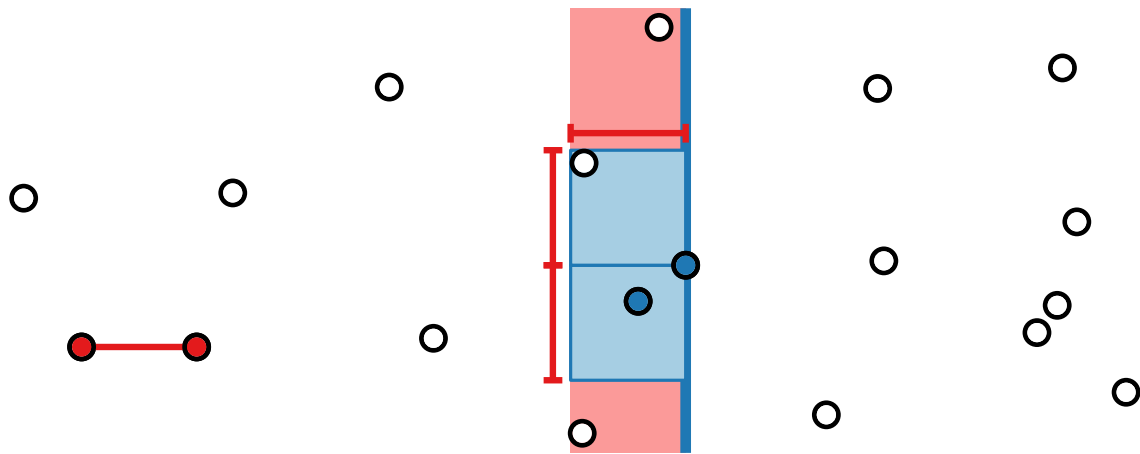


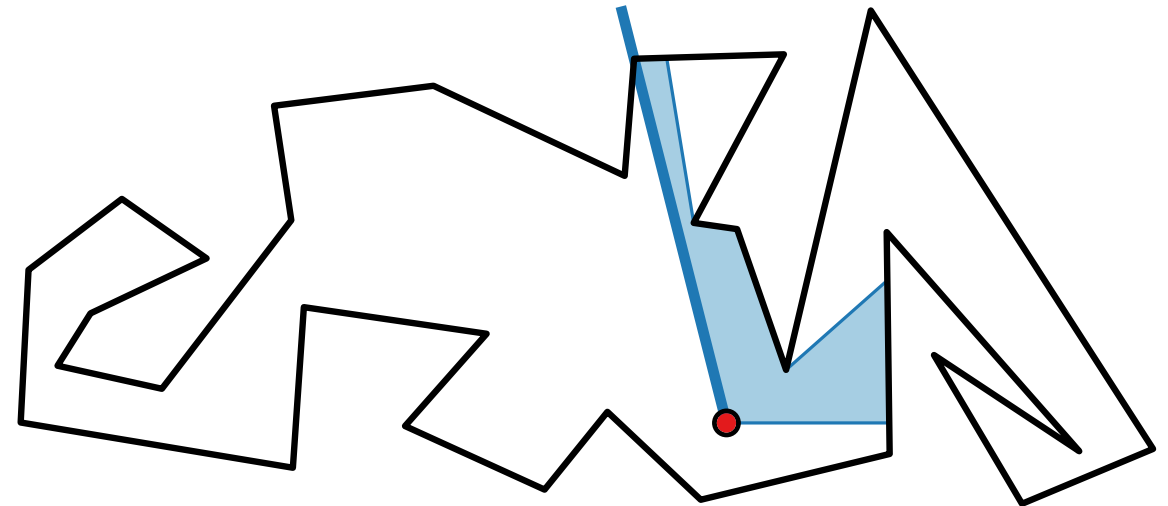
Advanced Algorithms

Computational Geometry

Sweep-Line Algorithms



Alexander Wolff



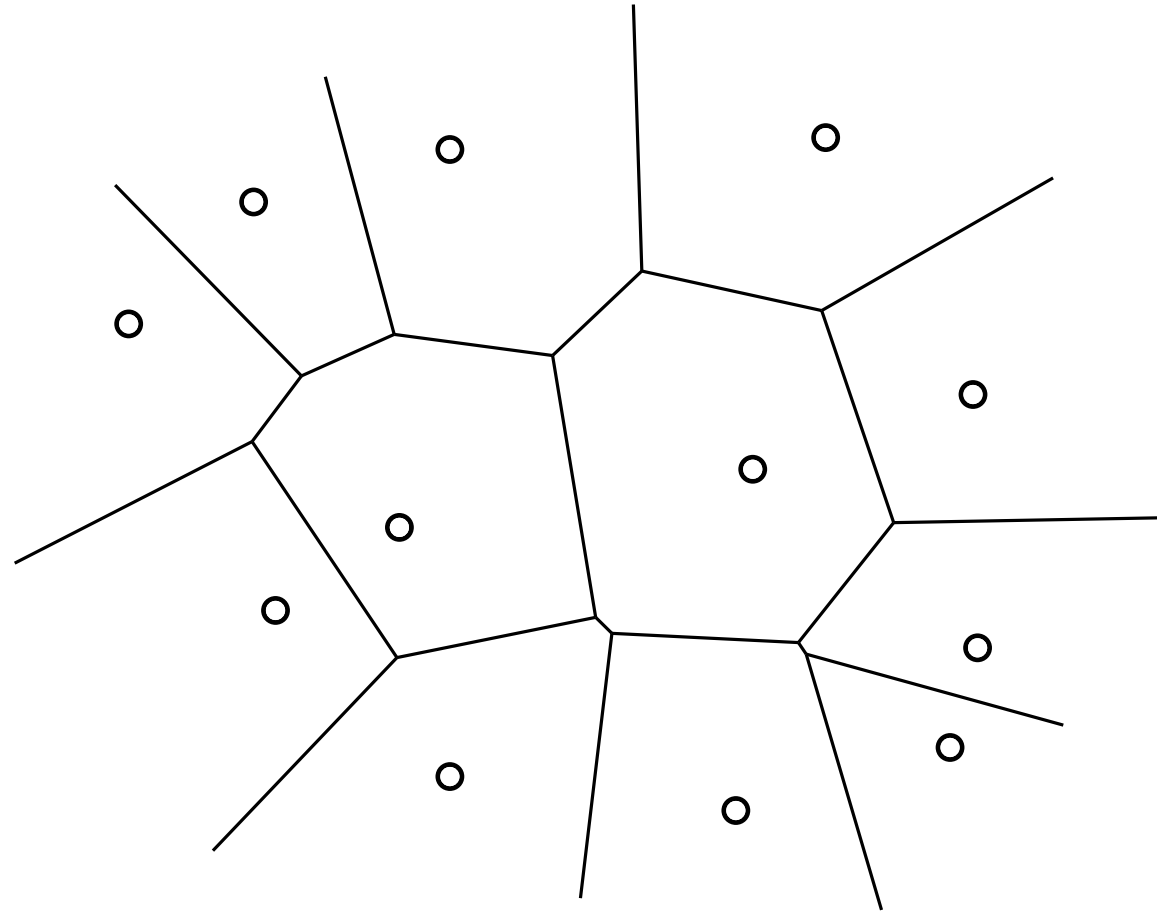
Summer term 2026

Introduction

Computational Geometry is about algorithmic problems that involve geometric objects such as points, line segments, lines, polygons, circles, planes, polyhedra, ...

Some problems:

- CLOSEST PAIR
- LINE SEGMENT INTERSECTION
- Determining visibility
- Guarding an art gallery
- Triangulating a polygon
- Motion planning
- Finding the closest post office
- and many more.

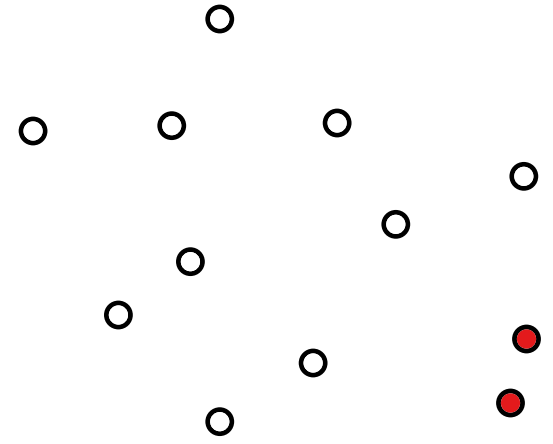


We offer an entire course on Computational Geometry in the winter term!

CLOSEST PAIR

Given: (multi-)set of points $P \subseteq \mathbb{R}^2$.

Task: Find a pair of distinct elements $p_a, p_b \in P$ such that the Euclidean distance $\|p_a - p_b\|$ is minimum.



Deterministic algorithms:

Brute-force	$\mathcal{O}(n^2)$	
Divide and conquer (recall from ADS)	$\mathcal{O}(n \log n)$	(optimal)
Sweep line	$\mathcal{O}(n \log n)$	(optimal) now!

Randomized algorithm:

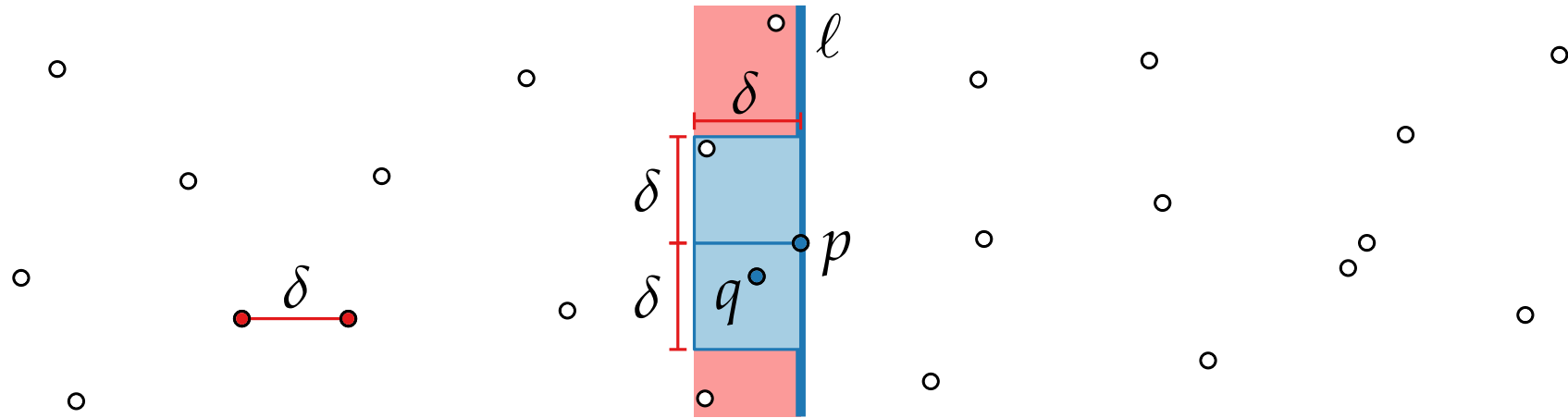
Randomized incremental construction	$\mathcal{O}(n)$	(expected runtime)
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later in
this course!

A Sweep Line Approach for CLOSEST PAIR

Assumption: The points in P have pairwise distinct x-coordinates.

Idea: Sweep the plane from left to right with a vertical line ℓ (the **sweep line**).



Invariant: a closest pair of the points to the left of ℓ and its distance δ is already known.

Observations:

- This partial solution can only change when ℓ sweeps a point p of P .
- Each new closest pair consists of p and a point q with distance $< \delta$ to ℓ .
- q needs to be located in a $\delta \times 2\delta$ rectangle R to the left of p .
- R contains $\mathcal{O}(1)$ points of $P \setminus \{p\}$ since their pairwise distance is $\geq \delta$. (packing argument)

Computing the Points in R Efficiently

Let S denote the vertical slab of width δ to the left of ℓ .

Assume that the points $P \cap S$ are stored in a **linked list** \mathcal{L} sorted according to their y-coordinates.

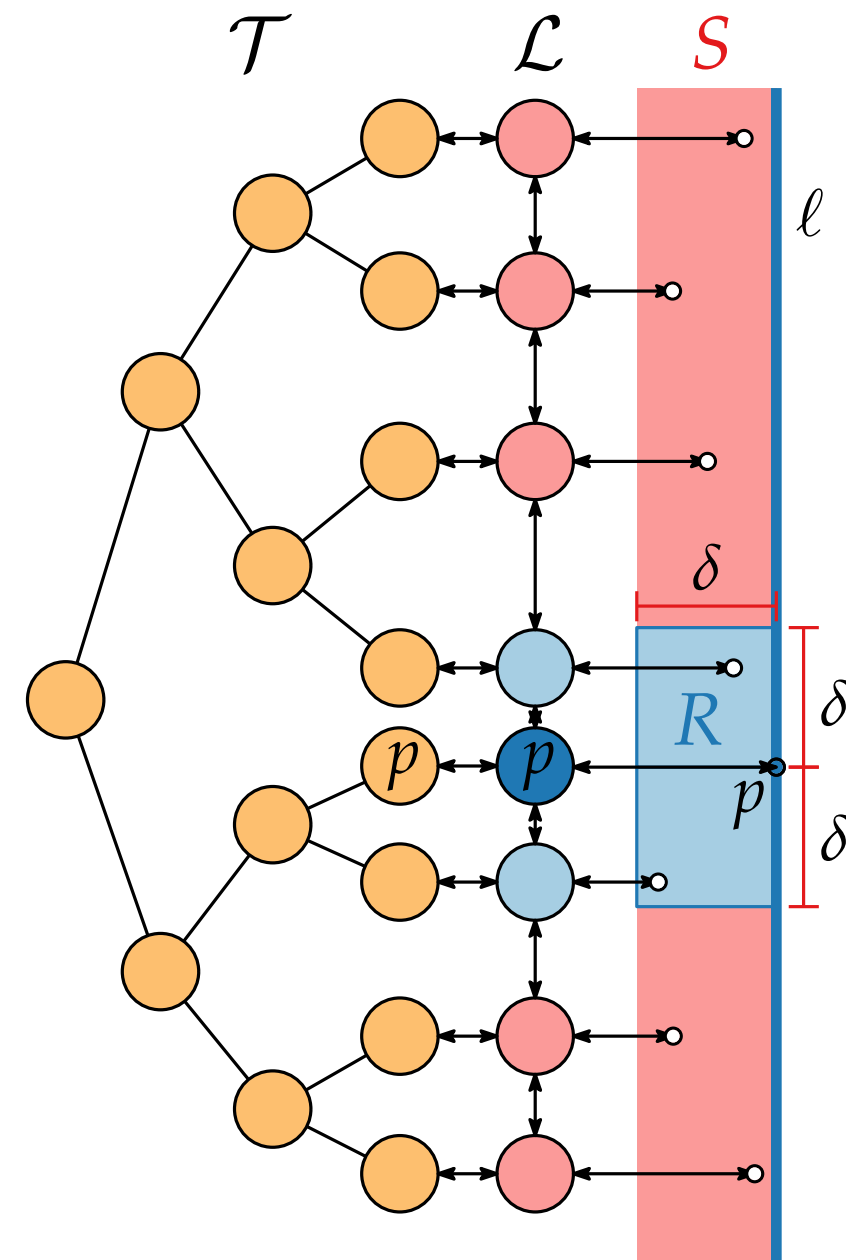
\Rightarrow Given a pointer to p , we can determine the points in R by searching the interval $[y(p) - \delta, y(p) + \delta]$.
This takes $\mathcal{O}(1)$ time since R contains $\mathcal{O}(1)$ points.

To ensure that \mathcal{L} can be updated efficiently, we additionally store the points $P \cap S$ in a **balanced binary search tree** \mathcal{T} using the y-coordinates as keys.

The corresponding elements in \mathcal{L} and \mathcal{T} are linked.

\Rightarrow When a point is inserted in \mathcal{T} in $\mathcal{O}(\log n)$ time, its according position in \mathcal{L} can be determined in $\mathcal{O}(1)$ time.

Invariant 2: When we reach a point p , \mathcal{T} and \mathcal{L} contain exactly the points in $P \cap S$.



Algorithm

p_1, p_2, \dots, p_n = points of P sorted according to their x-coordinates

$P_{\min} = \text{nil}$ // current closest pair

$\delta = \infty$ // distance of current closest pair

$k = 1$ // index of the left-most point in \mathcal{L} and \mathcal{T}

initialize \mathcal{L} and \mathcal{T} with p_1

for $i = 2, 3, \dots, n$ **do**

 insert p_i into \mathcal{L} and \mathcal{T}

for $p_j \in [y(p_i) - \delta, y(p_i) + \delta] \setminus \{p_i\}$ **do**

if $\|p_j - p_i\| < \delta$ **do**

$P_{\min} = \{p_j, p_i\}; \delta = \|p_j - p_i\|$

while $x(p_k) < x(p_{i+1}) - \delta$ **do**

 delete p_k from \mathcal{L} and \mathcal{T}

$k = k + 1$

return P_{\min}

Algorithm

$p_1, p_2, \dots, p_n =$ points of P sorted according to their x-coordinates $\mathcal{O}(n \log n)$
 $P_{\min} = \text{nil}$ // current closest pair
 $\delta = \infty$ // distance of current closest pair \Rightarrow **Total runtime:** $\mathcal{O}(n \log n)$
 $k = 1$ // index of the left-most point in \mathcal{L} and \mathcal{T}
 initialize \mathcal{L} and \mathcal{T} with p_1

for $i = 2, 3, \dots, n$ **do** $\mathcal{O}(n)$

 insert p_i into \mathcal{L} and \mathcal{T} $\mathcal{O}(\log n)$

for $p_j \in [y(p_i) - \delta, y(p_i) + \delta] \setminus \{p_i\}$ **do** $\mathcal{O}(1)$ $\mathcal{O}(n \log n)$

if $\|p_j - p_i\| < \delta$ **do** $\mathcal{O}(1)$

$P_{\min} = \{p_j, p_i\}; \delta = \|p_j - p_i\|$

while $x(p_k) < x(p_{i+1}) - \delta$ **do** $\mathcal{O}(n)$ in total $\mathcal{O}(n \log n)$ in total

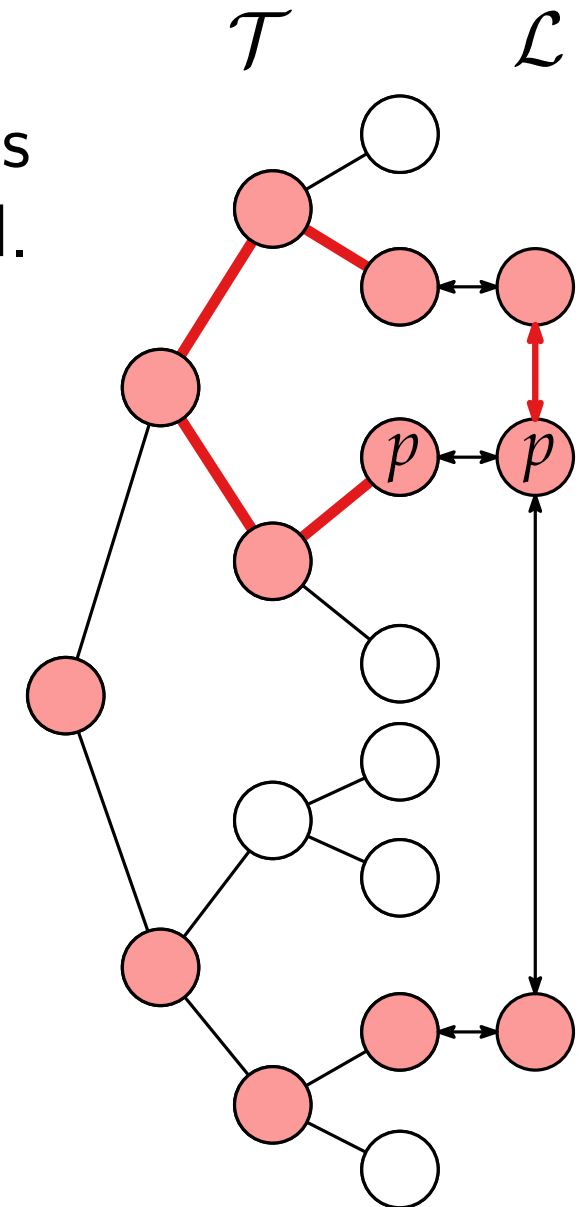
 delete p_k from \mathcal{L} and \mathcal{T} $\mathcal{O}(\log n)$

$k = k + 1$

return P_{\min}

Remarks on the Implementation

- The list \mathcal{L} is actually not necessary: given a point p in \mathcal{T} , its neighbors in the ordering can be determined in $\mathcal{O}(\log n)$ time.
- The tree \mathcal{T} does not need to be dynamic! A static tree on all points suffices if each point currently in \mathcal{S} and all its ancestors are marked. \rightarrow simple and space efficient (1 bit of extra information / node).
- We assumed that the points in P have pairwise distinct x-coordinates. This situation can be established by rotating P or tilting ℓ slightly.
Simply visit the points in lexicographical order!



Summary and Discussion

The **sweep line approach** is an important design paradigm (like divide and conquer, prune and search, dynamic programming, greedy, ...) in computational geometry.

Main idea: Sweep the plane with a line ℓ while maintaining two invariants:

- A **partial solution** for the input to the left of ℓ is known.
- The part of the input to the left of ℓ that is still relevant for updating the partial solution is encoded in a suitable data structure (**sweep line status**).

The partial solution and the sweep line status only change at specific positions (**events**) that may be part of the input or arise during the execution of the algorithm.

The sweep line paradigm is **powerful** and leads to **simple** algorithms for many problems: computing Voronoi diagrams, crossings in an arrangement of line segments, intersection/union of two polygons, decompositions of polygons, certain triangulations, visibility polygons, ...

Outlook: Computing Visibility Polygons

The sweep “line” does not always have to move from left to right!

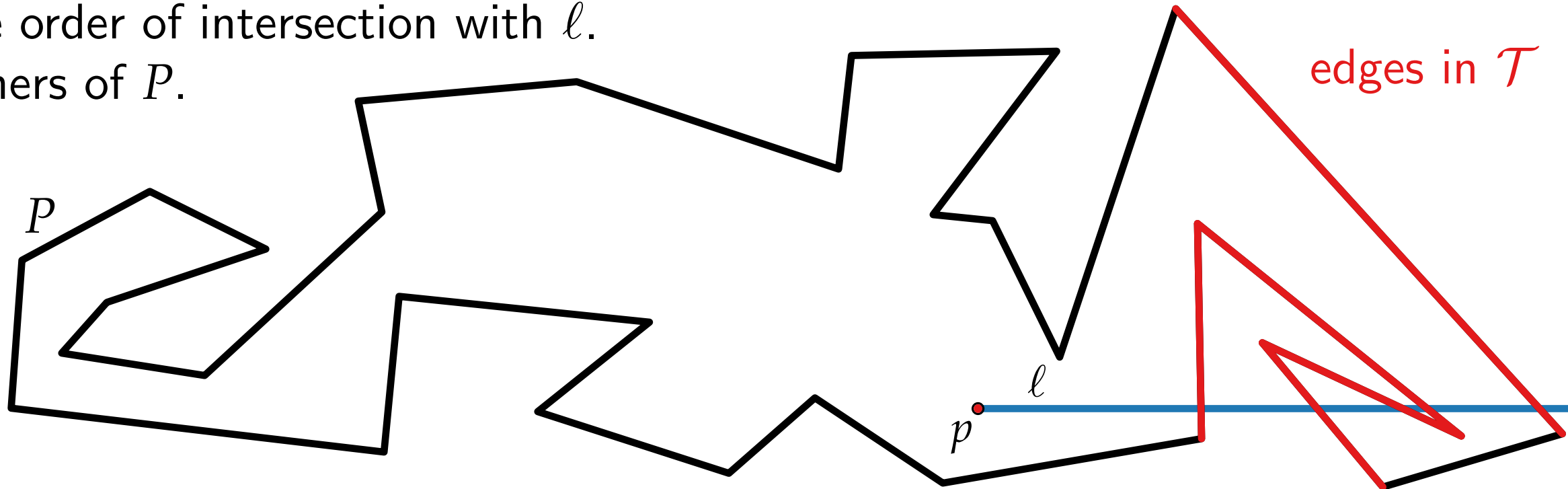
Given: A polygon P with n corners and a point p in its interior.

Task: Compute the visibility polygon of p with respect to P .

Idea: Sweep a **ray** ℓ radially around p .

Sweep line status: Edges of P intersected by ℓ are stored in a balanced binary search tree \mathcal{T} in the order of intersection with ℓ .

Events: Corners of P .



Outlook: Computing Visibility Polygons

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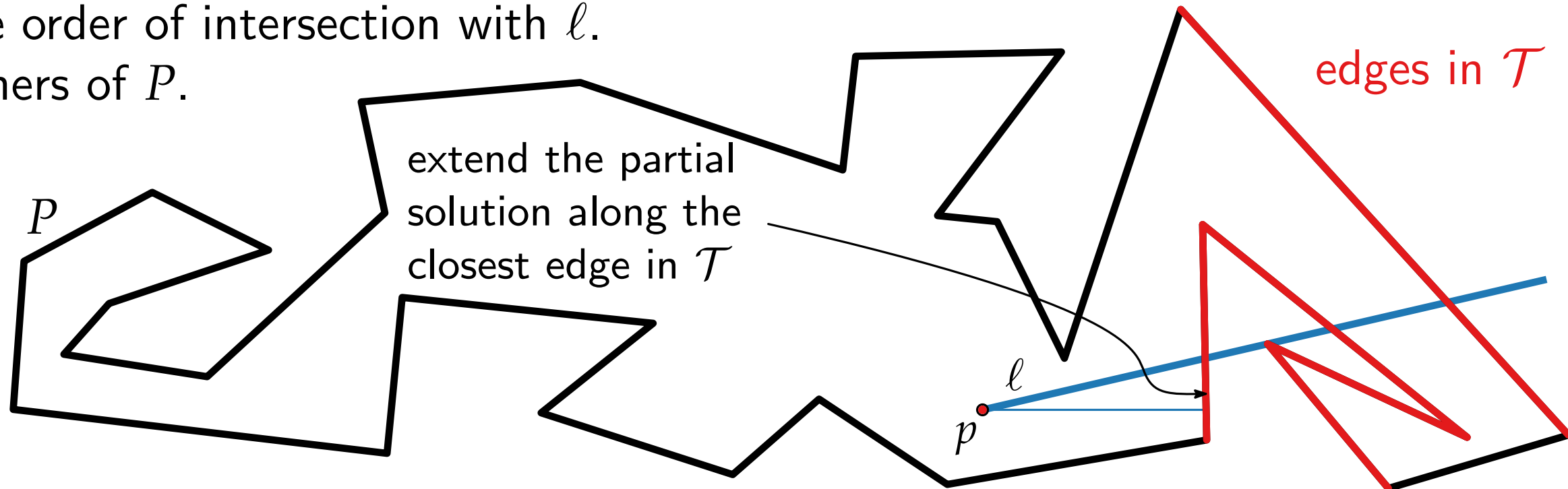
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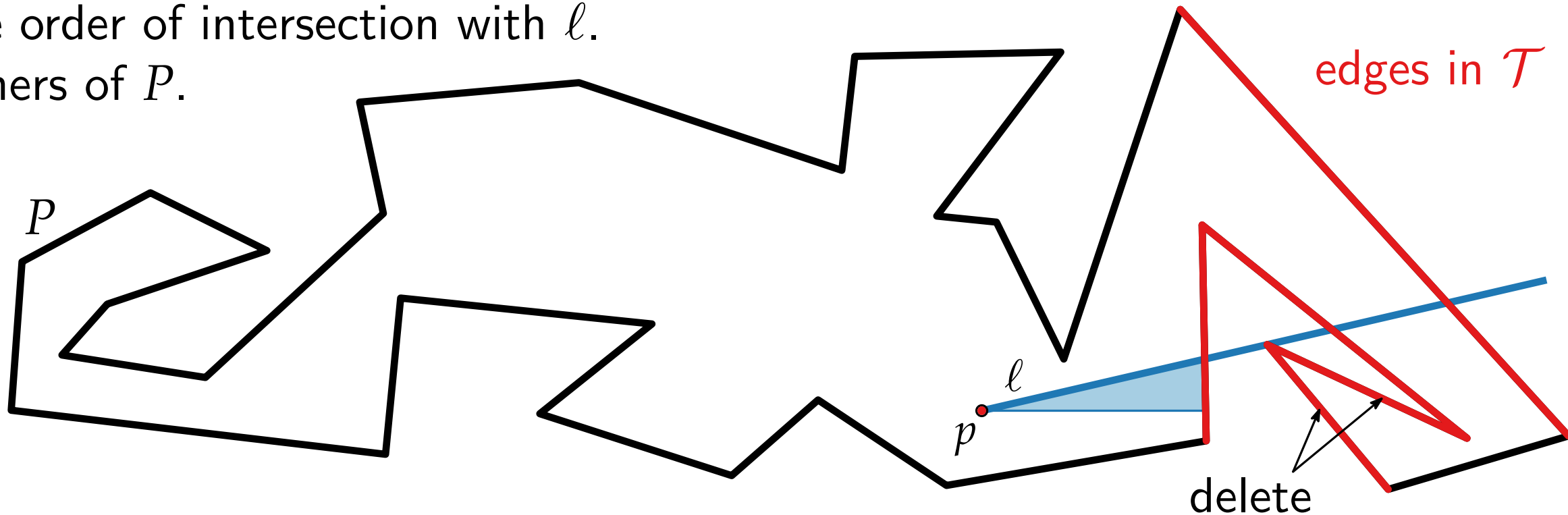
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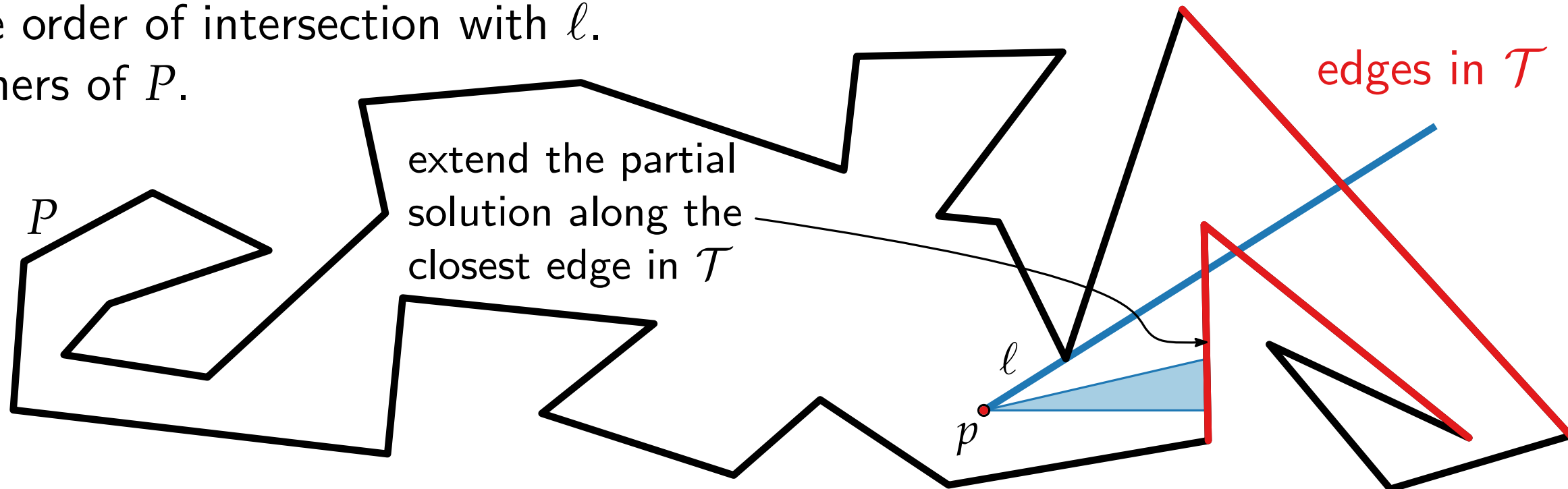
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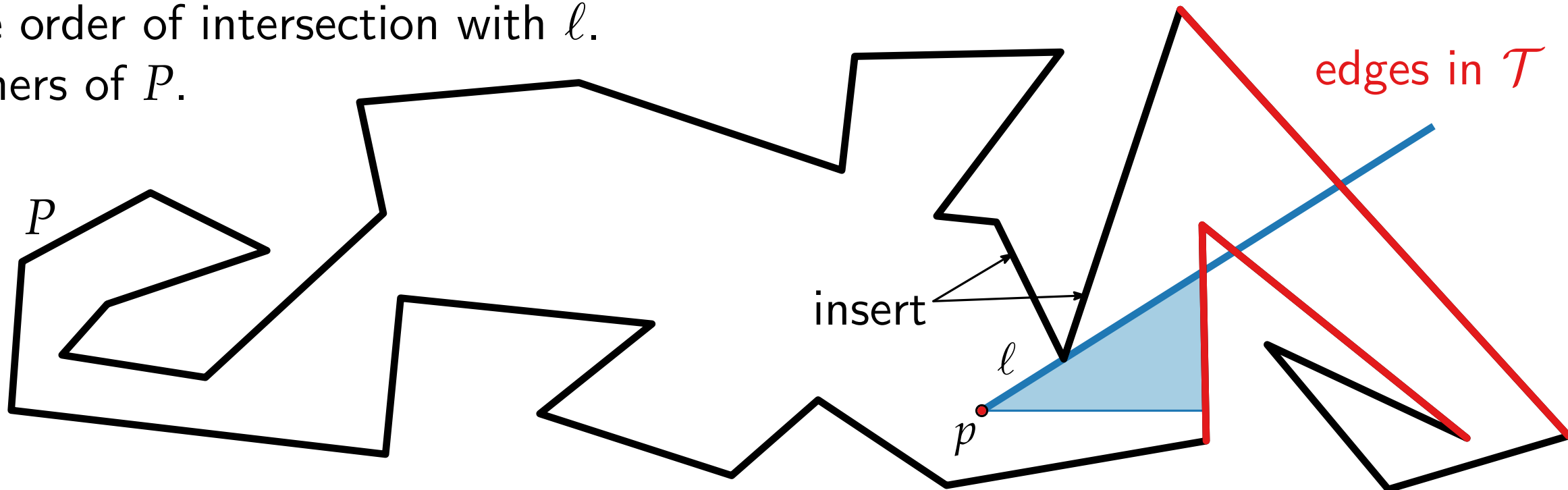
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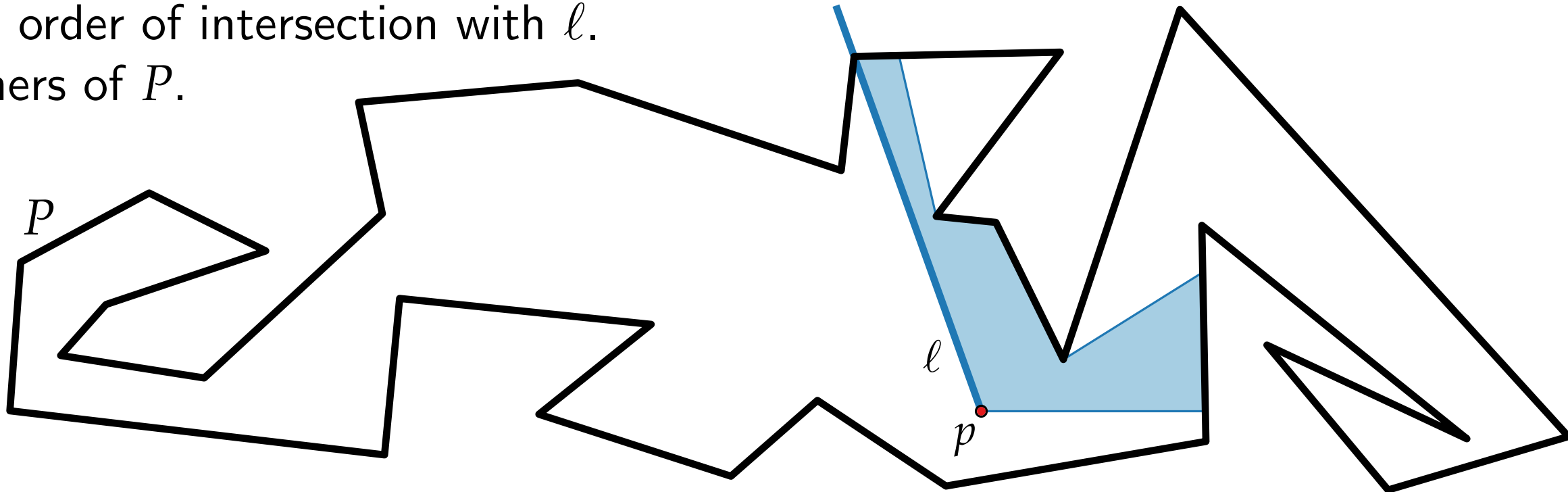
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Total runtime: $\mathcal{O}(n \log n)$

Sweep line status: Edges of P intersected by ℓ are stored in a balanced binary search tree \mathcal{T} in the order of intersection with ℓ .

Events: Corners of P .



Literature

Rolf Klein. Algorithmische Geometrie: Grundlagen, Methoden, Anwendungen. Springer 2005.

Otfried Cheong, Mark de Berg, Mark Overmars, Marc van Kreveld:
Computational Geometry: Algorithms and Applications, 3rd edition, Springer 2008.