

# Approximation Algorithms

## Lecture 9: An Approximation Scheme for EUCLIDEAN TSP

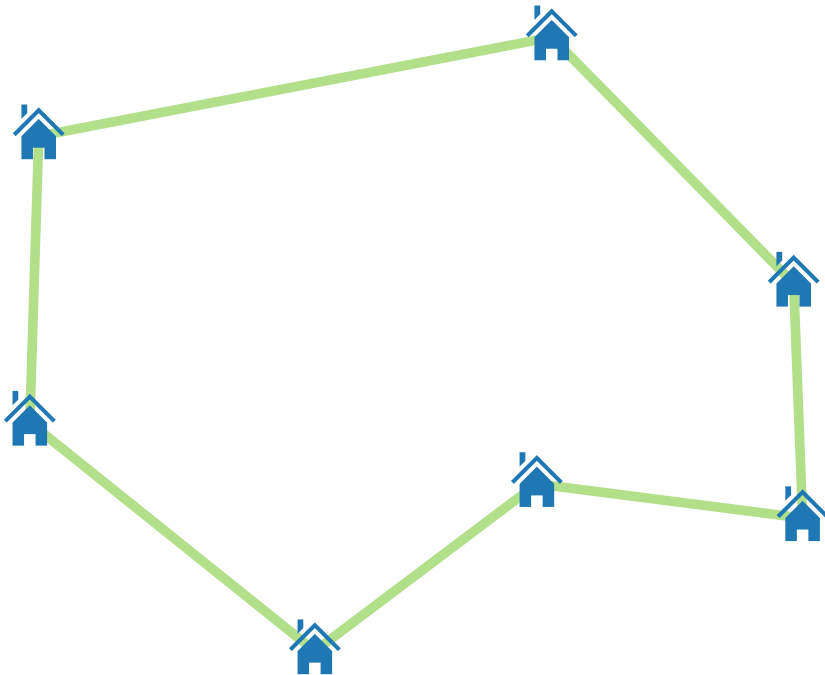
### Part I: The TRAVELING SALESMAN PROBLEM

# TRAVELING SALESMAN PROBLEM (TSP)

**Question:** What's the fastest way to deliver all parcels to their destination?

**Given:** A set of  $n$  houses (points) in  $\mathbb{R}^2$ .

**Task:** Find a tour (Hamiltonian cycle) of min. length.



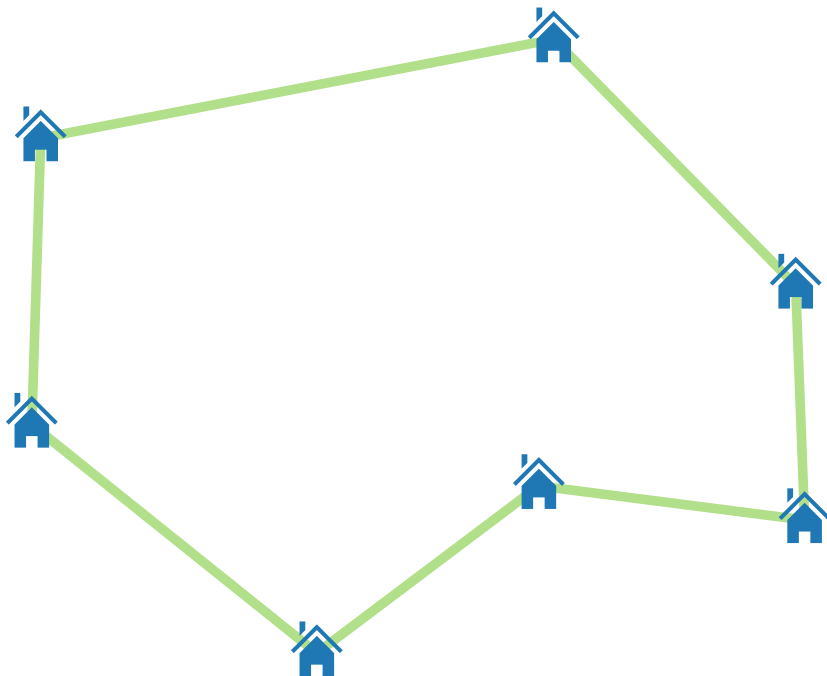
# TRAVELING SALESMAN PROBLEM (TSP)

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Distance between two points?



For every polynomial  $p(n)$ , TSP cannot be approximated within a factor of  $2^{p(n)}$  (unless  $P = NP$ ).

There is a  $3/2$ -approximation algorithm for METRIC TSP [Christofides'76]

METRIC TSP cannot be approximated within factor  $123/122$  (unless  $P = NP$ ).

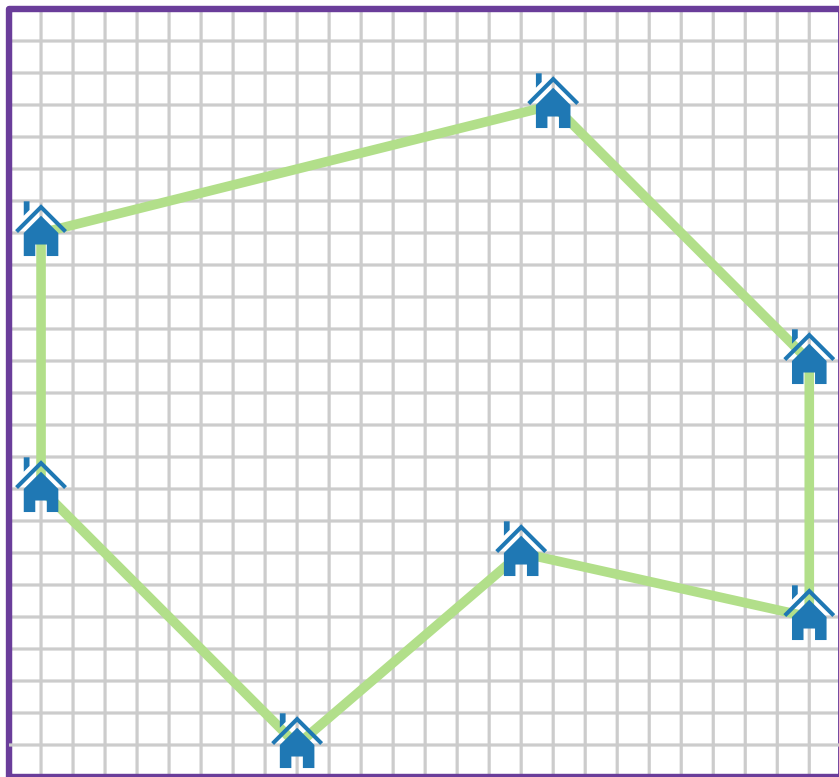
# TRAVELING SALESMAN PROBLEM (TSP)

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Let's assume that the salesman flies  $\Rightarrow$  Euclidean distances.



## Simplifying Assumptions

- Houses inside  $(L \times L)$ -square
  - $L := 4n^2 = 2^k$ ;  
 $k = 2 + 2 \log_2 n$
  - integer coordinates
- (“justification”: homework)

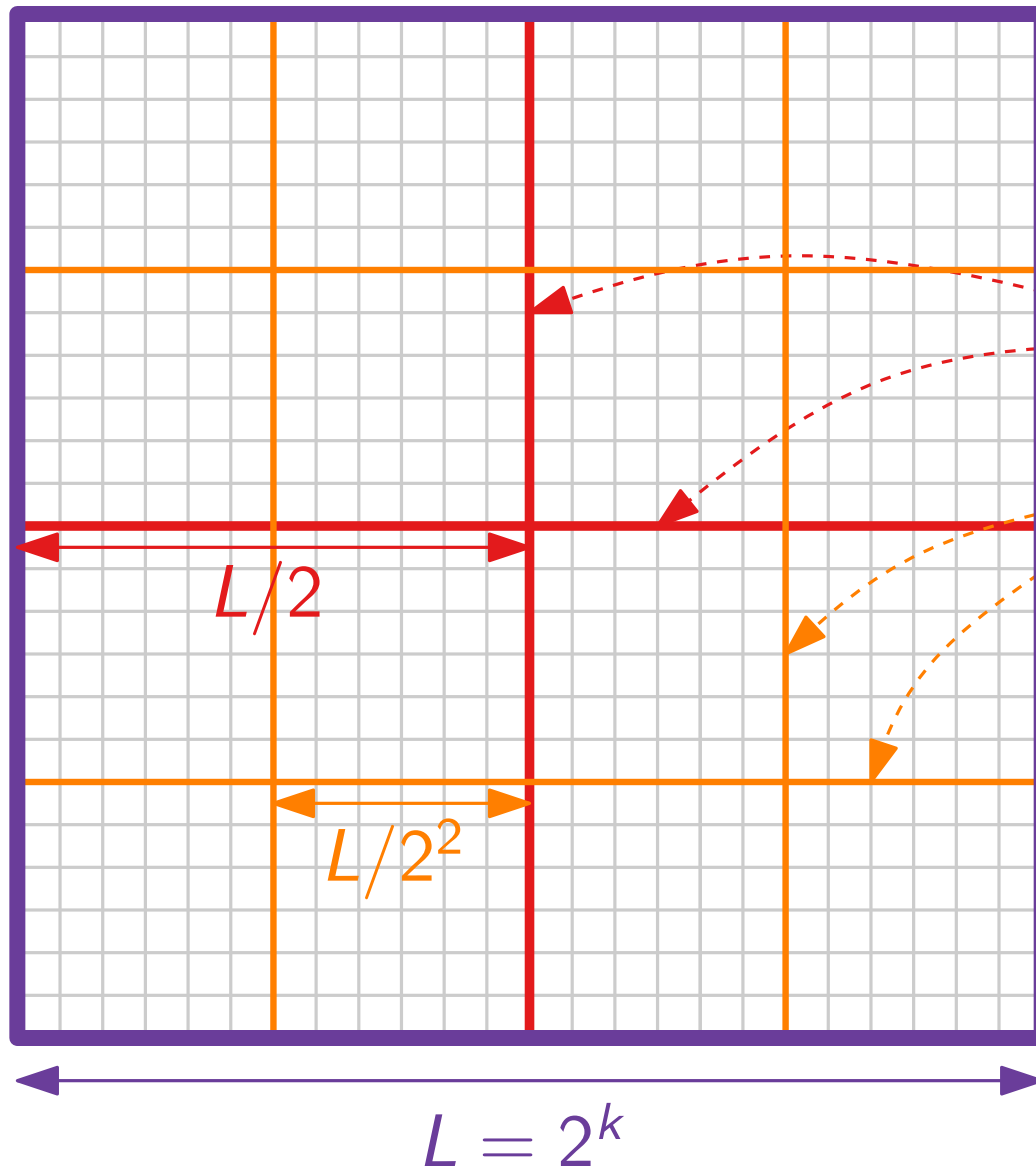
Goal:  $(1 + \varepsilon)$ -approximation!

# Approximation Algorithms

## Lecture 9: A PTAS for EUCLIDEAN TSP

### Part II: Dissection

# Basic Dissection



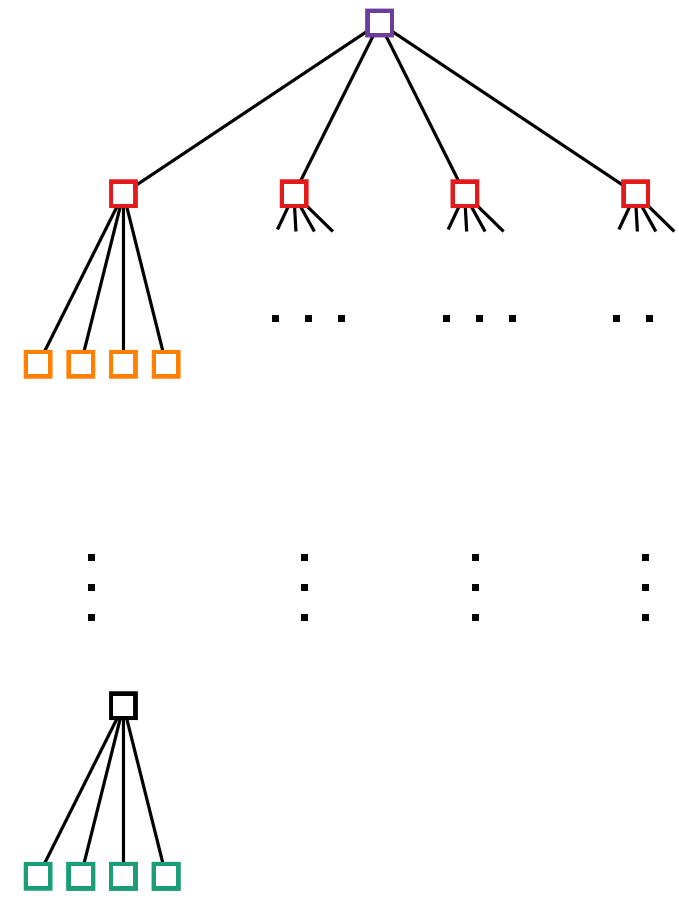
Level 0

Level 1

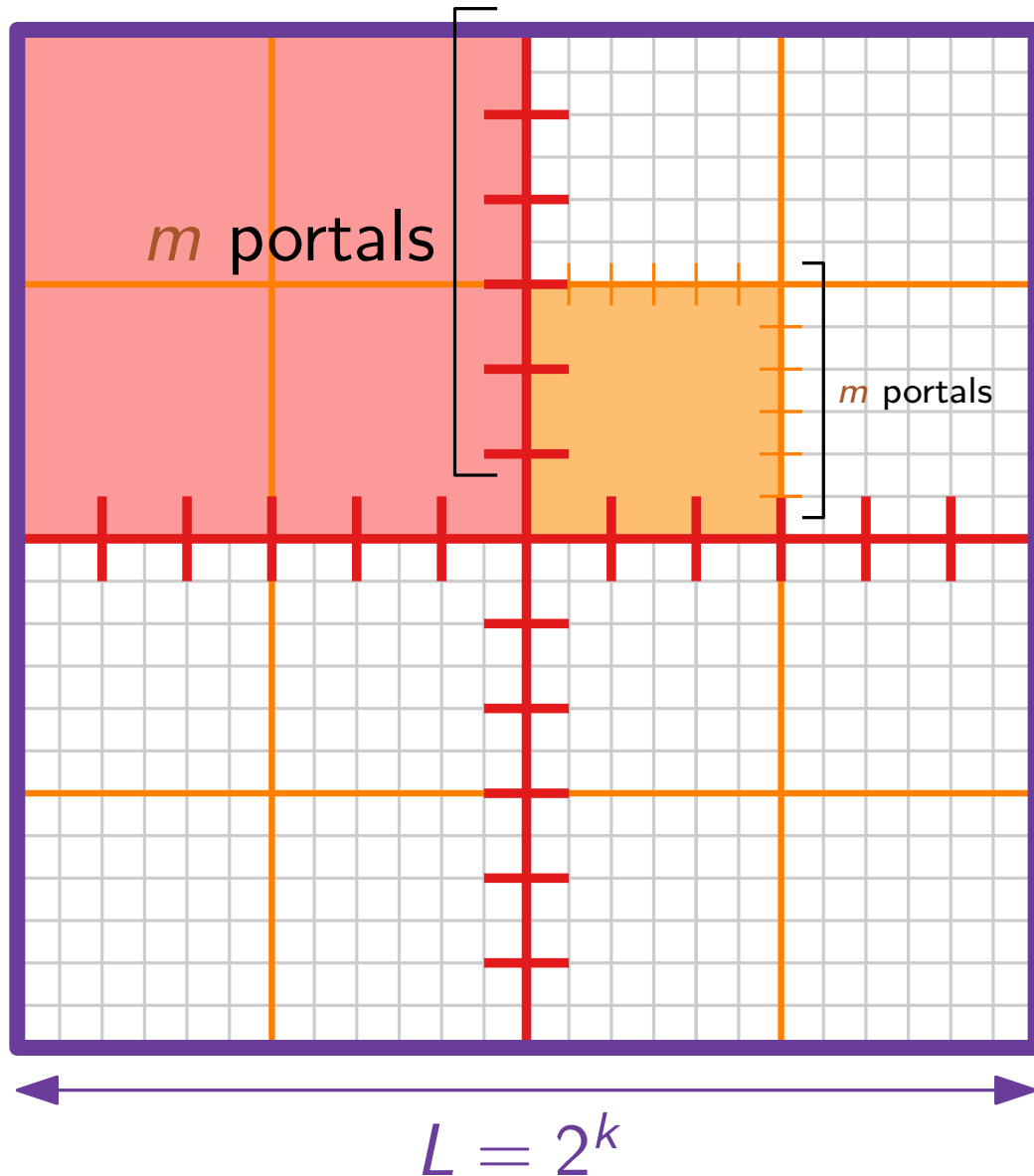
Level 2

Level  $k$

(squares of size  $1 \times 1$ )



# Portals



- Let  $m$  be a power of 2 in the interval  $[k/\varepsilon, 2k/\varepsilon]$ .  
Recall that  $k = 2 + 2 \log_2 n$ .  
 $\Rightarrow m \in O((\log n)/\varepsilon)$
- **Portals** on level- $i$  line are at a distance of  $L/(2^i m)$ .
- Every level- $i$  square has size  $L/2^i \times L/2^i$ .
- A level- $i$  square has  $\leq 4m$  portals on its boundary.

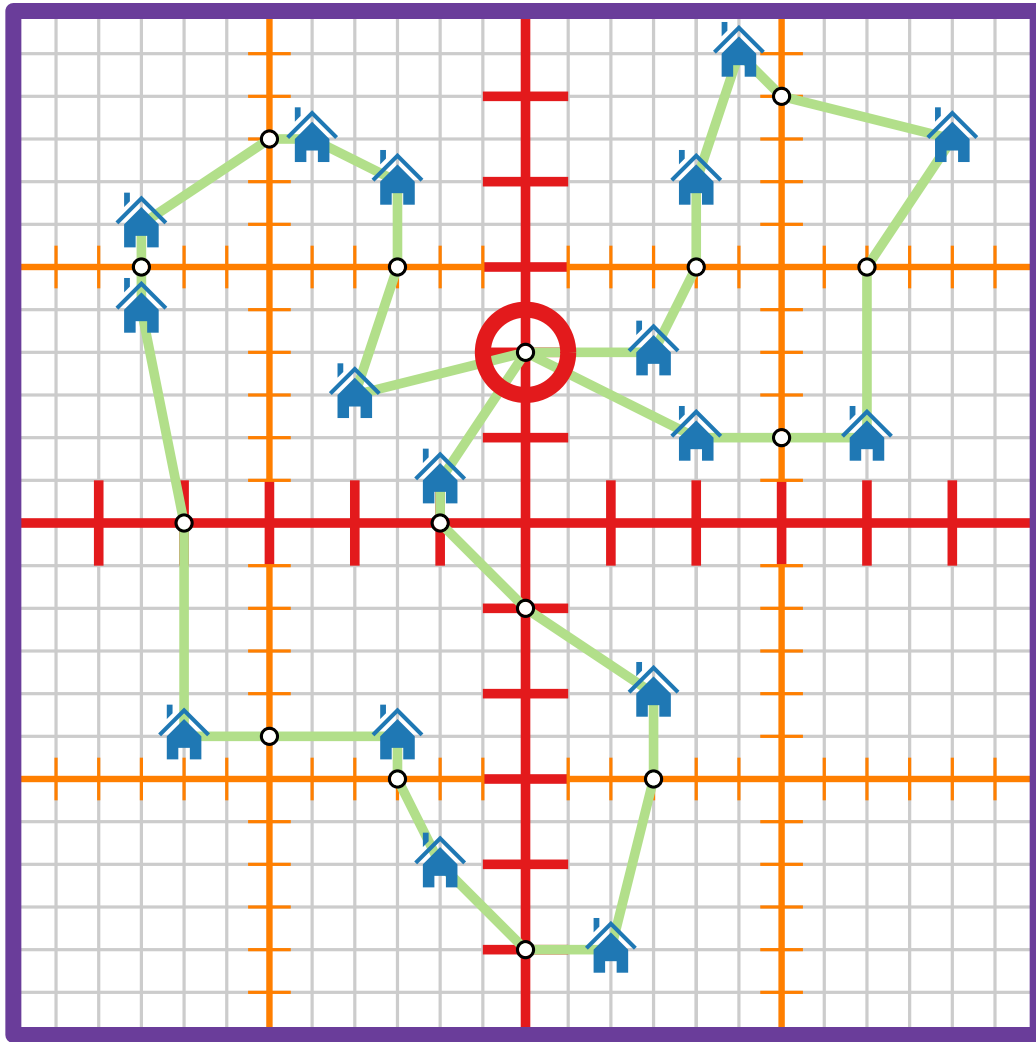
# Approximation Algorithms

## Lecture 9: A PTAS for EUCLIDEAN TSP

### Part III: Well-Behaved Tours



# Well-Behaved Tours



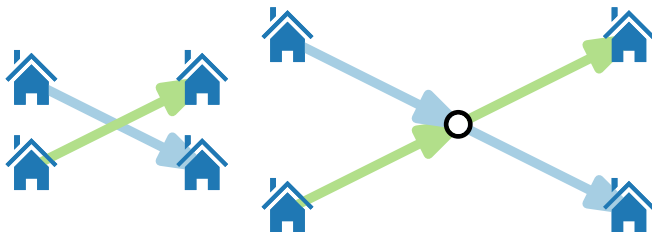
A tour is *well-behaved* if

- it involves all houses and a subset of the portals,
- no edge of the tour crosses a line of the basic dissection,
- it is crossing-free.

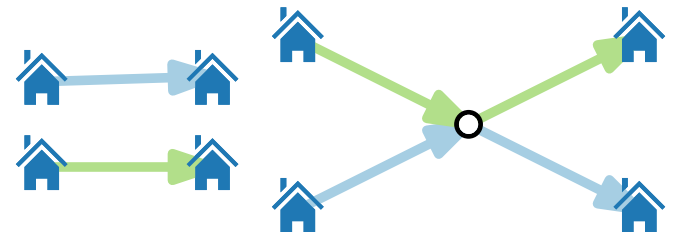
W.l.o.g. (**homework**):

No portal visited more than twice

Crossing



No crossing



# Computing a Well-Behaved Tour

**Lemma.** An optimal well-behaved tour can be computed in  $2^{O(m)} = n^{O(1/\varepsilon)}$  time.

- Sketch.**
- Dynamic programming!
  - Compute sub-structure of an optimal tour for each square in the dissection tree.
  - These solutions can be efficiently propagated bottom-up through the dissection tree.

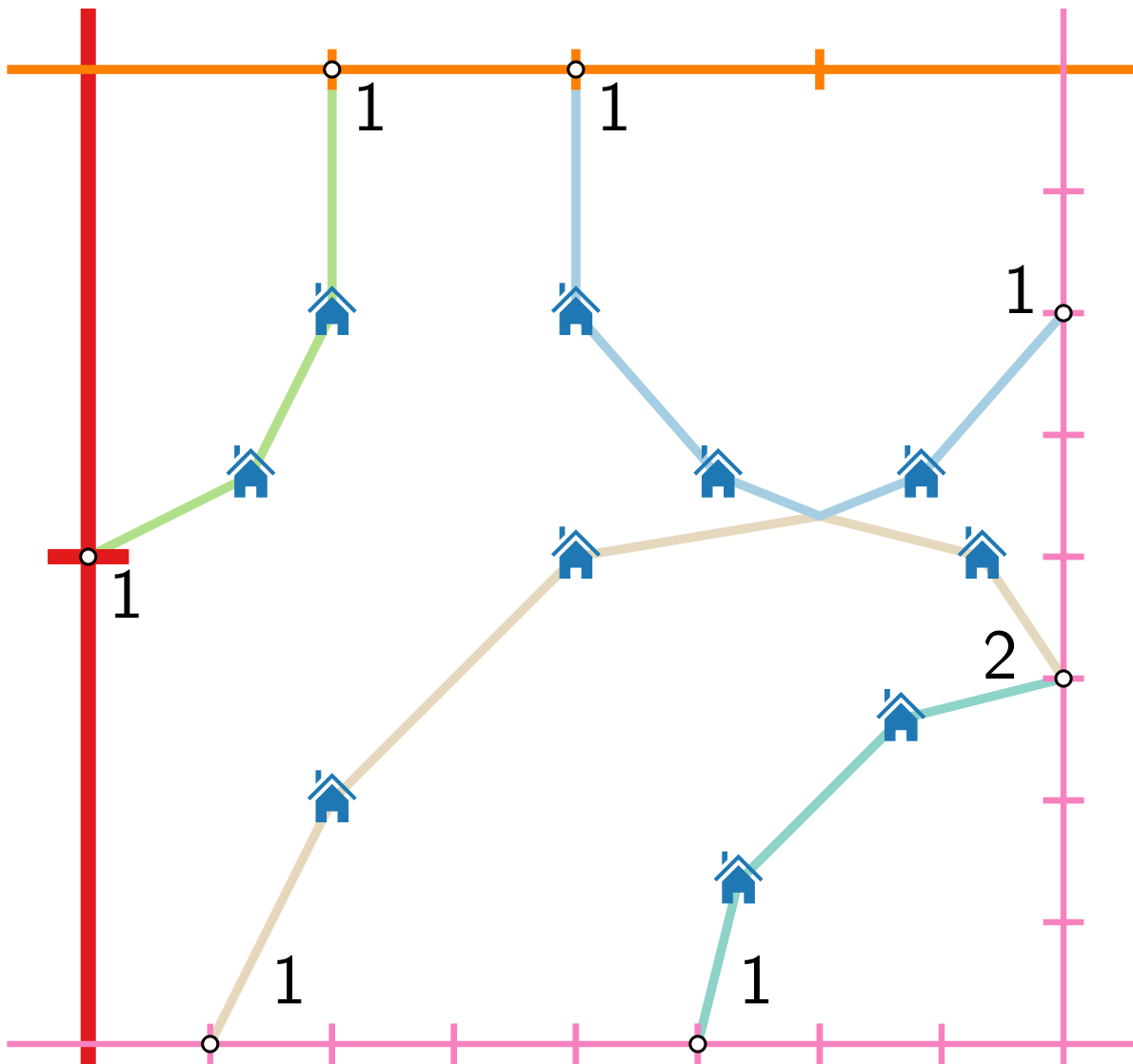


# Approximation Algorithms

## Lecture 9: A PTAS for EUCLIDEAN TSP

### Part IV: Dynamic Program

# Dynamic Program (I)



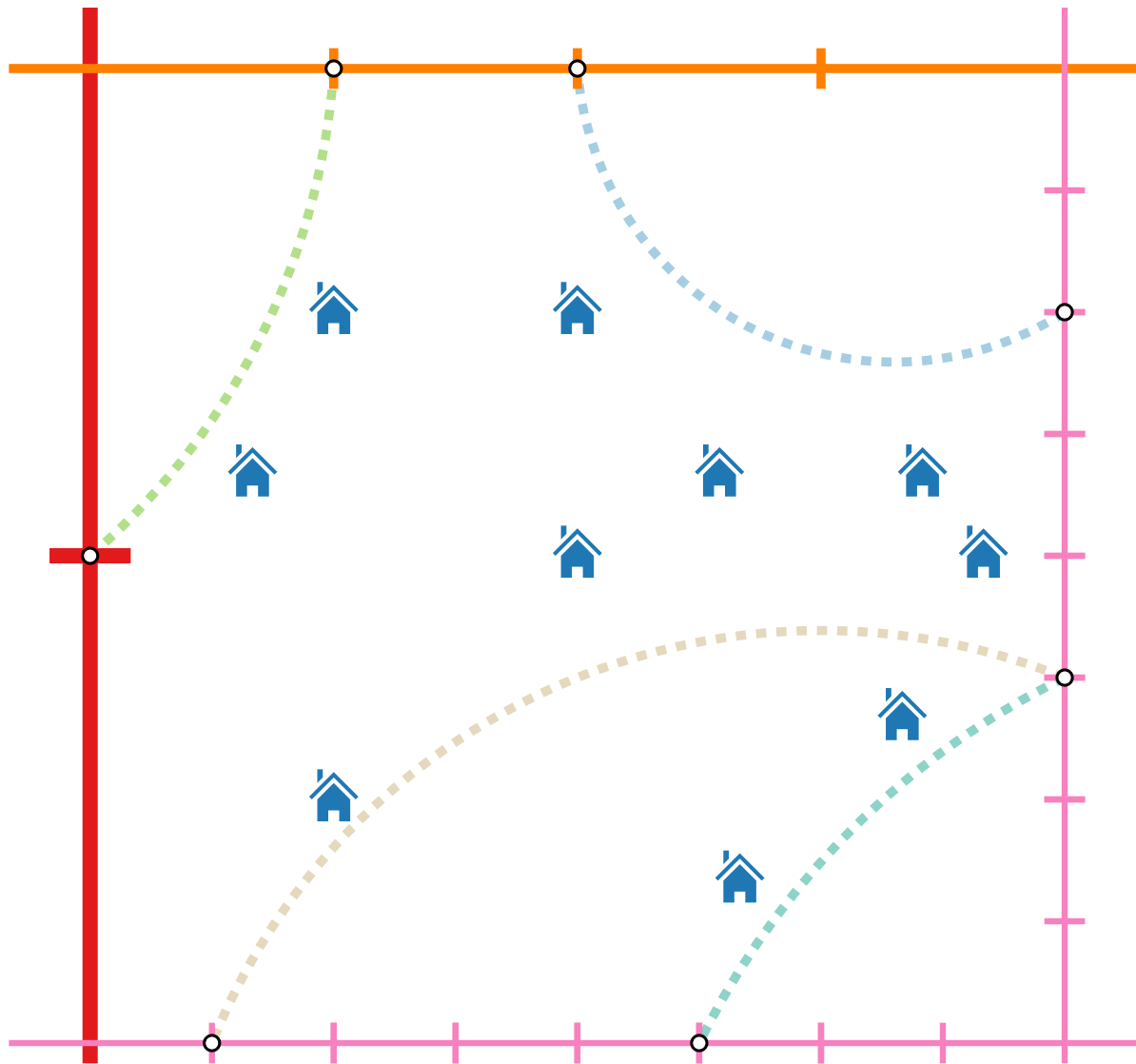
Each well-behaved tour induces the following in each square  $Q$  of the dissection:

- a path cover of the houses in  $Q$ ,
- ...such that each portal of  $Q$  is visited 0, 1 or 2 times,

$\Rightarrow$  at most  $3^{4m} \in 3^{O((\log n)/\epsilon)} = n^{O(1/\epsilon)}$  possibilities

$m = O((\log n)/\epsilon)$

# Dynamic Program (II)



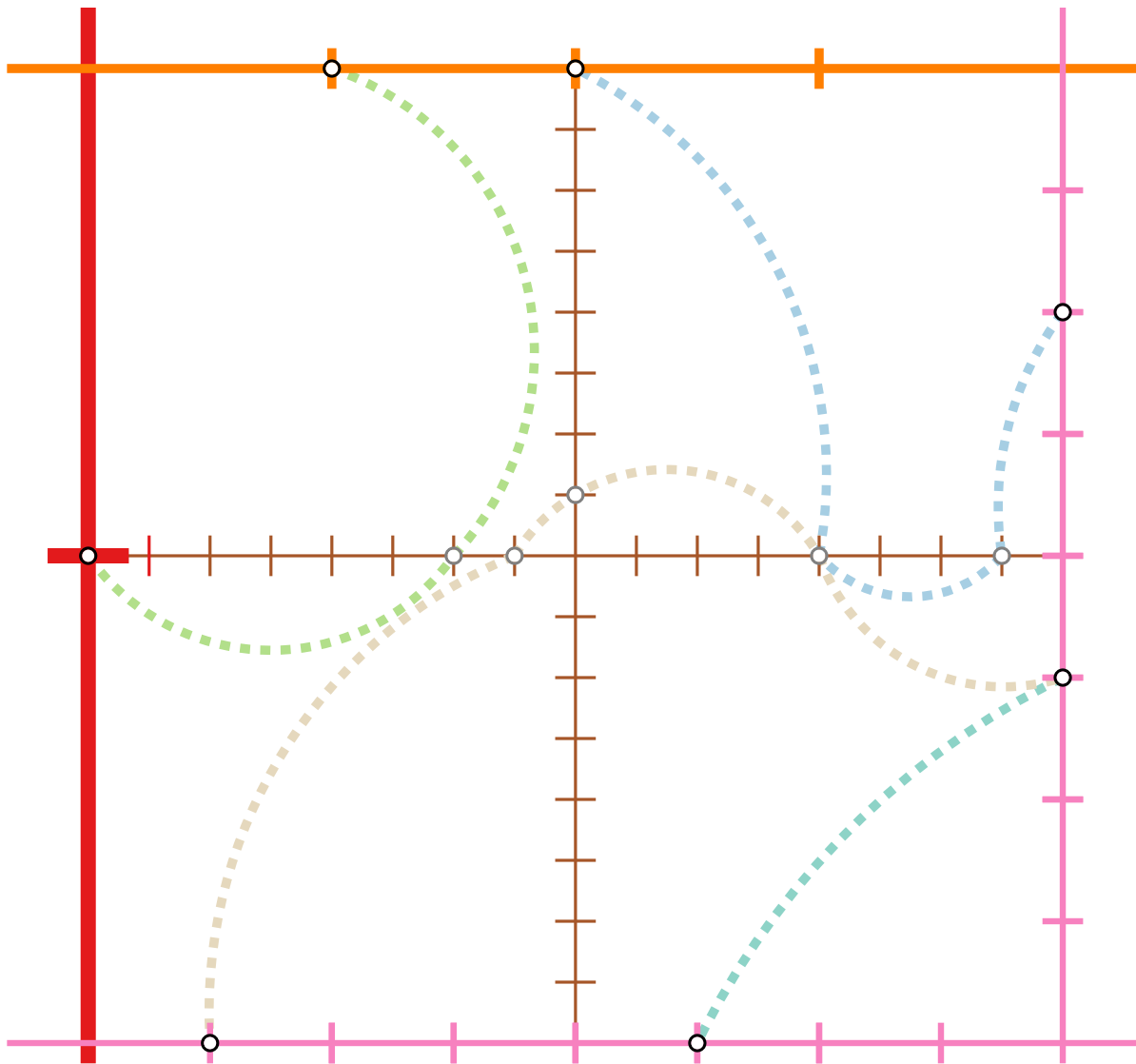
Compute

- for each square  $Q$  in the dissection and
- for each crossing-free pairing  $P$  of  $Q$ ,

an optimal path cover that respects  $P$ .

How?

# Dynamic Program (III)



For a given square  $Q$  and pairing  $P$ :

- Iterate over all  $(n^{O(1/\varepsilon)})^4 = n^{O(1/\varepsilon)}$  crossing-free pairings of the child squares.
- Minimize the cost over all such pairings that additionally respect  $P$ .
- Correctness follows by induction.

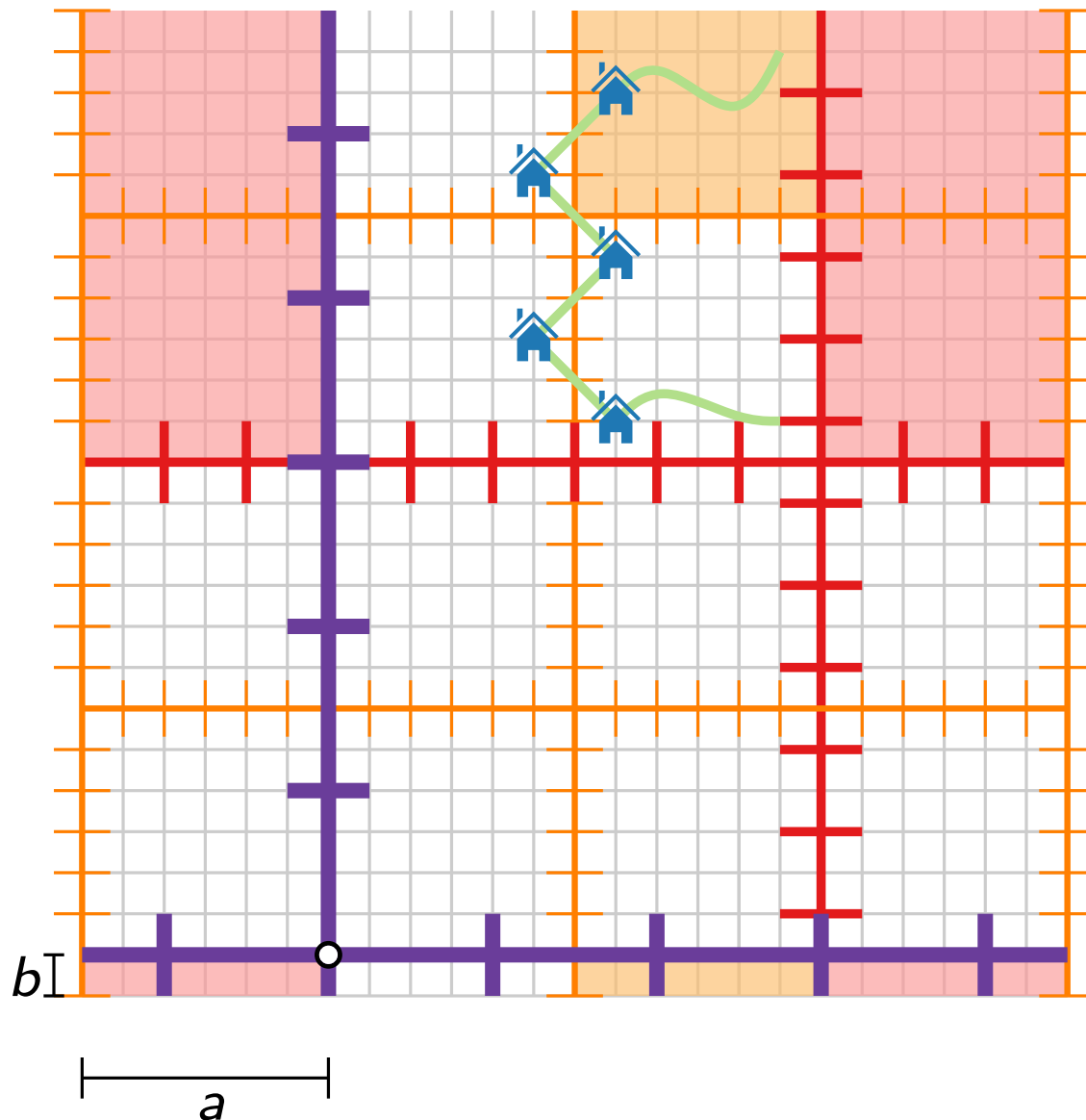
**Lemma.** An optimal well-behaved tour can be computed in  $2^{O(m)} = n^{O(1/\varepsilon)}$  time.

# Approximation Algorithms

## Lecture 9: A PTAS for EUCLIDEAN TSP

### Part V: Shifted Dissections

# Shifted Dissections



- The best well-behaved tour can be a bad approximation.
- Consider an  $(a, b)$ -shifted dissection:
 
$$x \mapsto (x + a) \bmod L$$

$$y \mapsto (y + b) \bmod L$$
- Squares in the dissection tree are “wrapped around”.
- Dynamic program must be modified accordingly.

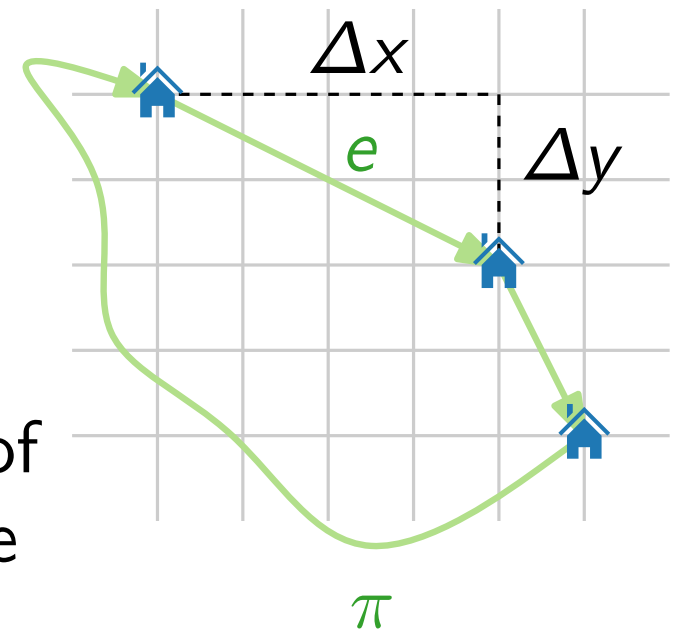


# Shifted Dissections (II)

**Lemma.** If  $\pi$  is an optimal tour and  $N(\pi)$  is the number of crossings of  $\pi$  with the lines of the  $(L \times L)$ -grid, then  $N(\pi) \leq \sqrt{2} \cdot \text{OPT}$ .

**Proof.**

- Consider a tour as an ordered cyclic sequence.
- Each edge  $e$  generates  $N_e \leq \Delta x + \Delta y$  crossings.
- Crossings at the endpoint of an edge are counted for the next edge.



$$0 \leq (\Delta x - \Delta y)^2$$

- $N_e^2 \leq (\Delta x + \Delta y)^2 \leq 2(\Delta x^2 + \Delta y^2) = 2|e|^2$ .
- $N(\pi) = \sum_{e \in \pi} N_e \leq \sum_{e \in \pi} \sqrt{2|e|^2} = \sqrt{2} \cdot \text{OPT}$ .

□

# Approximation Algorithms

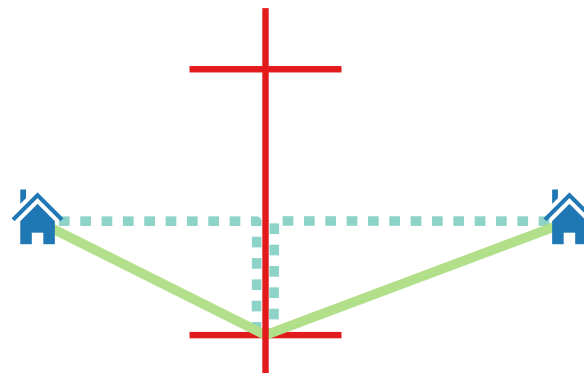
## Lecture 9: A PTAS for EUCLIDEAN TSP

### Part VI: Approximation Factor

# Shifted Dissections (III)

**Theorem.** Let  $a, b \in [0, L - 1]$  be chosen independently and uniformly at random. Then the expected cost of an optimal well-behaved tour with respect to the  $(a, b)$ -shifted dissection is at most  $(1 + 2\sqrt{2}\varepsilon)\text{OPT}$ .

**Proof.** Consider optimal tour  $\pi$ . Make  $\pi$  well-behaved by moving each intersection point with the  $(L \times L)$ -grid to the nearest portal.



$\Rightarrow$  Detour per intersection  $\leq$  inter-portal distance.

# Shifted Dissections (III)

- Consider an intersection point between  $\pi$  and a line  $l$  of the  $(L \times L)$ -grid.
- With probability at most  $2^i/L = 2^{i-k}$ , line  $l$  is a level- $i$  line.  
 $\Rightarrow$  Increase in tour length  $\leq L/(2^i m)$  (inter-portal distance).
- Thus, the expected increase in tour length due to this intersection is at most:

$$m \in [k/\varepsilon, 2k/\varepsilon]$$

$$\sum_{i=0}^k \frac{2^i}{L} \cdot \frac{L}{2^i m} \leq \frac{k+1}{m} \leq \frac{k+1}{k} \cdot \varepsilon \leq 2\varepsilon.$$

- Summing over all  $N(\pi) \leq \sqrt{2} \cdot \text{OPT}$  intersection points and applying linearity of expectation yields the claim.

# Polynomial-Time Approximation Scheme

**Theorem.** Let  $a, b \in [0, L - 1]$  be chosen independently and uniformly at random. Then the expected cost of an optimal well-behaved tour with respect to the  $(a, b)$ -shifted dissection is at most  $(1 + 2\sqrt{2}\varepsilon)\text{OPT}$ .

**Theorem.** There is a *deterministic* algorithm (PTAS) for EUCLIDEAN TSP that provides, for every  $\varepsilon > 0$ , a  $(1 + \varepsilon)$ -approximation in  $n^{O(1/\varepsilon)}$  time.

**Proof.** Try all  $L^2 \in O(n^4)$  many  $(a, b)$ -shifted dissections. By the previous theorem and the pigeon-hole principle, one of them is good enough.  $\square$

# Literature

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- Sanjeev Arora: Nearly linear time approximation schemes for Euclidean TSP and other geometric problems.  
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Randomized,  $O\left(n(\log n)^{O(1/\varepsilon)}\right)$  time

# Literature (cont'd)

Runtime  $O\left(n^{O(1/\varepsilon^2)}\right)$

- Sanjeev Arora, Michelangelo Grigni, David Karger, Philip Klein, Andrzej Wołoszyn: Polynomial time approximation scheme for Weighted Planar Graph TSP. *Proc. SODA*, p. 33–41, 1998.
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Best paper award!