Approximation Algorithms

Lecture 11: MaxSat via Randomized Rounding

Part I: Maximum Satisfiability (MaxSat)

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Maximum Satisfiability (MaxSat)

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E.g., $(x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee \overline{x_4})$.

Maximum Satisfiability (MaxSat)

Given: Boolean variables $x_1, \ldots, x_n$ and
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Task: Find an assignment of the variables $x_1, \ldots, x_n$ such that the total weight of the *satisfied* clauses is **maximized**.

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Problem is NP-hard since Satisfiability (Sat) is NP-hard:
Is a given formula in conjunctive normal form satisfiable?
E.g., $(x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee \overline{x_4})$.

Approximation Algorithms

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Part II:
A Simple Randomized Algorithm

A Simple Randomized Algorithm

Theorem. Independently setting each variable to 1 (`true`) with probability provides an expected -approximation for MaxSat.

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Proof.

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Let $Y_j \in \{0, 1\}$ be a random variable for the truth value of clause C_j.
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$$\mathbf{E}[W] = \mathbf{E}\left[\sum_{j=1}^{m} w_j Y_j\right] =$$

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Thus, $\mathbf{E}[W] \geq \frac{1}{2} \sum_{j=1}^{m} w_j \geq \text{OPT}/2.$ □

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Part III:
Derandomization by Conditional Expectation

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$$\big(\mathbf{E}[W \mid x_1 = b_1, \ldots, x_i = b_i, x_{i+1} = 0]$$
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Consider a partial assignment $x_1 = b_1, \ldots, x_i = b_i$
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If C_j is already satisfied, then it contributes exactly to
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If C_j is not yet satisfied and contains k unassigned variables,
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The conditional expectation is simply the sum of the contributions from each clause. $\square$

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The algorithm simply chooses the best option at each step.

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Using *Conditional Expectation* is a standard procedure with which many randomized algorithms can be derandomized.

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Global optimization?

Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part IV:
Randomized Rounding

An ILP

Let $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x_i}$ for $j = 1, \ldots, m$.

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An ILP ... and Its Relaxation

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Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation.

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Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation. Independently setting each variable x_i to 1 with probability ___ provides a (___)-approximation for MaxSat.

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Let $\quad C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x_i} \quad$ for $j = 1, \ldots, m$.

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Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation. Independently setting each variable x_i to 1 with probability y_i^* provides a ( )-approximation for MAXSAT.

An ILP . . . and Its Relaxation

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Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation. Independently setting each variable x_i to 1 with probability y_i^* provides a $(1 - 1/e)$-approximation for MaxSat.

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Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation. Independently setting each variable x_i to 1 with probability y_i^* provides a $0.63 \approx (1 - 1/e)$-approximation for MaxSat.

Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part V:
Randomized Rounding – Proof

Mathematical Toolkit

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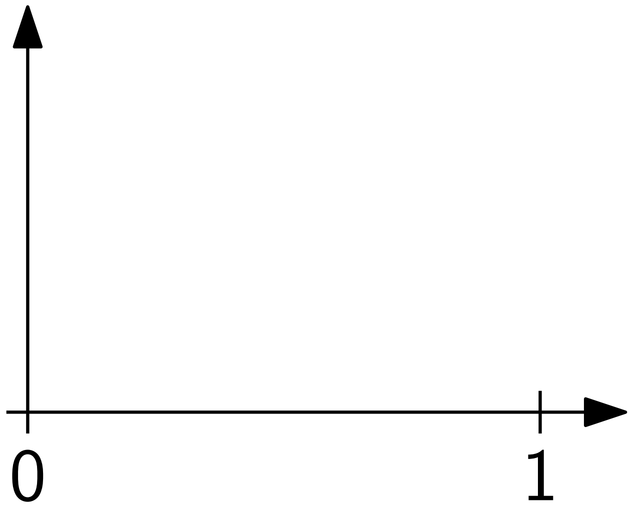

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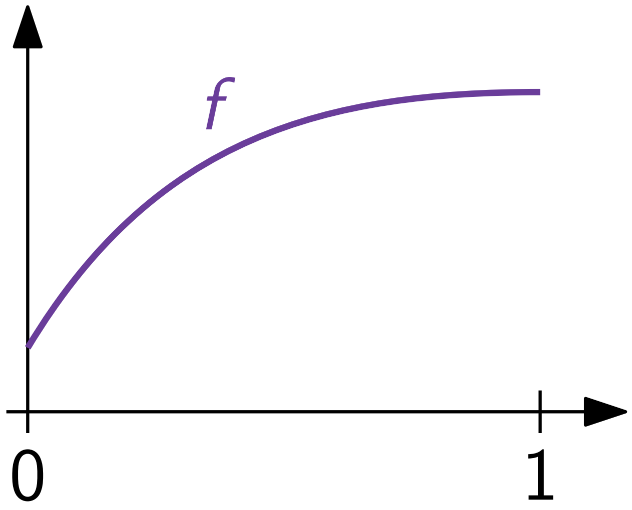

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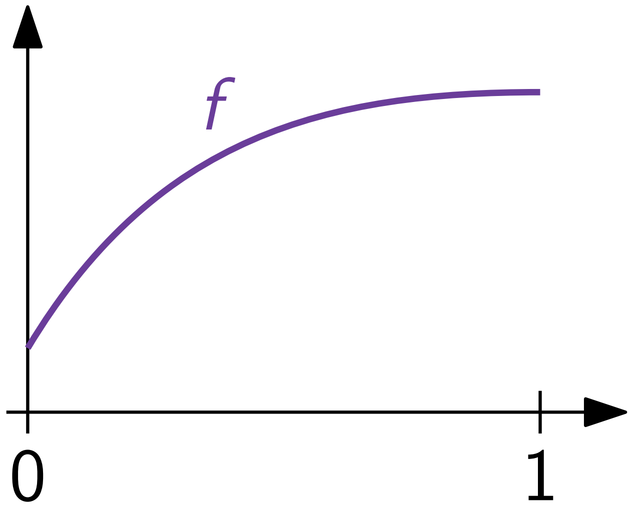

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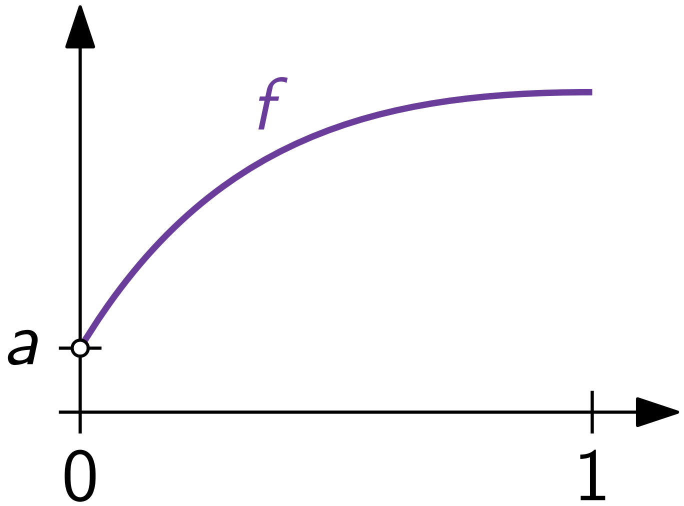

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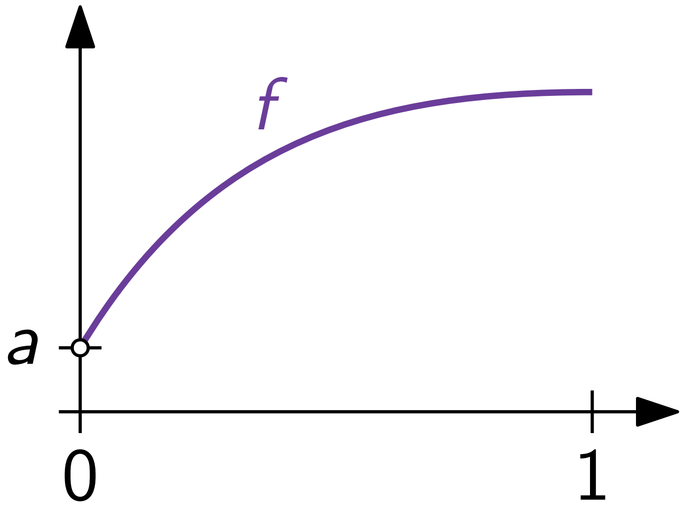

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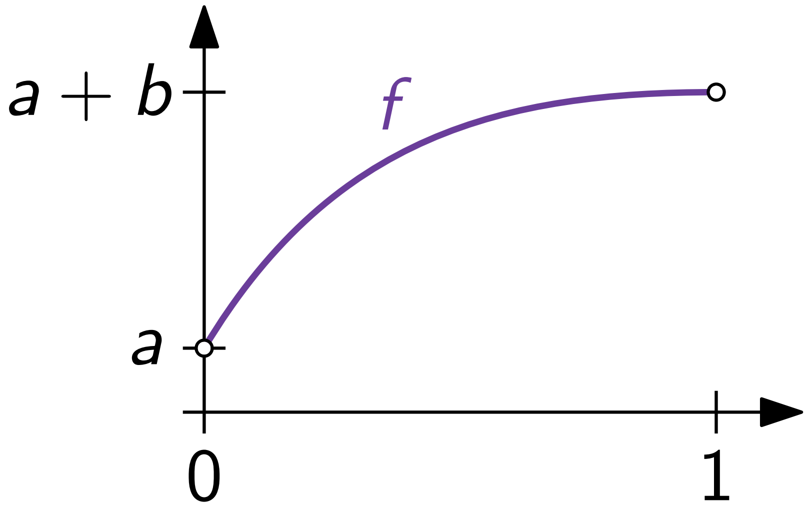

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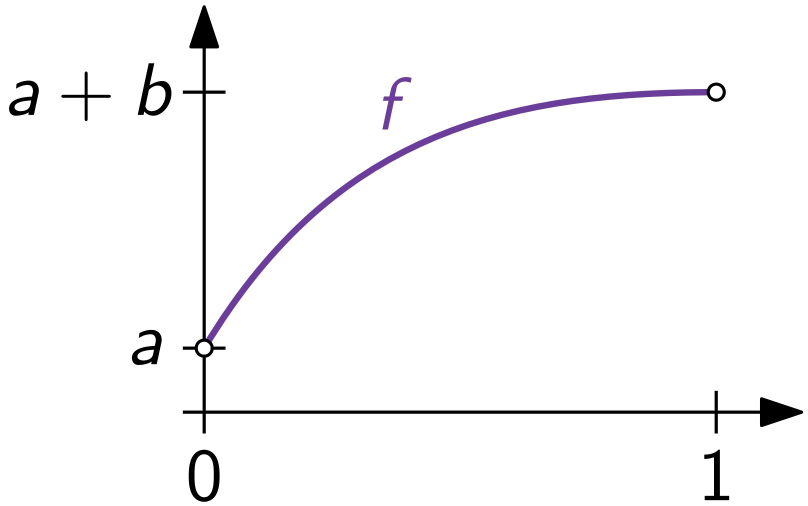

$\Rightarrow f(x) \geq bx + a$ for $x \in [0,1]$.

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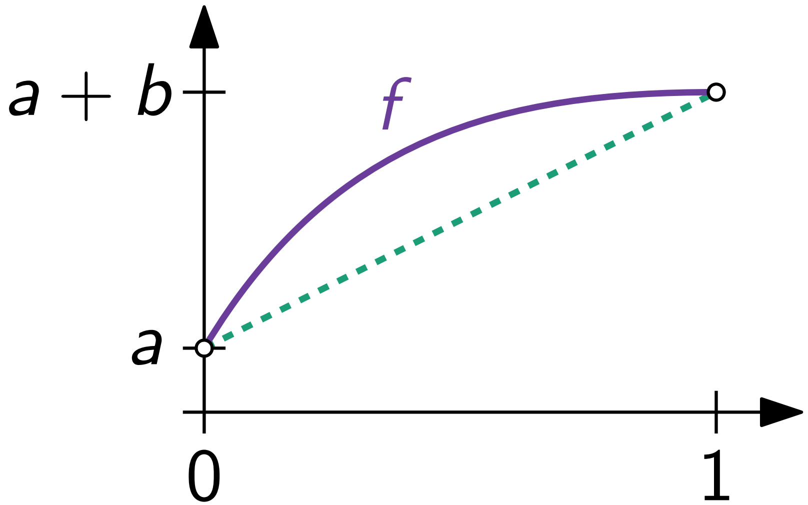

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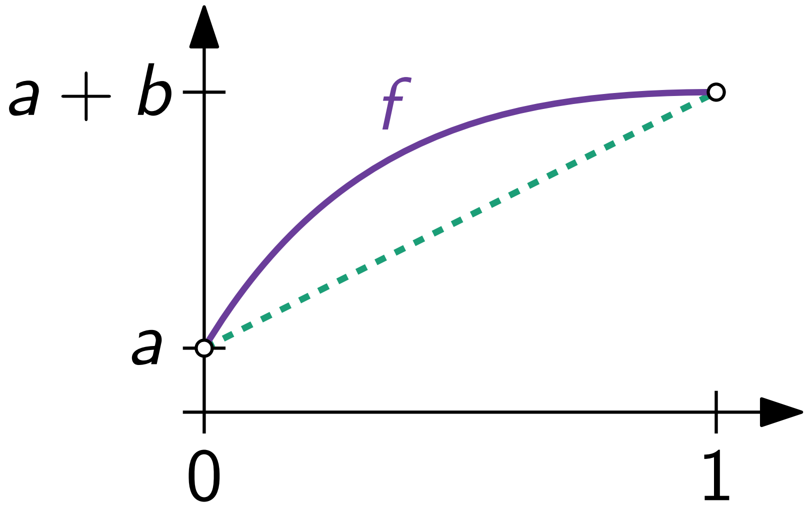

Arithmetic–Geometric Mean Inequality (AGMI):

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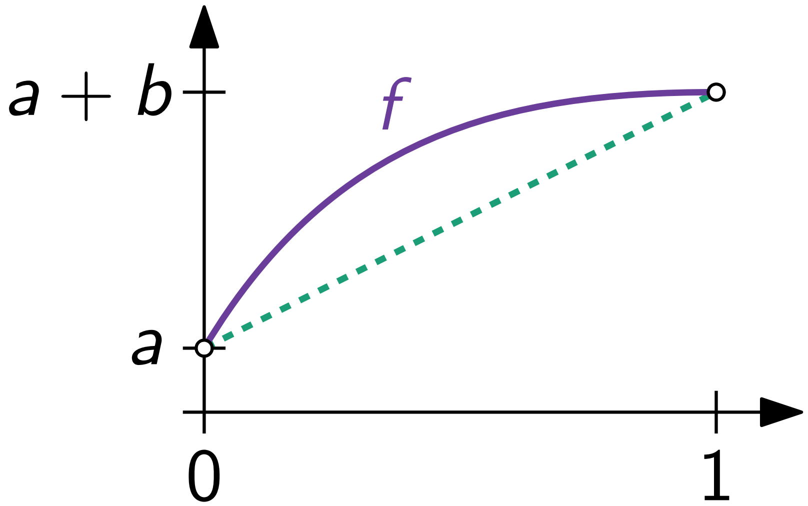

Arithmetic–Geometric Mean Inequality (AGMI):

For all non-negative numbers $a_1, \ldots, a_k$:

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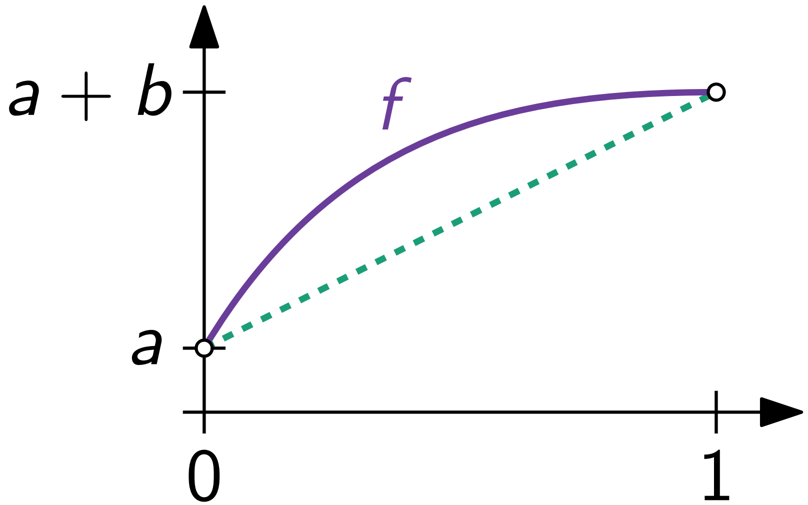

Arithmetic–Geometric Mean Inequality (AGMI):

For all non-negative numbers $a_1, \ldots, a_k$:

$$\leq \quad \frac{1}{k}\left(\sum_{i=1}^{k} a_i\right)$$

Mathematical Toolkit

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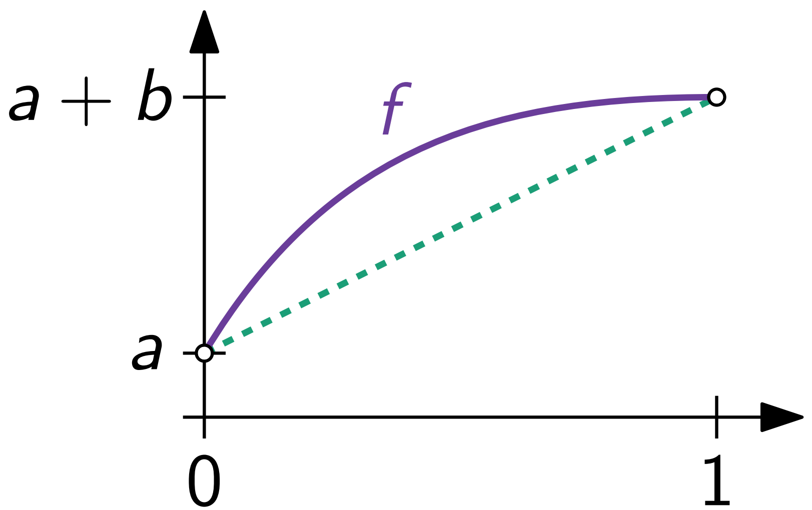

Arithmetic–Geometric Mean Inequality (AGMI):

For all non-negative numbers $a_1, \ldots, a_k$:

$$\left(\prod_{i=1}^{k} a_i\right)^{1/k} \leq \frac{1}{k}\left(\sum_{i=1}^{k} a_i\right)$$

Randomized Rounding (Proof)

Consider a fixed clause C_j of length ℓ_j. Then we have:

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AGMI

$$\leq \left(\sum_{i \in P_j} (1 - y_i^*) + \sum_{i \in N_j} y_i^* \right)$$

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AGMI

$$\leq \left[\frac{1}{\ell_j}\left(\sum_{i\in P_j}(1 - y_i^*) + \sum_{i\in N_j} y_i^*\right)\right]^{\ell_j}$$

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AGMI

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Randomized Rounding (Proof)

Consider a fixed clause C_j of length ℓ_j. Then we have:

$$\mathbf{Pr}[C_j \text{ not sat.}] = \prod_{i \in P_j} (1 - y_i^*) \prod_{i \in N_j} y_i^*$$

$$\left(\prod_{i=1}^{k} a_i \right)^{1/k} \leq \frac{1}{k} \left(\sum_{i=1}^{k} a_i \right)$$

AGMI

$$\leq \left[\frac{1}{\ell_j} \left(\sum_{i \in P_j} (1 - y_i^*) + \sum_{i \in N_j} y_i^* \right) \right]^{\ell_j}$$

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&\geq \left(1 - \frac{1}{e}\right) z_j^*
\end{aligned}$$

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Therefore

$$\mathbf{E}[W] = \sum_{j=1}^{m} \mathbf{Pr}[C_j \text{ satisfied}] \cdot w_j$$

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$$\geq \left(1 - \frac{1}{e}\right) \boxed{\sum_{j=1}^{m} w_j z_j^*}$$ LP objective function

$$=$$

Randomized Rounding (Proof)

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$$\geq \left(1 - \frac{1}{e}\right) \mathrm{OPT} \qquad \square$$

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Therefore

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&= \left(1 - \frac{1}{e}\right) \mathrm{OPT}_{\mathrm{LP}} \\
&\geq \left(1 - \frac{1}{e}\right) \mathrm{OPT}
\end{aligned}$$

□

Theorem. The previous algorithm can be derandomized by the method of conditional expectation.

Approximation Algorithms

Lecture 11: MaxSat via Randomized Rounding

Part VI: Combining the Algorithms

Take the better of the two solutions!

Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a   -approximation for MaxSat.

Take the better of the two solutions!

Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a $3/4$-approximation for MaxSat.

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Proof.

We use another probabilistic argument.
With probability $1/2$, choose the solution of the first algorithm; otherwise the solution of the second algorithm.

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Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a $3/4$-approximation for MaxSat.

Proof.

We use another probabilistic argument.
With probability $1/2$, choose the solution of the first algorithm; otherwise the solution of the second algorithm.

The better solution is at least as good as the expectation of the above randomized algorithm.

Take the better of the two solutions!

The probability that clause C_j is satisfied is at least:

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$$\frac{1}{2}\left[\qquad\qquad\qquad + \qquad\qquad \right].$$

Take the better of the two solutions!

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$$\frac{1}{2}\left[\underbrace{\left(1-\left(1-\frac{1}{\ell_j}\right)^{\ell_j}\right)z_j^*}_{\text{LP-rounding}} + \qquad\qquad\right].$$

Take the better of the two solutions!

The probability that clause C_j is satisfied is at least:

$$\frac{1}{2}\left[\underbrace{\left(1-\left(1-\frac{1}{\ell_j}\right)^{\ell_j}\right) z_j^*}_{\text{LP-rounding}} + \underbrace{\left(1-2^{-\ell_j}\right)}_{\text{rand. alg.}}\right].$$

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For $\ell_j \geq 3$, $1-(1-1/\ell_j)^{\ell_j} \geq$ and $1-2^{-\ell_j} \geq$

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Thus, we have at least:

$$\frac{1}{2}\left[\left(1-\frac{1}{e}\right)+\frac{7}{8}\right] z_j^* \approx$$

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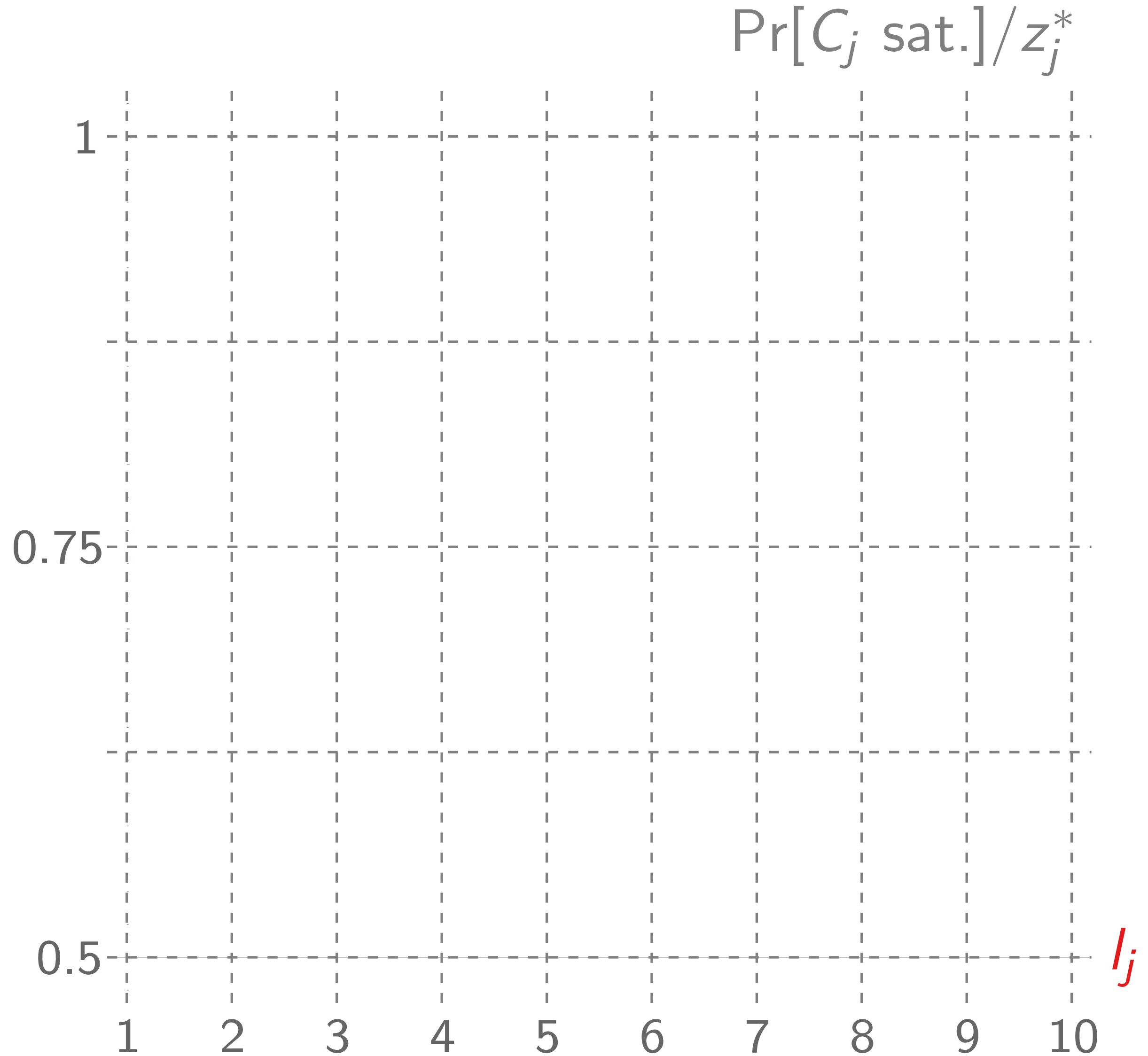
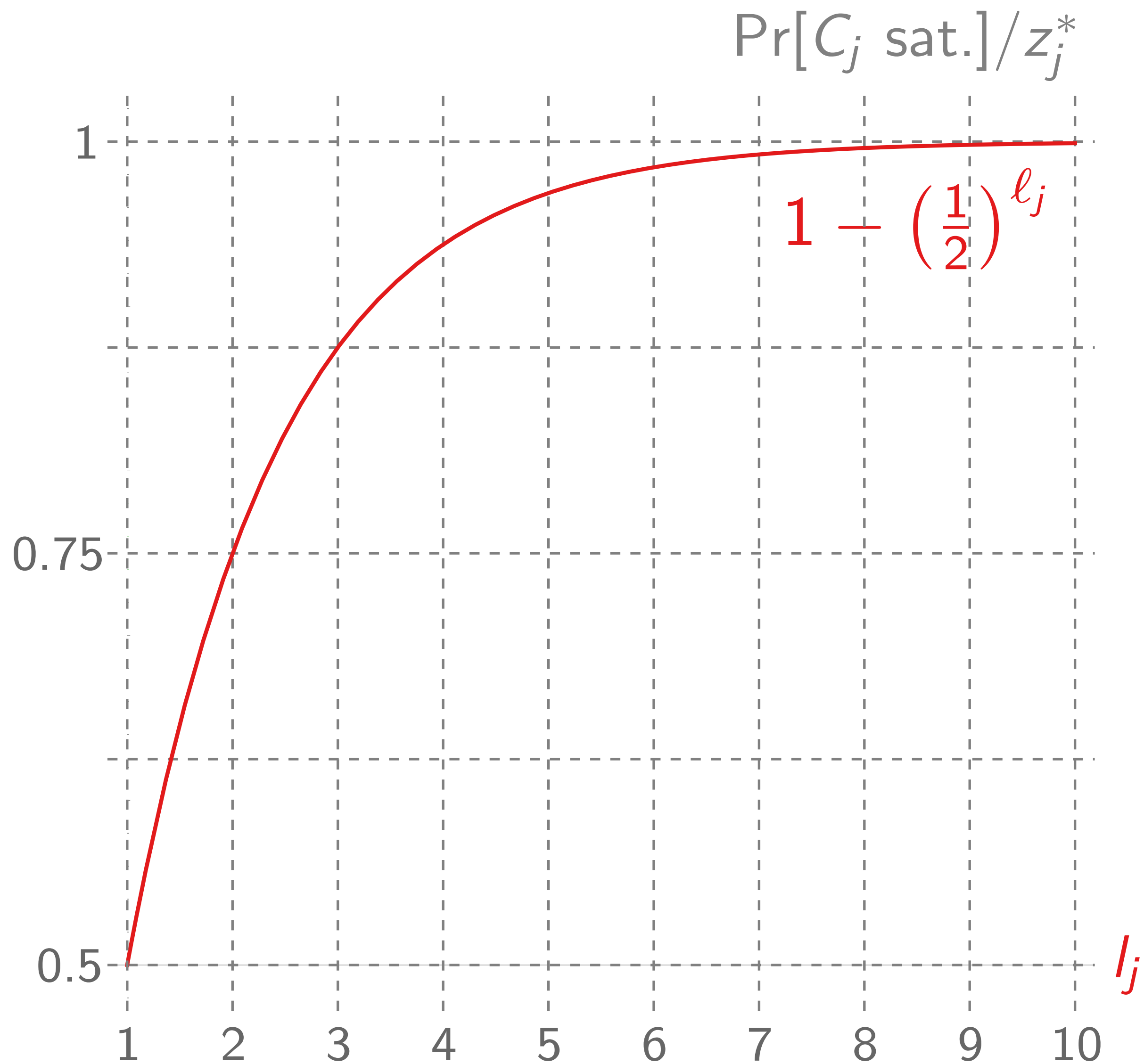
Visualization and Derandomization

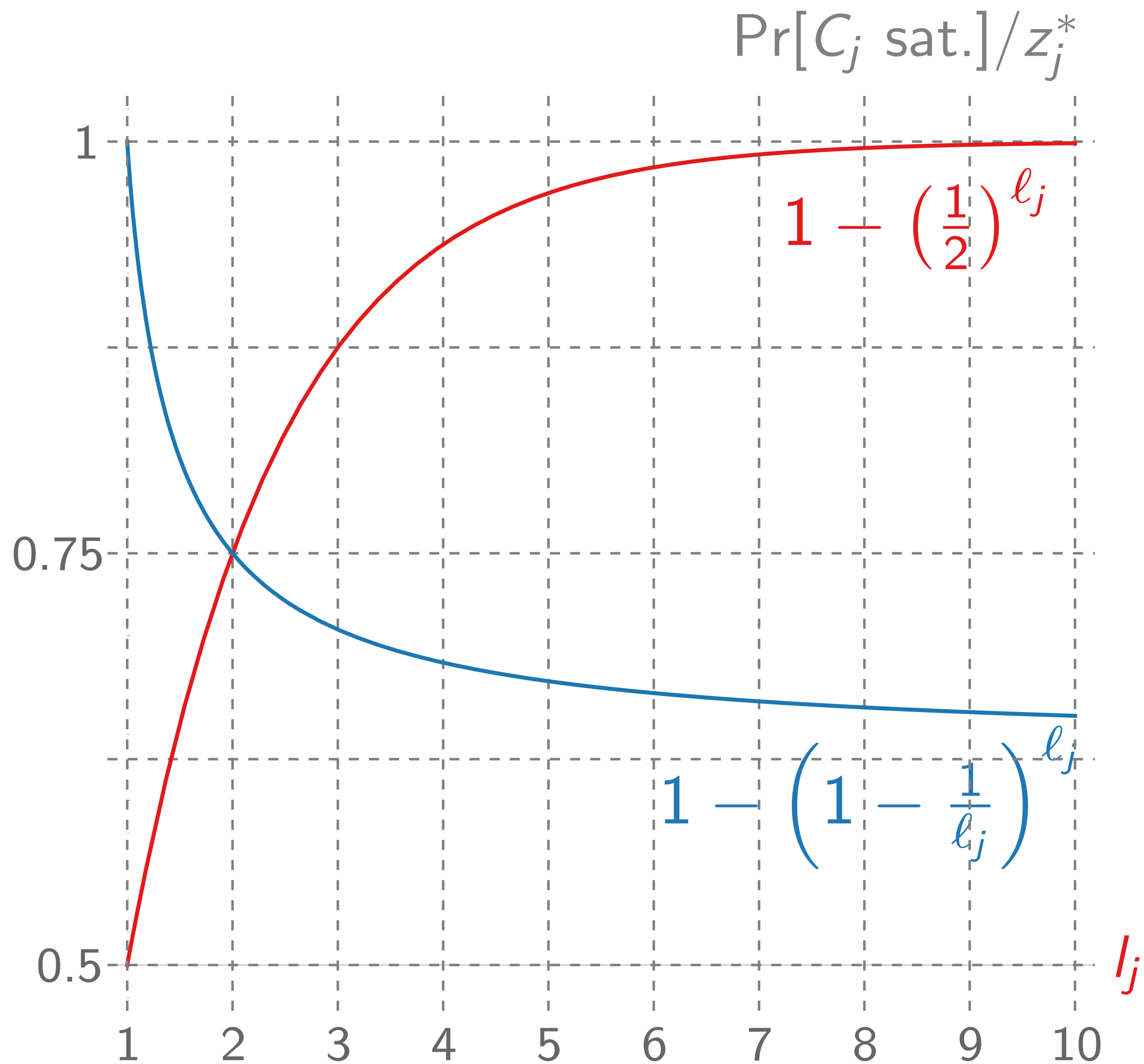
Visualization and Derandomization



Visualization and Derandomization

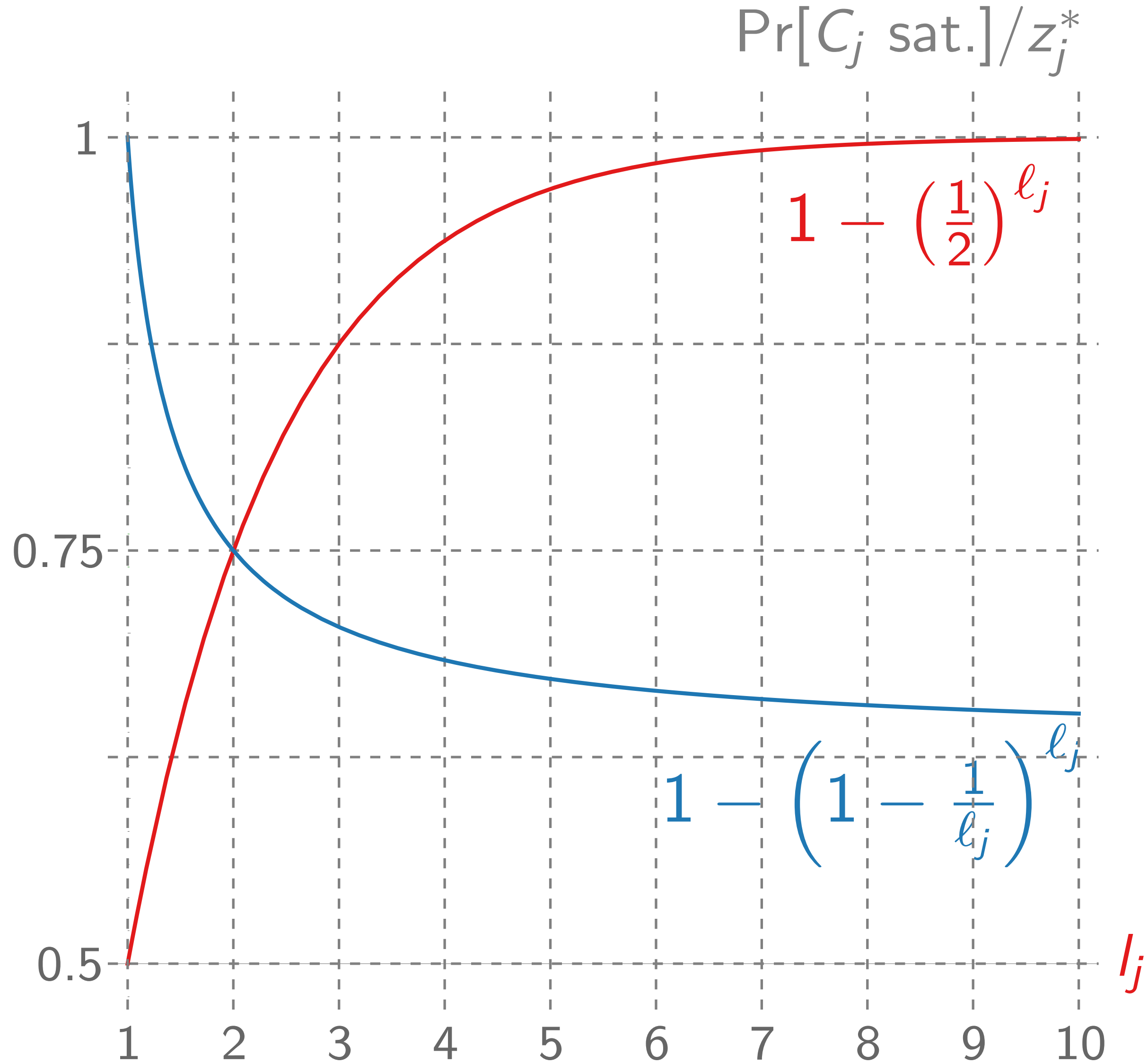


Visualization and Derandomization



Visualization and Derandomization

– Randomized alg. is better for large values of ℓ_j.



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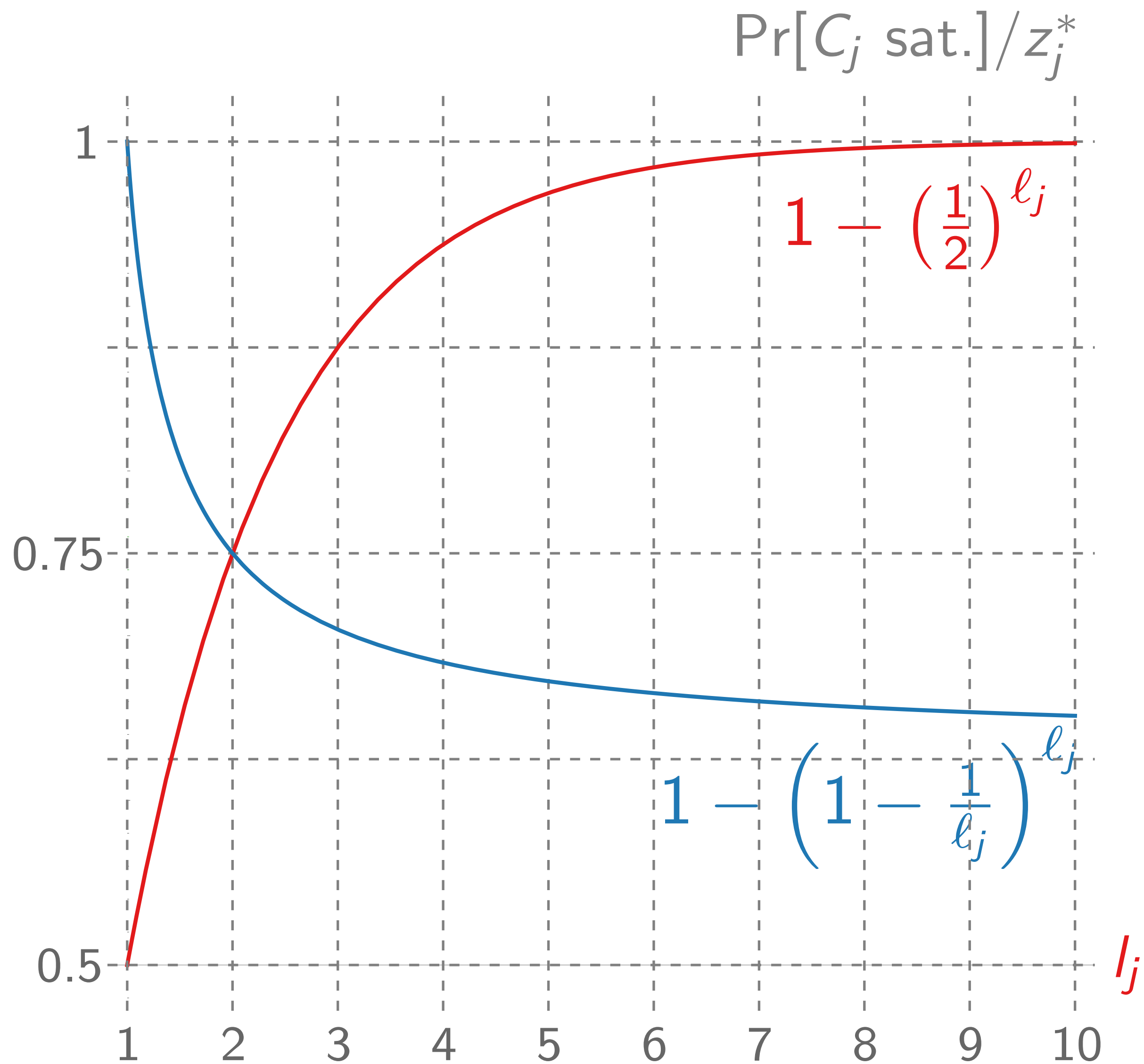

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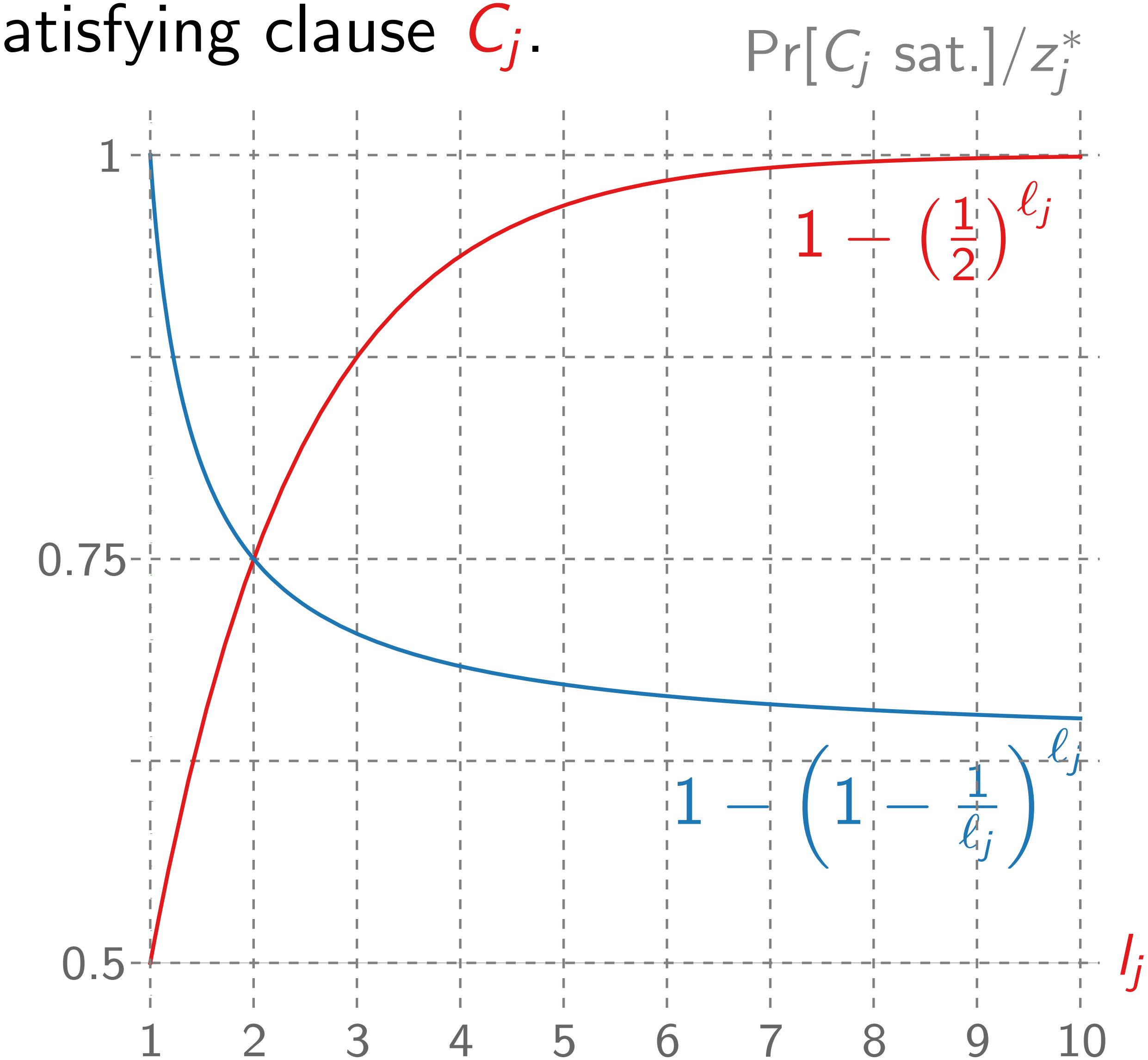

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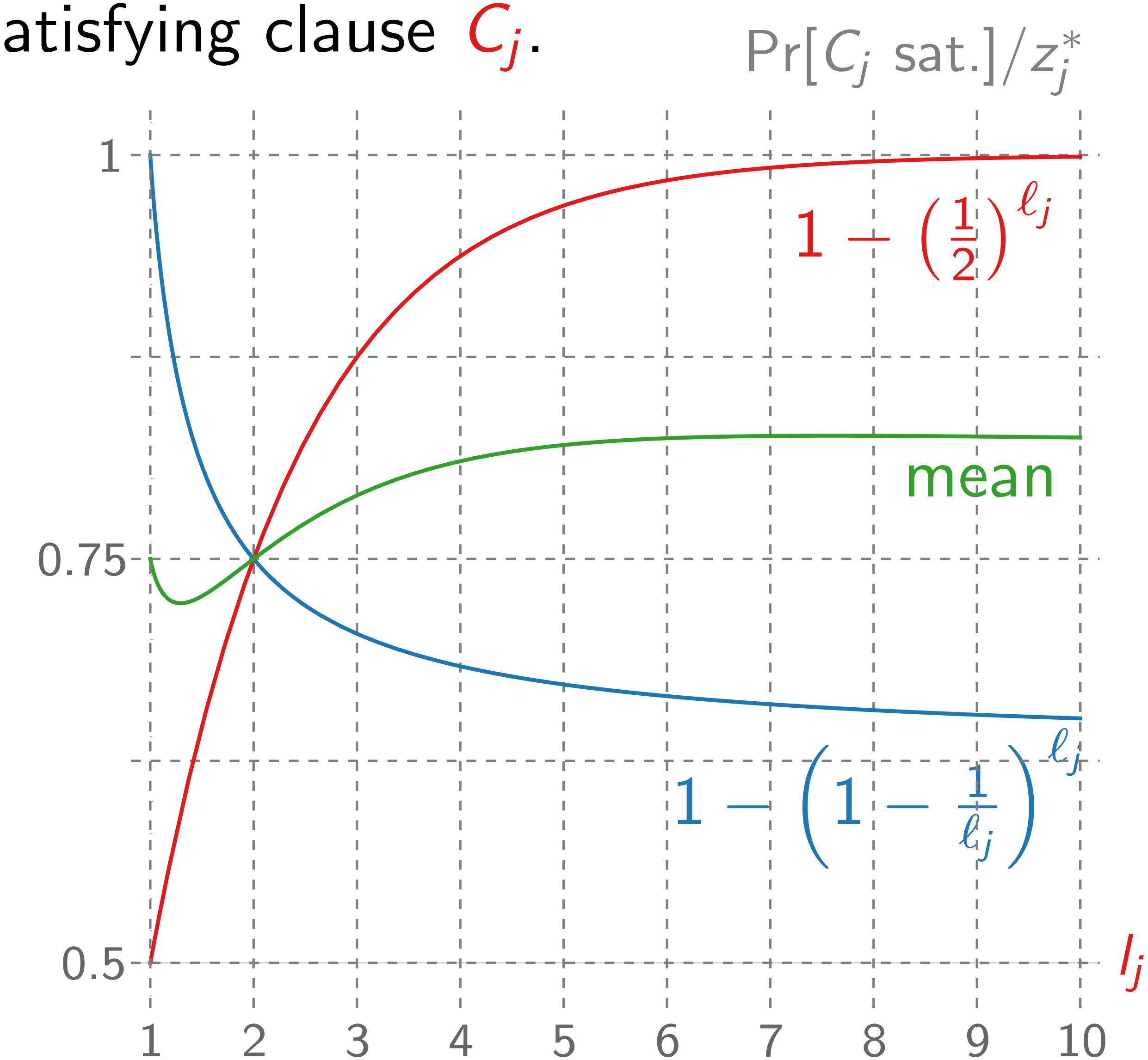

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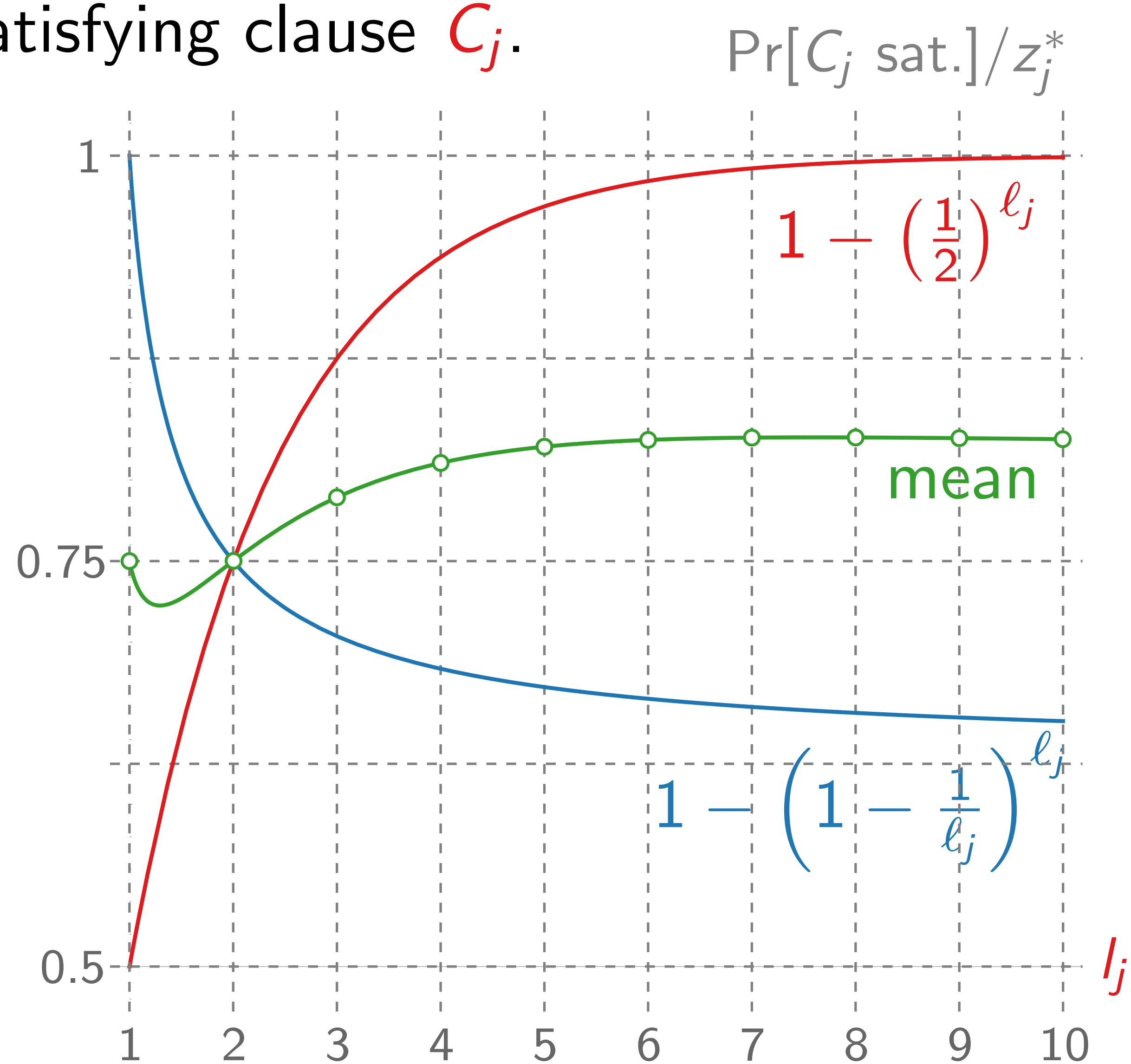

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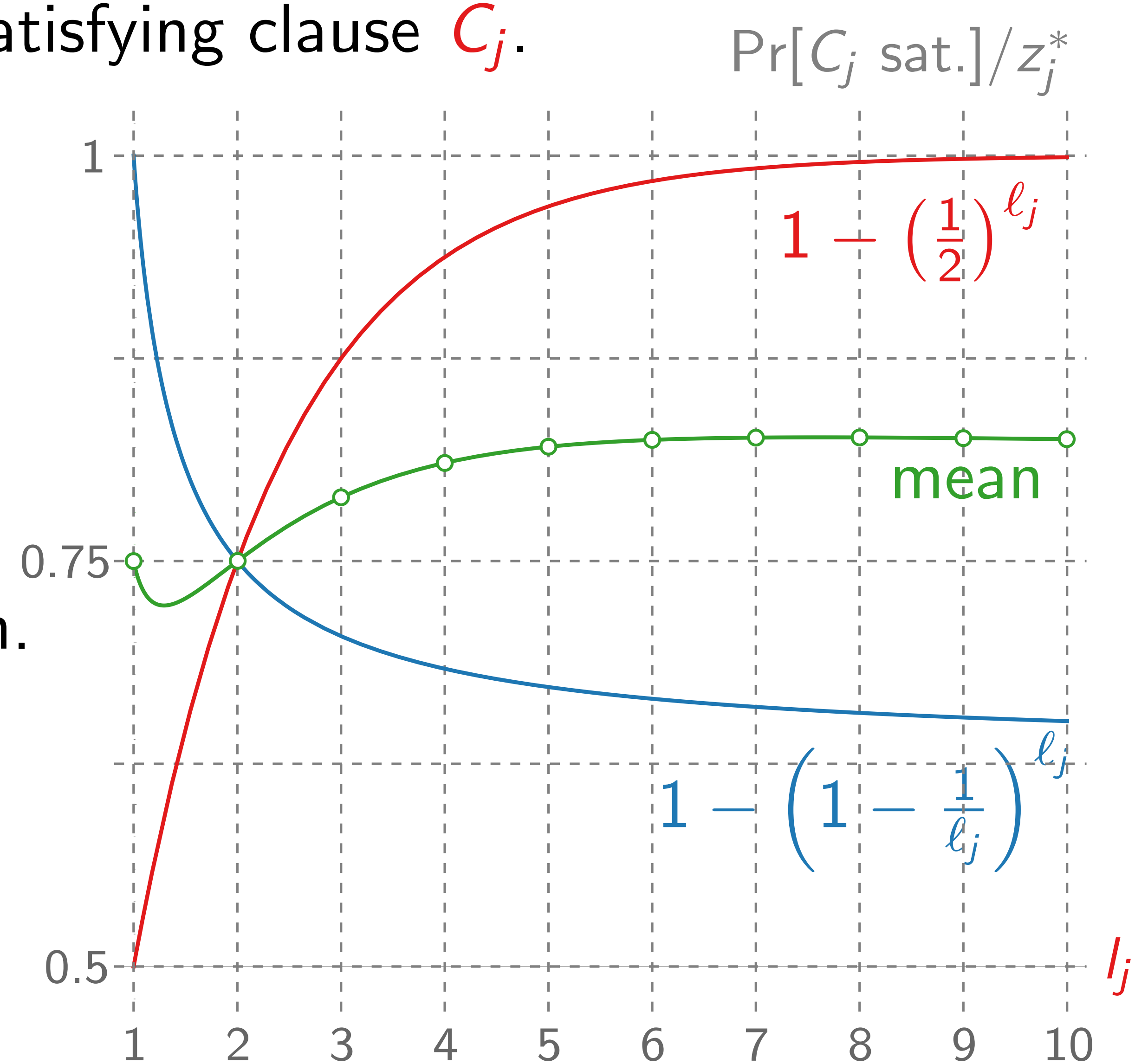

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This algorithm, too, can be derandomized by conditional expectation.