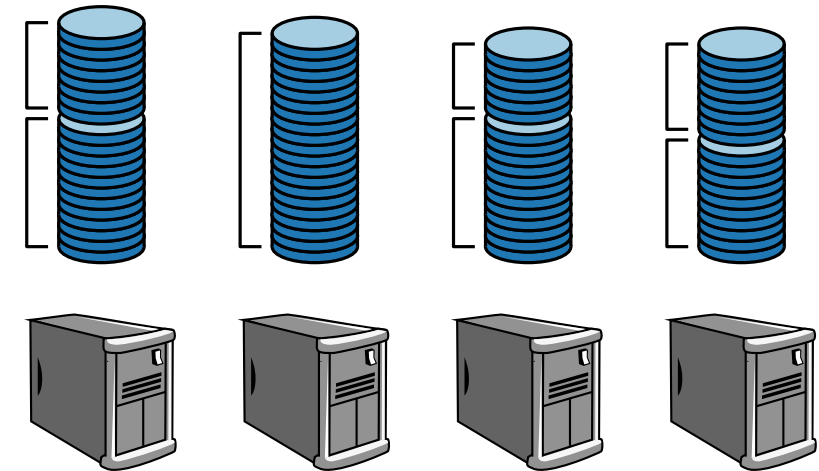
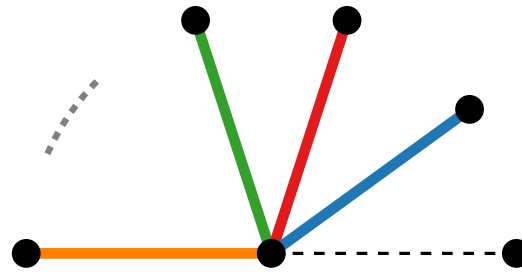
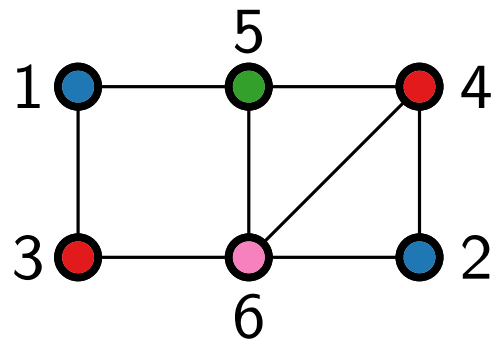


Advanced Algorithms

Approximation Algorithms

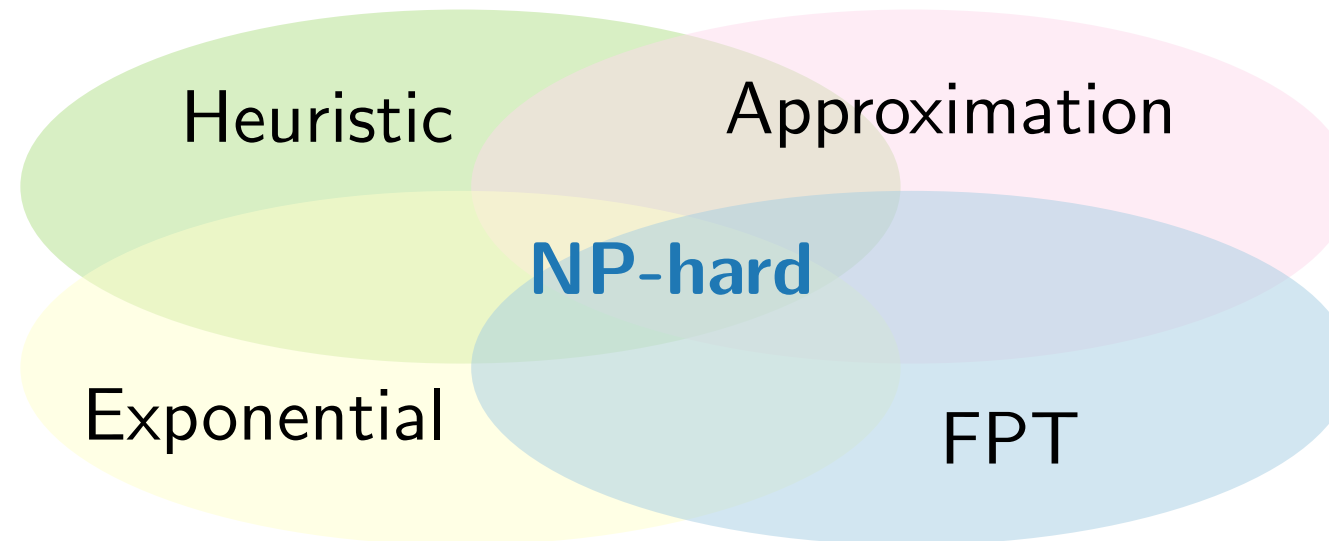
Coloring and Scheduling Problems



Dealing with NP-Hard Optimization Problems

What should we do?

- Sacrifice optimality for speed
 - Heuristics
 - Approximation algorithms ← *this lecture*
- Optimal solutions
 - Exact exponential-time algorithms
 - Fine-grained analysis – parameterized algorithms



Approximation Algorithms

Problem.

- For NP-hard optimization problems, we cannot compute the optimal solution of every instance efficiently (unless $P = NP$).
- Heuristics offer no guarantee on the quality of their solutions.

Goal.

- Design **approximation algorithms**:
 - run in polynomial time and
 - compute solutions of guaranteed quality.
- Study techniques for the design and analysis of approximation algorithms.

Overview.

- Approximation algorithms that compute solutions
 - with additive guarantee, ■ with relative guarantee, ■ that are “arbitrarily good”.

PTAS (*polynomial-time approximation scheme*)



Approximation with Additive Guarantee

Definition.

Let Π be an optimization problem,
let \mathcal{A} be a polynomial-time algorithm for Π ,
let I be an instance of Π , and
let $\text{ALG}(I)$ be the value of the objective function of
the solution that \mathcal{A} computes given I .

Then \mathcal{A} is called an **approximation algorithm with additive guarantee** δ (which can depend on I) if

$$|\text{OPT}(I) - \text{ALG}(I)| \leq \delta$$

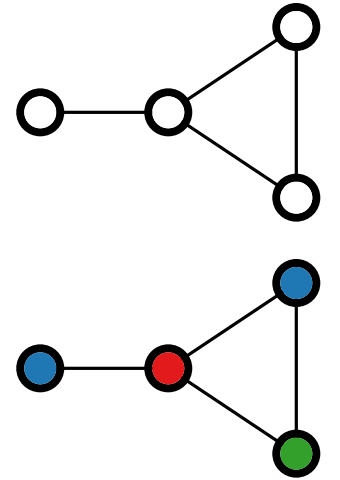
for every instance I of Π .

- Most problems that we know do not admit an approximation algorithm with additive guarantee.

Minimum Vertex Coloring

Input. A graph G . Let $\Delta(G)$ be the maximum degree of G .

Output. A **minimum vertex coloring**, that is, an assignment of the vertices of G to colors such that (i) no two adjacent vertices get the same color and (ii) the number of colors is minimum.



- Minimum Vertex Coloring is NP-hard.
- Even Vertex 3-Coloring is NP-complete.

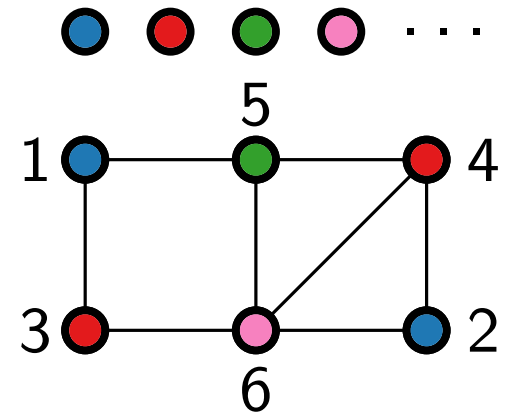
`GreedyVertexColoring(graph G , not edgeless)`

Color vertices in some order with the lowest feasible color.

Theorem 1.

`GreedyVertexColoring(G)` computes a vertex coloring with at most $\Delta(G) + 1$ colors in $\mathcal{O}(V(G) + E(G))$ time.

Thus, it has an additive approximation guarantee of $\Delta(G) - 1$.



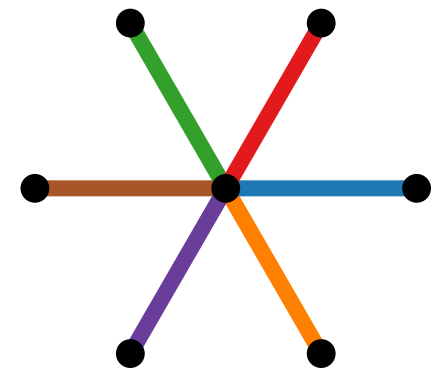
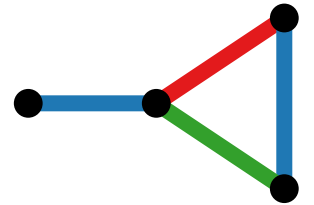
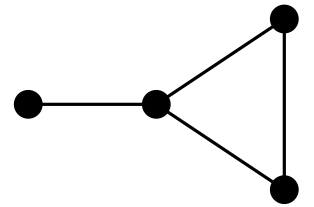
We can even get $\Delta(G) - 2$ if we return a 2-coloring whenever G is bipartite.

Minimum Edge Coloring

Input. A graph G . Let $\Delta(G)$ be the maximum degree of G .

Output. A **minimum edge coloring**, that is, an assignment of colors to the edges of G such that no two adjacent edges get the same color and the number of colors is minimum.

- Minimum Edge Coloring is NP-hard.
- Even Edge 3-Coloring is NP-complete.
- The minimum number of colors needed for an edge coloring of G is called the **chromatic index** $\chi'(G)$.
- $\chi'(G)$ is lowerbounded by $\Delta(G)$.
- We show that $\chi'(G) \leq \Delta(G) + 1$.



Minimum Edge Coloring – Upper Bound

Vizing's Theorem.

For every graph G , it holds that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Proof by induction on $m = |E|$.

- Base case $m = 1$ is trivial.

Let G be a graph on m edges, and let $e = uv$ be an edge of G .

- By induction, $G - e$ has a $(\Delta(G - e) + 1)$ -edge coloring.
- If $\Delta(G) > \Delta(G - e)$, color e with color $\Delta(G) + 1$.
- If $\Delta(G) = \Delta(G - e)$, change the coloring such that u and v miss the same color α .
- Then color e with α .



Vadim G. Vizing
(Kiev 1937 – 2017 Odessa)

Minimum Edge Coloring – Recoloring

Lemma 2.

Let G be a graph with a $(\Delta(G) + 1)$ -edge coloring c , let u, v be non-adjacent vertices, both of degree $< \Delta(G)$. Then c can be changed s.t. u and v miss the same color.

Proof. Note that every vertex is **missing** a color. Let u miss β and v miss α_1 ; apply the following algorithm:

VizingRecoloring(G, c, u, α_1)

$i \leftarrow 1$

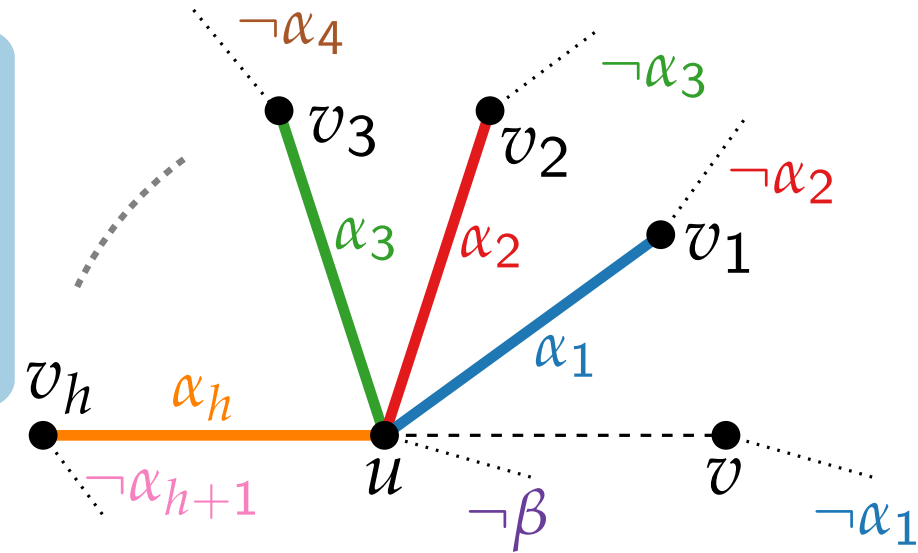
while $\exists w \in N(u) : c(uw) = \alpha_i \wedge w \notin \{v_1, \dots, v_{i-1}\}$ **do**

$v_i \leftarrow w$

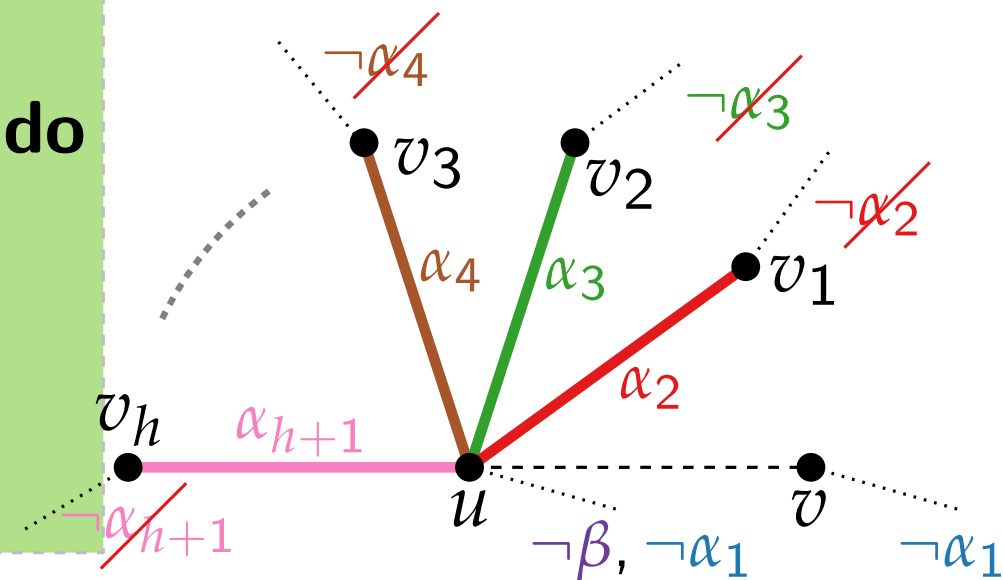
$\alpha_{i+1} \leftarrow$ min color missing at w

$i \leftarrow i + 1$

return $v_1, \dots, v_i; \alpha_1, \dots, \alpha_{i+1}$



Case 1: u misses α_{h+1} .



Minimum Edge Coloring – Recoloring

Lemma 2.

Let G be a graph with a $(\Delta(G) + 1)$ -edge coloring c , let u, v be non-adjacent vertices, both of degree $< \Delta(G)$. Then c can be changed s.t. u and v miss the same color.

Proof. Note that every vertex is **missing** a color.

Let u miss β and v miss α_1 ; apply the following algorithm:

VizingRecoloring(G, c, u, α_1)

$i \leftarrow 1$

while $\exists w \in N(u) : c(uw) = \alpha_i \wedge w \notin \{v_1, \dots, v_{i-1}\}$ **do**

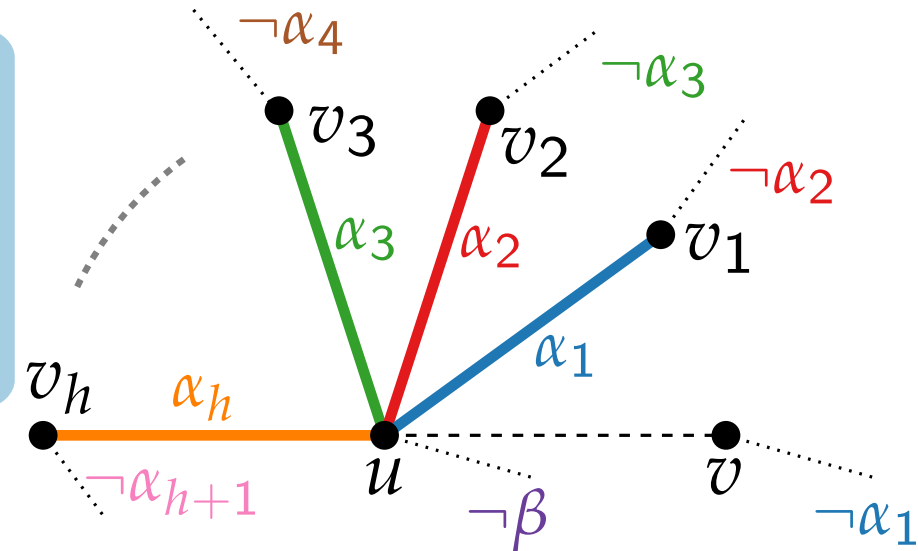
$v_i \leftarrow w$

$\alpha_{i+1} \leftarrow$ min color missing at w

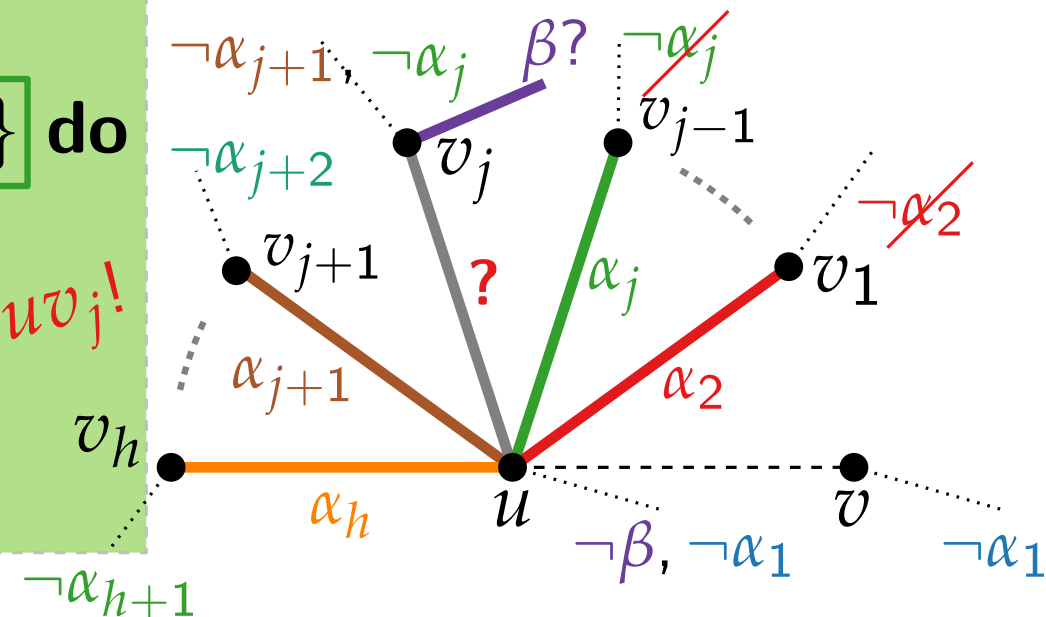
$i \leftarrow i + 1$

return $v_1, \dots, v_i; \alpha_1, \dots, \alpha_{i+1}$

Need color for edge $uv_j!$



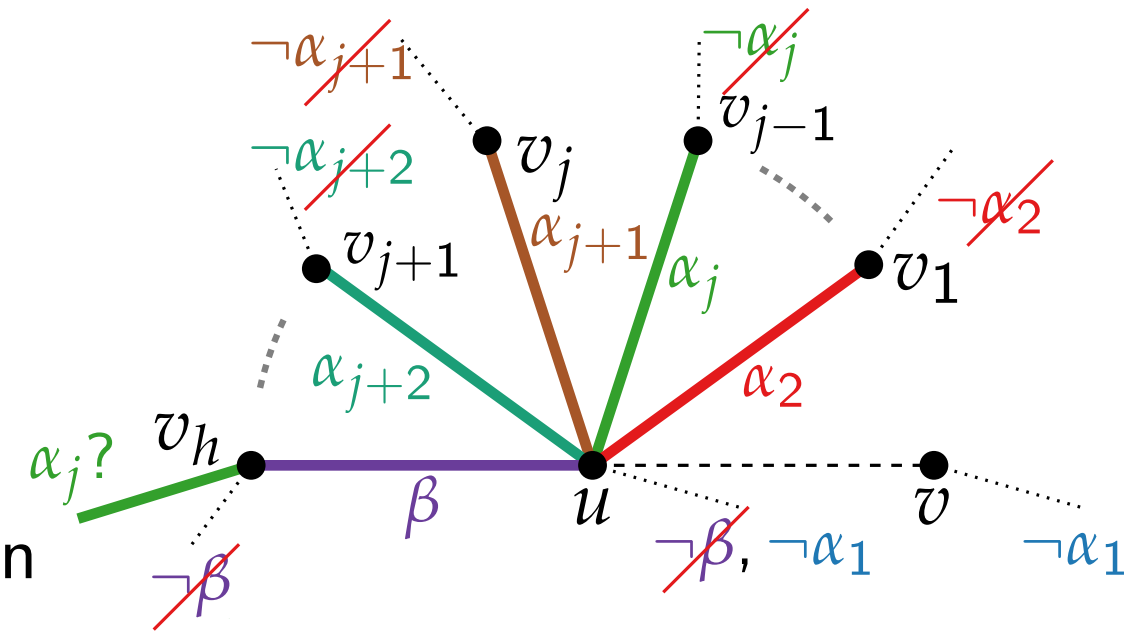
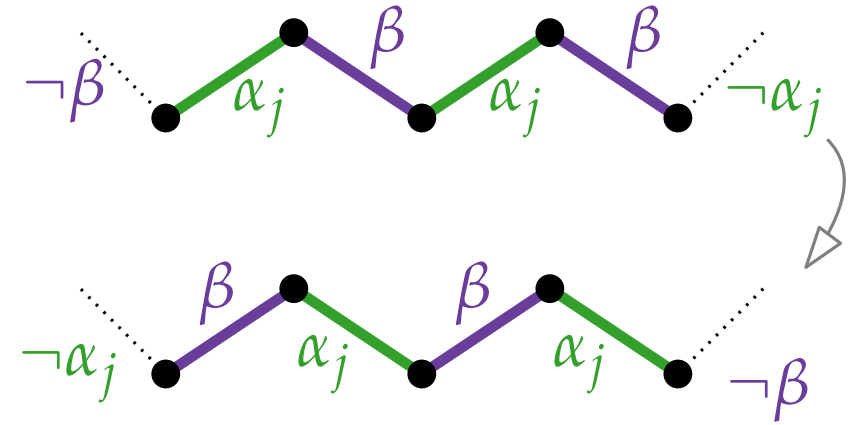
Case 2: $\alpha_{h+1} = \alpha_j, j < h.$



Minimum Edge Coloring – Recoloring

Proof continued for **Case 2**: $\alpha_{h+1} = \alpha_j$, $j < h$,
and we need to find a color for edge uv_j .

- Consider subgraph G' of G induced by the edges of colors β and α_j .
- Since $\Delta(G') \leq 2$, we can recolor components.
- Nodes u , v_j , v_h are all leaves in G' .
 \Rightarrow They are not all in the same component of G' .
- If u and v_j are not in the same component:
 - re-color component ending at v_j ,
 - v_j now misses β ; color uv_j with β .
- What if u and v_j are in the same component?
 - re-color component ending at v_h if there is β
 - color uv_h with β ; color uv_j with α_{j+1} and so on



Minimum Edge Coloring – Algorithm

VizingEdgeColoring(graph G , coloring $c \equiv 0$)

if $E(G) \neq \emptyset$ **then**

Let $e = uv$ be an arbitrary edge of G .

$G_e \leftarrow G - e$

VizingEdgeColoring(G_e, c)

if $\Delta(G_e) < \Delta(G)$ **then**

└ Color e with lowest free color.

else

└ Recolor G_e as in Lemma 2.

└ Color e with color now missing at u and v .

Theorem 4.

VIZINGEDGECOLORING is an approximation algorithm with additive approximation guarantee $\text{ALG}(G) - \text{OPT}(G) \leq 1$.

Approximation with Relative Factor

- An additive approximation guarantee can rarely be achieved; but sometimes, there is a multiplicative approximation!

Definition. **maximization**

Let Π be a minimization problem, and let $\alpha \in \mathbb{Q}^+$.

A **factor- α approximation algorithm** for Π is a polynomial-time algorithm \mathcal{A} that computes, for every instance I of Π , a solution of value $\text{ALG}(I)$ such that

$$\frac{\text{ALG}(I)}{\text{OPT}(I)} \stackrel{>}{\leq} \alpha.$$

We call α the **approximation factor** of \mathcal{A} .

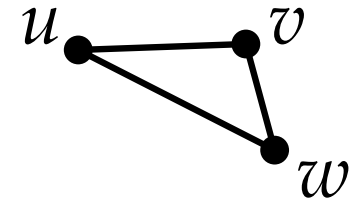
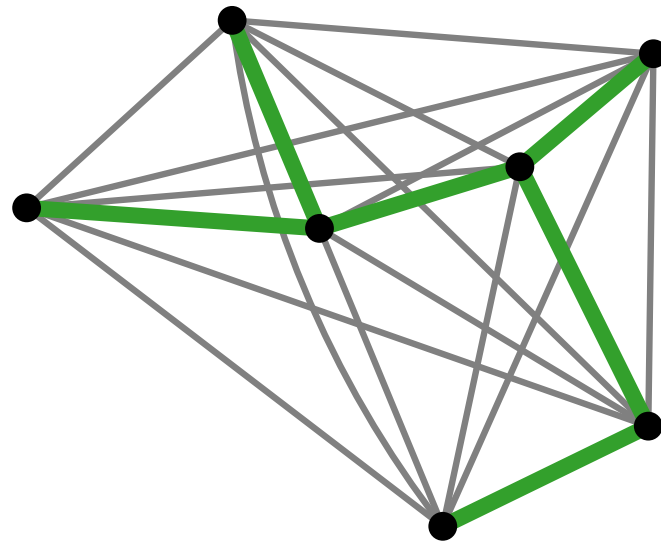
2-Approximation for Metric TSP (from AGT)

Input. Complete graph G and a distance function $d: E(G) \rightarrow \mathbb{R}_{\geq 0}$ that satisfies the triangle inequality, i.e., $\forall u, v, w \in V(G): d(u, w) \leq d(u, v) + d(v, w)$.

Output. A shortest Hamiltonian cycle in G .

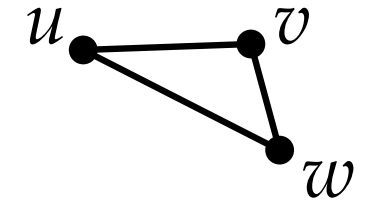
Algorithm.

- Compute MST.



2-Approximation for Metric TSP (from AGT)

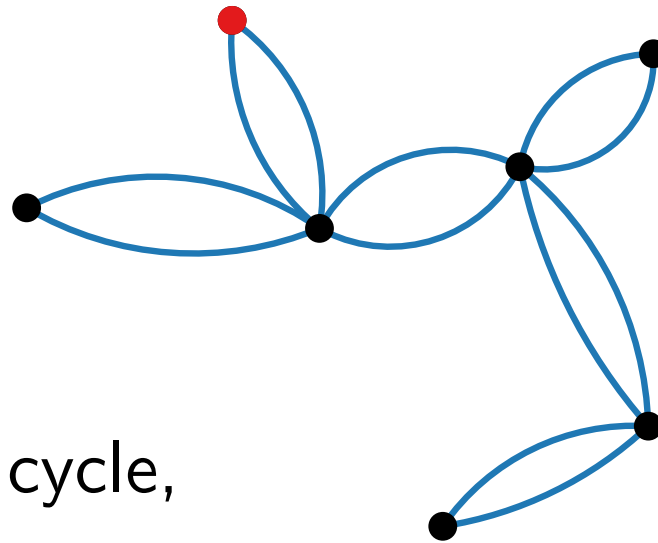
Input. Complete graph G and a distance function $d: E(G) \rightarrow \mathbb{R}_{\geq 0}$ that satisfies the triangle inequality, i.e., $\forall u, v, w \in V(G): d(u, w) \leq d(u, v) + d(v, w)$.



Output. A shortest Hamiltonian cycle in G .

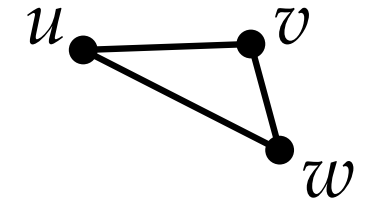
Algorithm.

- Compute MST.
- Double edges.
 \Rightarrow Eulerian cycle
- Walk along Eulerian cycle,



2-Approximation for Metric TSP (from AGT)

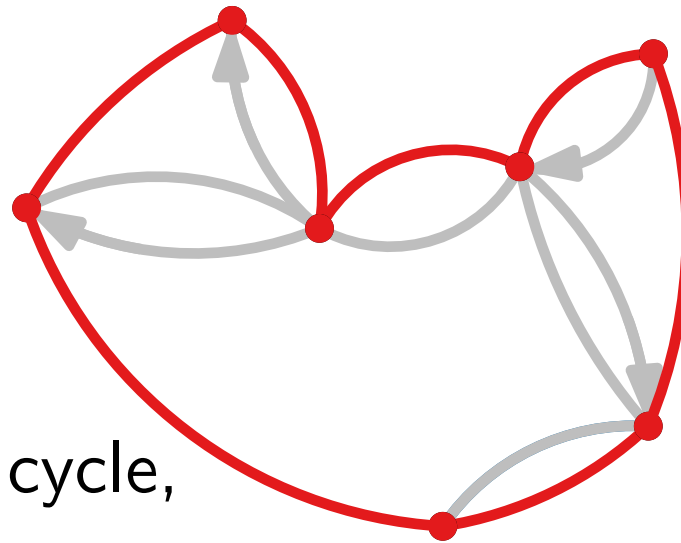
Input. Complete graph G and a distance function $d: E(G) \rightarrow \mathbb{R}_{\geq 0}$ that satisfies the triangle inequality, i.e., $\forall u, v, w \in V(G): d(u, w) \leq d(u, v) + d(v, w)$.



Output. A shortest Hamiltonian cycle in G .

Algorithm.

- Compute MST.
- Double edges.
⇒ Eulerian cycle
- Walk along Eulerian cycle,
skipping visited vertices
and adding shortcuts.



Theorem 5.

The MST edge doubling algorithm is a 2-approximation algorithm for metric TSP.

Proof.

$$\text{ALG} \leq d(\text{cycle}) = 2d(\text{MST}) \leq 2\text{OPT}.$$

Nearest Addition Algorithm for Metric TSP

NearestAdditionAlgorithm($G = (V, E), d$)

Find closest pair, say i and k .

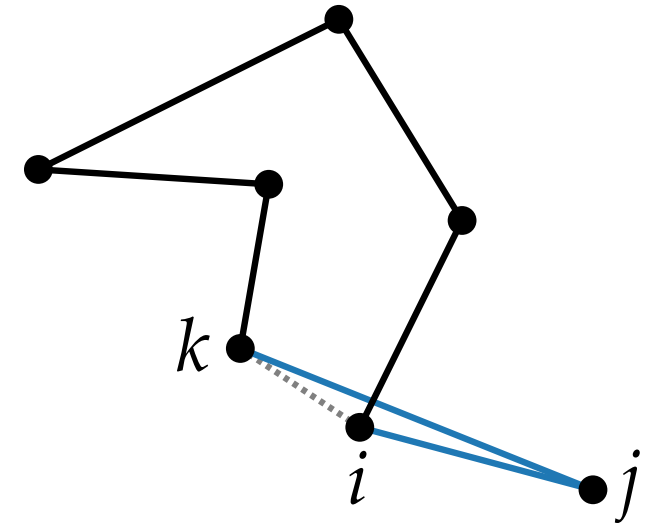
Set tour T to go from i to k to i (clockwise).

while $T \subsetneq V$ **do**

 Find pair $(i, j) \in T \times (V \setminus T)$ minimizing $d(i, j)$.

 Let k be vertex after i in T .

 Add j between i and k .



Theorem 6.

NearestAdditionAlgorithm is a 2-approximation algorithm for metric TSP.

Proof.

■ Exercise.

■ Hint:

Consider the algorithm of Jarník–Prim.

Approximation Schemes

- In some cases, we can get arbitrarily good approximations.

Definition. **maximization**

Let Π be a minimization problem. An algorithm \mathcal{A} is called a **polynomial-time approximation scheme (PTAS)** if \mathcal{A} computes, for every input (I, ε) (consisting of an instance I of Π and a real $\varepsilon > 0$), a value $\text{ALG}(I)$ such that:

- $\text{ALG}(I) \geq (1 - \varepsilon) \cdot \text{OPT}(I)$, and
- $\text{ALG}(I) \leq (1 + \varepsilon) \cdot \text{OPT}(I)$, and
- the runtime of \mathcal{A} is polynomial in $|I|$ for every $\varepsilon > 0$.

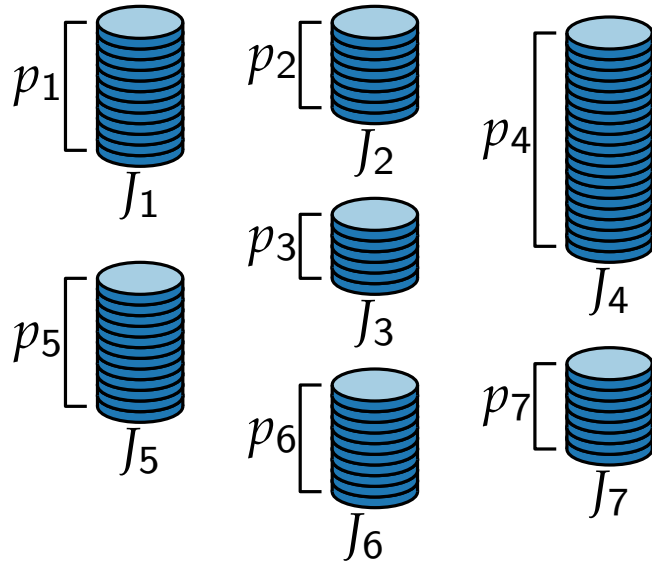
\mathcal{A} is called a **fully polynomial-time approximation scheme (FPTAS)** if it runs in time polynomial in $|I|$ and $1/\varepsilon$.

Examples.

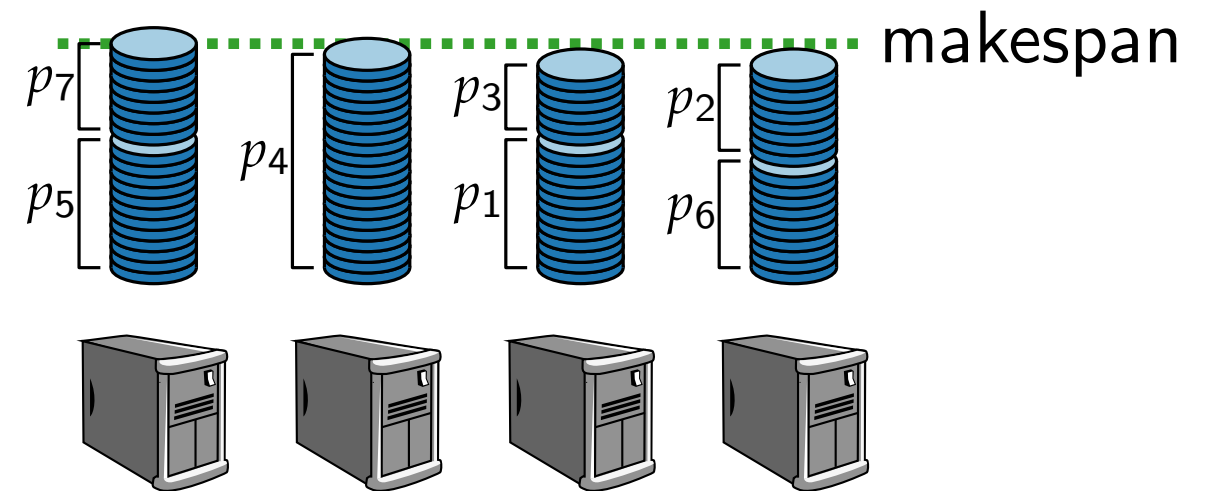
- $\mathcal{O}\left(2^{\frac{n}{\varepsilon}}\right) \Rightarrow$ no PTAS
- $\mathcal{O}\left(n^2 \cdot 3^{\frac{1}{\varepsilon}}\right) \Rightarrow$ PTAS but no FPTAS
- $\mathcal{O}\left(n^2 + n^{\frac{1}{\varepsilon}}\right) \Rightarrow$ PTAS but no FPTAS
- $\mathcal{O}\left(n^4 \cdot \left(\frac{1}{\varepsilon}\right)^2\right) \Rightarrow$ FPTAS

Multiprocessor Scheduling

Input. ■ n jobs J_1, \dots, J_n with durations p_1, \dots, p_n .



■ m identical machines ($m < n$)



Output. Assignment of jobs to machines such that the time when all jobs have been processed is minimum.

This is called the **makespan** of the assignment.

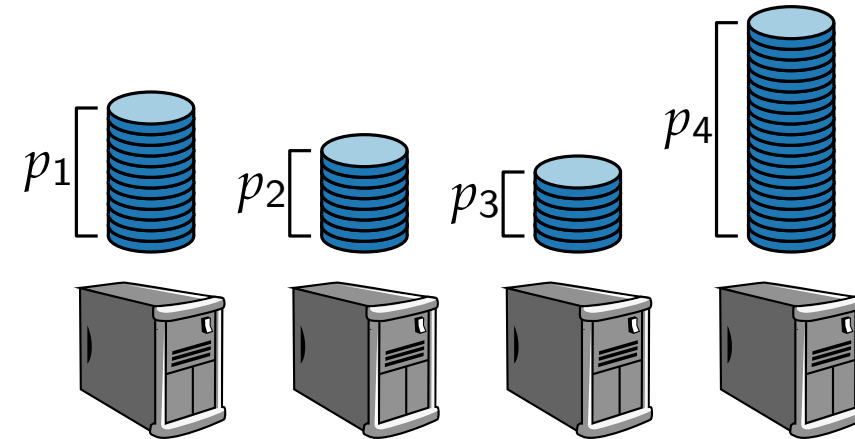
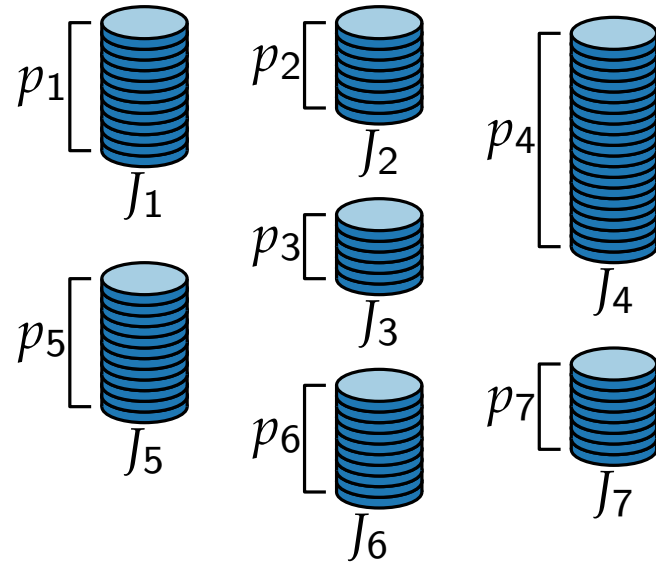
■ Multiprocessor scheduling is NP-hard.

Multiprocessor Scheduling – List Scheduling

LISTSCHEDULING(J_1, \dots, J_n, m)

Put the first m jobs on the m machines.
Put the next job on the first free machine.

Example.



Multiprocessor Scheduling – List Scheduling

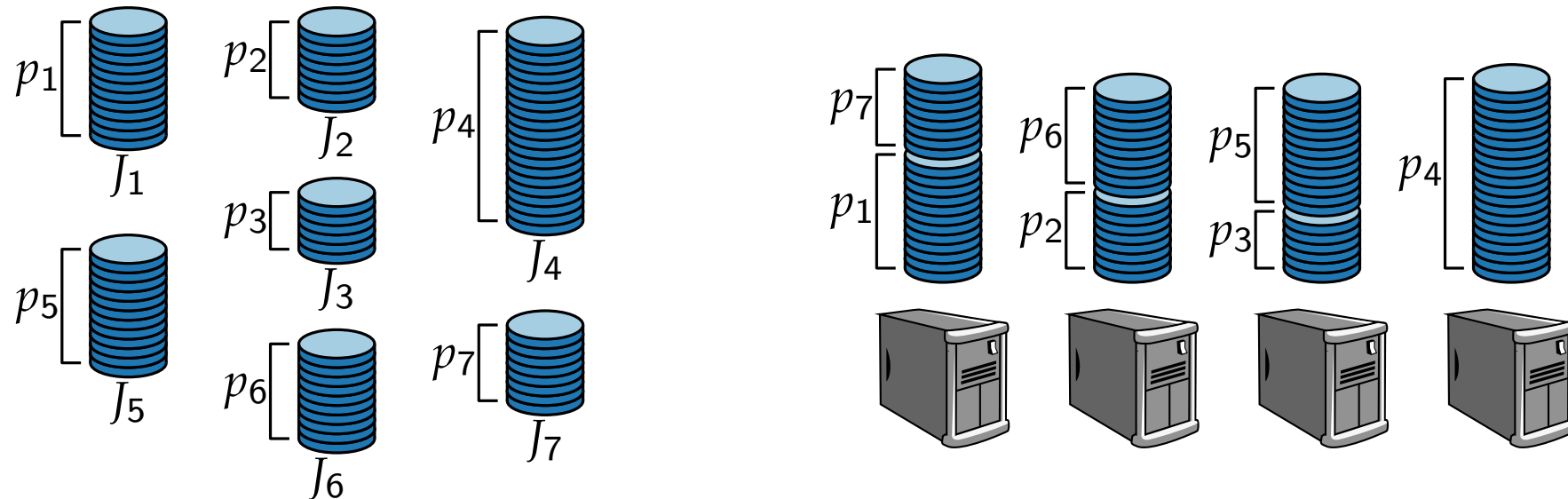
LISTSCHEDULING(J_1, \dots, J_n, m)

Put the first m jobs on the m machines.
Put the next job on the first free machine.

Theorem 7.

LISTSCHEDULING is a factor-
 $\left(2 - \frac{1}{m}\right)$ approximation algorithm.

Example.



- **LISTSCHEDULING** runs in $\mathcal{O}(n \log m)$ time.

Iterate over n jobs while maintaining a priority queue for the machines where each machine has its current completion time as its priority.

Multiprocessor Scheduling – List Scheduling (Proof)

LISTSCHEDULING(J_1, \dots, J_n, m)

Put the first m jobs on the m machines.
Put the next job on the first free machine.

Theorem 7.

LISTSCHEDULING is a $(2 - \frac{1}{m})$ -approximation alg.

Proof. Let $J_k = (S_k, T_k)$ be the last job, that is, T_k determines the makespan.

■ No machine idles at time S_k .

$$S_k \leq \frac{1}{m} \sum_{i \neq k} p_i \quad \begin{array}{l} \text{weight of all jobs but } J_k \\ \text{evenly distributed on } m \text{ machines} \end{array}$$

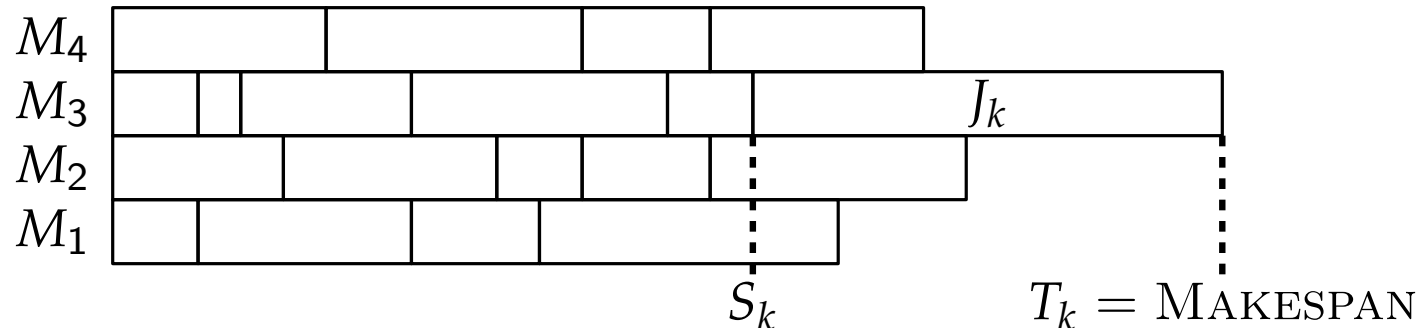
■ For the optimal makespan T_{OPT} , we have:

■ $T_{\text{OPT}} \geq p_k$

■ $T_{\text{OPT}} \geq \frac{1}{m} \sum_{i=1}^n p_i$ weight of all jobs evenly distributed

■ Hence:

$$\begin{aligned} T_k &= S_k + p_k \\ &\leq \frac{1}{m} \cdot \sum_{i \neq k} p_i + p_k \\ &= \frac{1}{m} \cdot \sum_{i=1}^n p_i + \left(1 - \frac{1}{m}\right) \cdot p_k \\ &\leq T_{\text{OPT}} + \left(1 - \frac{1}{m}\right) \cdot T_{\text{OPT}} \\ &= \left(2 - \frac{1}{m}\right) \cdot T_{\text{OPT}} \quad \square \end{aligned}$$



Multiprocessor Scheduling – PTAS

For a constant ℓ ($1 \leq \ell \leq n$) define the algorithm \mathcal{A}_ℓ as follows.

$\mathcal{A}_\ell(J_1, \dots, J_n, m)$

Sort jobs in descending order of runtime.

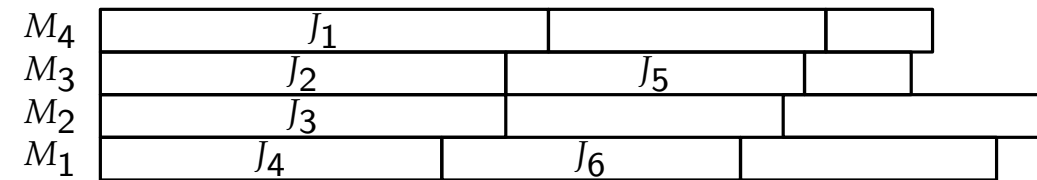
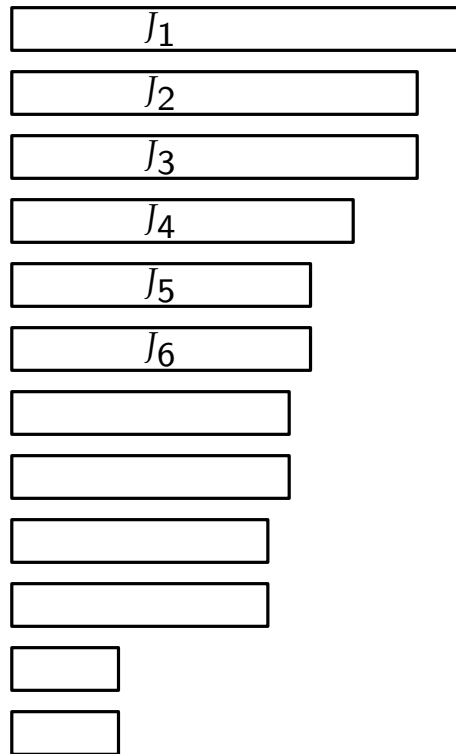
Schedule the ℓ longest jobs J_1, \dots, J_ℓ optimally.

Use LISTSCHEDULING for the remaining jobs $J_{\ell+1}, \dots, J_n$.

Example.

$\ell = 6$

sorted jobs



Multiprocessor Scheduling – PTAS

For a constant ℓ ($1 \leq \ell \leq n$) define the algorithm \mathcal{A}_ℓ as follows.

$\mathcal{A}_\ell(J_1, \dots, J_n, m)$

Sort jobs in descending order of runtime.

Schedule the ℓ longest jobs J_1, \dots, J_ℓ optimally.

Use `LISTSCHEDULING` for the remaining jobs $J_{\ell+1}, \dots, J_n$.

$\mathcal{O}(n \log n)$

$\mathcal{O}(m^\ell)$

$\mathcal{O}(n \log m)$

■ Polynomial time for constant ℓ :

$\mathcal{O}(m^\ell + n \log n)$

Theorem 8.

For constant $\ell \in \{1, \dots, n\}$, algorithm \mathcal{A}_ℓ

is a $\left(1 + \frac{1 - \frac{1}{m}}{1 + \lfloor \frac{\ell}{m} \rfloor}\right)$ -approximation algorithm.

- For $\varepsilon > 0$, choose ℓ such that $\mathcal{A}_\varepsilon = \mathcal{A}_{\ell(\varepsilon)}$ is a $(1 + \varepsilon)$ -approximation algorithm.
- $\{\mathcal{A}_\varepsilon \mid \varepsilon > 0\}$ is not an FPTAS since the running time is not polynomial in $\frac{1}{\varepsilon}$.

Corollary 9.

For a constant number of machines, $\{\mathcal{A}_\varepsilon \mid \varepsilon > 0\}$ is a PTAS.

Multiprocessor Scheduling – PTAS (Proof)

Theorem 8.

For constant $1 \leq \ell \leq n$, the algorithm \mathcal{A}_ℓ is a $1 + \frac{1 - \frac{1}{m}}{1 + \lfloor \frac{\ell}{m} \rfloor}$ -approximation algorithm.

$\mathcal{A}_\ell(J_1, \dots, J_n, m)$

Sort jobs in descending order of runtime.

Schedule the ℓ longest jobs J_1, \dots, J_ℓ optimally.

Use LISTSCHEDULING for the remaining jobs $J_{\ell+1}, \dots, J_n$.

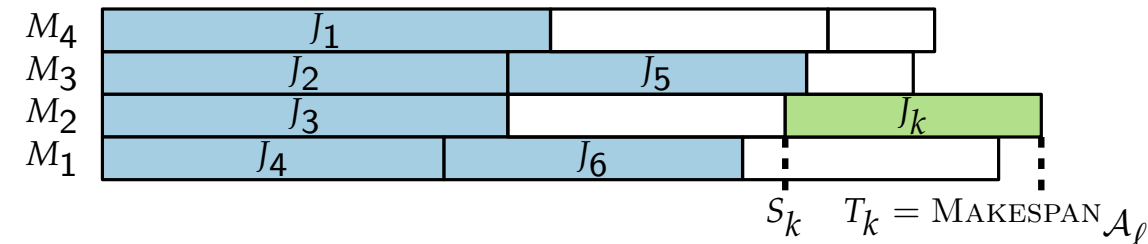
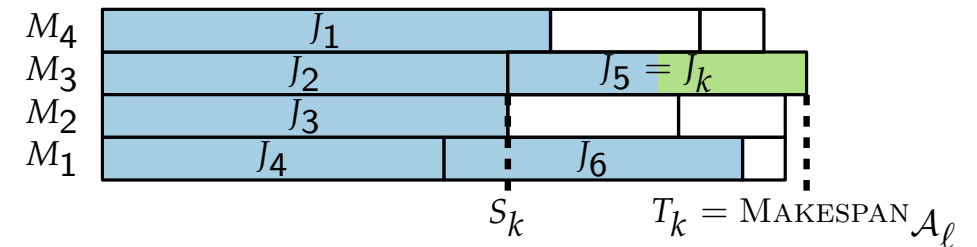
Proof. Let $J_k = (S_k, T_k)$ be the last job, that is, T_k determines the makespan.

Case 1. J_k is one of the longest ℓ jobs J_1, \dots, J_ℓ .

- Solution is optimal for J_1, \dots, J_k
- Hence, solution is optimal for J_1, \dots, J_n

Case 2. J_k is not one of the longest ℓ jobs J_1, \dots, J_ℓ .

- Similar analysis to LISTSCHEDULING
- Use that there are $\ell + 1$ jobs that are at least as long as J_k (including J_k).



Multiprocessor Scheduling – PTAS (Proof)

Theorem 8.

For constant $1 \leq \ell \leq n$, the algorithm \mathcal{A}_ℓ is a $1 + \frac{1 - \frac{1}{m}}{1 + \lfloor \frac{\ell}{m} \rfloor}$ -approximation algorithm.

$\mathcal{A}_\ell(J_1, \dots, J_n, m)$

Sort jobs in descending order of runtime.

Schedule the ℓ longest jobs J_1, \dots, J_ℓ optimally.

Use LISTSCHEDULING for the remaining jobs $J_{\ell+1}, \dots, J_n$.

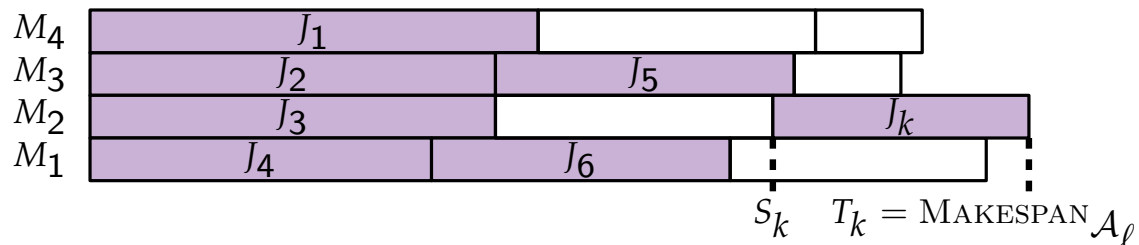
Proof of Case 2.

$$\blacksquare S_k \leq \frac{1}{m} \sum_{i \neq k} p_i \quad \blacksquare T_{\text{OPT}} \geq \frac{1}{m} \sum_{i=1}^n p_i$$

■ Consider only J_1, \dots, J_ℓ, J_k :

$T_{\text{OPT}} \geq p_k \cdot \left(1 + \left\lfloor \frac{\ell}{m} \right\rfloor\right)$ one machine has this many jobs* each has length $\geq p_k$

- * on average, each machine has more than $\frac{\ell}{m}$ of the $\ell + 1$ jobs
- at least one machine achieves the average



$$\begin{aligned} T_k &= S_k + p_k \\ &\leq \frac{1}{m} \cdot \sum_{i \neq k} p_i + p_k \\ &= \frac{1}{m} \cdot \sum_{i=1}^m p_i + \left(1 - \frac{1}{m}\right) \cdot p_k \\ &\leq T_{\text{OPT}} + \frac{1 - \frac{1}{m}}{1 + \left\lfloor \frac{\ell}{m} \right\rfloor} \cdot T_{\text{OPT}} \end{aligned}$$

Discussion

- Only “easy” NP-hard problems admit FPTAS (PTAS).
- Some problems cannot be approximated very well (e.g., Maximum Clique).
- Study of approximability of NP-hard problems yields more fine-grained classifications.
- Approximation algorithms exist also for non-NP-hard problems.
- Approximation algorithms can be of various types:
greedy, local search, geometric, DP, ...
- One important technique is LP-relaxation (more later in this lecture).
- Minimum Vertex Coloring on planar graphs can be approximated with an additive approximation guarantee of 2.
- Christofides’ approximation algorithm for Metric TSP has approximation factor 1.5.
- There is a whole lecture on approximation algorithms in the winter term!

Literature

Main references

- [Jansen & Margraf, 2008: Chapter 3]
“Approximative Algorithmen und Nichtapproximierbarkeit”
- [Williamson & Shmoys, 2011: Chapter 3]
“The Design of Approximation Algorithms”

Another book recommendation:

- [Vazirani, 2013] “Approximation Algorithms”

