Approximation Algorithms

Lecture 10: Minimum-Degree Spanning Tree via Local Search

Part I: Minimum-Degree Spanning Tree

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Minimum-Degree Spanning Tree

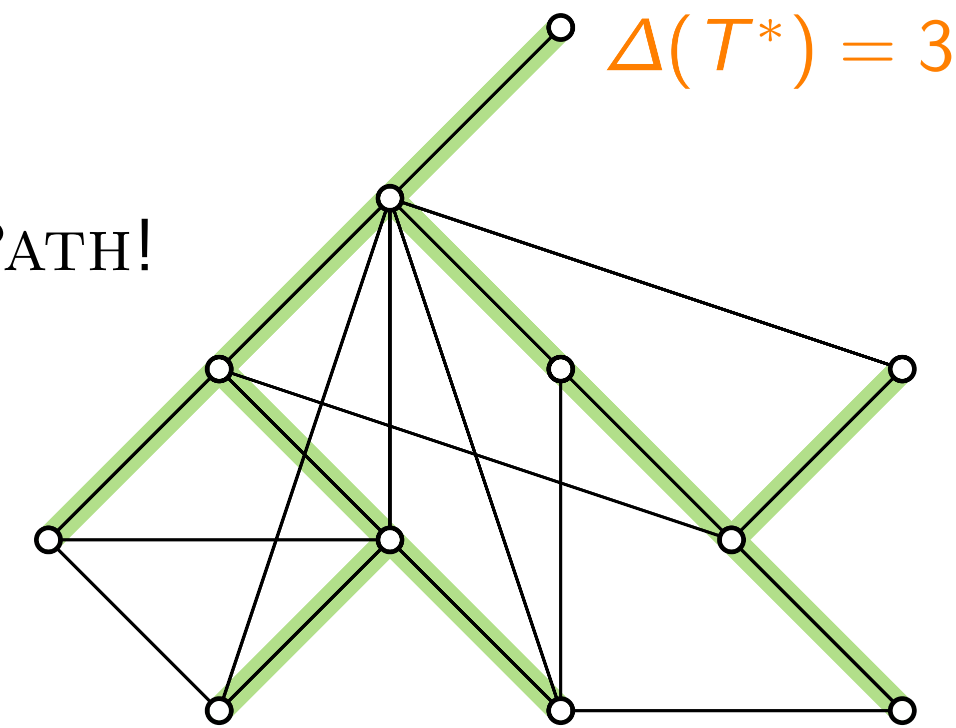
Given: A connected graph G.

Task: Find a spanning tree T that has the smallest maximum degree $\Delta(T)$ among all spanning trees of G.

NP-hard. ☹

Why?

Special case of Hamiltonian Path!



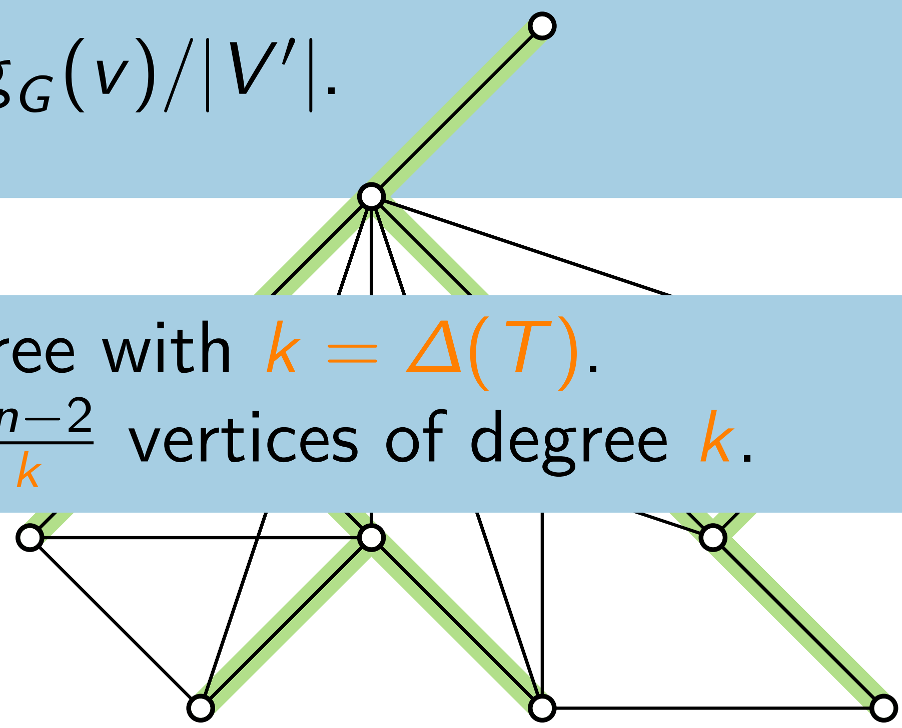
Warm-up

Obs. 1. A spanning tree T has...

- n vertices and $n-1$ edges,
- sum of degrees $\sum_{v \in V(G)} \deg_T(v) = 2n-2$,
- average degree < 2.

Obs. 2. Let $V' \subseteq V(G)$.
Then $\Delta(G) \geq \sum_{v \in V'} \deg_G(v)/|V'|$.

Obs. 3. Let T be a spanning tree with $k = \Delta(T)$.
Then T has at most $\frac{2n-2}{k}$ vertices of degree k.



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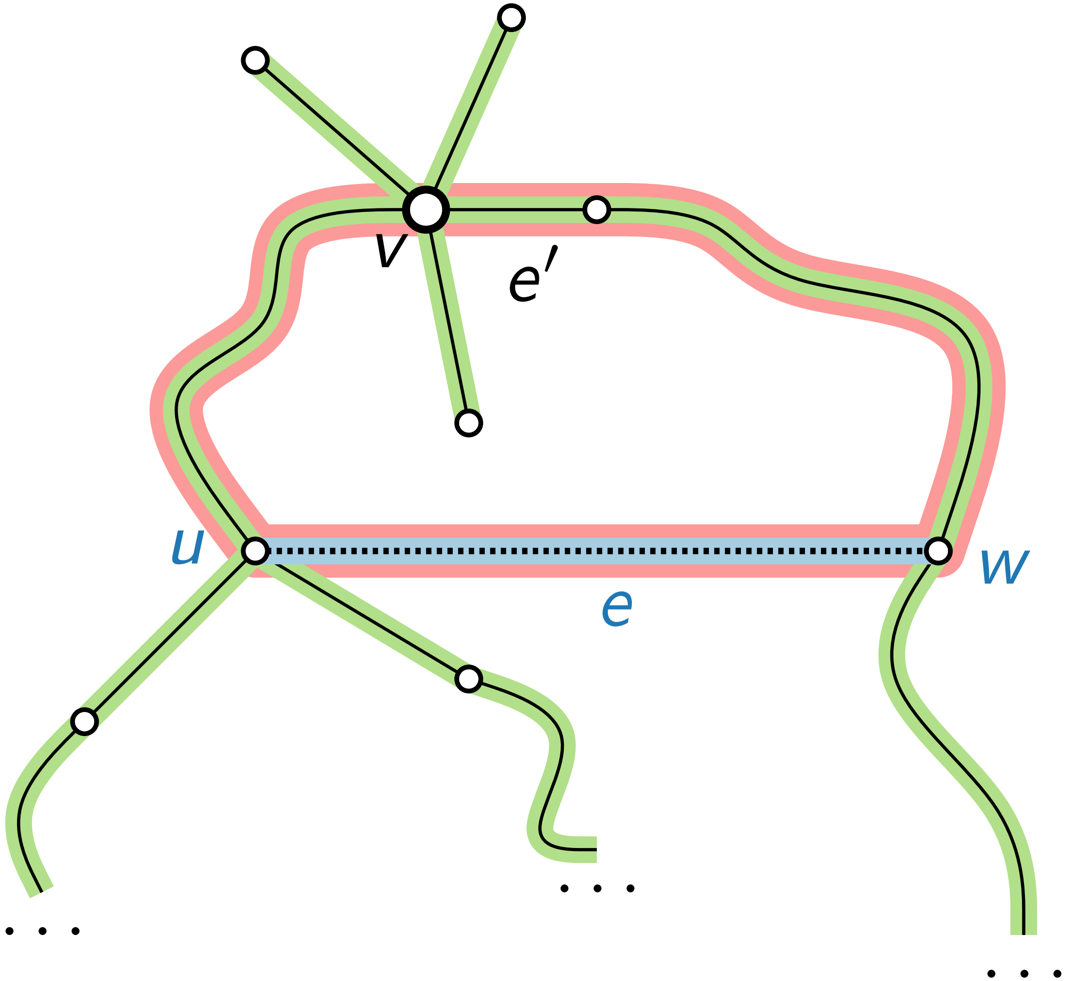
Part II:
Edge Flips and Local Search

Edge Flips



$T + e$

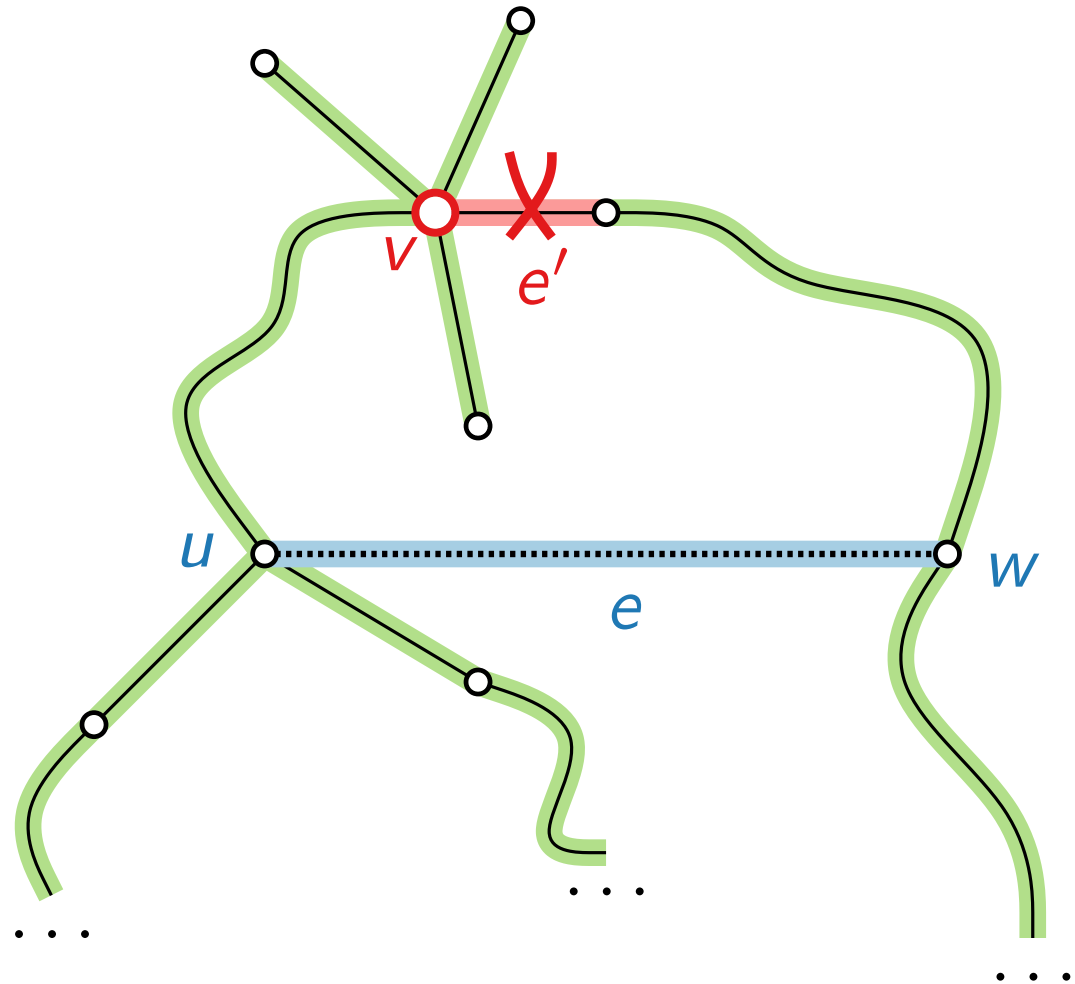
contains a cycle!

$E(T)$

$E(G) - E(T)$

Edge Flips

Def. An **improving flip** in T for a vertex v and an edge $uw \in E(G) \setminus E(T)$ is a flip with $\deg_T(v) > \max\{\deg_T(u), \deg_T(w)\} + 1$.



$T + e - e'$

is a new spanning tree.

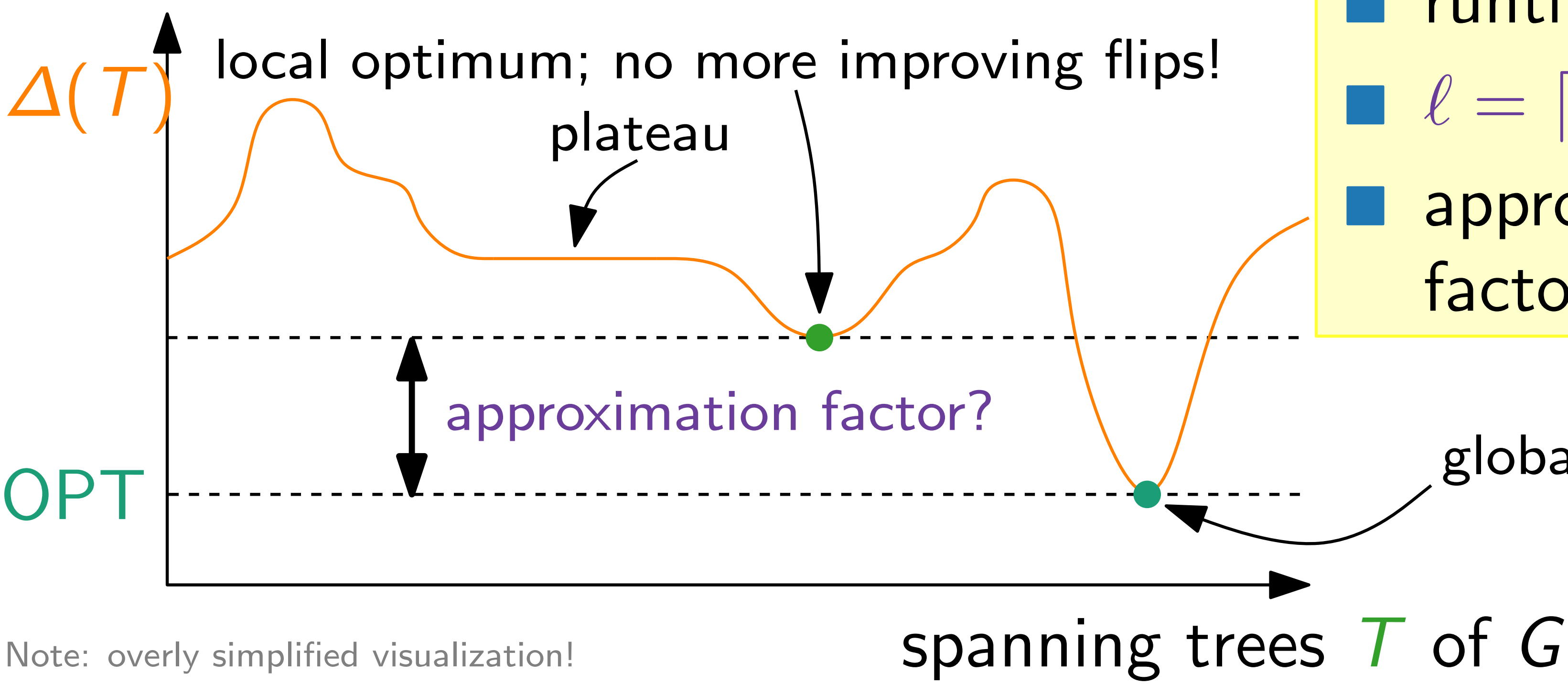
$E(T)$

$E(G) - E(T)$

Local Search

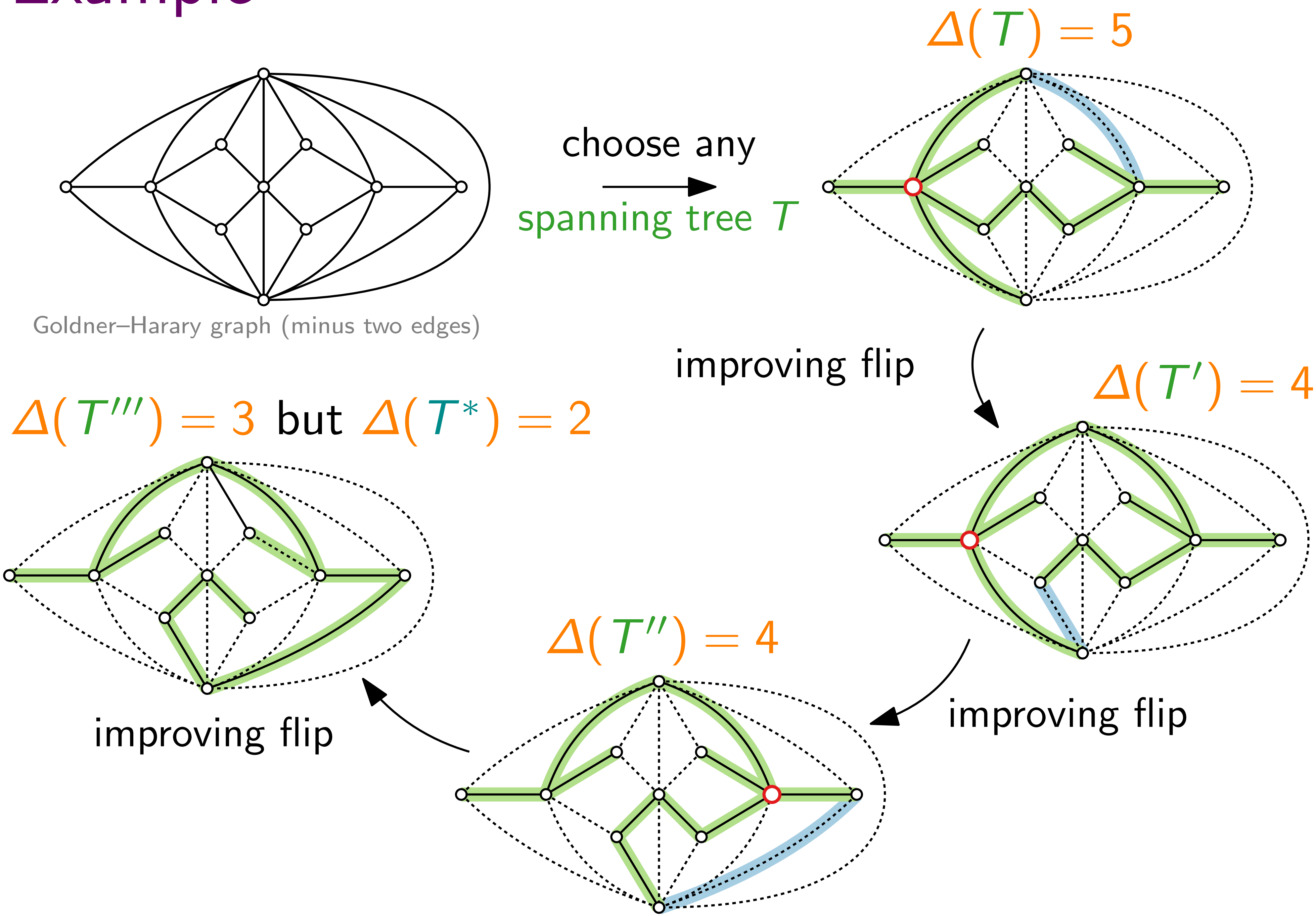
MinDegSpanningTreeLocalSearch(graph G)
  $T \leftarrow$ any spanning tree of G
  while $\exists$ improving flip in T for a vertex v
      with $\deg_T(v) \geq \Delta(T) - \ell$ **do**
    do the improving flip
  return T

- Termination?
- runtime?
- $\ell = \lceil \log_2 n \rceil$
- approximation factor?



Note: overly simplified visualization!

Example



Goldner–Harary graph (minus two edges)
choose any
spanning tree T
$\Delta(T) = 5$
improving flip
$\Delta(T') = 4$
improving flip
$\Delta(T'') = 4$
improving flip
$\Delta(T''') = 3$ but $\Delta(T^*) = 2$

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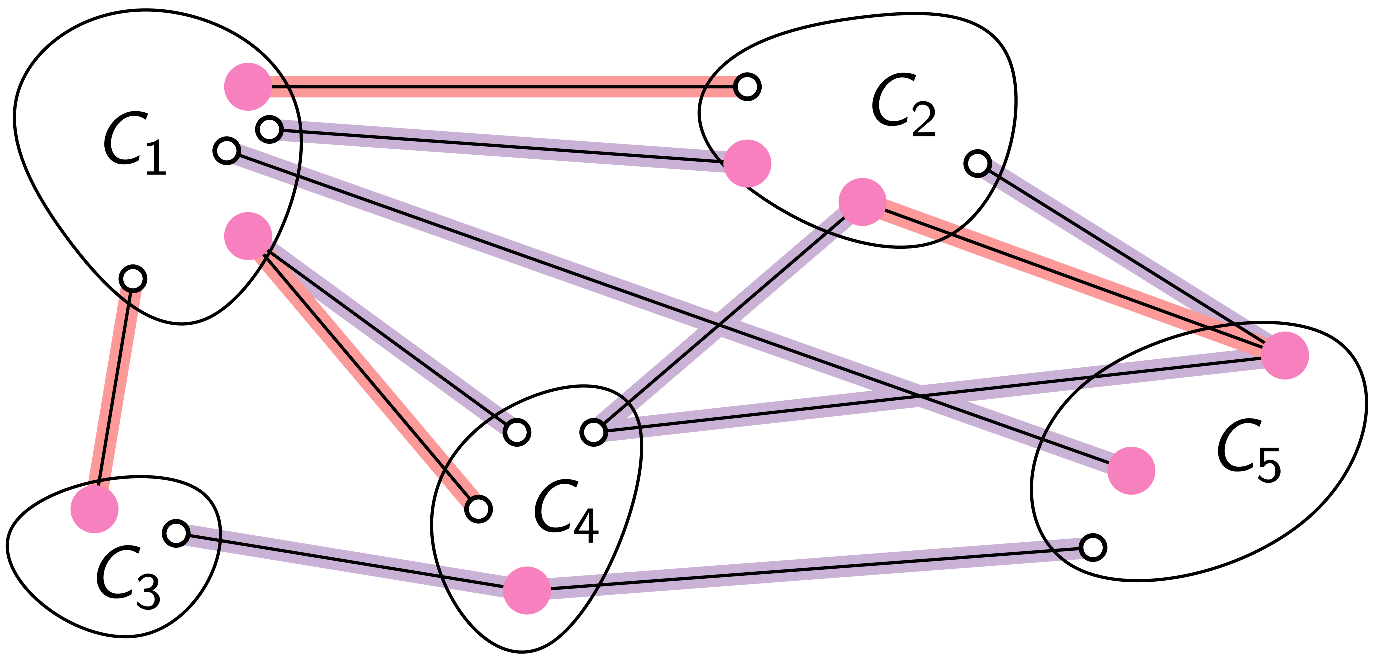
Part III:
Lower Bound

Decomposition $\Rightarrow$ Lower Bound for OPT

- Removing k edges decomposes T into $k+1$ components.
- $E' = \{$edges in G between different components $C_i \neq C_j\}$.
- $S :=$ vertex cover of E'.

spanning tree T



- For any spanning tree T', $|E(T') \cap E'| \geq k$,
- $\sum_{v \in S} \deg_{T'}(v) \geq k$, and $\Delta(T') \geq k/|S|$. (Obs. 2)
- Consider the optimal spanning tree T^*.

Lemma 1.
$\Rightarrow$ OPT $\geq k/|S|$

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Part IV:
Structure of a Decomposition

Structure of a Decomposition

$\Rightarrow S_1 \supseteq S_2 \supseteq \dots$
$\Rightarrow S_1 = V(G)$
$\Rightarrow E_1 = E(T)$

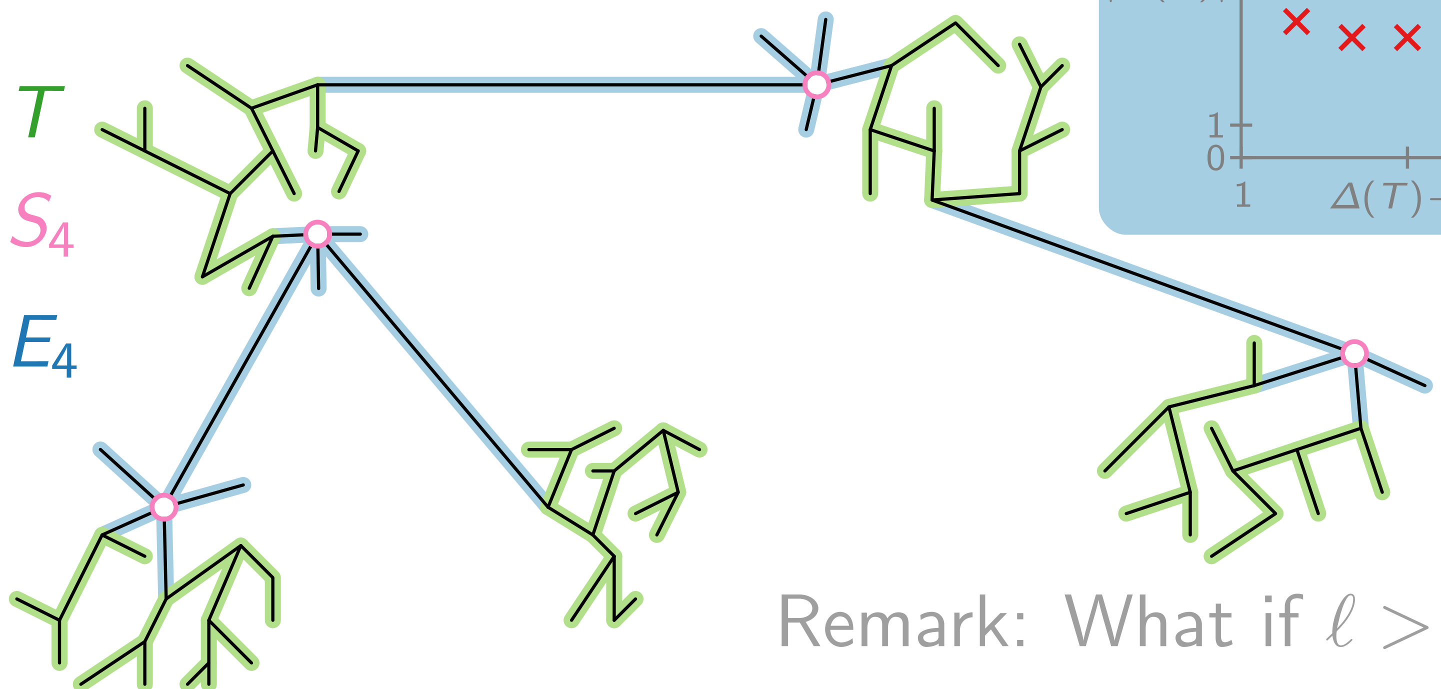
Let S_i be the set of vertices v in T with $\deg_T(v) \geq i$.
Let E_i be the set of edges in T incident to S_i.

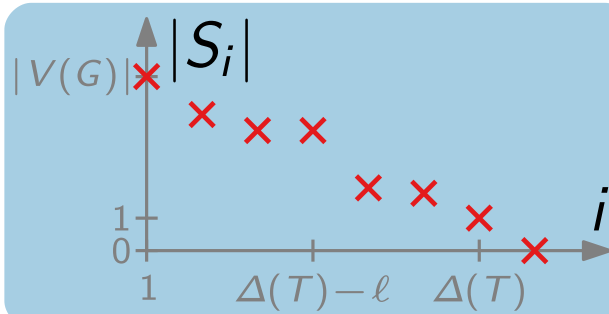
Lemma 2. $\exists i$ s.t. $\Delta(T) - \ell + 1 \leq i \leq \Delta(T)$ with $|S_{i-1}| \leq 2|S_i|$.

Proof. $|S_{\Delta(T)-\ell}| > 2^\ell |S_{\Delta(T)}| = 2^{\lceil \log_2 n \rceil} |S_{\Delta(T)}| \geq n \cdot |S_{\Delta(T)}|$ ↯

Otherwise

$\ell = \lceil \log_2 n \rceil$





Remark: What if $\ell > \Delta(T)$?

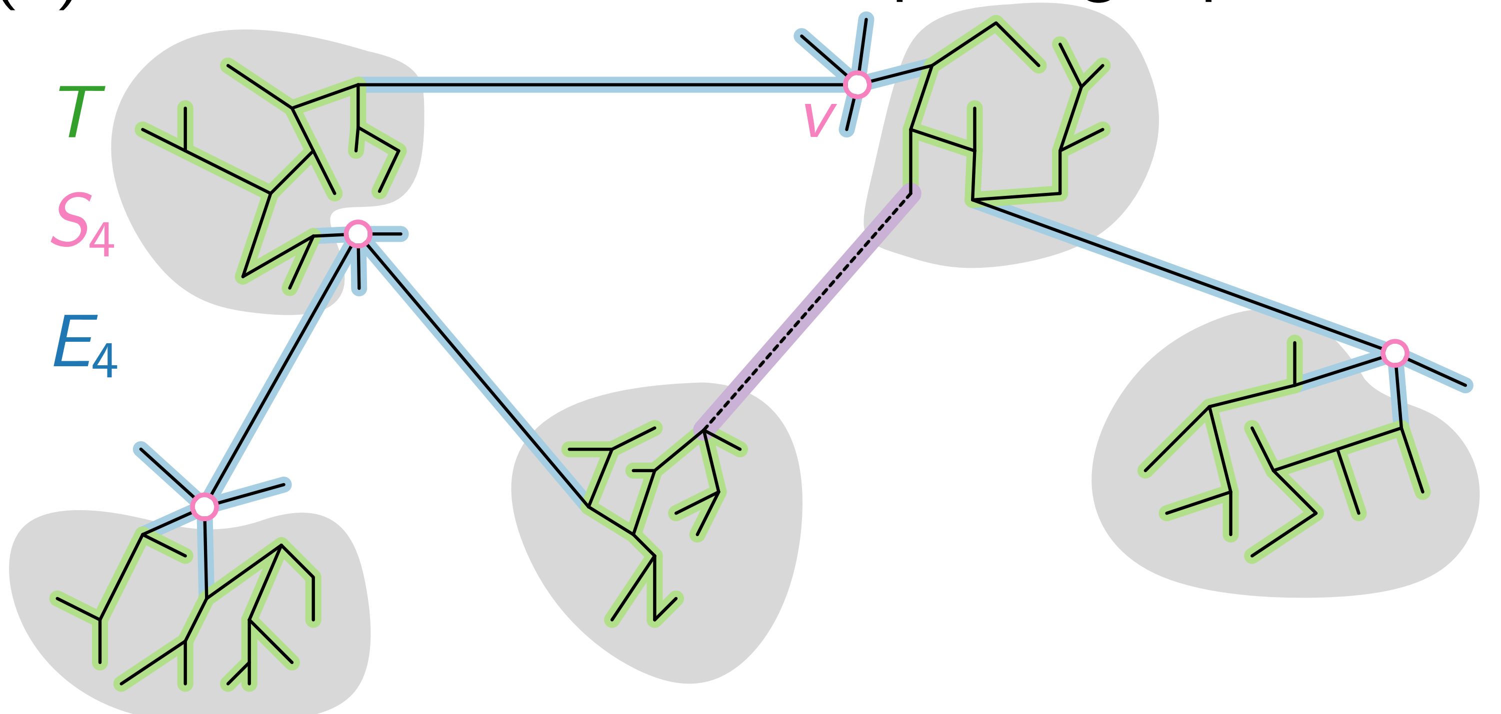
Structure of a Decomposition

Lemma 3. For locally opt. spanning tree T, $i \geq \Delta(T) - \ell + 1$:
(i) $|E_i| \geq (i-1)|S_i| + 1$,
(ii) Each edge $e \in E(G) \setminus E_i$ connecting distinct components of $T \setminus E_i$ is incident to a node of S_{i-1}.

Proof. (i) $|E_i| \geq i|S_i| - (|S_i| - 1) = (i-1)|S_i| + 1$

vertex-deg   counted twice?

(ii) Otherwise, there is an improving flip for some $v \in S_i$.



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Part V:
Approximation Factor

Approximation Factor

[Fürer & Raghavachari: SODA'92, JA'94]

Theorem. Let T be a locally optimal spanning tree.
Then $\Delta(T) \leq 2 \cdot \mathrm{OPT} + \ell$, where $\ell = \lceil \log_2 n \rceil$.

Proof. Let S_i be the vertices v in T with $\deg_T(v) \geq i$.
Let E_i be the edges in T incident to S_i.

Lemma 1. $\mathrm{OPT} \geq k/|S|$ if $k = |\text{removed edges}|$, S vertex cover.

Lemma 2. $\exists i$ s.t. $\Delta(T) - \ell + 1 \leq i \leq \Delta(T)$ with $|S_{i-1}| \leq 2|S_i|$.

Lemma 3. For $i \geq \Delta(T) - \ell + 1$,
(i) $|E_i| \geq (i-1)|S_i| + 1$,
(ii) Each edge $e \in E(G) \setminus E_i$ connecting distinct components of $T \setminus E_i$ is incident to a node of S_{i-1}.

Remove E_i for this i! $\Rightarrow$ S_{i-1} covers edges between comp.

$$\mathrm{OPT} \underset{\text{Lemma 1}}{\geq} \frac{k}{|S|} = \frac{|E_i|}{|S_{i-1}|} \underset{\text{Lemma 3}}{\geq} \frac{(i-1)|S_i|+1}{|S_{i-1}|} \underset{\text{Lemma 2}}{\geq} \frac{(i-1)|S_i|+1}{2|S_i|} > \frac{(i-1)}{2} \underset{\text{Lemma 2}}{\geq} \frac{\Delta(T)-\ell}{2} \quad \square$$

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Part VI:
Termination, Running Time & Extensions

Termination and Running Time

Theorem. The algorithm finds a locally optimal spanning tree after at most $O(n^4)$ iterations.

Proof. Via potential function $\Phi(T)$ measuring the value of a solution where (hopefully): $\Phi(T) = \sum_{v \in V(G)} 3^{\deg_T(v)}$

- Each iteration decreases the potential of a solution.

Lemma. After each flip $T \to T'$, $\Phi(T') \leq (1 - \frac{2}{27n^3})\Phi(T)$.

Homework

- The function is bounded both from above and below.

Lemma. For every spanning tree T, $\Phi(T) \in [3n, n3^n]$.

- Executing $f(n)$ iterations would exceed the lower bound.

  Let $f(n) = \frac{27}{2} n^4 \cdot \ln 3$. How does $\Phi(T)$ change?

  $\Phi(T)$ decreases by: $(1 - \frac{2}{27n^3})^{f(n)} \leq (e^{-\frac{2}{27n^3}})^{f(n)} = e^{-n \ln 3} = 3^{-n}$

  Goal: After $f(n)$ iterations: $\Phi(T) = n < 3n$. $\square$

Extensions

Corollary. For any constant $b > 1$ and $\ell = \lceil \log_b n \rceil$, the local search algorithm runs in polynomial time and produces a spanning tree T with $\Delta(T) \leq b \cdot \mathrm{OPT} + \ell$.

Proof. Similar to previous pages. Homework □

- A variant of this algorithm yields the following result:

[Fürer & Raghavachari: SODA'92, JA'94]

Theorem. There is a local search algorithm that runs in $O(EV\alpha(E,V)\log V)$ time and produces a spanning tree T with $\Delta(T) \leq \mathrm{OPT} + 1$.

- Further variants for directed graphs and Steiner tree.