Approximation Algorithms

Lecture 10:
Minimum-Degree Spanning Tree
via Local Search

Part I:
Minimum-Degree Spanning Tree

Alexander Wolff

Winter term 2025

Minimum-Degree Spanning Tree

Given: A connected graph G.

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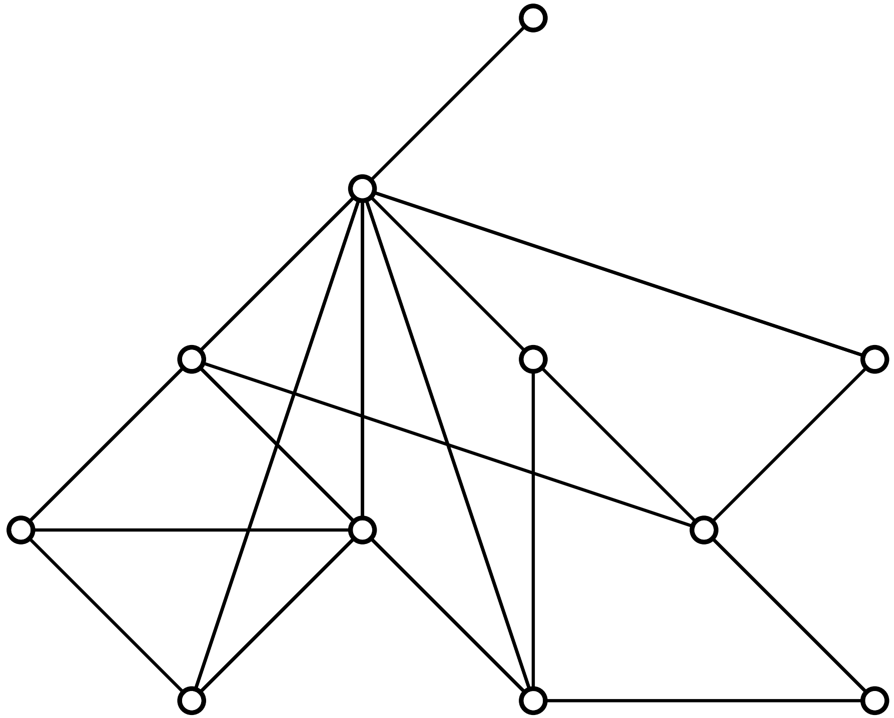
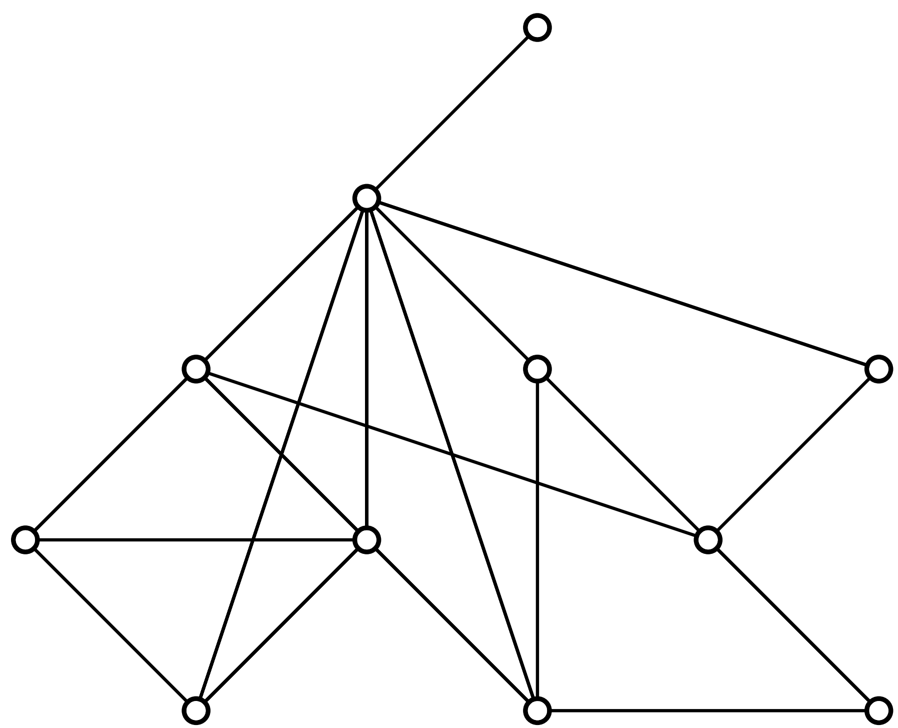


Minimum-Degree Spanning Tree

Given: A connected graph G.

Task: Find a spanning tree T that has the smallest maximum degree $\Delta(T)$ among all spanning trees of G.



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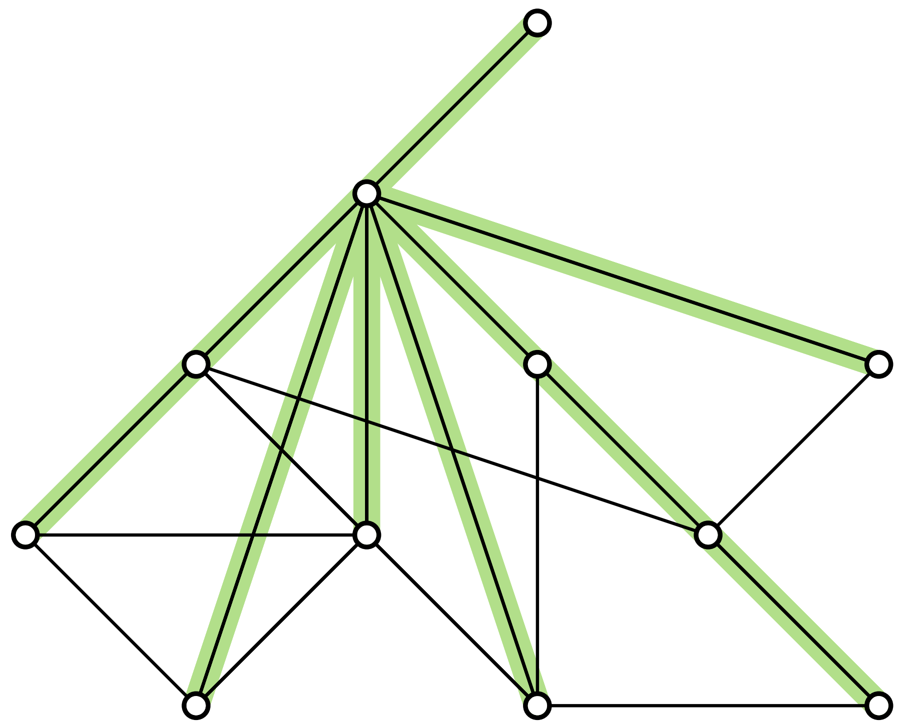

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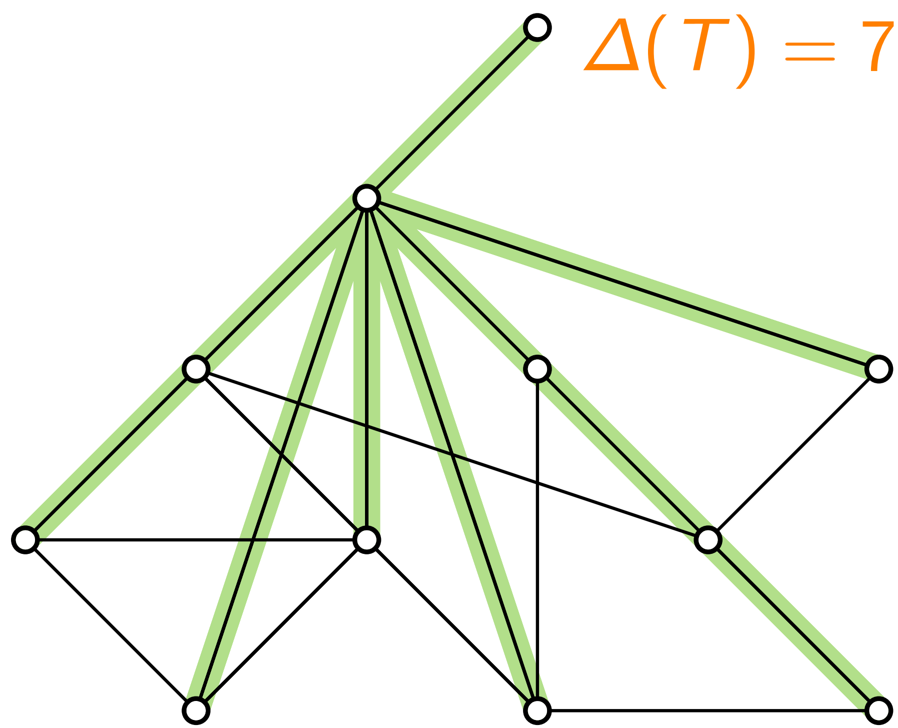

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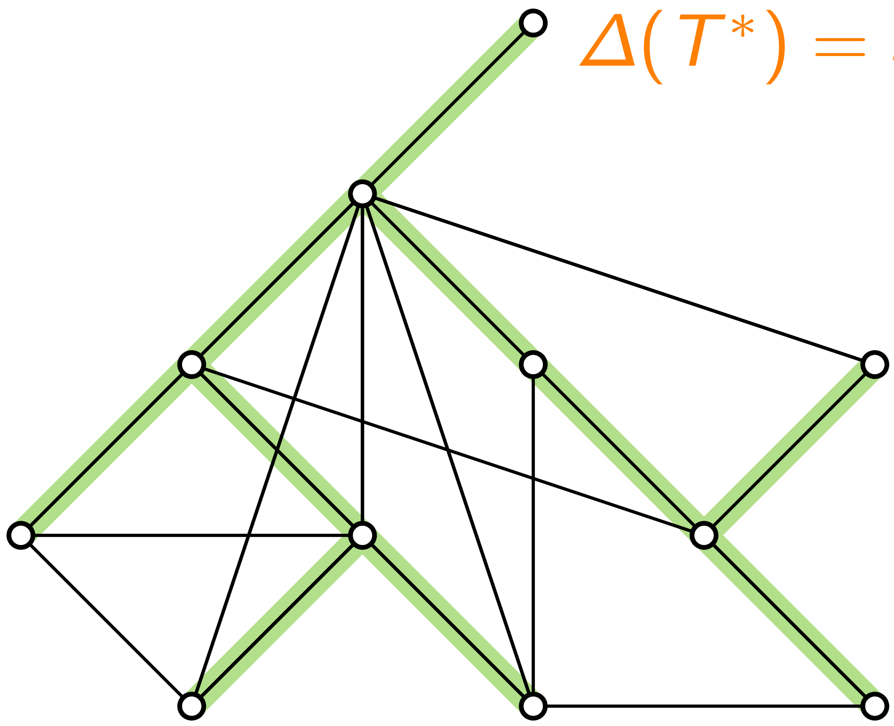

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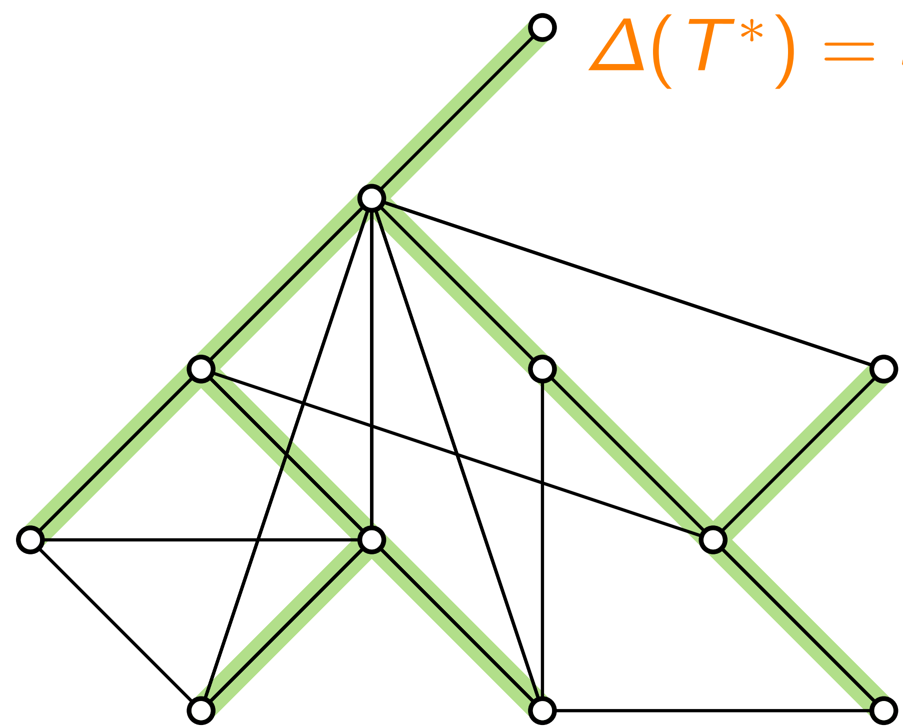
NP-hard.



$\Delta(T^*) = 3$

Minimum-Degree Spanning Tree

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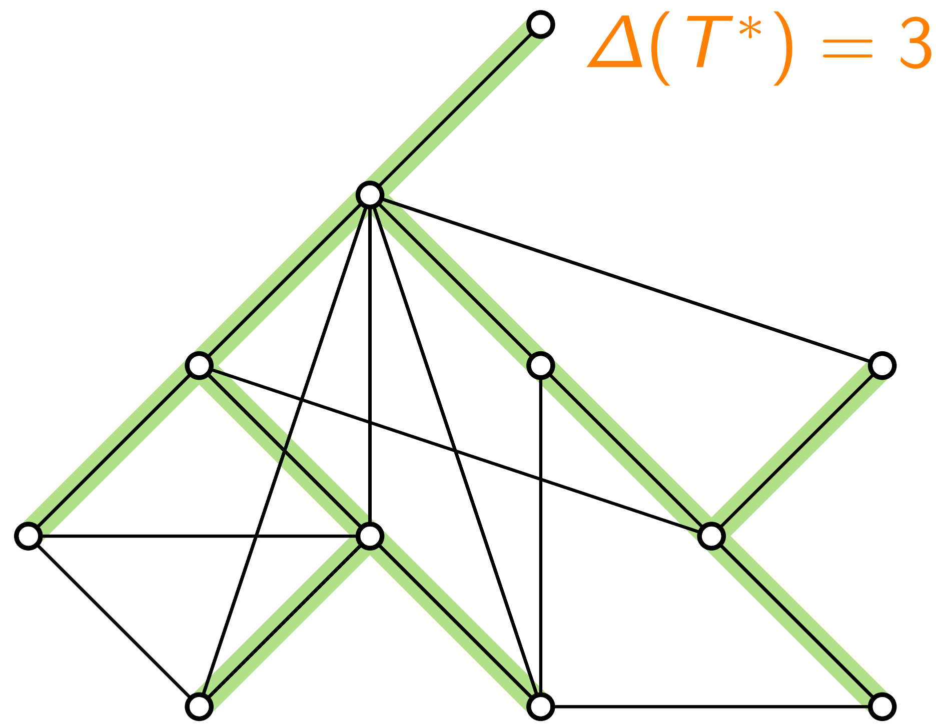
NP-hard. 

Why?



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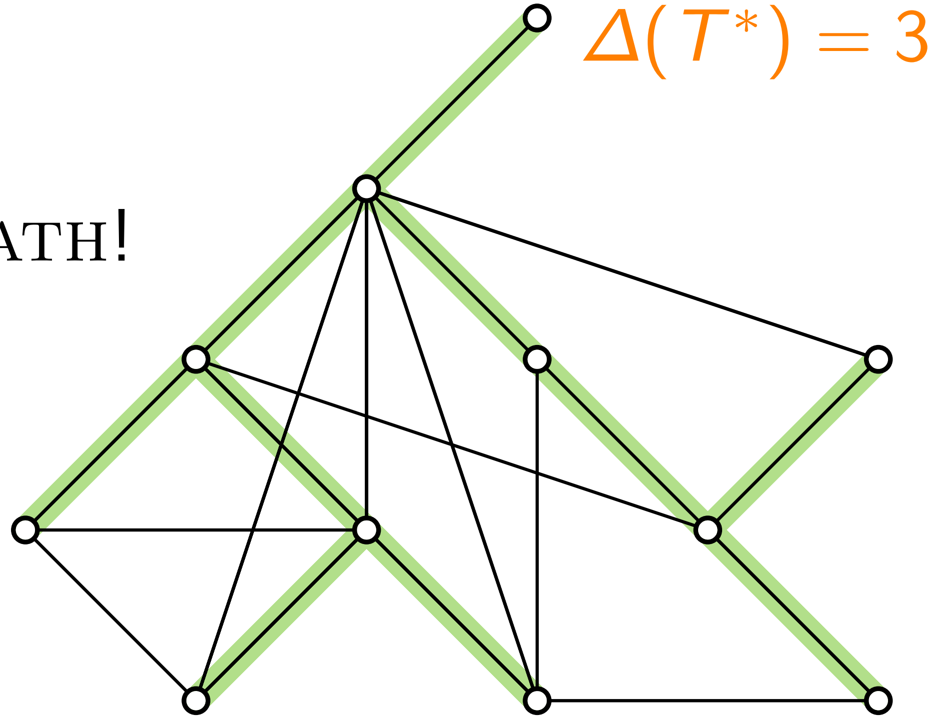
NP-hard. 

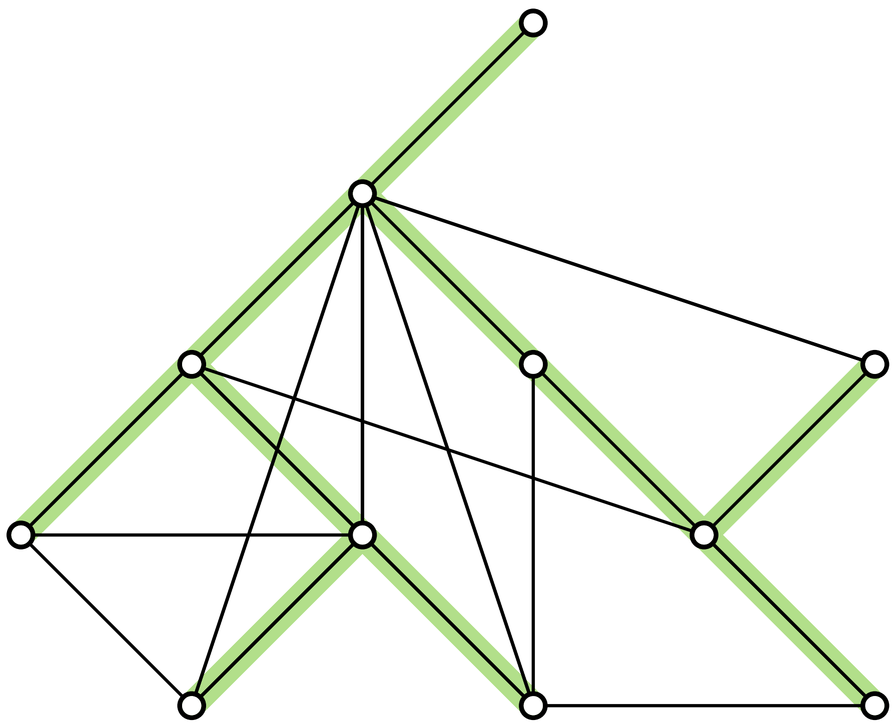
Why?

Special case of Hamiltonian Path!



Warm-up

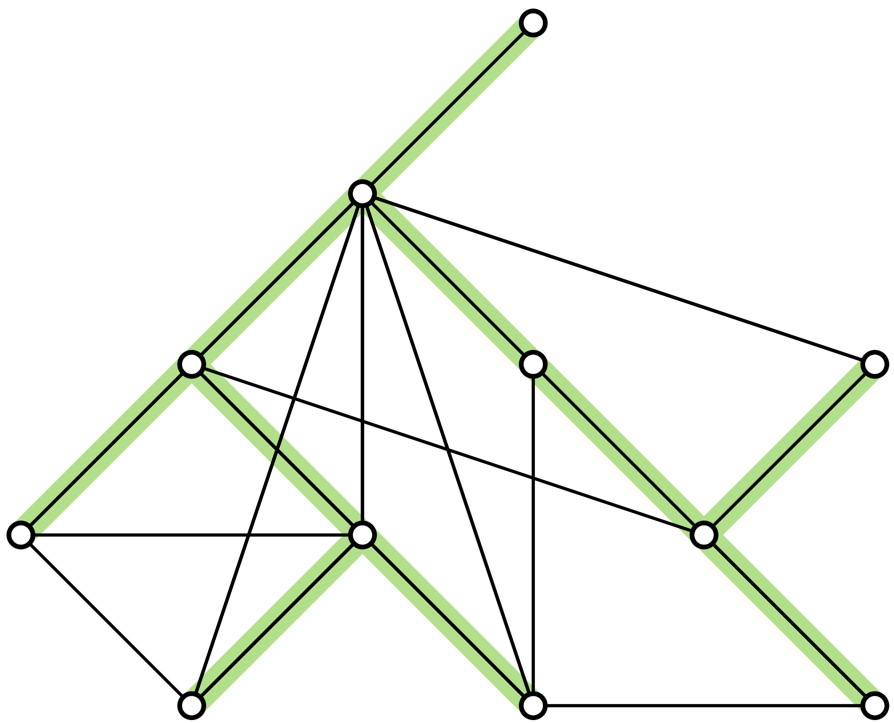
Obs. 1. A spanning tree T has...



Warm-up

Obs. 1. A spanning tree T has. . .

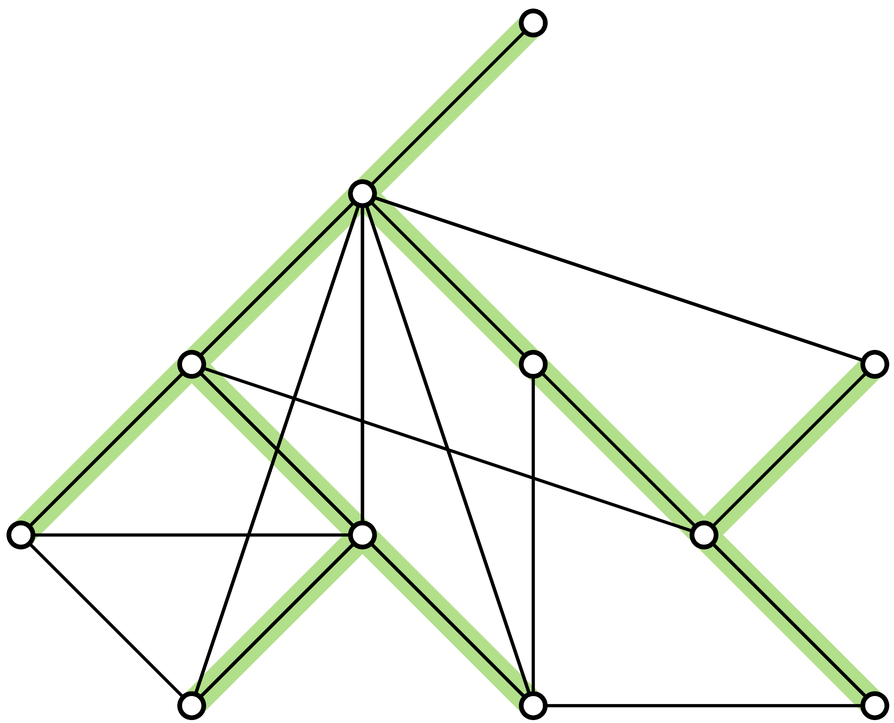
- n vertices and ? edges,
- sum of degrees $\sum_{v \in V(G)} \deg_T(v) =$?
- average degree $<$?



Warm-up

Obs. 1. A spanning tree T has...

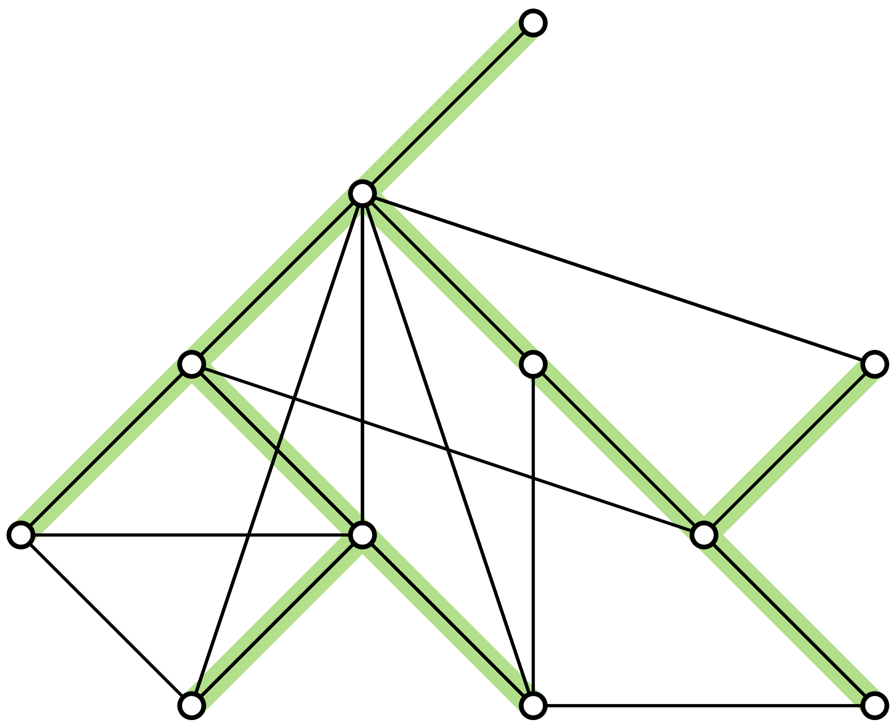
- n vertices and $n-1$ edges,
- sum of degrees $\sum_{v \in V(G)} \deg_T(v) =$?
- average degree $<$?



Warm-up

Obs. 1. A spanning tree T has...

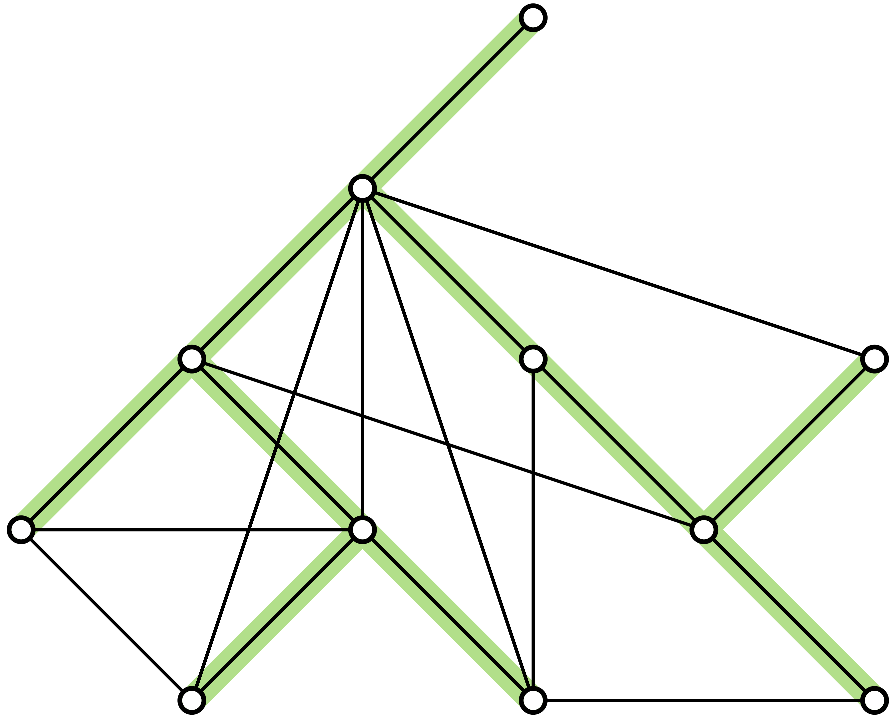
- n vertices and $n-1$ edges,
- sum of degrees $\sum_{v \in V(G)} \deg_T(v) = 2n-2$,
- average degree $<$?



Warm-up

Obs. 1. A spanning tree T has...

- n vertices and $n-1$ edges,
- sum of degrees $\sum_{v \in V(G)} \deg_T(v) = 2n-2$,
- average degree < 2.

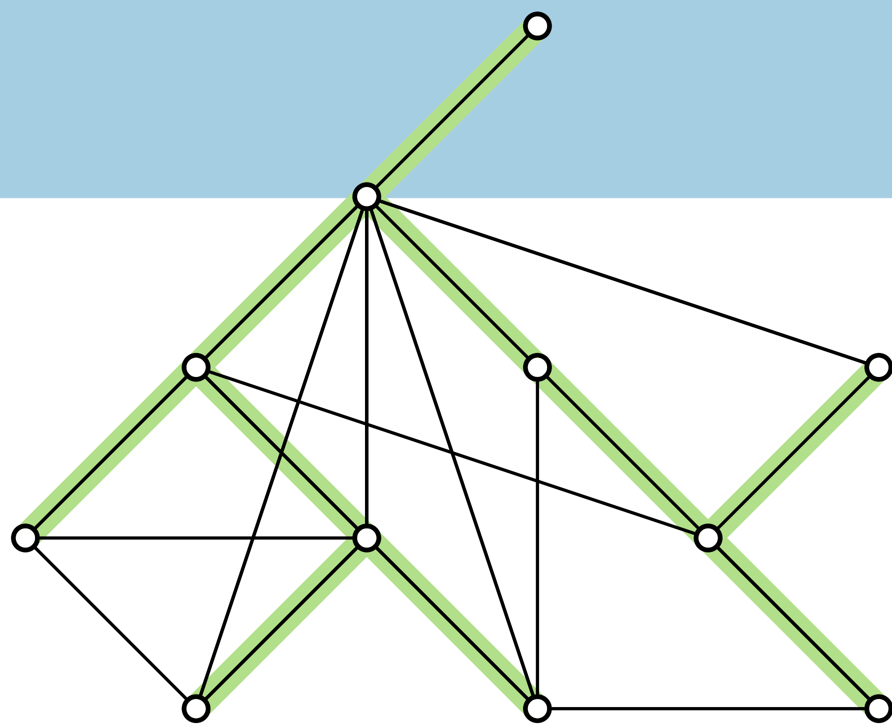


Warm-up

Obs. 1. A spanning tree T has...

- n vertices and $n-1$ edges,
- sum of degrees $\sum_{v \in V(G)} \deg_T(v) = 2n - 2$,
- average degree < 2.

Obs. 2. Let $V' \subseteq V(G)$.
Then $\Delta(G) \geq$?



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Obs. 1. A spanning tree T has...

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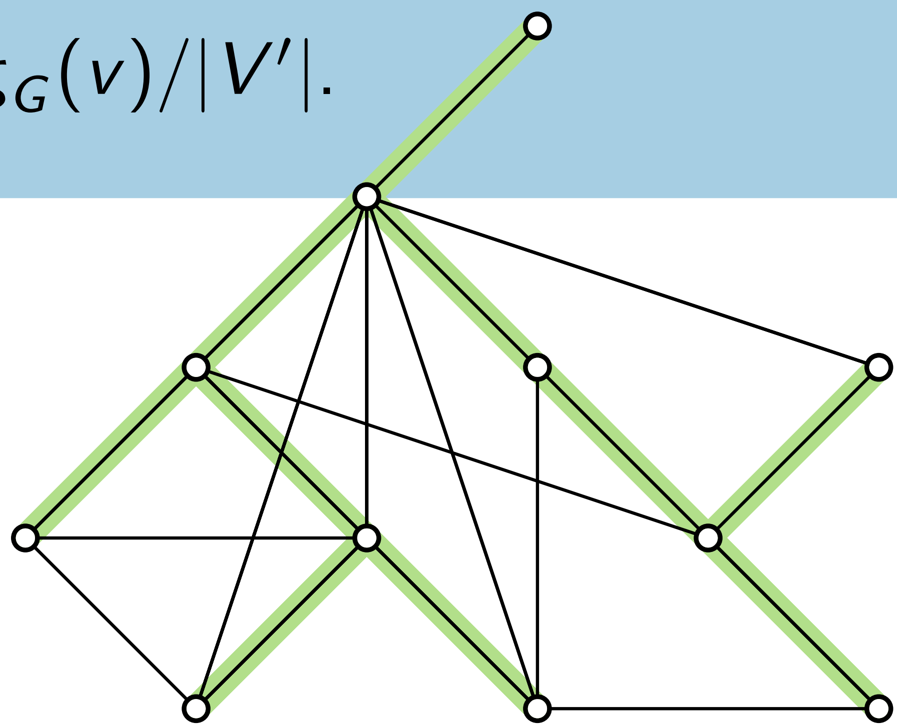
Obs. 2. Let $V' \subseteq V(G)$.
Then $\Delta(G) \geq \sum_{v \in V'} \deg_G(v)/|V'|$.



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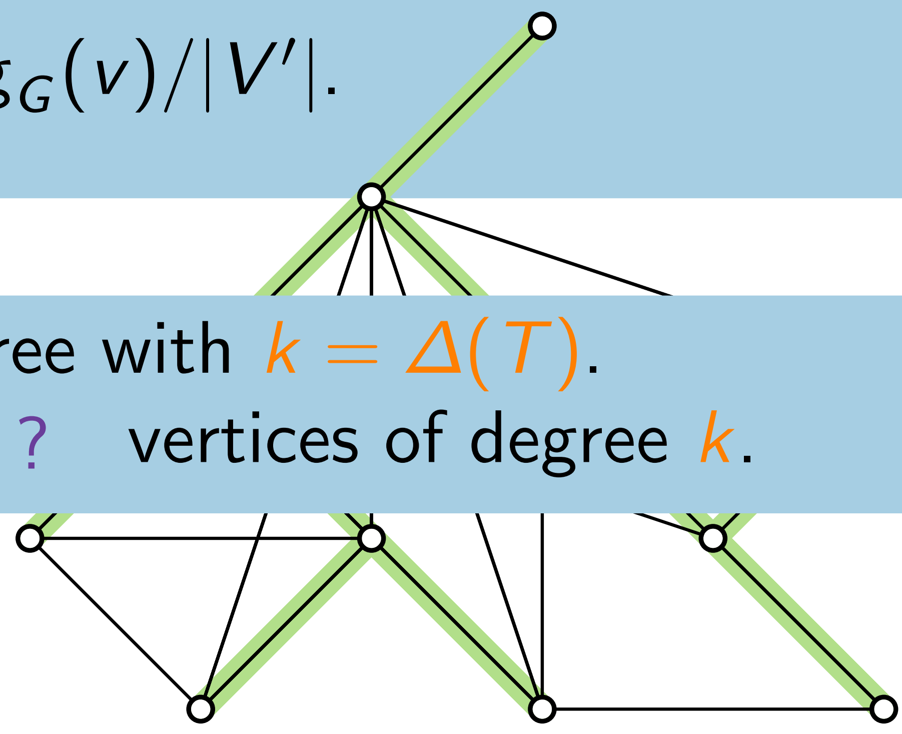
Obs. 3. Let T be a spanning tree with $k = \Delta(T)$.
Then T has at most ? vertices of degree k.



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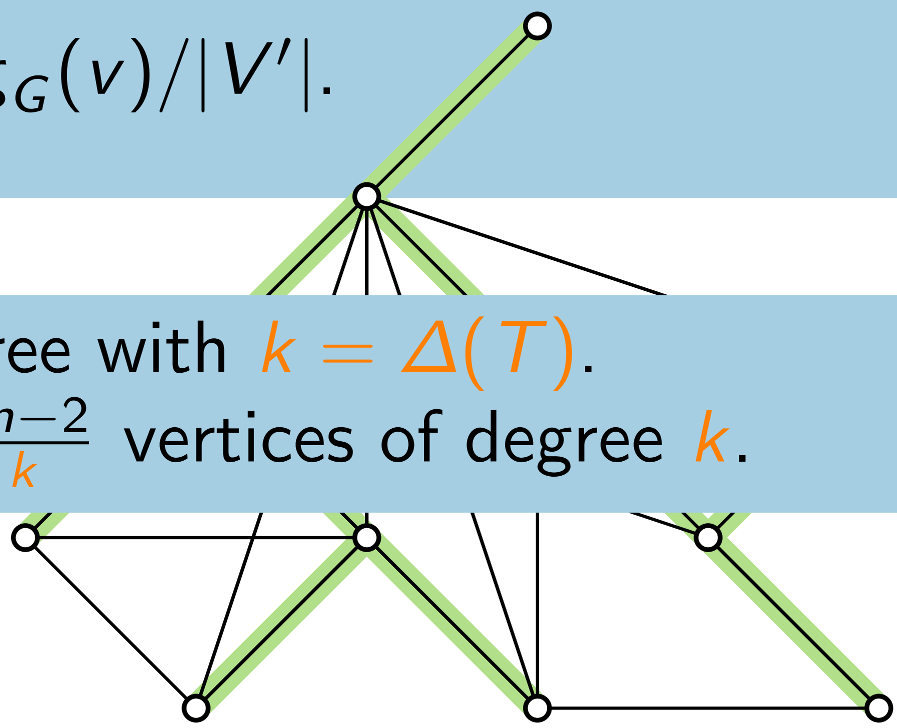
Then $\Delta(G) \geq \sum_{v \in V'} \deg_G(v)/|V'|$.

Obs. 3. Let T be a spanning tree with $k = \Delta(T)$.
Then T has at most $\frac{2n-2}{k}$ vertices of degree k.



Approximation Algorithms

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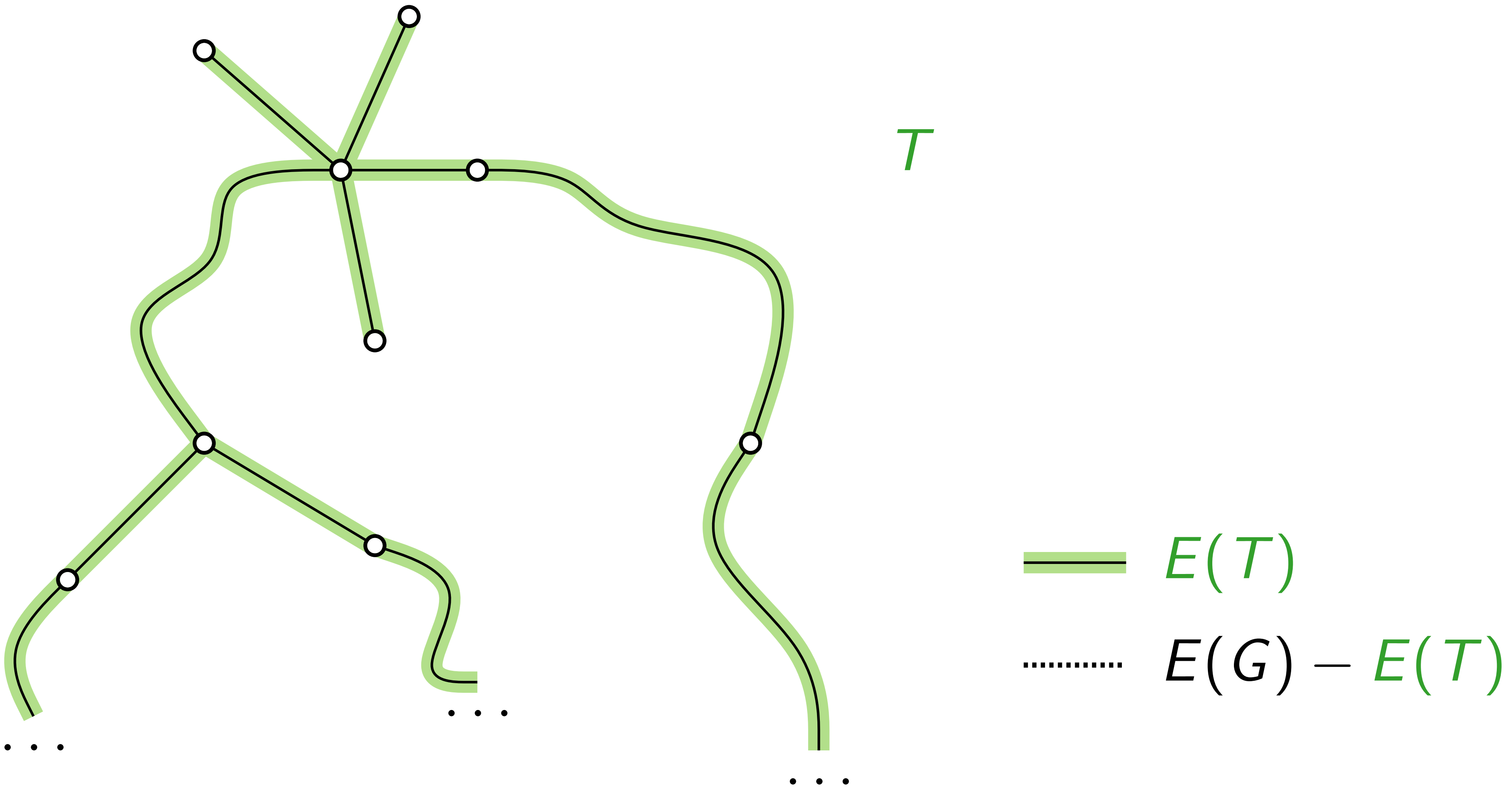
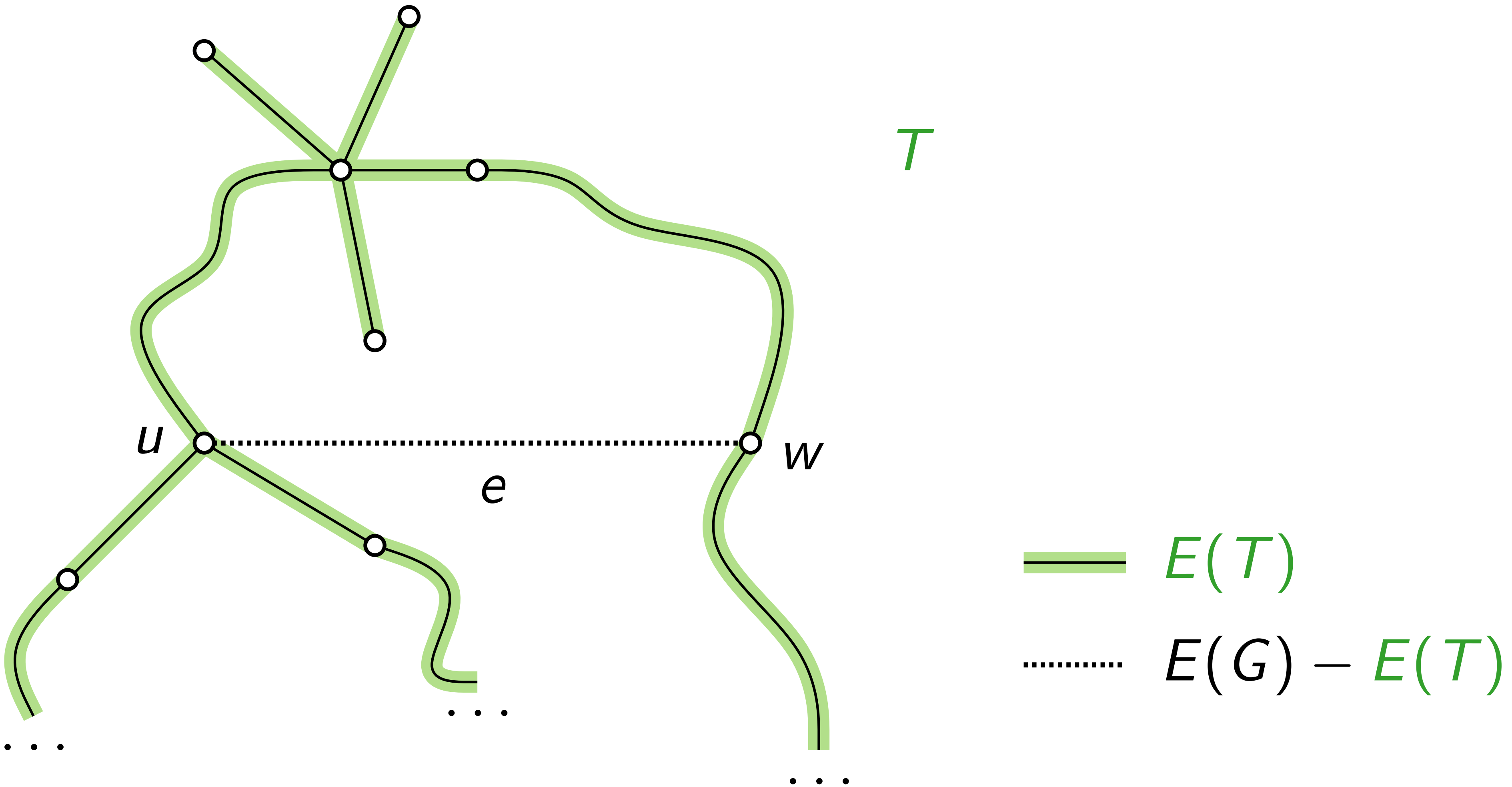
Part II:
Edge Flips and Local Search

Edge Flips



Edge Flips



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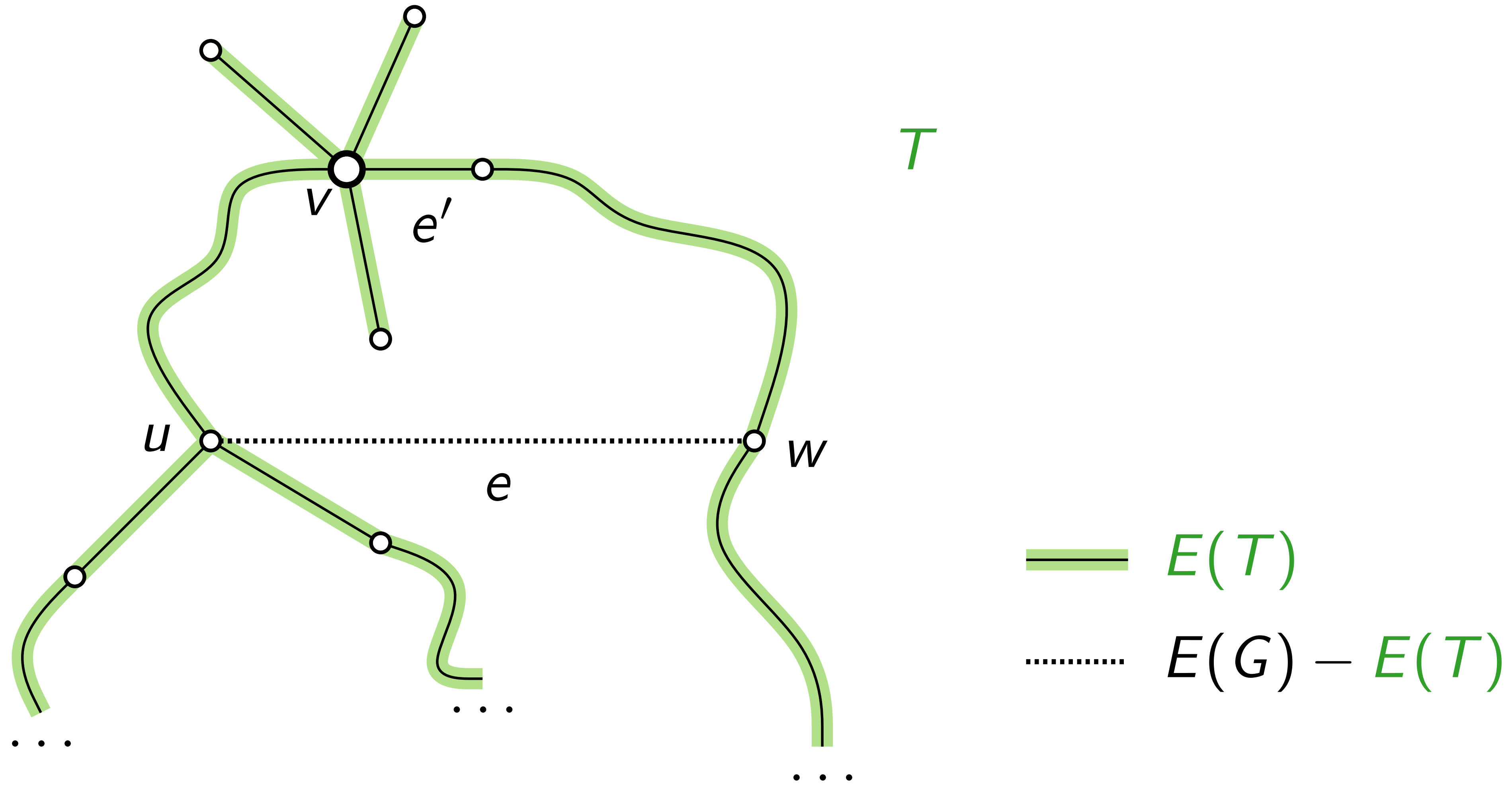

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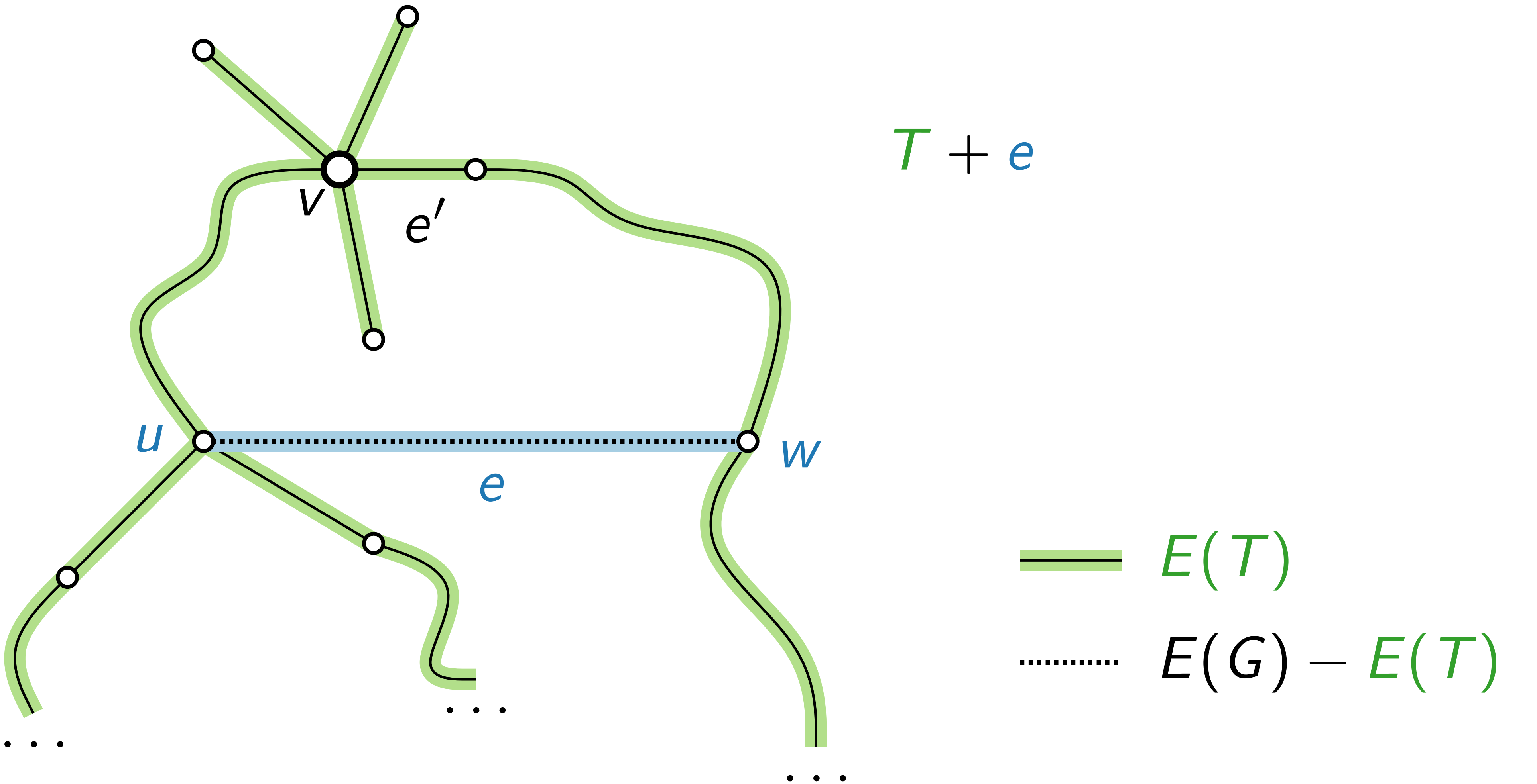

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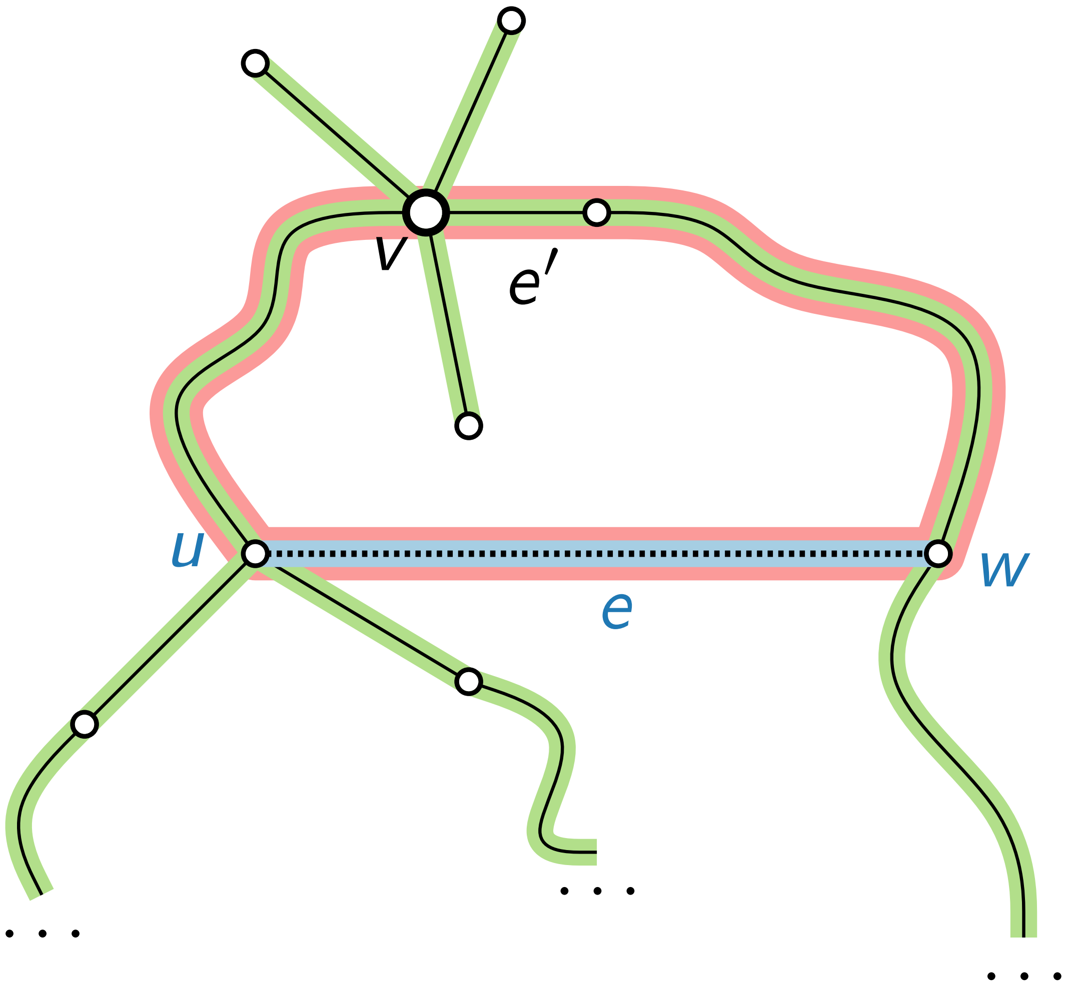


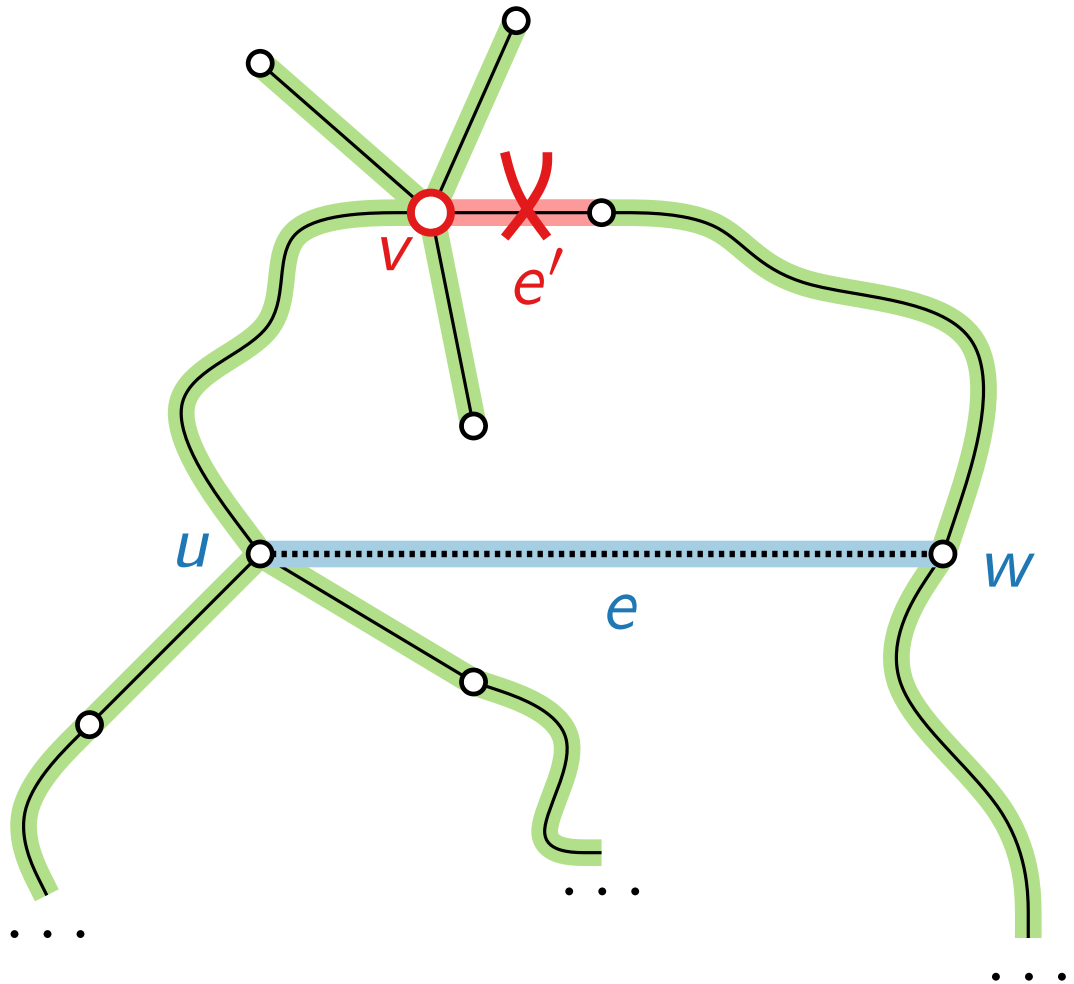
$T + e$

contains a cycle!

$E(T)$

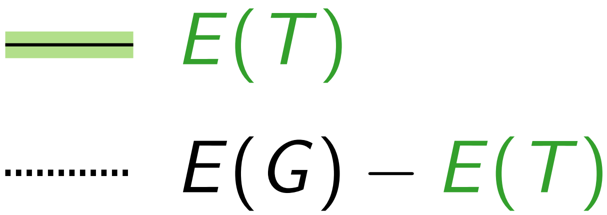
$E(G) - E(T)$

Edge Flips



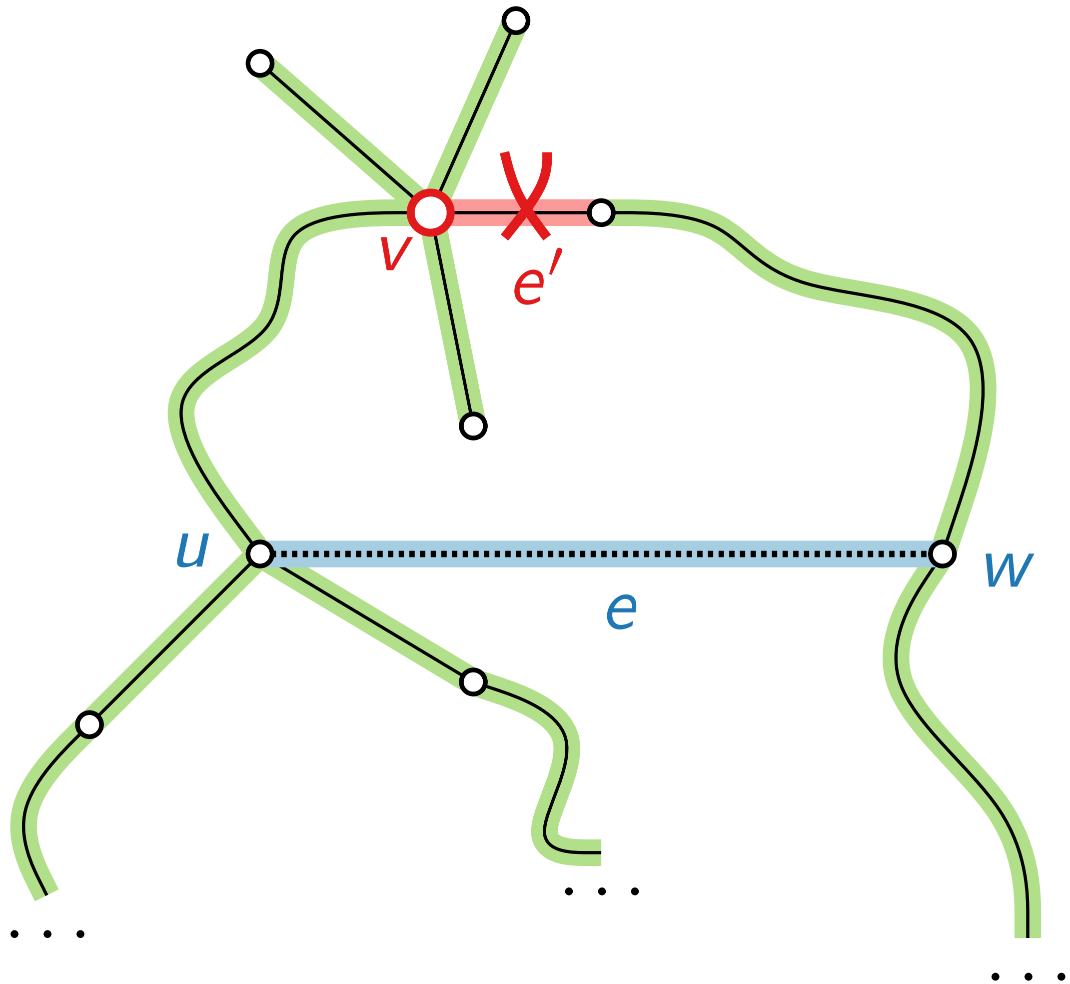
$T + e - e'$

is a new spanning tree.



Edge Flips

Def. An **improving flip** in T for a vertex v and an edge $uw \in E(G) \setminus E(T)$ is a flip with $\deg_T(v) >$



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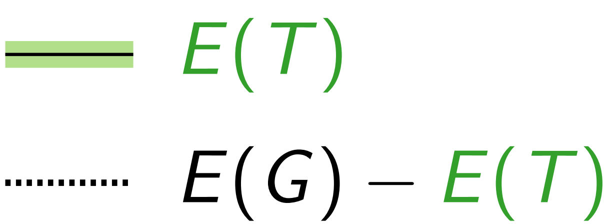
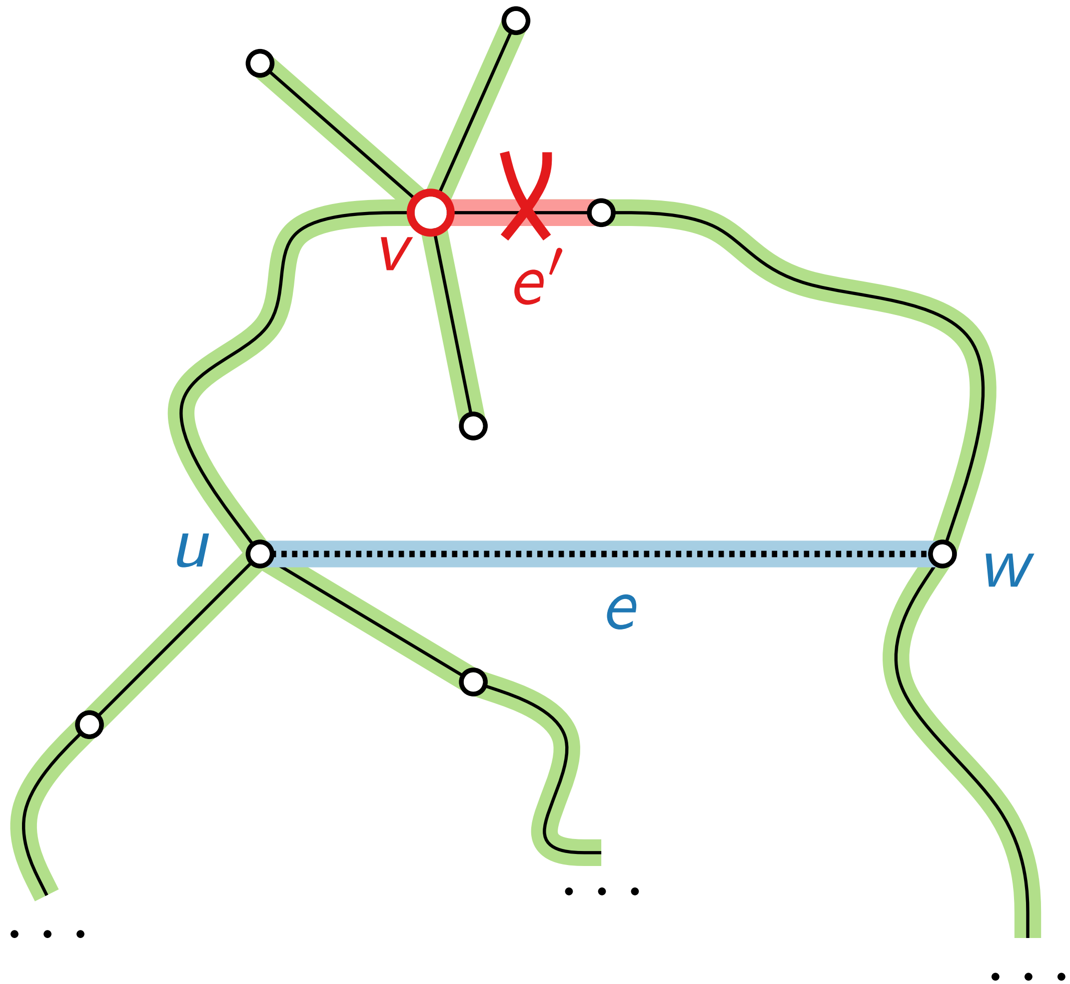


Edge Flips

Def. An **improving flip** in T for a vertex v and an edge $uw \in E(G) \setminus E(T)$ is a flip with $\deg_T(v) > \max\{\deg_T(u), \deg_T(w)\} + 1$.



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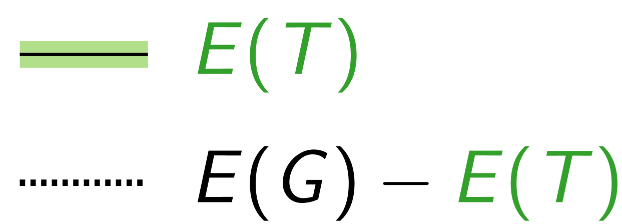

Local Search

MinDegSpanningTreeLocalSearch(graph G)
  $T \leftarrow$ any spanning tree of G
  while $\exists$ improving flip in T for a vertex v
      with $\deg_T(v) \geq \Delta(T) - \ell$ **do**
    do the improving flip
  return T

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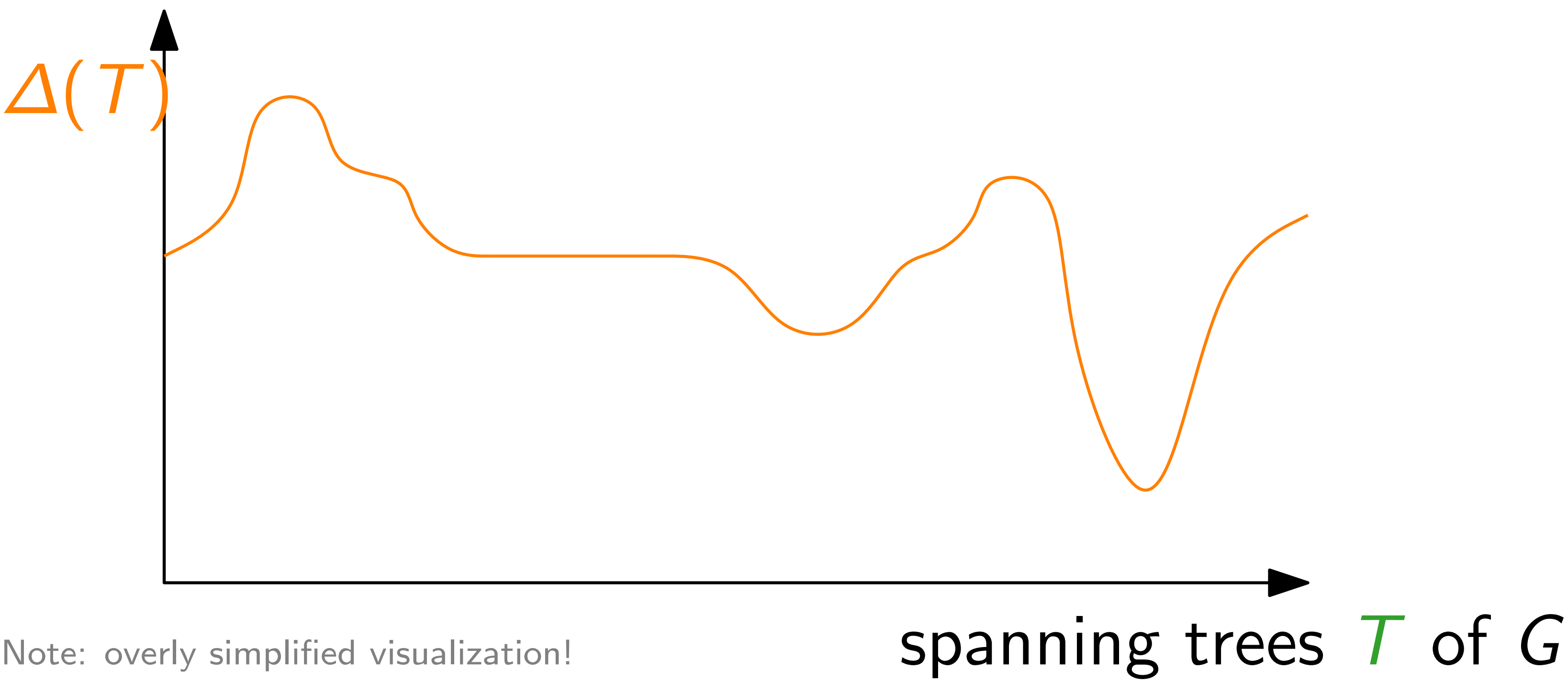

Note: overly simplified visualization!

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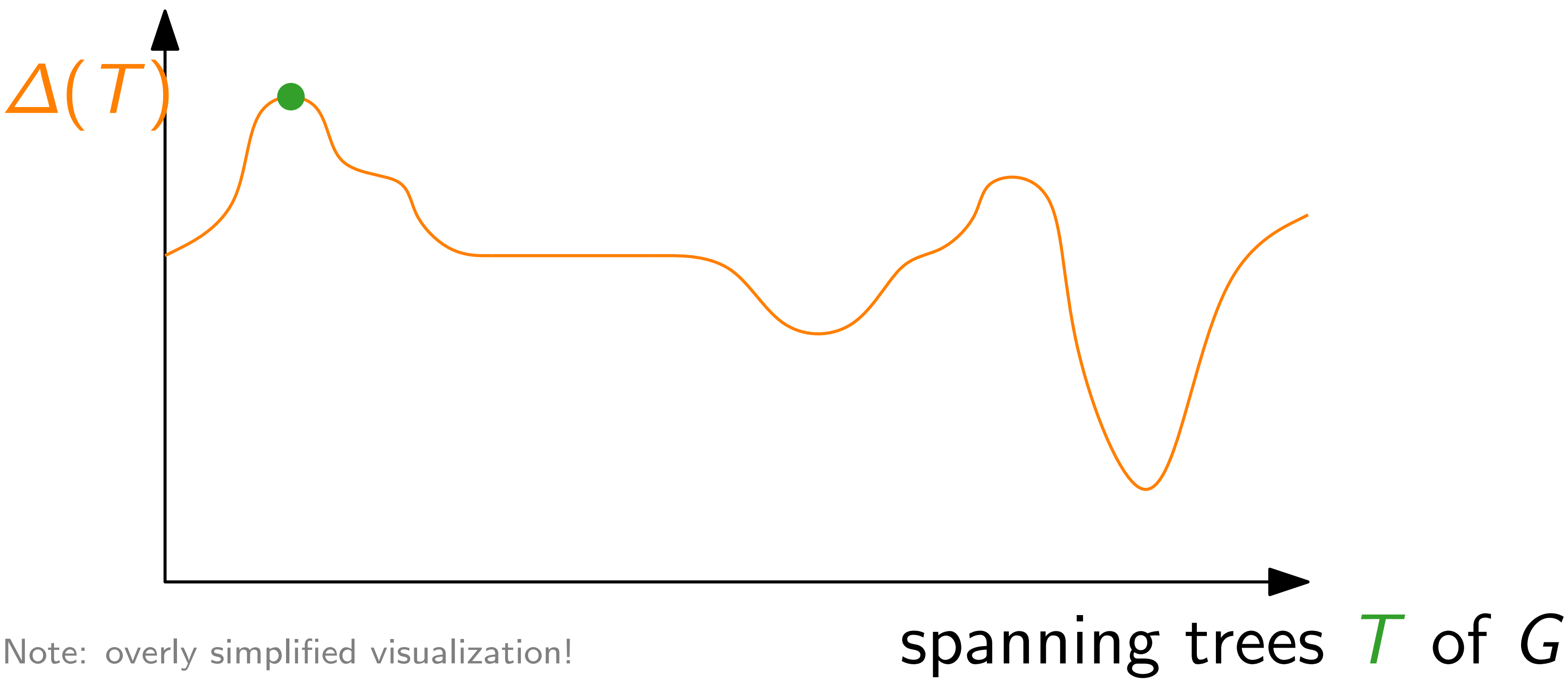

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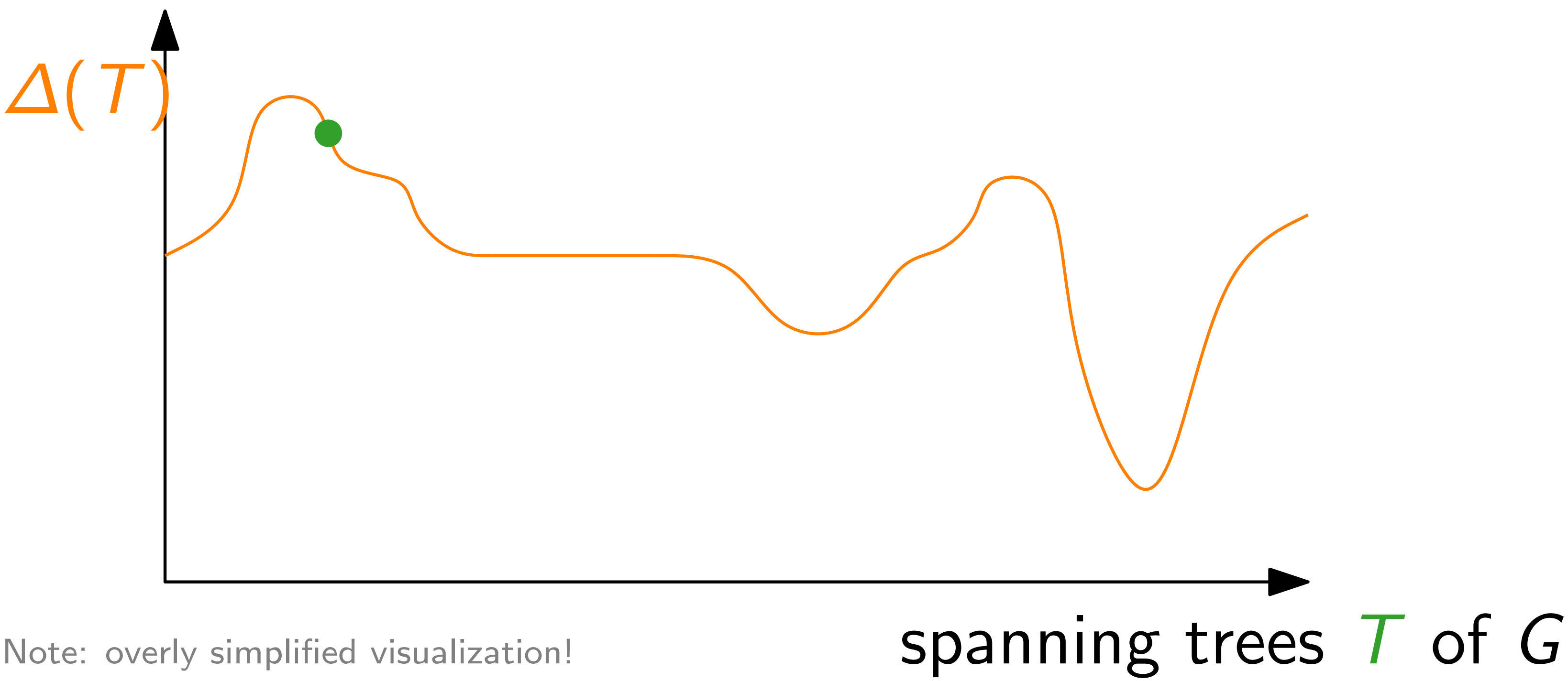

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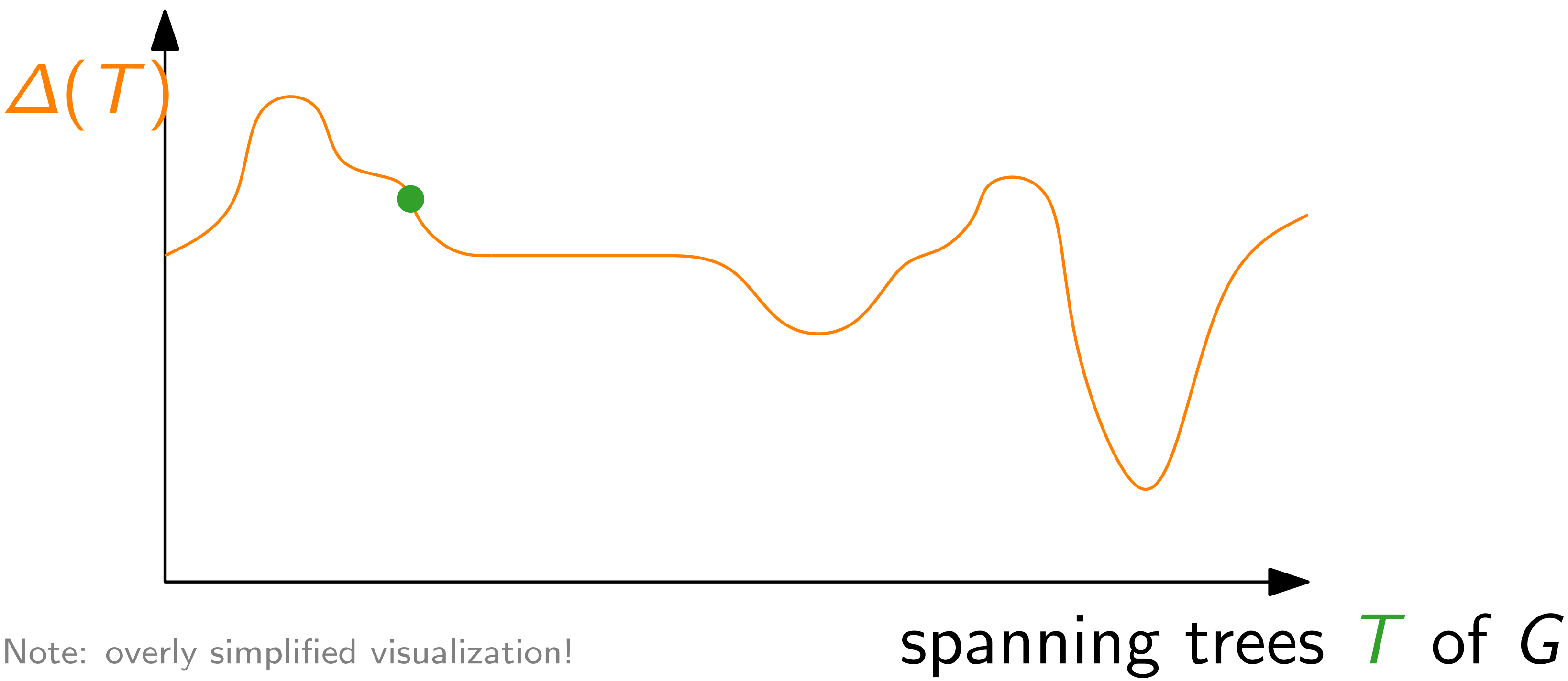

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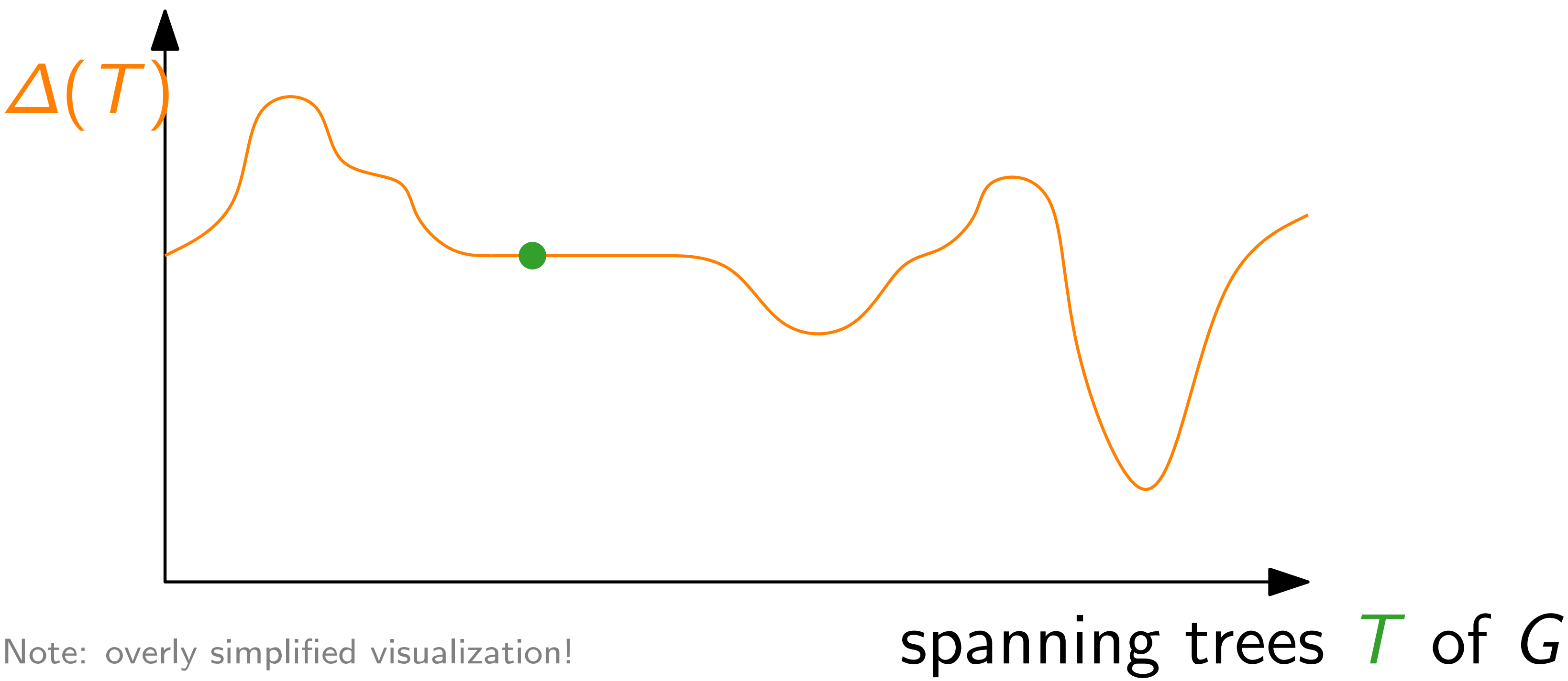

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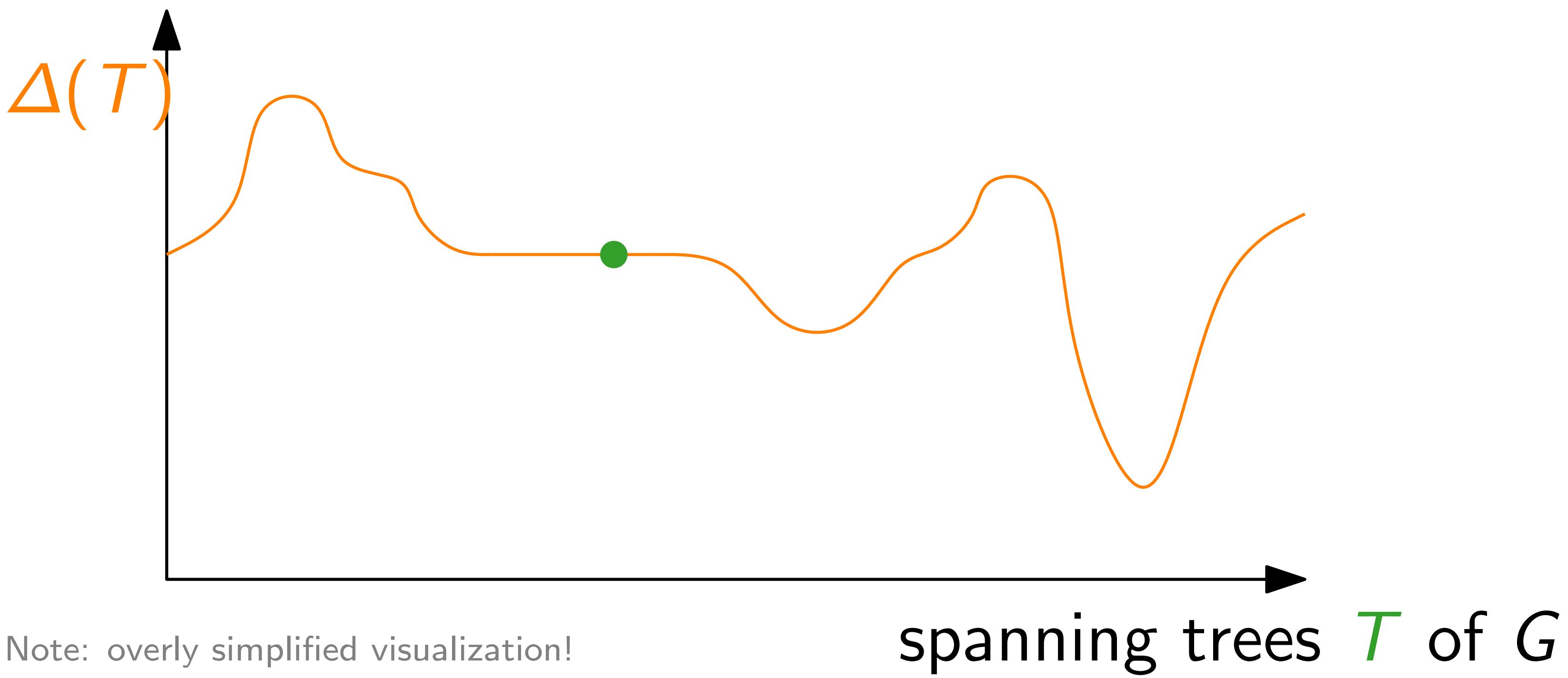

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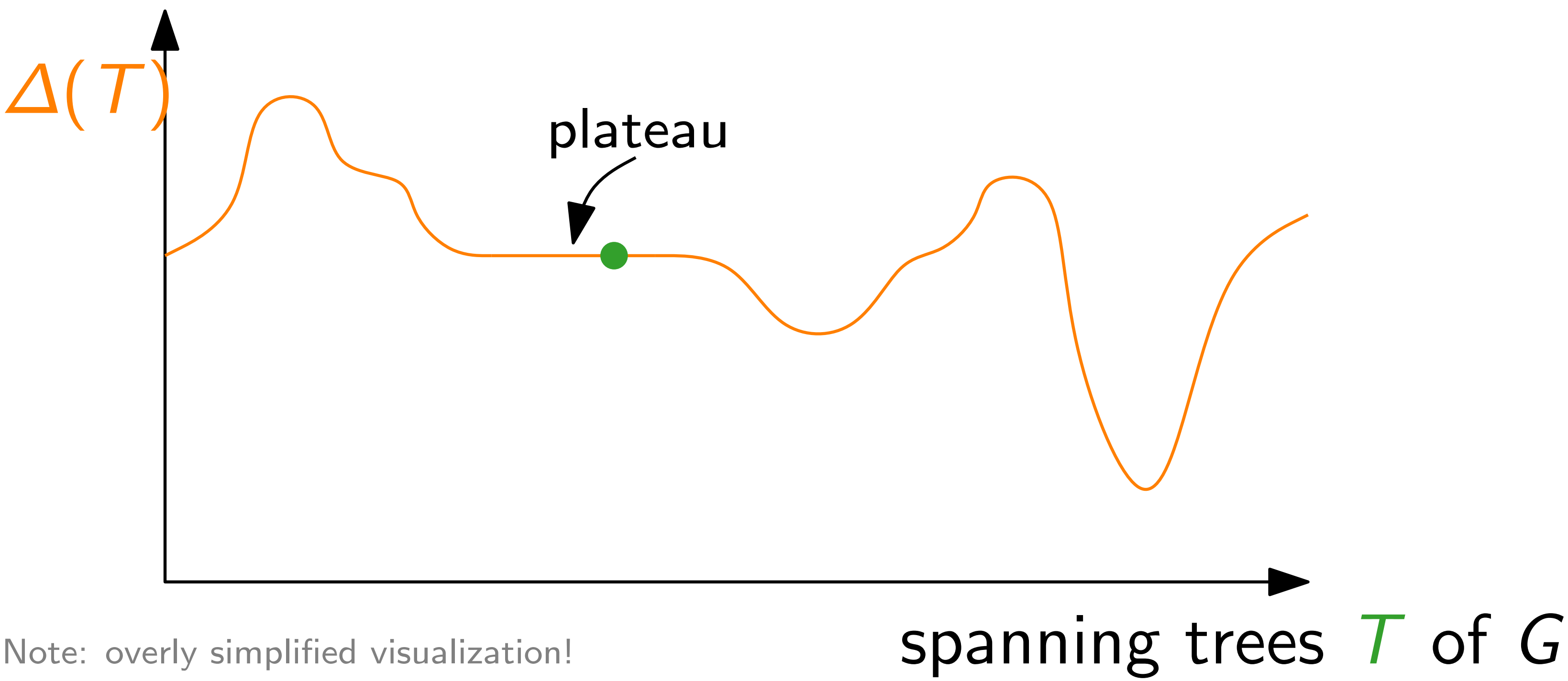

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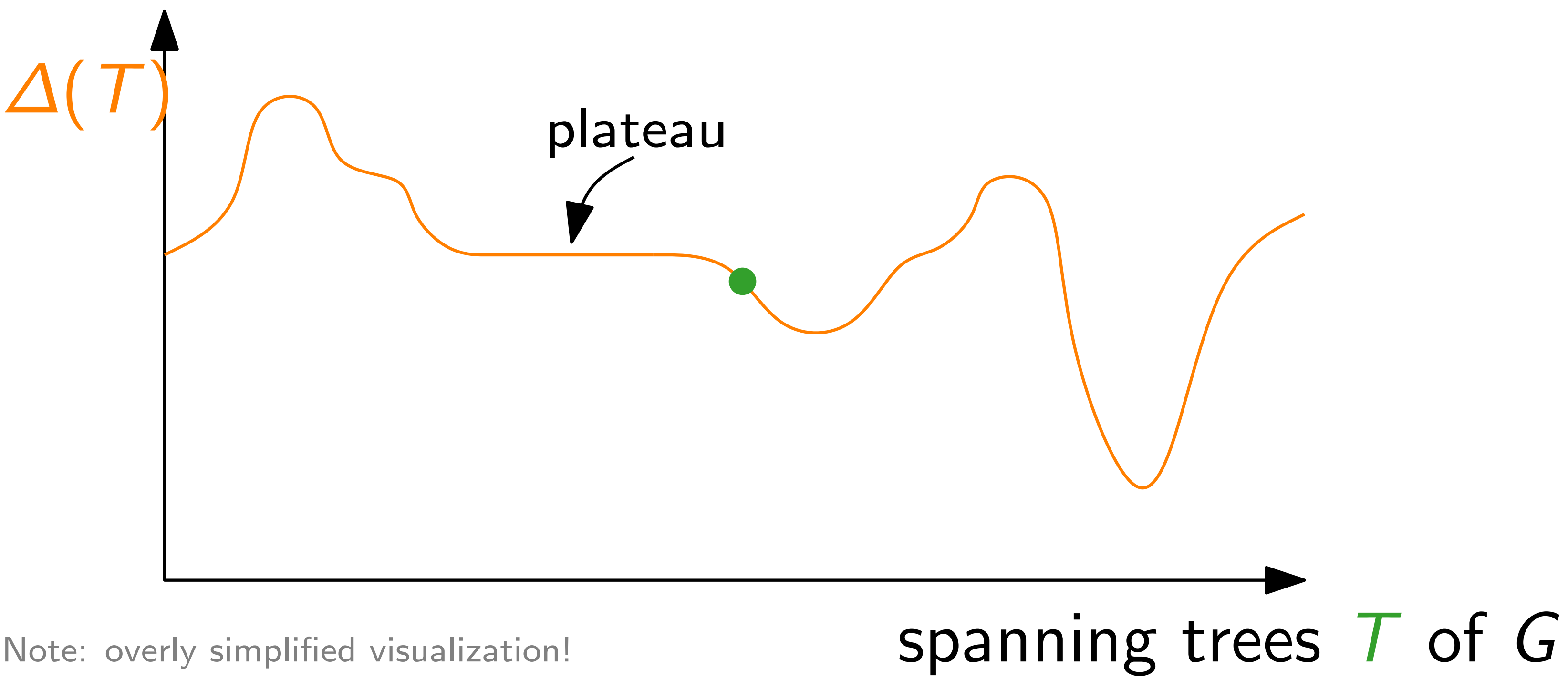

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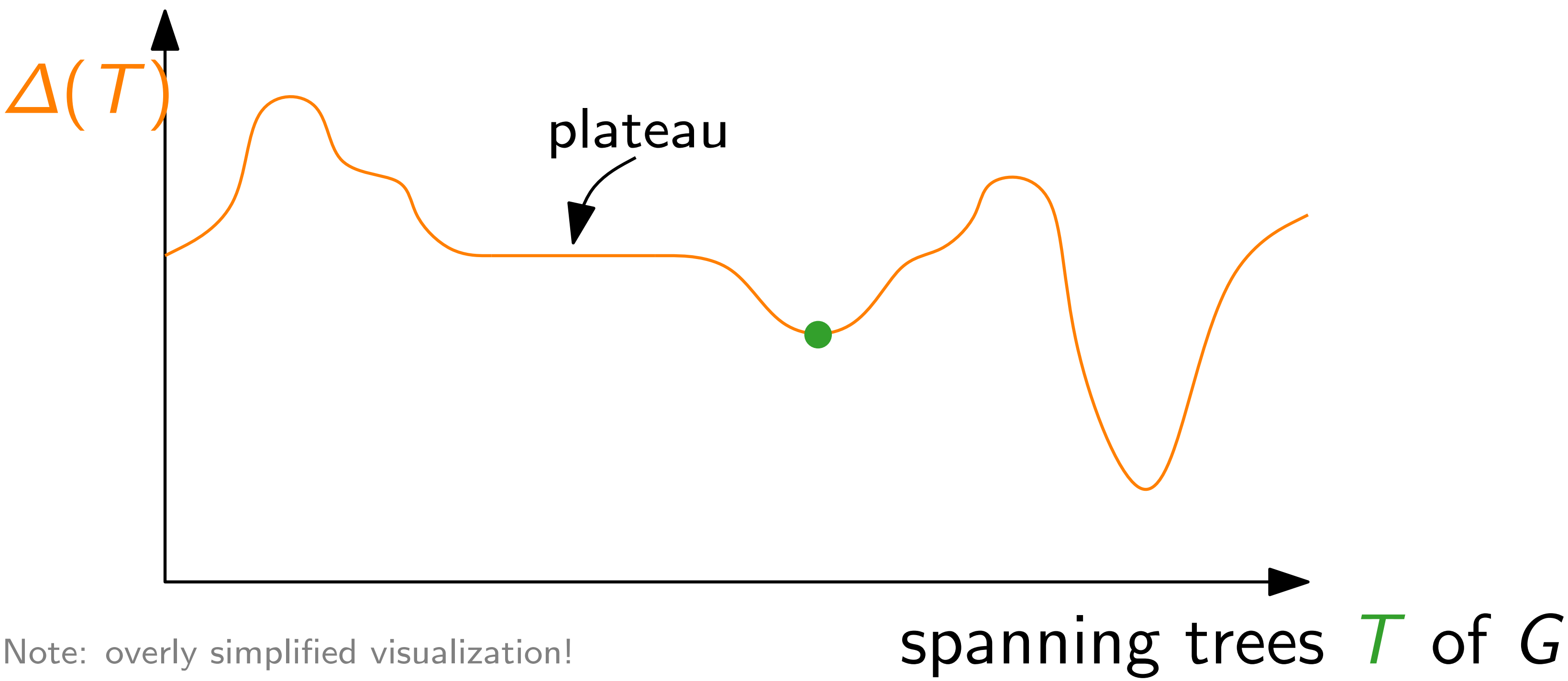

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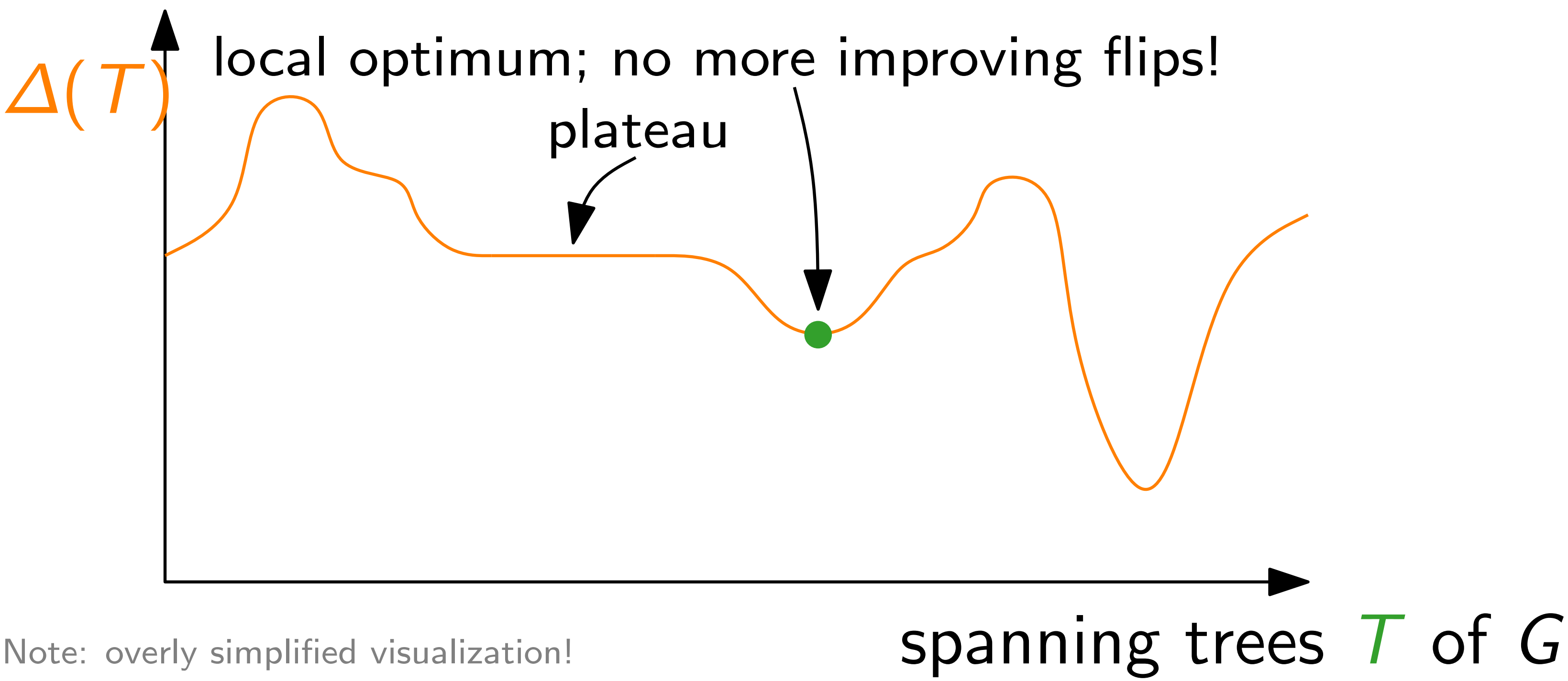

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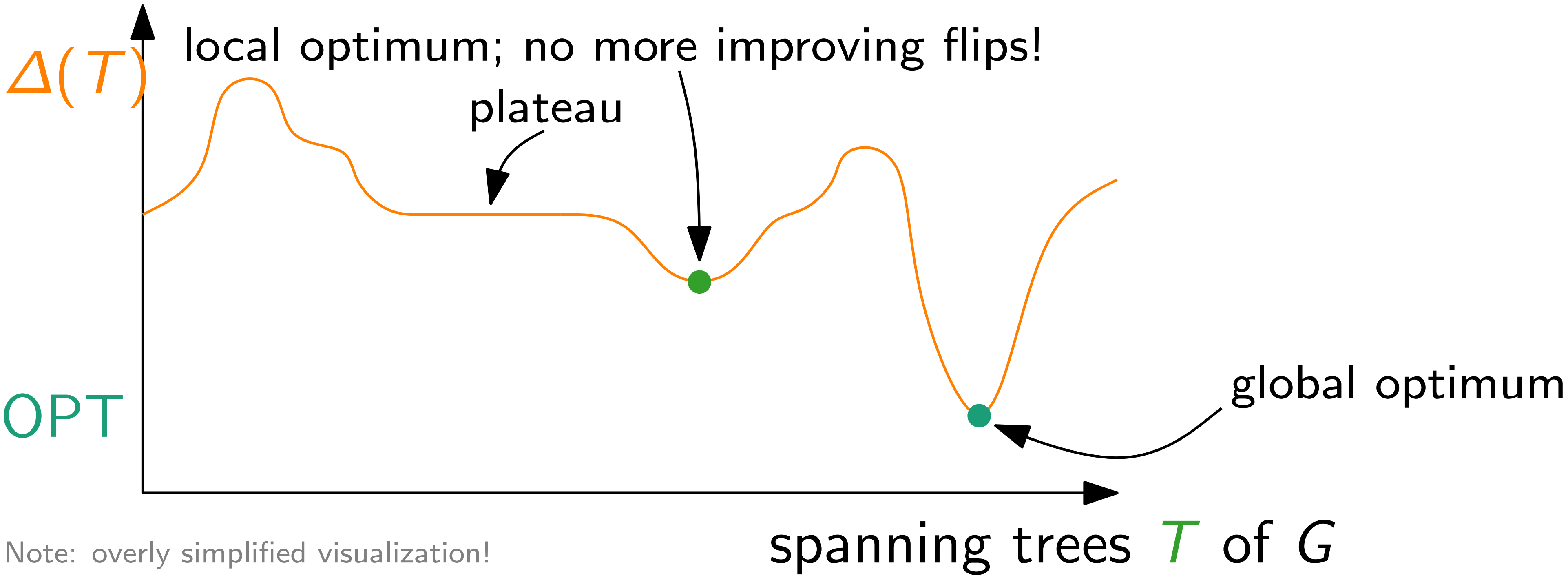

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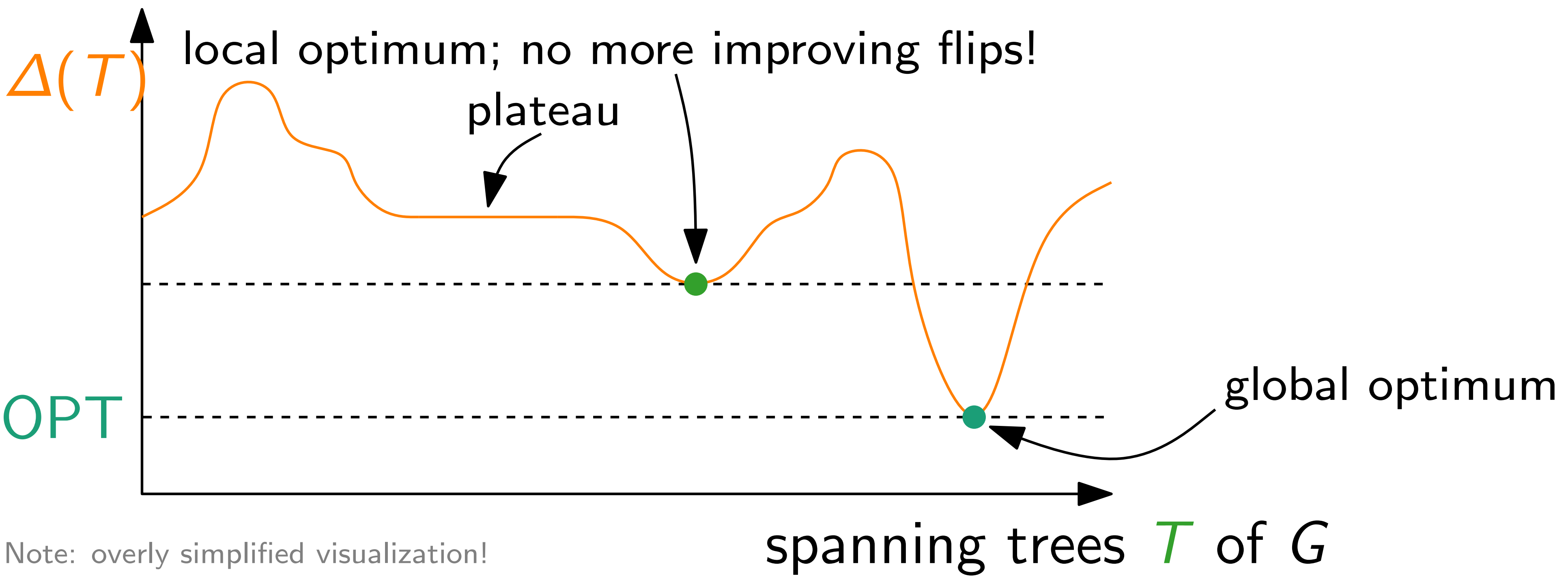

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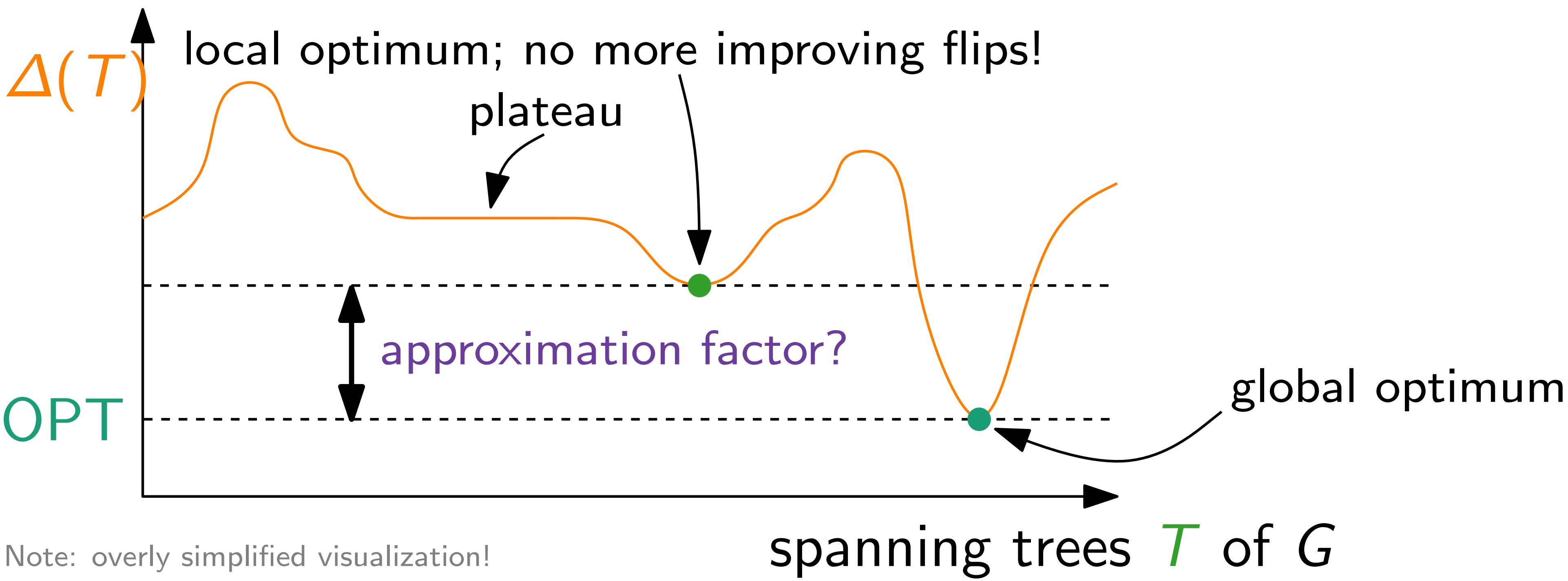

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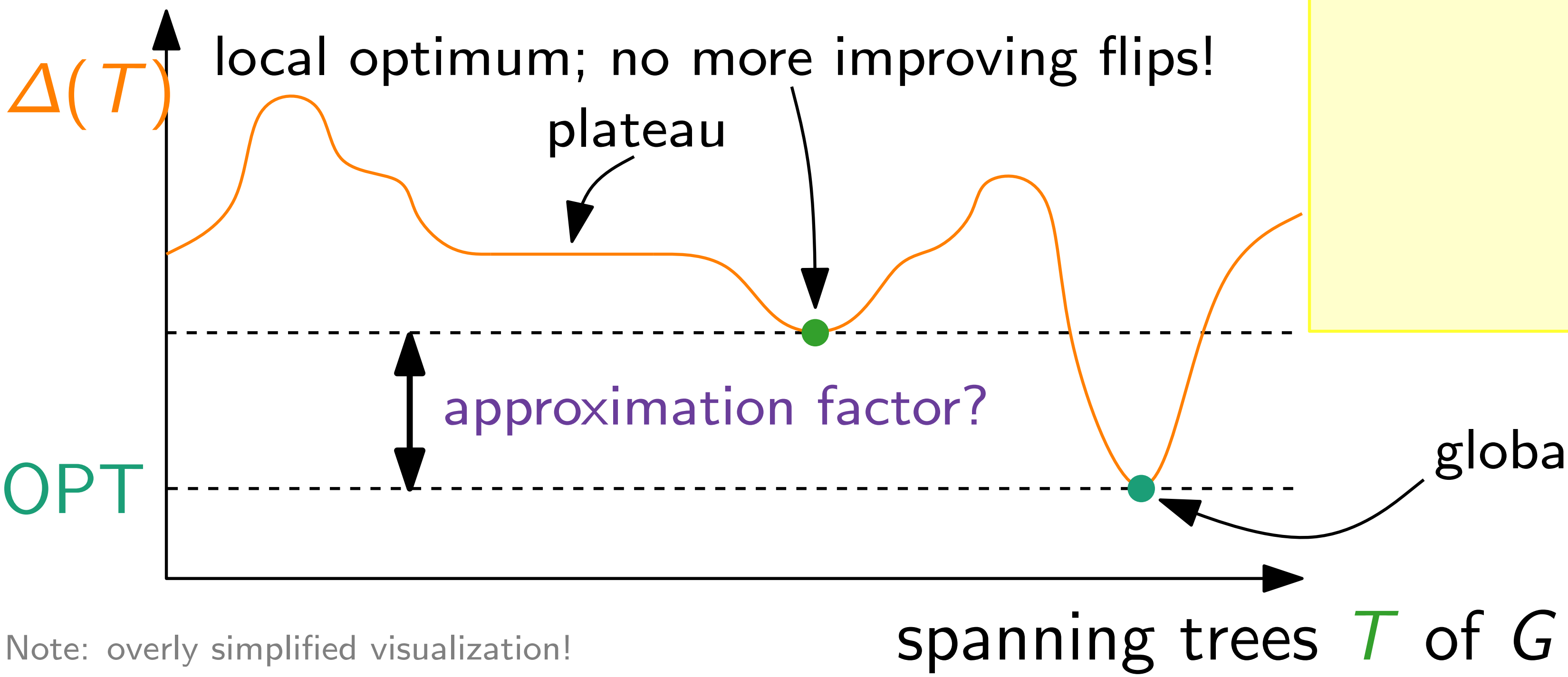
- Termination?



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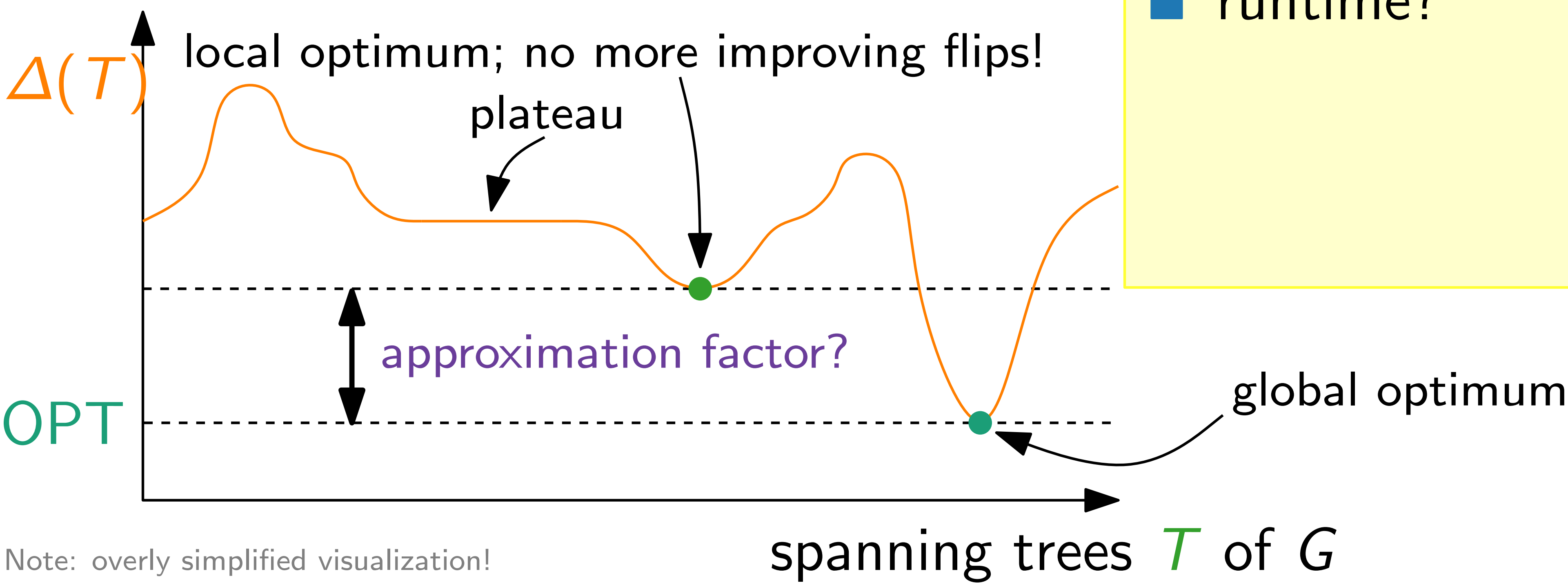
- Termination?
- runtime?



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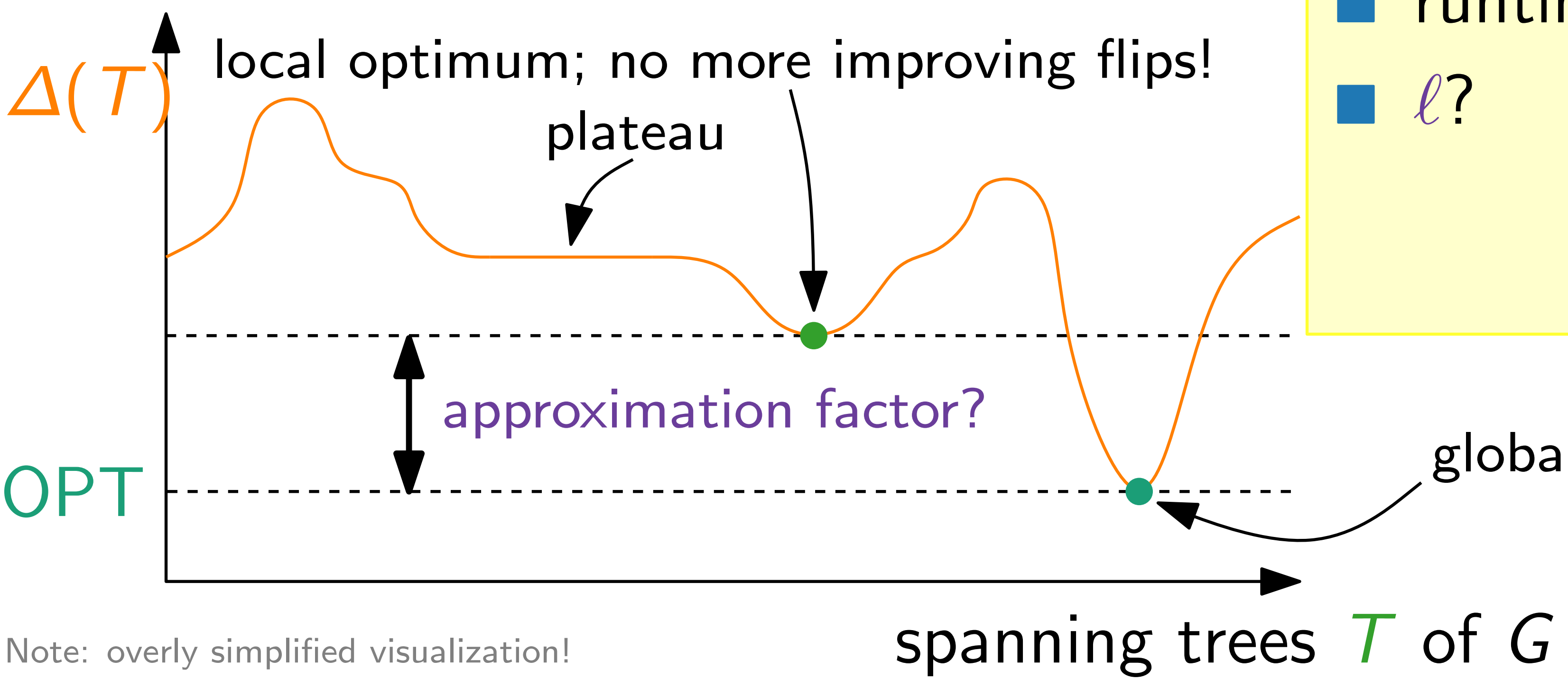
- Termination?
- runtime?
- ℓ?



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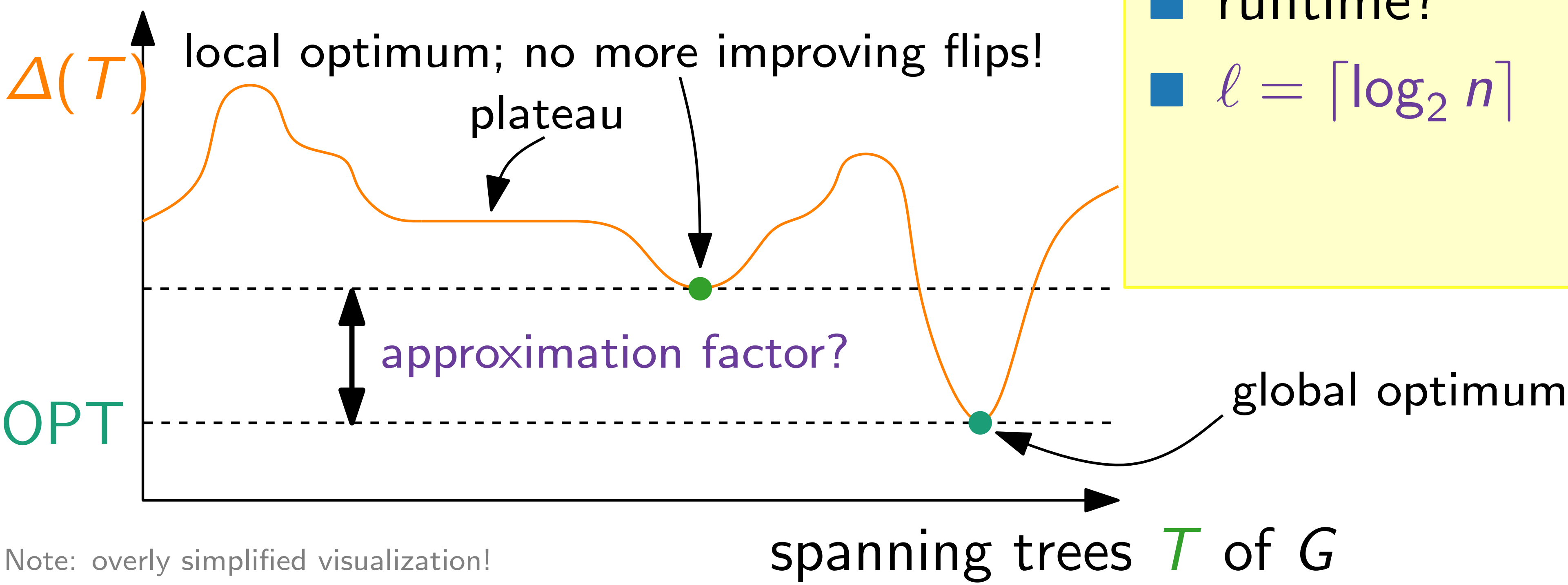
- Termination?
- runtime?
- $\ell = \lceil \log_2 n \rceil$



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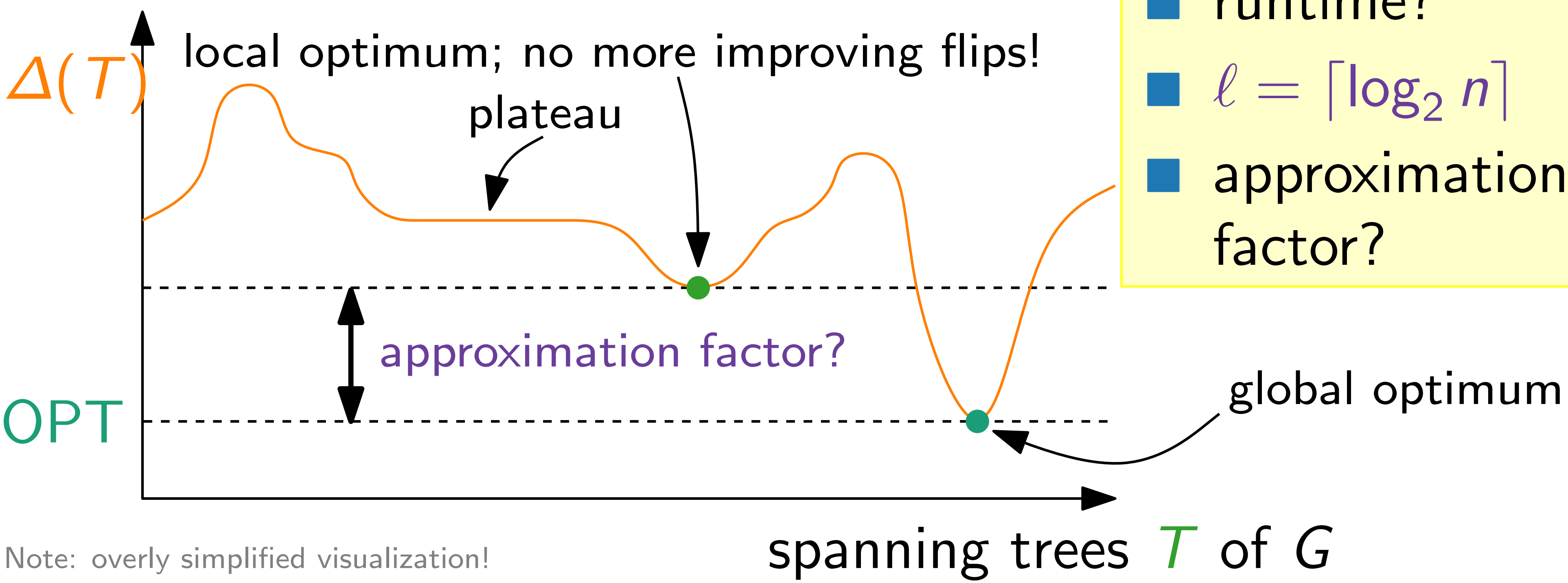
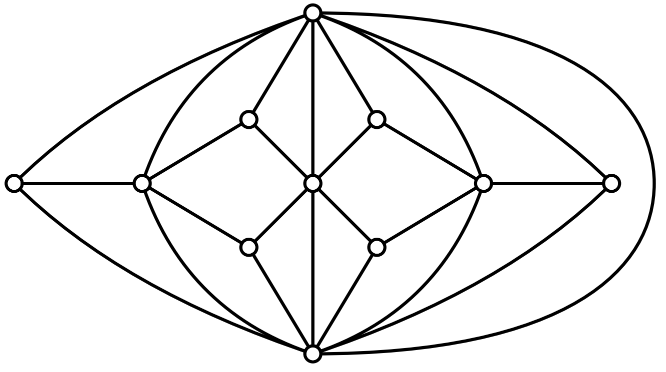
- Termination?
- runtime?
- $\ell = \lceil \log_2 n \rceil$
- approximation factor?

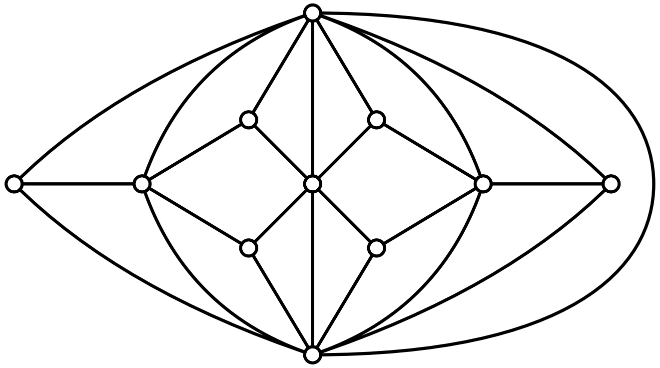


Note: overly simplified visualization!

Example

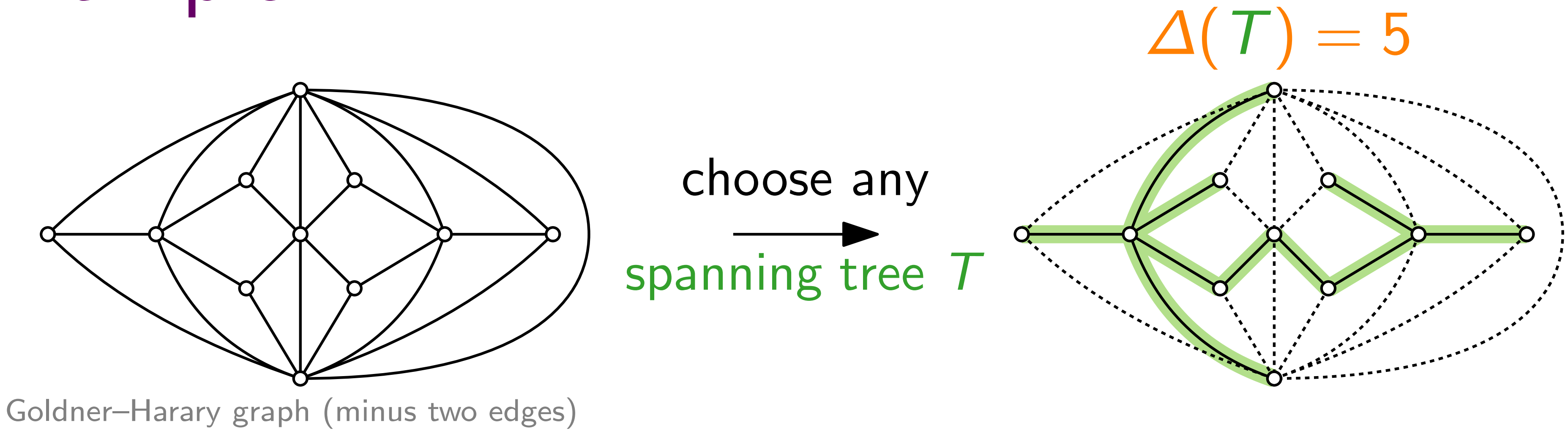


Example

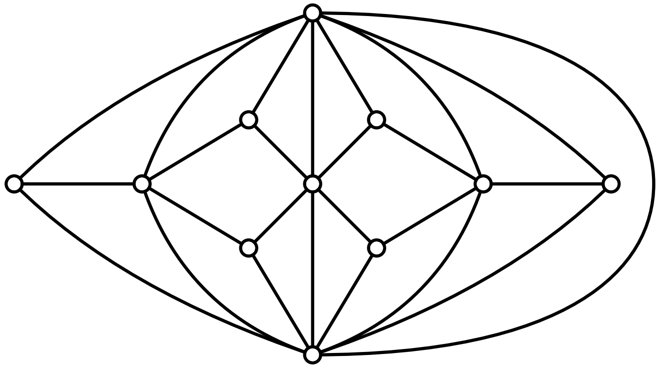


Goldner–Harary graph (minus two edges)

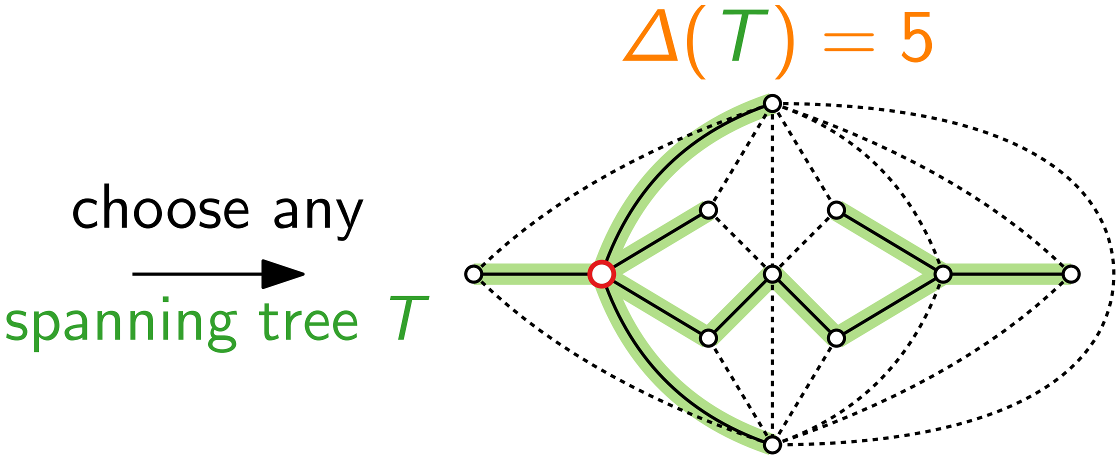
Example


$\Delta(T) = 5$
choose any
spanning tree T
Goldner–Harary graph (minus two edges)

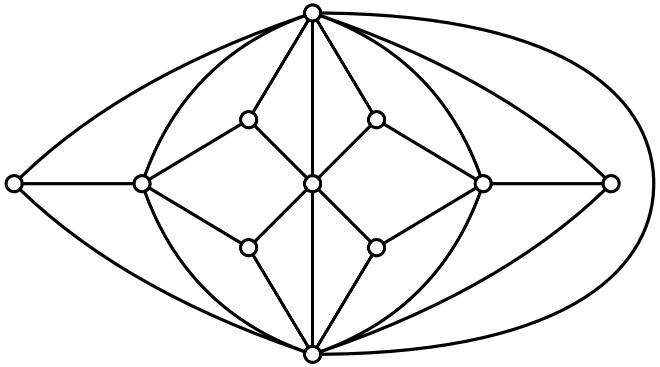
Example



Goldner–Harary graph (minus two edges)



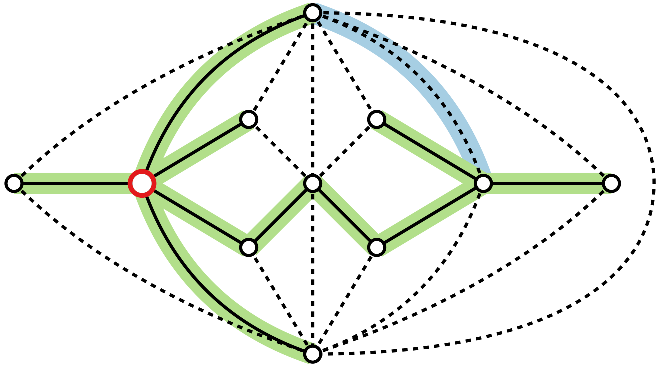
Example



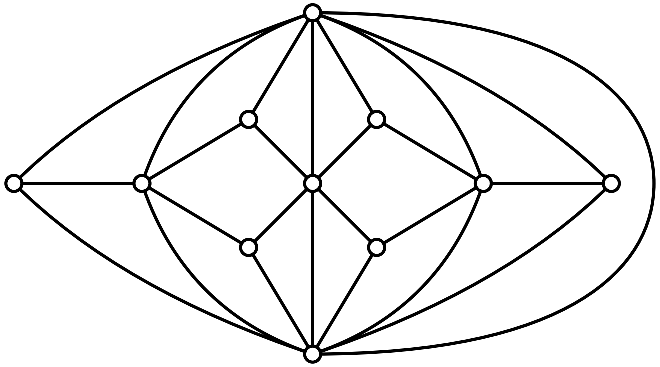
Goldner–Harary graph (minus two edges)

choose any → spanning tree T

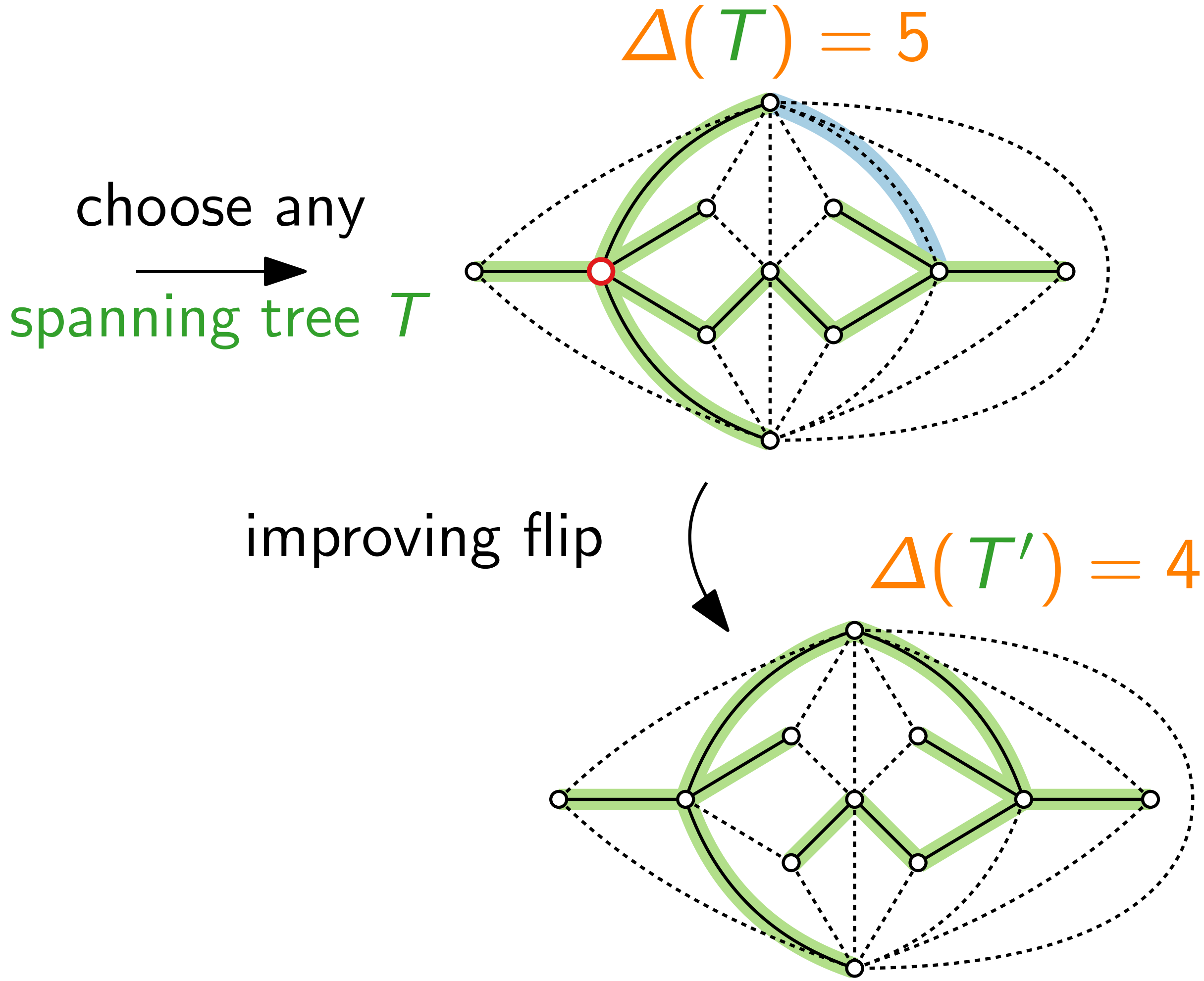
$\Delta(T) = 5$



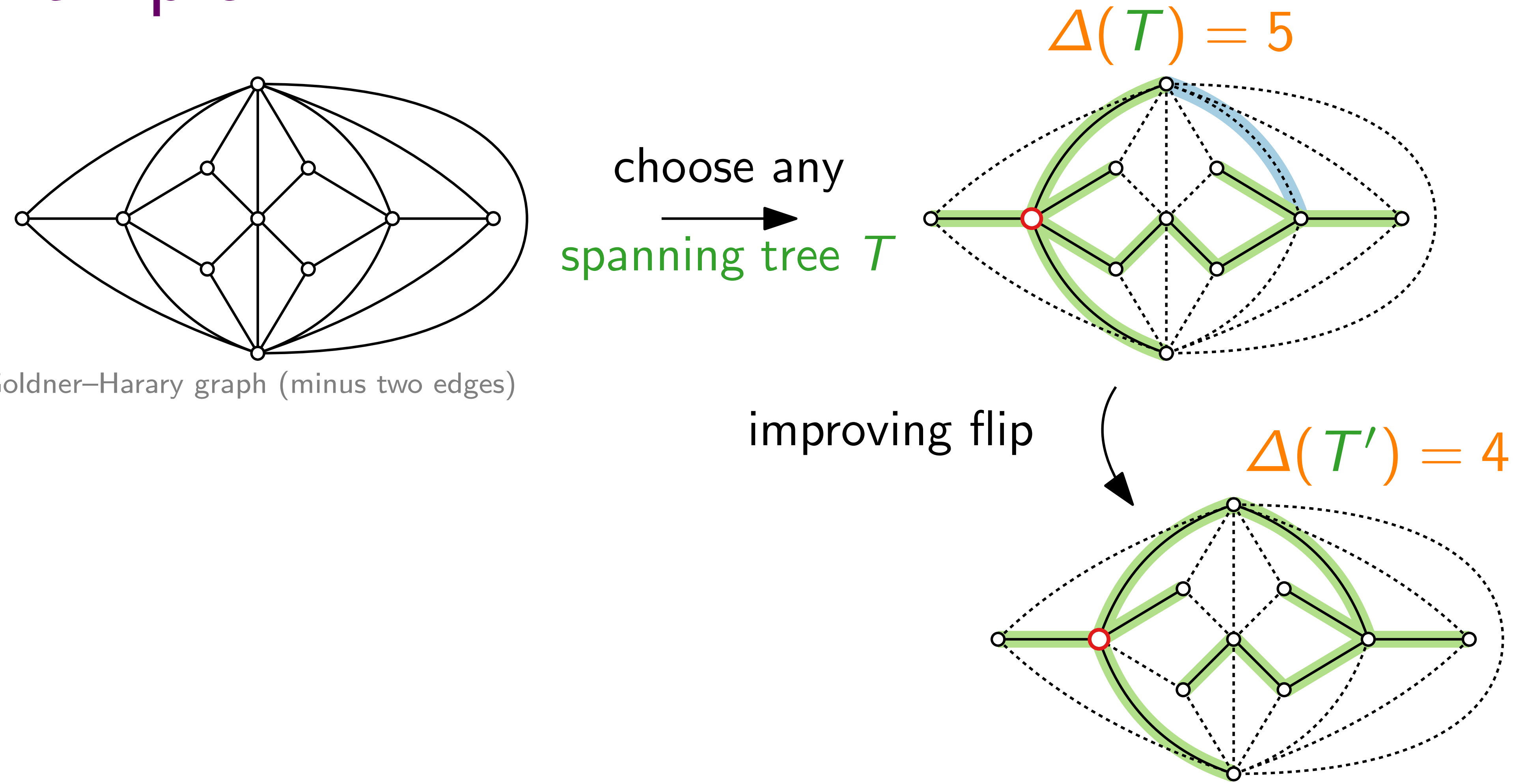
Example



Goldner–Harary graph (minus two edges)

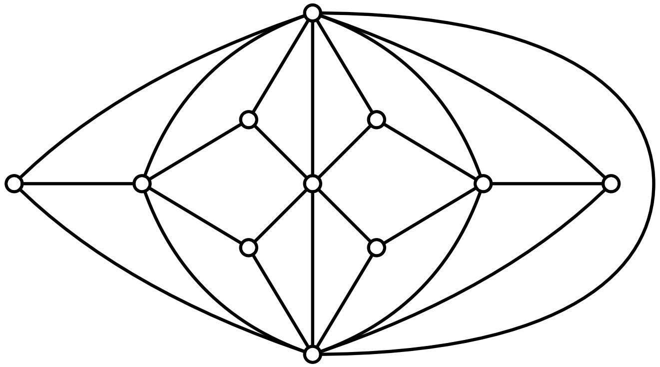


Example

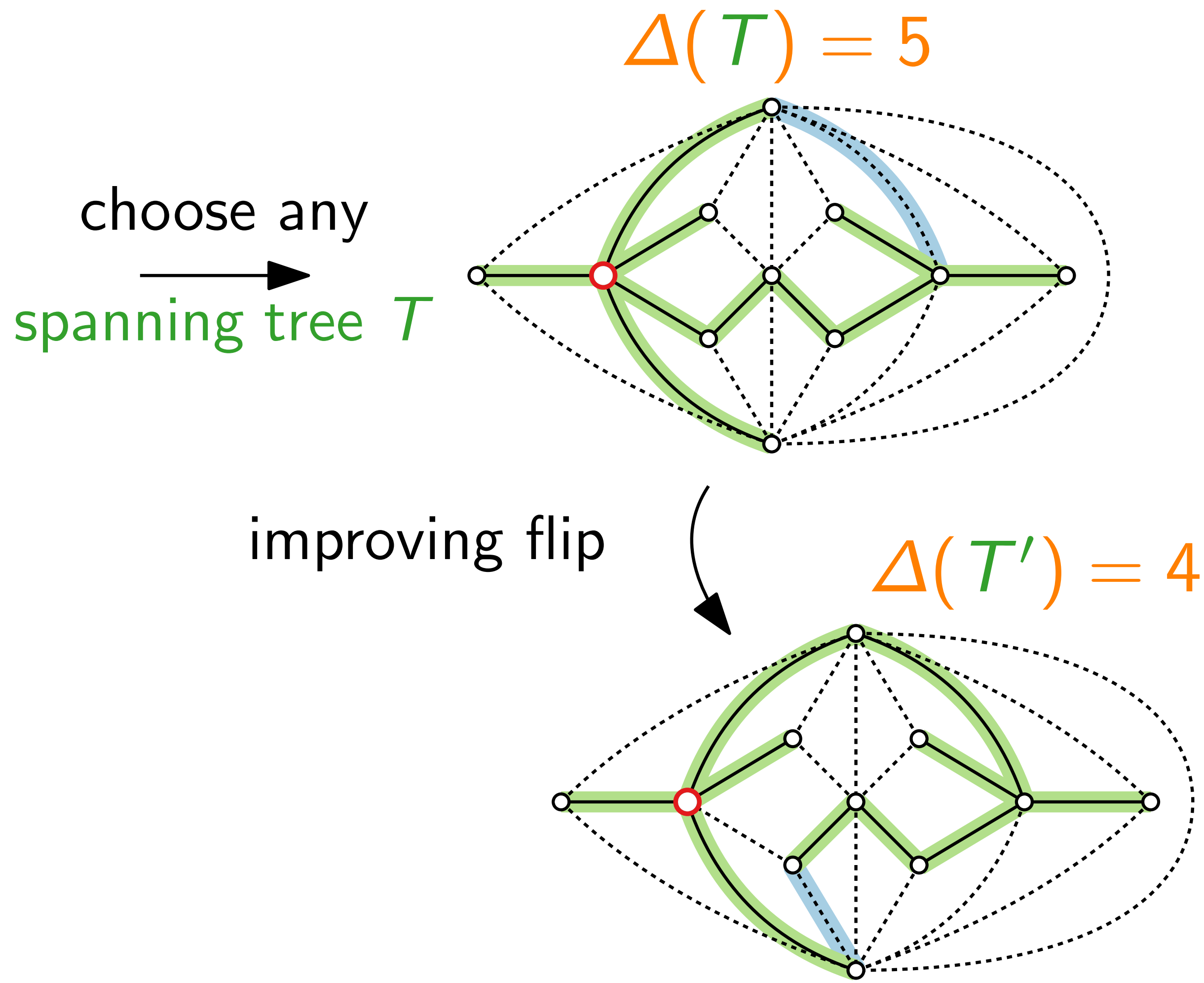


Goldner–Harary graph (minus two edges)
choose any
spanning tree T
$\Delta(T) = 5$
improving flip
$\Delta(T') = 4$

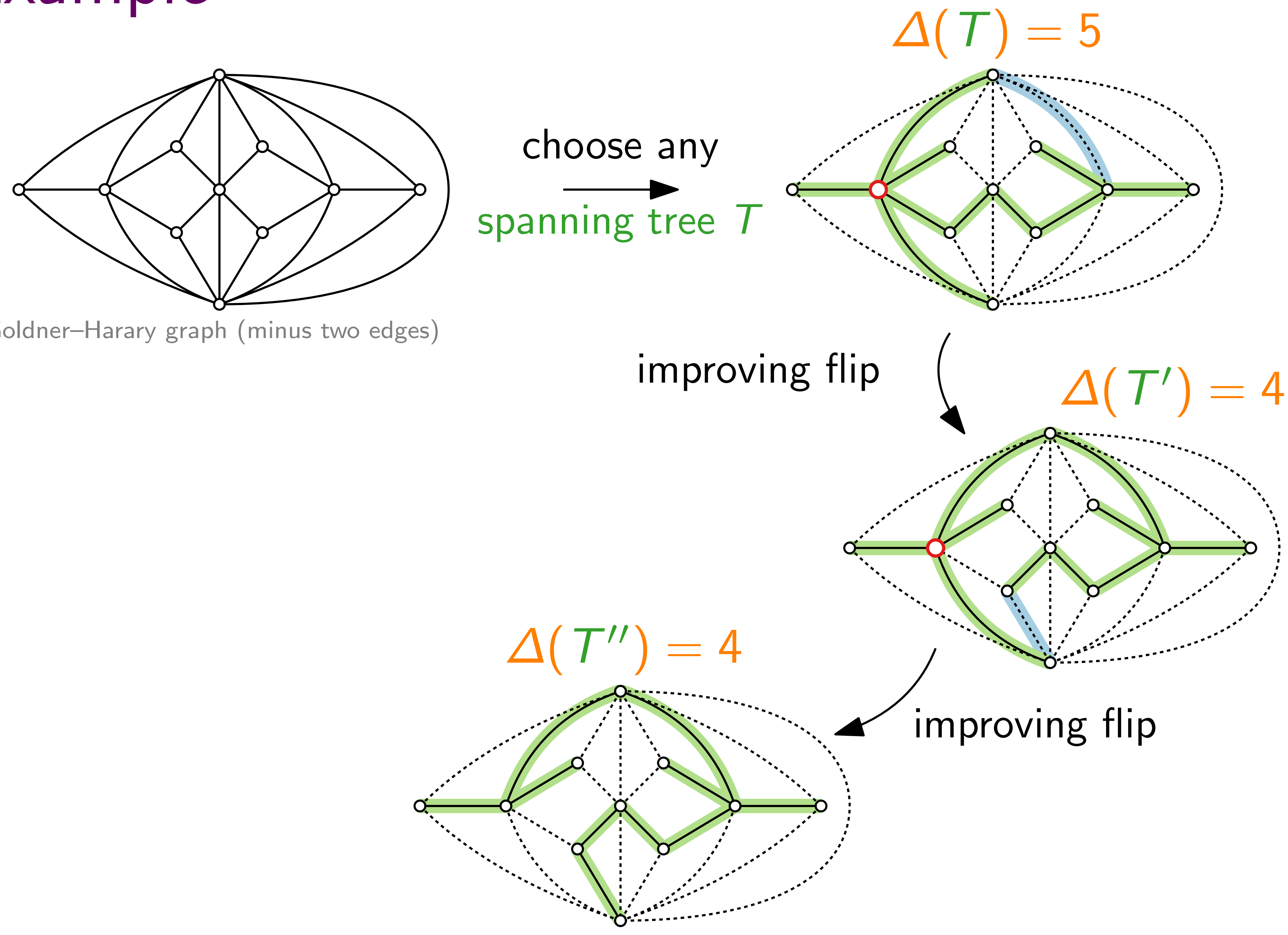
Example



Goldner–Harary graph (minus two edges)

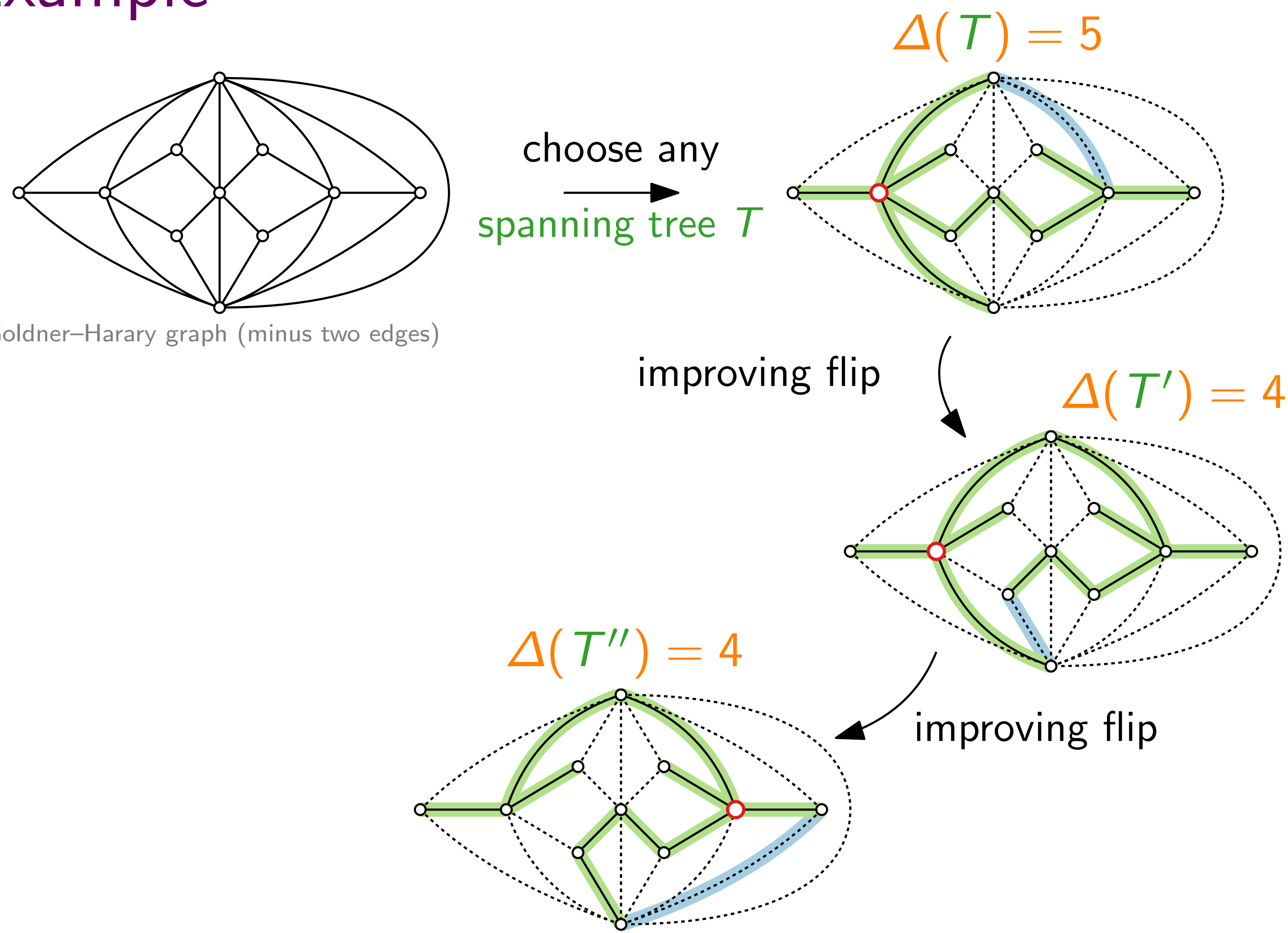


Example



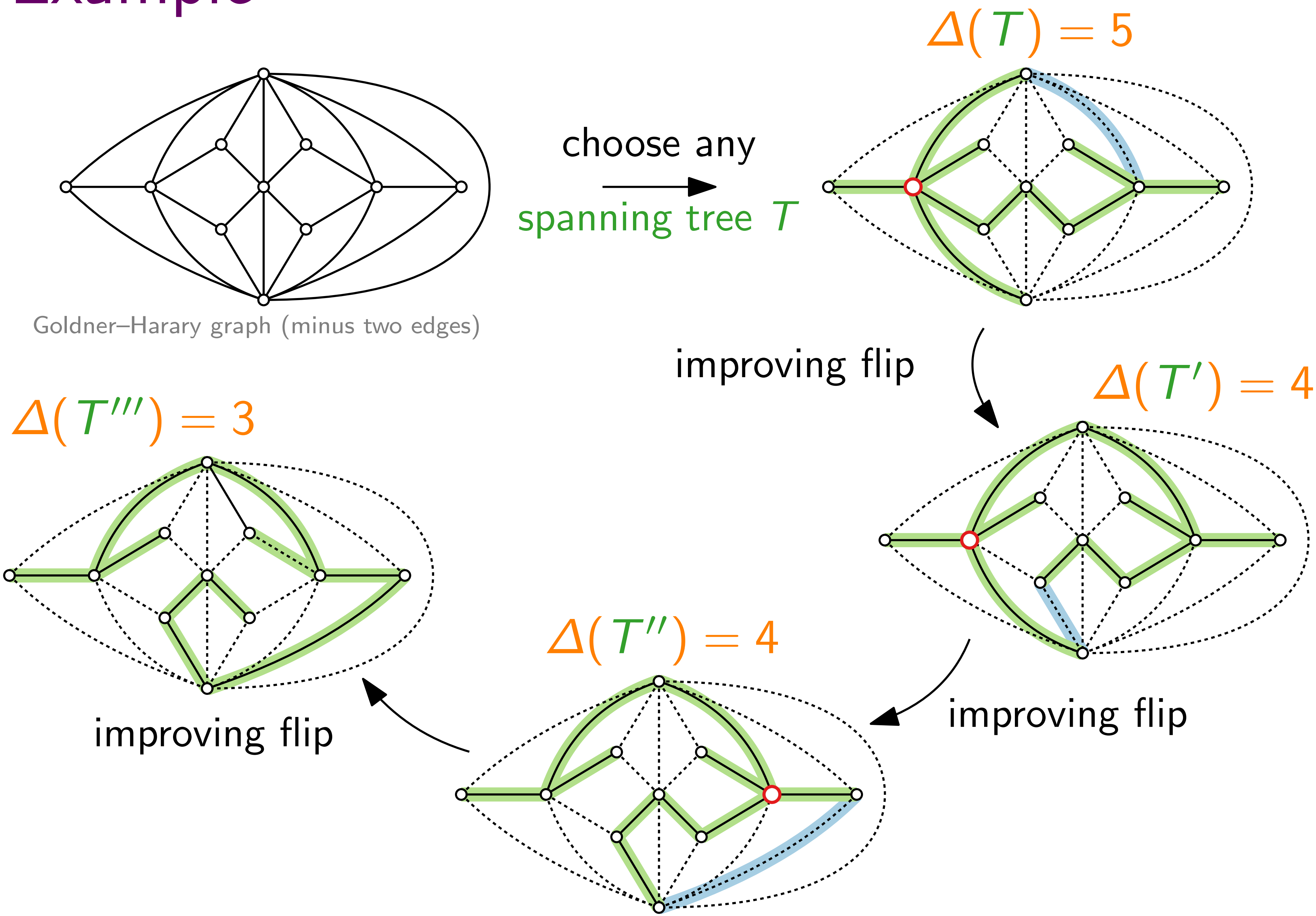
Goldner–Harary graph (minus two edges)
choose any
spanning tree T
$\Delta(T) = 5$
improving flip
$\Delta(T') = 4$
improving flip
$\Delta(T'') = 4$

Example



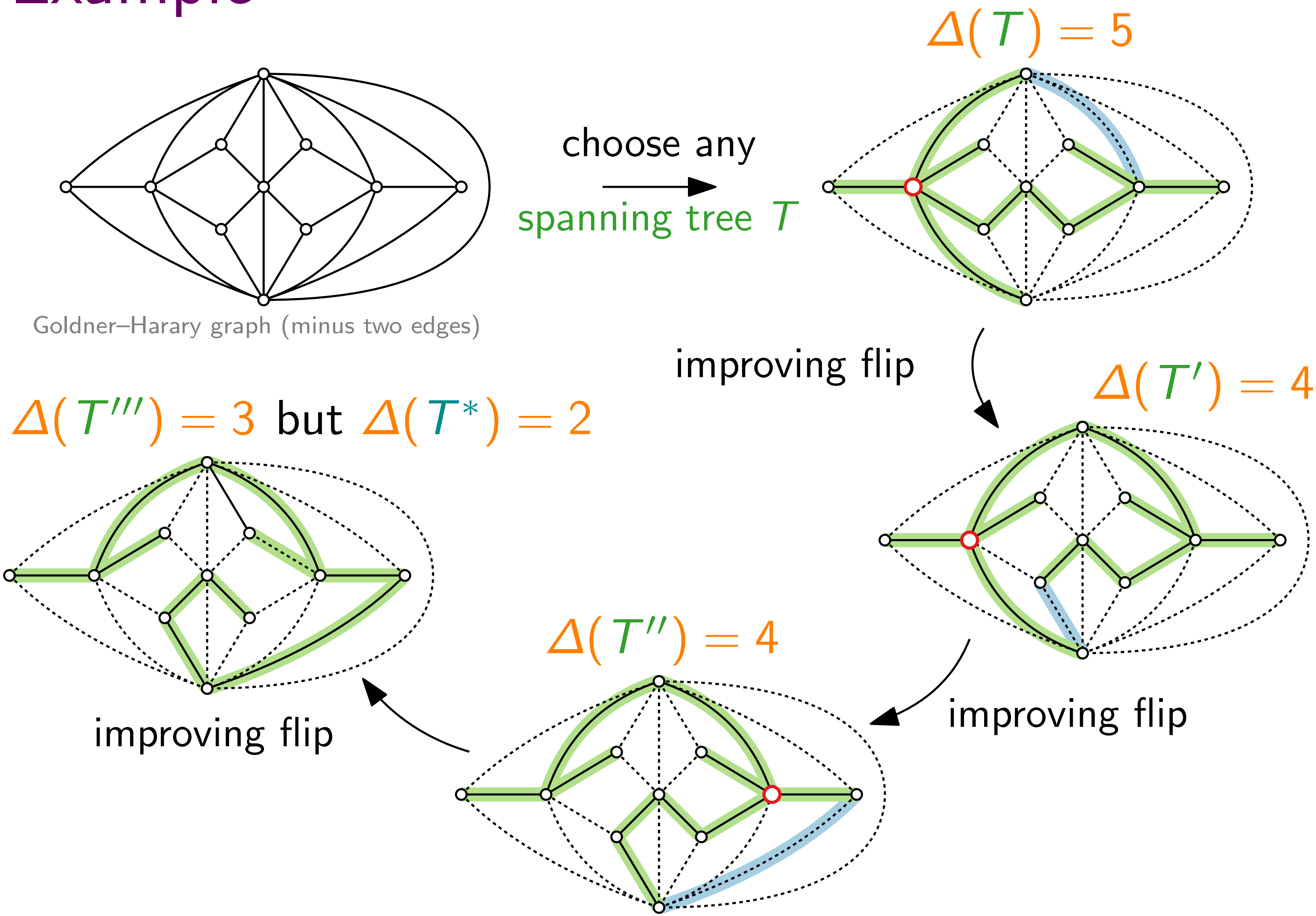
Goldner–Harary graph (minus two edges)

Example



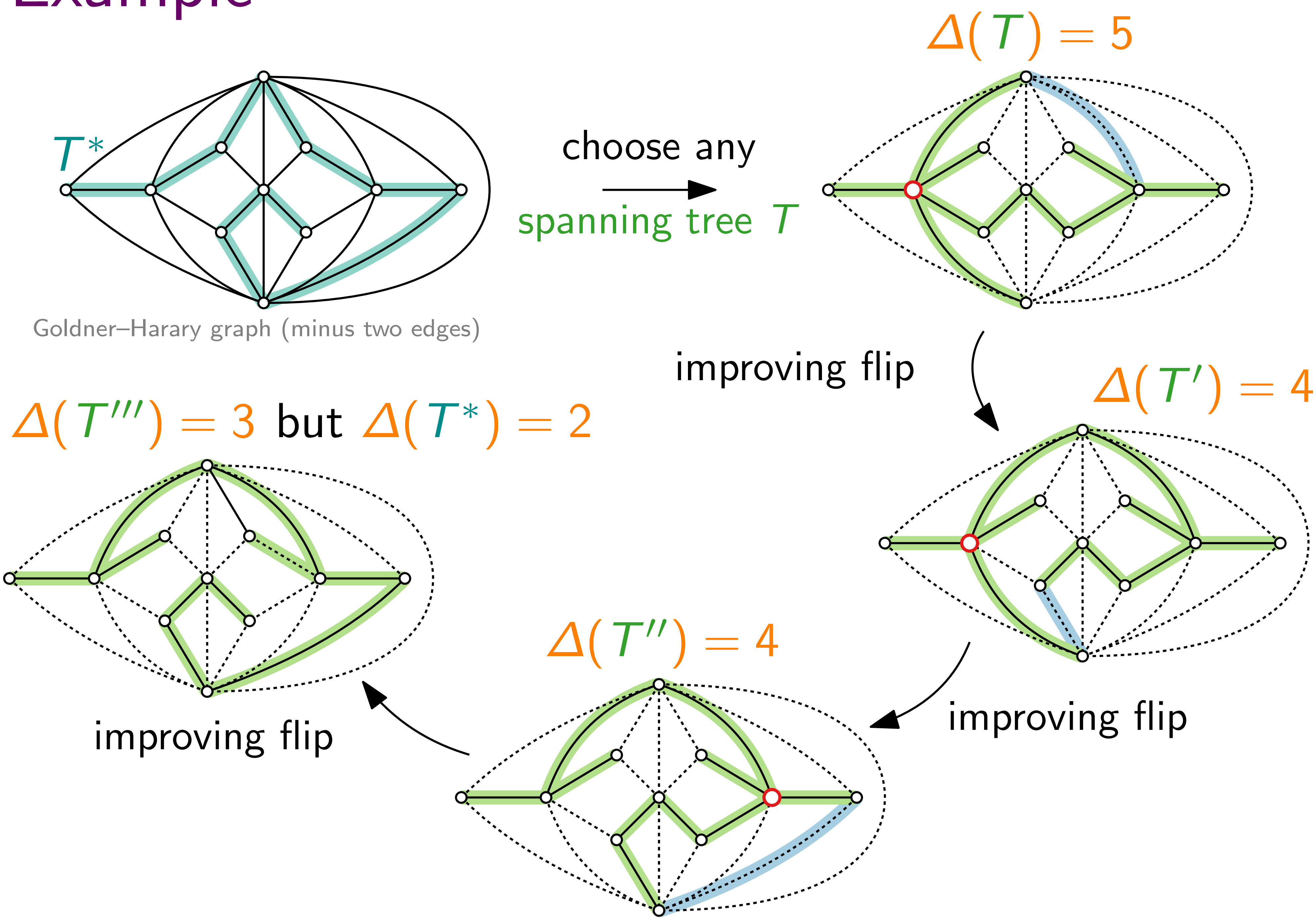
Goldner–Harary graph (minus two edges)
choose any
spanning tree T
$\Delta(T) = 5$
improving flip
$\Delta(T') = 4$
improving flip
$\Delta(T'') = 4$
improving flip
$\Delta(T''') = 3$

Example



Goldner–Harary graph (minus two edges)
choose any
spanning tree T
$\Delta(T) = 5$
improving flip
$\Delta(T') = 4$
improving flip
$\Delta(T'') = 4$
improving flip
$\Delta(T''') = 3$ but $\Delta(T^*) = 2$

Example



T^*
Goldner–Harary graph (minus two edges)
choose any
spanning tree T
$\Delta(T) = 5$
improving flip
$\Delta(T') = 4$
improving flip
$\Delta(T'') = 4$
improving flip
$\Delta(T''') = 3$ but $\Delta(T^*) = 2$

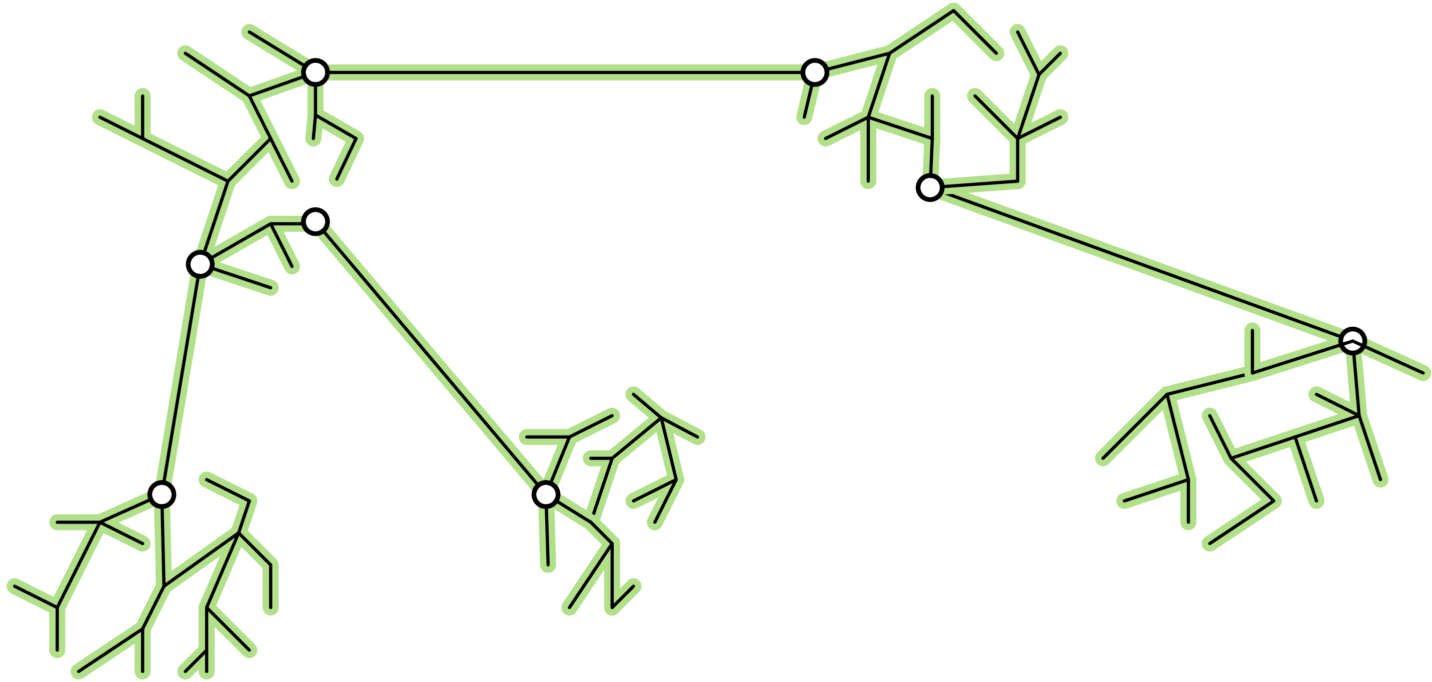
Approximation Algorithms

Lecture 10:
MINIMUM-DEGREE SPANNING TREE
via Local Search

Part III:
Lower Bound

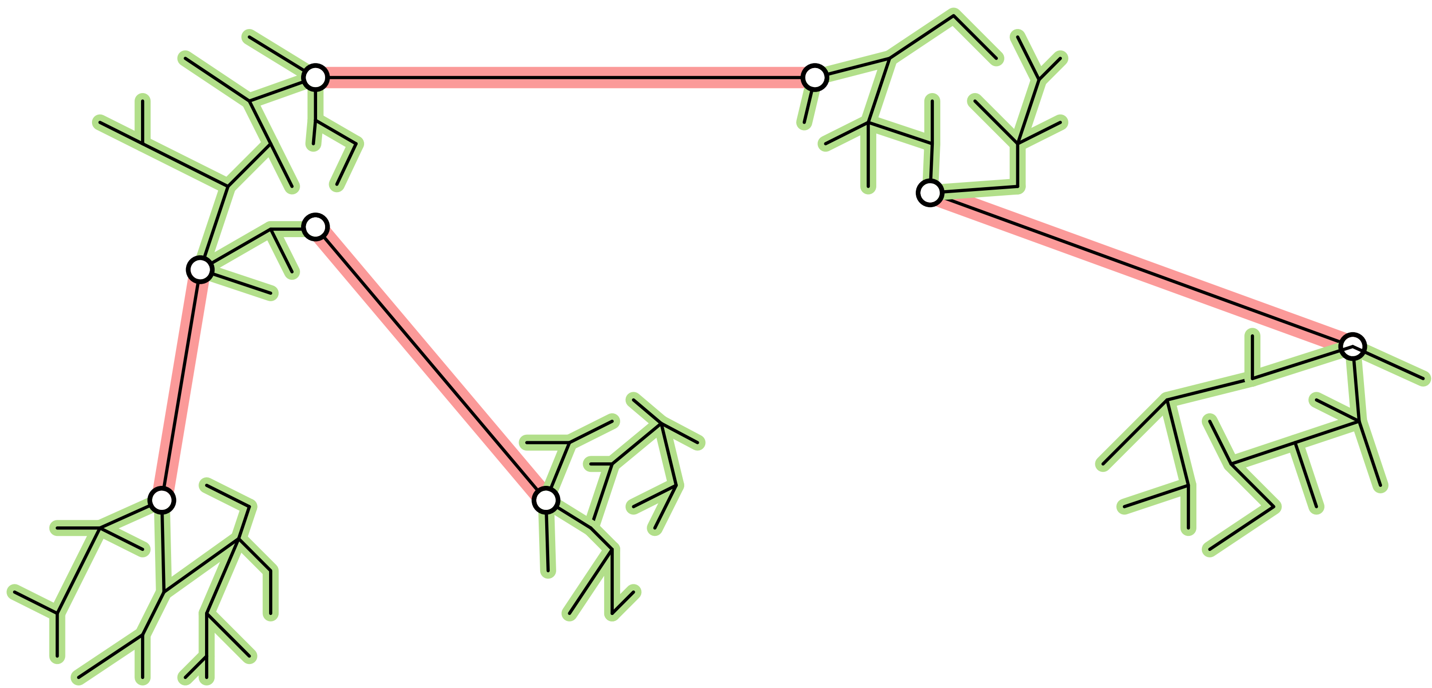
Decomposition

spanning
tree *T*



Decomposition

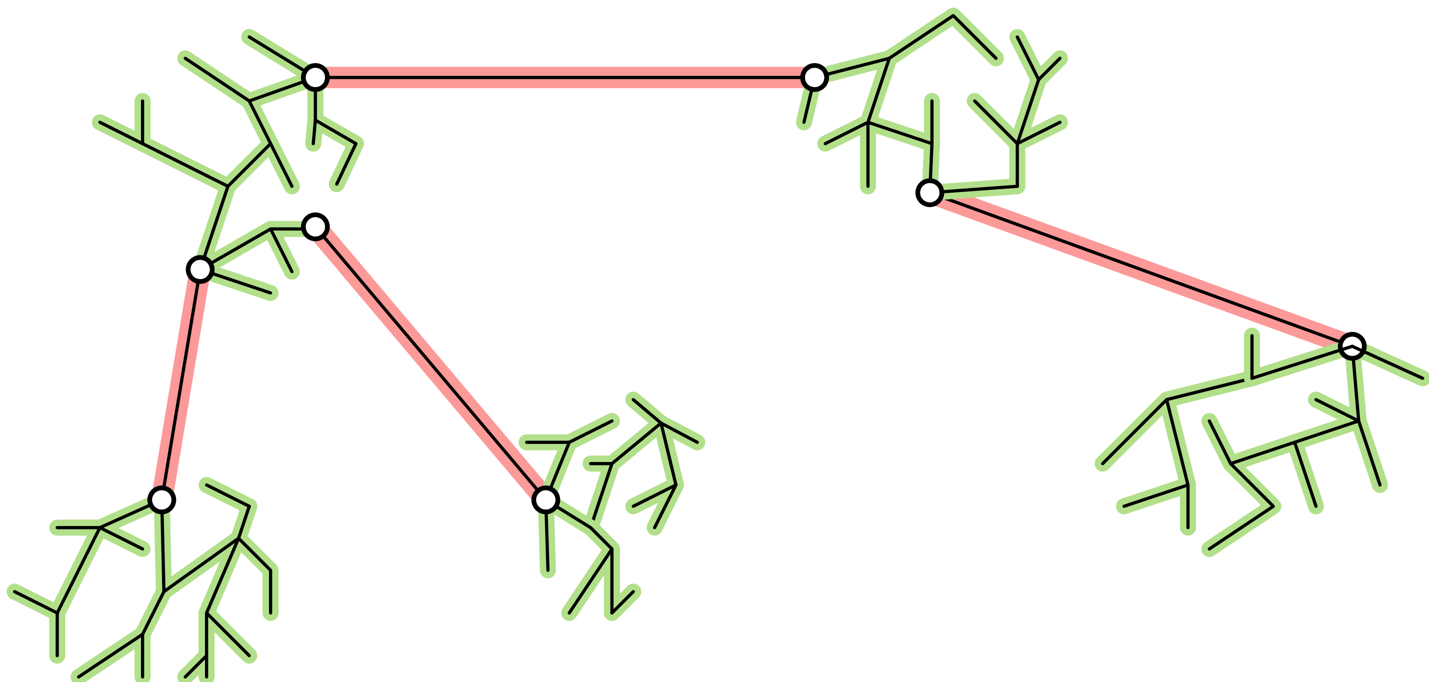
spanning tree T



Decomposition

- Removing k edges decomposes T into $k+1$ components.

spanning tree T



Decomposition

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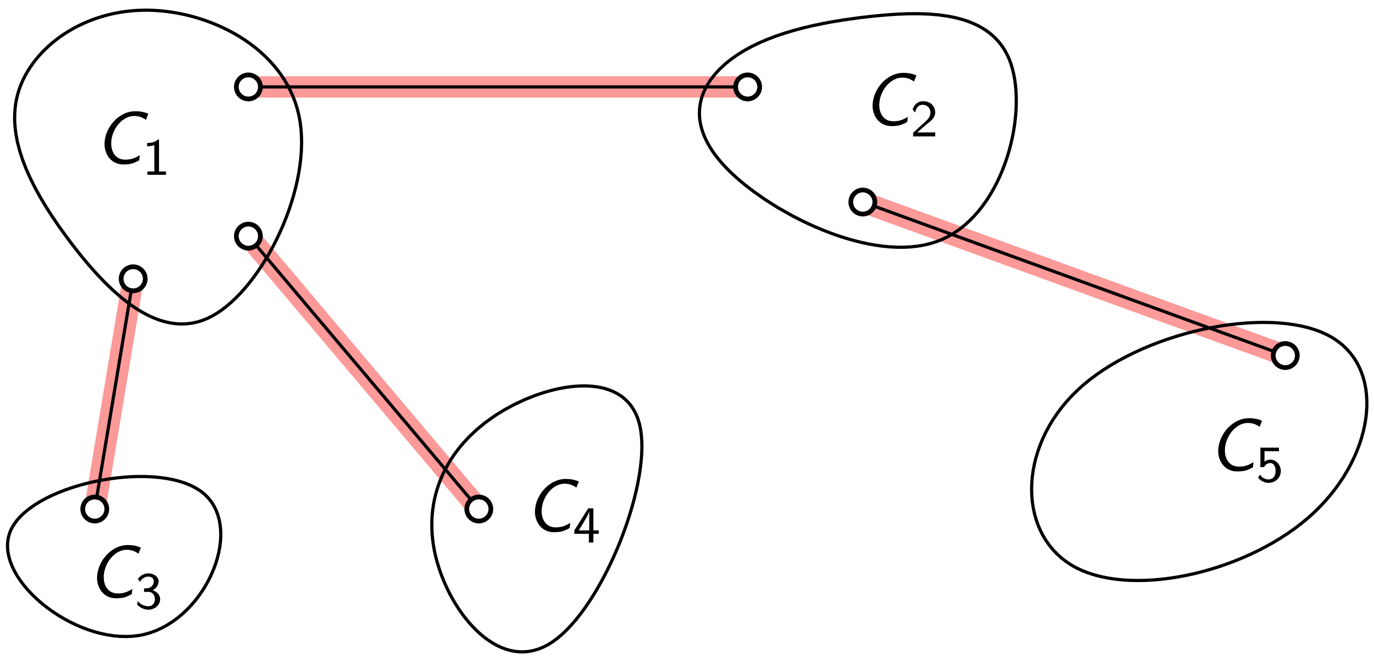
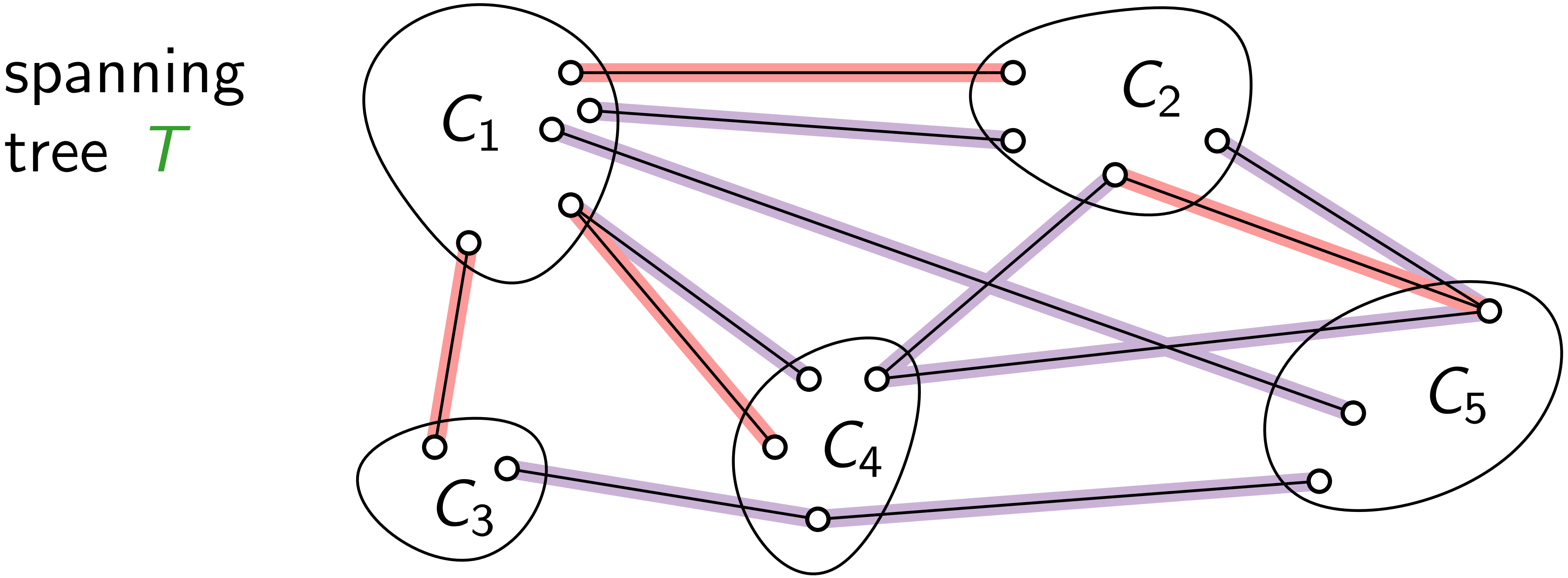
spanning tree T



Decomposition

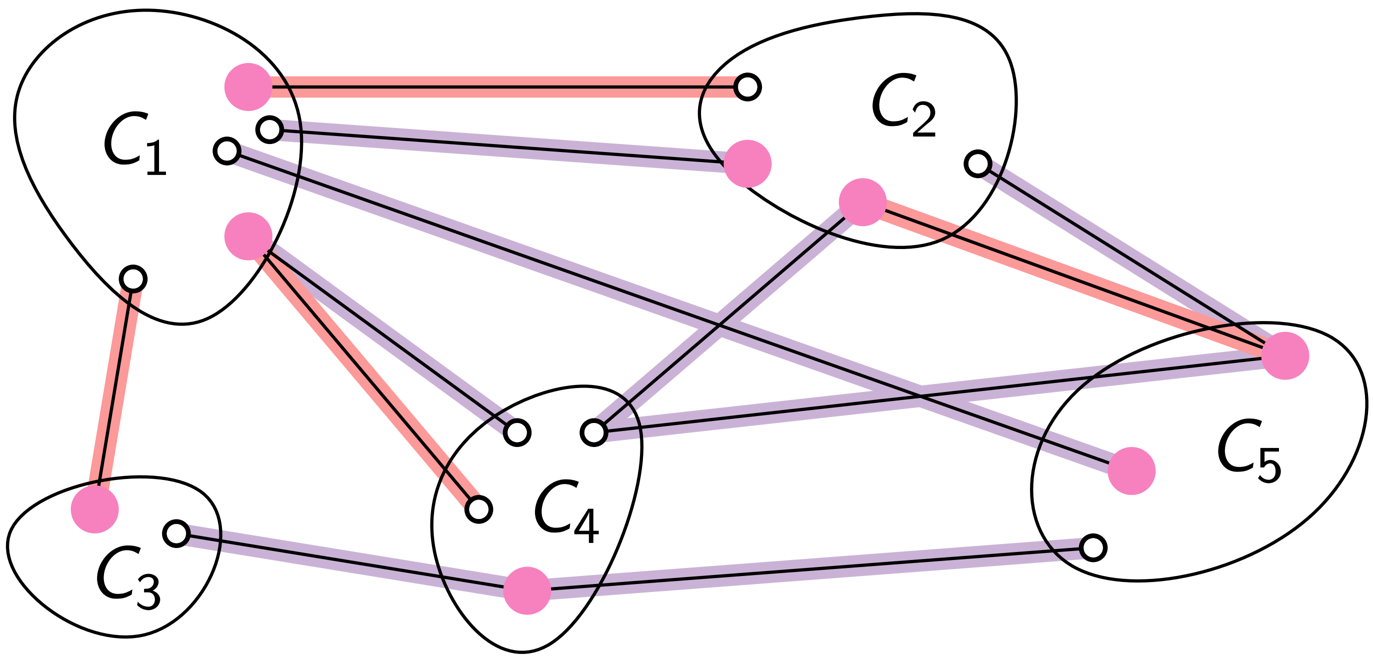
- Removing k edges decomposes T into $k+1$ components.
- $E' = \{\text{edges in } G \text{ between different components } C_i \neq C_j\}$.



Decomposition

- Removing k edges decomposes T into $k+1$ components.
- $E' = \{$edges in G between different components $C_i \neq C_j\}$.
- $S :=$ vertex cover of E'.

spanning tree T



Decomposition

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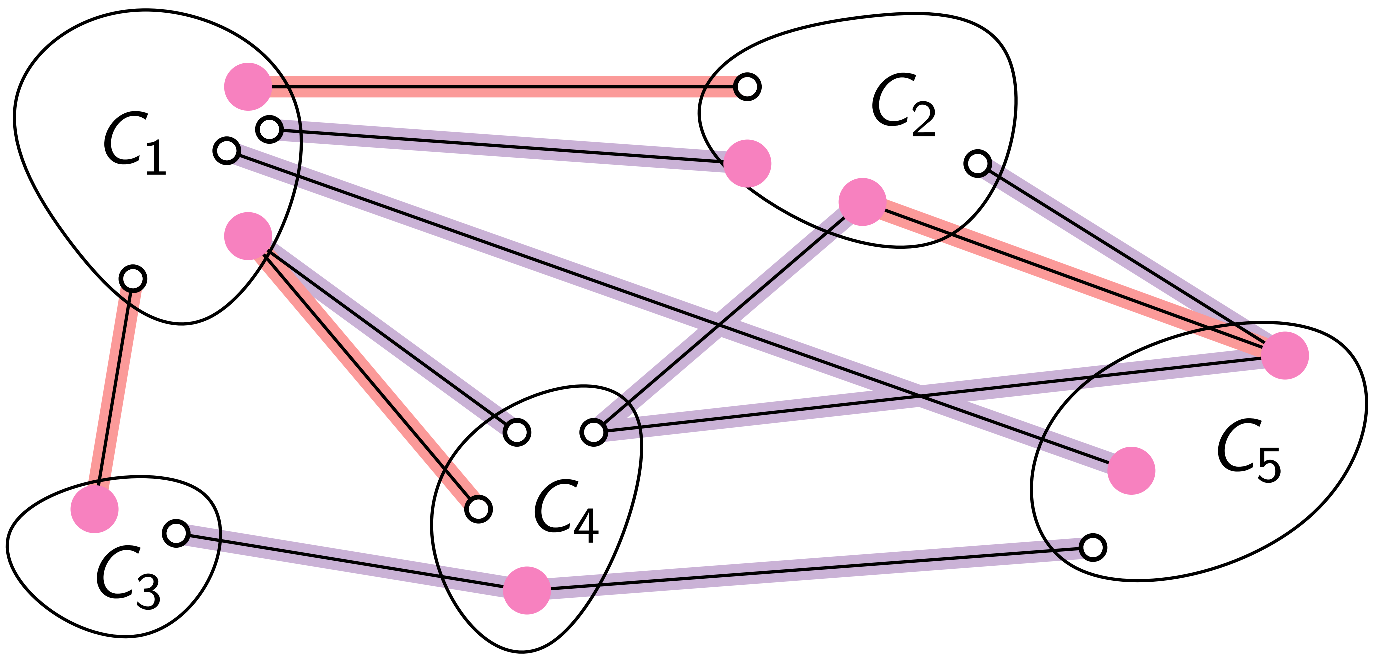
spanning tree T



- For any spanning tree T', $|E(T') \cap E'| \geq$

Decomposition

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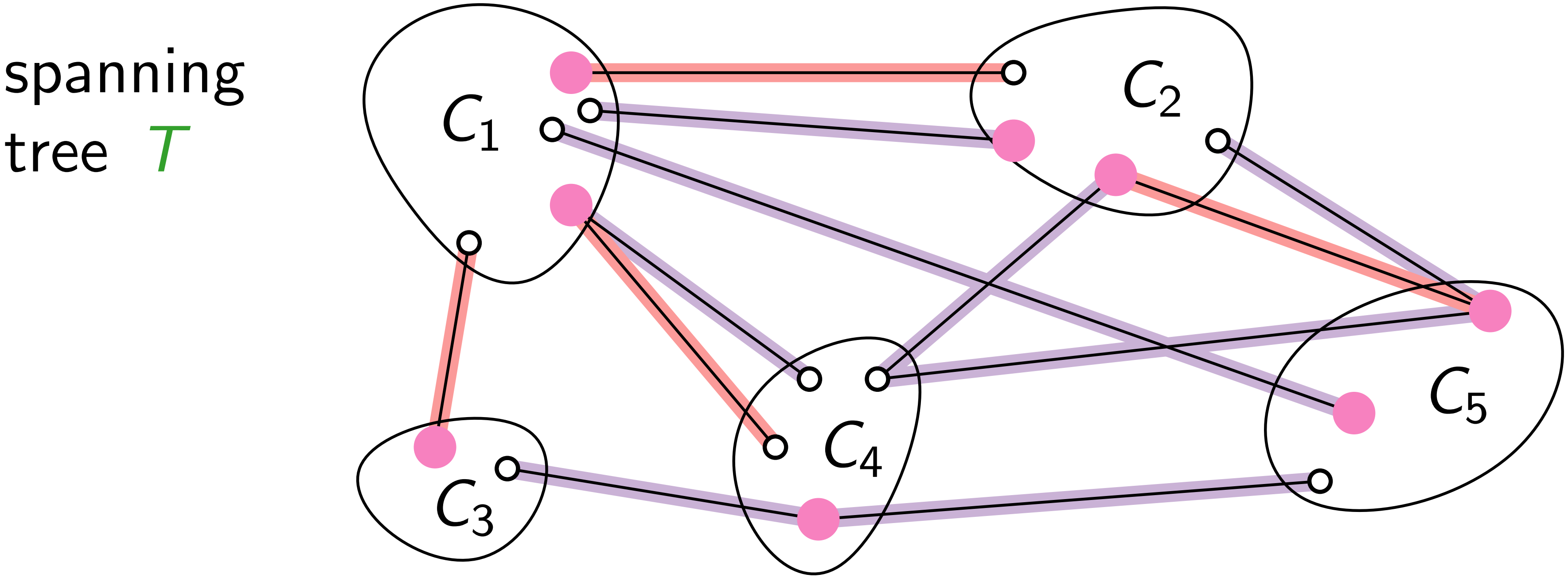

- For any spanning tree T', $|E(T') \cap E'| \geq k$,

Decomposition

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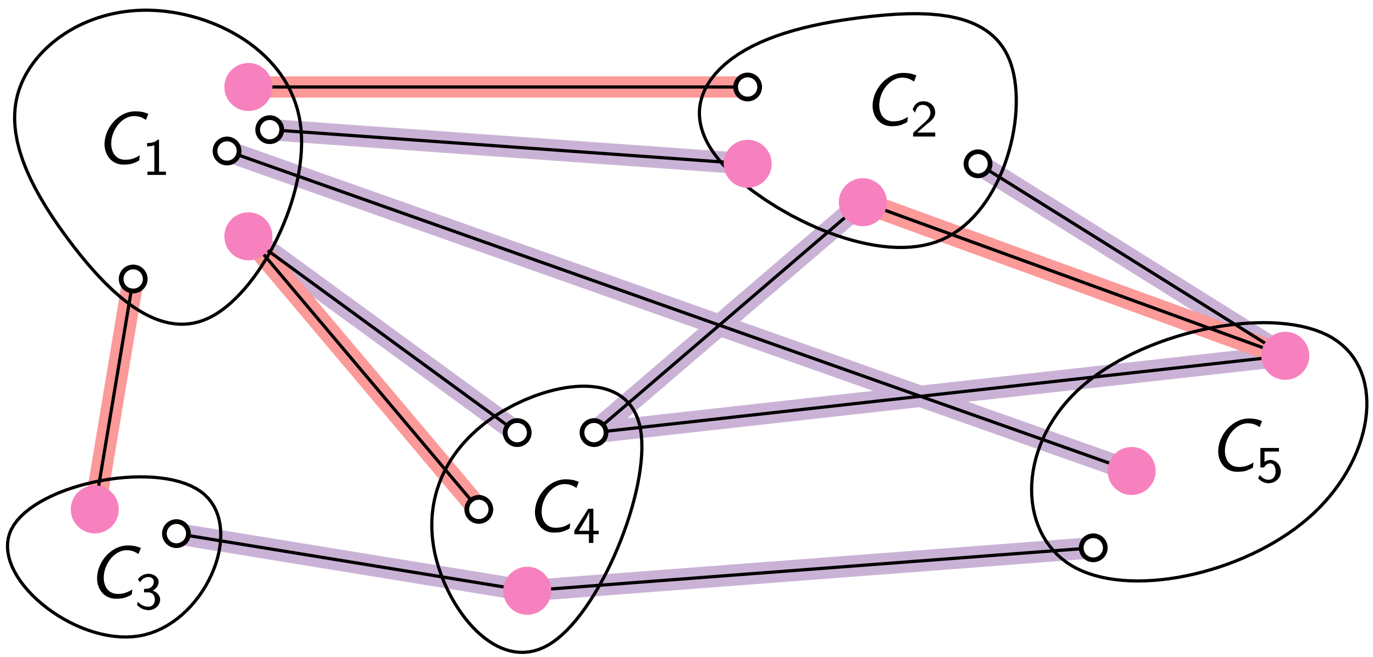
spanning tree T



- For any spanning tree T', $|E(T') \cap E'| \geq k$,
- $\sum_{v \in S} \deg_{T'}(v) \geq$

Decomposition

- Removing k edges decomposes T into $k+1$ components.
- $E' = \{$edges in G between different components $C_i \neq C_j\}$.
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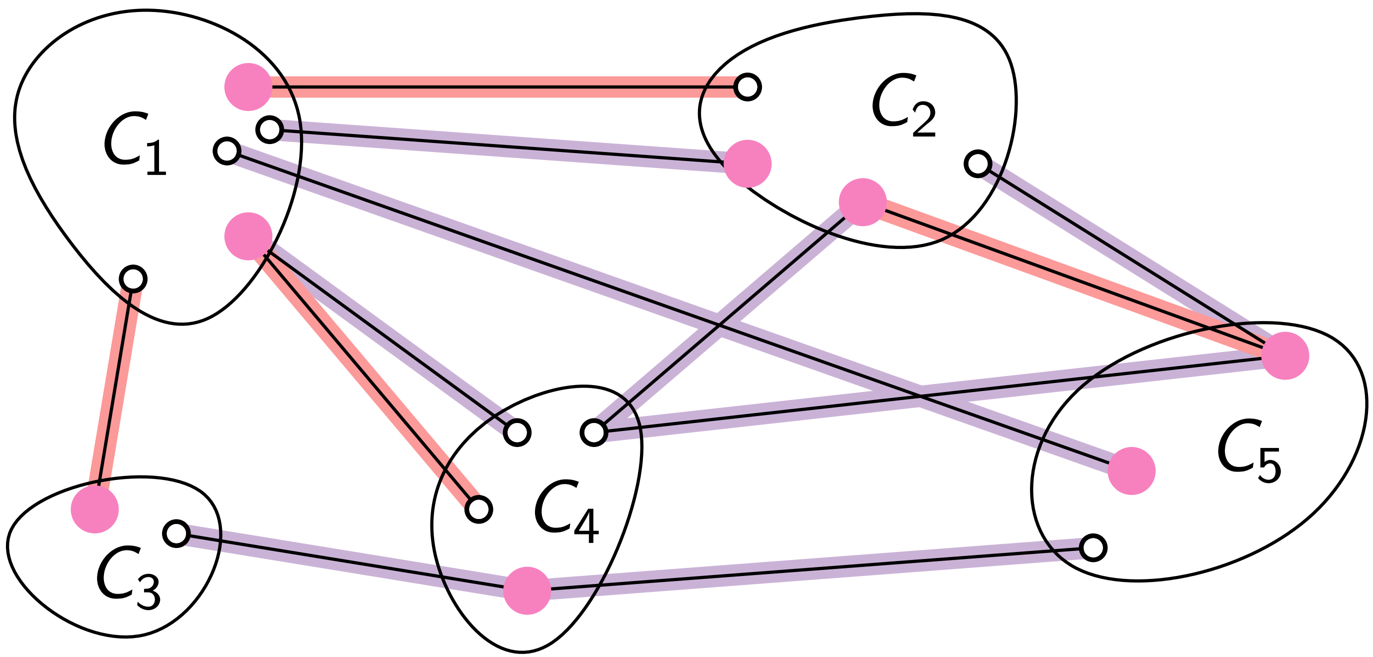
spanning tree T



- For any spanning tree T', $|E(T') \cap E'| \geq k$,
- $\sum_{v \in S} \deg_{T'}(v) \geq k$, and (Obs. 2)

Decomposition

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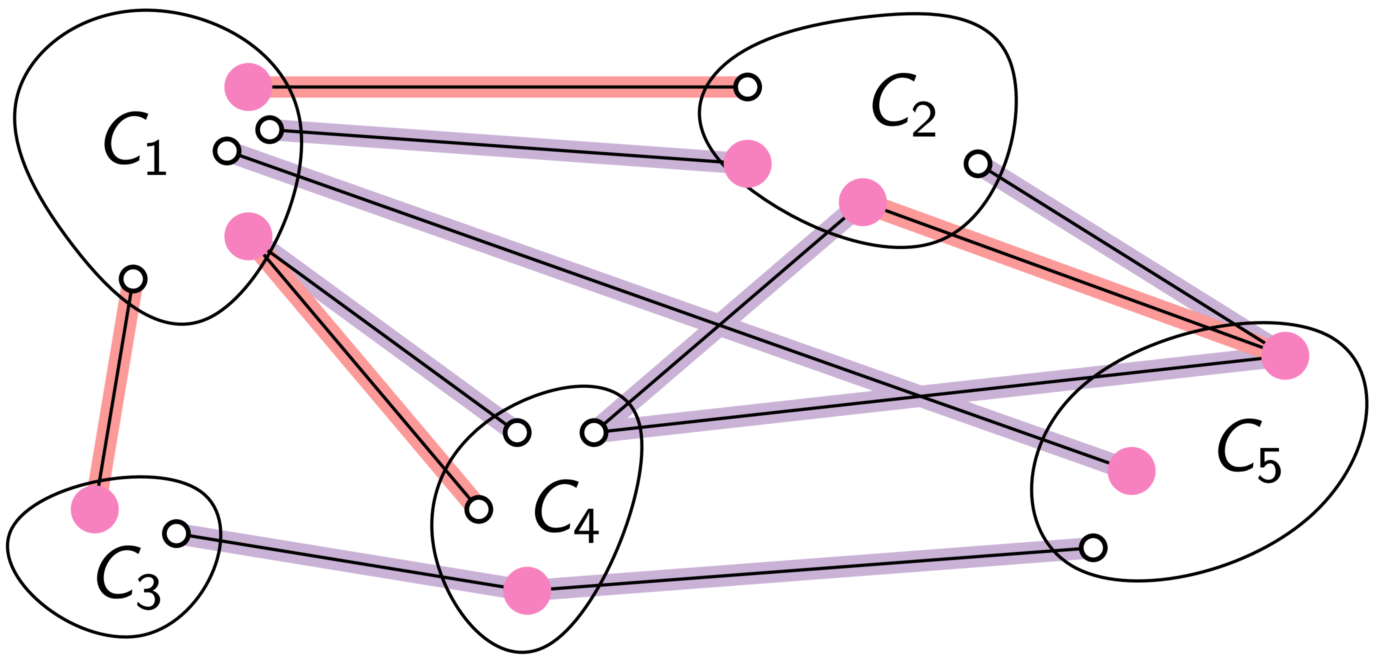
spanning tree T



- For any spanning tree T', $|E(T') \cap E'| \geq k$,
- $\sum_{v \in S} \deg_{T'}(v) \geq k$, and $\Delta(T') \geq$
  (Obs. 2)

Decomposition

- Removing k edges decomposes T into $k+1$ components.
- $E' = \{$edges in G between different components $C_i \neq C_j\}$.
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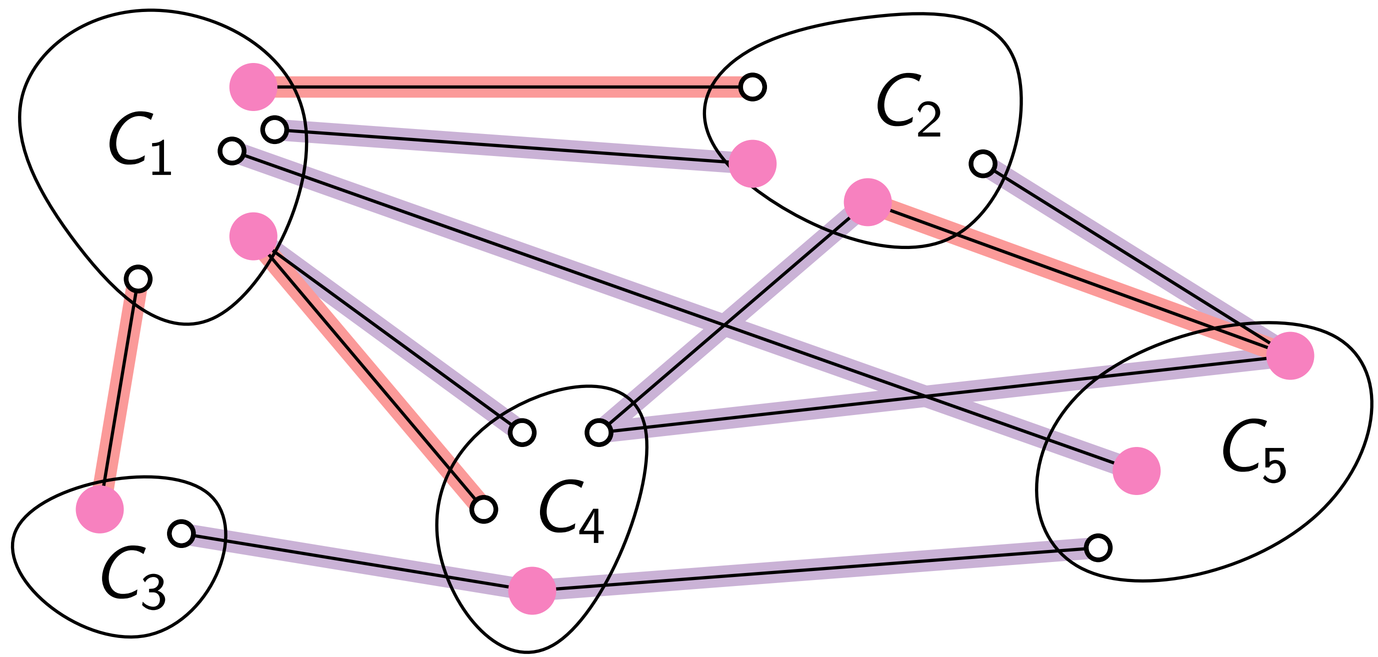
spanning tree T



- For any spanning tree T', $|E(T') \cap E'| \geq k$,
- $\sum_{v \in S} \deg_{T'}(v) \geq k$, and $\Delta(T') \geq k/|S|$.

(Obs. 2)

Decomposition

- Removing k edges decomposes T into $k+1$ components.
- $E' = \{$edges in G between different components $C_i \neq C_j\}$.
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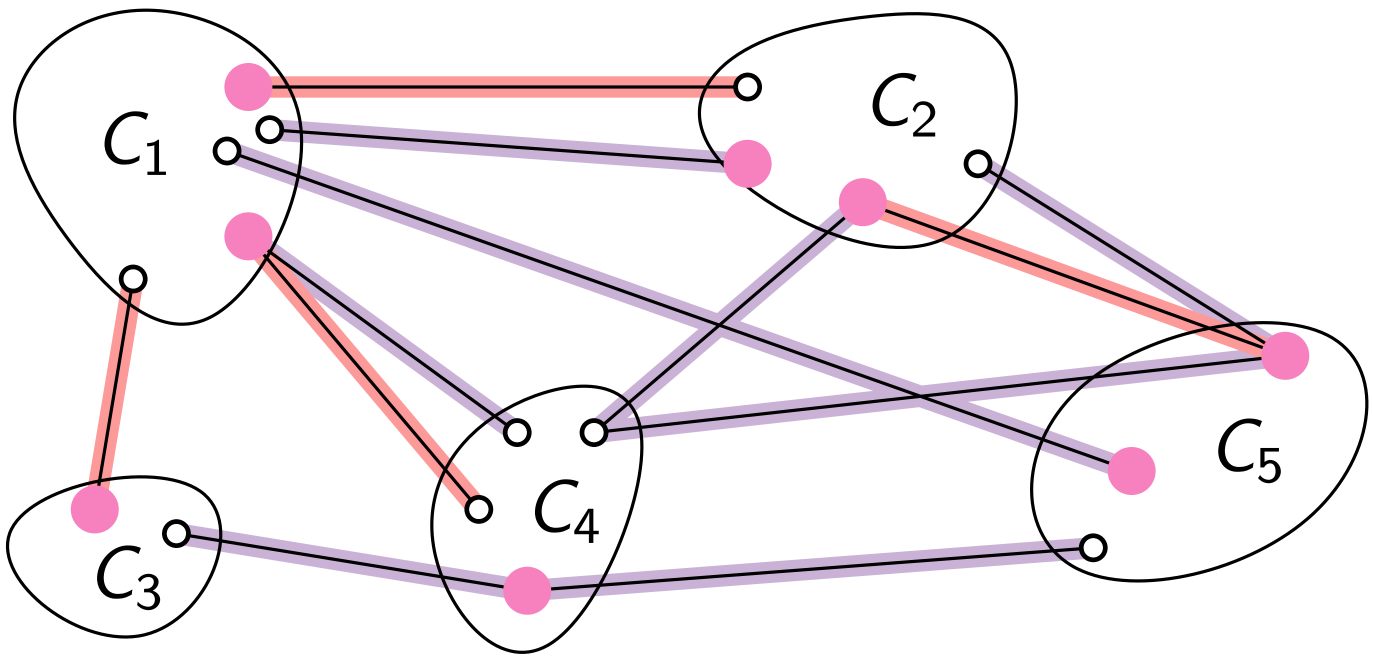
spanning tree T



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  (Obs. 2)
- Consider the optimal spanning tree T^*.

Decomposition $\Rightarrow$ Lower Bound for OPT

- Removing k edges decomposes T into $k+1$ components.
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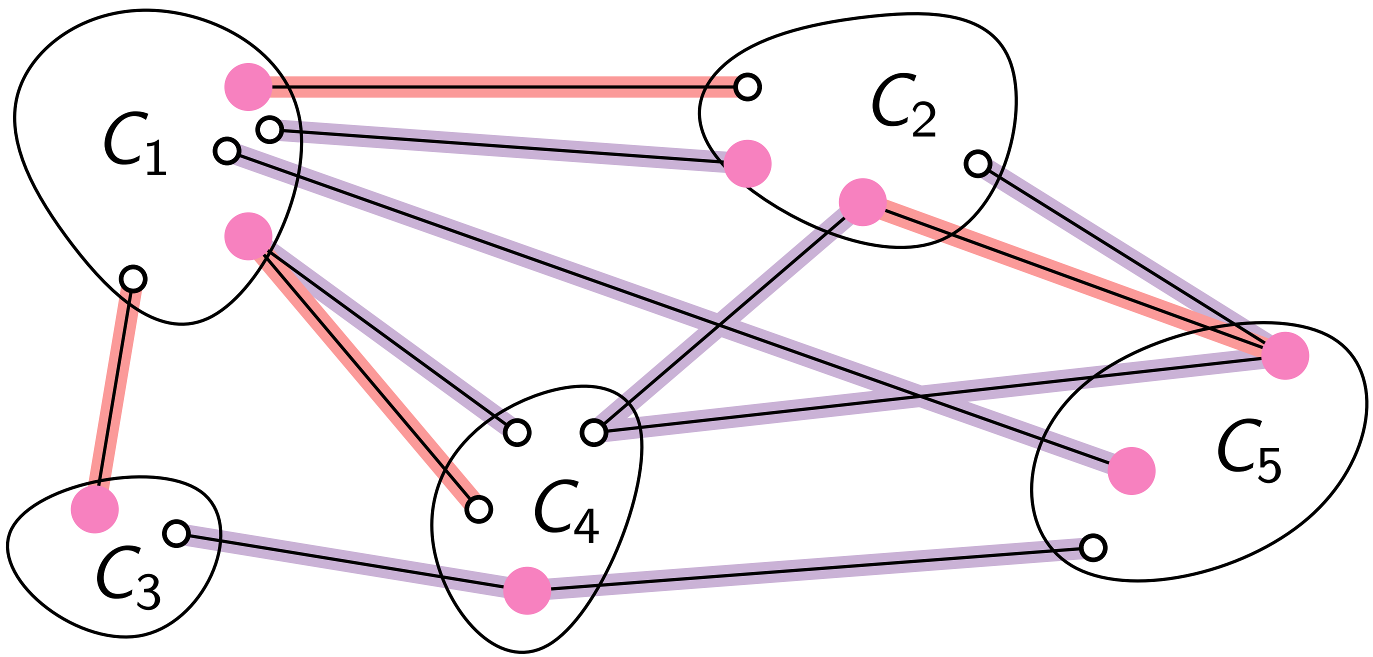
spanning tree T



- For any spanning tree T', $|E(T') \cap E'| \geq k$,
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- Consider the optimal spanning tree T^*.

Lemma 1.
$\Rightarrow$ OPT $\geq$

Decomposition $\Rightarrow$ Lower Bound for OPT

- Removing k edges decomposes T into $k+1$ components.
- $E' = \{$edges in G between different components $C_i \neq C_j\}$.
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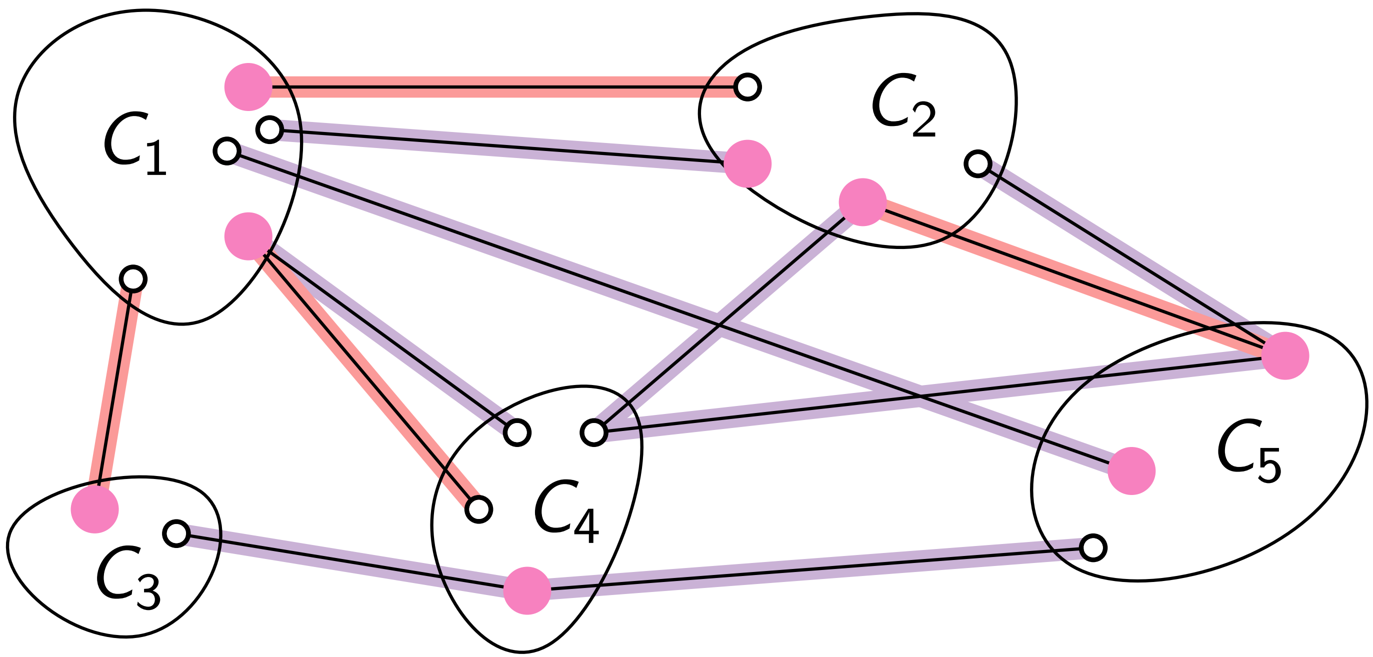
spanning tree T



- For any spanning tree T', $|E(T') \cap E'| \geq k$,
- $\sum_{v \in S} \deg_{T'}(v) \geq k$, and $\Delta(T') \geq k/|S|$. (Obs. 2)
- Consider the optimal spanning tree T^*.

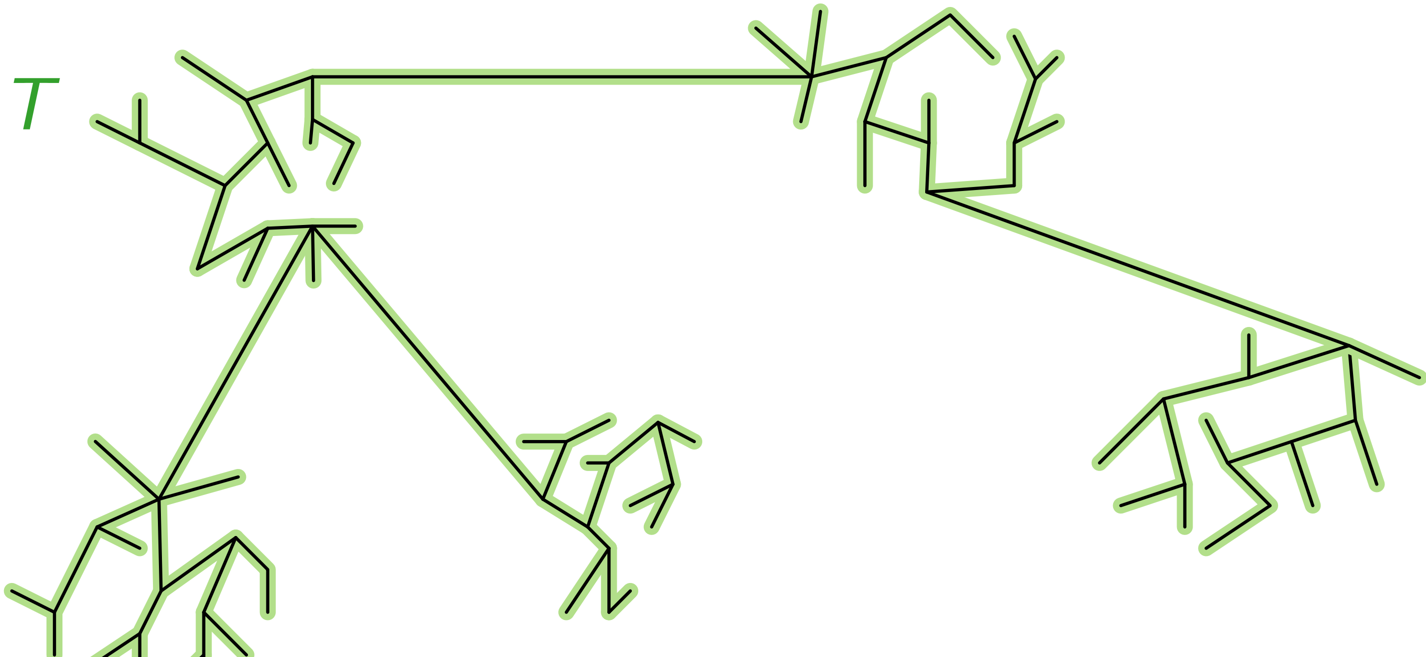
Lemma 1.
$\Rightarrow \mathrm{OPT} \geq k/|S|$

Approximation Algorithms

Lecture 10:
Minimum-Degree Spanning Tree
via Local Search

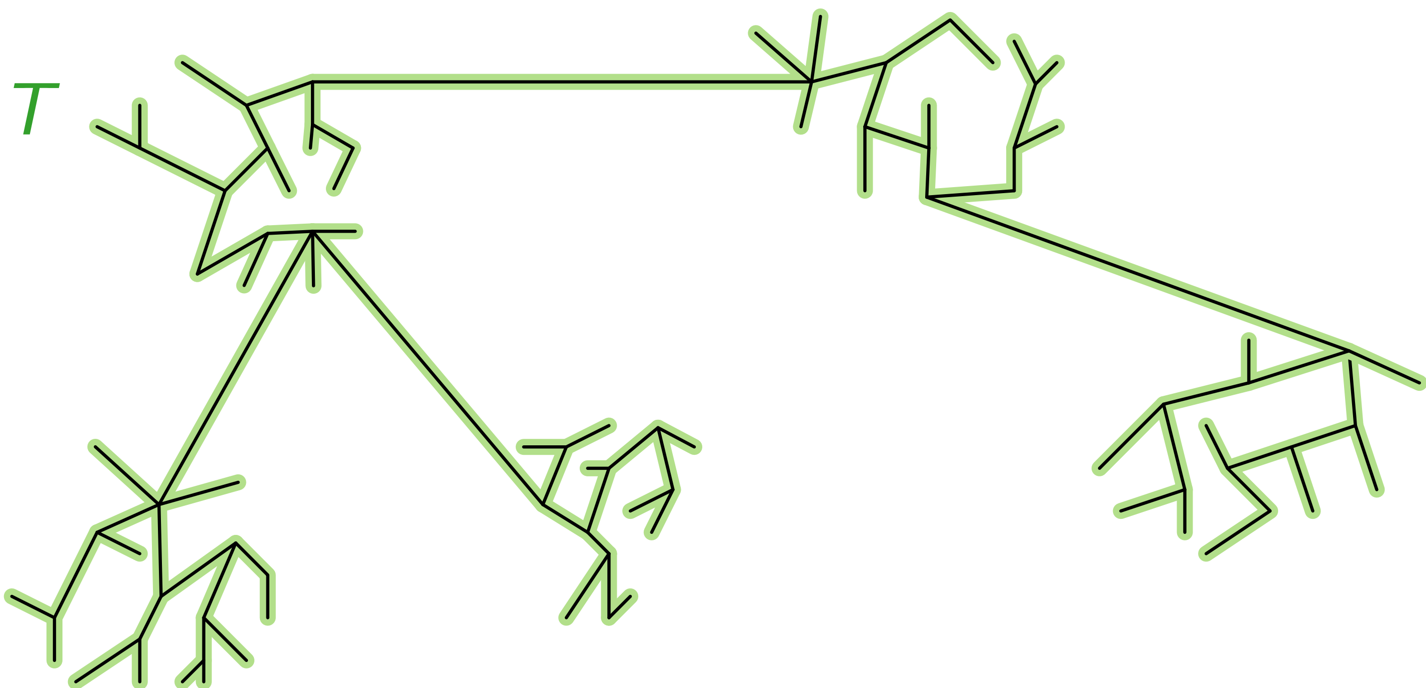
Part IV:
Structure of a Decomposition

Structure of a Decomposition



Structure of a Decomposition

Let S_i be the set of vertices v in T with $\deg_T(v) \geq i$.



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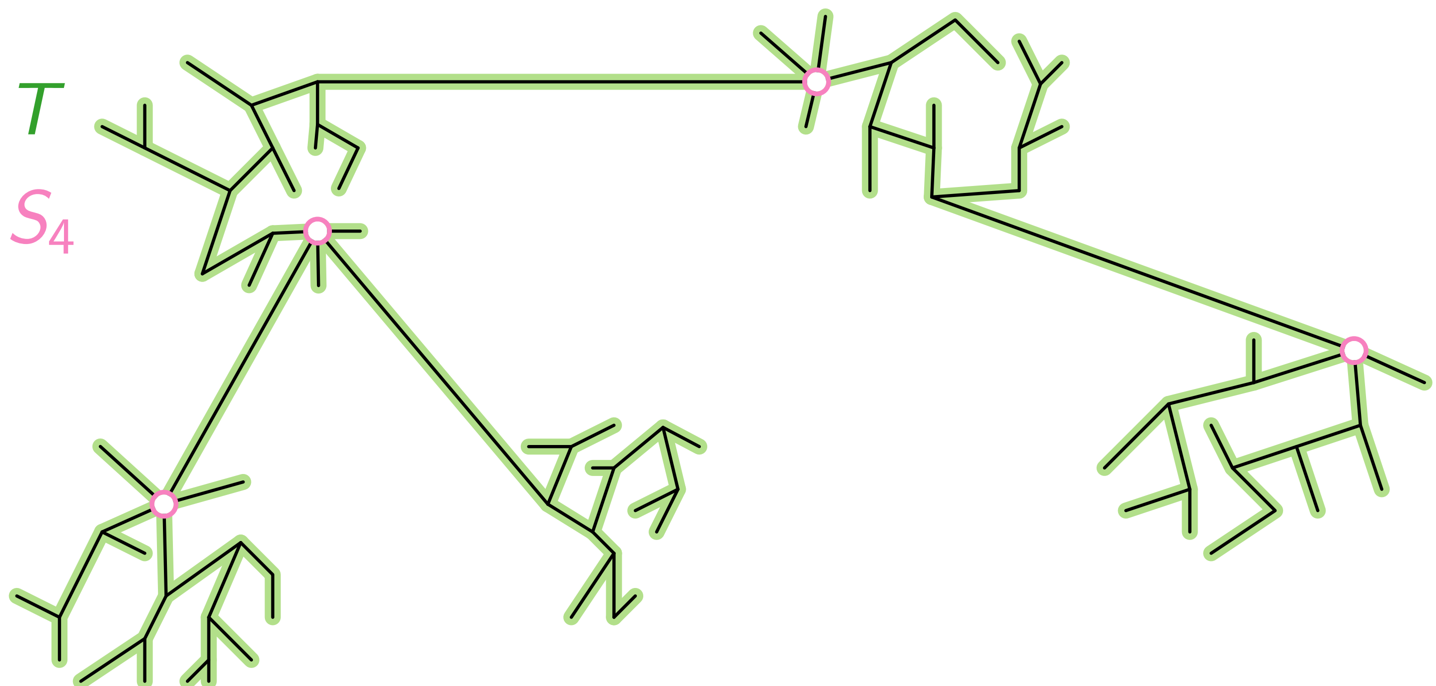
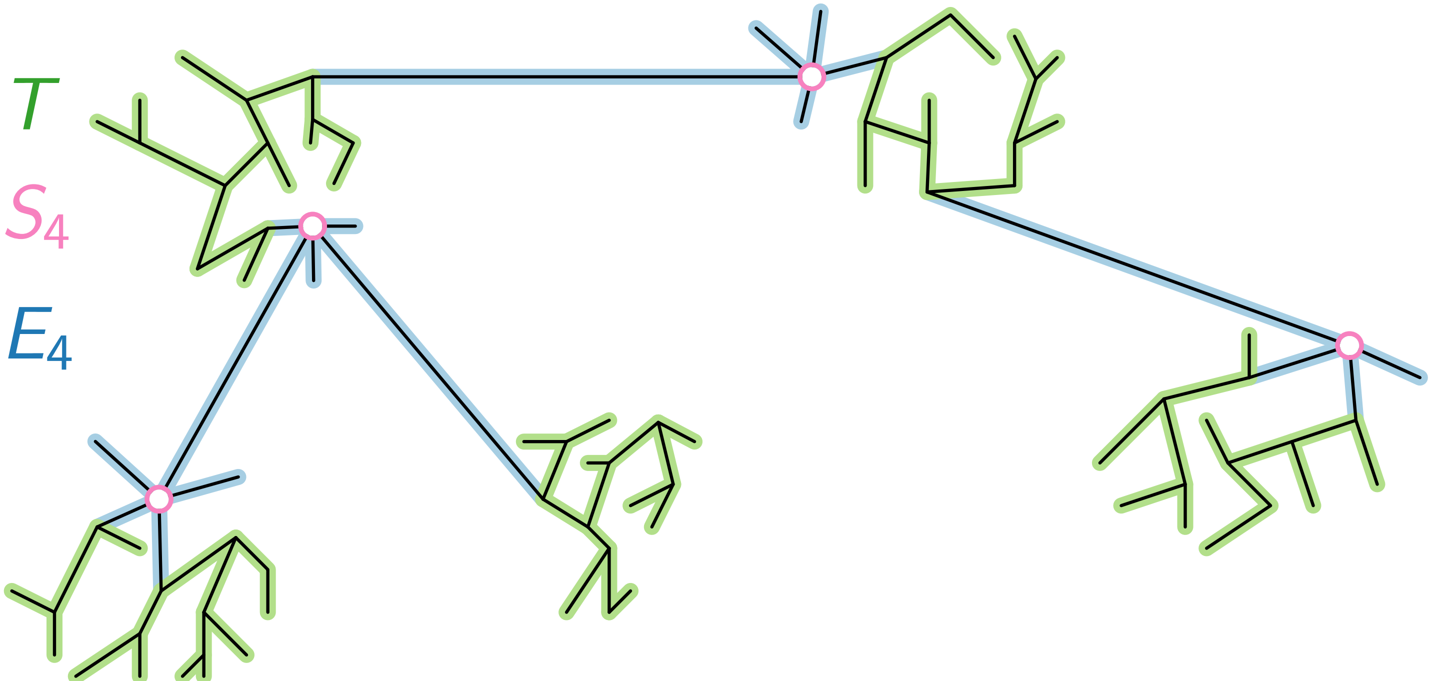


Structure of a Decomposition

Let S_i be the set of vertices v in T with $\deg_T(v) \geq i$.
Let E_i be the set of edges in T incident to S_i.



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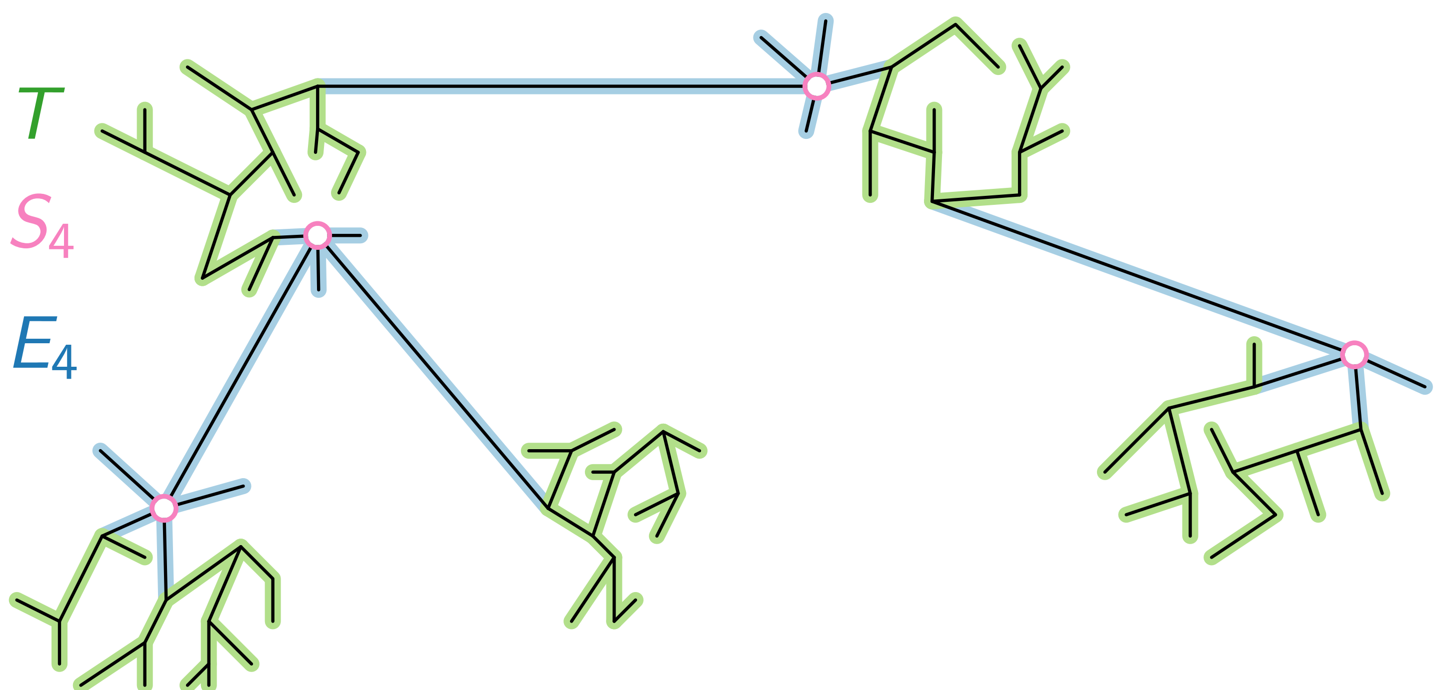

Structure of a Decomposition

$\Rightarrow S_1 \supseteq S_2 \supseteq \dots$
$\Rightarrow S_1 = V(G)$

Let S_i be the set of vertices v in T with $\deg_T(v) \geq i$.
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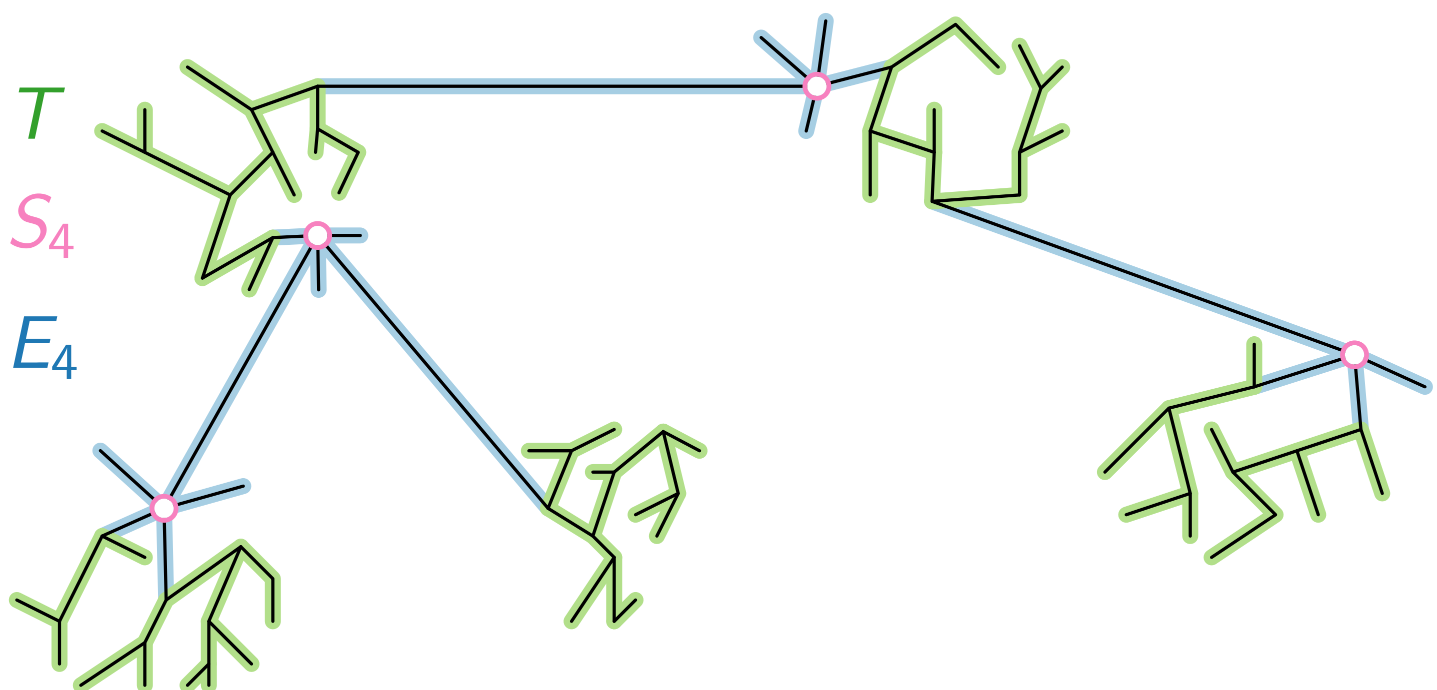

Structure of a Decomposition

$\Rightarrow S_1 \supseteq S_2 \supseteq \ldots$
$\Rightarrow S_1 = V(G)$
$\Rightarrow E_1 = E(T)$

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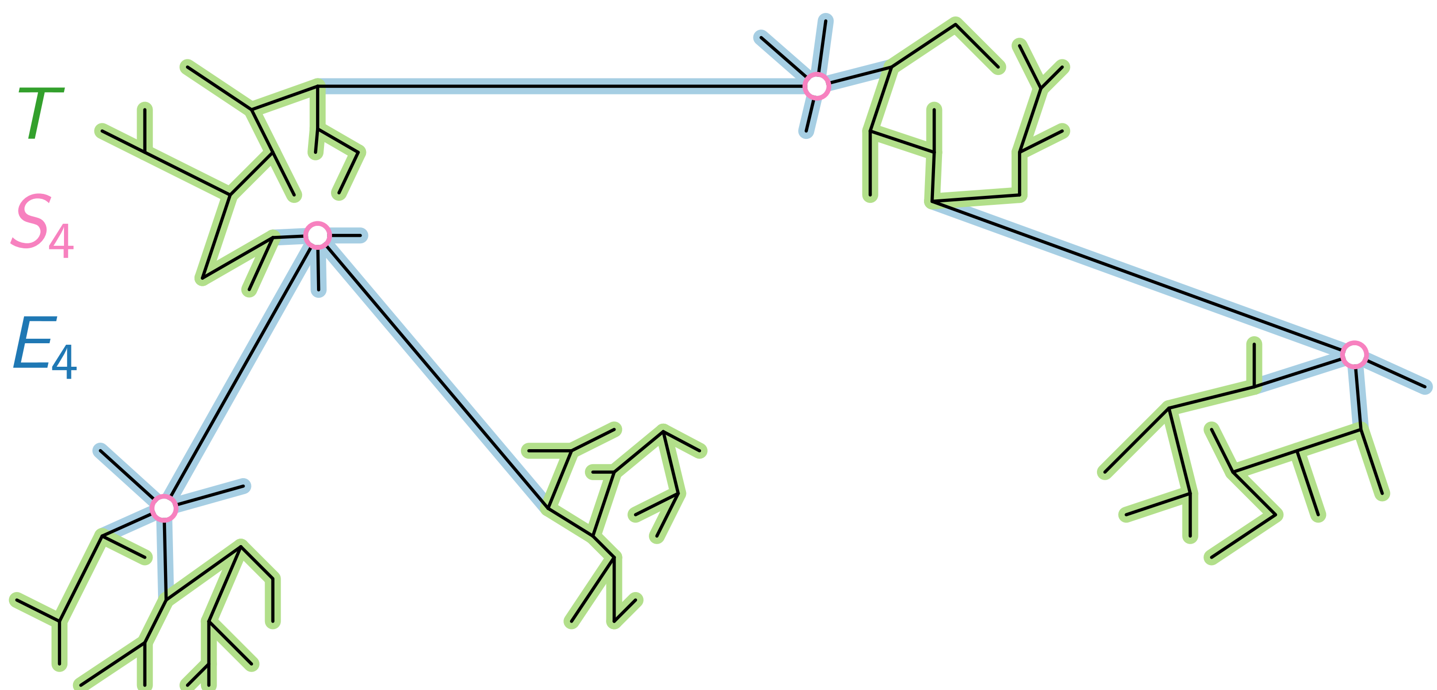

Structure of a Decomposition

$\Rightarrow S_1 \supseteq S_2 \supseteq \dots$
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Let S_i be the set of vertices v in T with $\deg_T(v) \geq i$.
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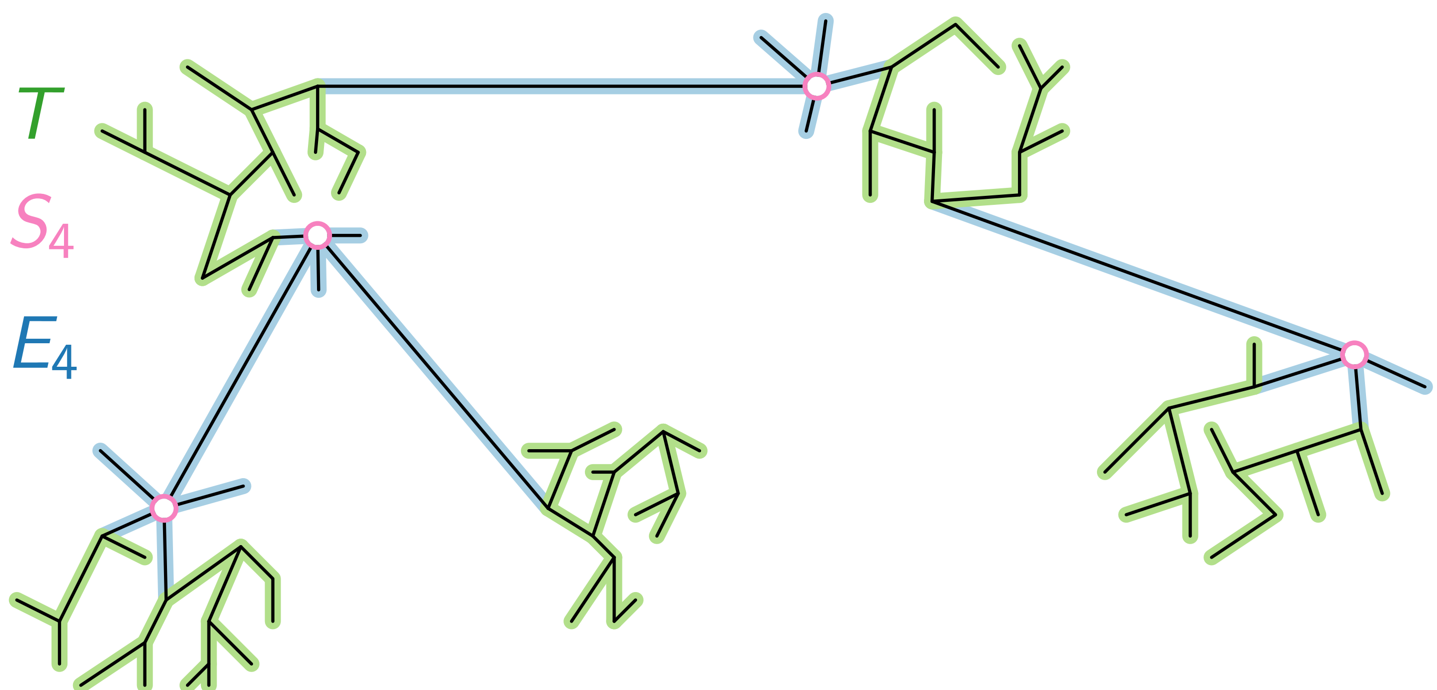
Lemma 2. $\exists i$ s.t. $\Delta(T) - \ell + 1 \leq i \leq \Delta(T)$ with $|S_{i-1}| \leq 2|S_i|$.



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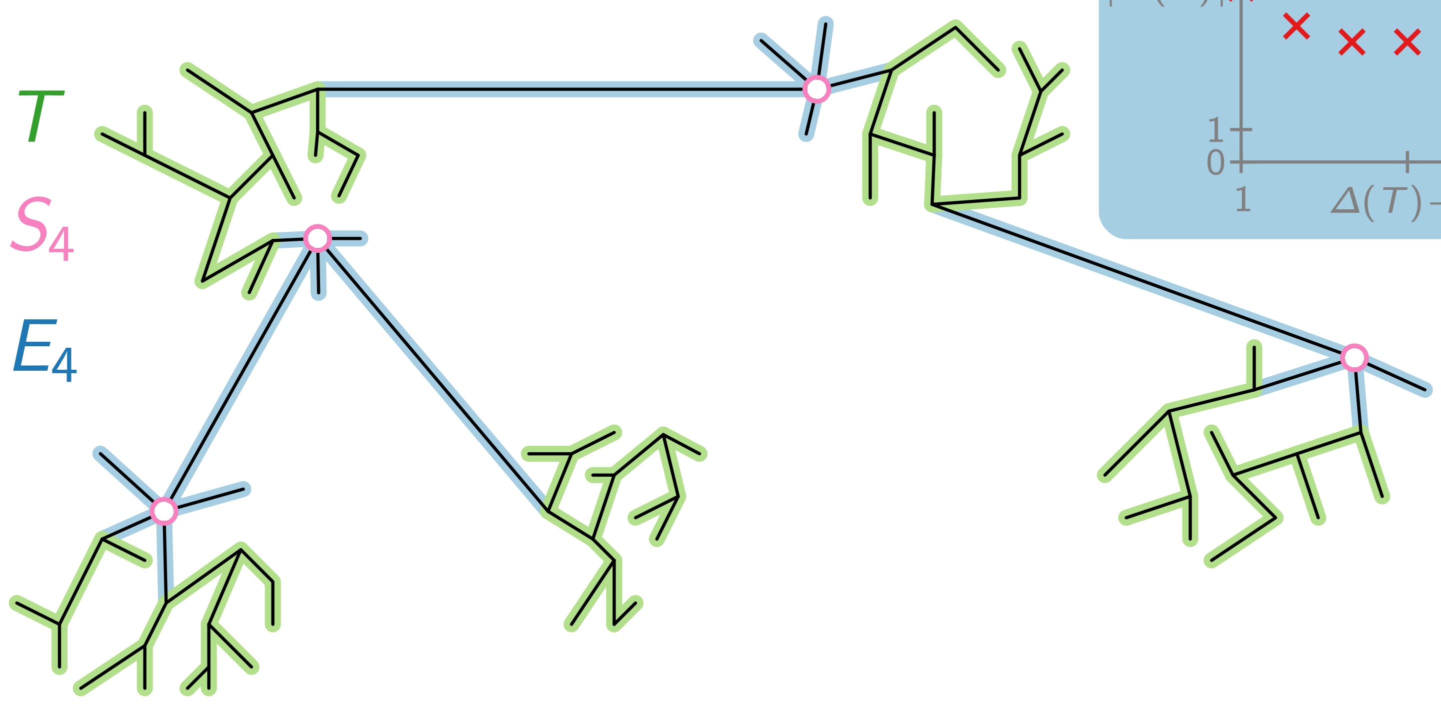

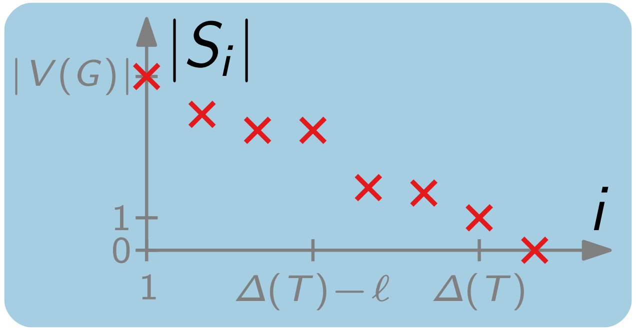

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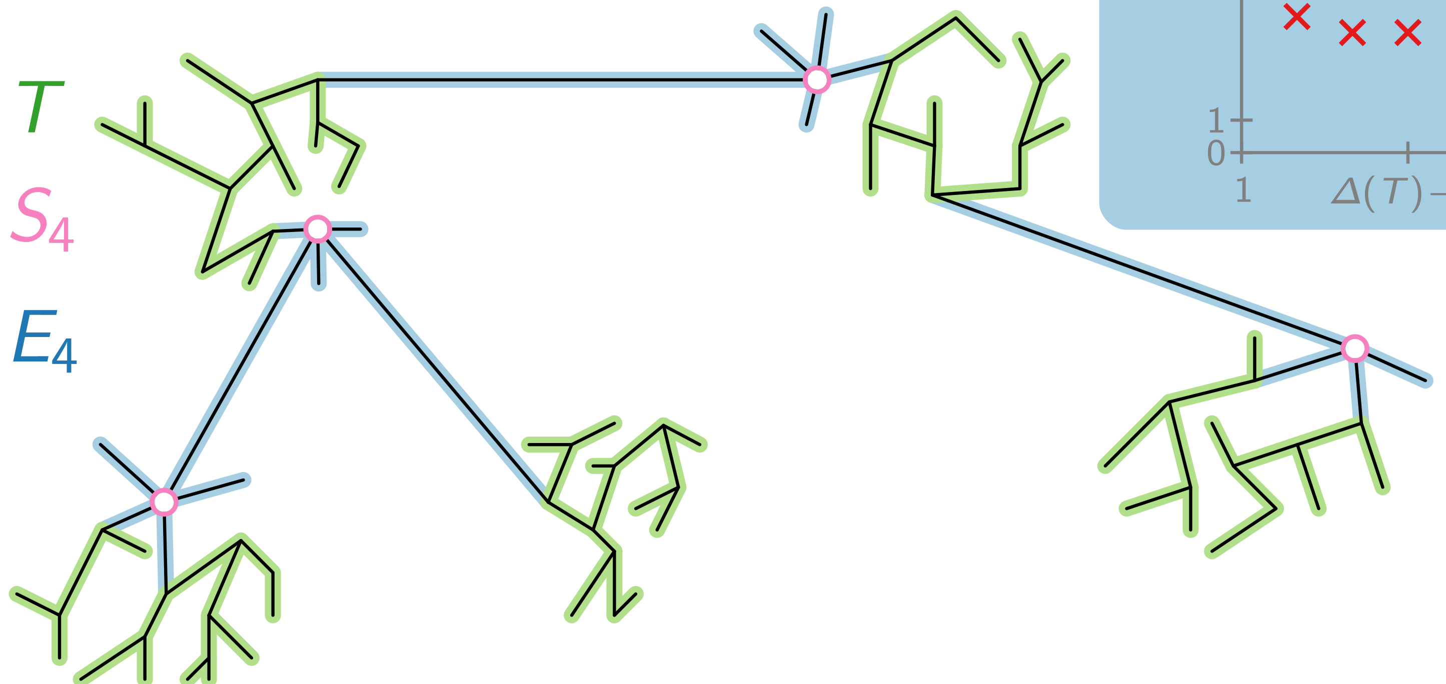
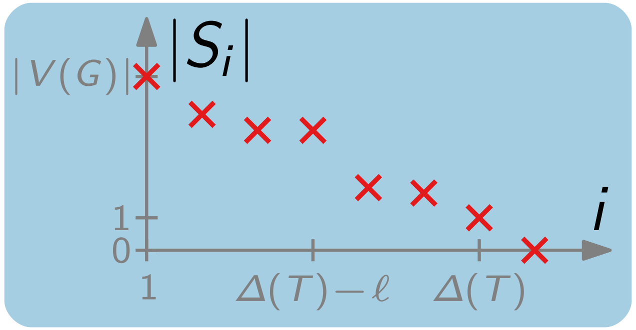
Lemma 2. $\exists i$ s.t. $\Delta(T) - \ell + 1 \leq i \leq \Delta(T)$ with $|S_{i-1}| \leq 2|S_i|$.

Proof. $|S_{\Delta(T)-\ell}| > 2^\ell |S_{\Delta(T)}|$

Otherwise





Structure of a Decomposition

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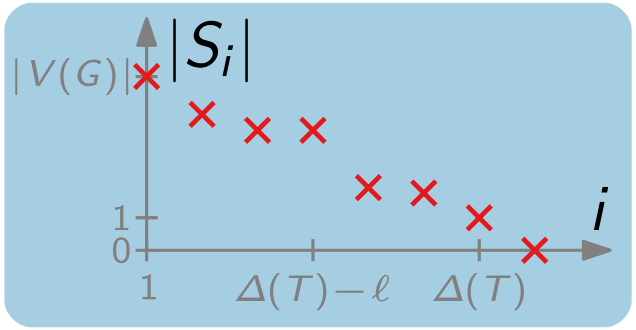
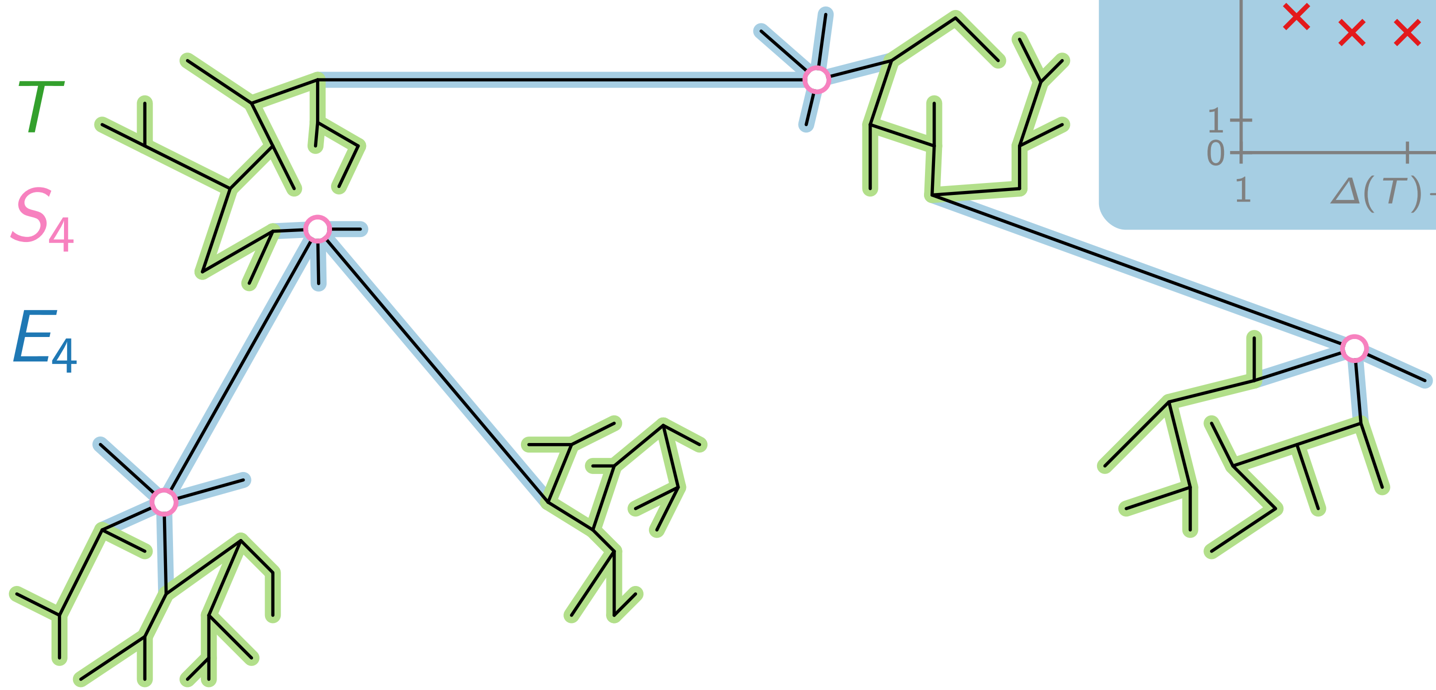
Proof. Otherwise $|S_{\Delta(T)-\ell}| > 2^{\ell}|S_{\Delta(T)}| = 2^{\lceil \log_2 n \rceil}|S_{\Delta(T)}| \geq$

$\ell = \lceil \log_2 n \rceil$





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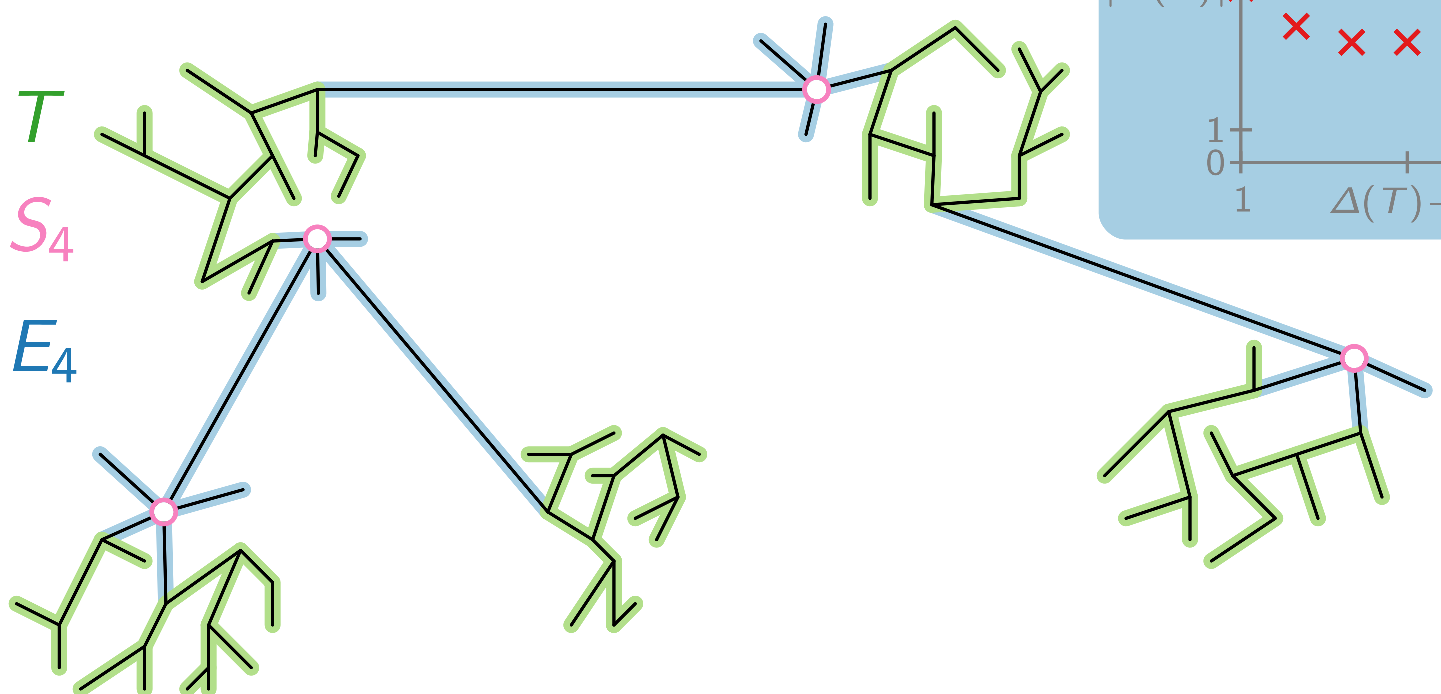
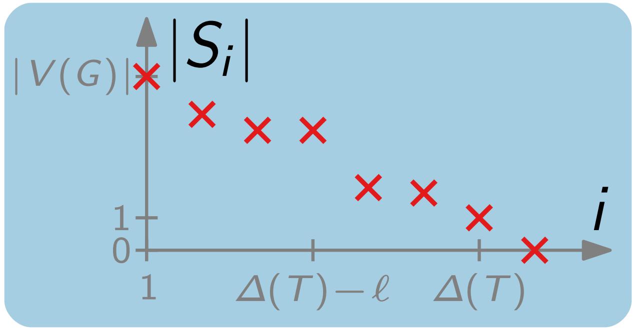
Proof. $|S_{\Delta(T)-\ell}| > 2^\ell |S_{\Delta(T)}| = 2^{\lceil \log_2 n \rceil} |S_{\Delta(T)}| \geq n \cdot |S_{\Delta(T)}|$

$\ell = \lceil \log_2 n \rceil$

Otherwise





Structure of a Decomposition

$\Rightarrow S_1 \supseteq S_2 \supseteq \dots$
$\Rightarrow S_1 = V(G)$
$\Rightarrow E_1 = E(T)$

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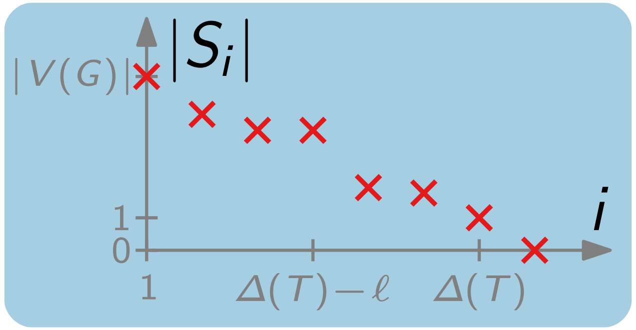
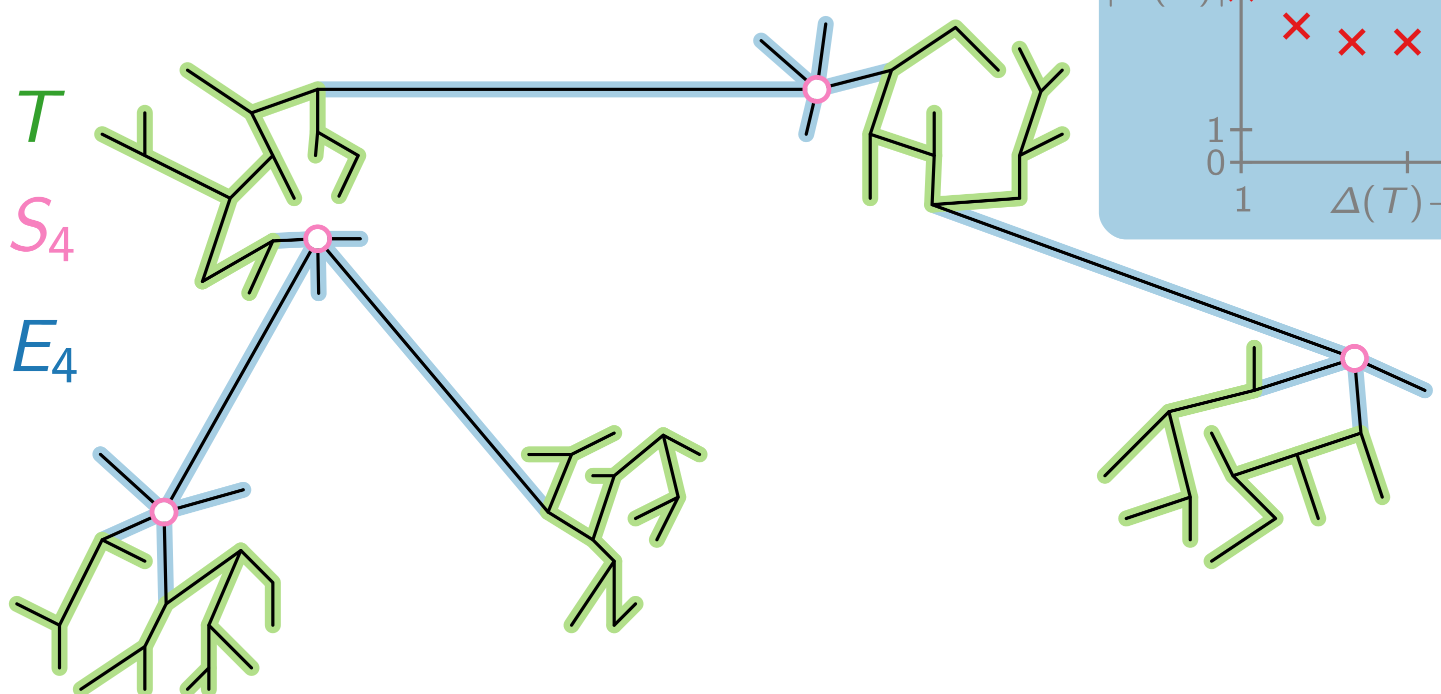
Lemma 2. $\exists i$ s.t. $\Delta(T) - \ell + 1 \leq i \leq \Delta(T)$ with $|S_{i-1}| \leq 2|S_i|$.

Proof. $|S_{\Delta(T)-\ell}| > 2^{\ell}|S_{\Delta(T)}| = 2^{\lceil \log_2 n \rceil}|S_{\Delta(T)}| \geq n \cdot |S_{\Delta(T)}|$ ↯

Otherwise

$\ell = \lceil \log_2 n \rceil$





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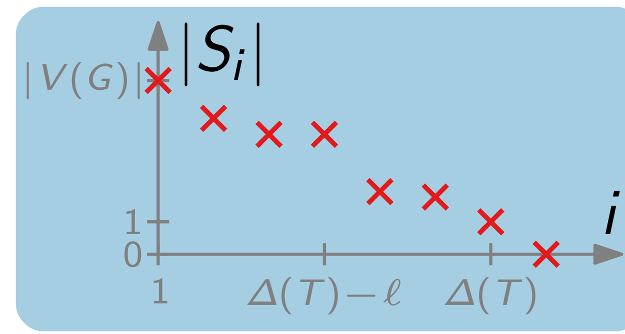
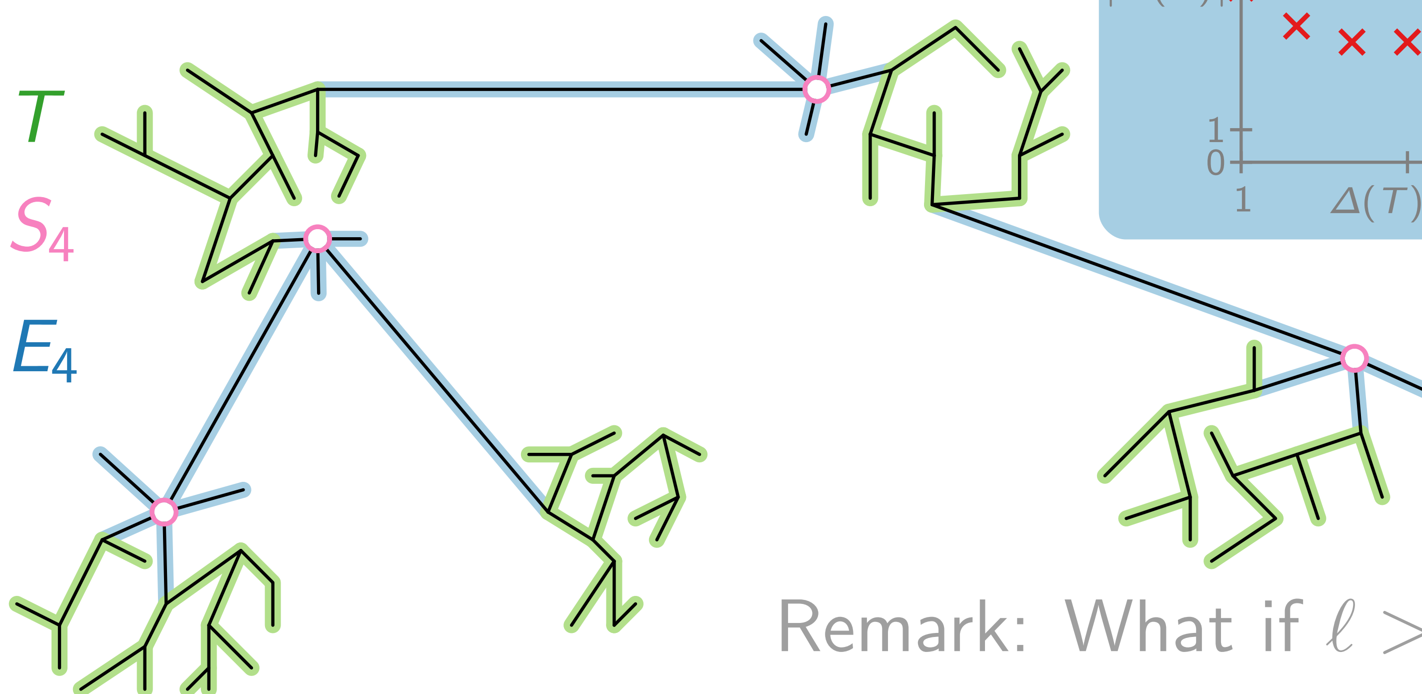
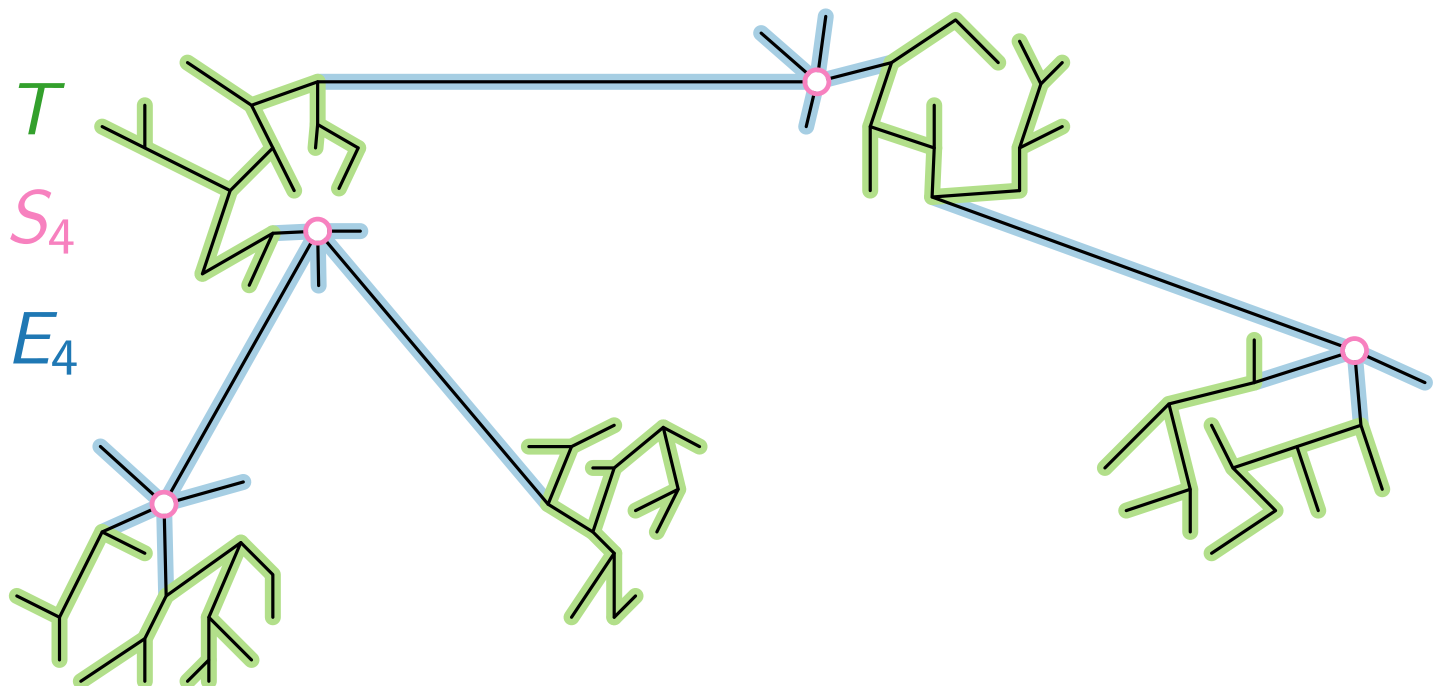




Remark: What if $\ell > \Delta(T)$?

Structure of a Decomposition

Lemma 3. For locally opt. spanning tree T, $i \geq \Delta(T) - \ell + 1$:



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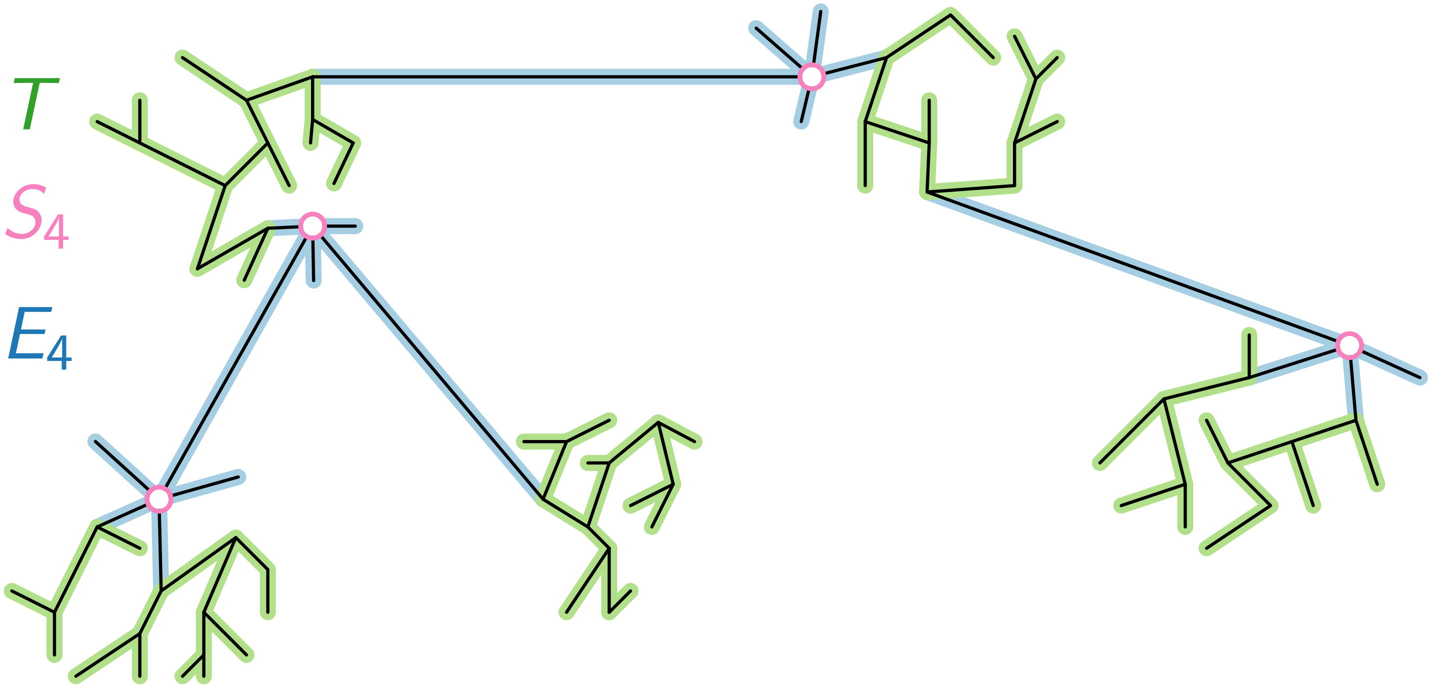

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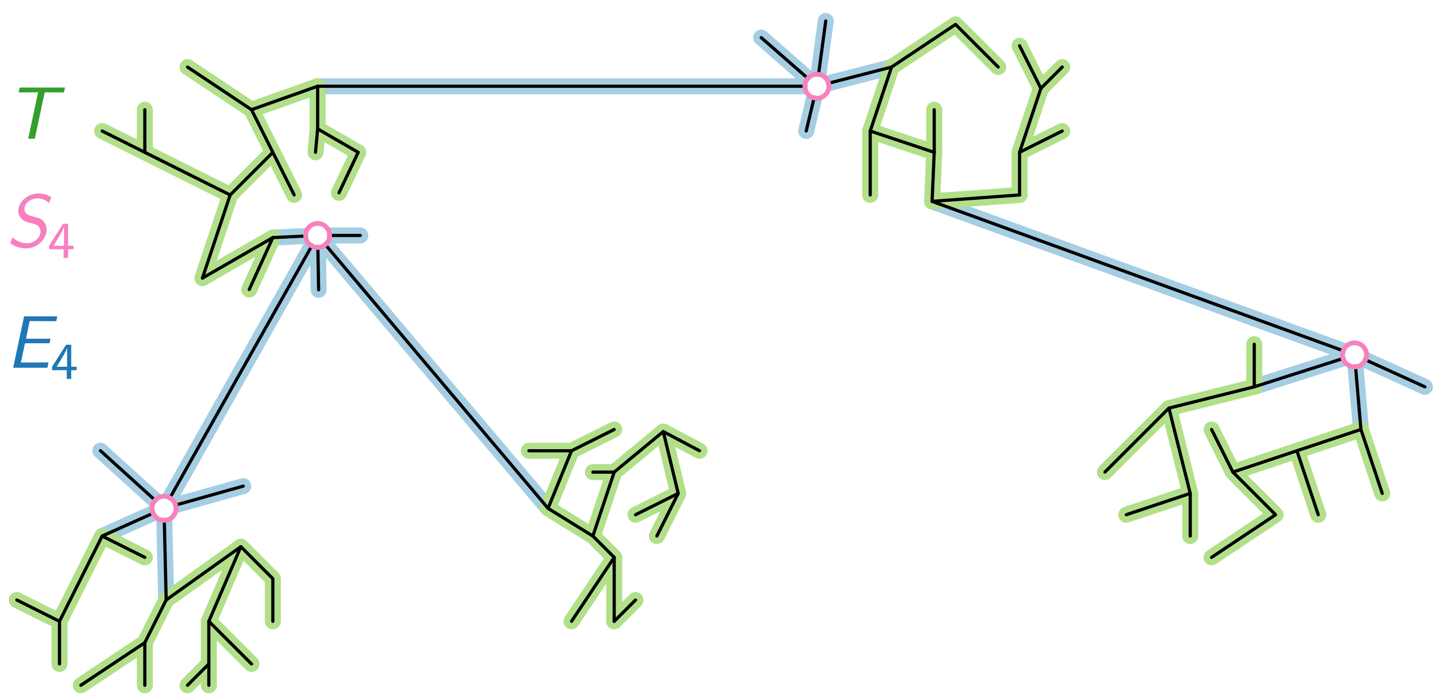

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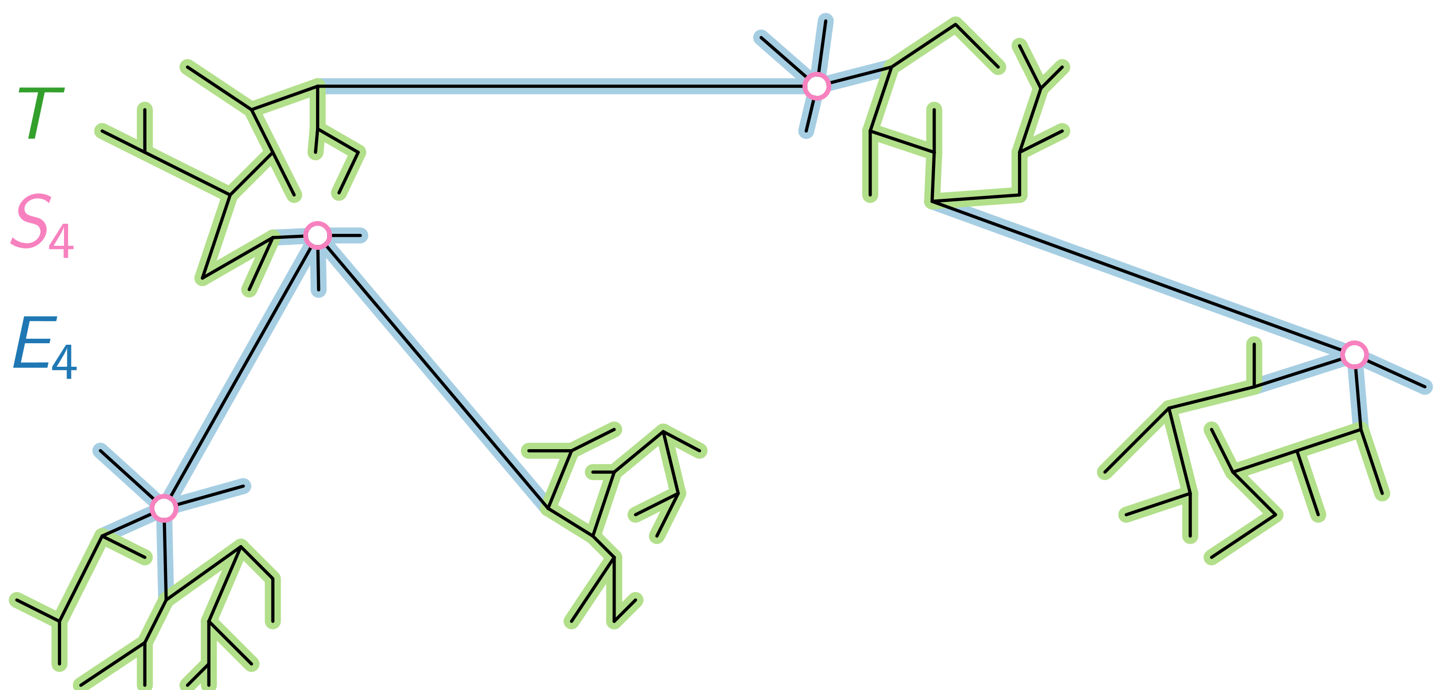

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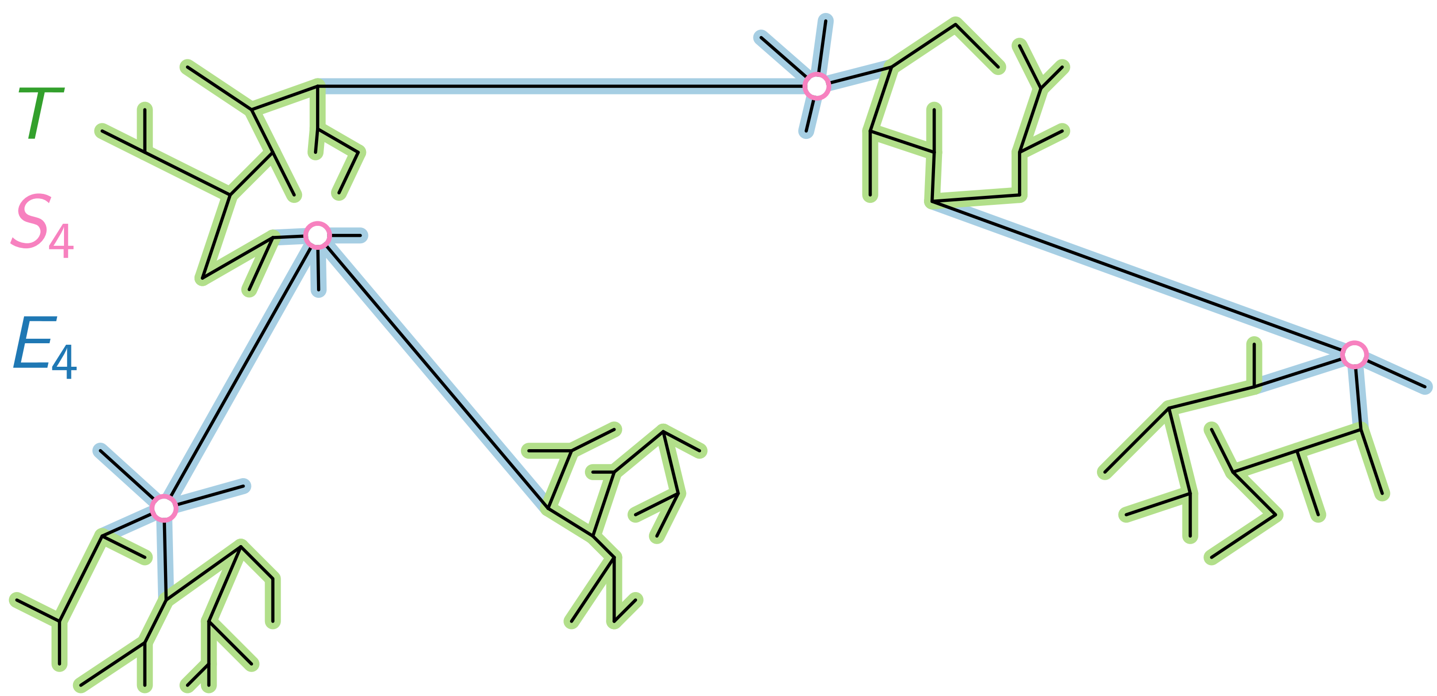

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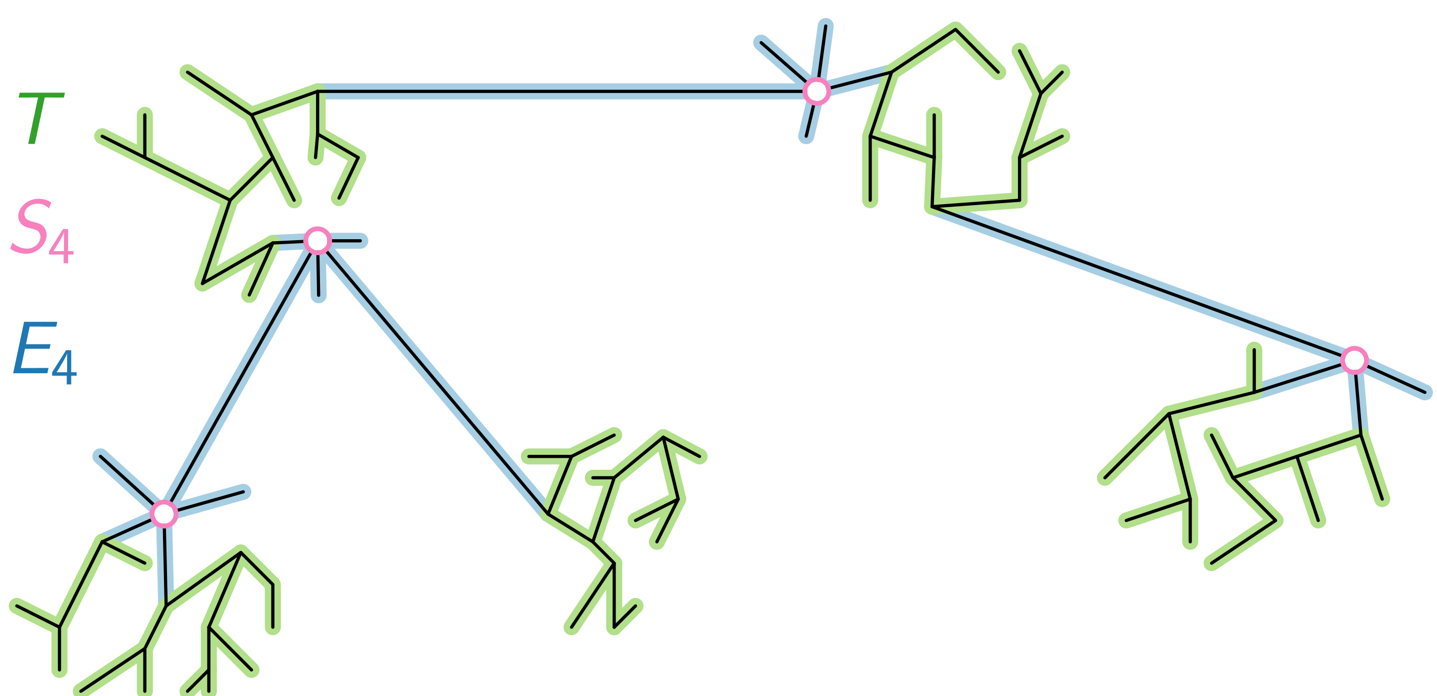
vertex-deg counted twice?



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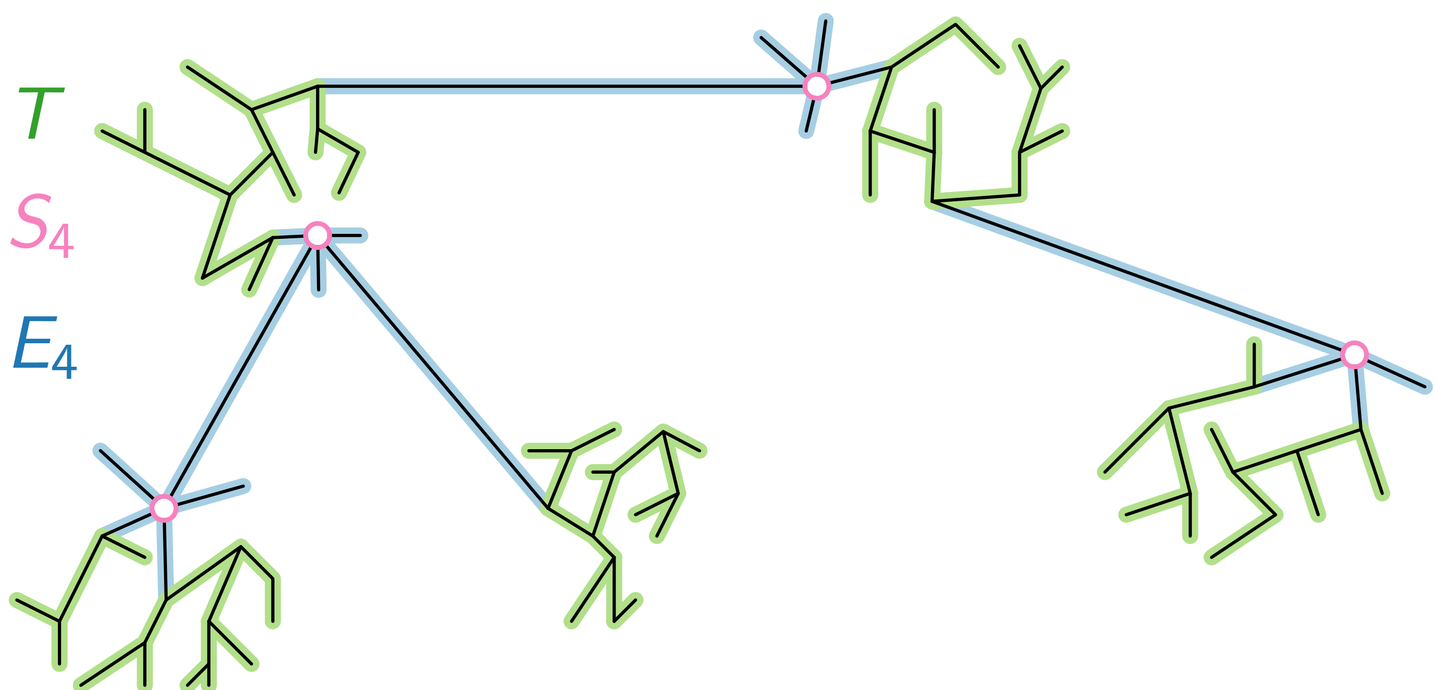

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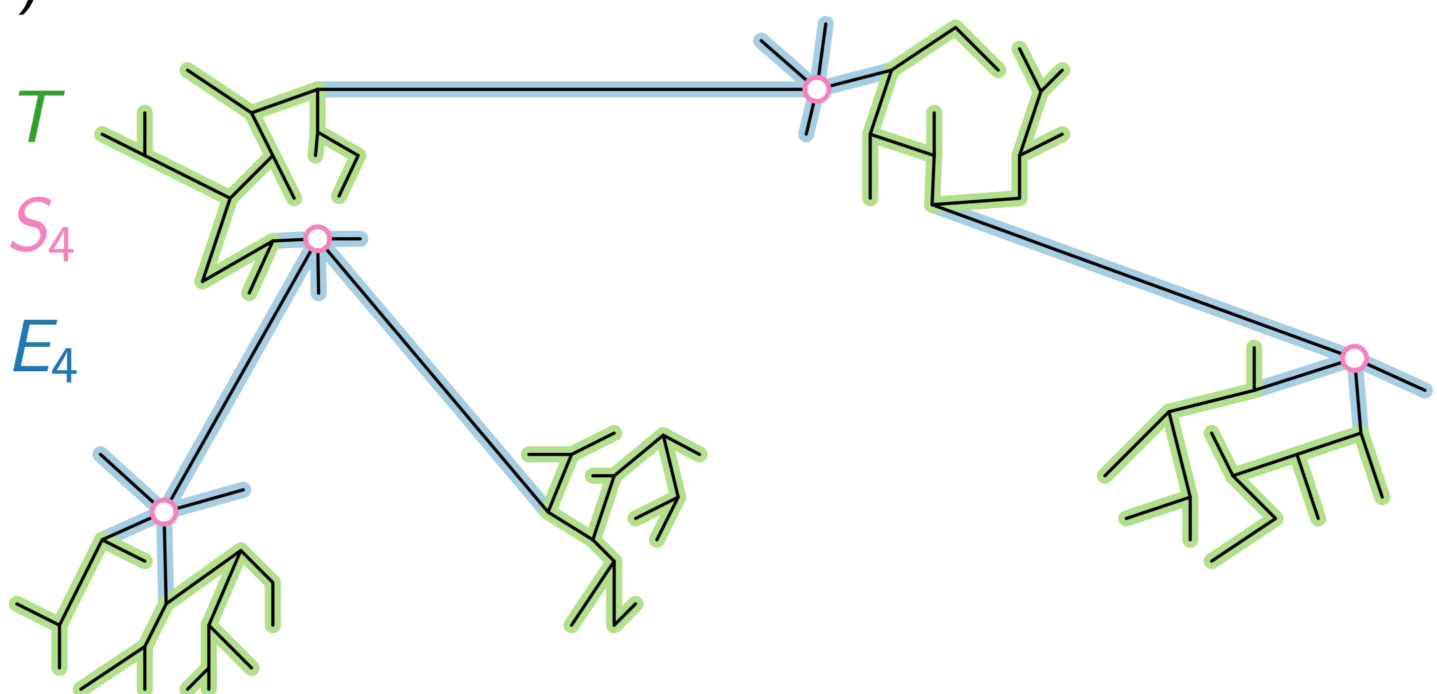

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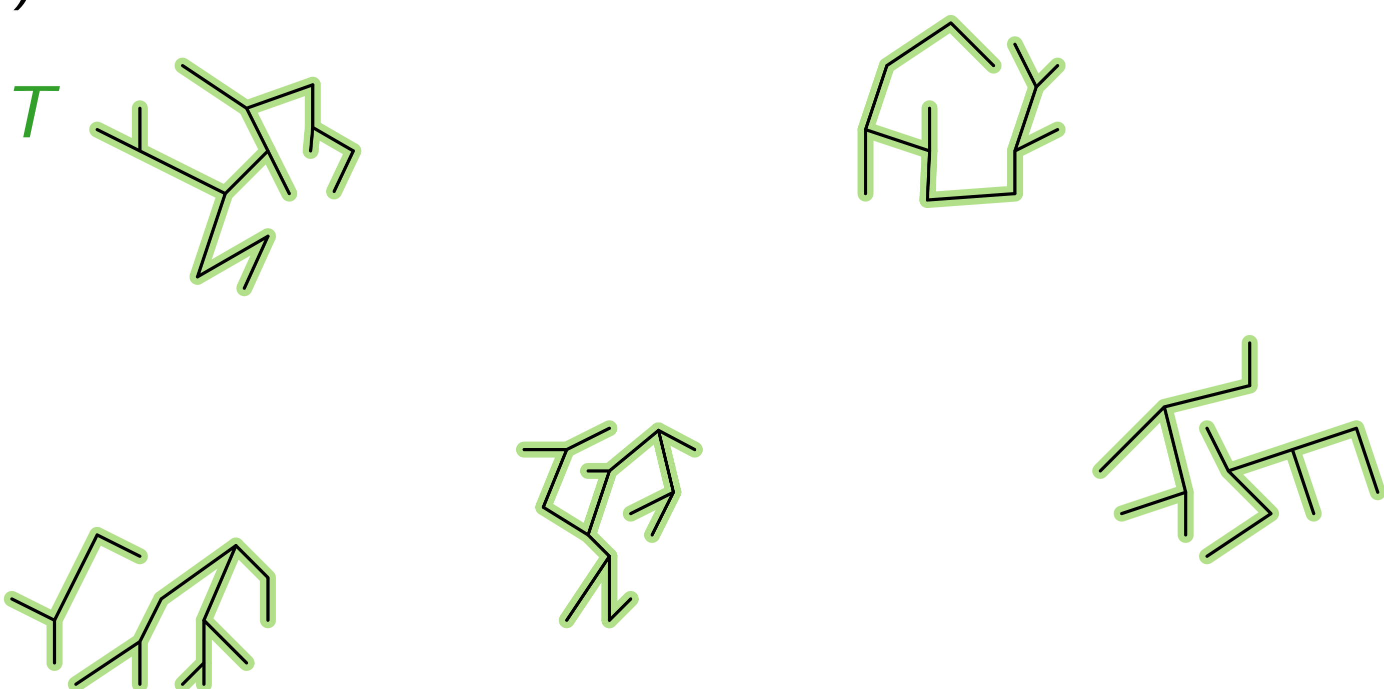

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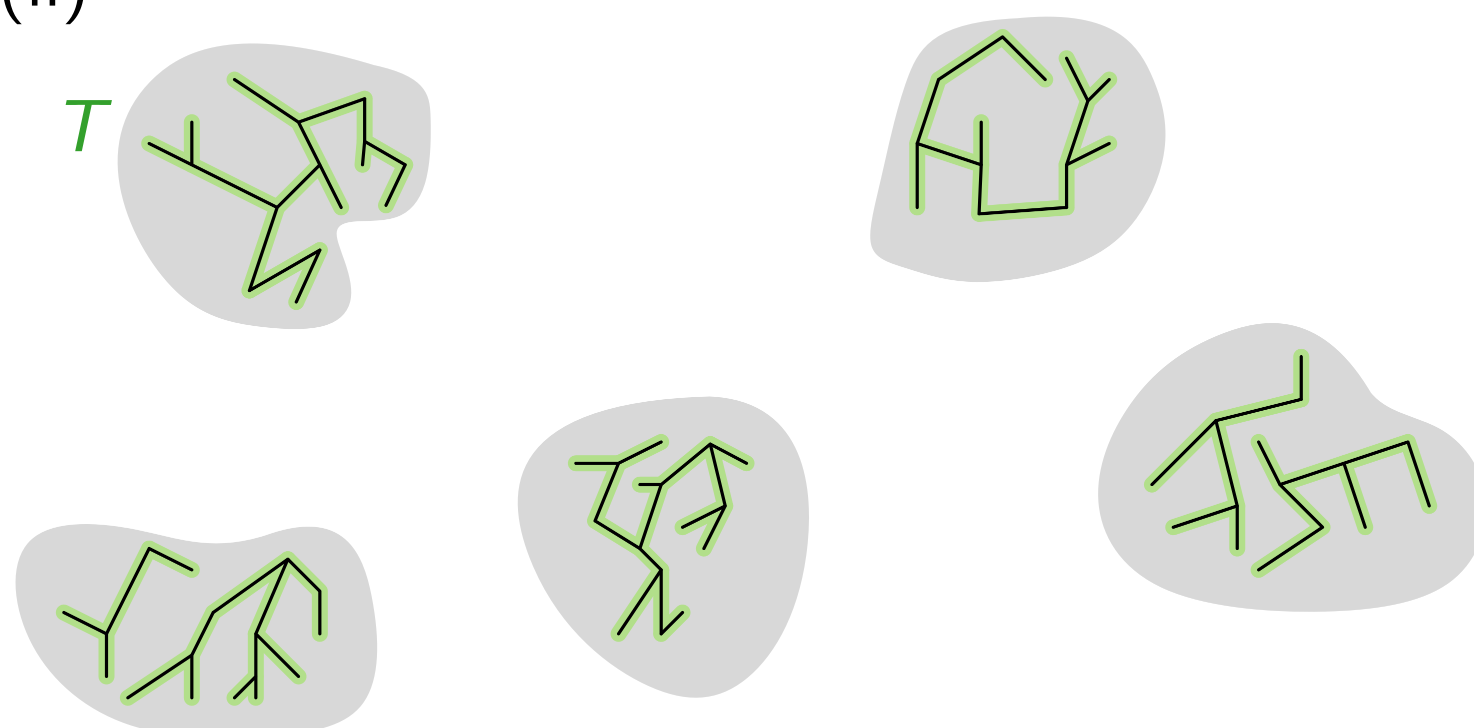

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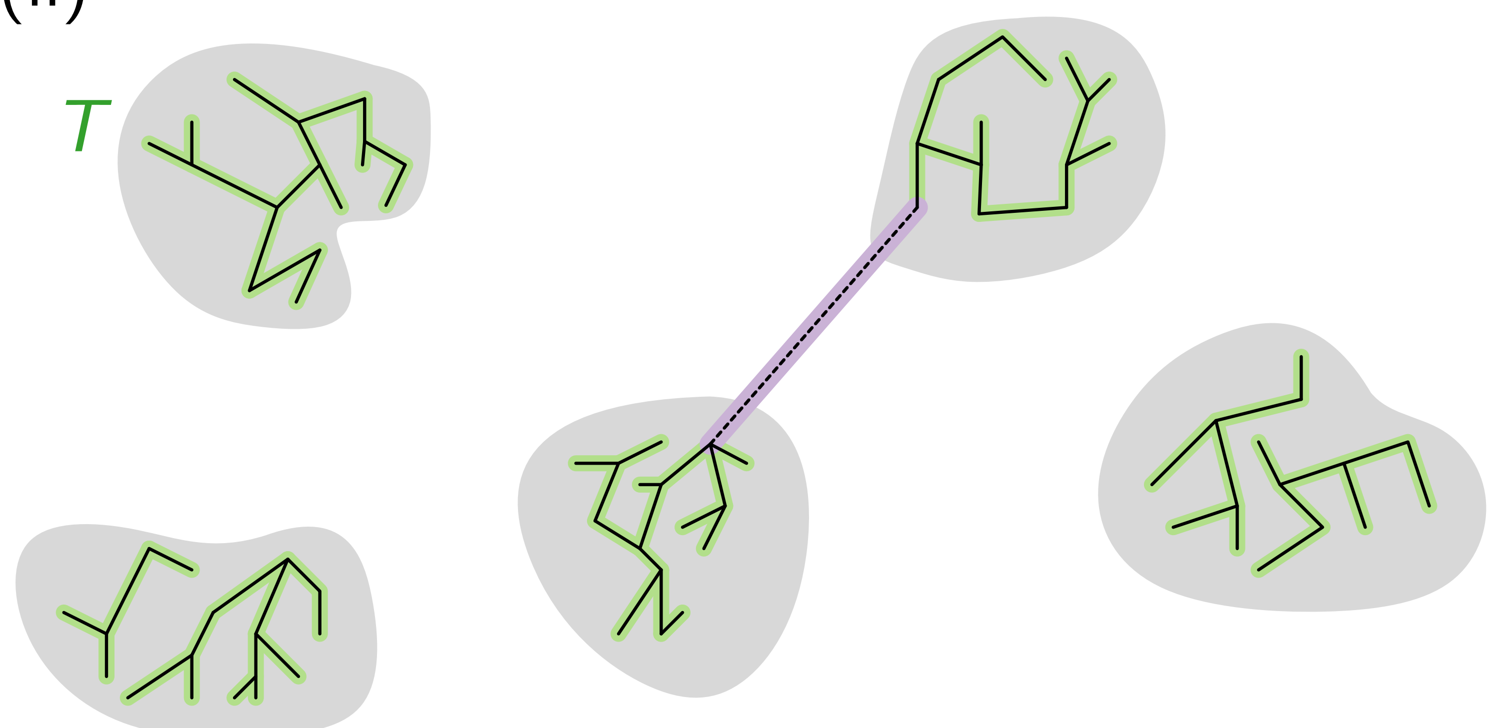

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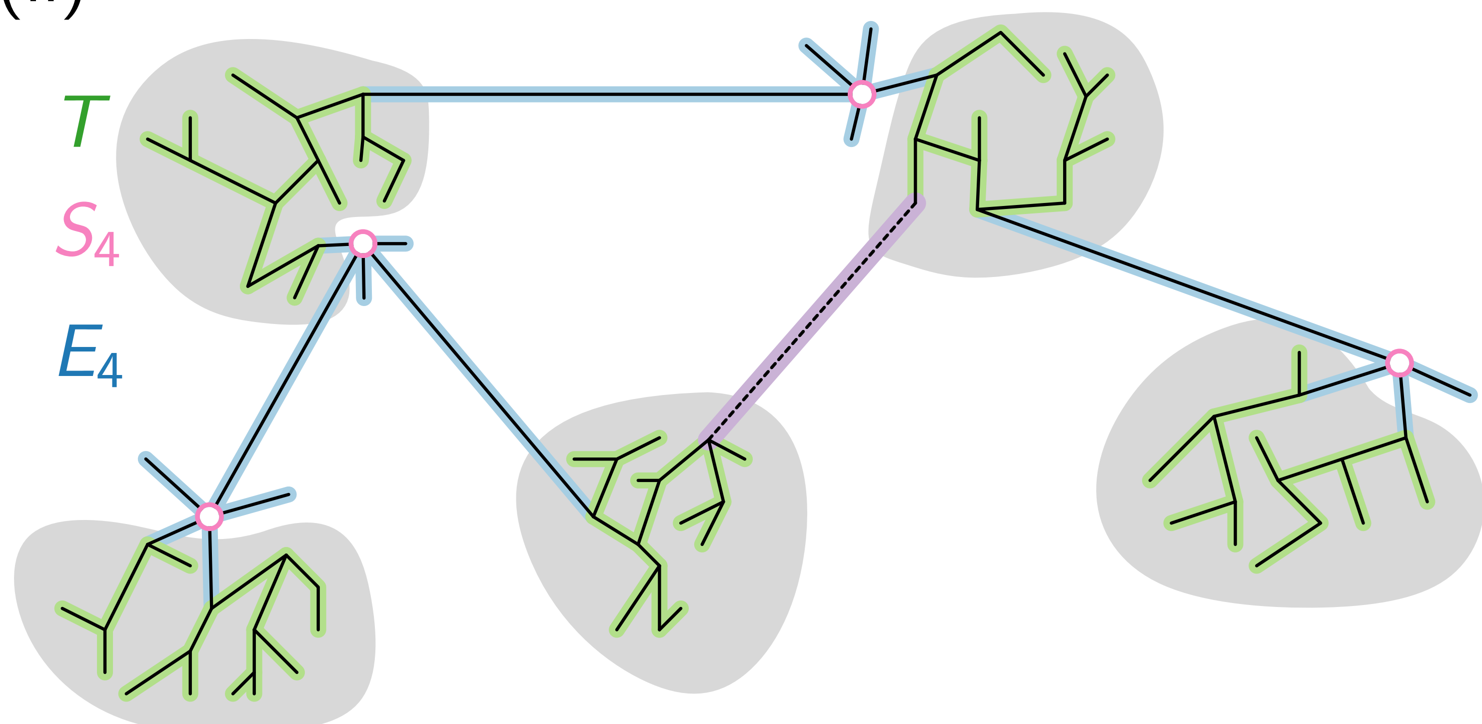

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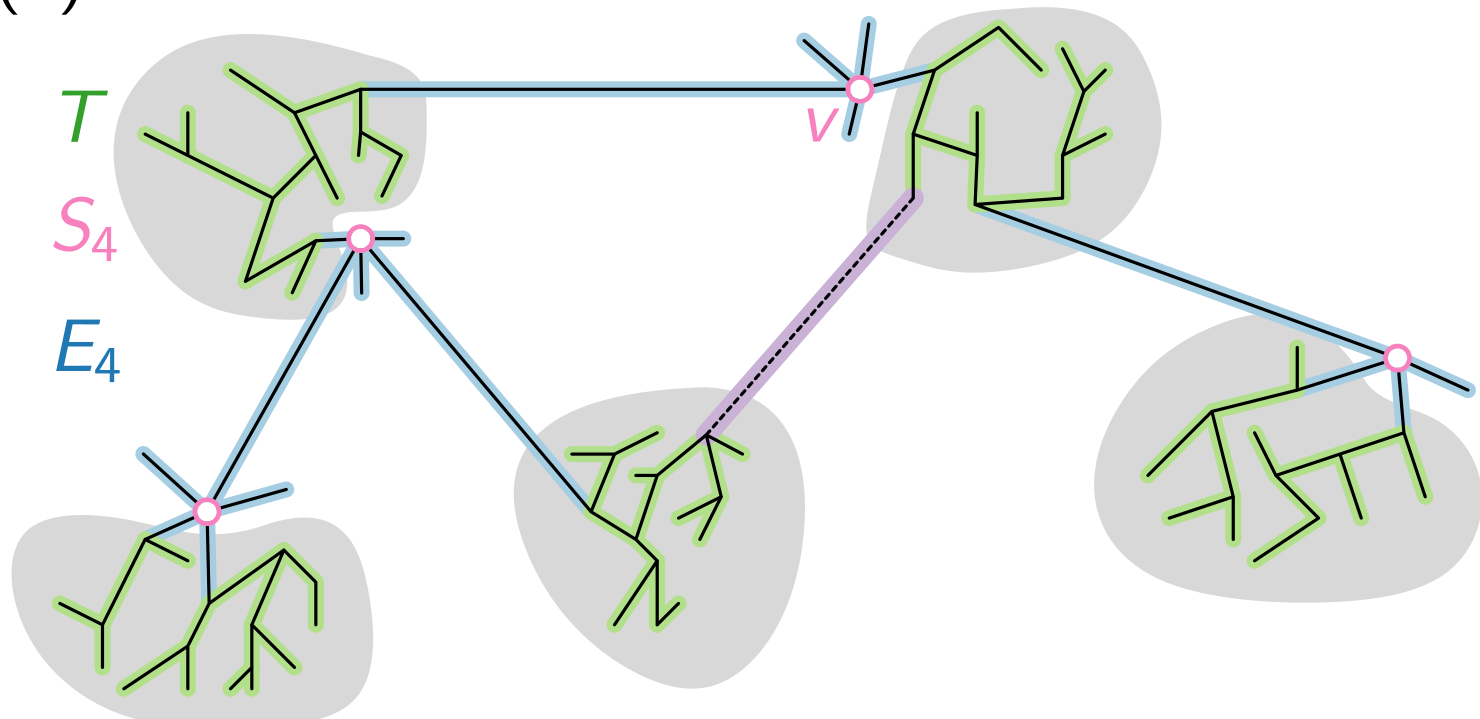

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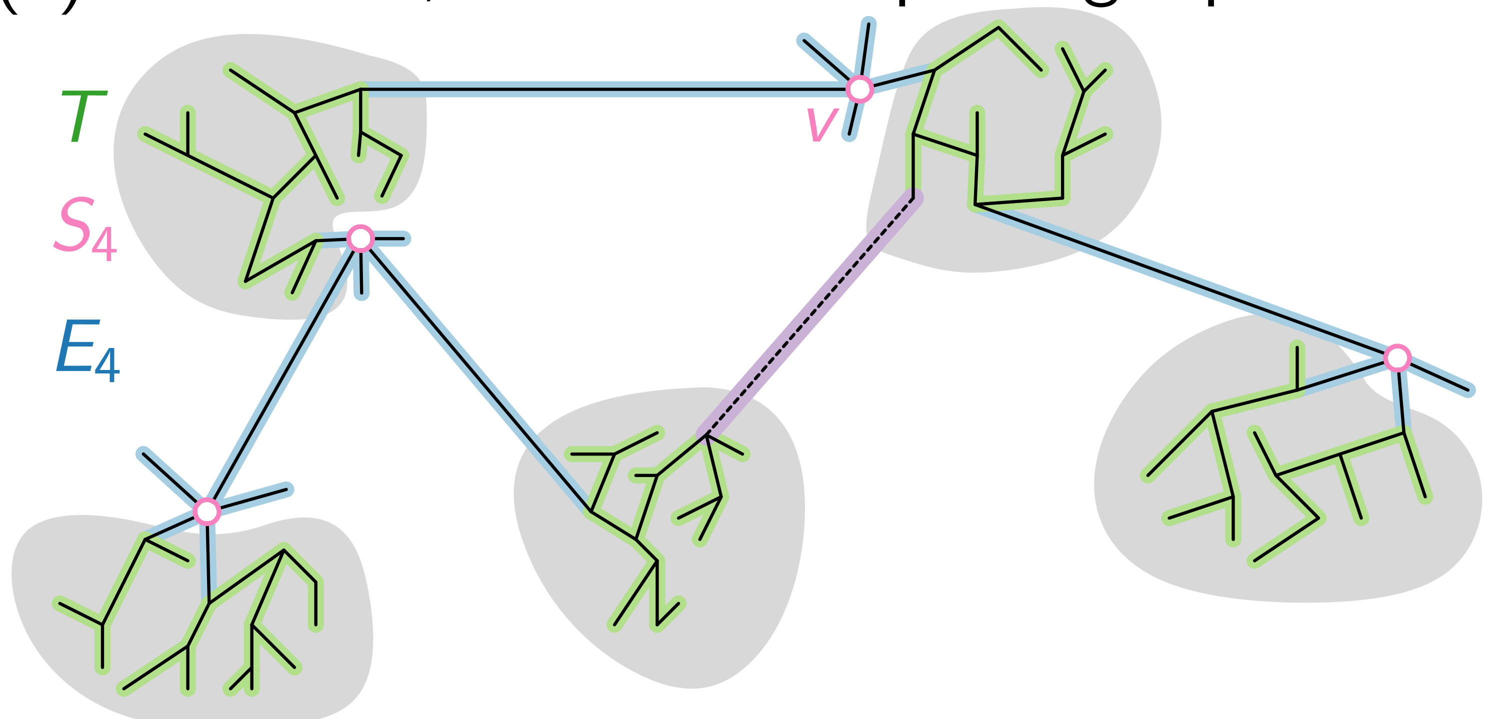
(ii) Otherwise, there is an improving flip for some $v \in S_i$.



Approximation Algorithms

Lecture 10:
Minimum-Degree Spanning Tree
via Local Search

Part V:
Approximation Factor

Approximation Factor

Approximation Factor

[Fürer & Raghavachari: SODA'92, JA'94]

Theorem. Let T be a locally optimal spanning tree.
Then $\Delta(T) \leq 2 \cdot \mathrm{OPT} + \ell$, where $\ell = \lceil \log_2 n \rceil$.

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$$\mathrm{OPT} \underset{\text{Lemma 1}}{\geq} \frac{k}{|S|} = \frac{|E_i|}{|S_{i-1}|} \underset{\text{Lemma 3}}{\geq} \frac{(i-1)|S_i|+1}{|S_{i-1}|} \underset{\text{Lemma 2}}{\geq} \frac{(i-1)|S_i|+1}{2|S_i|} > \frac{(i-1)}{2} \underset{\text{Lemma 2}}{\geq} \frac{\Delta(T)-\ell}{2} \qquad \square$$

Approximation Algorithms

Lecture 10:
Minimum-Degree Spanning Tree
via Local Search

Part VI:
Termination, Running Time & Extensions

Termination and Running Time

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Homework

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Theorem. The algorithm finds a locally optimal spanning tree after at most $O(n^4)$ iterations.

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- Further variants for directed graphs and Steiner tree.