Approximation Algorithms

Lecture 12: SteinerForest via Primal–Dual

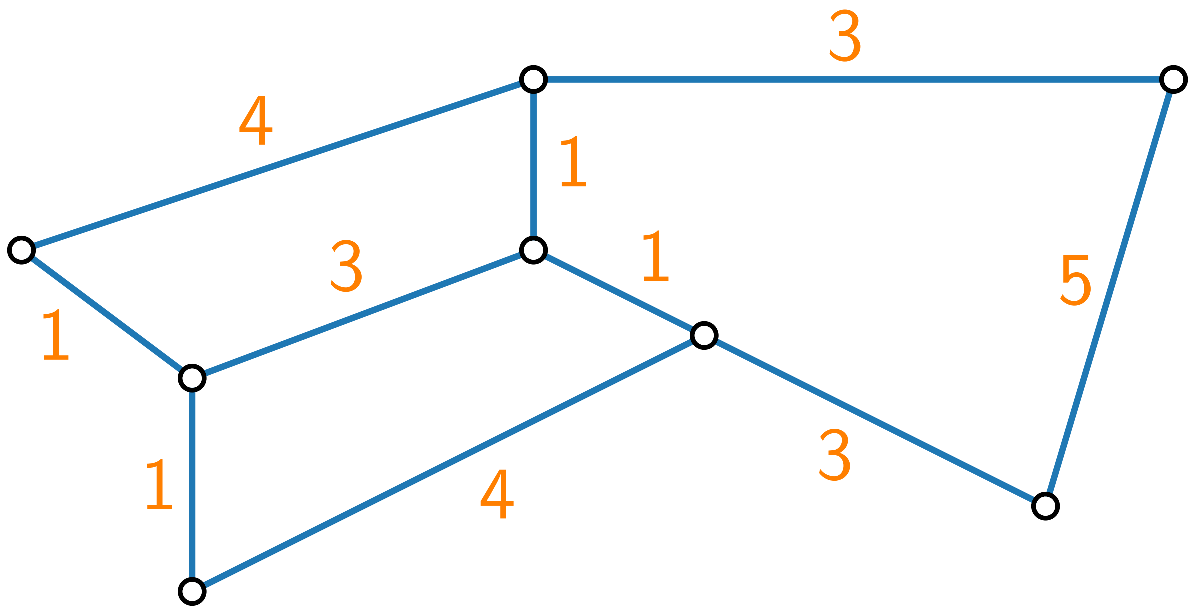
Part I: SteinerForest

Alexander Wolff

Winter term 2025

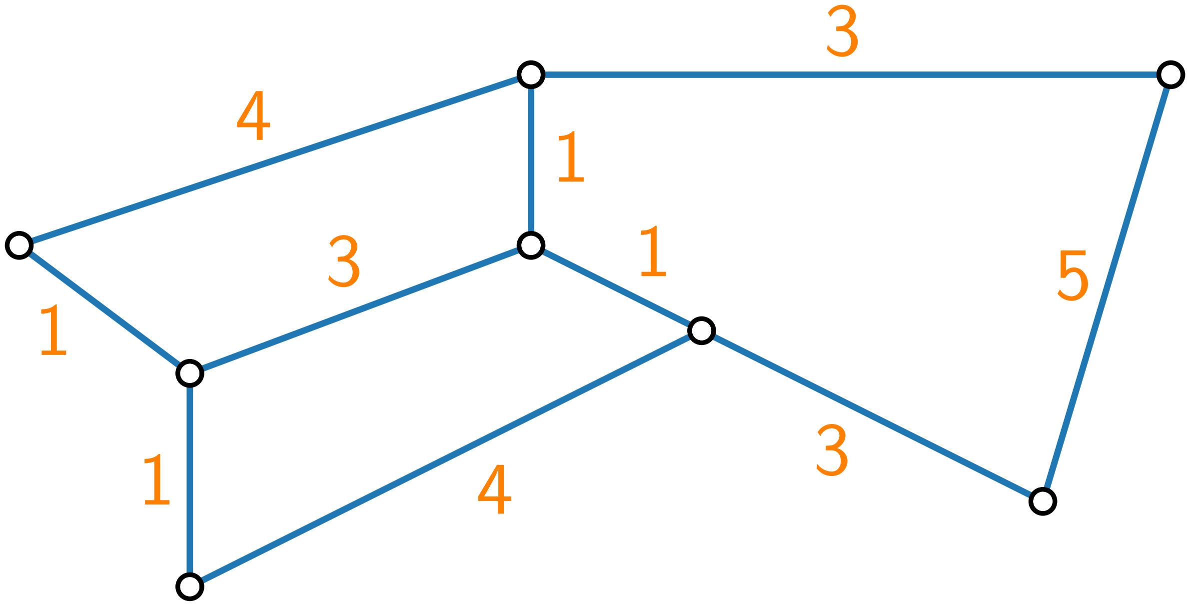
SteinerForest

Given: A graph G with edge costs $c\colon E(G) \to \mathbb{N}$



SteinerForest

Given: A graph G with edge costs $c \colon E(G) \to \mathbb{N}$
and a set $R = \{(s_1, t_1), \ldots, (s_k, t_k)\}$ of k vertex pairs.



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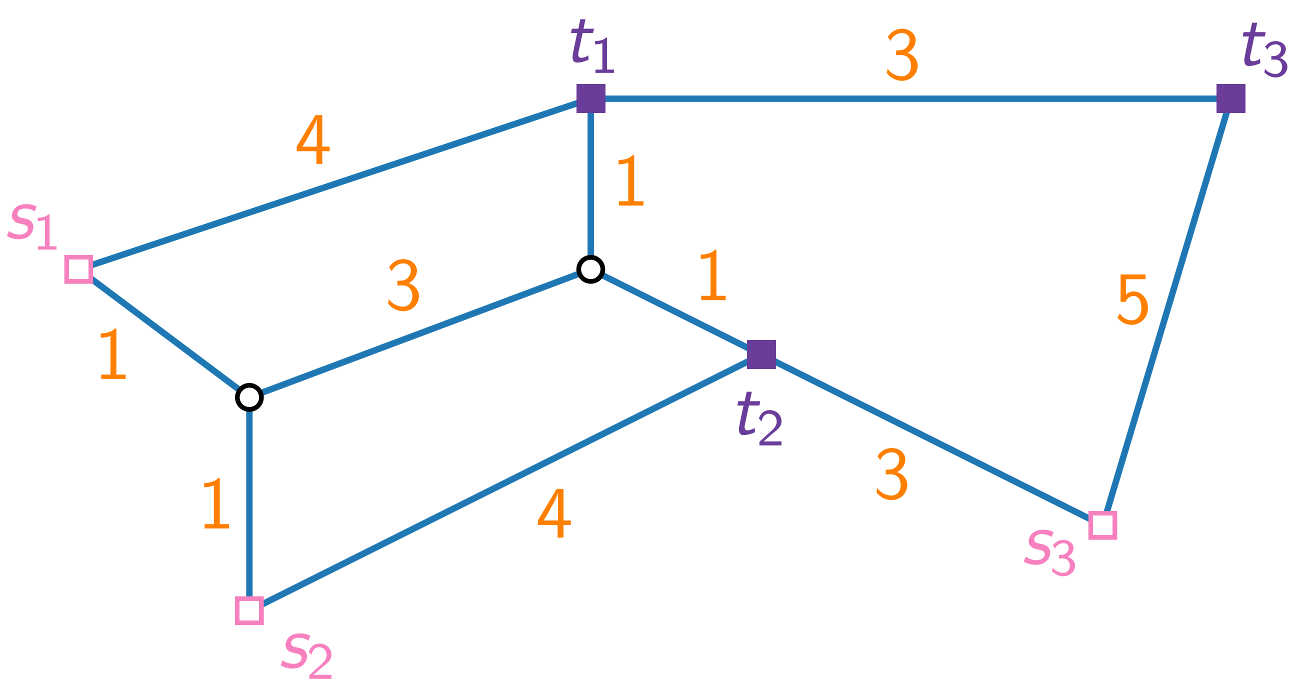

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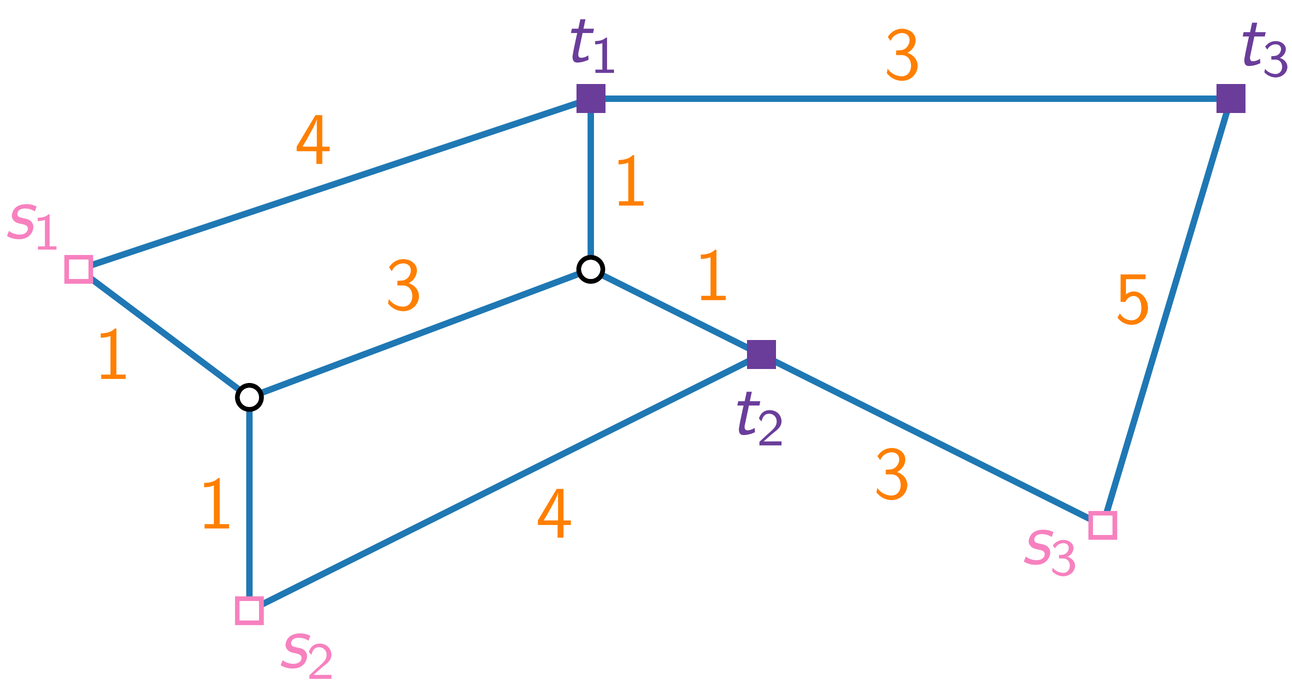
Task: Find an edge set $F \subseteq E(G)$ of minimum total cost



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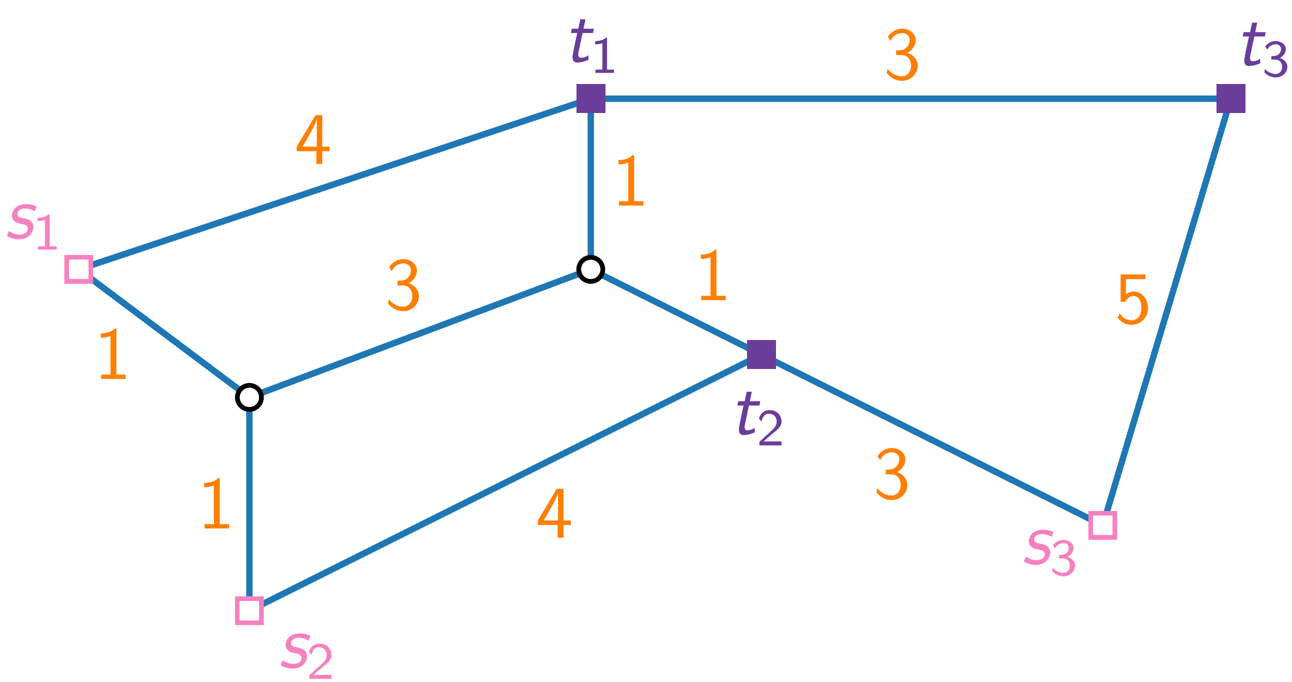
Task: Find an edge set $F \subseteq E(G)$ of minimum total cost $c(F)$ such that the subgraph $(V(G), F)$ connects all vertex pairs $(s_1, t_1), \ldots, (s_k, t_k)$.



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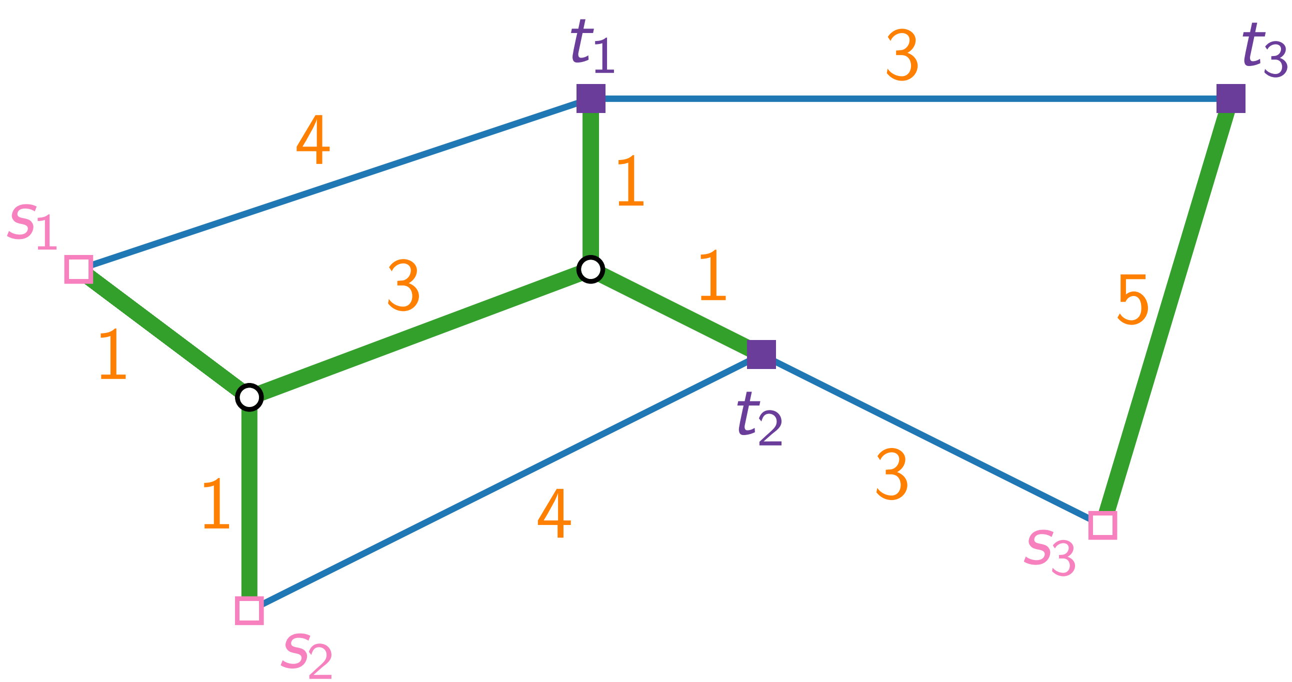

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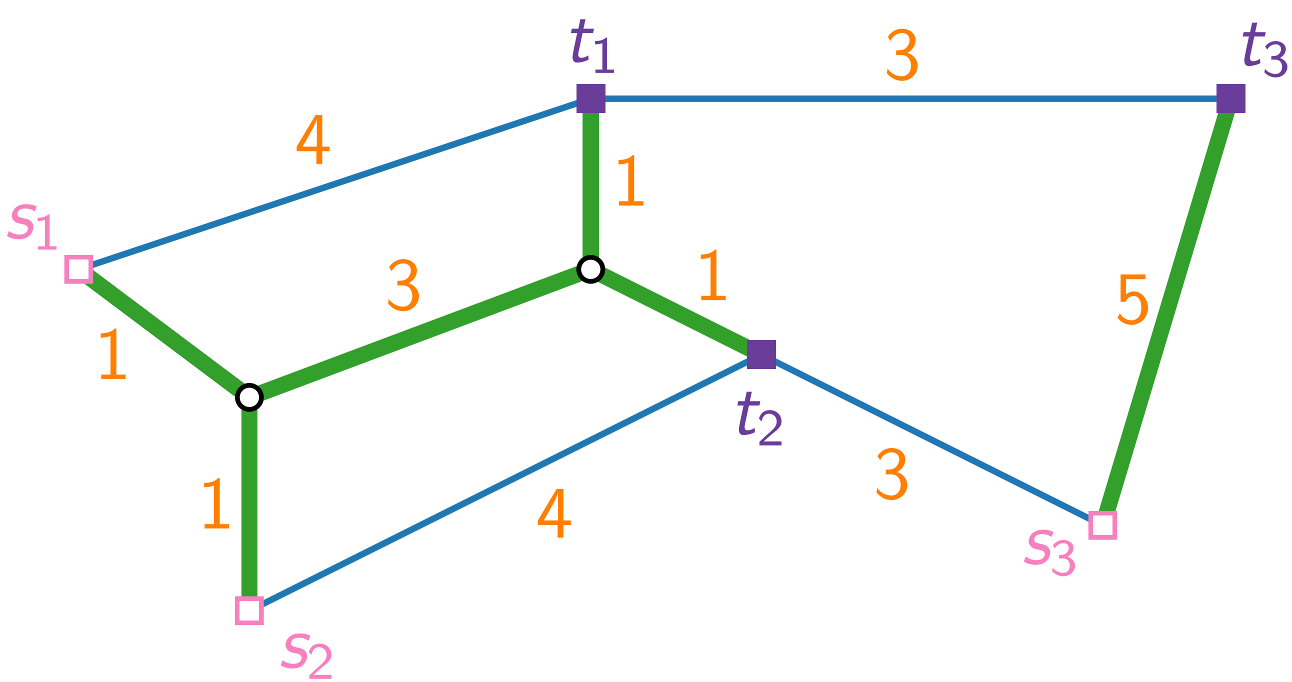

Special cases?

SteinerForest

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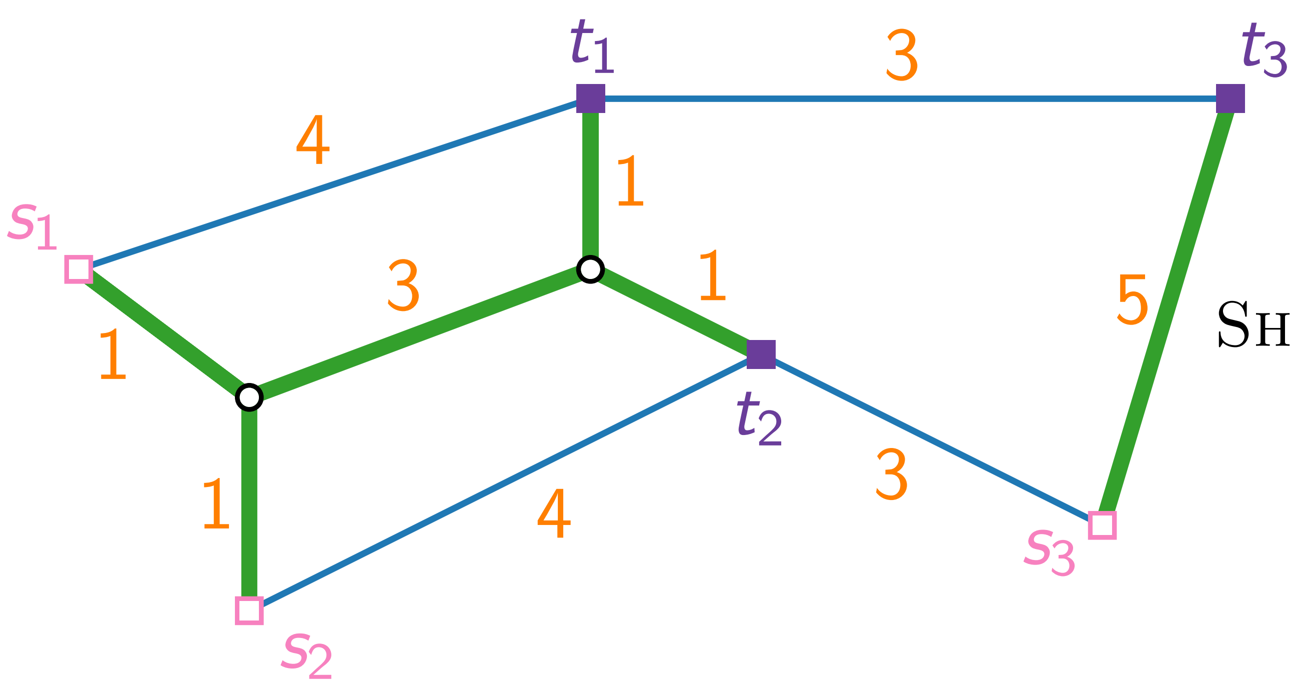

Special cases?

ShortestPath $(R = \{s, t\})$

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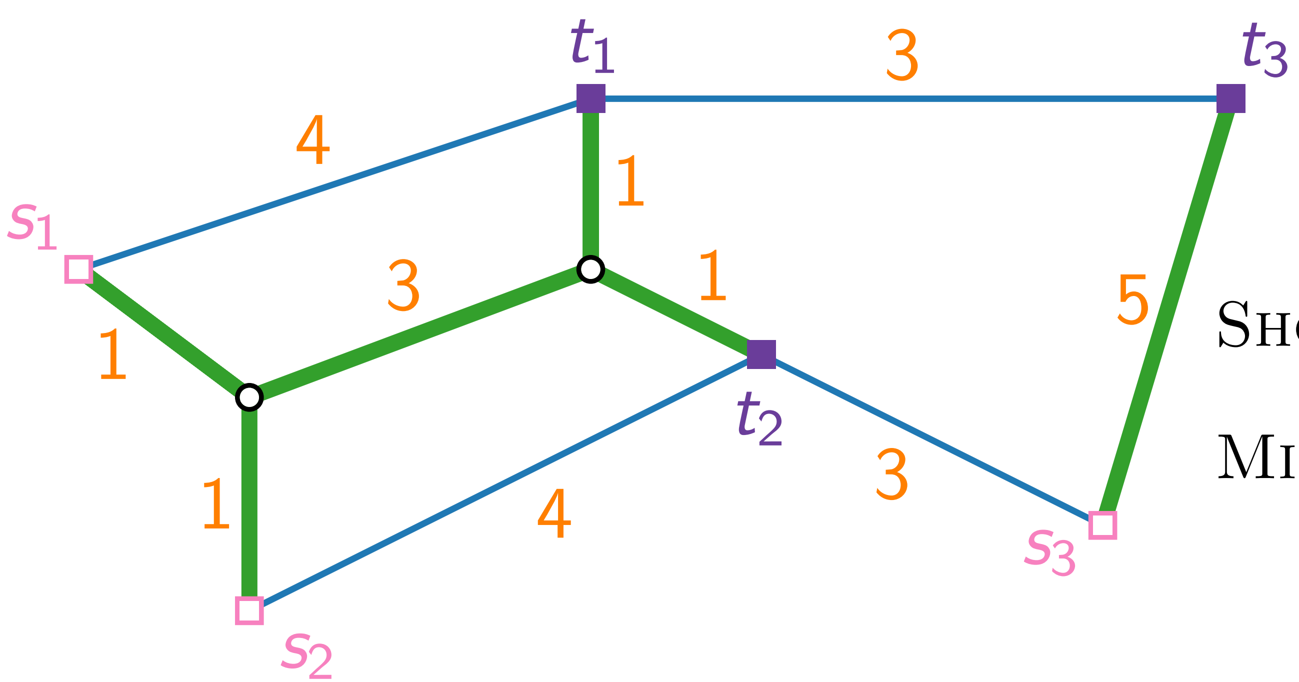

Special cases?

ShortestPath $(R = \{s, t\})$

MinSpannTree $(R = E(G))$

SteinerForest

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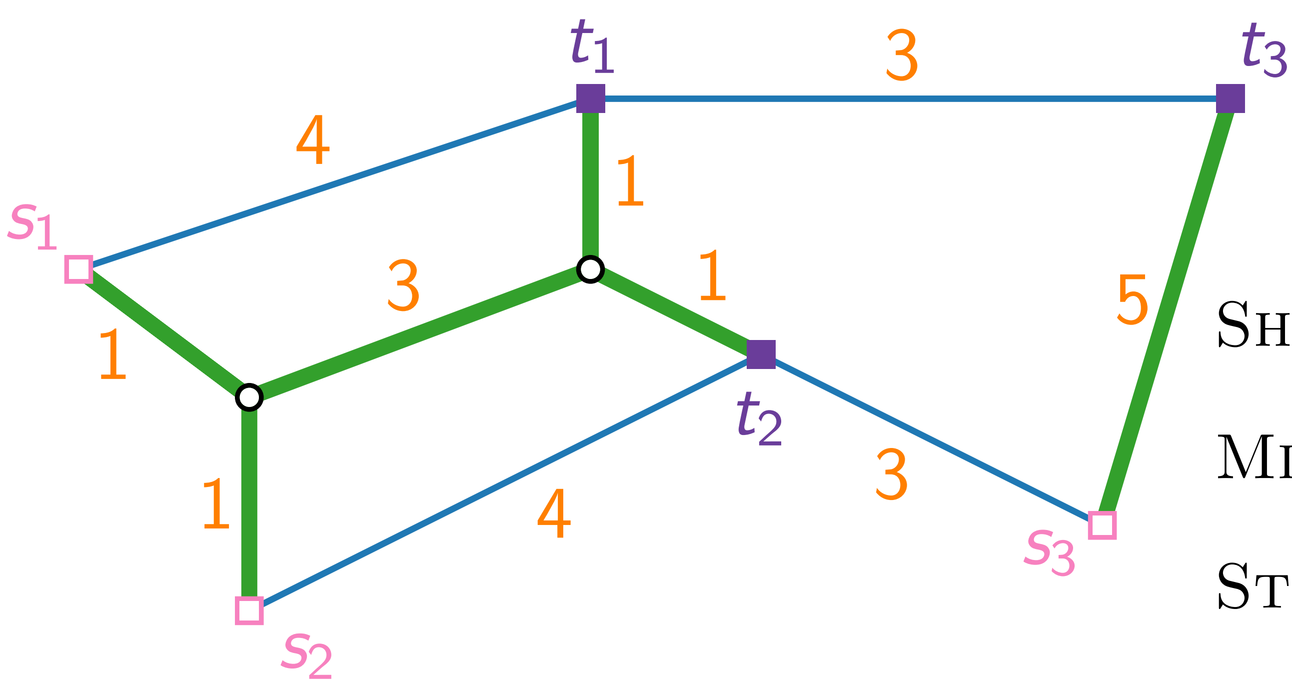


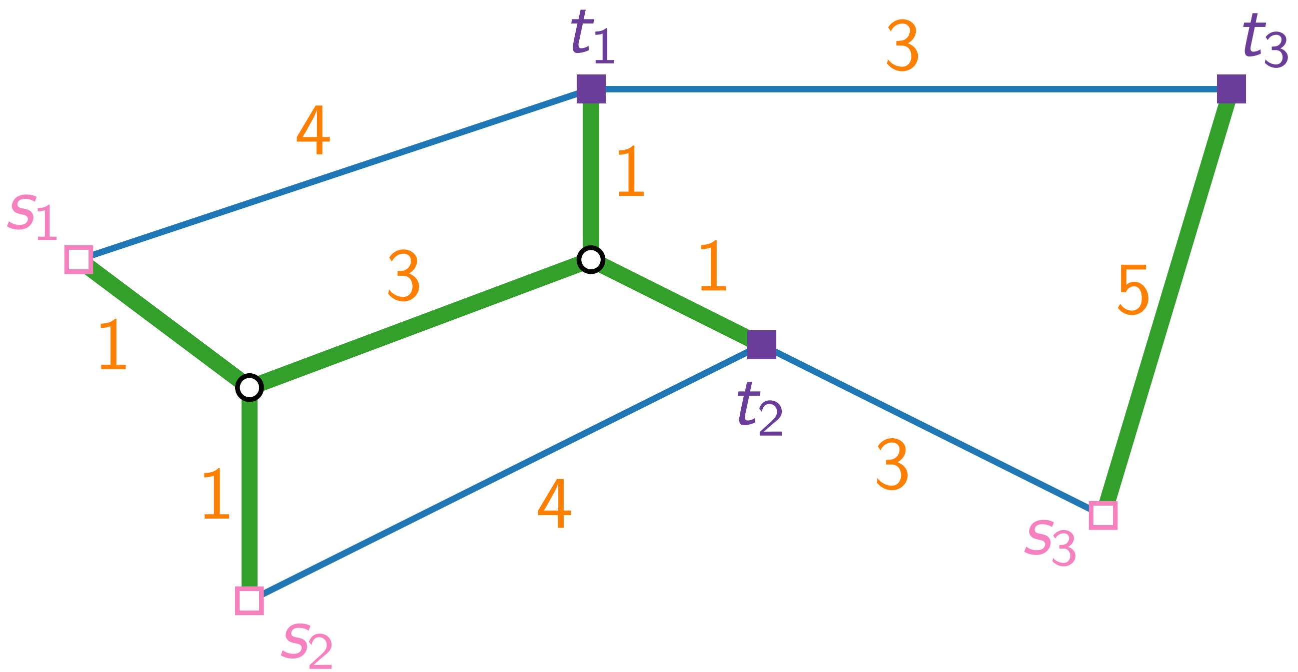
Special cases?

ShortestPath $(R = \{s, t\})$

MinSpannTree $(R = E(G))$

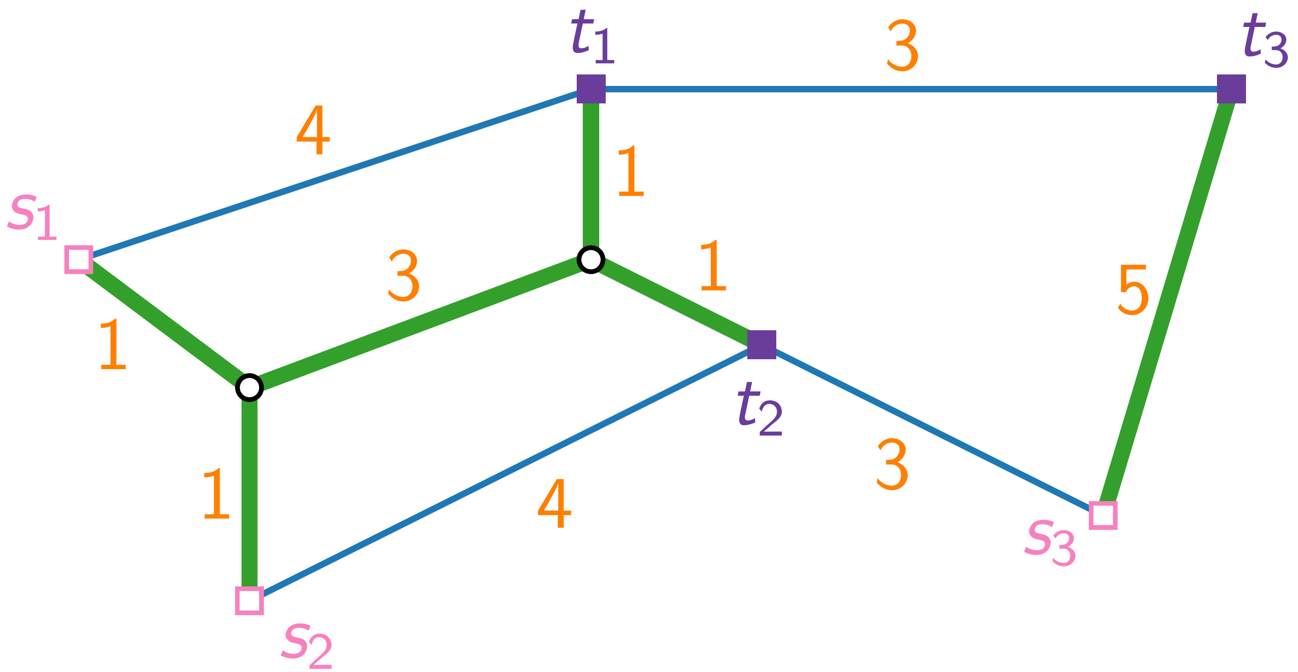
SteinerTree $(R = T \times T)$

Computational Approaches?



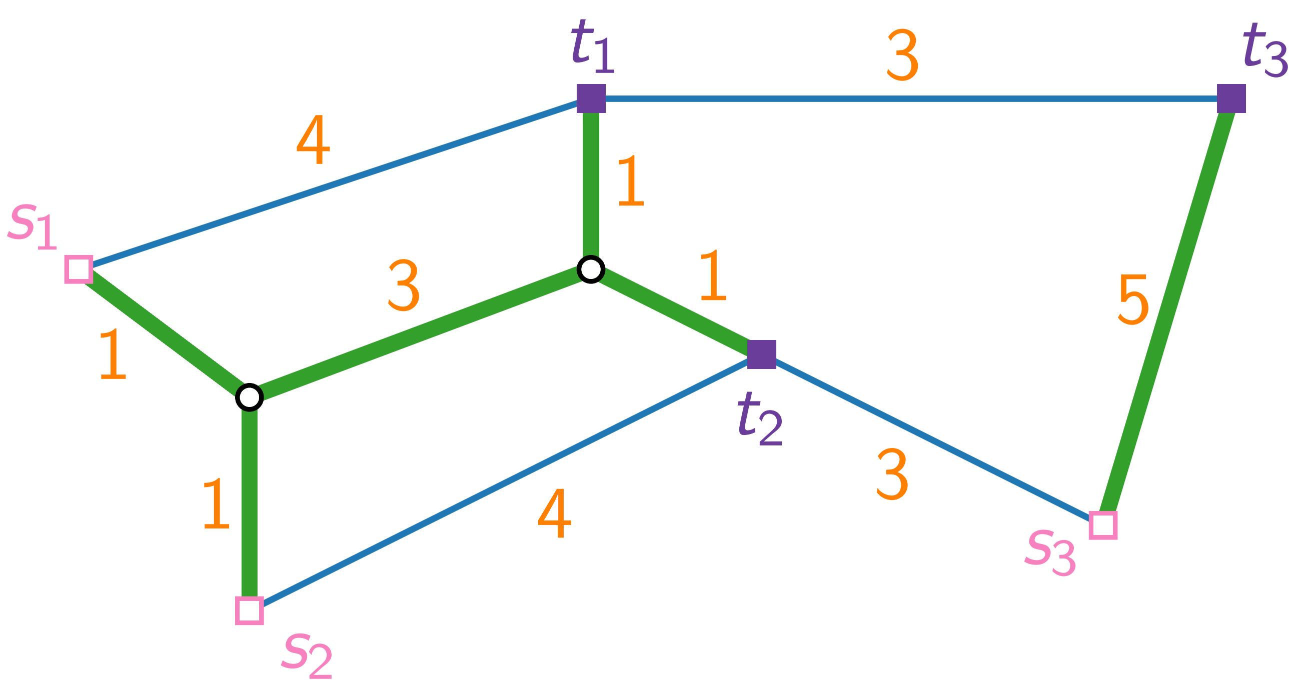
Computational Approaches?

- Merge k shortest s_i–t_i paths



Computational Approaches?

- Merge k shortest s_i–t_i paths
- SteinerTree on the set of terminals



Computational Approaches?

- Merge k shortest s_i–t_i paths
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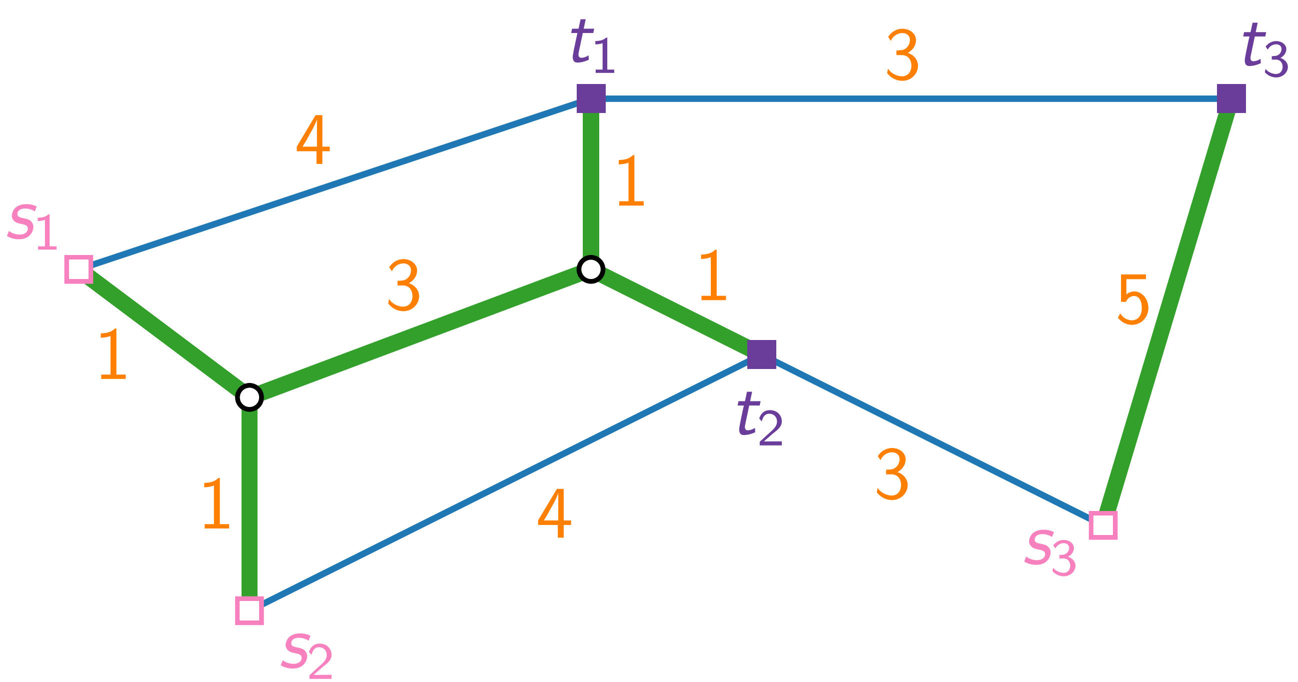
Homework: Both above approaches perform poorly :-(

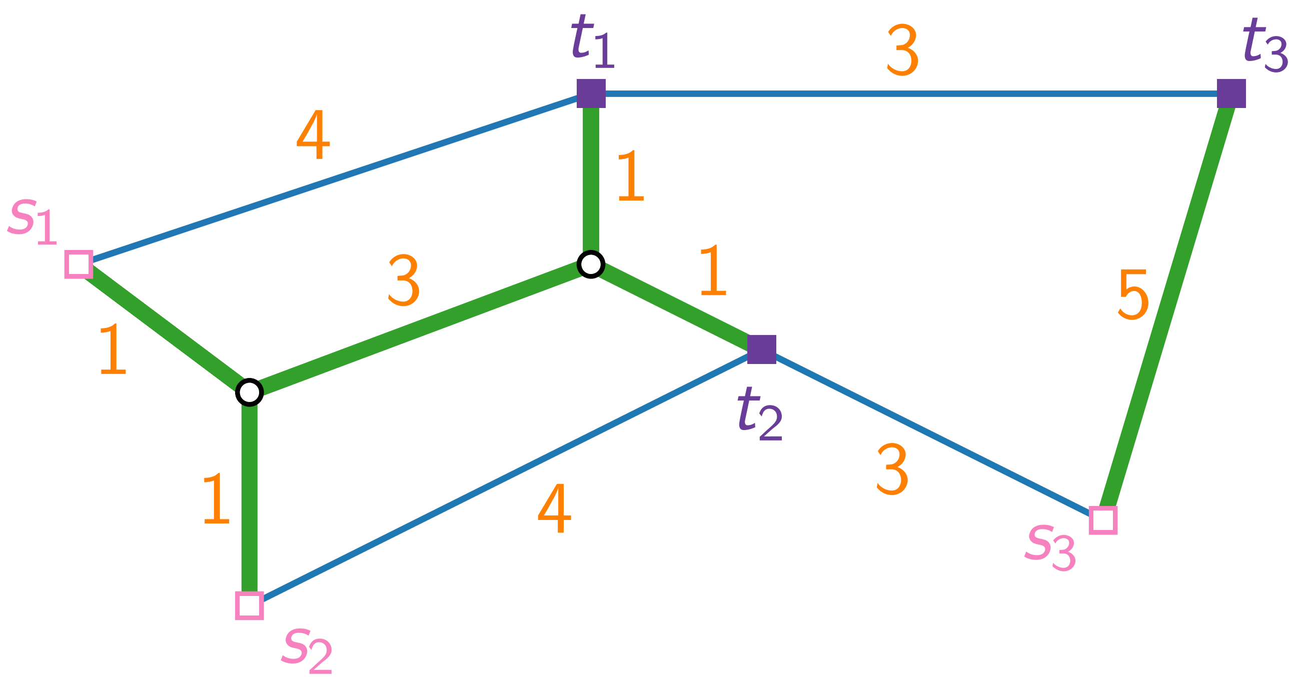


Computational Approaches?

- Merge k shortest s_i–t_i paths
- SteinerTree on the set of terminals

Homework: Both above approaches perform poorly :-(

Difficulty:
Which terminals belong to the same tree of the forest?



Approximation Algorithms

Lecture 12:
SteinerForest via Primal–Dual

Part II:
Primal and Dual LP

An ILP

minimize

subject to

An ILP

minimize

subject to

$$x_e \in \{0, 1\} \quad e \in E(G)$$

An ILP

minimize $\sum_{e \in E(G)} c_e x_e$

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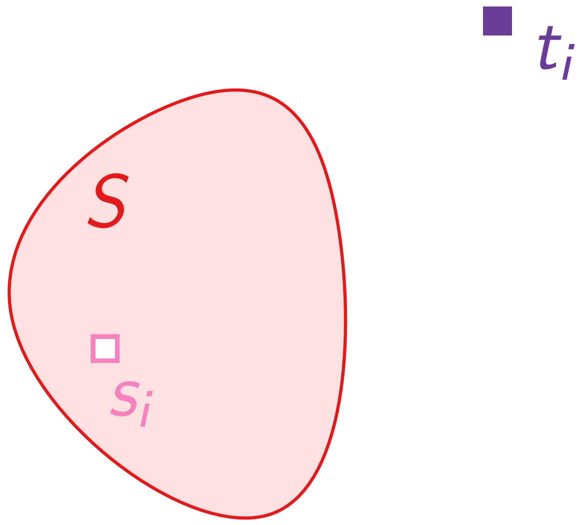

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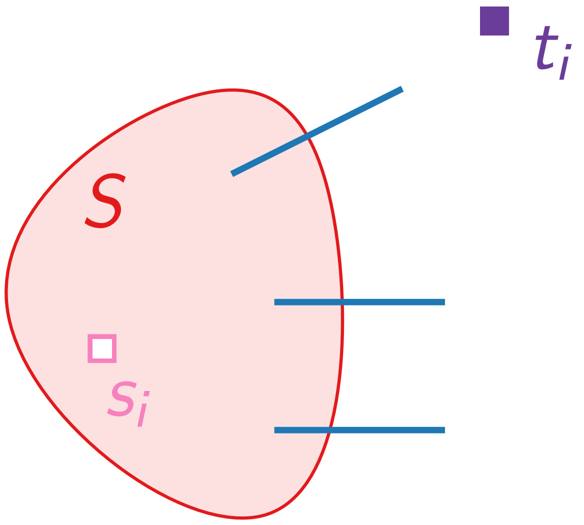

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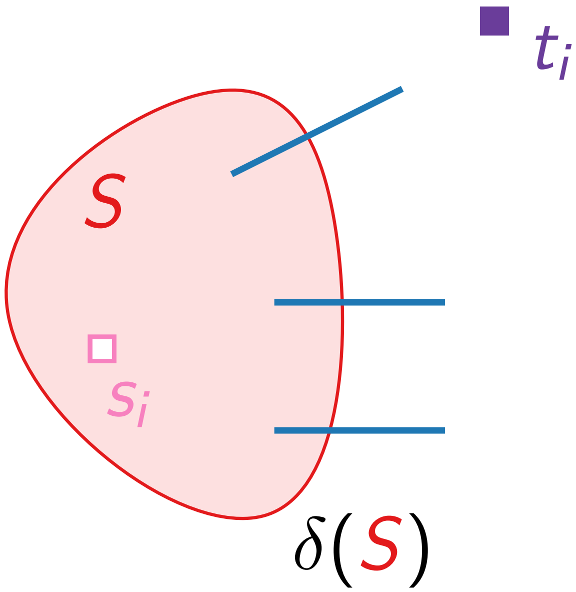


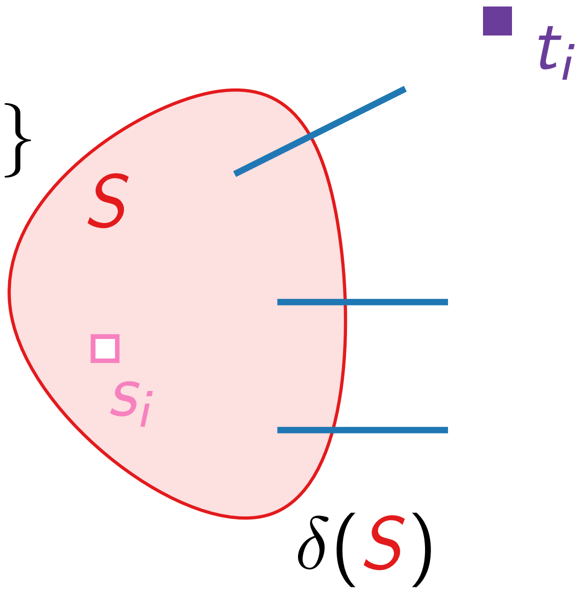
An ILP

minimize $\sum_{e \in E(G)} c_e x_e$

subject to

$x_e \in \{0, 1\} \quad e \in E(G)$

$\delta(S) := \{(u, v) \in E(G) \colon u \in S \text{ and } v \notin S\}$



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minimize $\sum_{e \in E(G)} c_e x_e$

subject to

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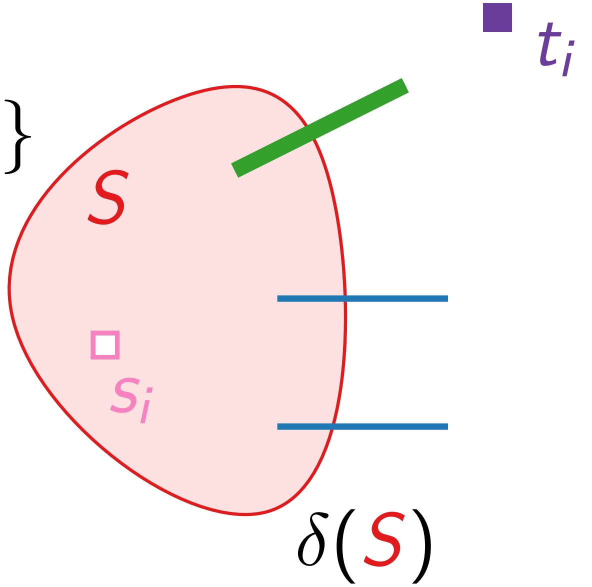

An ILP

minimize $$\sum_{e \in E(G)} c_e x_e$$

subject to $$\sum_{e \in \delta(S)} x_e \geq 1$$

$$x_e \in \{0,1\} \quad e \in E(G)$$

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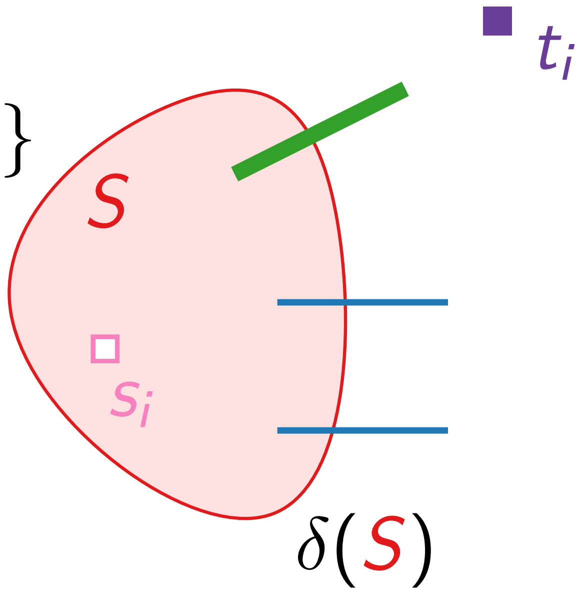

An ILP

minimize $\sum_{e \in E(G)} c_e x_e$

subject to $\sum_{e \in \delta(S)} x_e \geq 1 \qquad S \in \mathcal{S}_i,\ i \in \{1, \dots, k\}$

$x_e \in \{0, 1\} \qquad e \in E(G)$

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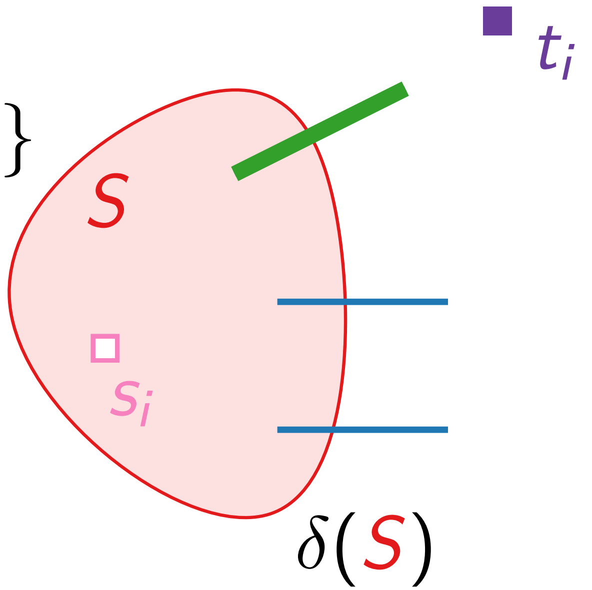

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$$\begin{aligned} \textbf{minimize} \quad & \sum_{e \in E(G)} c_e x_e \\ \textbf{subject to} \quad & \sum_{e \in \delta(S)} x_e \geq 1 \qquad S \in \mathcal{S}_i,\ i \in \{1, \dots, k\} \\ & x_e \in \{0, 1\} \qquad e \in E(G) \end{aligned}$$

where $\mathcal{S}_i := \{S \subseteq V(G) \colon s_i \in S, t_i \notin S\}$

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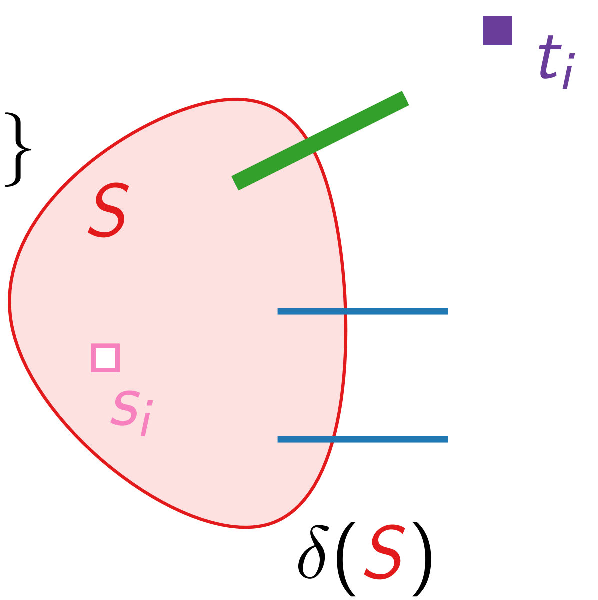

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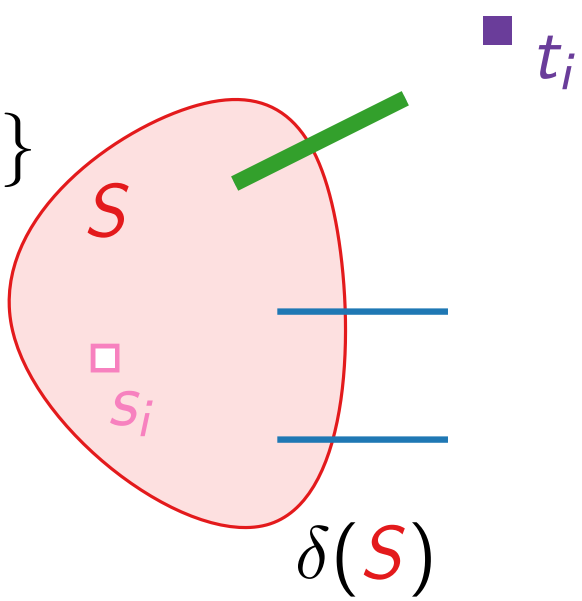
$\Rightarrow$ exponentially many constraints!



LP-Relaxation and Dual LP

$$\begin{aligned}
\textbf{minimize} \quad & \sum_{e \in E(G)} c_e x_e \\
\textbf{subject to} \quad & \sum_{e \in \delta(S)} x_e \geq 1 \qquad && S \in \mathcal{S}_i,\ i \in \{1, \ldots, k\} \\
& x_e \geq 0 \qquad && e \in E(G)
\end{aligned}$$

LP-Relaxation and Dual LP

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$x_e \geq 0 \qquad e \in E(G)$

maximize

subject to

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$$x_e \geq 0 \qquad e \in E(G)$$

maximize $$\sum_{\substack{S \in \mathcal{S}_i \\ i \in \{1,\ldots,k\}}} y_S$$

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\end{array}$$

The graph is a network of **bridges**, spanning the **moats**.

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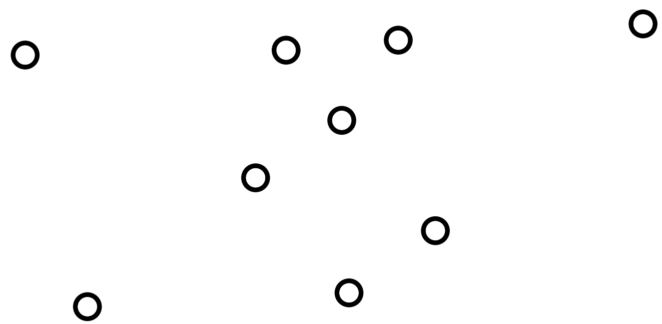

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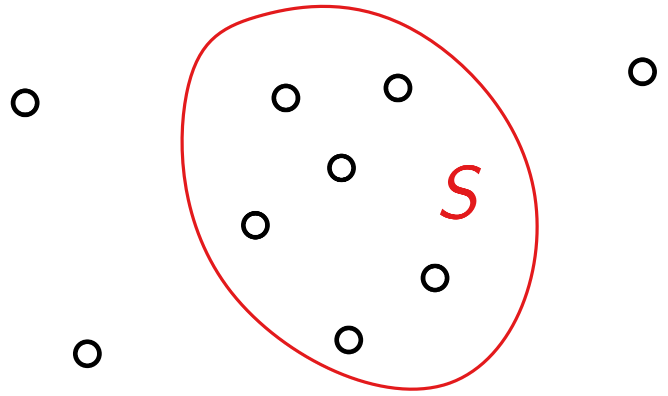

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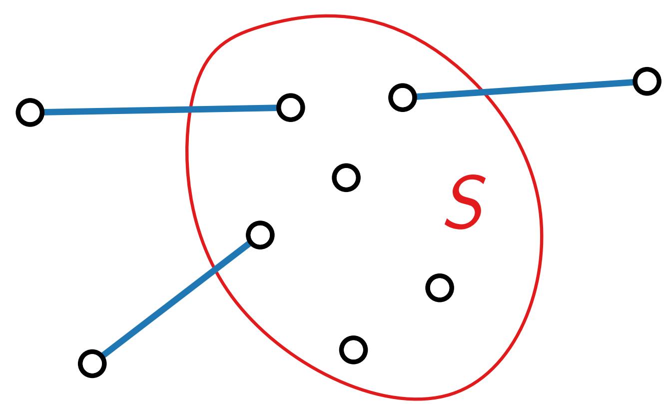

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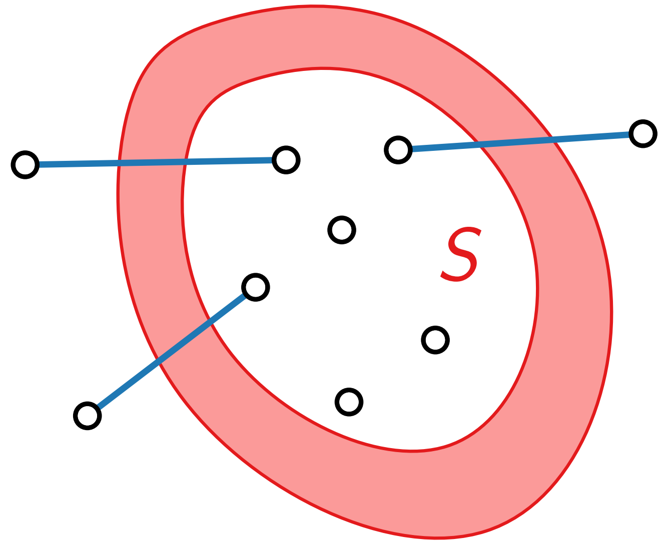

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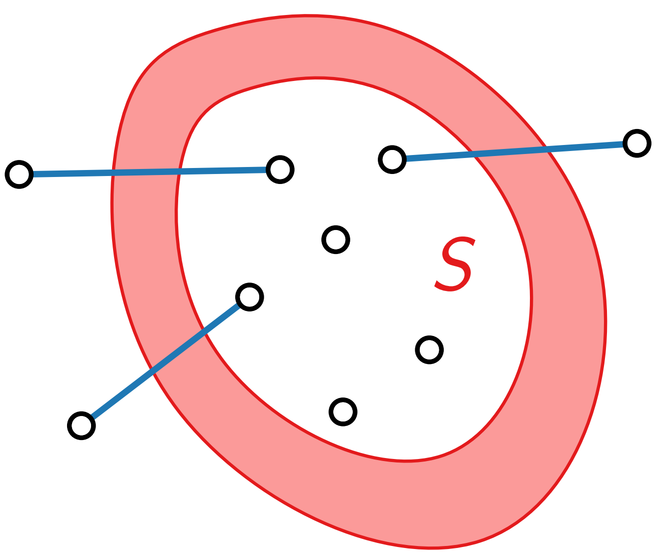

$\delta(S)$ = set of edges / bridges over the moat around S

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$$\text{maximize} \quad \sum_{\substack{S \in \mathcal{S}_i \\ i \in \{1,\dots,k\}}} y_S$$

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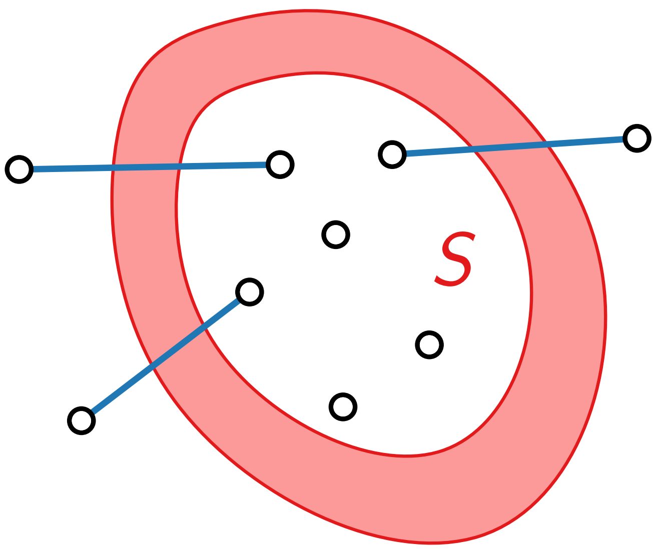

$\delta(S)$ = set of edges / bridges over the moat around S

y_S = width of the **moat** around S

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$$\begin{array}{llll} \textbf{maximize} & \displaystyle\sum_{\substack{S \in \mathcal{S}_i \\ i \in \{1,\ldots,k\}}} y_S & & \\ \textbf{subject to} & \displaystyle\sum_{S:\ e \in \delta(S)} y_S \le c_e & & e \in E(G) \\ & y_S \ge 0 & & S \in \mathcal{S}_i,\ i \in \{1,\ldots,k\} \end{array}$$

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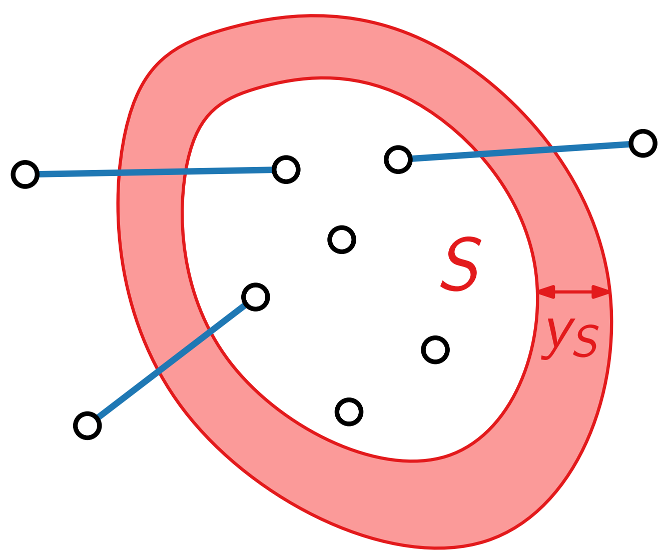

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subject to $\sum_{S:\ e \in \delta(S)} y_S \leq c_e \qquad e \in E(G)$

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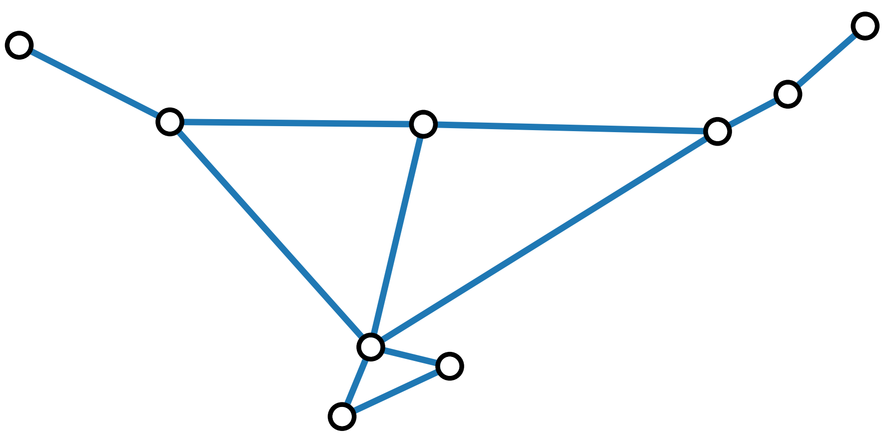

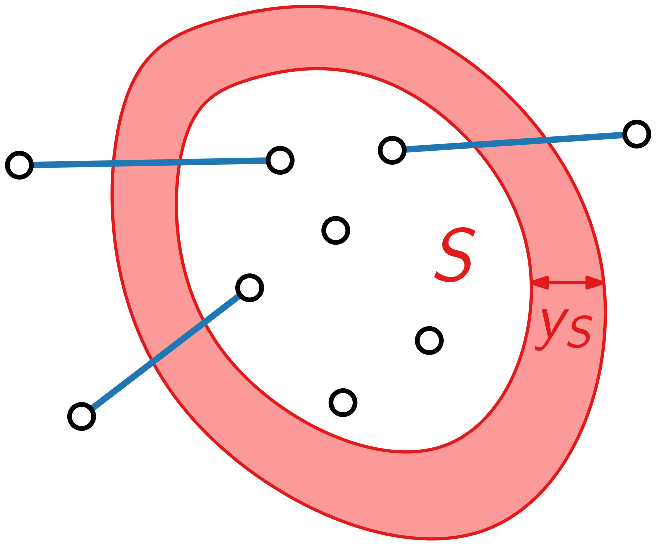

$\delta(S)$ = set of edges / bridges over the moat around S

y_S = width of the **moat** around S

Intuition for the Dual

$$\begin{aligned}
\textbf{maximize} \quad & \sum_{\substack{S \in \mathcal{S}_i \\ i \in \{1,\dots,k\}}} y_S \\
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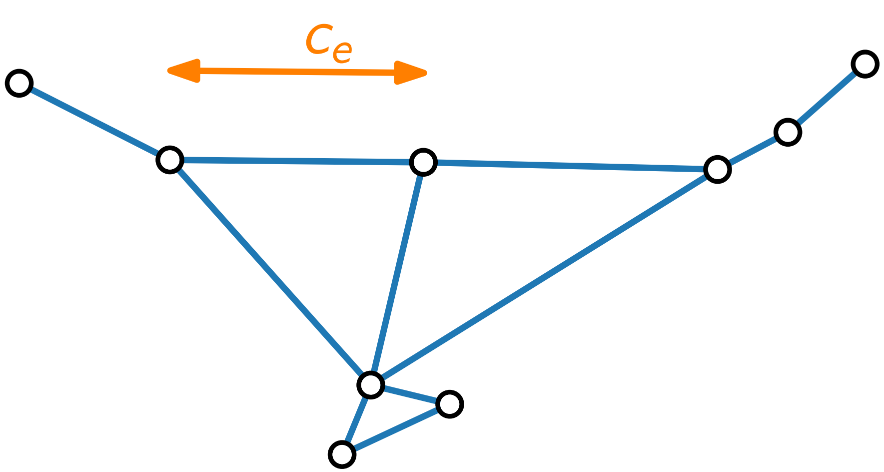

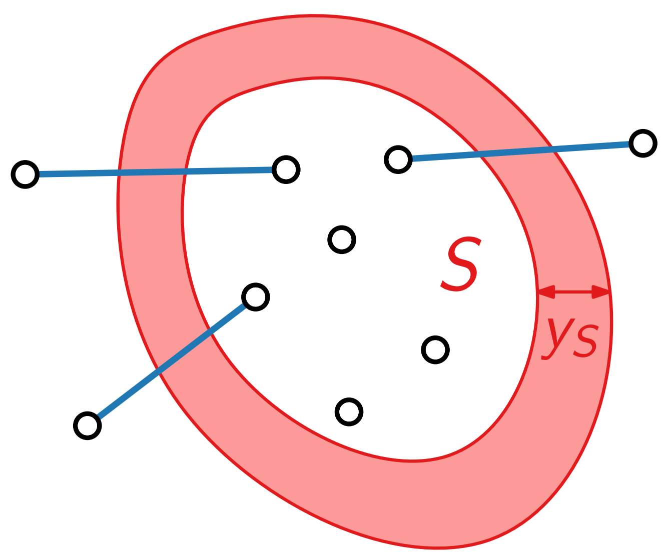

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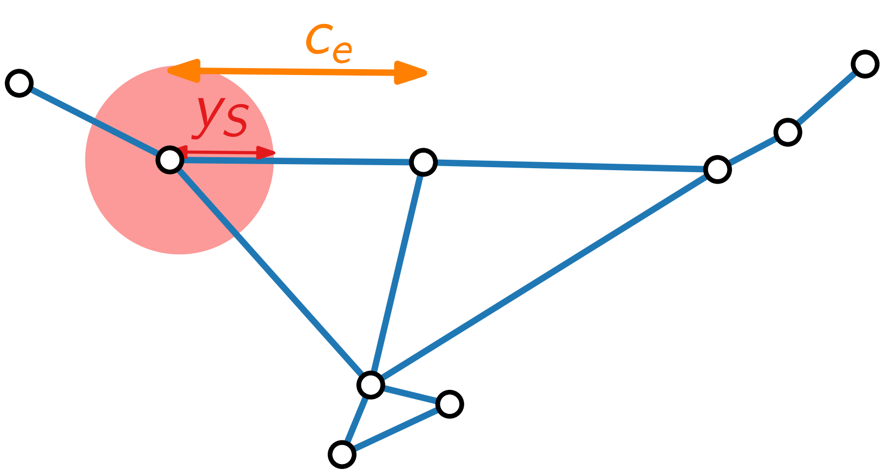

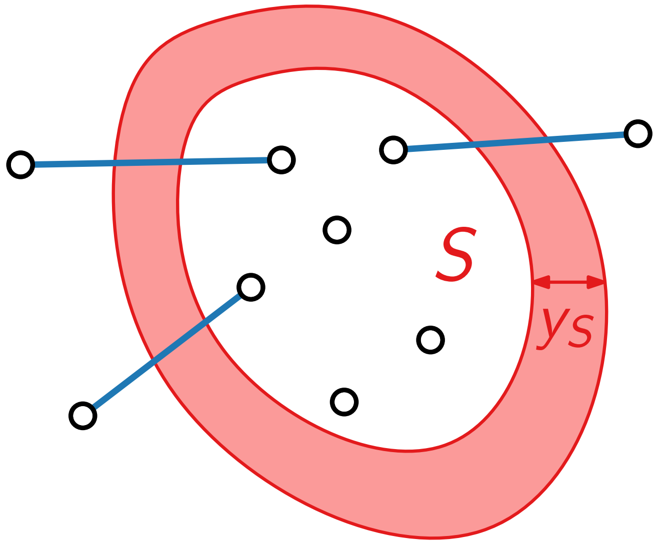

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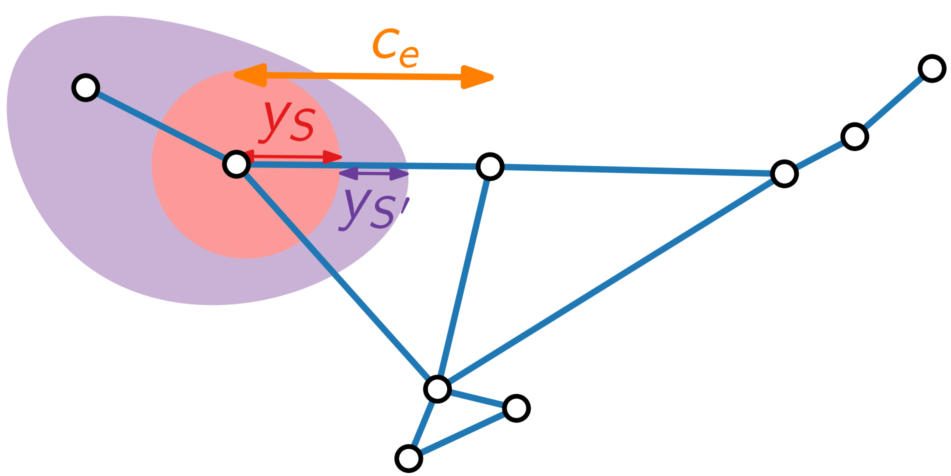

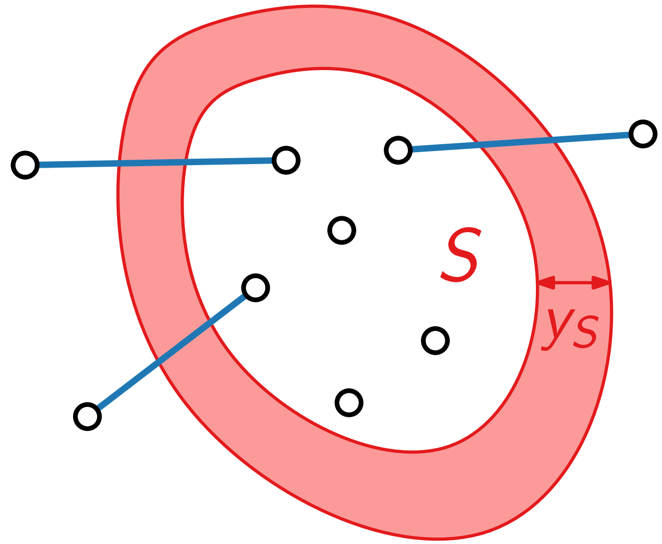

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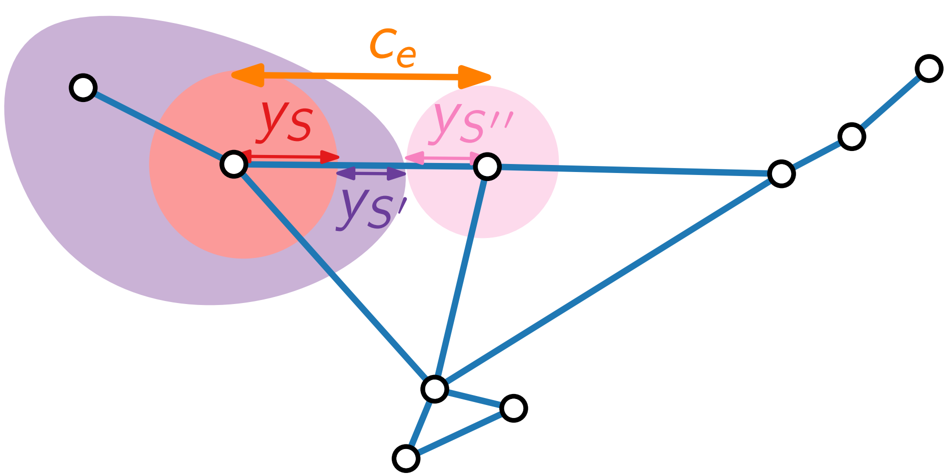

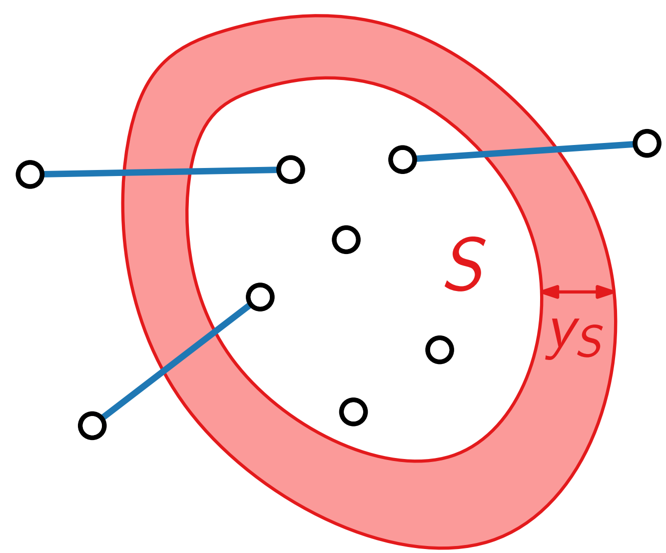

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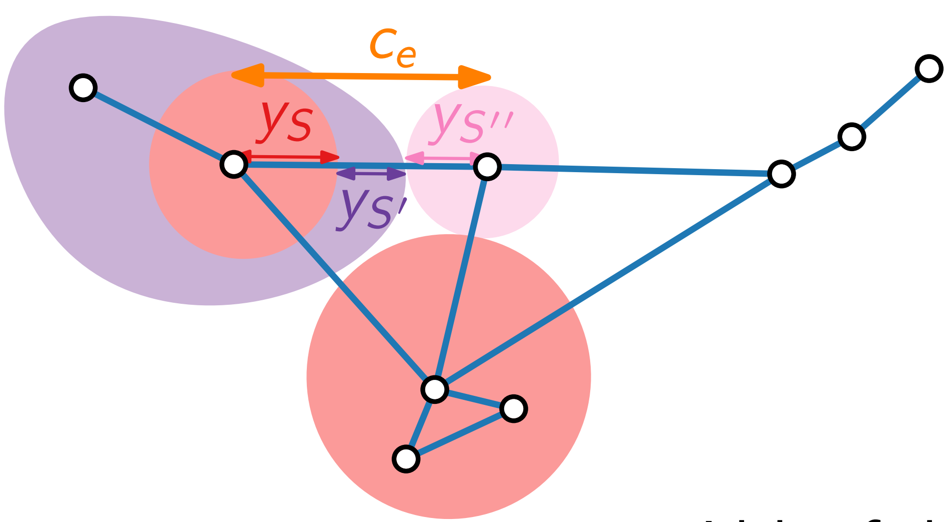

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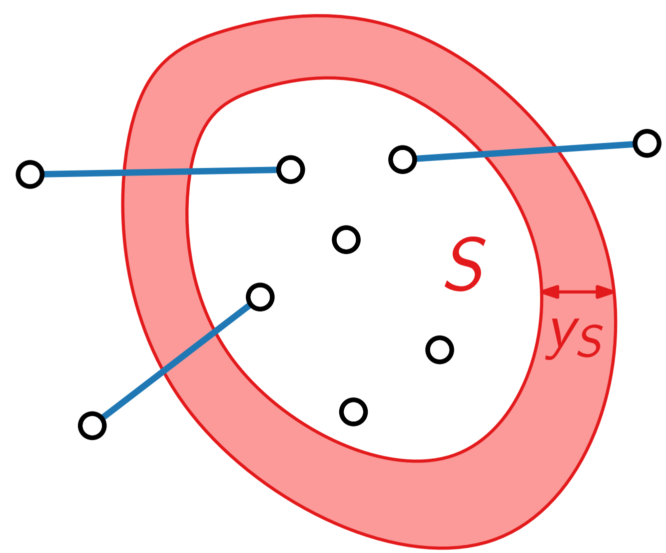

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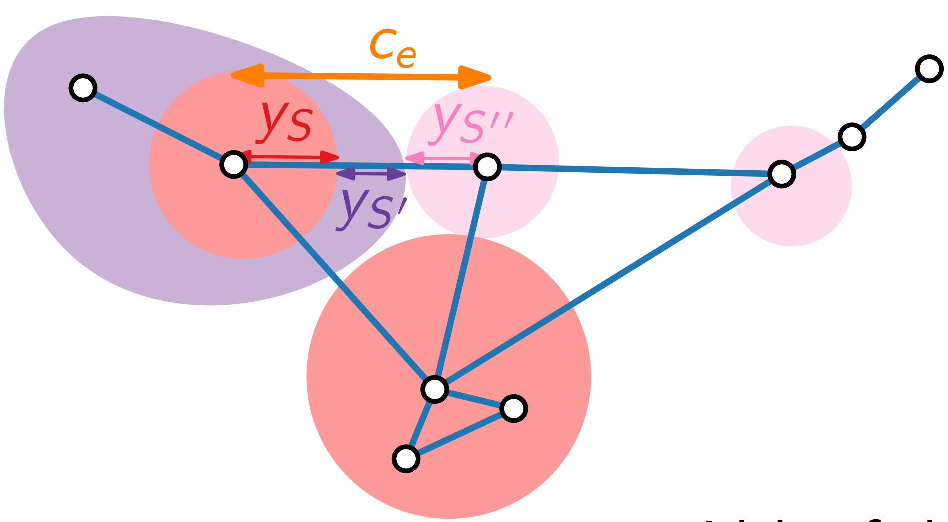

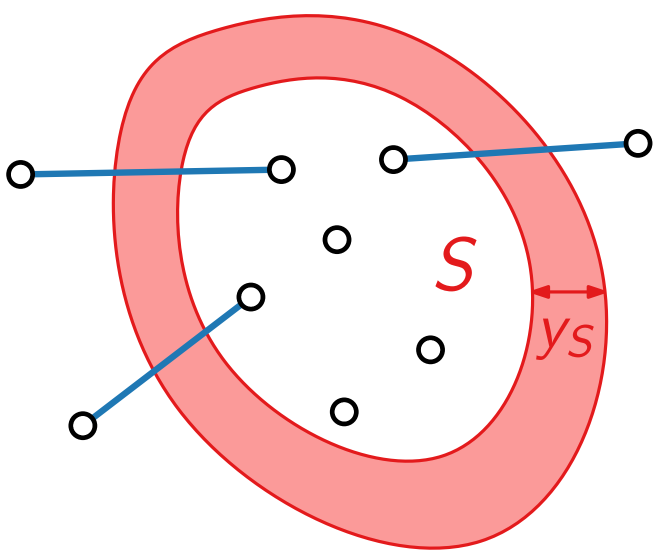

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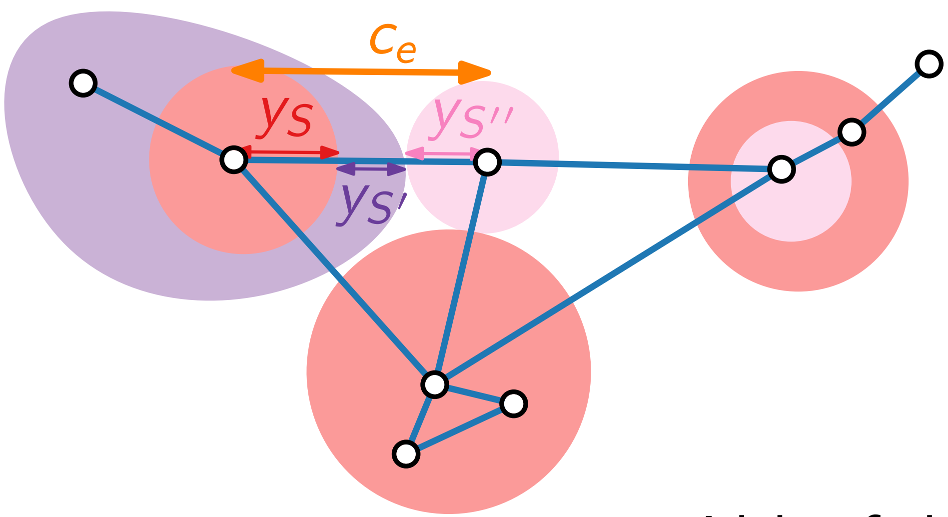

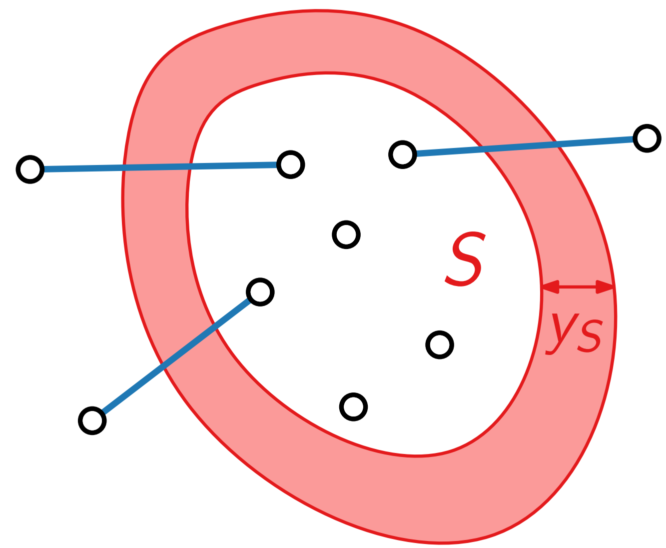

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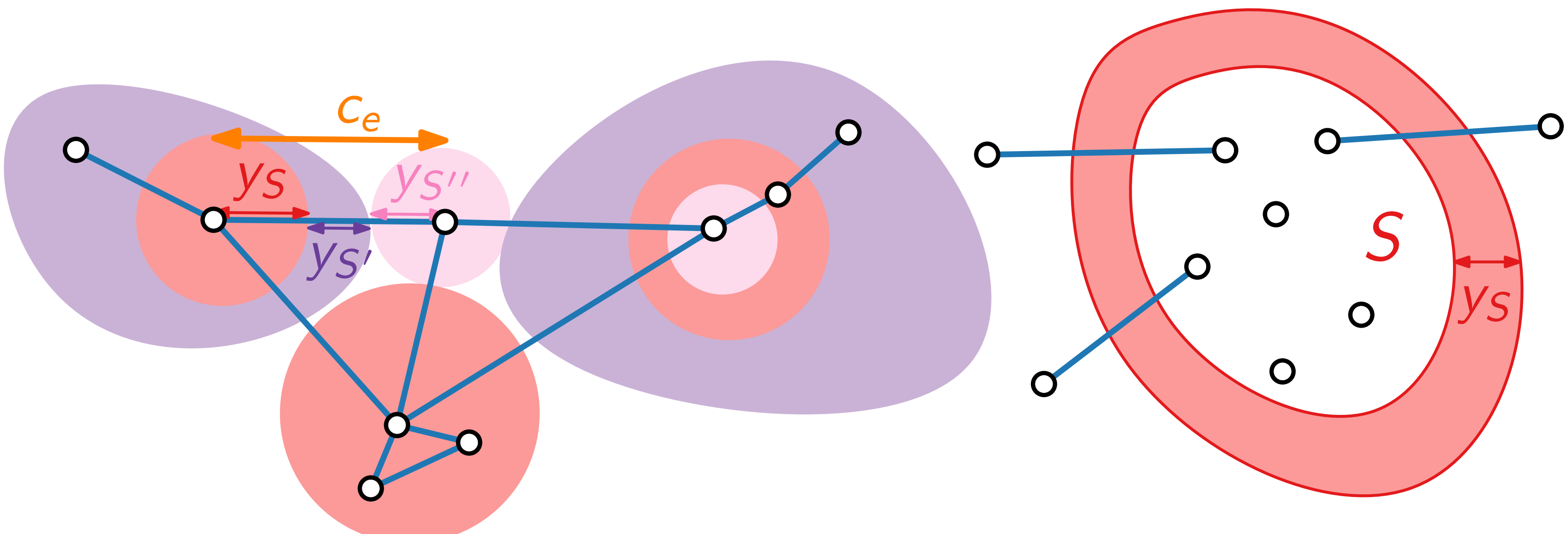

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Approximation Algorithms

Lecture 12:
SteinerForest via Primal–Dual

Part III:
A First Primal–Dual Approach

Complementary Slackness (Reminder)

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Complementary Slackness (Reminder)

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

Theorem. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program (resp.). Then x and y are optimal if and only if the following conditions are met:

Primal CS:
For each $j = 1, \ldots, n$: either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
For each $i = 1, \ldots, m$: either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

A First Primal–Dual Approach

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$\Rightarrow$ Pick "critical" edges (and only these)!

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How to find a violated primal constraint? $\left(\sum_{e\in\delta(S)} x_e < 1\right)$

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How do we iteratively improve the dual solution?

- Increase y_C (until some edge in $\delta(C)$ becomes critical)!

A First Primal–Dual Approach

PrimalDualSteinerForestNaive(graph G, costs c, pairs R)

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Trick: Handle all y_S with $y_S = 0$ implicitly.

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(Homework!)

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But: Average degree of "active components" is less than 2.

$\Rightarrow$ Increase y_C for all active components C simultaneously!

Approximation Algorithms

Lecture 12:
SteinerForest via Primal–Dual

Part IV:
Primal–Dual with Synchronized Increases

Primal–Dual with Synchronized Increases

PrimalDualSteinerForest(graph G, edge costs c, pairs R)

$y \leftarrow 0$, $F \leftarrow \emptyset$, $\ell \leftarrow 0$

while $\exists(s, t) \in R$ not connected in $(V(G), F)$ **do**

$\quad \ell \leftarrow \ell + 1$

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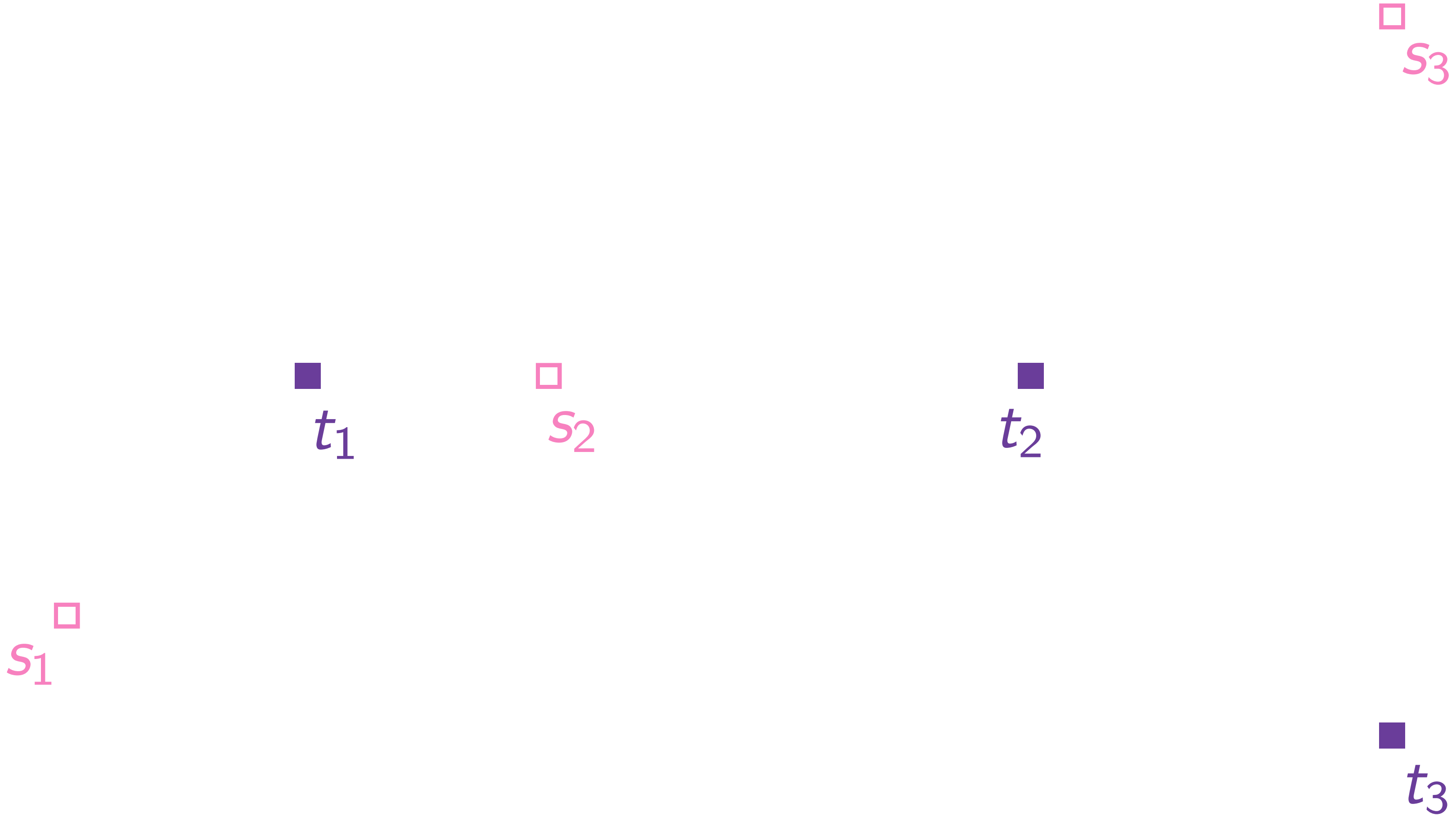
for $j \leftarrow \ell$ **downto** 1 **do**
  if $F' \setminus \{e_j\}$ is feasible solution **then**
    $F' \leftarrow F' \setminus \{e_j\}$
return F'

Illustration

$G = K_6$ with Euclidean edge costs



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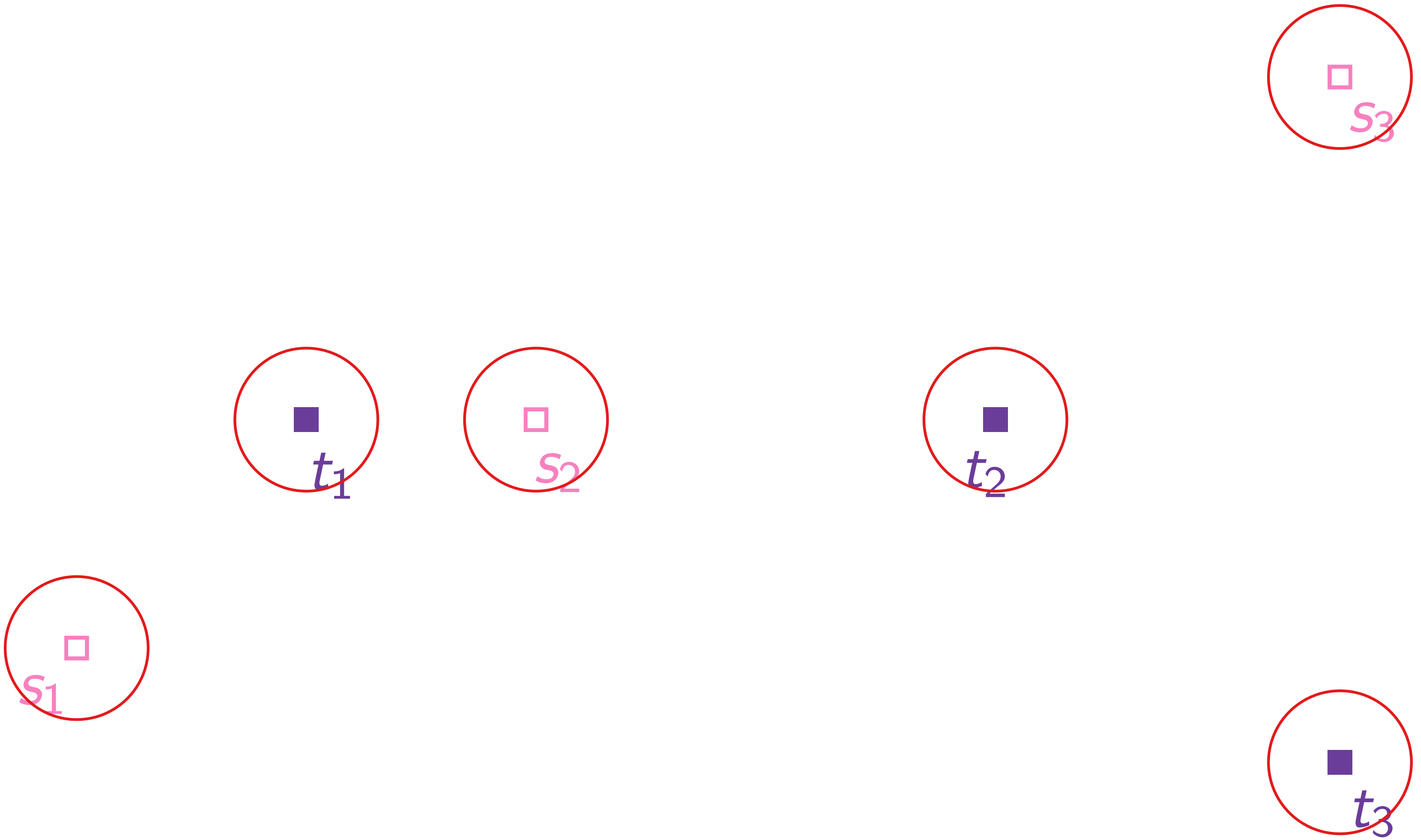

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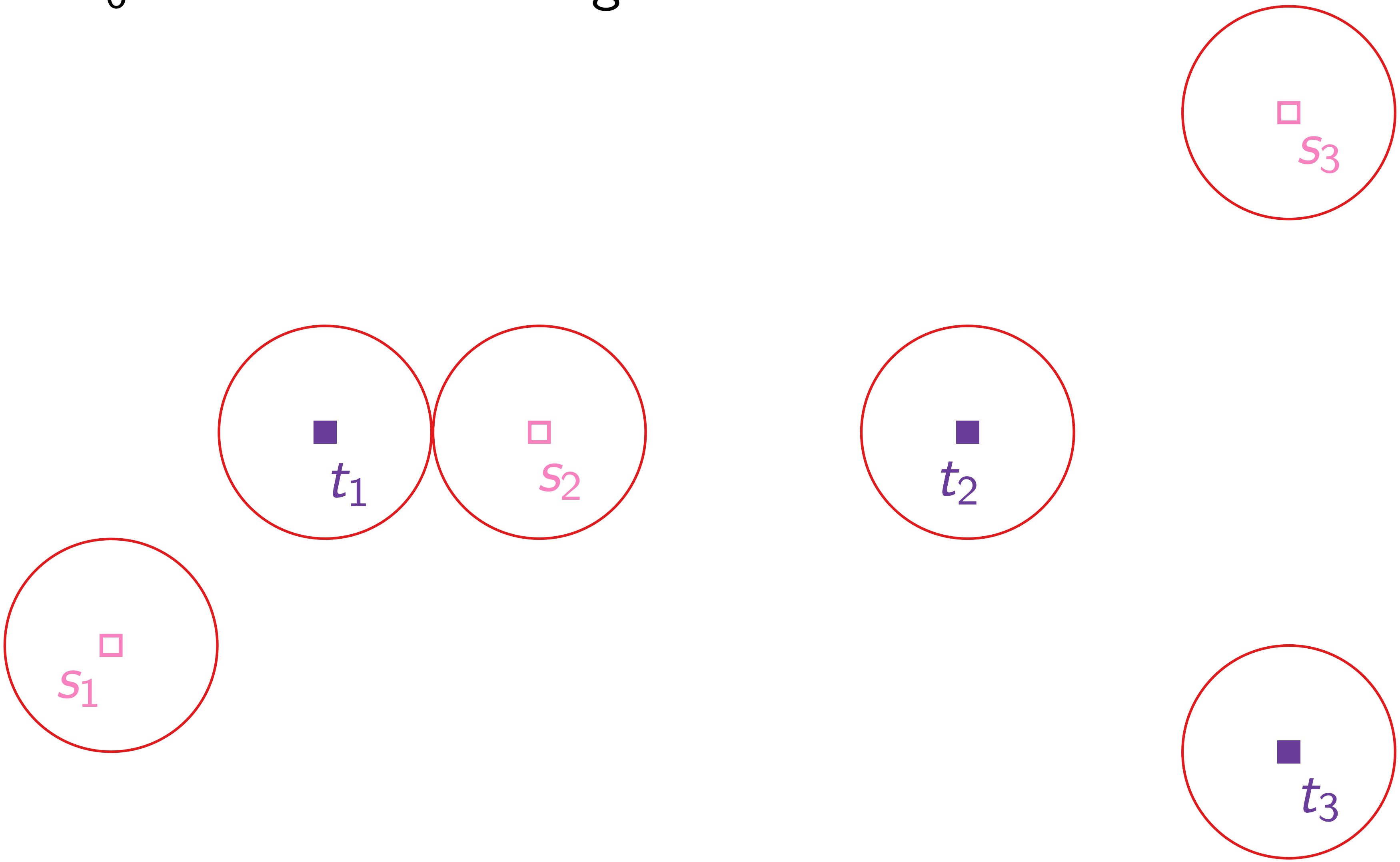

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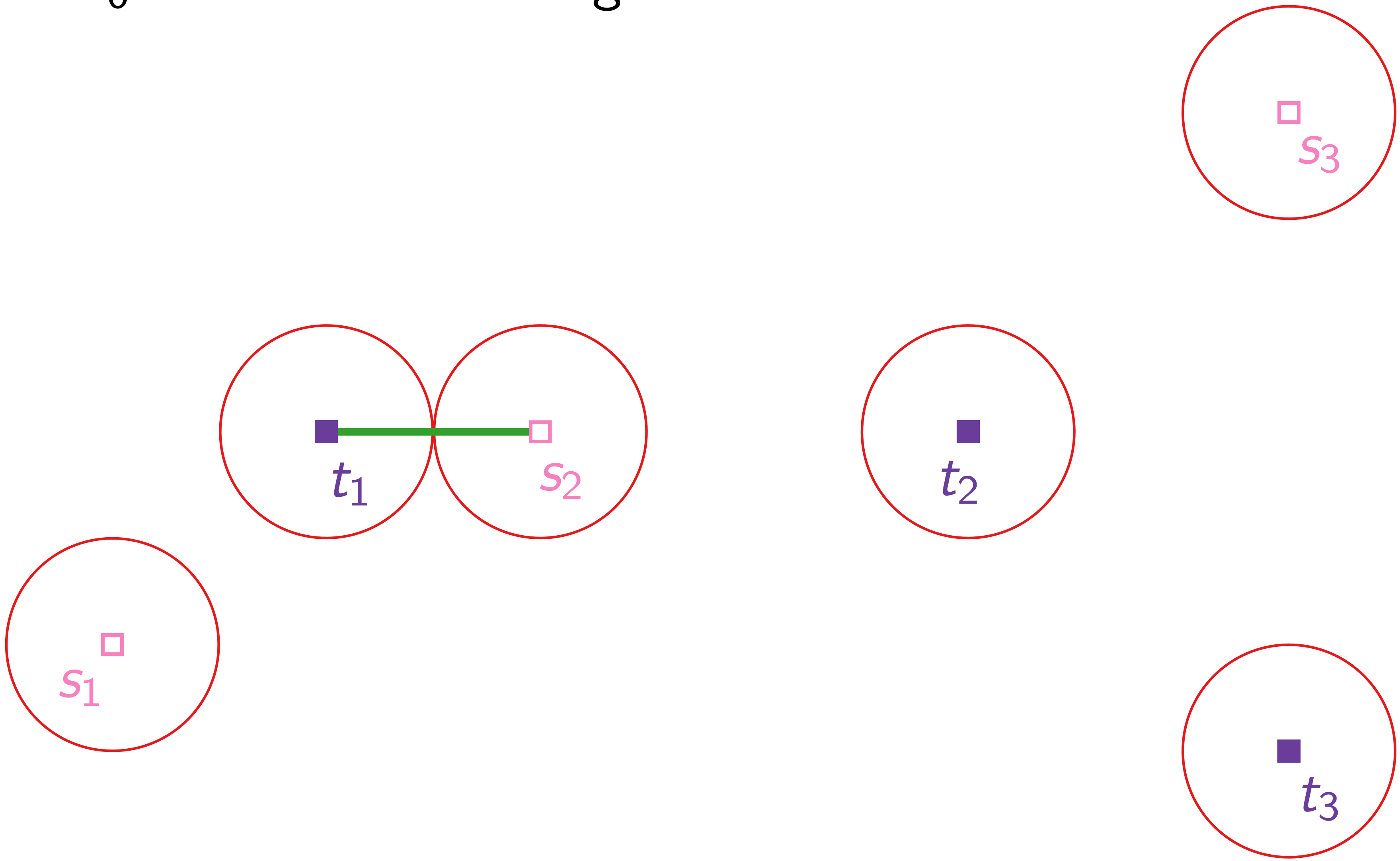

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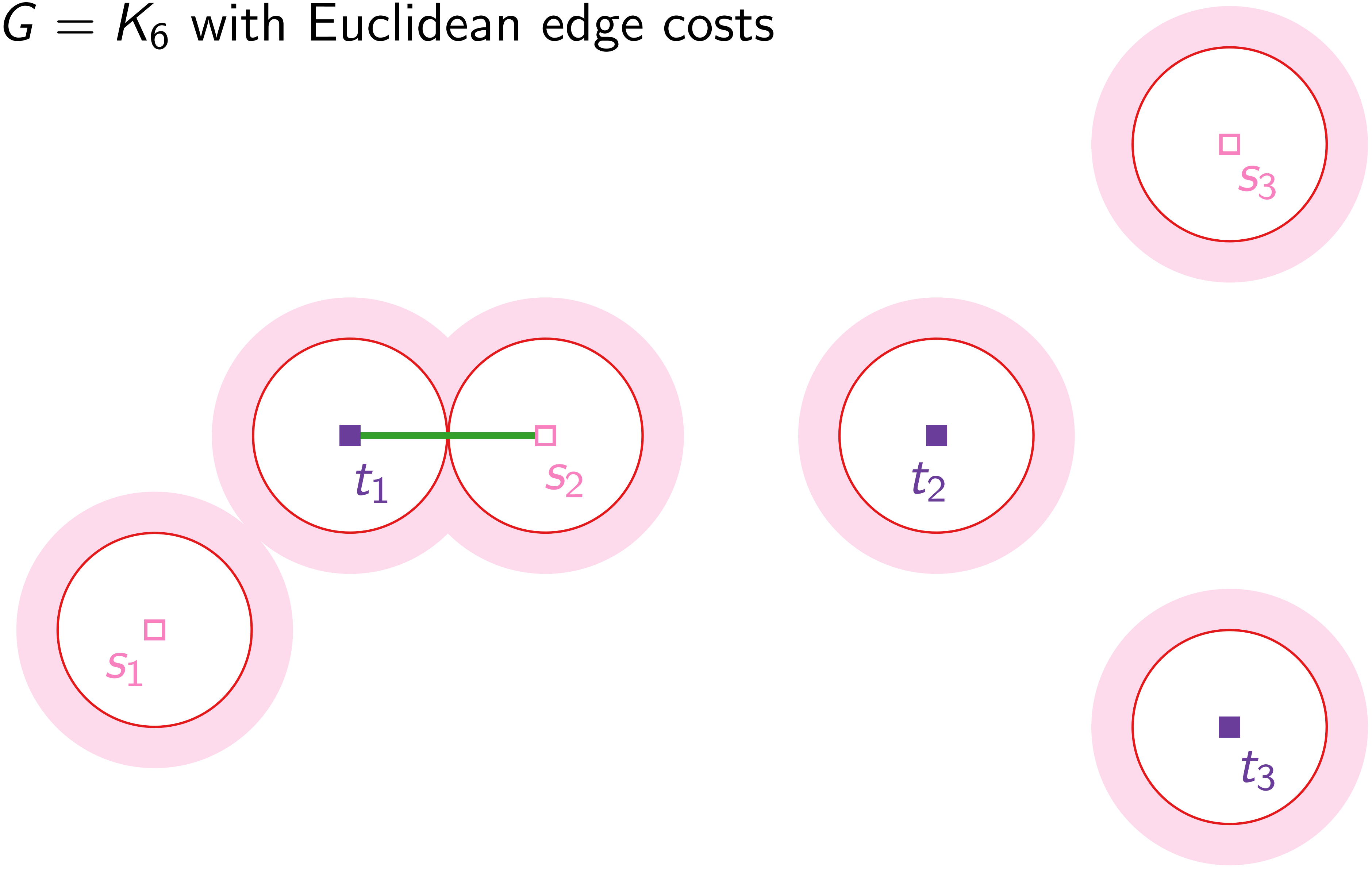

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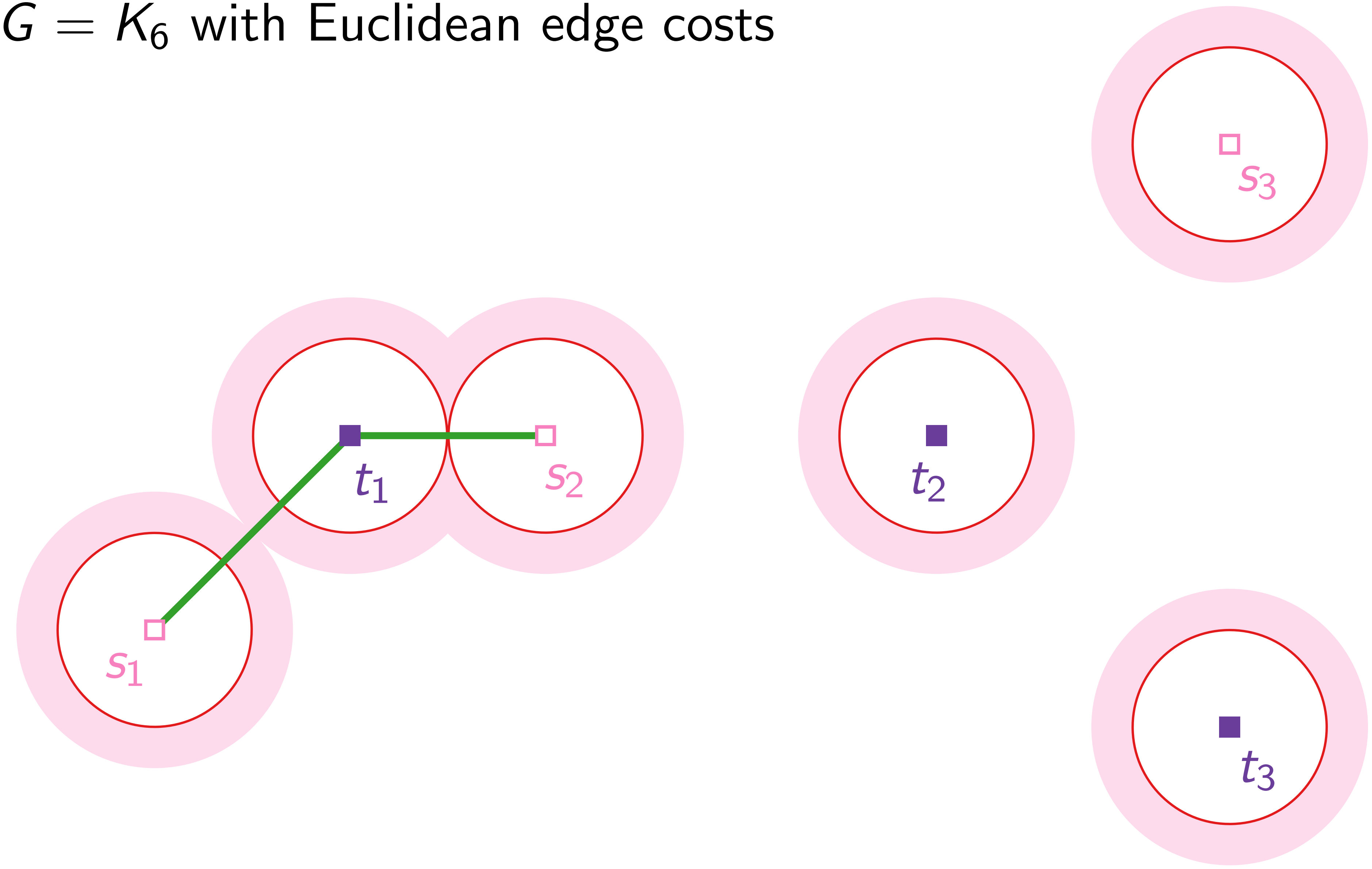

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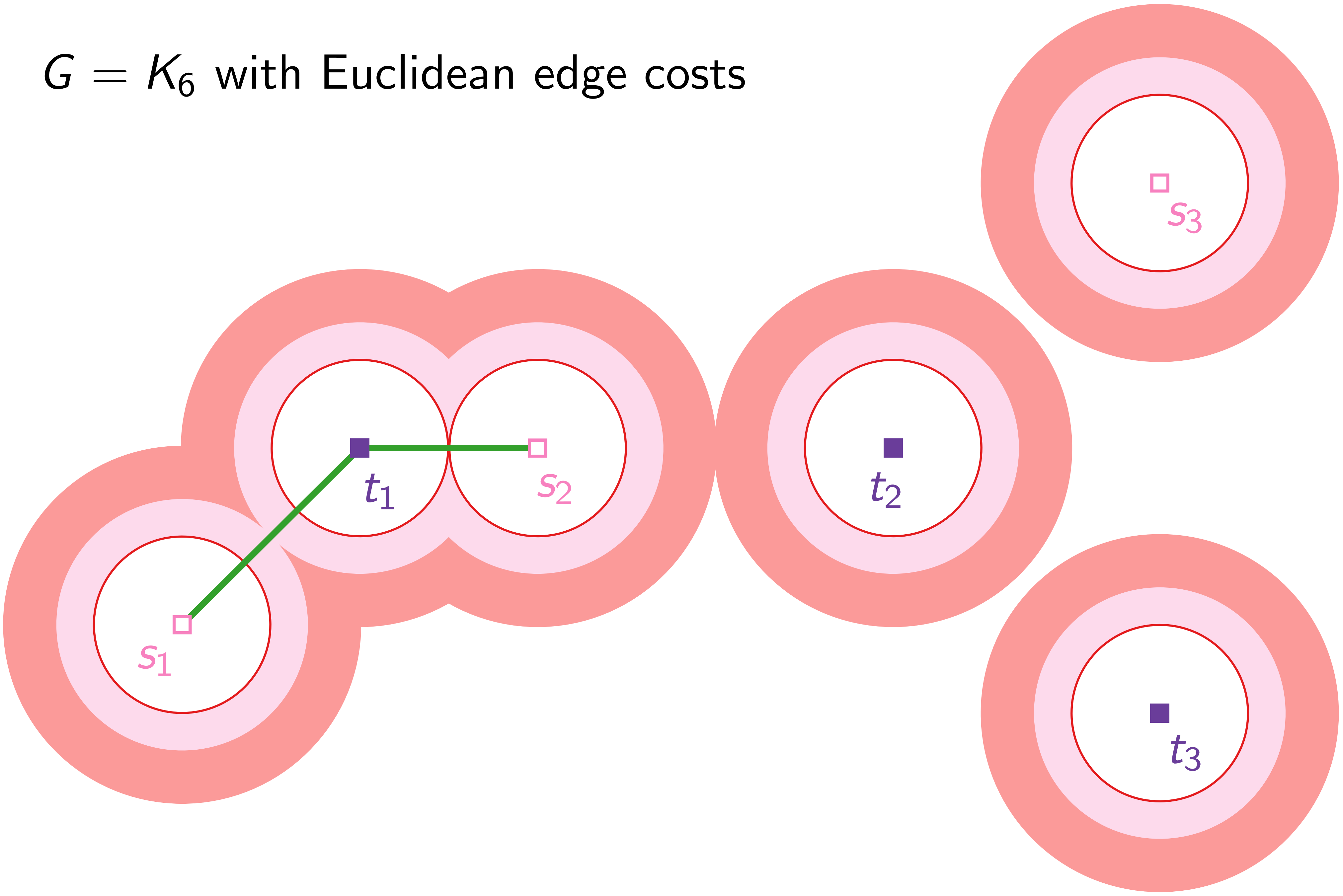

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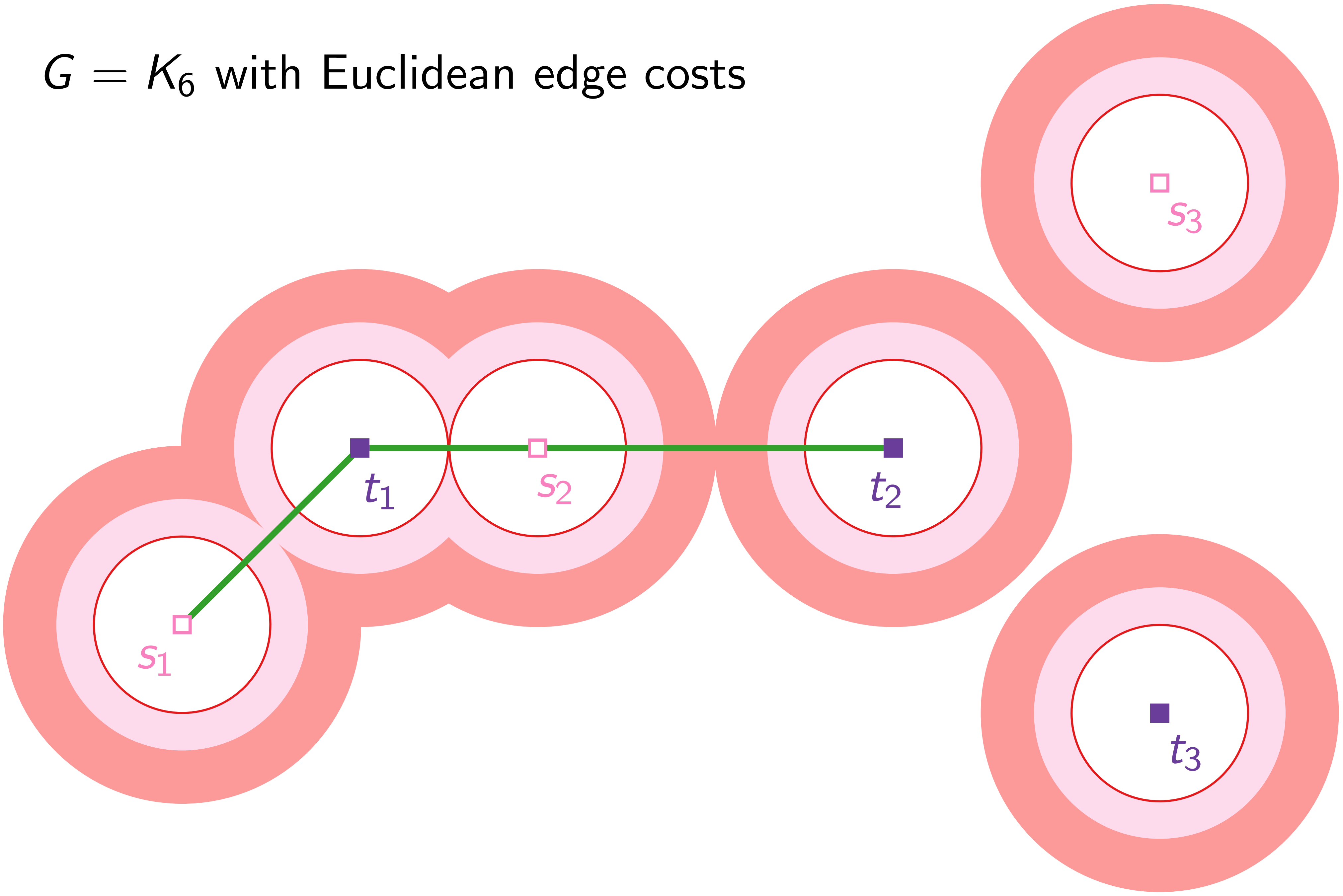

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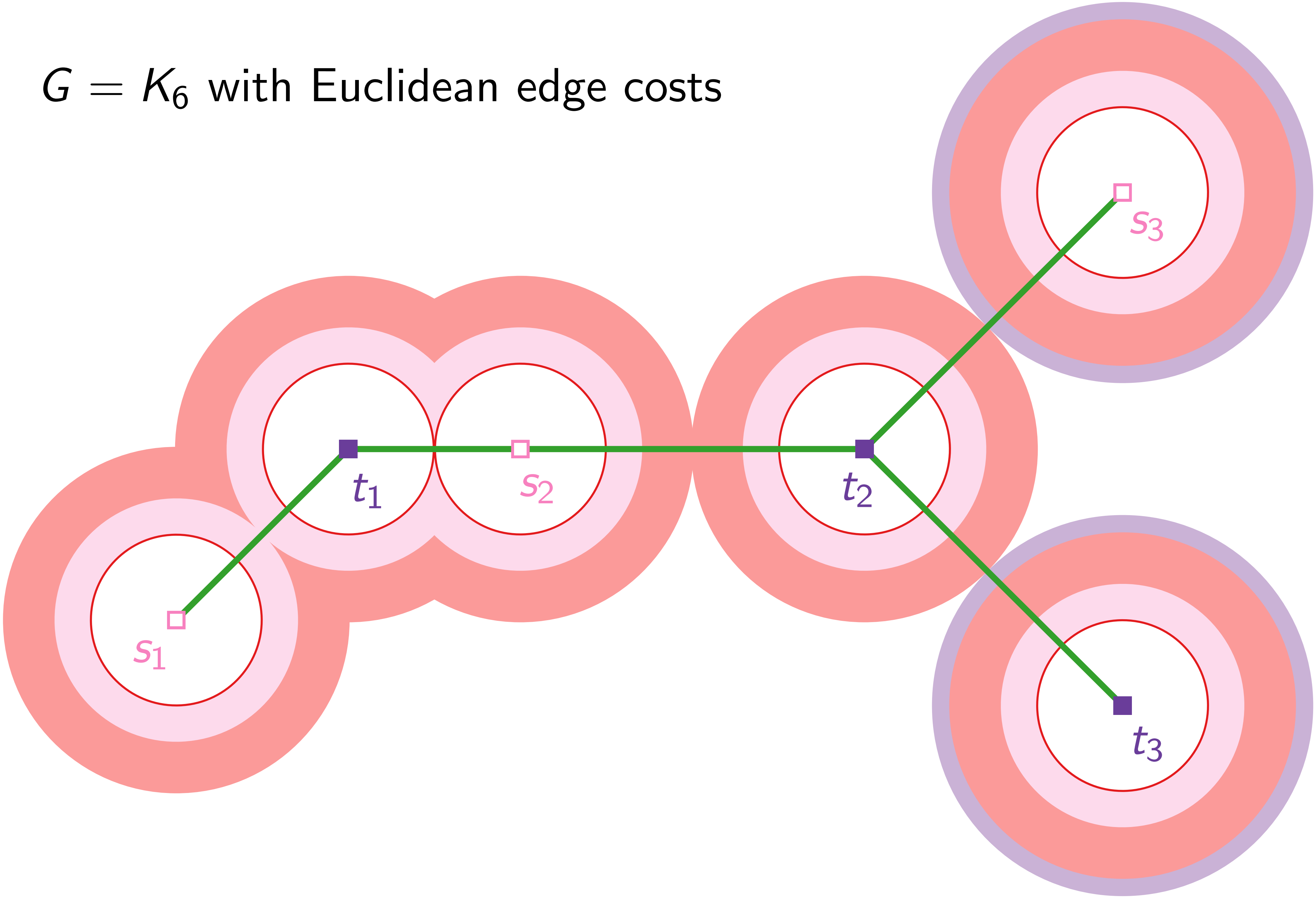

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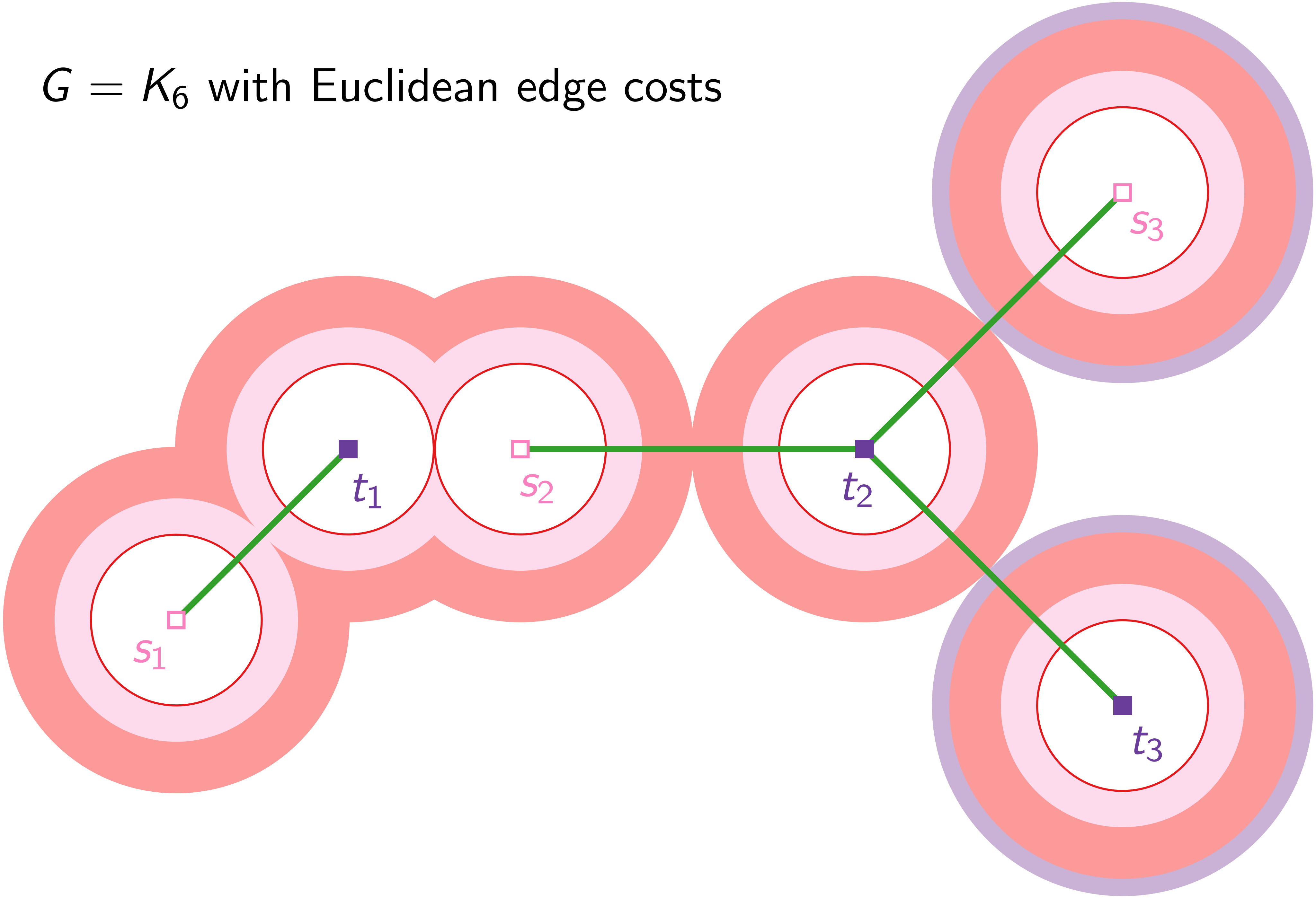

Approximation Algorithms

Lecture 12:
SteinerForest via Primal–Dual

Part V:
Structure Lemma

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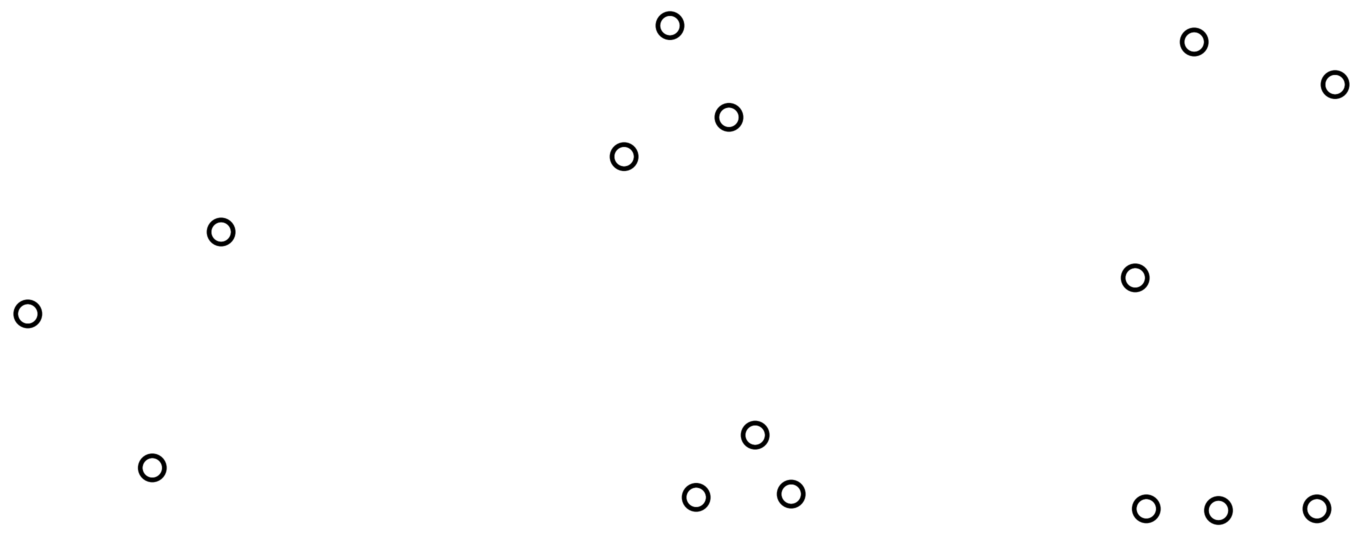

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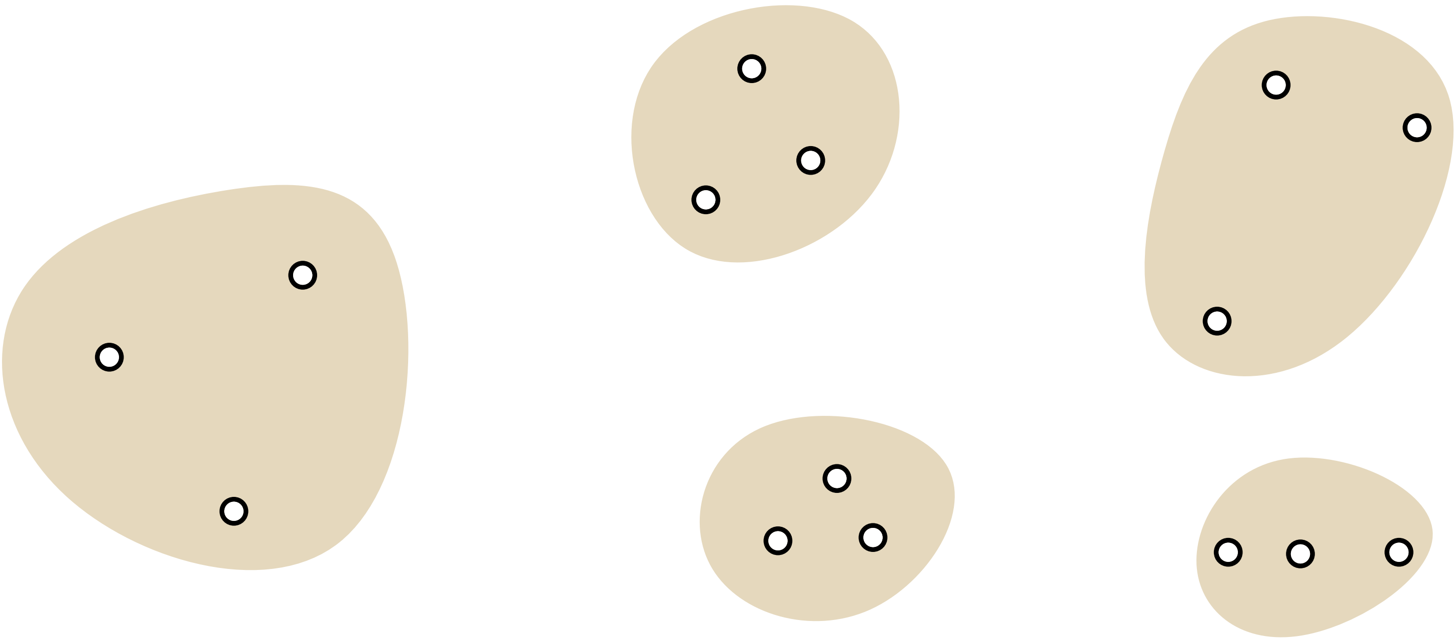

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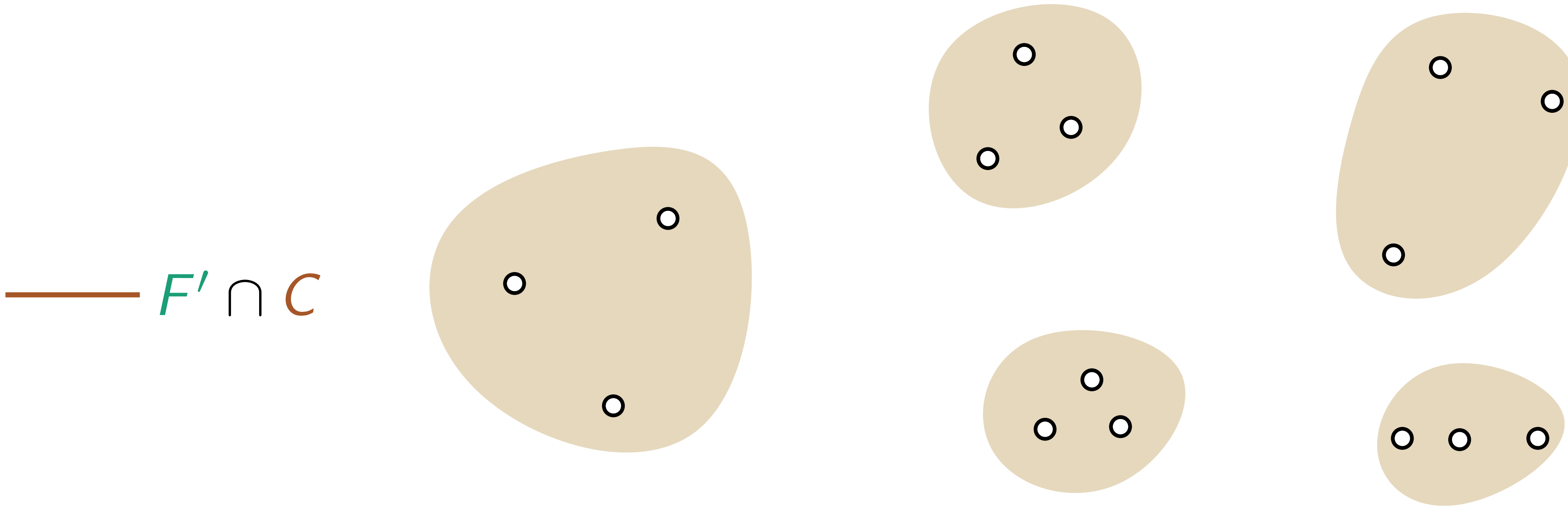

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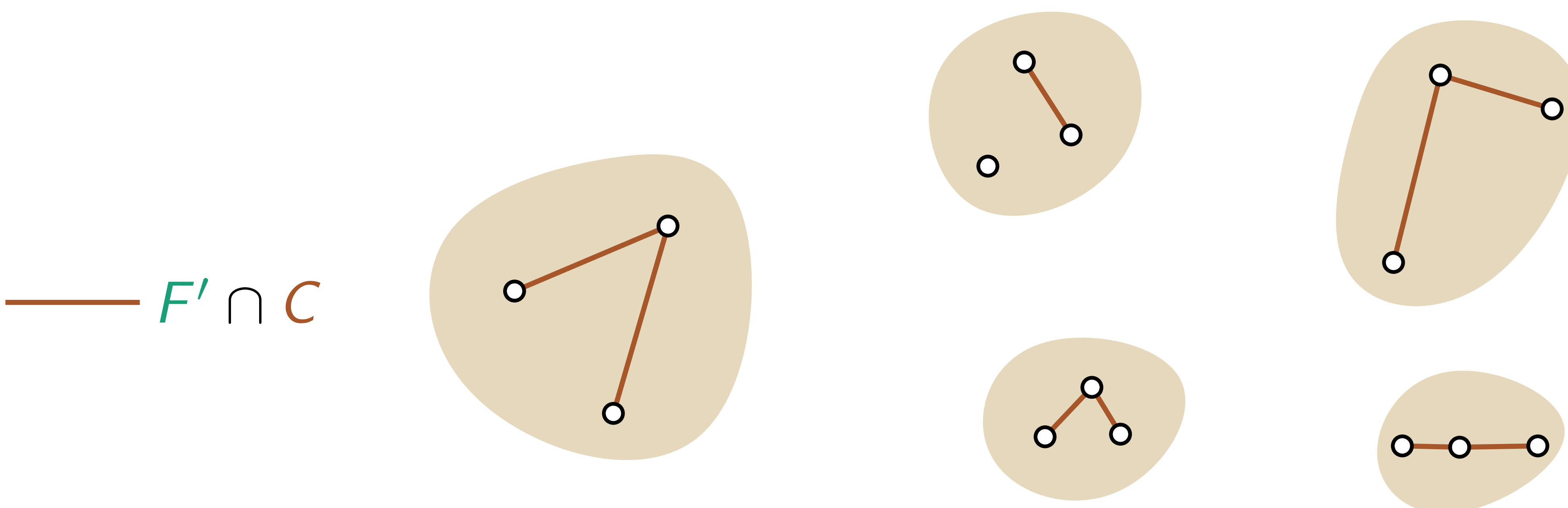

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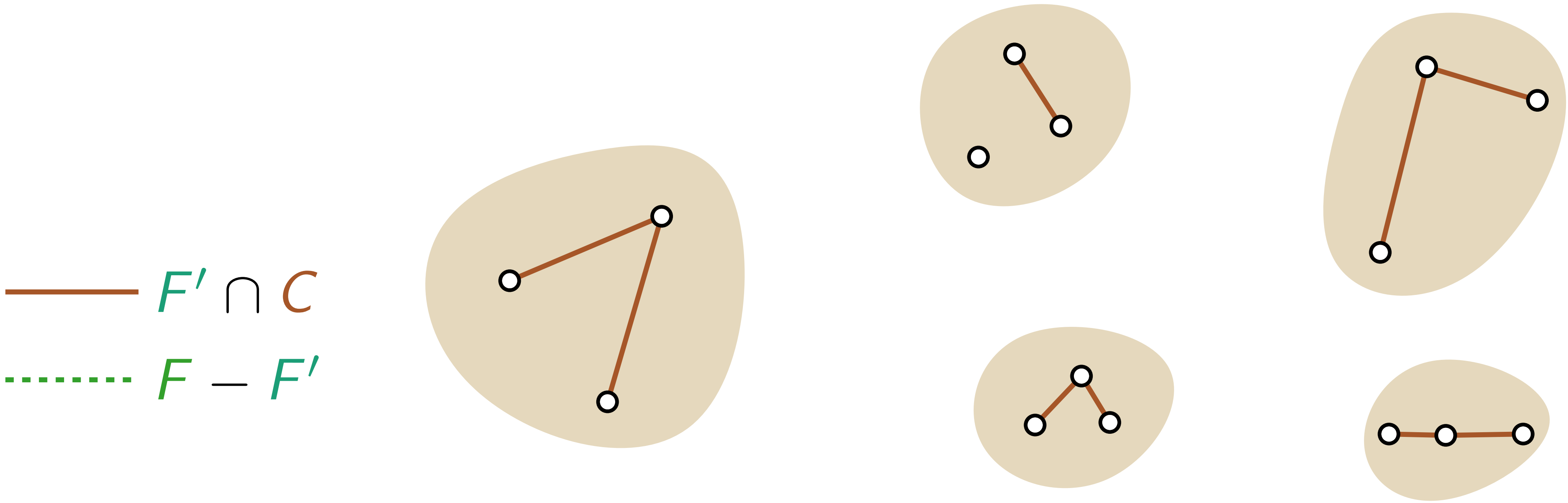

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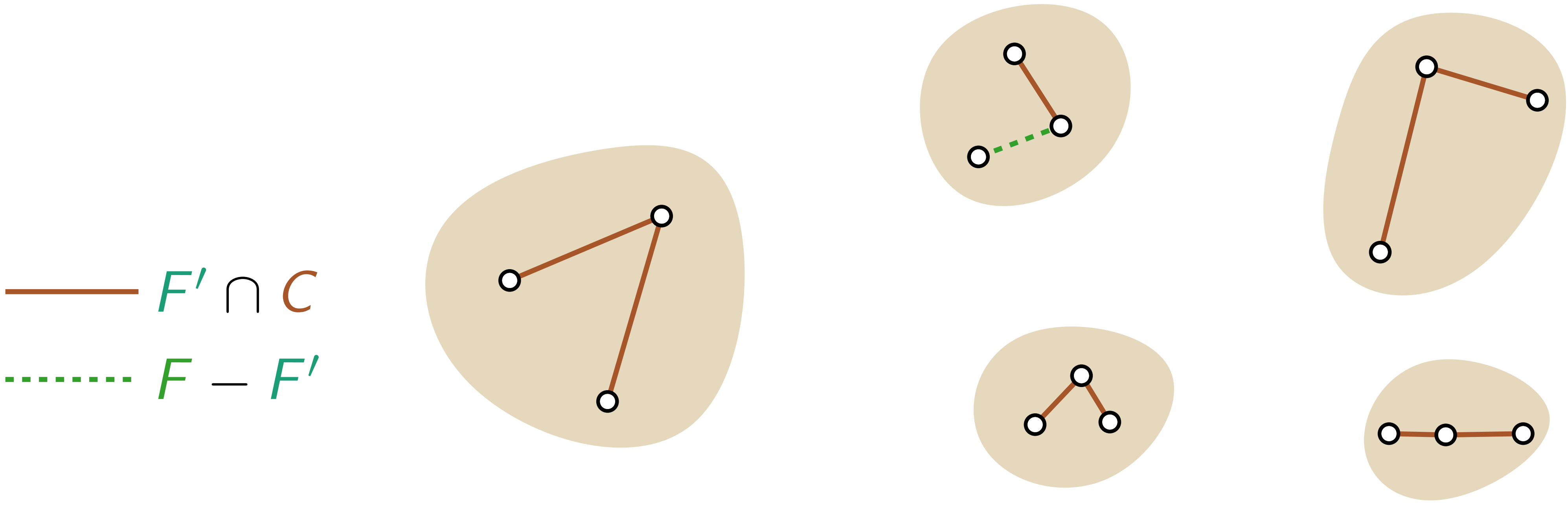

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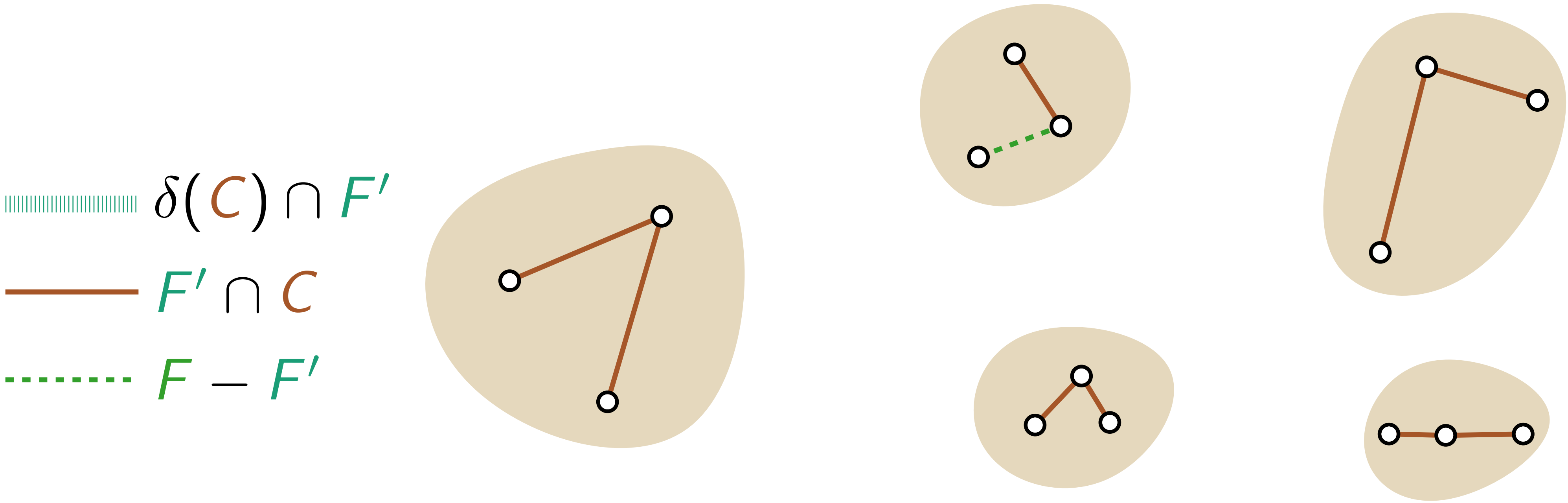

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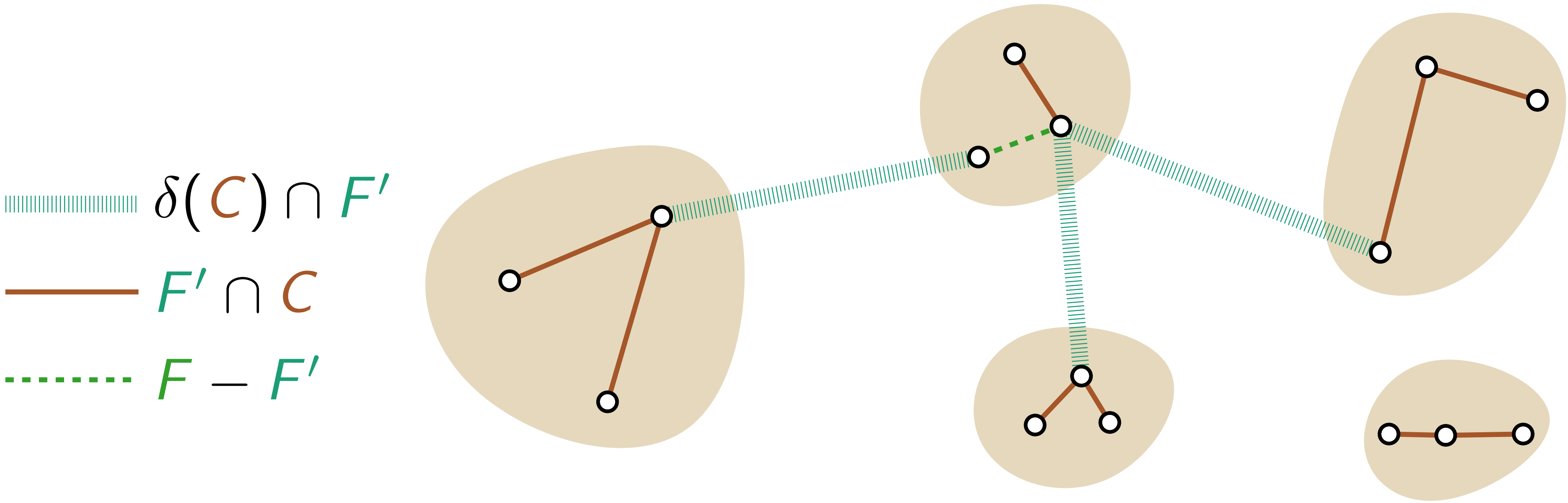

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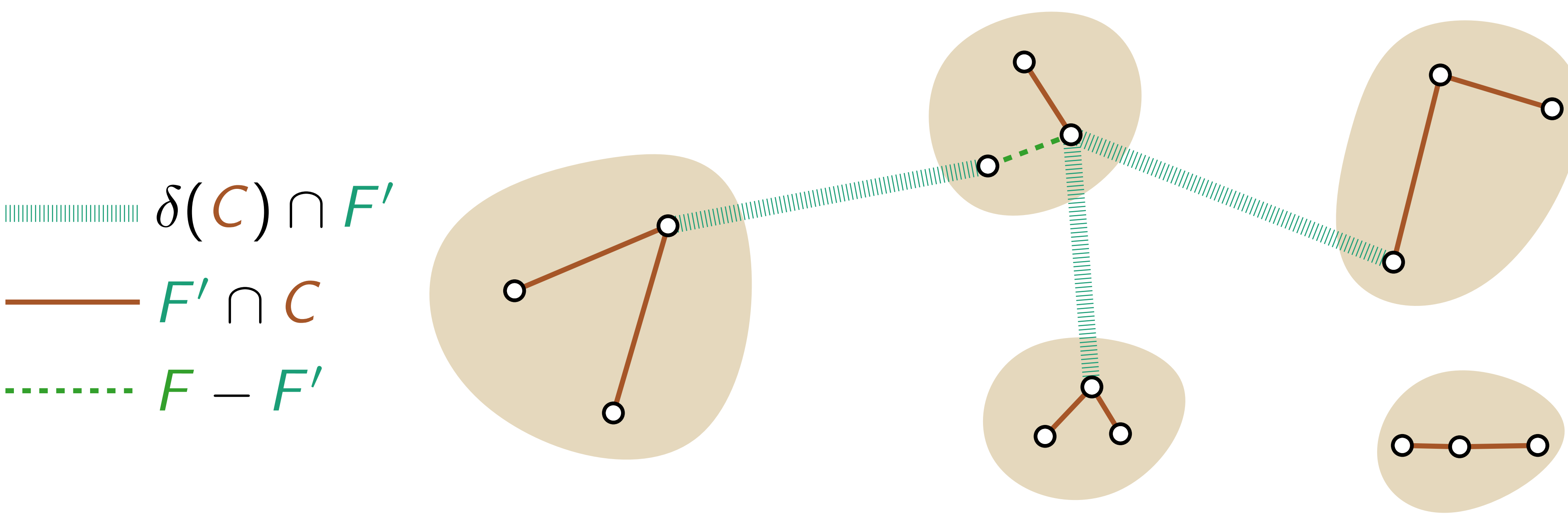

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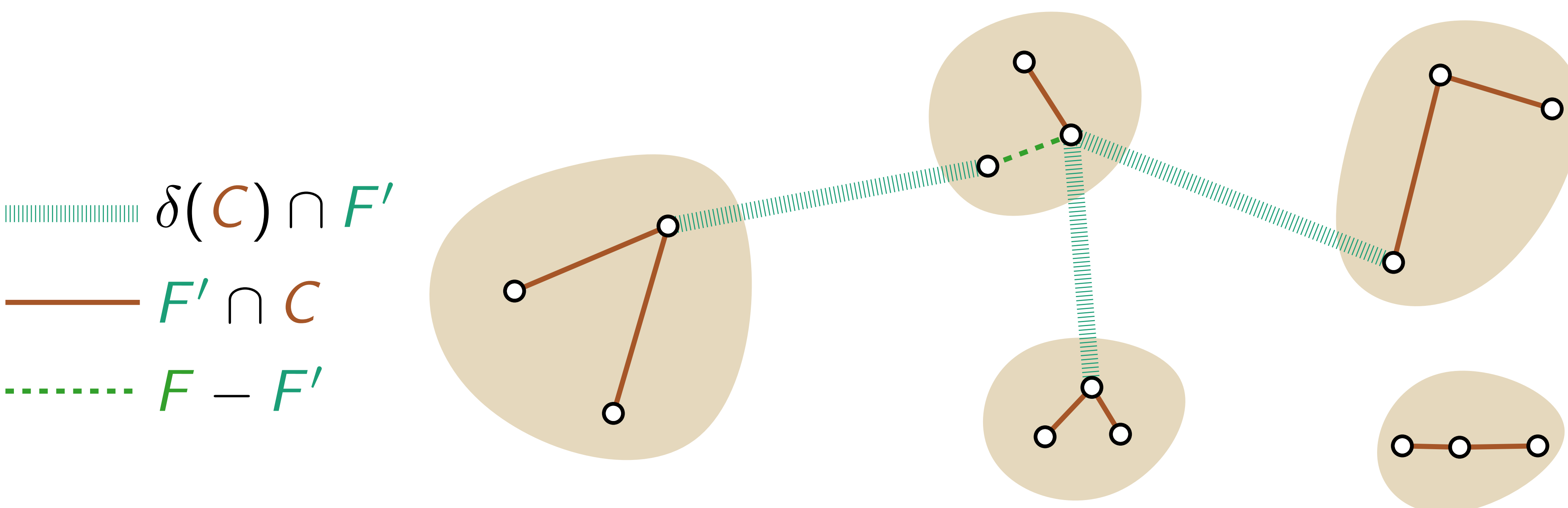

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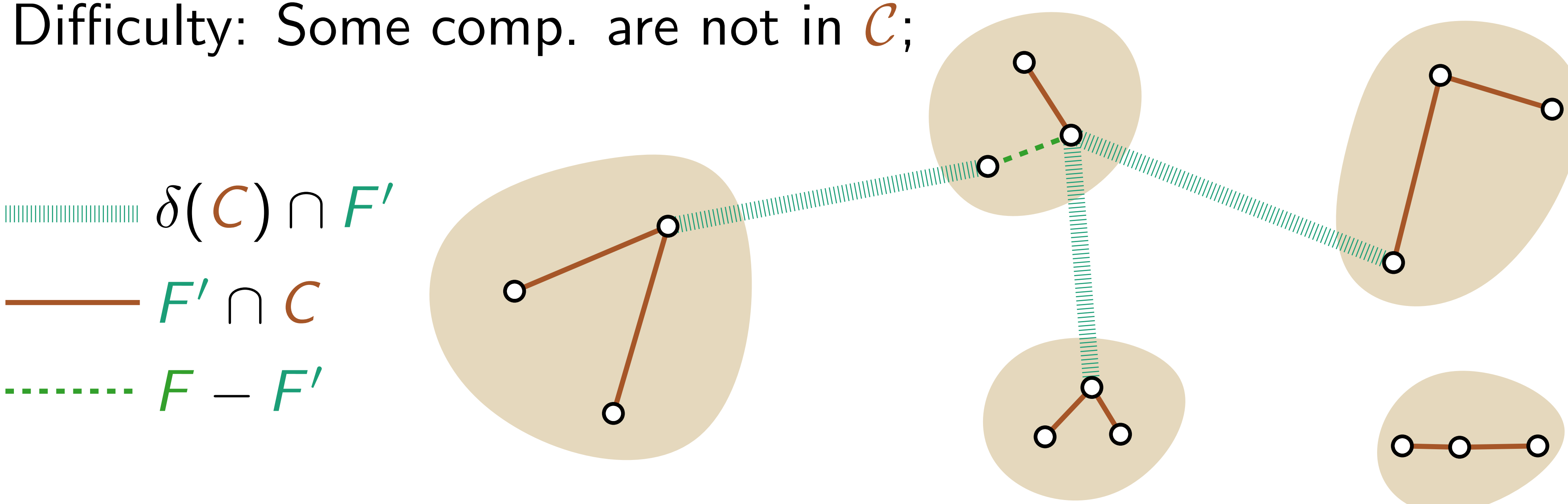

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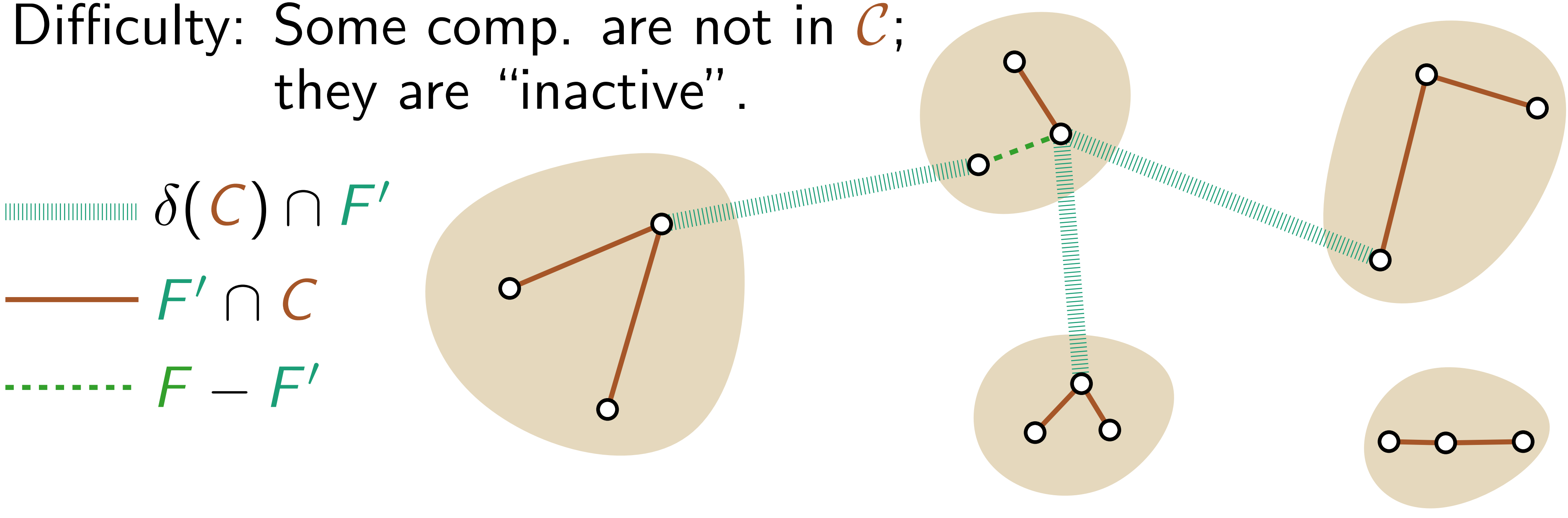

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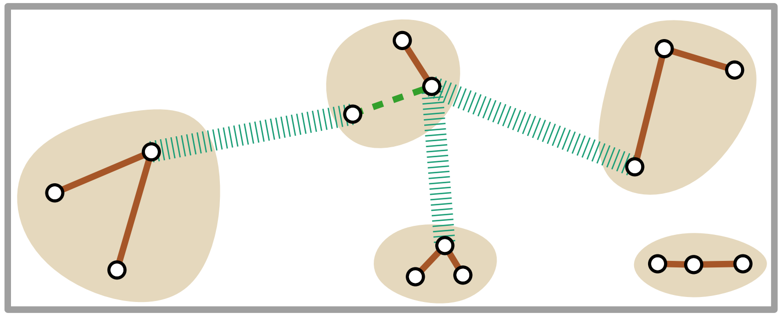

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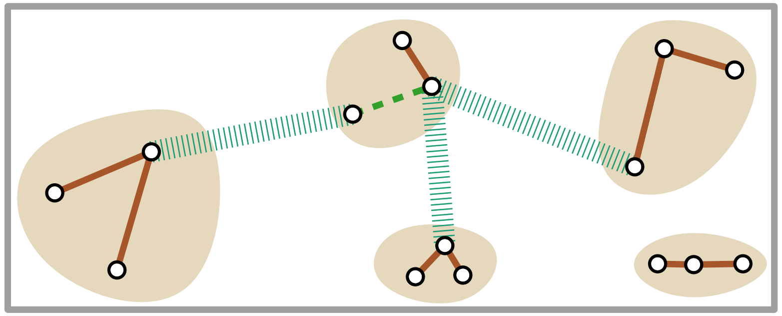

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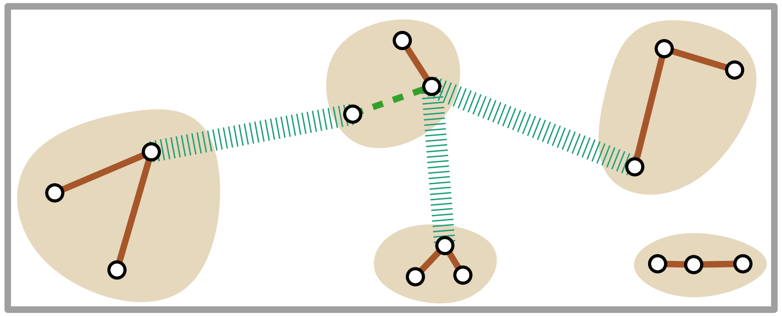

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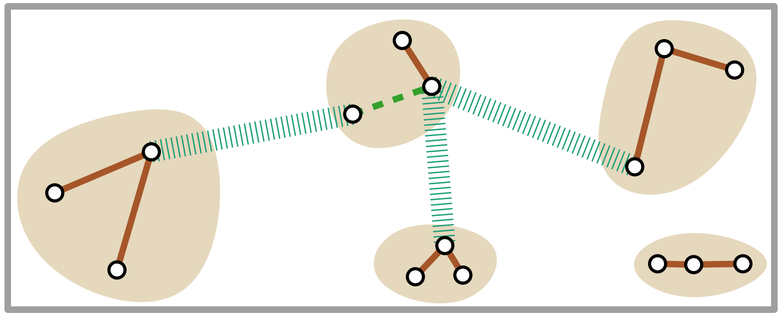

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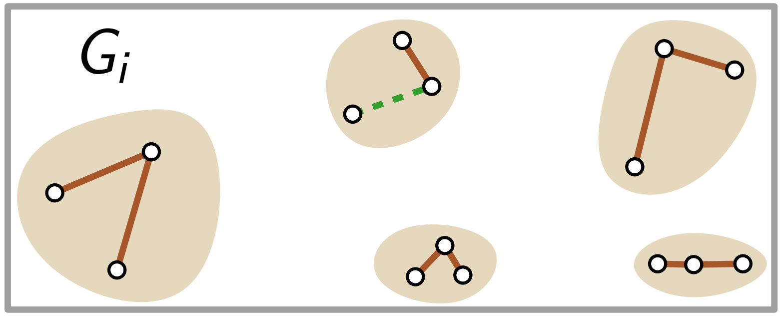

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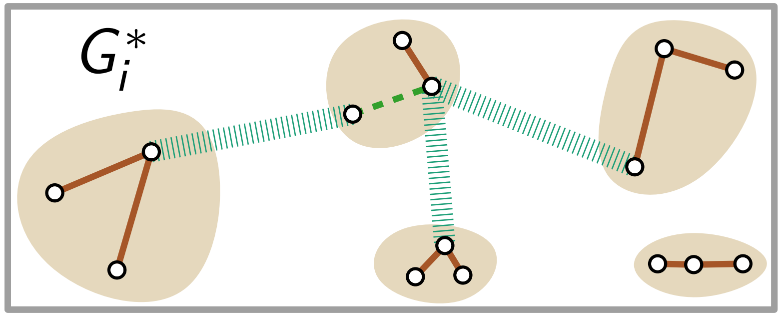

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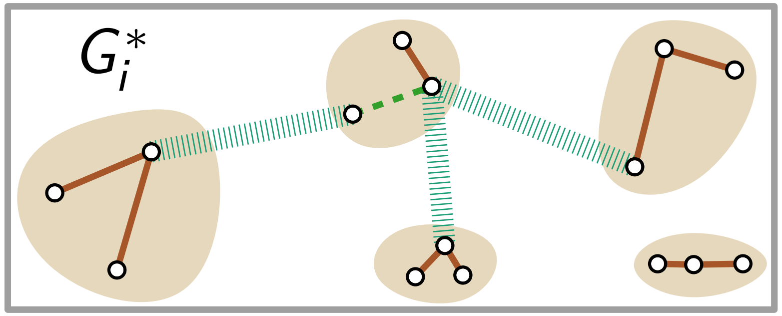

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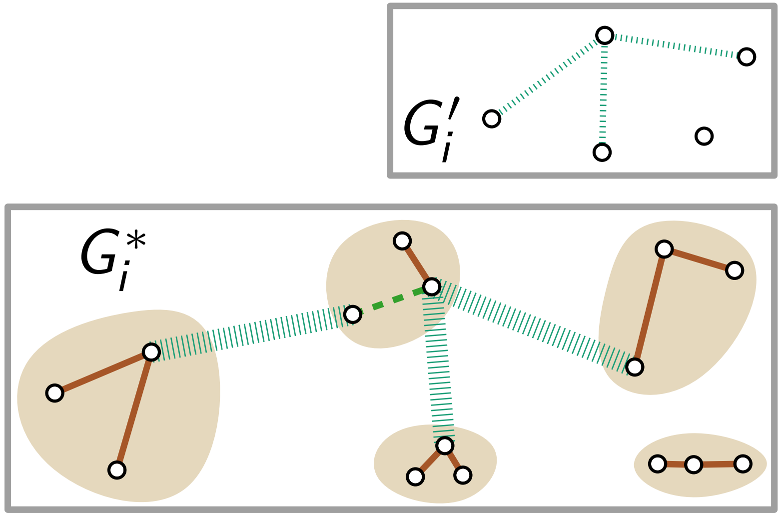

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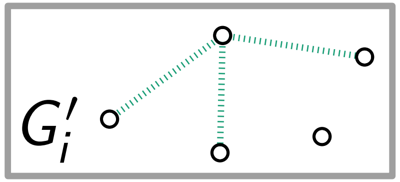

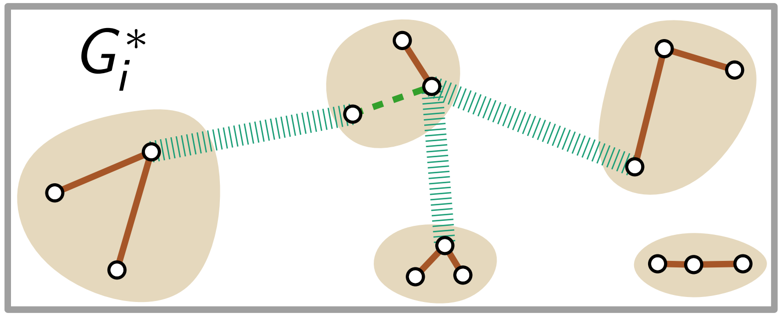

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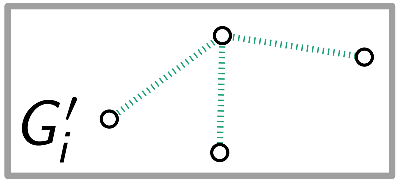

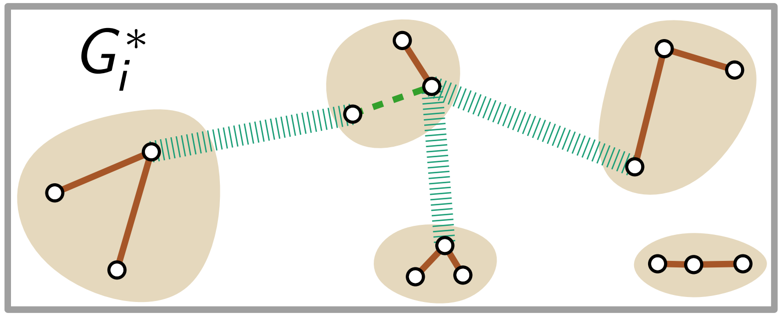

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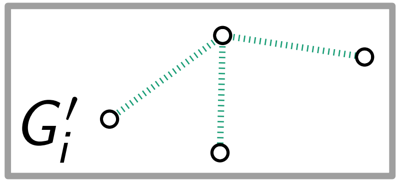

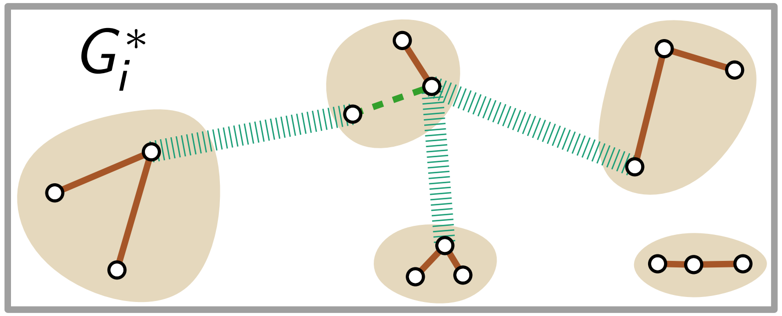

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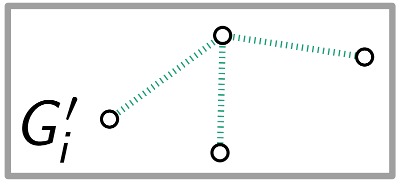

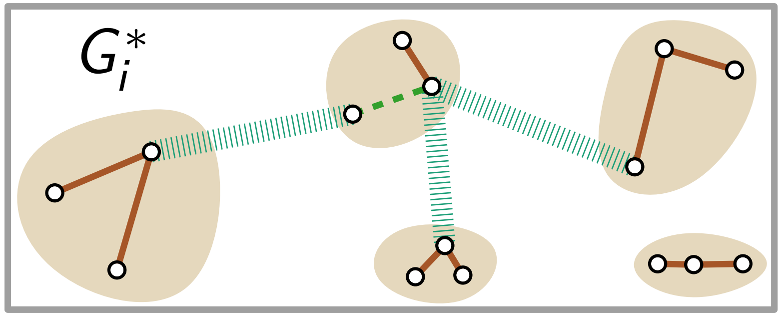

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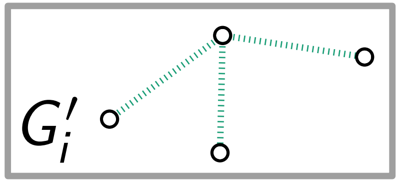

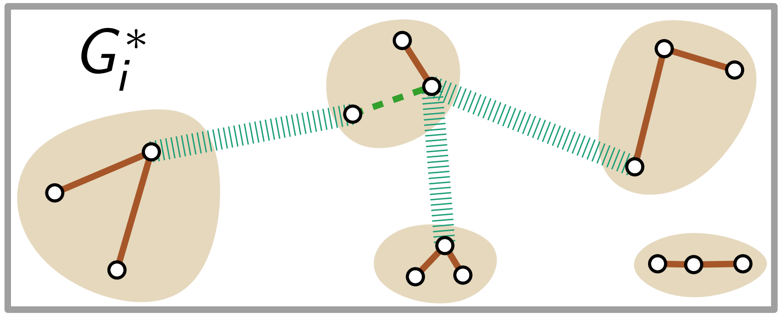

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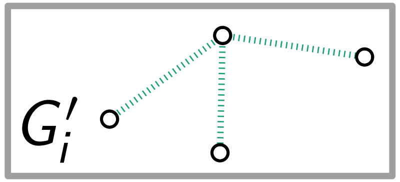

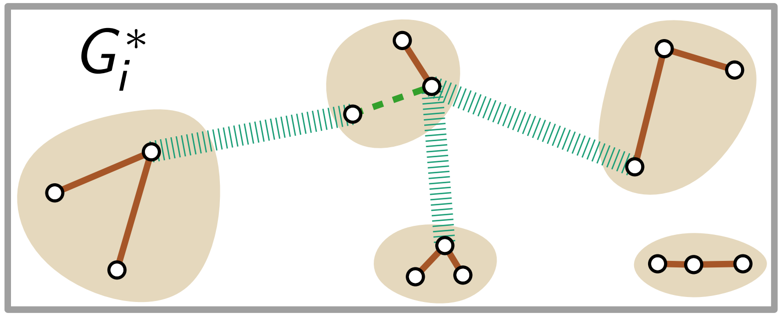

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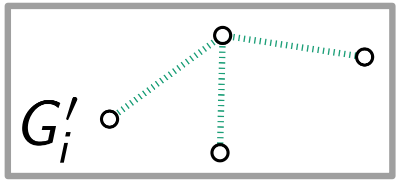

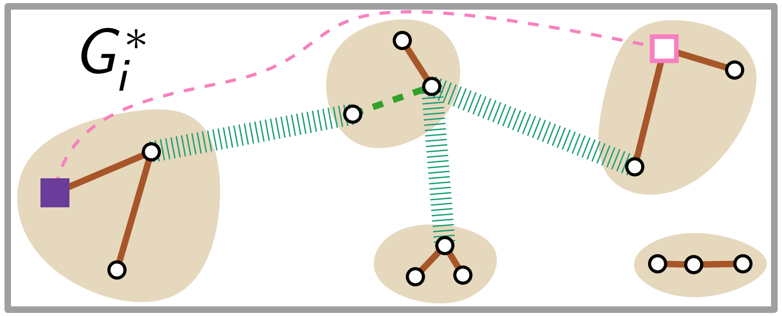

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Lemma. In any iteration of the algorithm, it holds that

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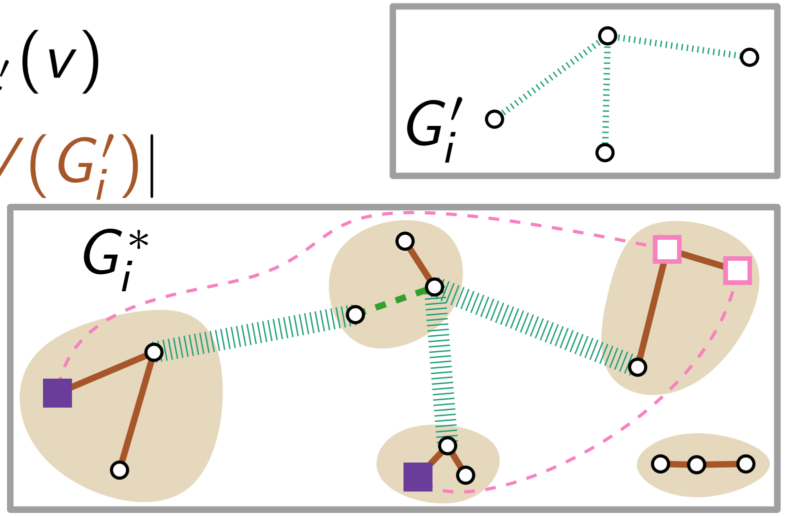

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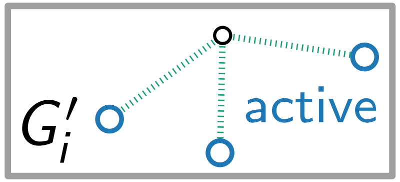

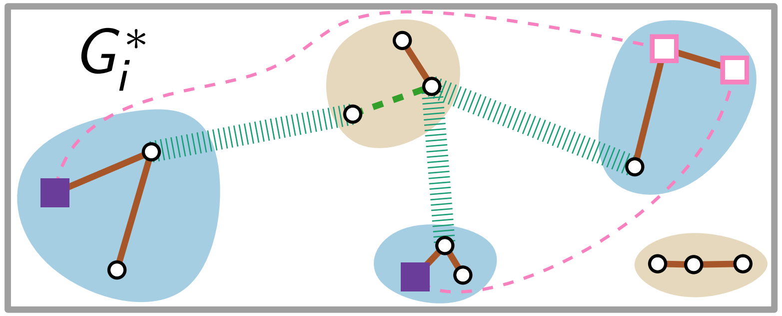

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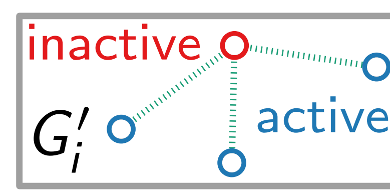

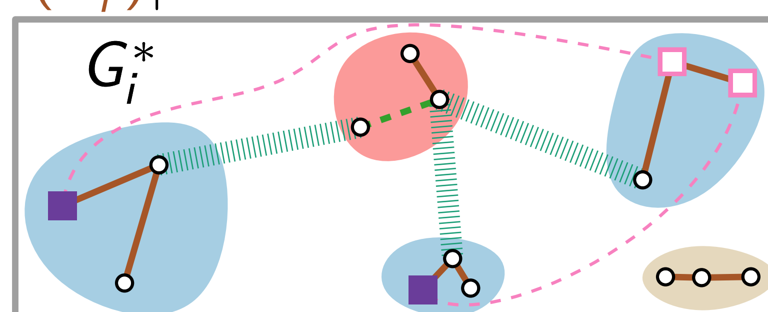

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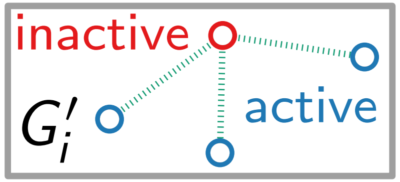

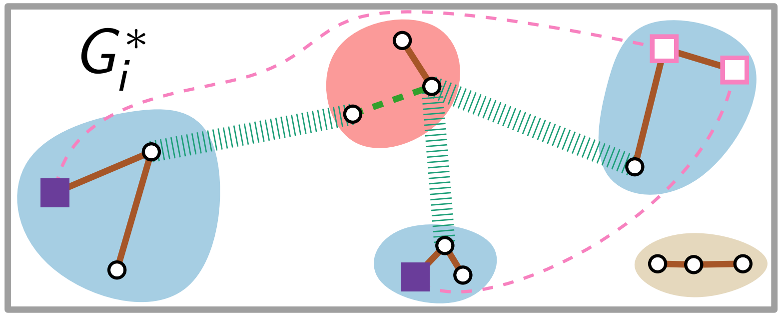

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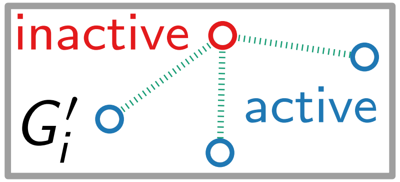

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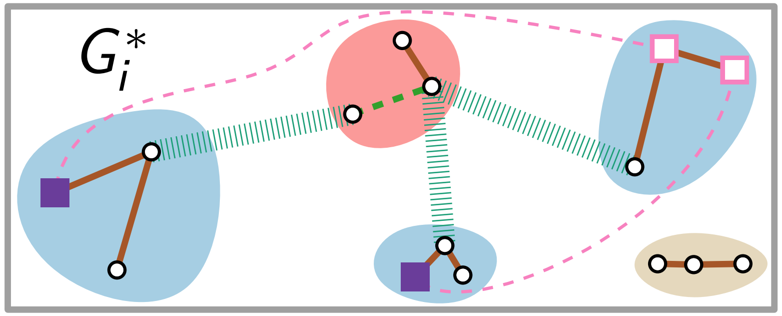

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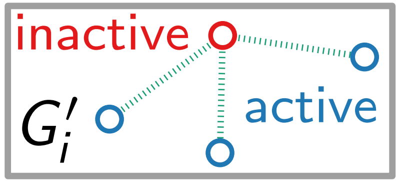

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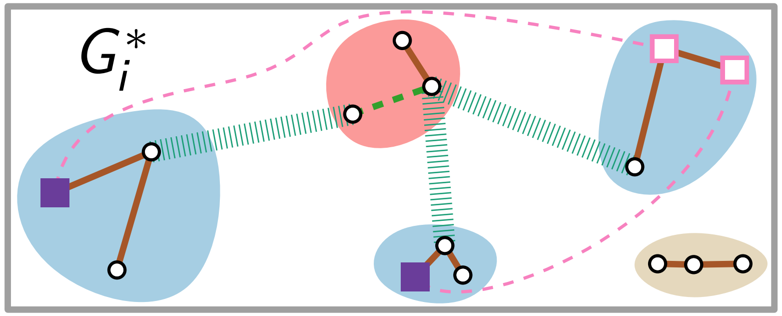

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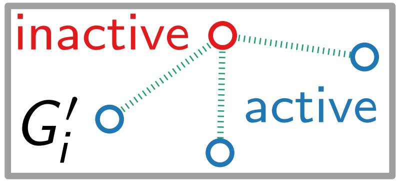

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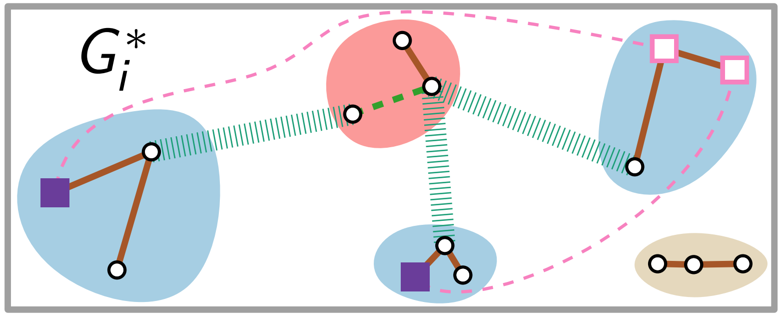

Approximation Algorithms

Lecture 12:
SteinerForest via Primal–Dual

Part VI:
Analysis

Analysis

Theorem. The Primal–Dual algorithm with synchronized increases yields a 2-approximation for SteinerForest.

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As mentioned before,

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From that, the claim of the theorem follows.

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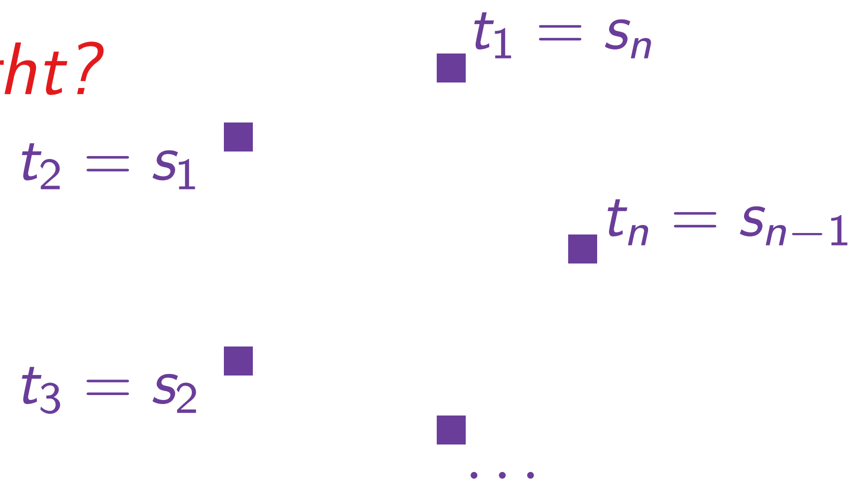

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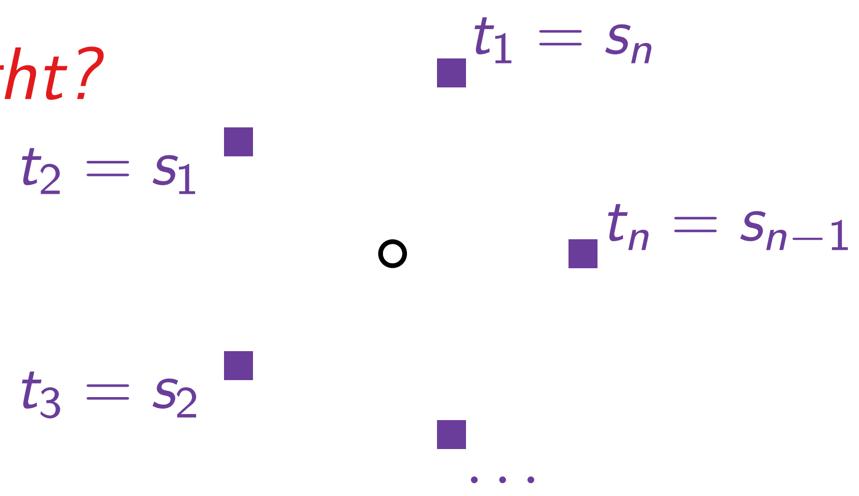

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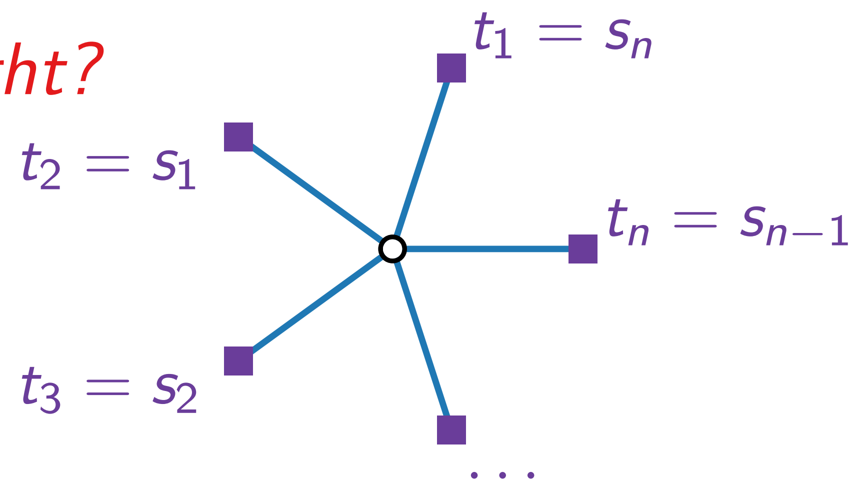

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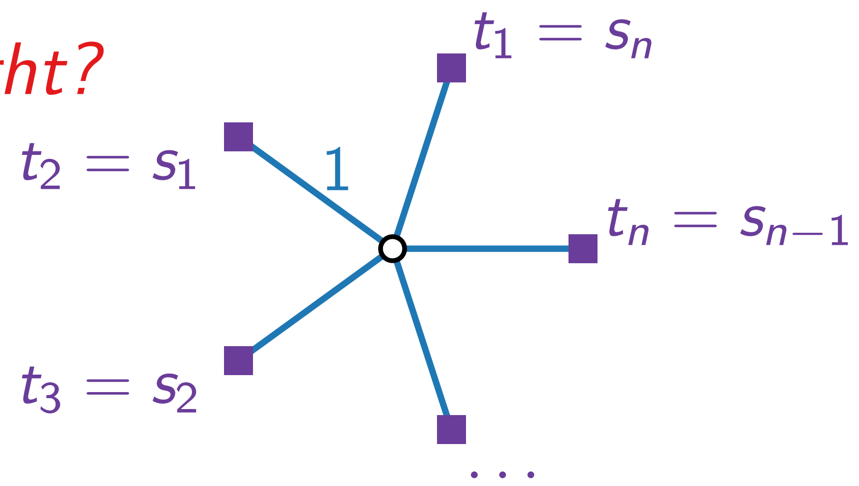

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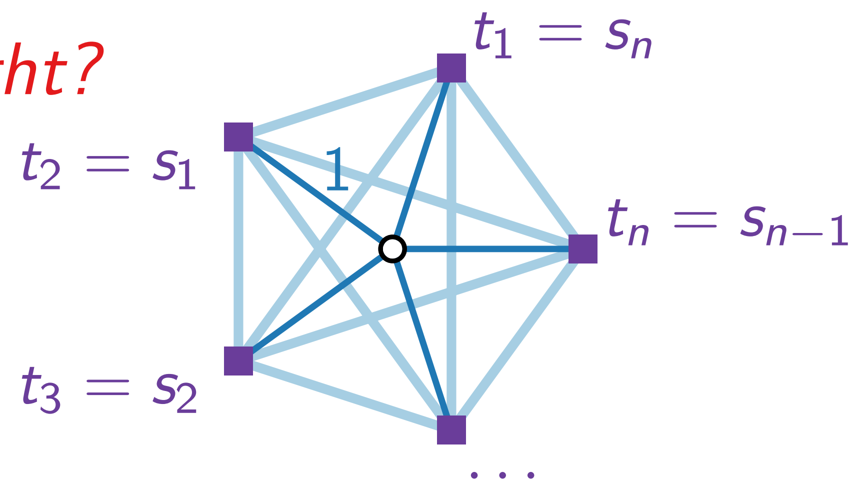

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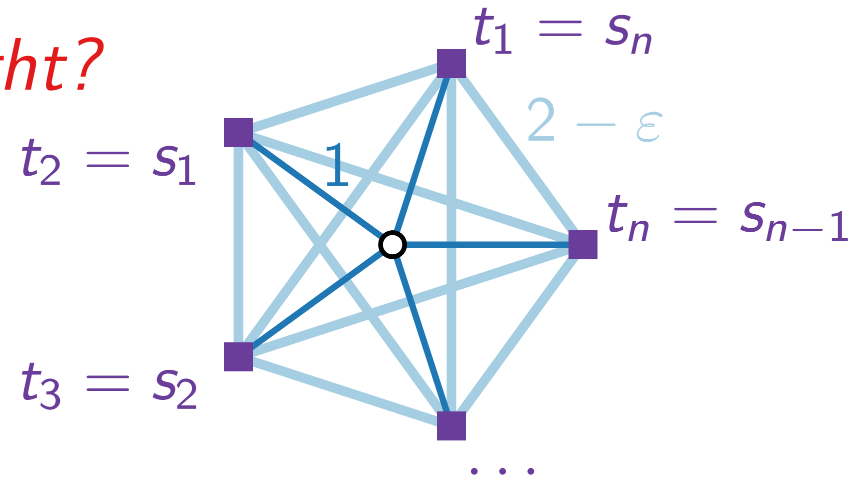

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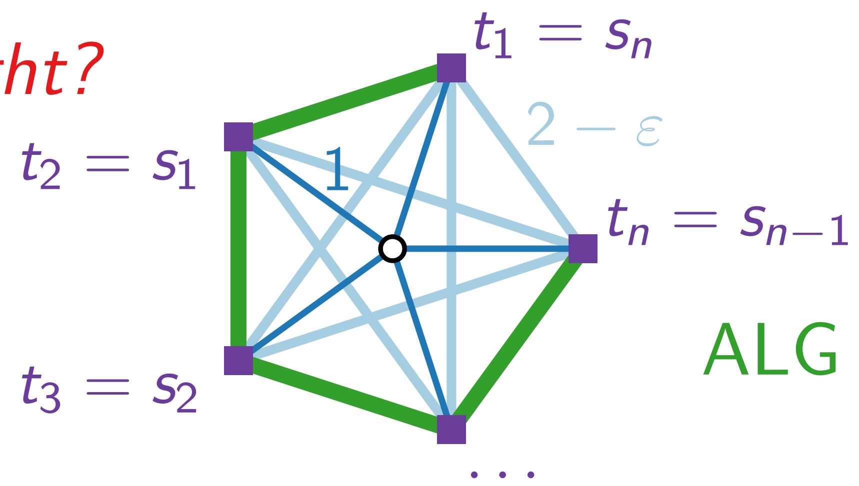

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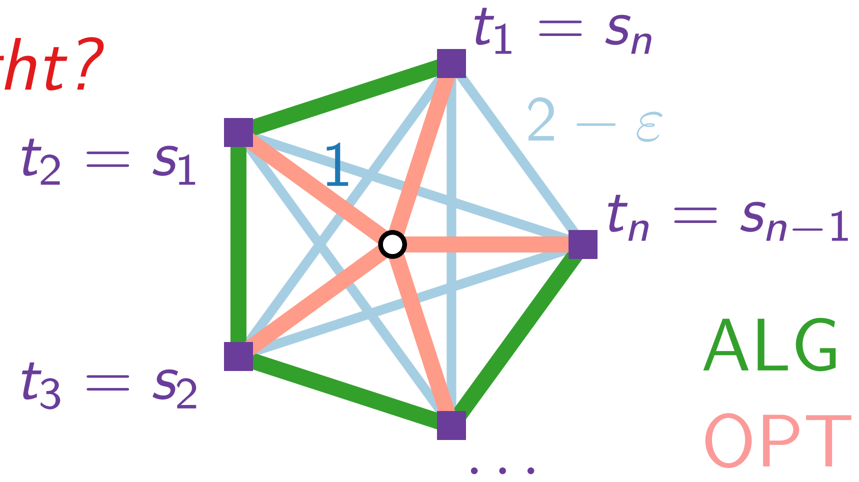

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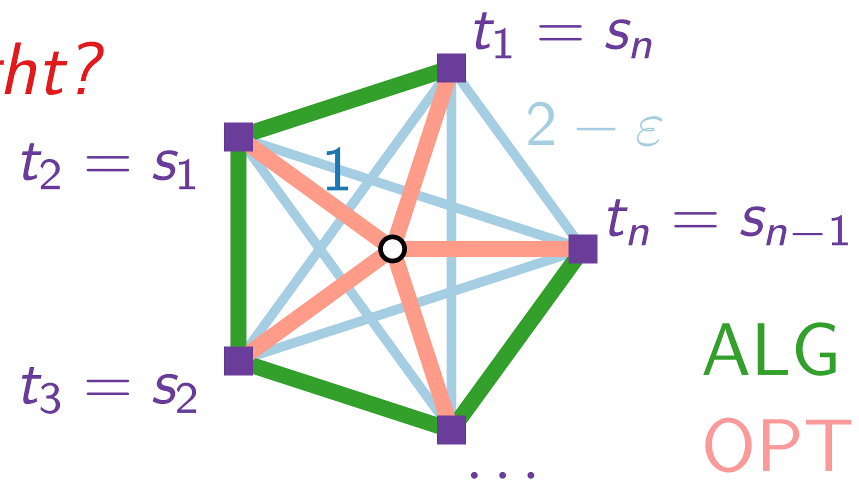

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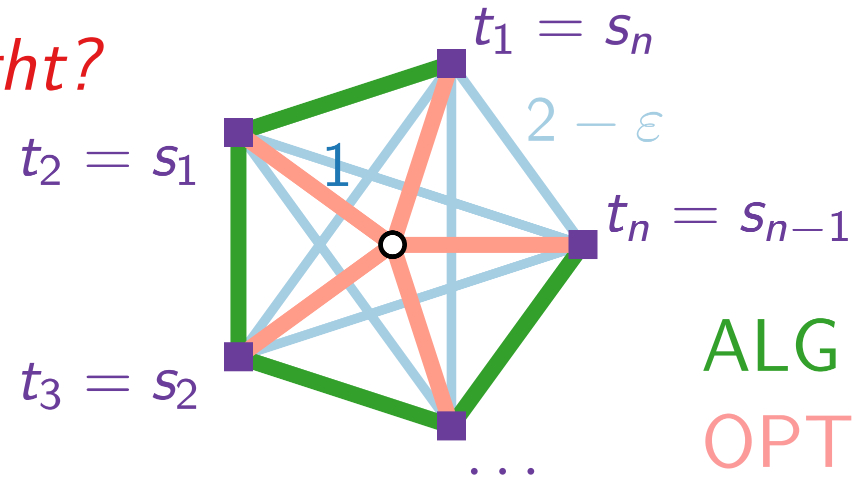

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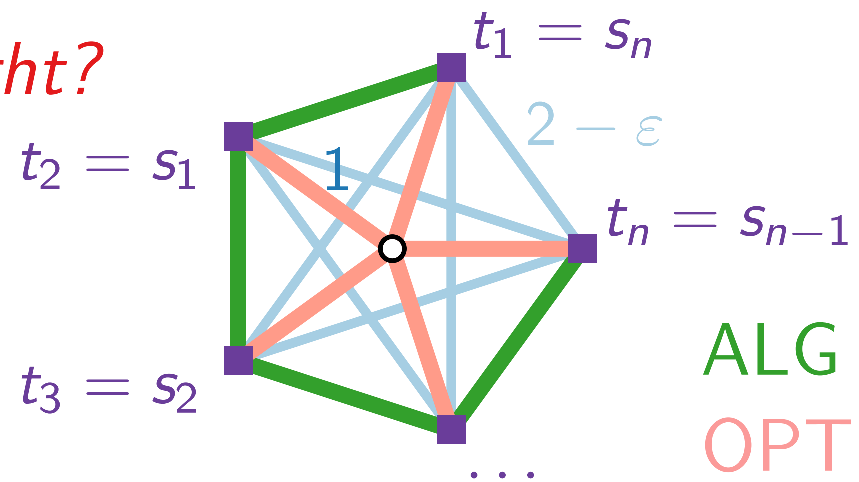

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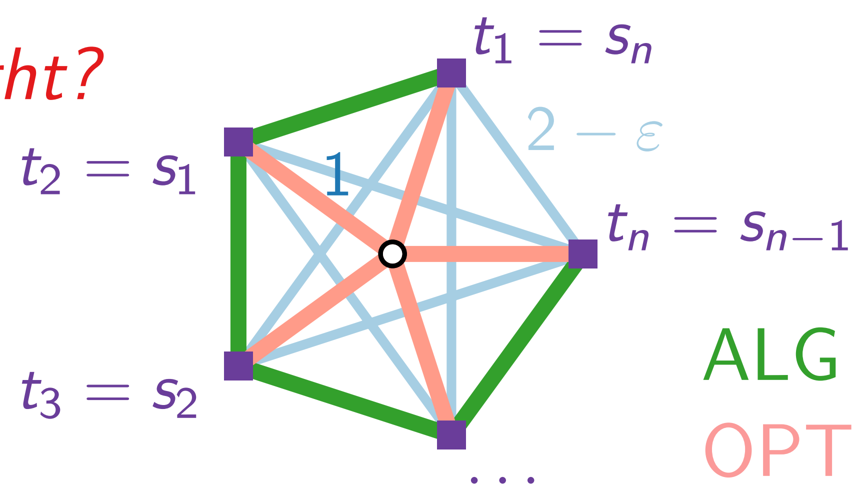

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