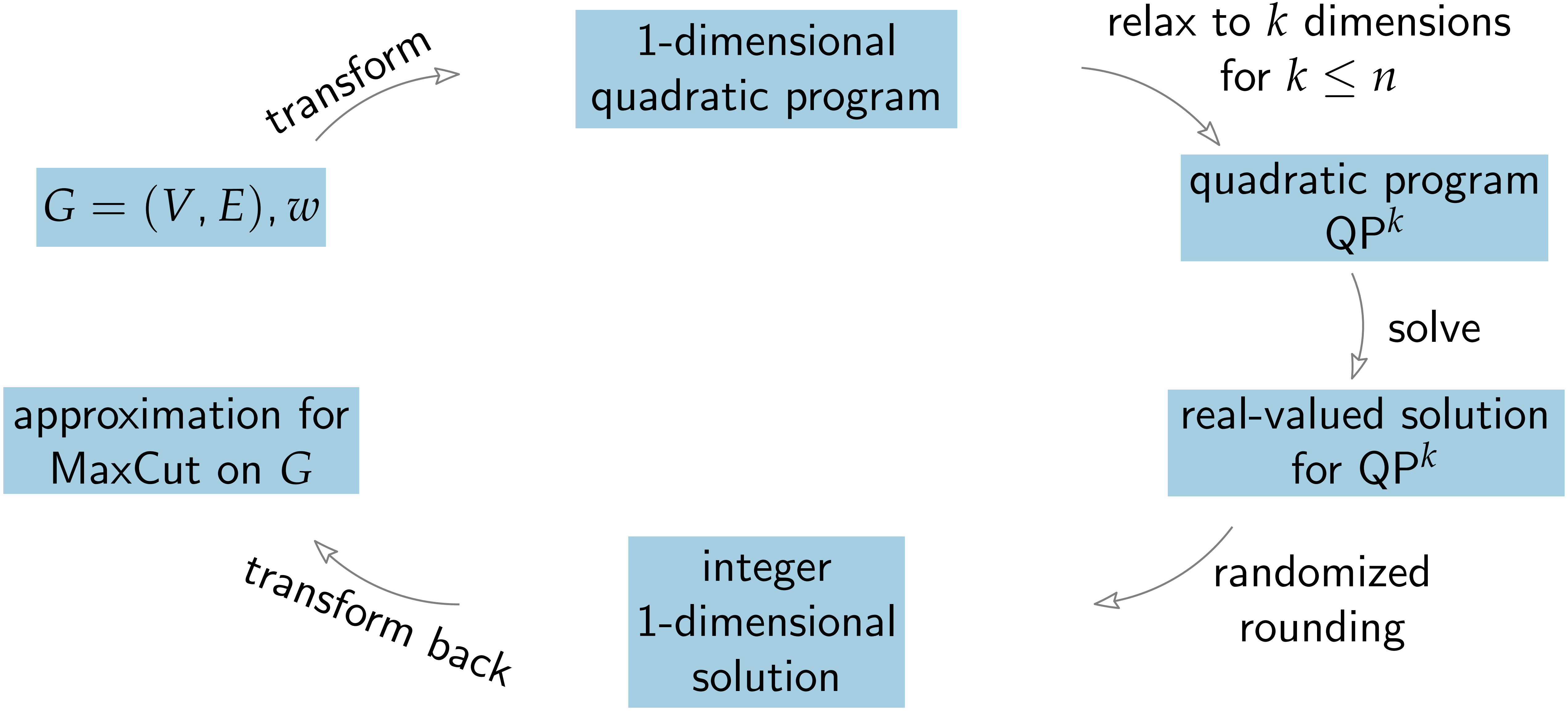
- The variables are 2-dimensional vectors.
- We maximize angles α_{ij} since larger α_{ij} increases the contribution of w_{ij}.
- Hence, our objective is:

$$\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij}(1 - \cos(\alpha_{ij}))$$





Goemans–Williamson Algorithm for MaxCut



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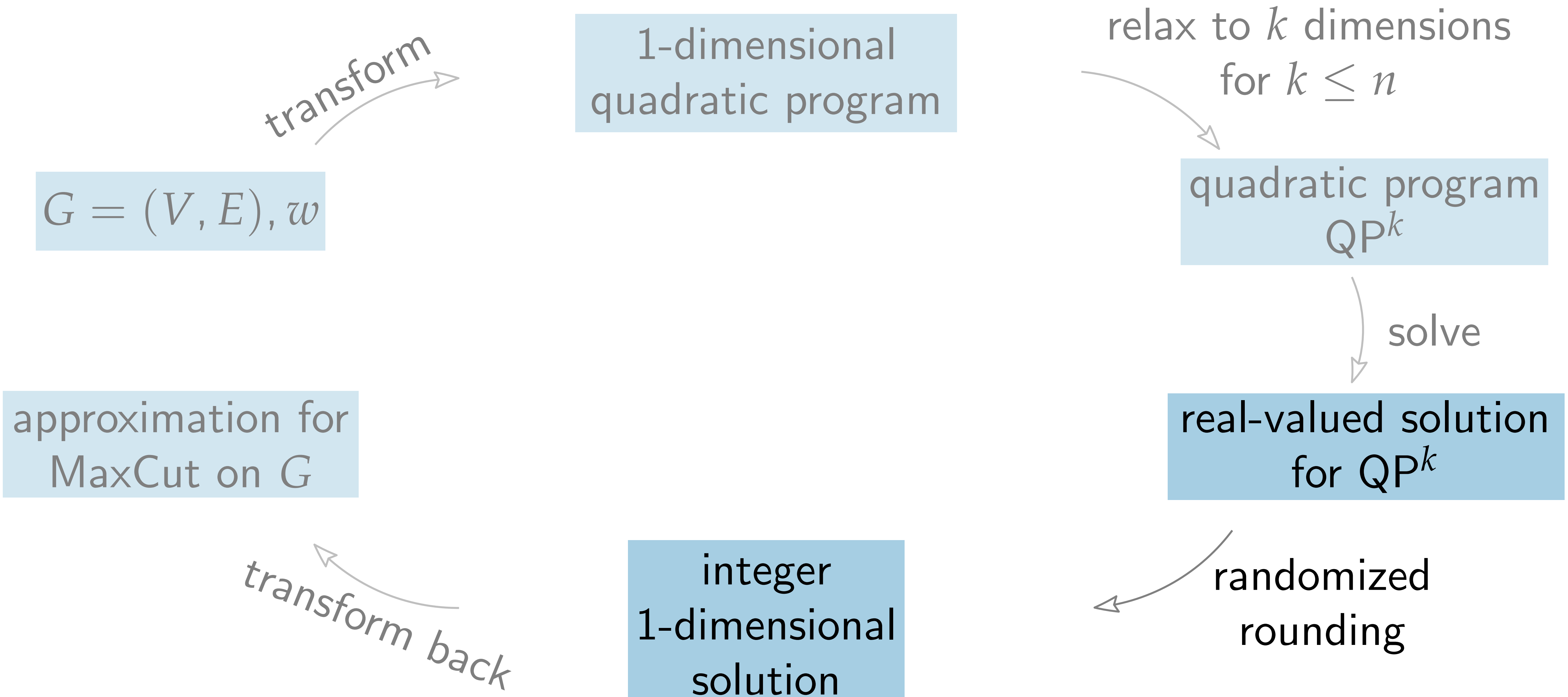

- Here again just for $k = 2$.

Algorithm RandomizedMaxCut

RandomizedMaxCut(G, w)

Compute optimal solution $(\tilde{x}^1, \ldots, \tilde{x}^n)$ for QP$^2(G, w)$

Pick random vector $r \in \mathbb{R}^2$

$S \leftarrow \{v_i \in V(G) : \tilde{x}^i \cdot r \geq 0\}$

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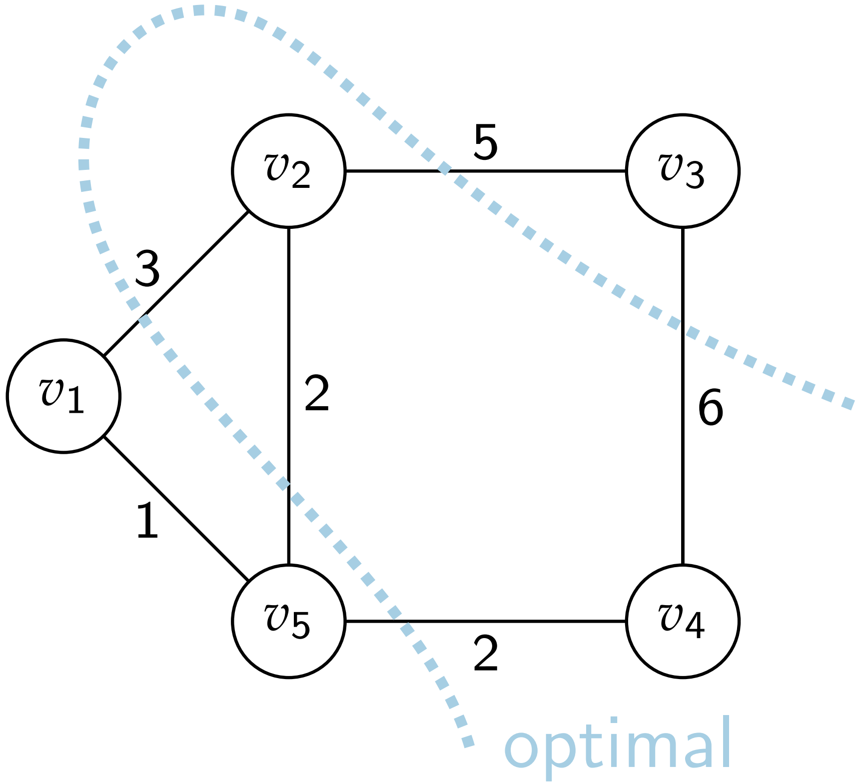

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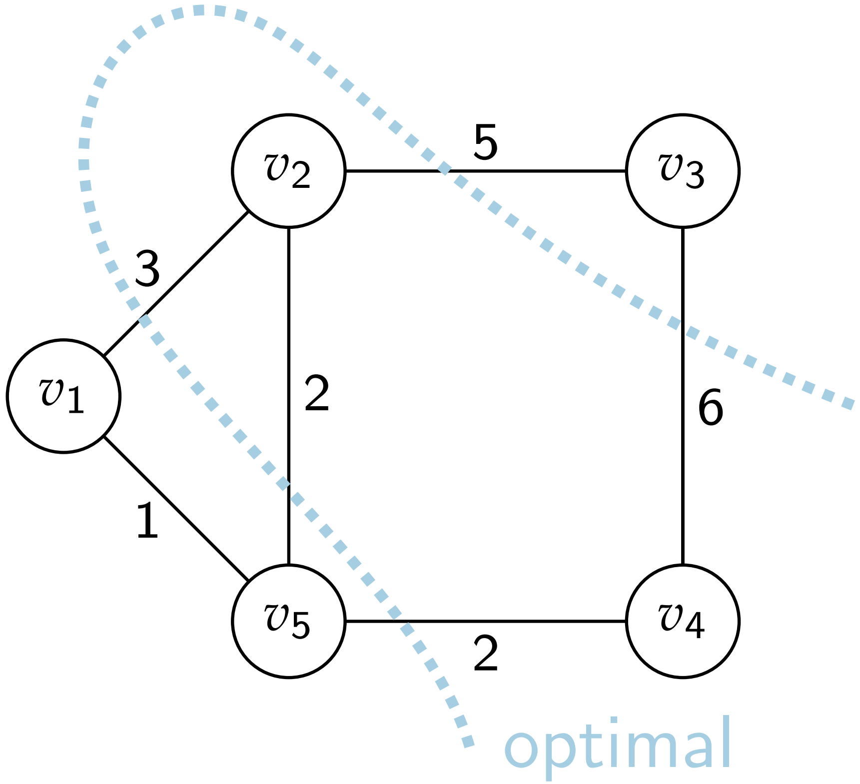

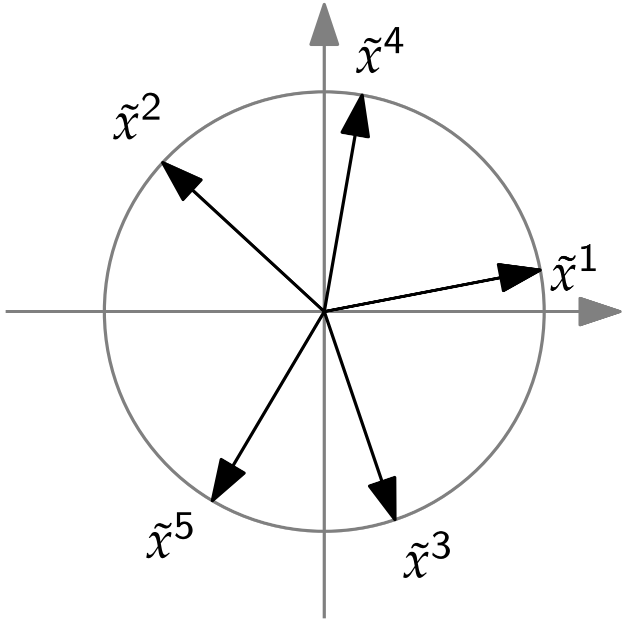

(Sketch)

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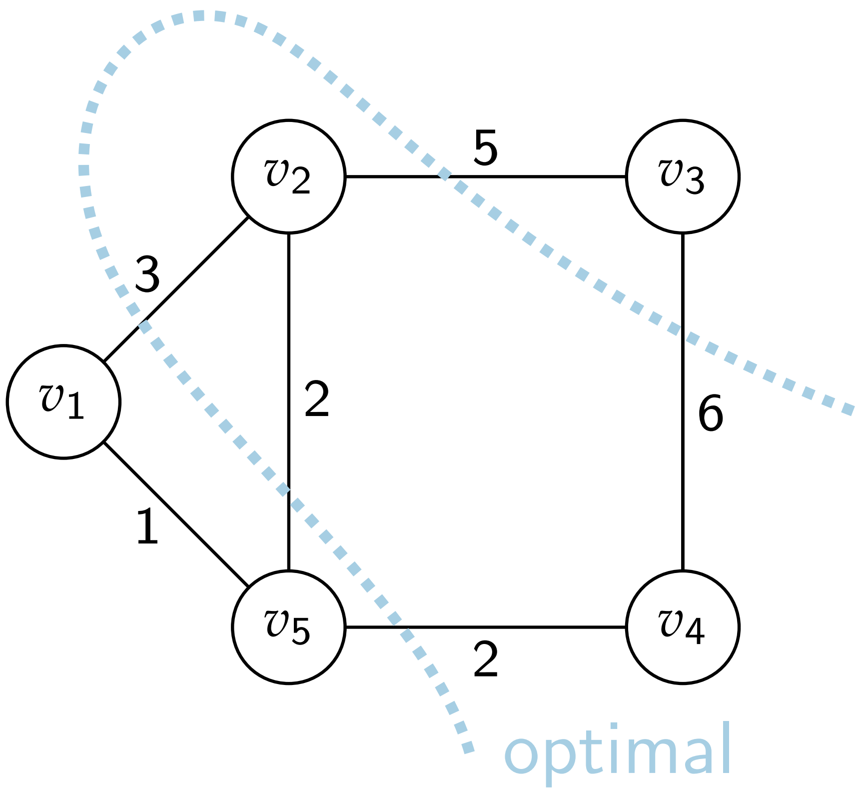

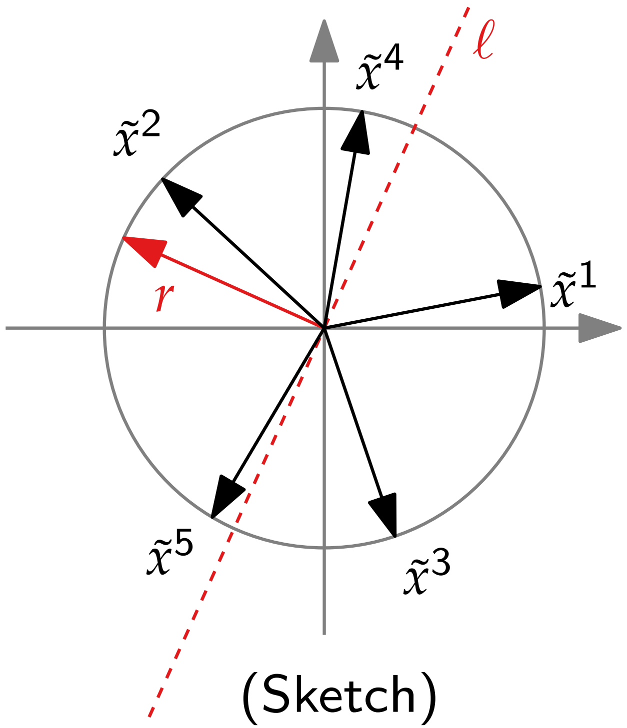

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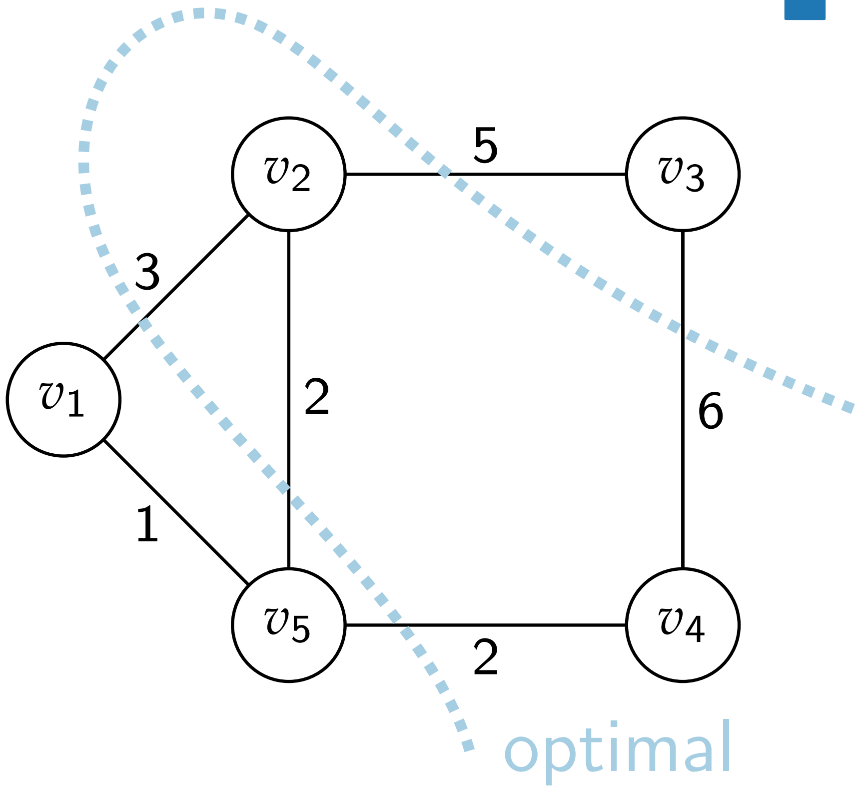
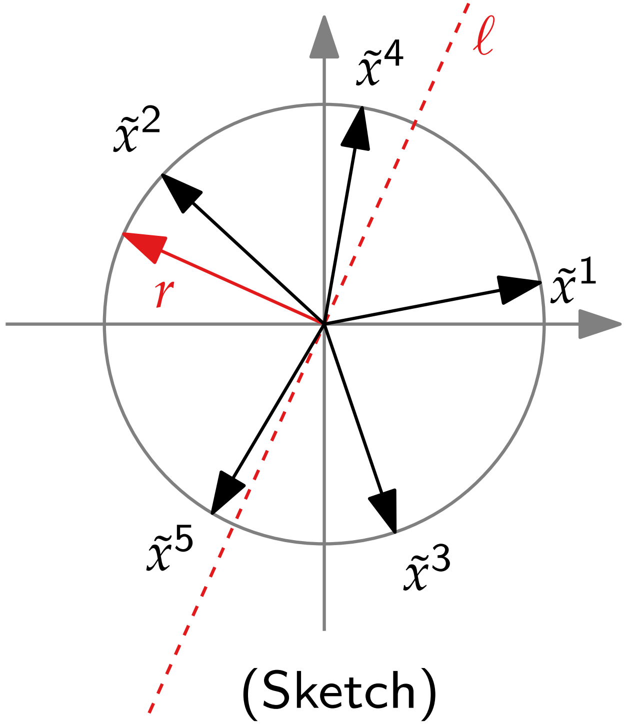
- $\tilde{x}^i$ lies above the line ℓ orthogonal to r





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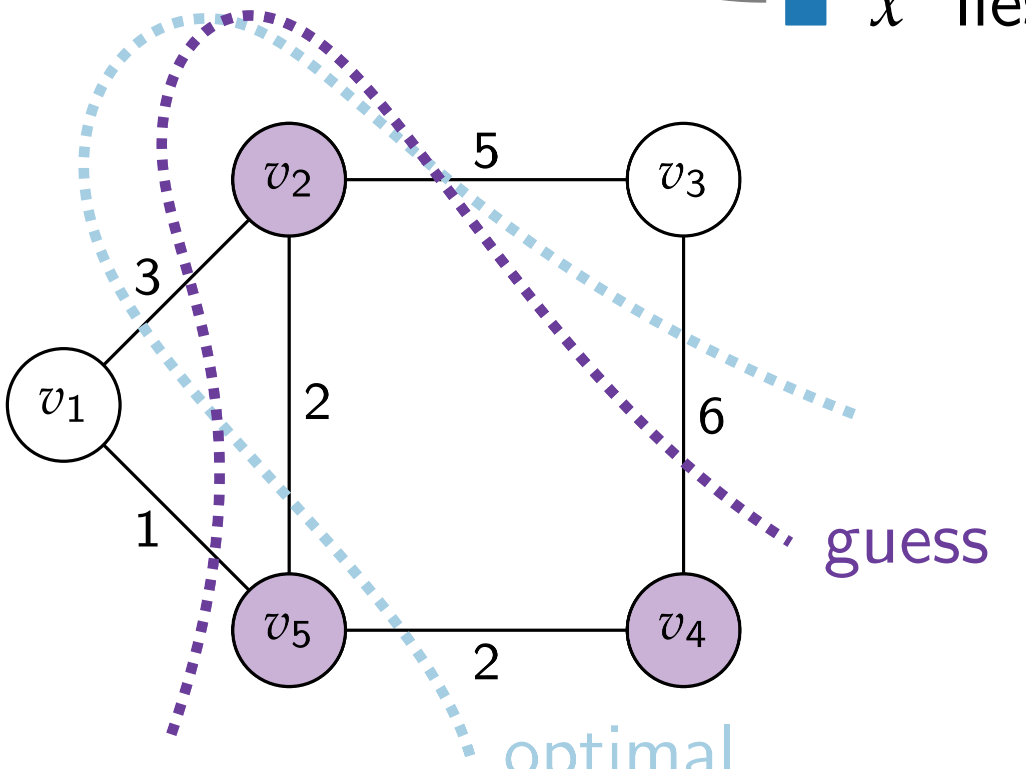

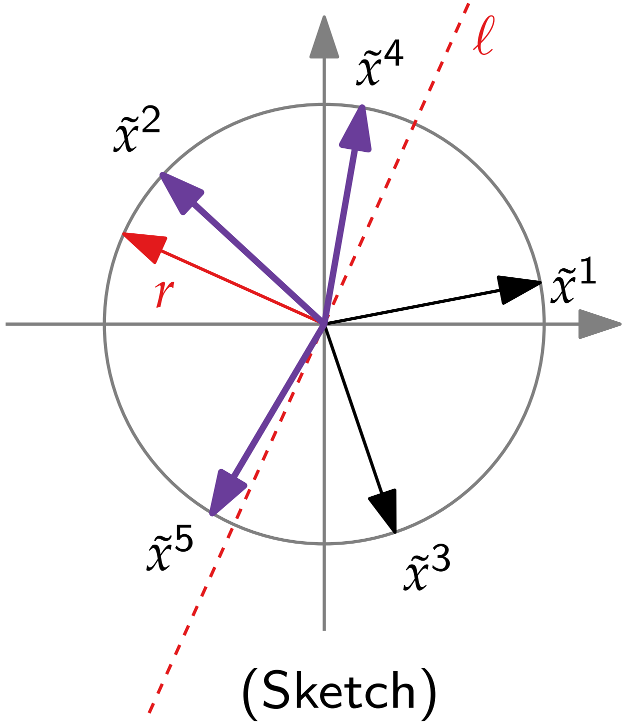

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$\textsc{RandomMaxCut}$ – Expected Value

Lemma 2.
Let X be the solution of $\textsc{RandomizedMaxCut}(G, w)$.
If r is picked uniformally at random, then

$$\mathrm{E}[X] = \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij} \frac{\alpha_{ij}}{\pi} .$$

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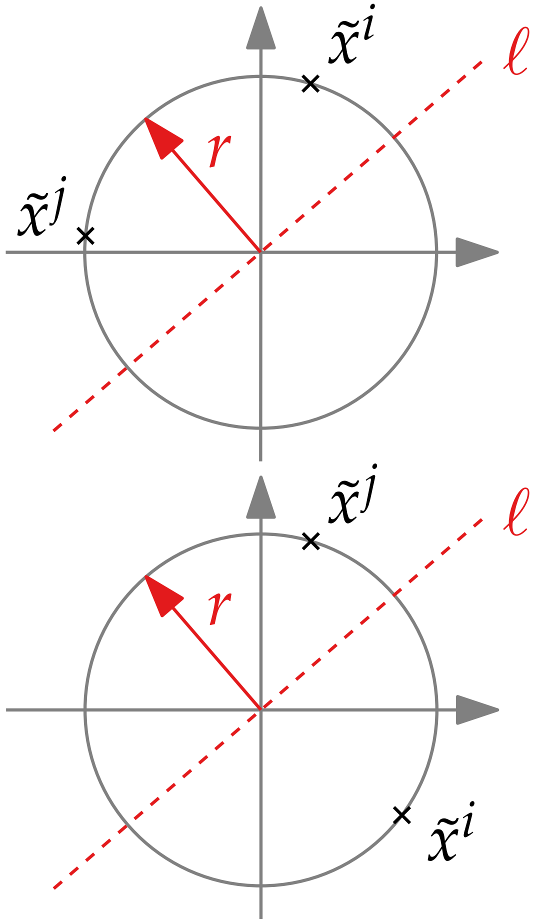

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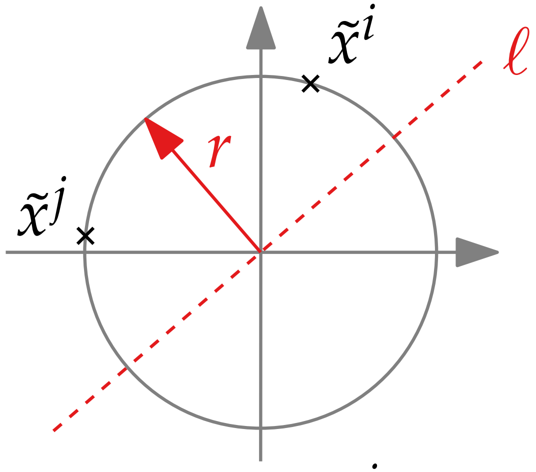

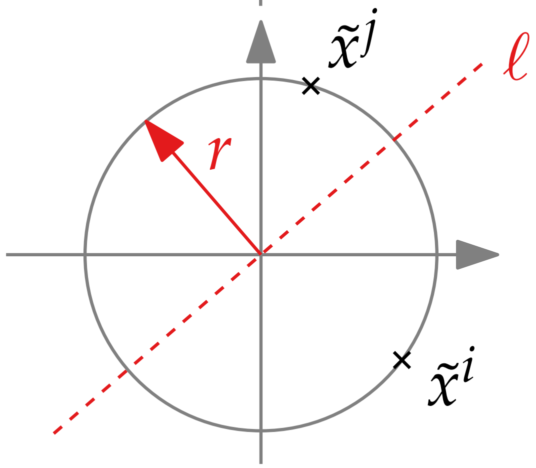

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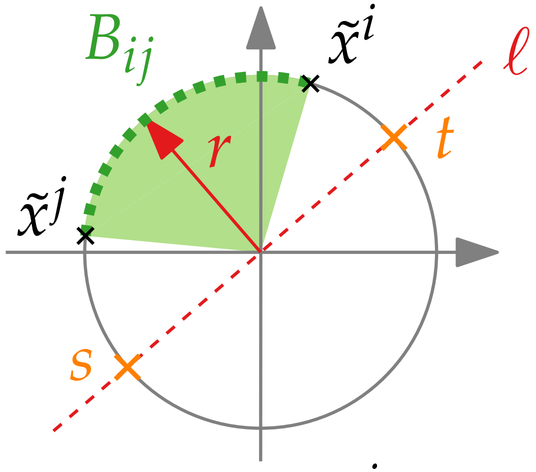

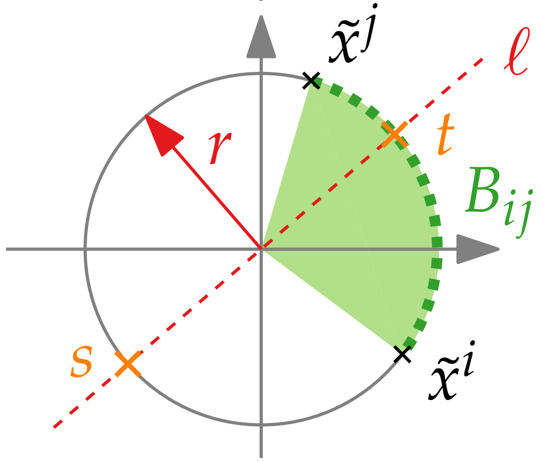

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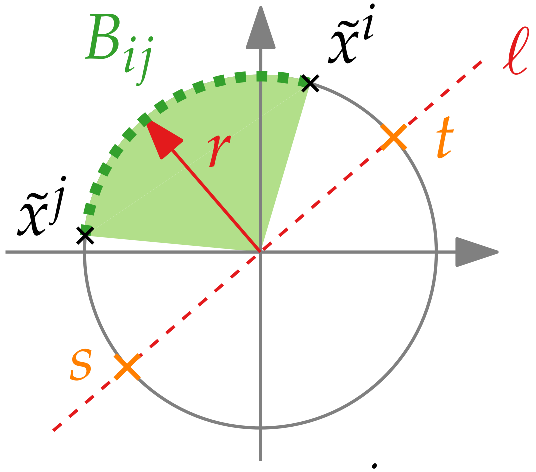

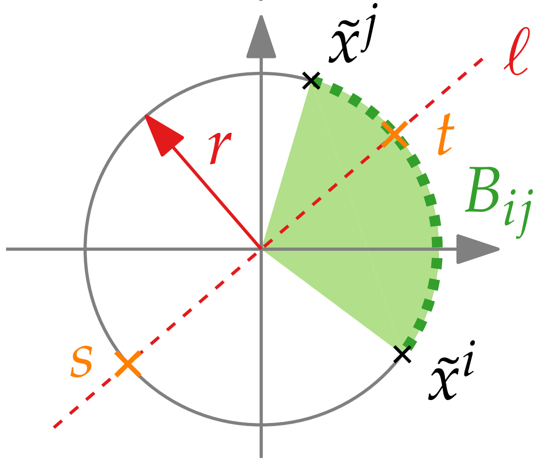

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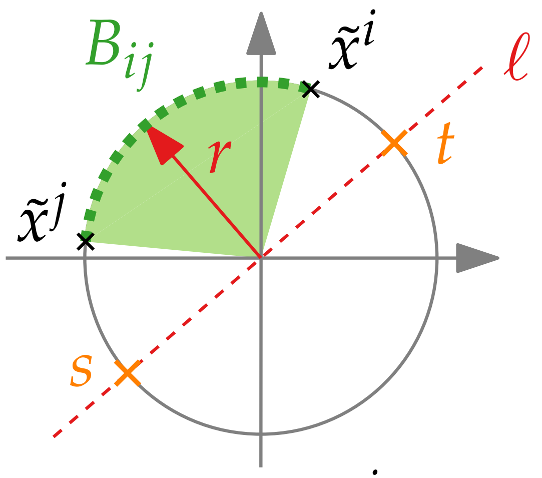

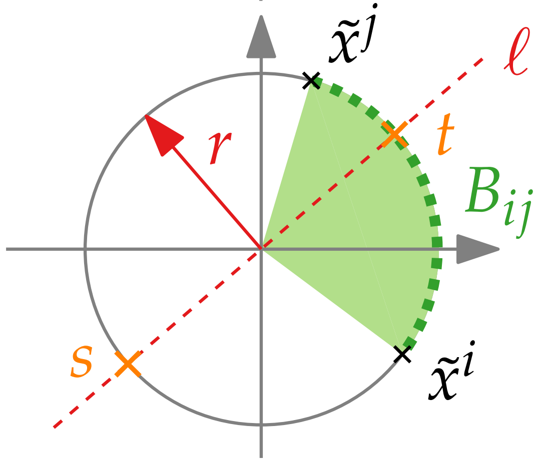

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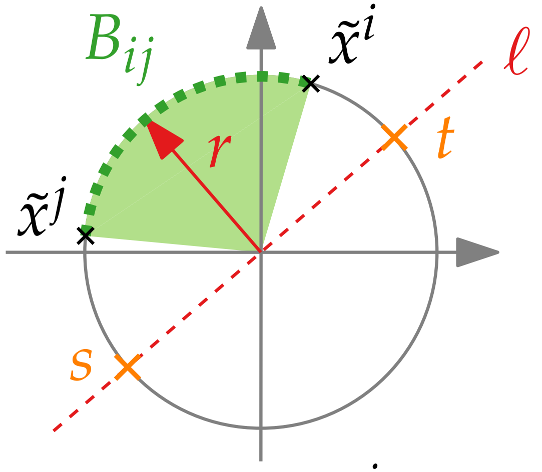

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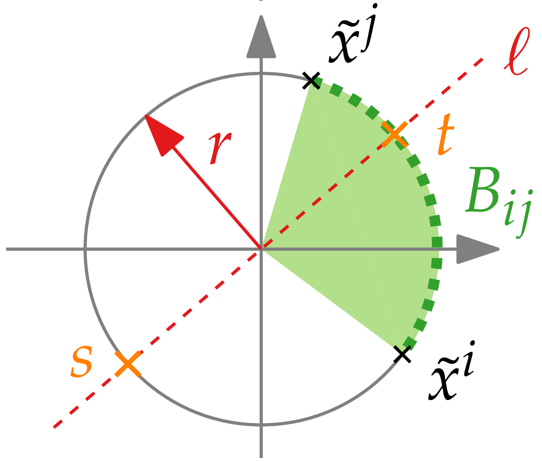

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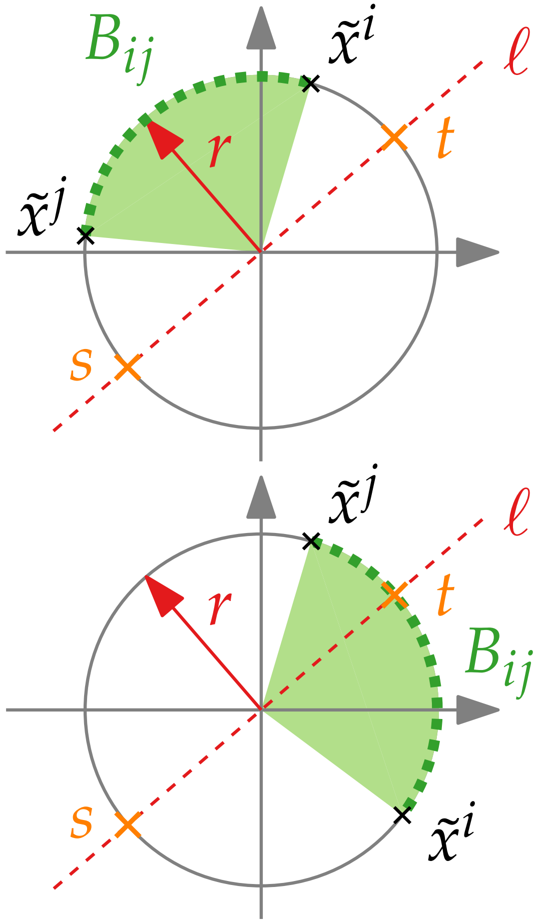

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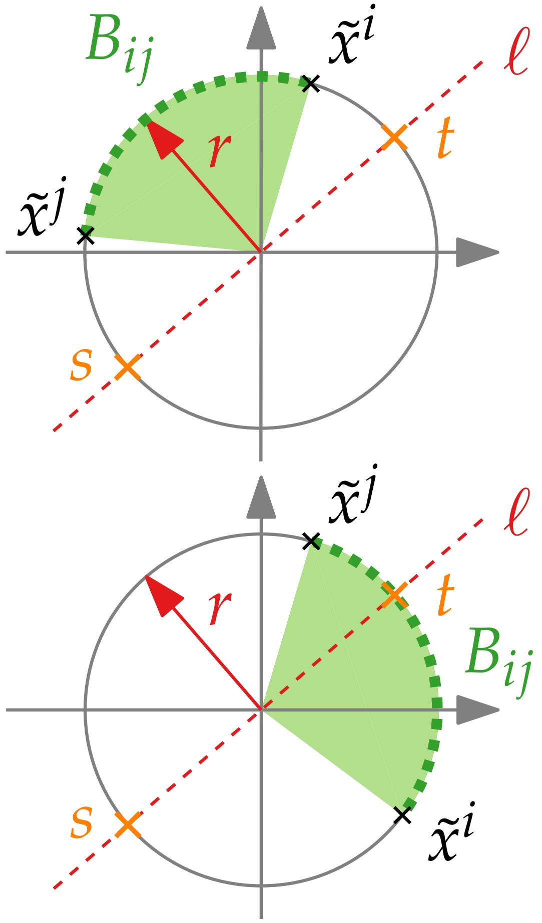
- $E[X] = \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij}\, \mathrm{P}[\ell \text{ separates } \tilde{x}^i, \tilde{x}^j] = \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij} \frac{\alpha_{ij}}{\pi}$
- $\mathrm{P}[\ell \text{ separates } \tilde{x}^i, \tilde{x}^j] = \mathrm{P}[s \in B_{ij} \text{ or } t \in B_{ij}] = \frac{\alpha_{ij}}{2\pi} + \frac{\alpha_{ij}}{2\pi} = \frac{\alpha_{ij}}{\pi}$
- B_{ij} has length α_{ij}.
- If $\tilde{x}^i$ (or $\tilde{x}^j$) lies $\leq \alpha_{ij}$ before s or t on the perimeter of the unit disk, then s or $t \in B_{ij}$.



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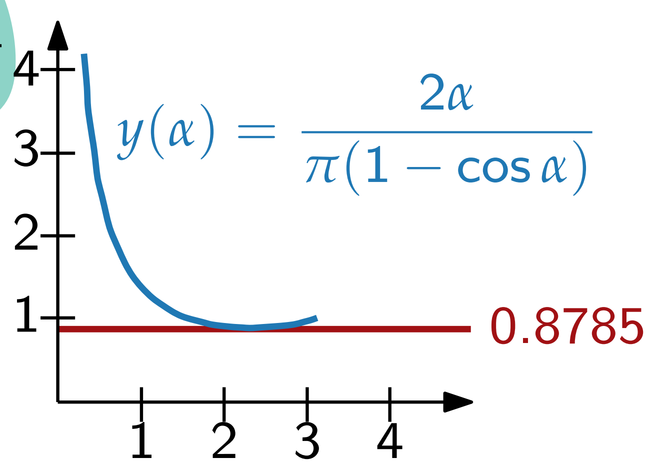

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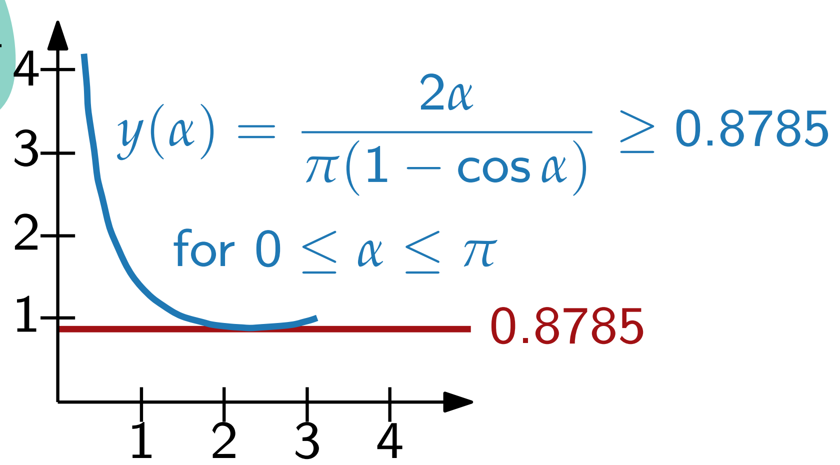

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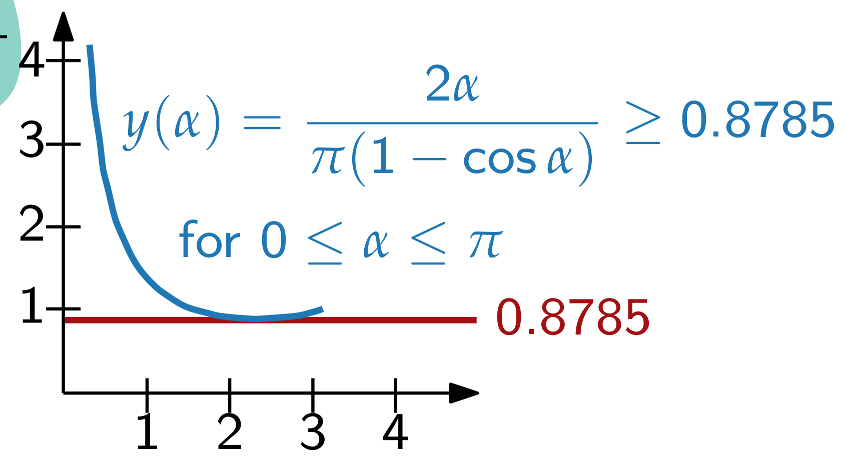

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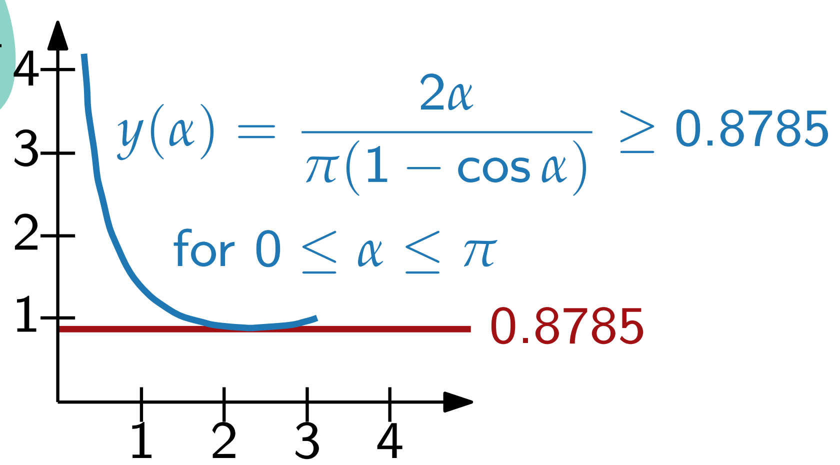

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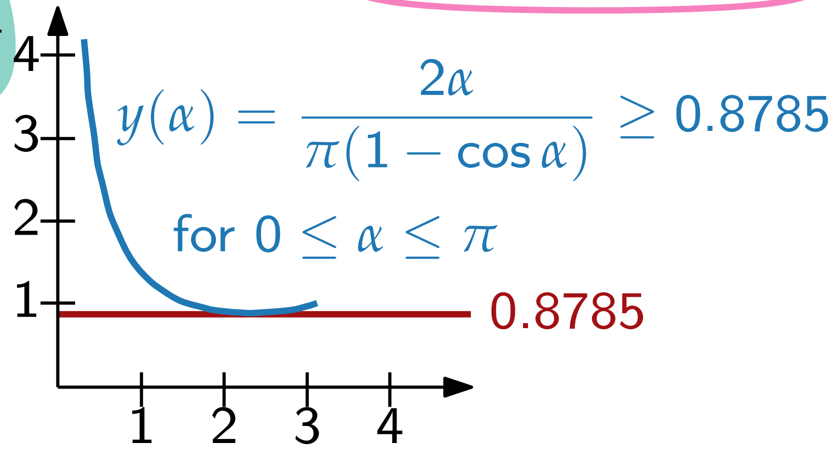

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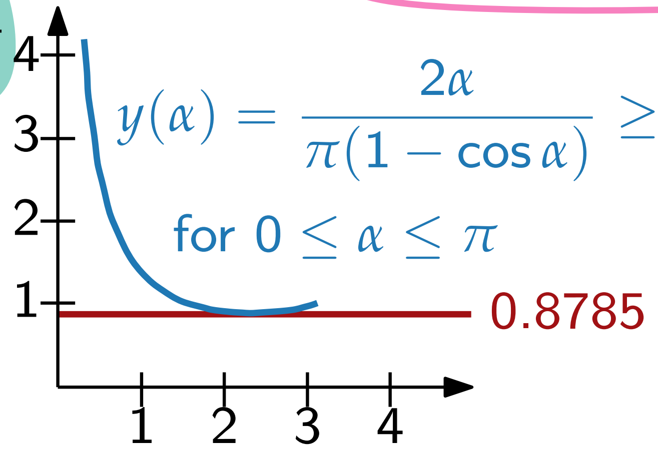
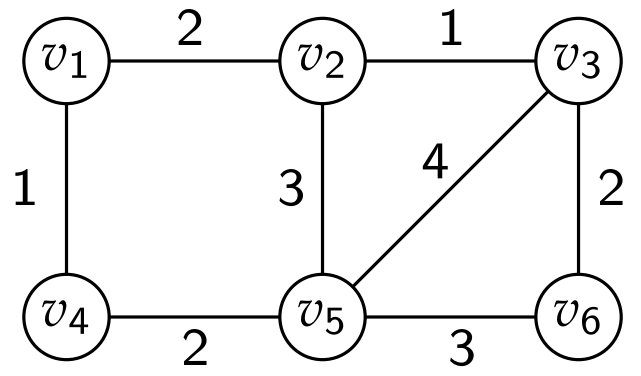


Example



Example

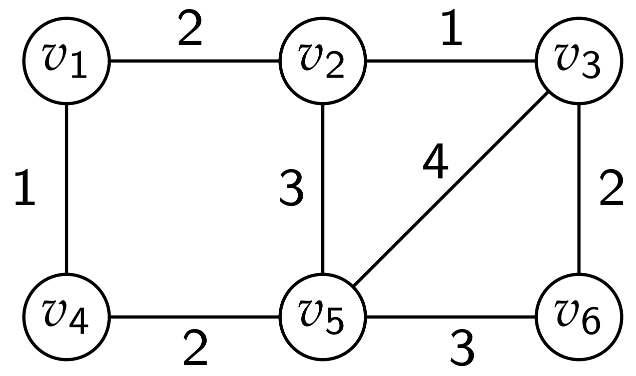
1. Step: Build QP

maximize $\frac{1}{2}\sum_{j=1}^{6}\sum_{i=1}^{j-1} w_{ij}(1 - x_i x_j)$

subject to $x_i^2 = 1$

Weight matrix w_{ij}

	1	2	3	4	5	6
1		2		1		
2	2		1		3	
3		1			4	2
4	1				2	
5		3	4	2		3
6			2		3	



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2. Step: Relax QP to QP^2

$$\begin{array}{ll} \textbf{maximize} & \frac{1}{2}\sum_{j=1}^{6}\sum_{i=1}^{j-1} w_{ij}(1 - x^i \cdot x^j) \\ \textbf{subject to} & x^i \cdot x^i = 1 \\ & x^i = (x_1^i, x_2^i) \in \mathbb{R}^2 \end{array}$$

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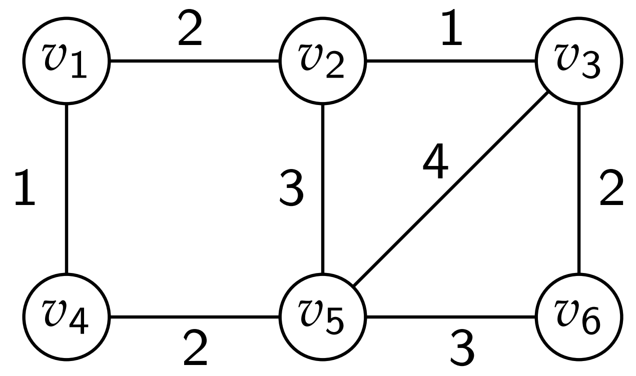

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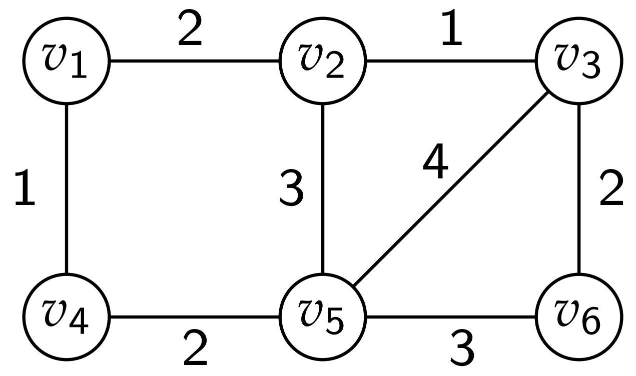
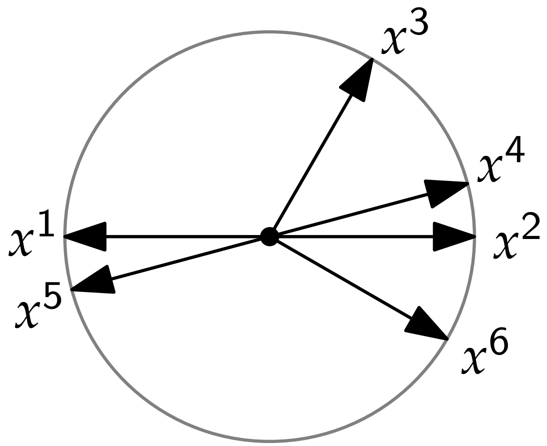
3. Step: Solve QP^2

Variable	x^1	x^2	x^3	x^4	x^5	x^6
Angle	0°	180°	120°	165°	345°	210°





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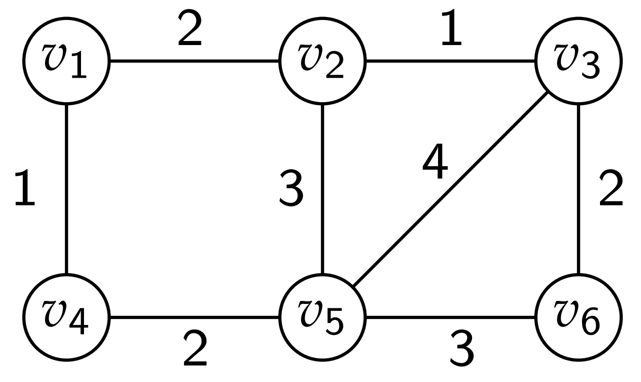

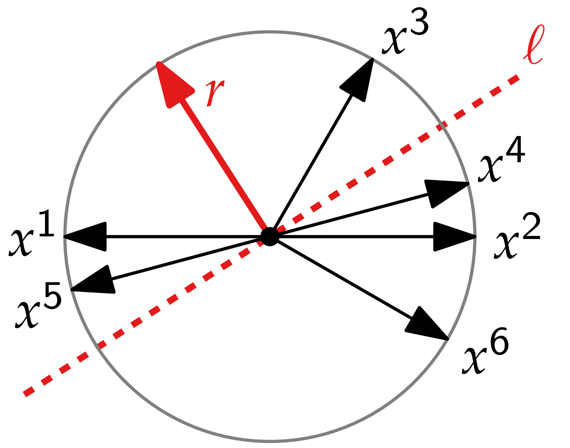

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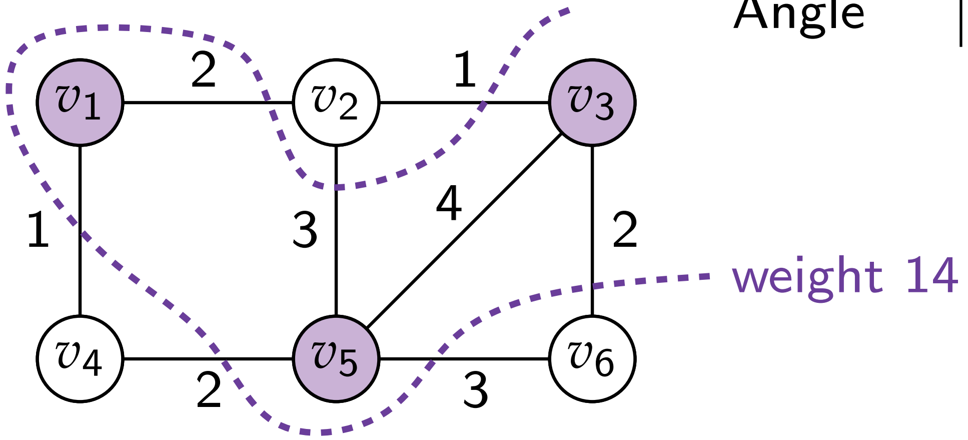

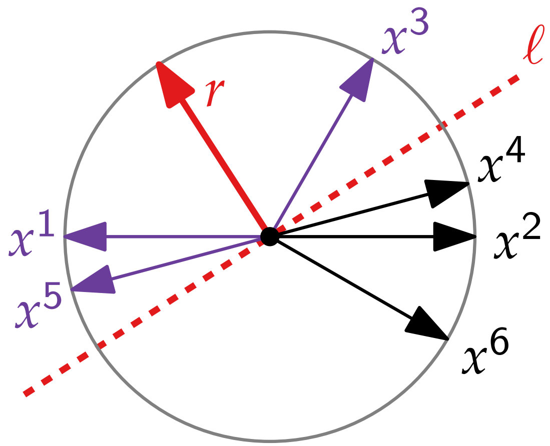

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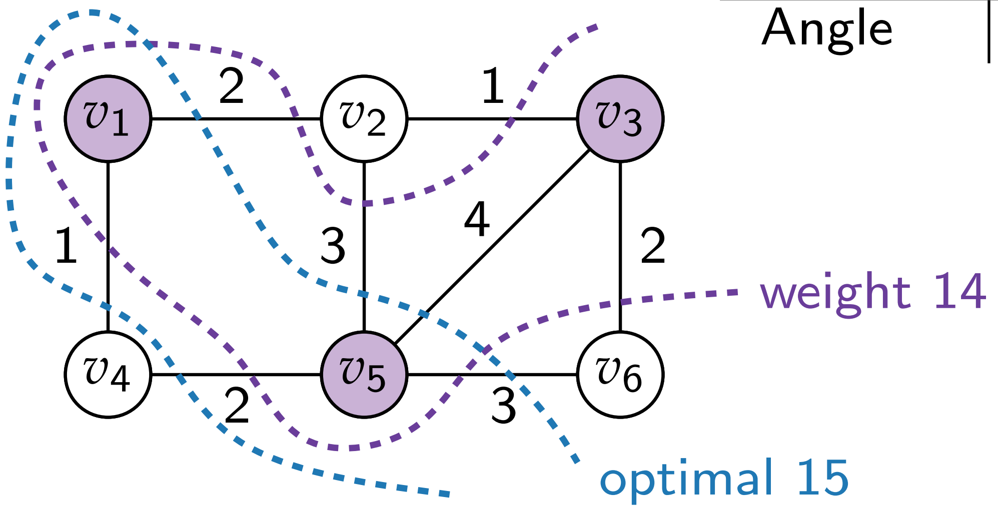
1		2		1		
2	2		1		3	
3		1			4	2
4	1				2	
5		3	4	2		3
6			2		3	

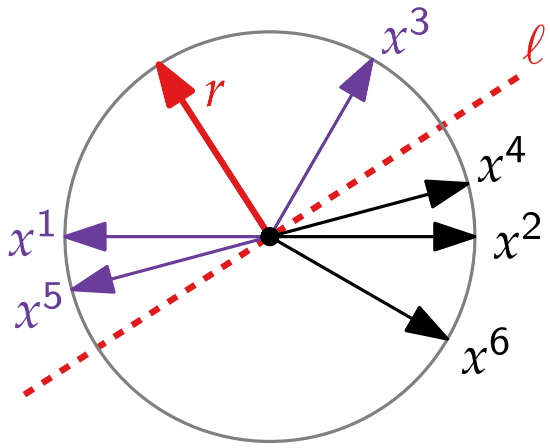
2. Step: Relax QP to QP^2

$$\text{maximize} \quad \frac{1}{2}\sum_{j=1}^{6}\sum_{i=1}^{j-1} w_{ij}(1 - x^i \cdot x^j)$$
$$\text{subject to} \quad x^i \cdot x^i = 1$$
$$x^i = (x^i_1, x^i_2) \in \mathbb{R}^2$$

3. Step: Solve QP^2

Variable	x^1	x^2	x^3	x^4	x^5	x^6
Angle	0°	180°	120°	165°	345°	210°

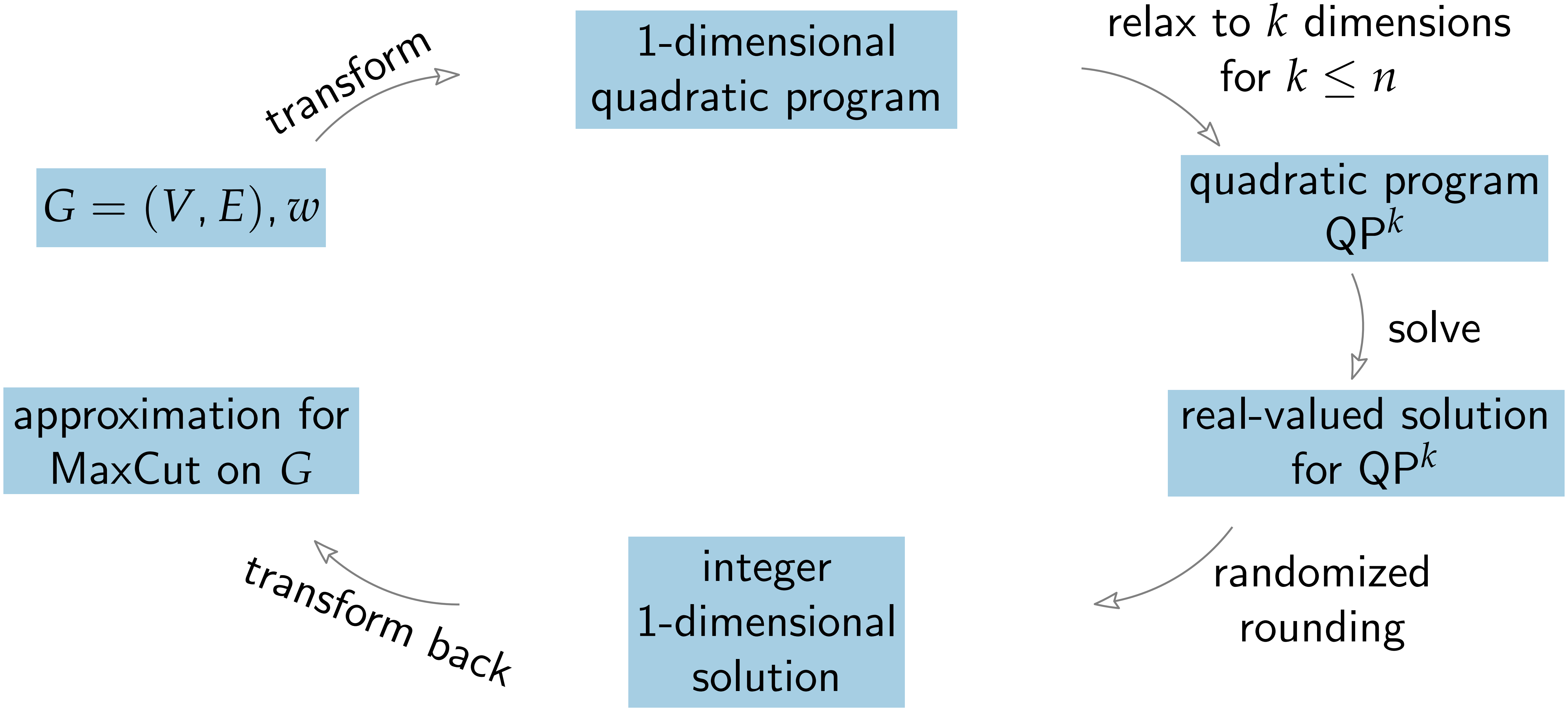




4. Step: Guess r

5. Step: Derive S

Goemans–Williamson Algorithm for MaxCut



Goemans–Williamson Algorithm for MaxCut

$G = (V, E), w$

transform

1-dimensional quadratic program

relax to k dimensions for $k \leq n$

quadratic program QP^k

solve

real-valued solution for QP^k

randomized rounding

integer 1-dimensional solution

transform back

approximation for MaxCut on G

- So far, $k = 2$.
- QP^n can be solved in polynomial time.

$\mathrm{QP}^n(G, w)$

QP$^2(G, w)$

$$\begin{array}{lrl} \textbf{maximize} & \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{j-1} w_{ij}(1 - x^i \cdot x^j) & \\ \textbf{subject to} & x^i \cdot x^i & = 1 \\ & x^i = (x^i_1, x^i_2) & \in \mathbb{R}^2 \end{array}$$

QP$^n(G, w)$

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QP^2 (G, w)

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$$x^i = (x_1^i, x_2^i) \in \mathbb{R}^2$$

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$$x^i \in \mathbb{R}^n$$

- A matrix M is called **positive semidefinite** if for any vector $v \in \mathbb{R}^n$:

$$v^\mathsf{T} \cdot M \cdot v \geq 0$$

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$\mathrm{QP}^2(G, w)$

maximize $\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij}(1 - x^i \cdot x^j)$

subject to
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$\mathrm{QP}^n(G, w)$

$\mathrm{QP}^2(G, w)$

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- A matrix M is called **positive semidefinite** if for any vector $v \in \mathbb{R}^n$:
  $$v^\intercal \cdot M \cdot v \geq 0$$
- $M = (m_{ij}) = (x^i \cdot x^j)$ is positive semidefinite.
- $\mathrm{QP}^n(G, w)$ becomes the problem $\textsc{SemiDefiniteCut}(G, w)$.
  - Can be approximated in time polynomial in (G, w) and $1/\varepsilon$ with additive guarantee ε.
  - Note that the approximation of $\mathrm{QP}(G, w)$ is an extra step we have seen before. (The approximation of $\mathrm{QP}(G, w)$ with factor 0.8785 works for $\mathrm{QP}^n(G, w)$, too)

Discussion

- If the *Unique Games Conjecture* is true, then the approximation ratio of ≈ 0.8785 achieved by $\textsc{SemiDefiniteCut}$ (and $\textsc{RandomizedMaxCut}$) is best possible.
- Otherwise, no approximation ratio better than $\frac{16}{17} \approx 0.941$ is possible.
  In particular no polynomial-time approximation scheme (PTAS) exists.
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- Otherwise, no approximation ratio better than $\frac{16}{17} \approx 0.941$ is possible.
  In particular no polynomial-time approximation scheme (PTAS) exists.
- On planar graphs, the MaxCut problem can be solved optimally in polynomial time.
- Semidefinite programming is a powerful tool to develop approximation algorithms.
- Using randomness is another tool to design approximation algorithms.
  $\rightarrow$ See future lectures, in particular the next lecture!

Literature

Original paper:

- [GW '95] "Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming"

Source:

- [Vazirani Ch26] "Approximation Algorithms"

Whole book on this topic:

- [Gärtner, Matoušek] "Approximation Algorithms and Semidefinite Progamming"