Approximation Algorithms

Lecture 9: An Approximation Scheme for Euclidean TSP

Part I: The Traveling Salesman Problem

Alexander Wolff

Winter term 2025

Traveling Salesman Problem (TSP)

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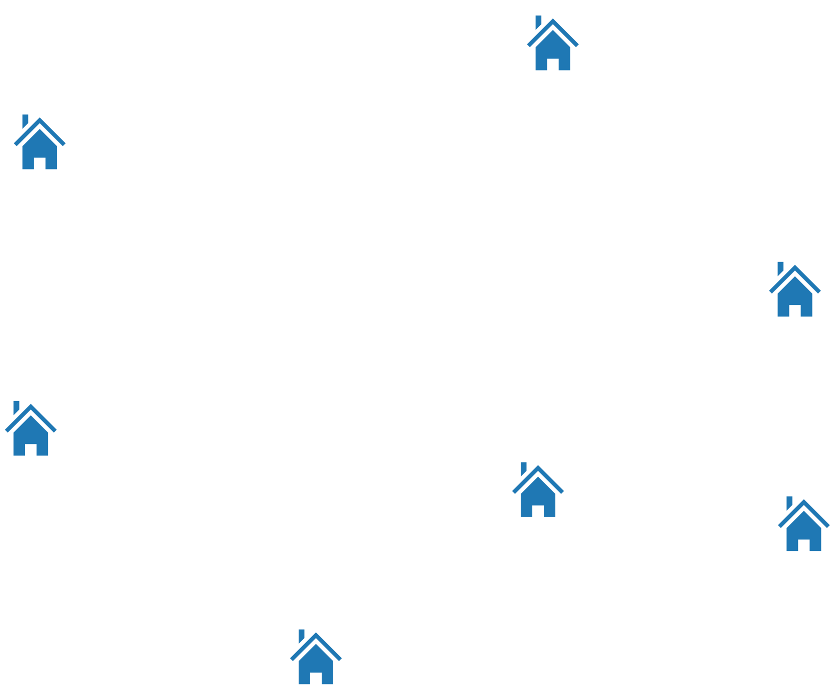

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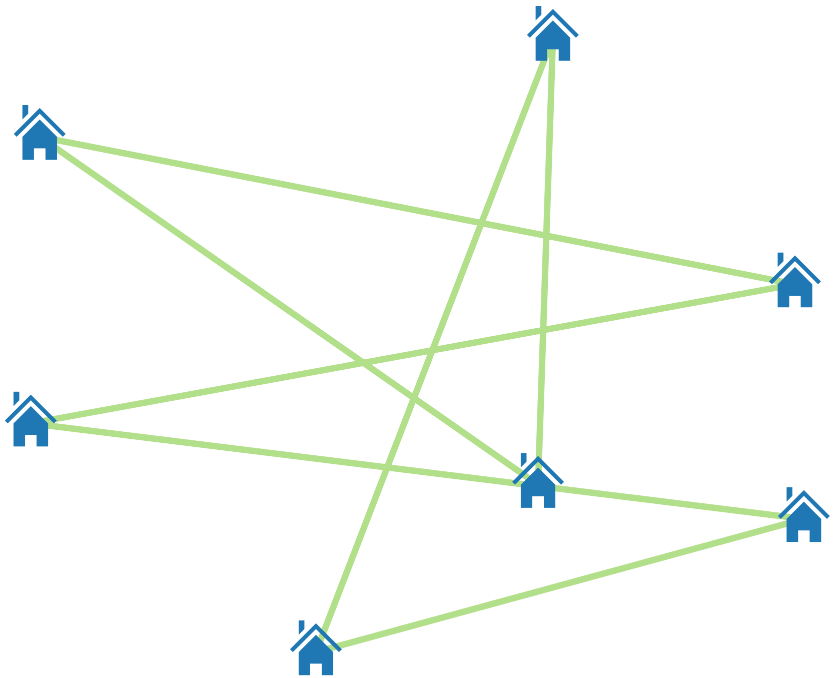

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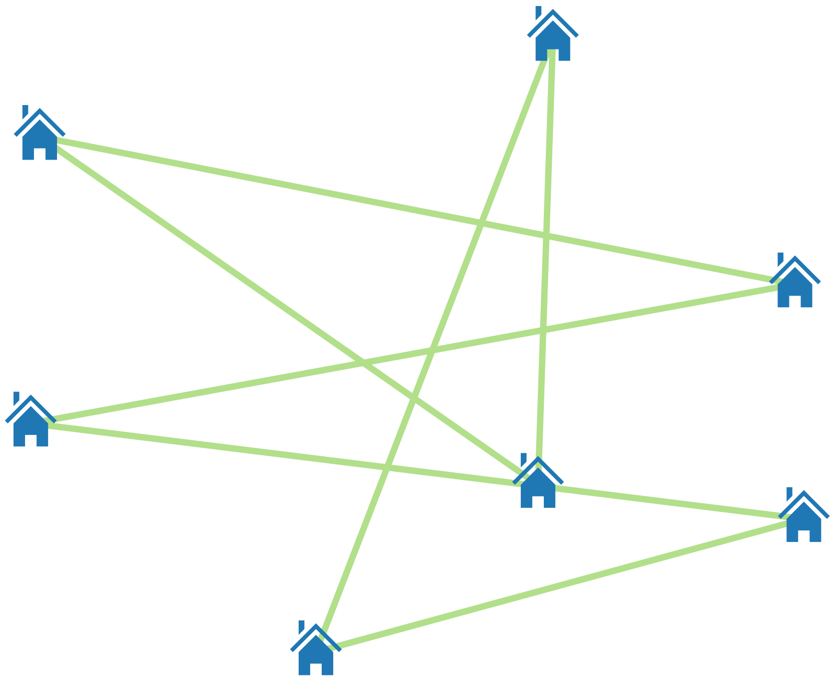

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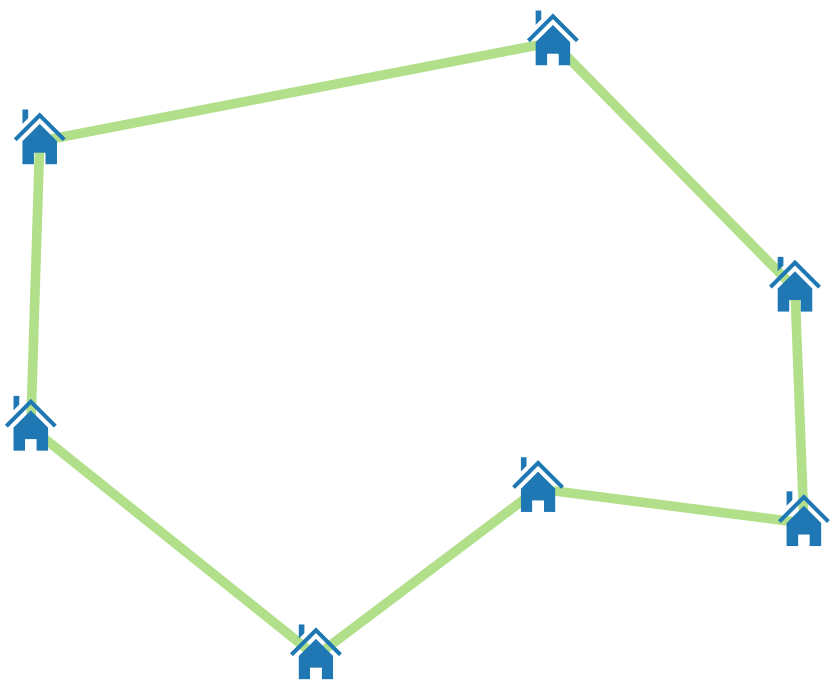

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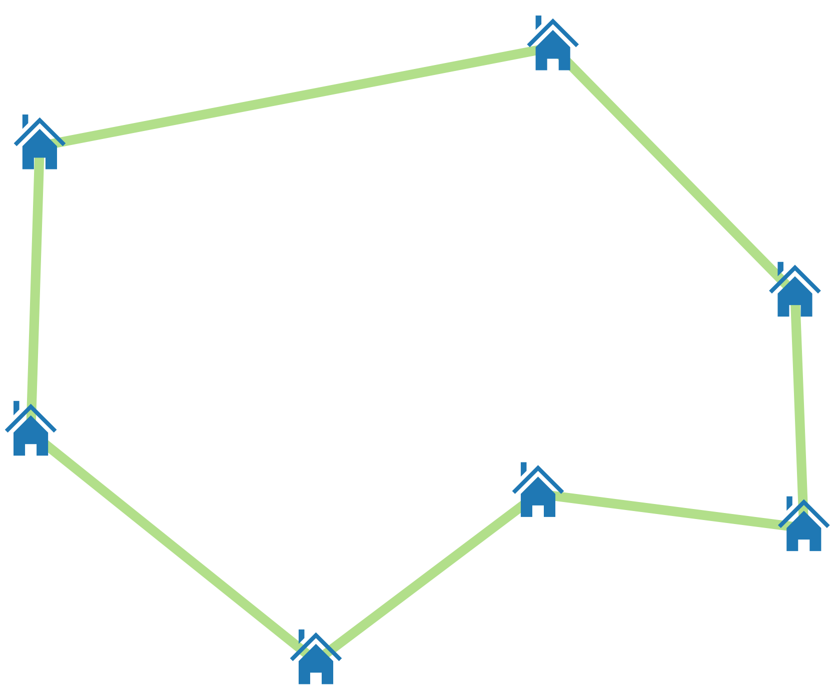

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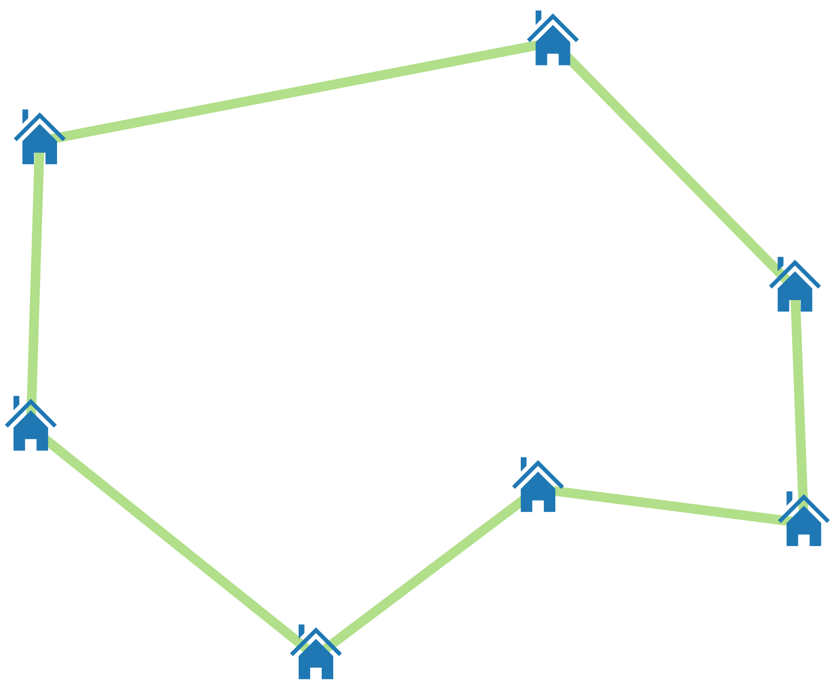

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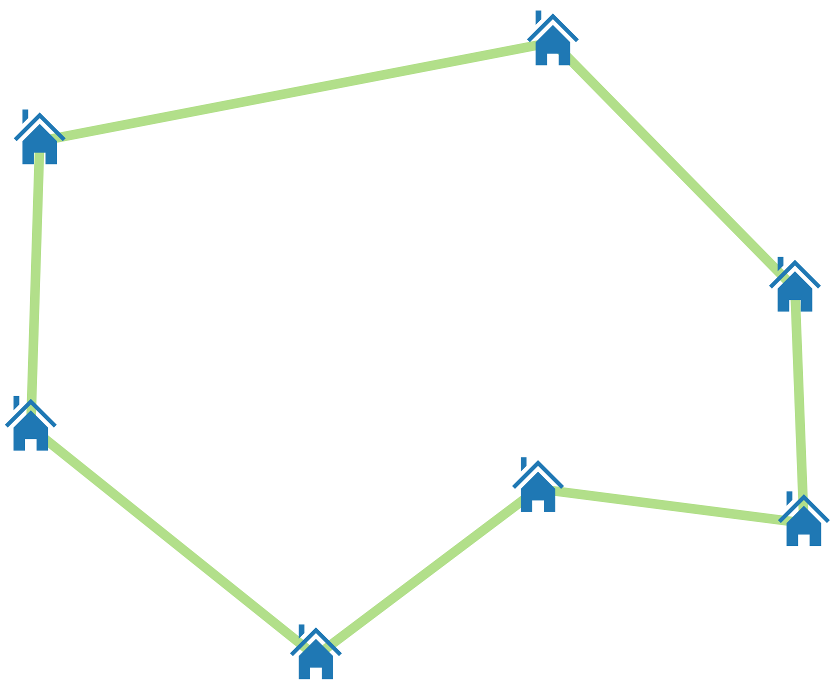

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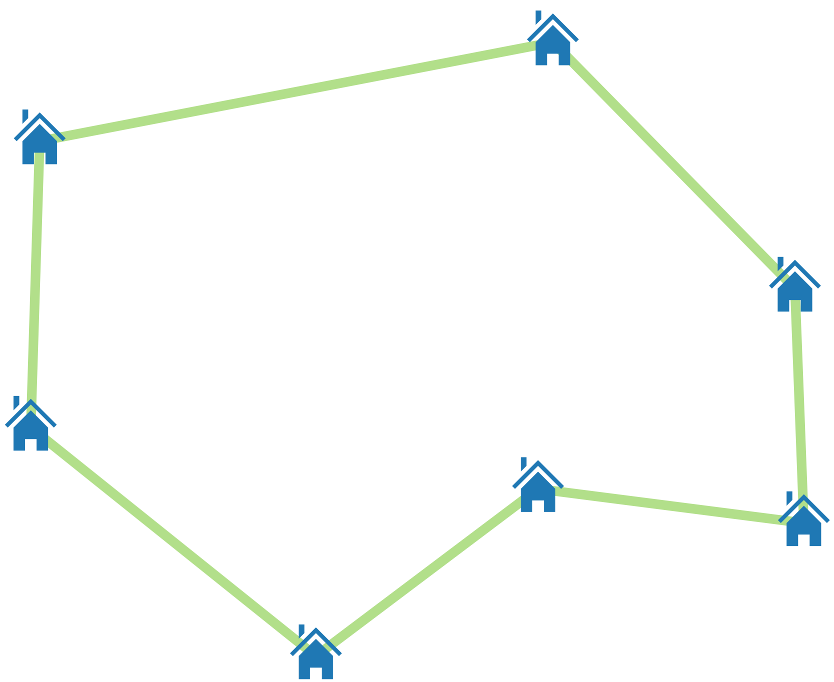

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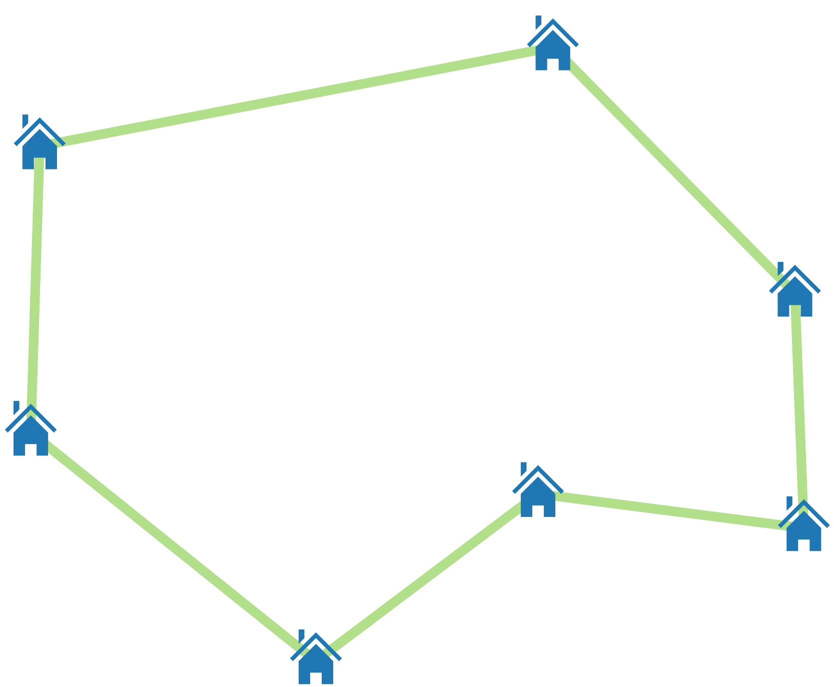

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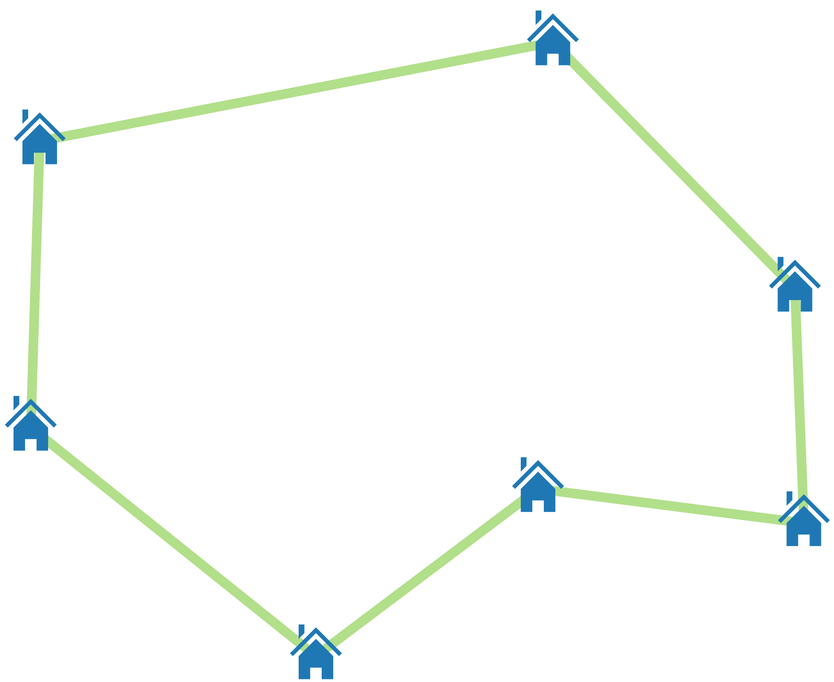

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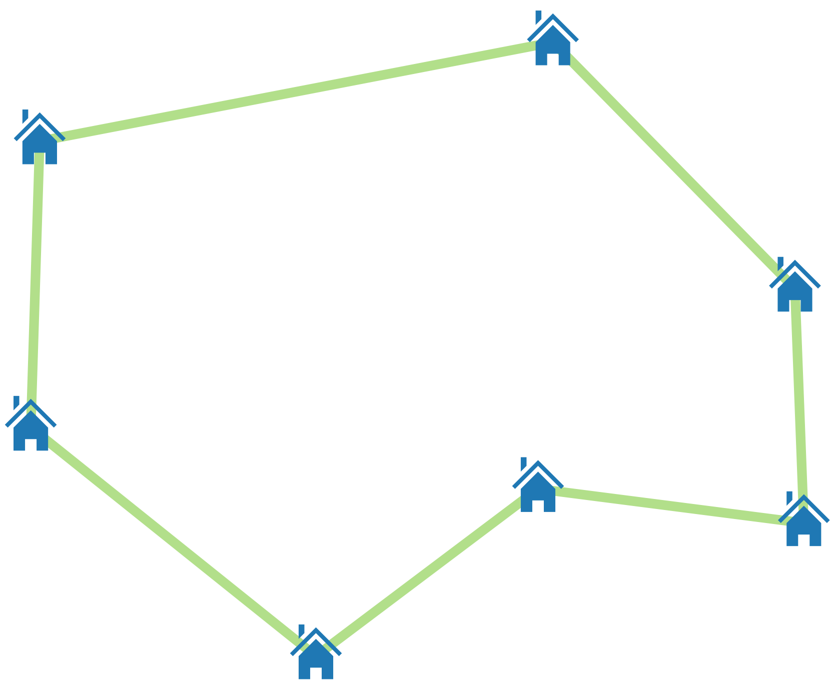

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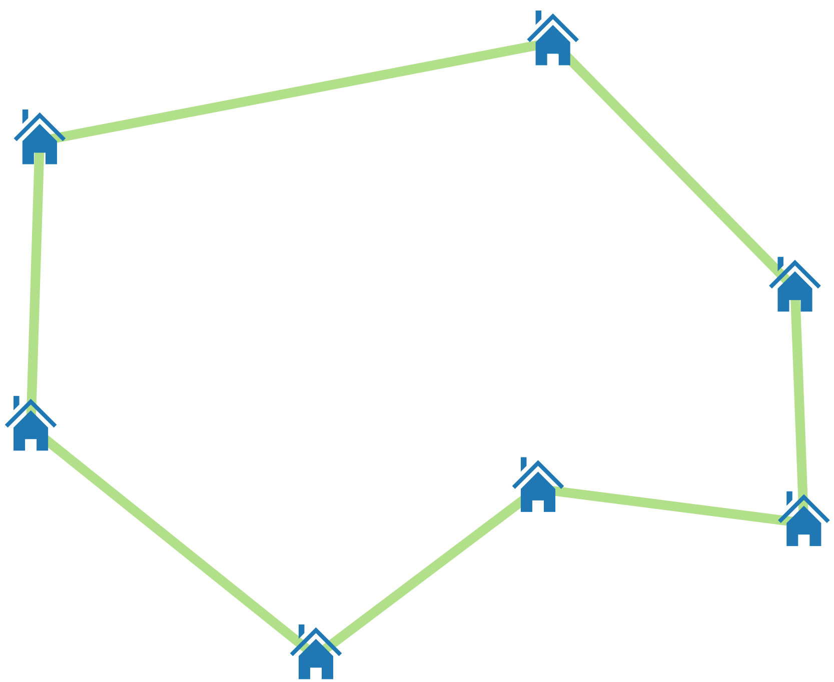

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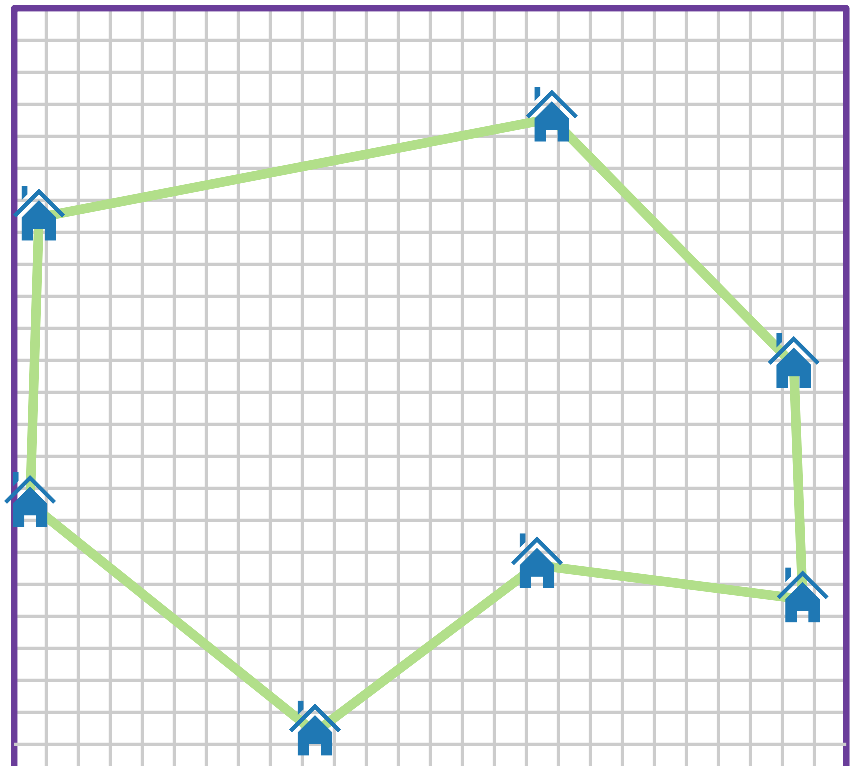

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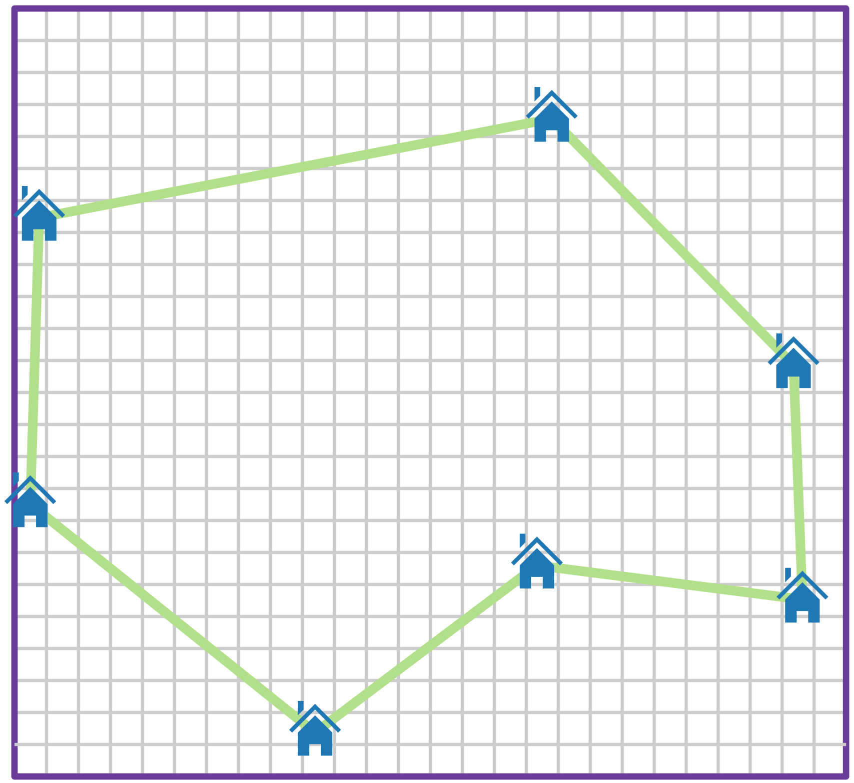

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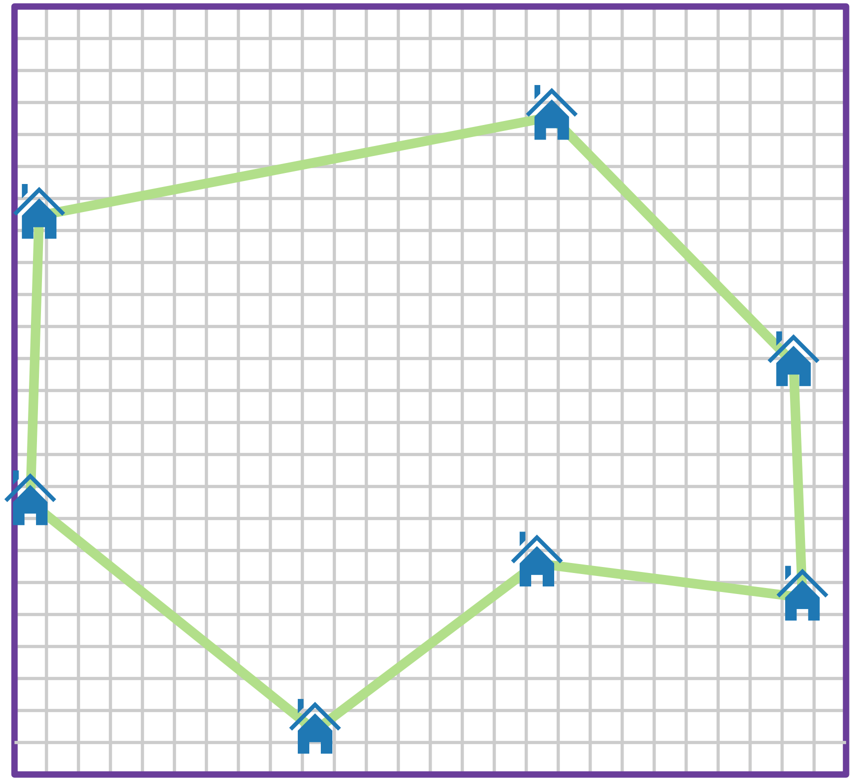

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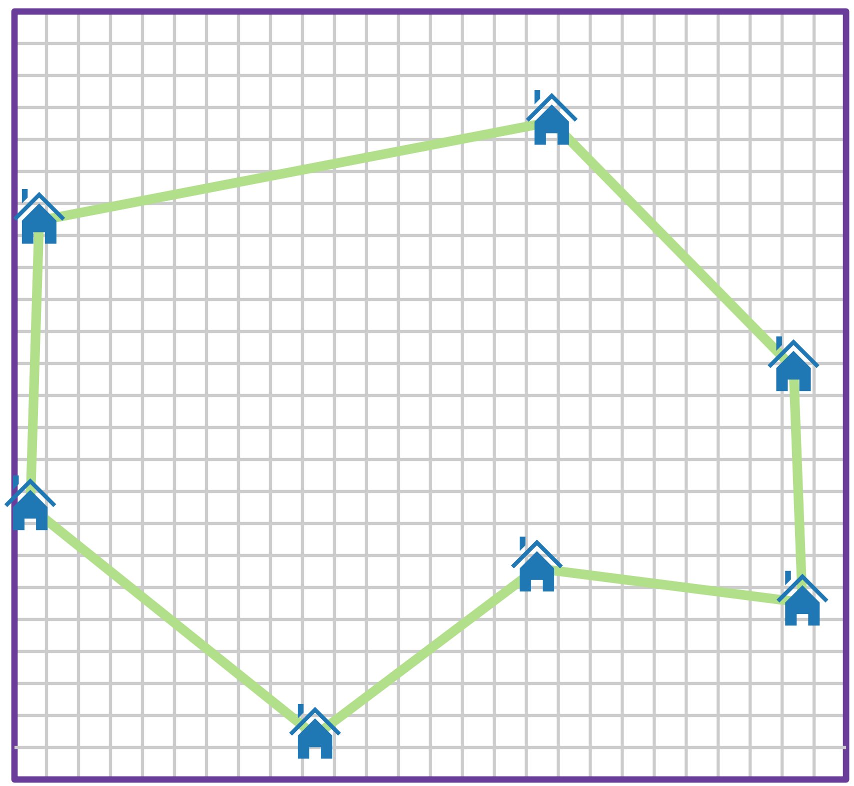

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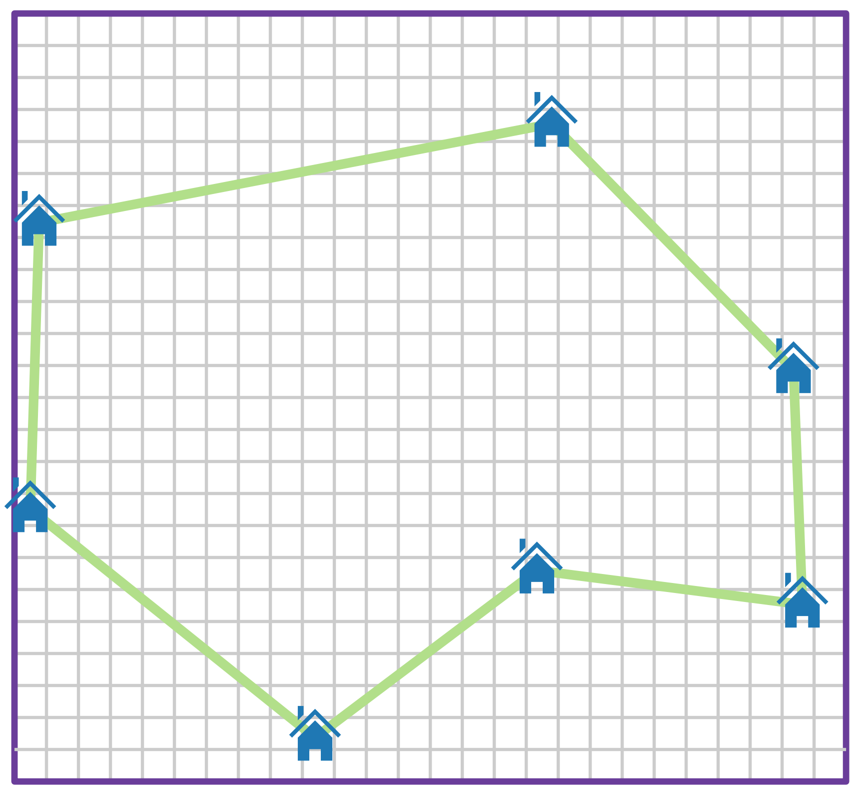

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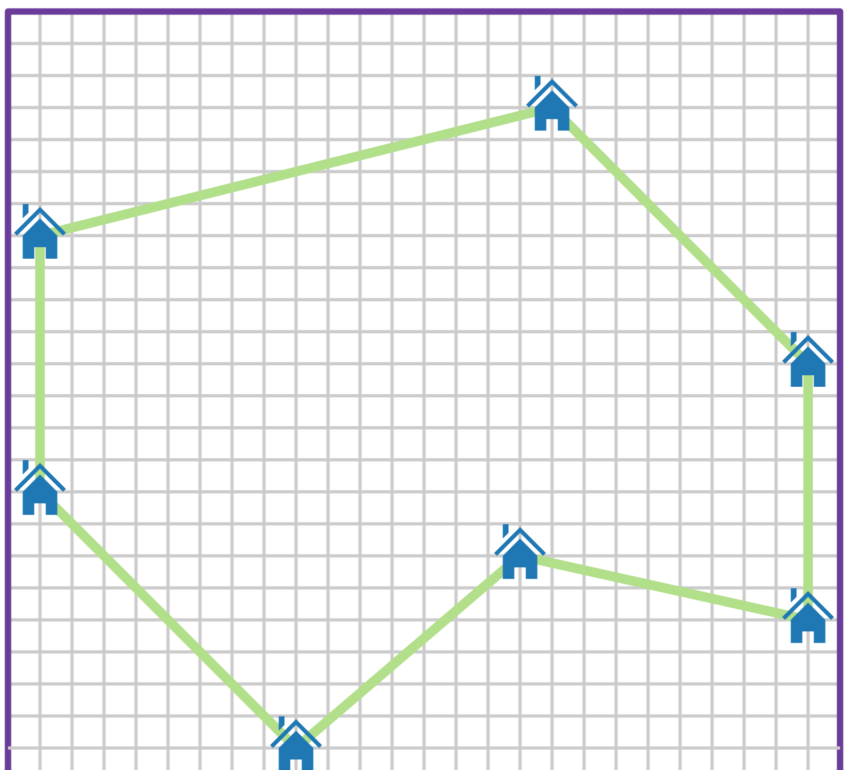

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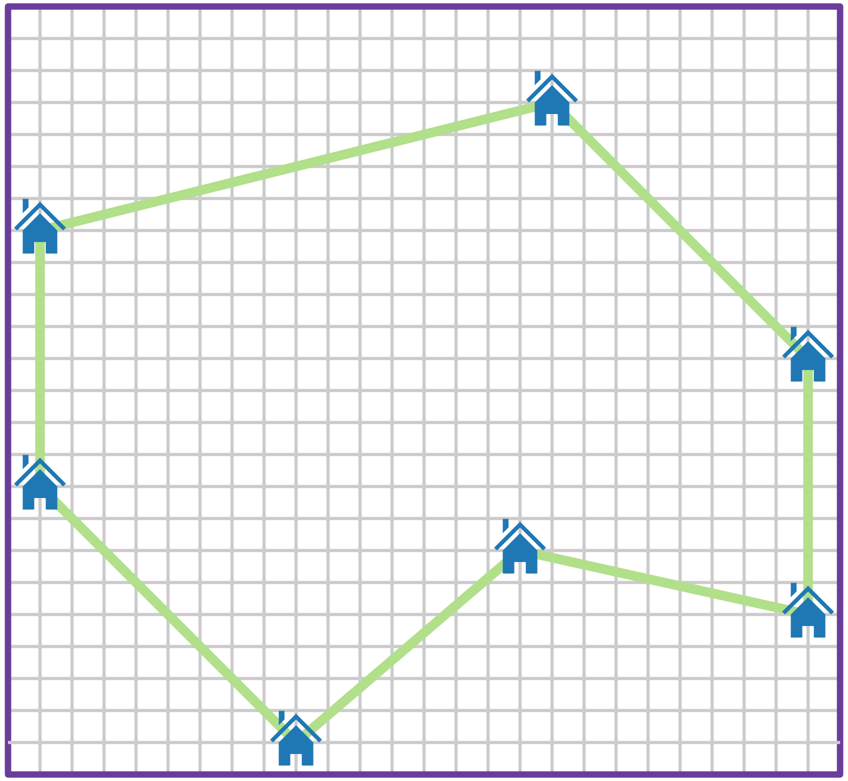

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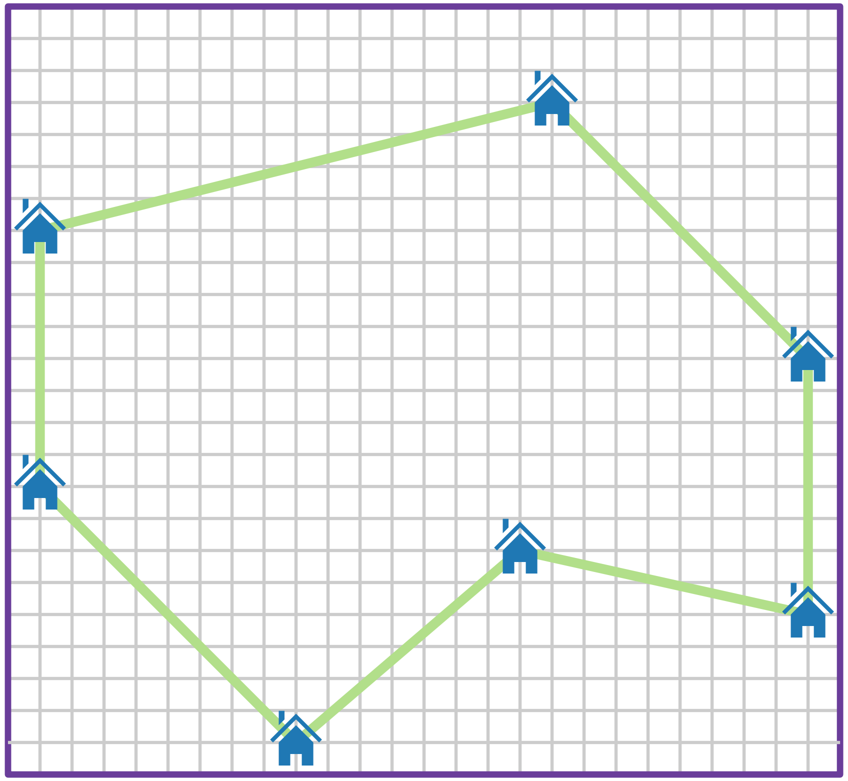

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Goal: $(1+\varepsilon)$-approximation!

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Lecture 9:
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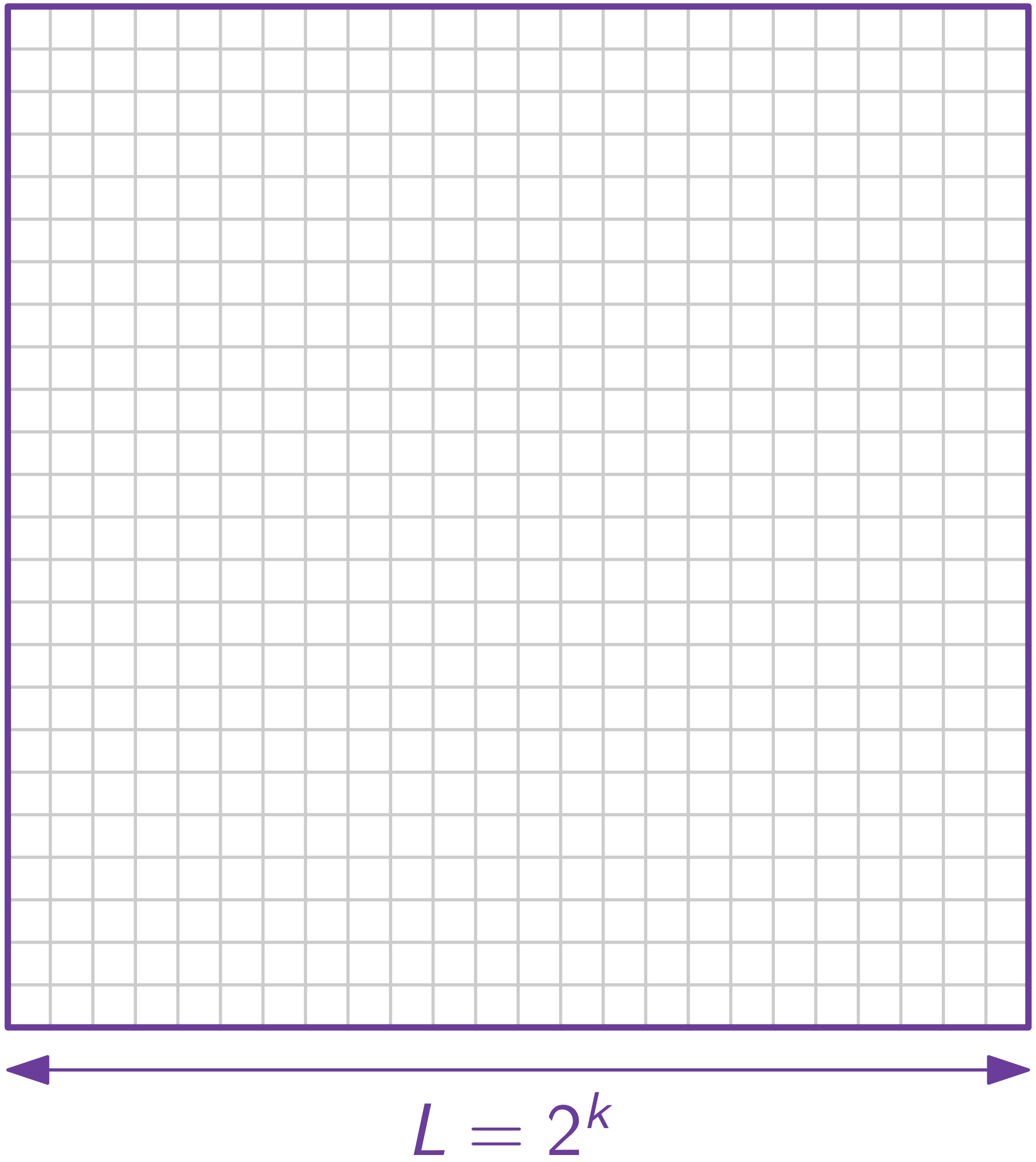
Part II:
Dissection

Basic Dissection



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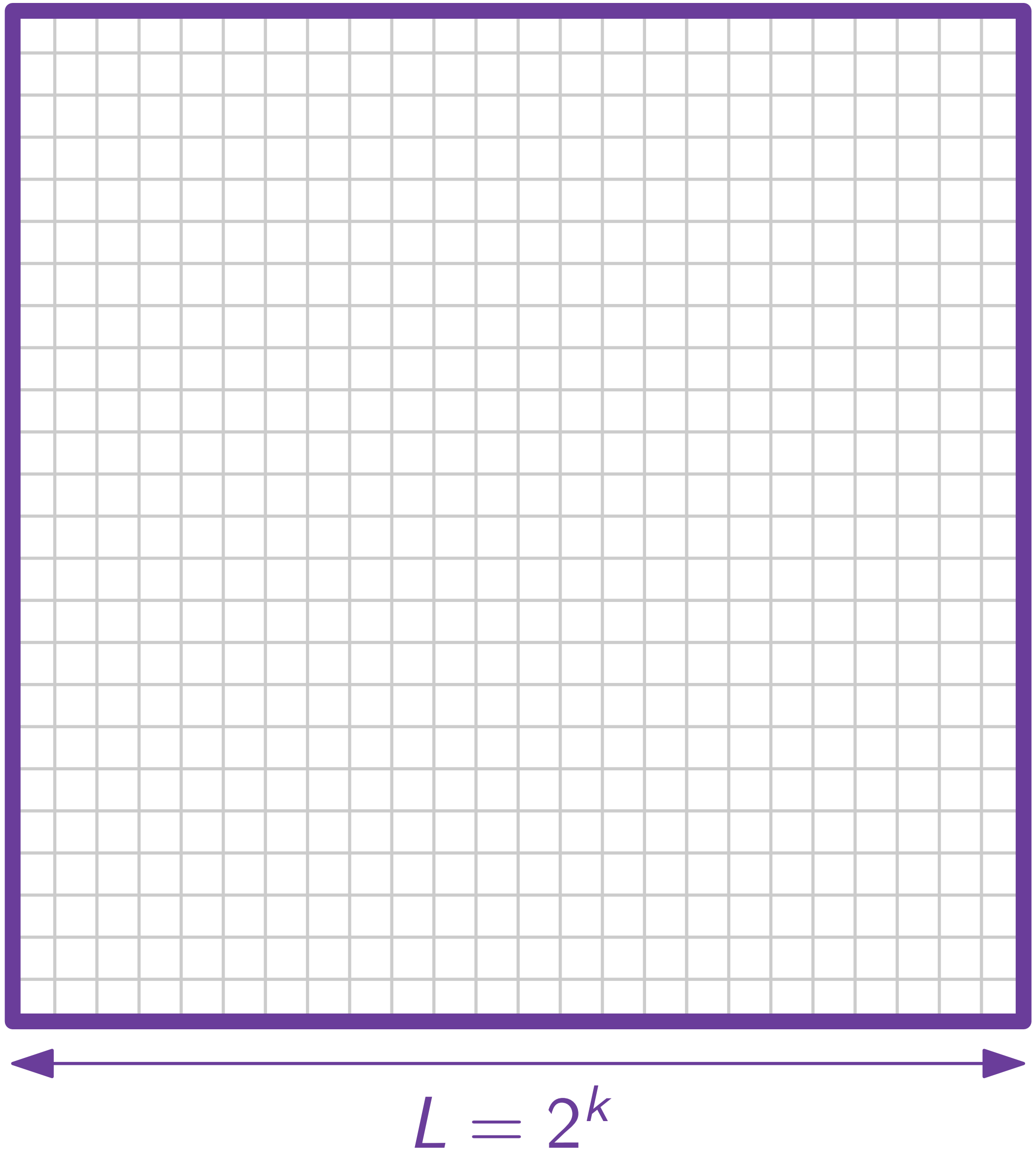
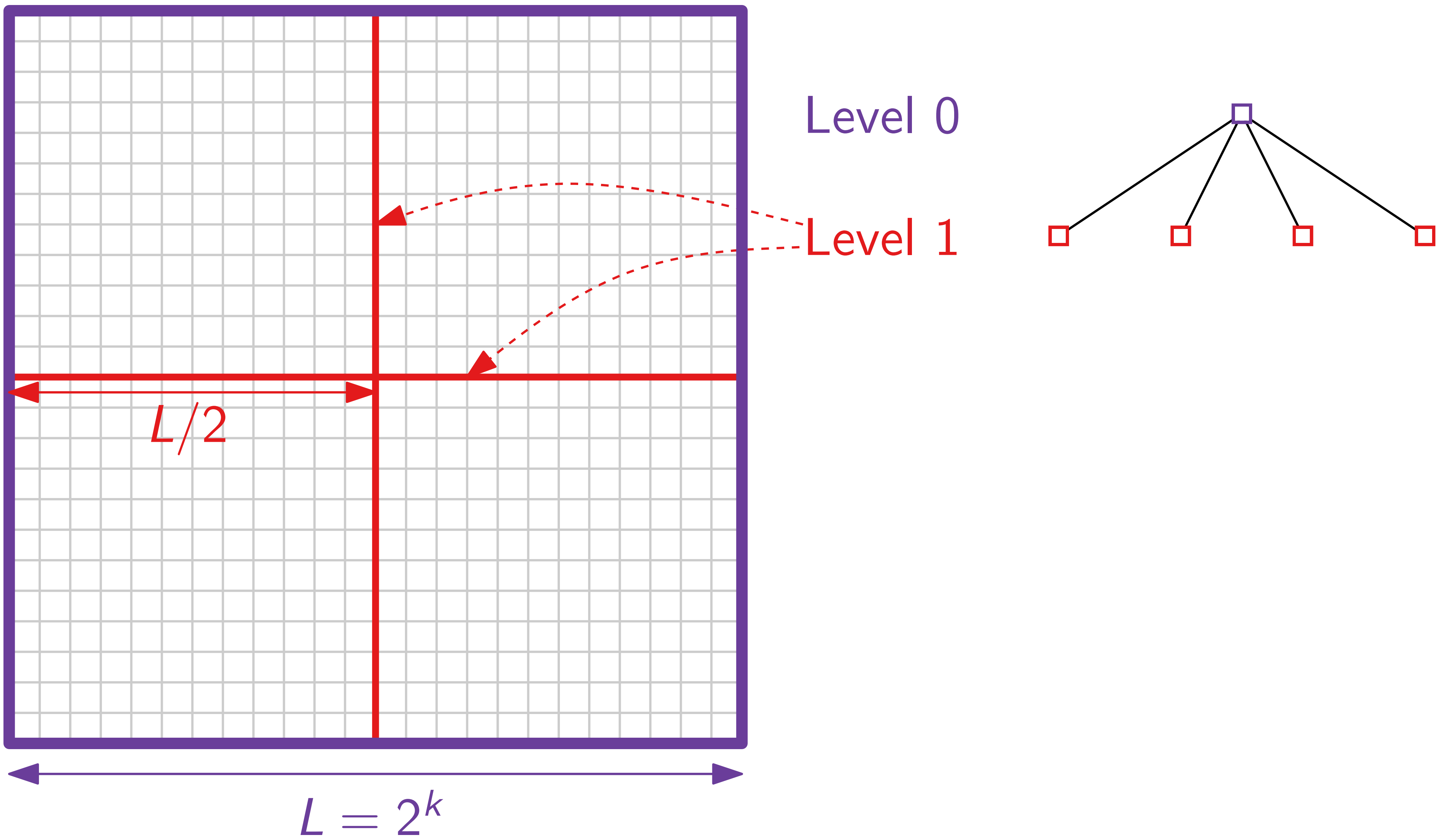


Level 0 ◻

Basic Dissection



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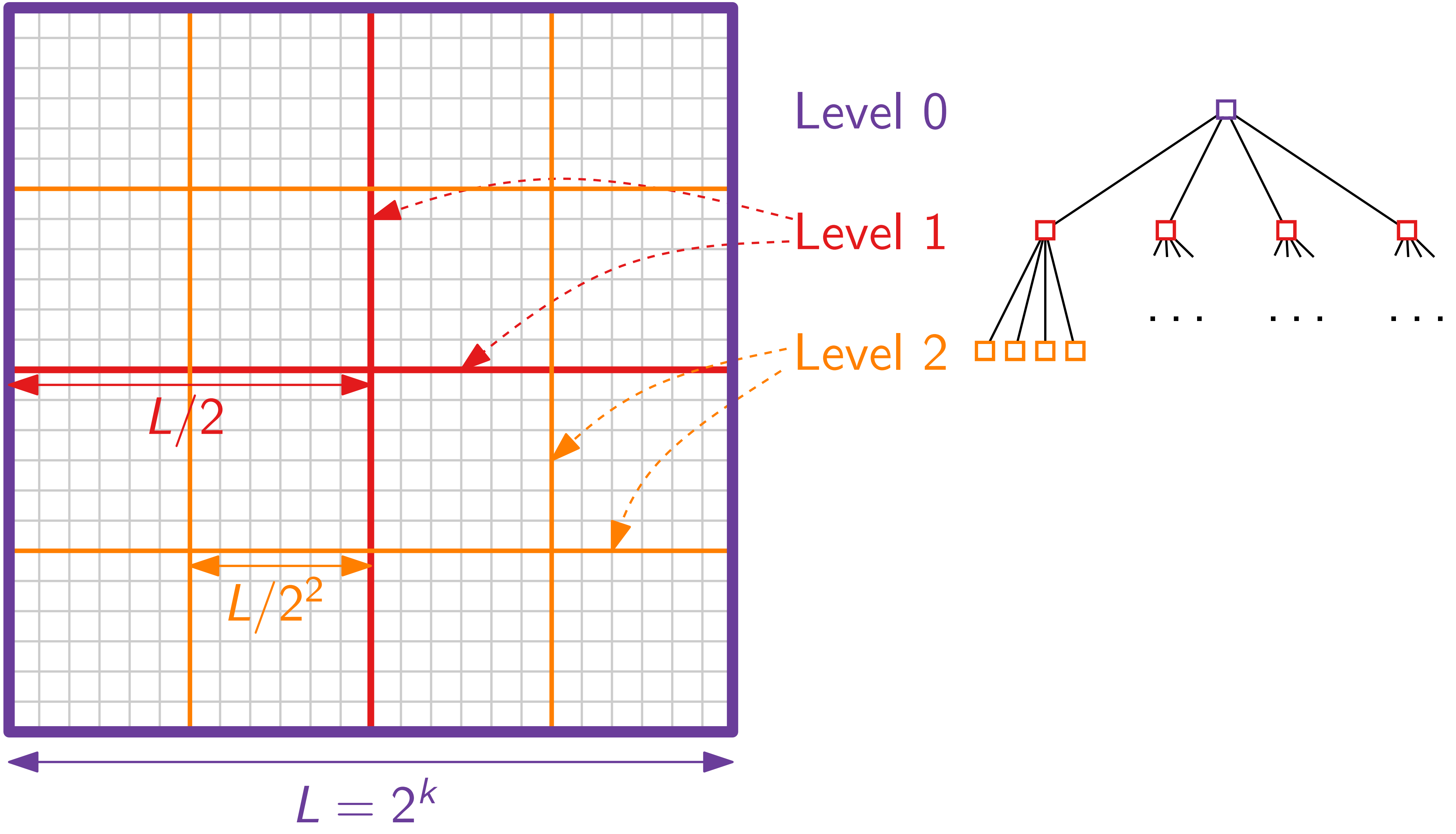

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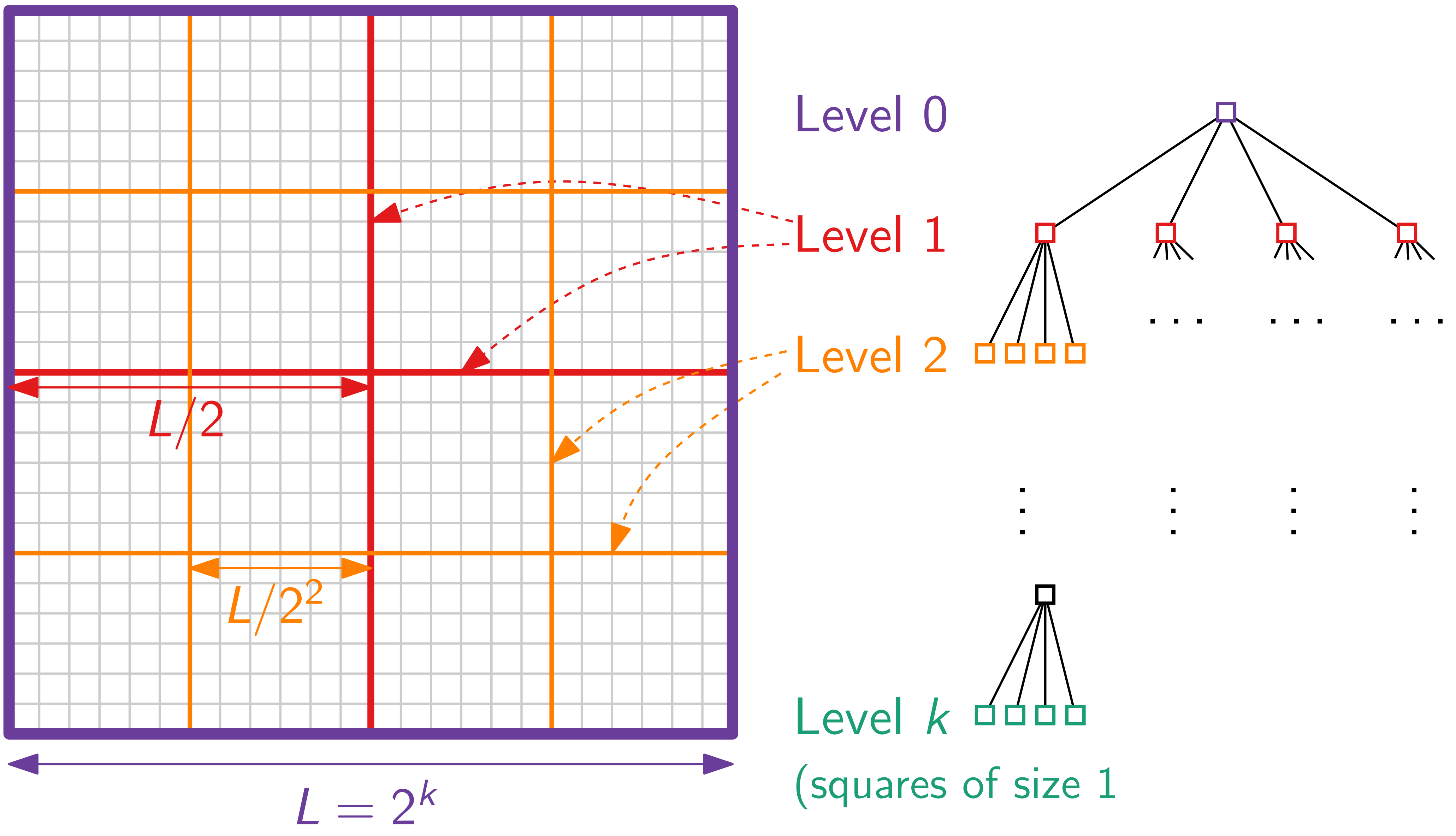

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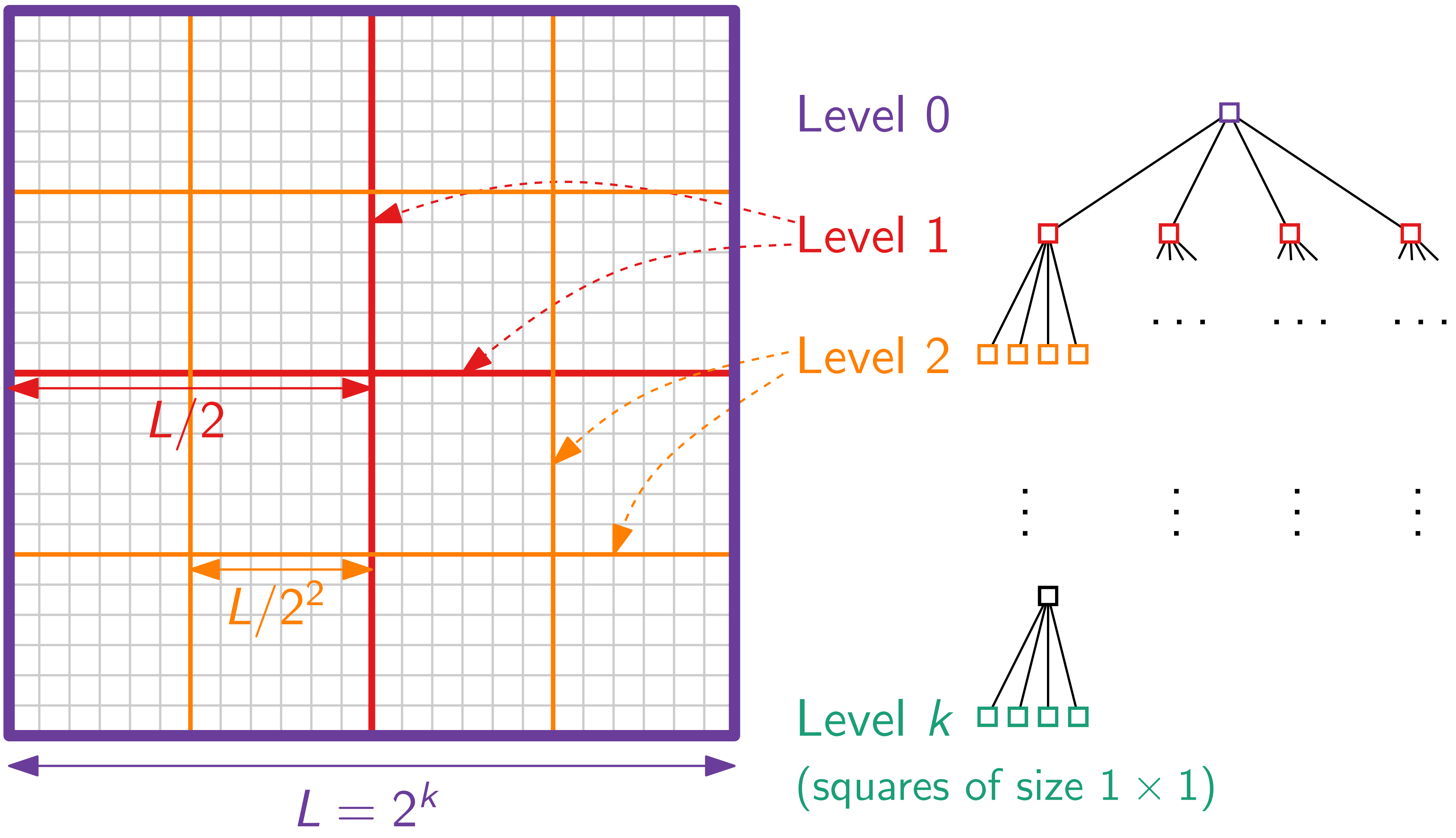

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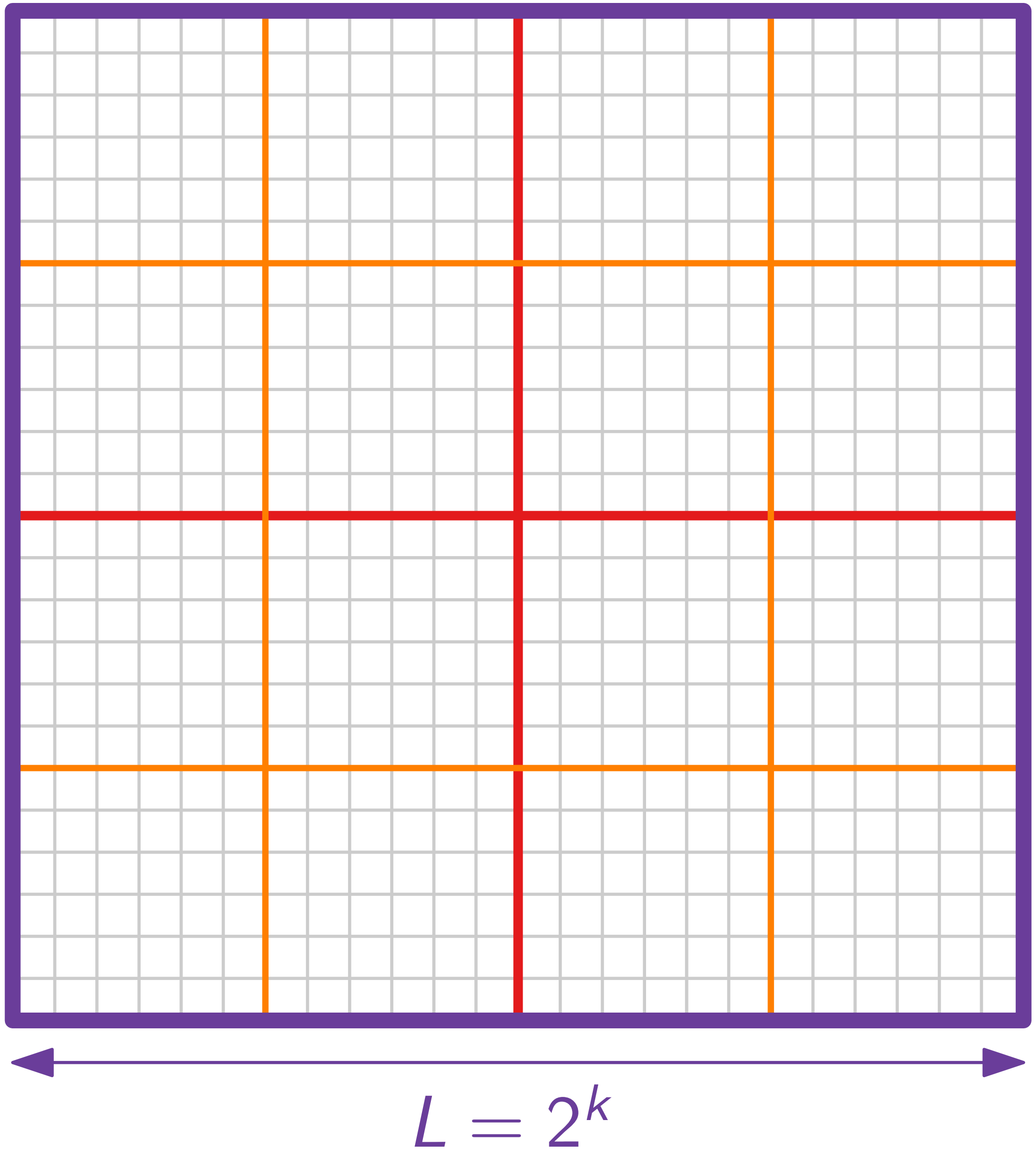

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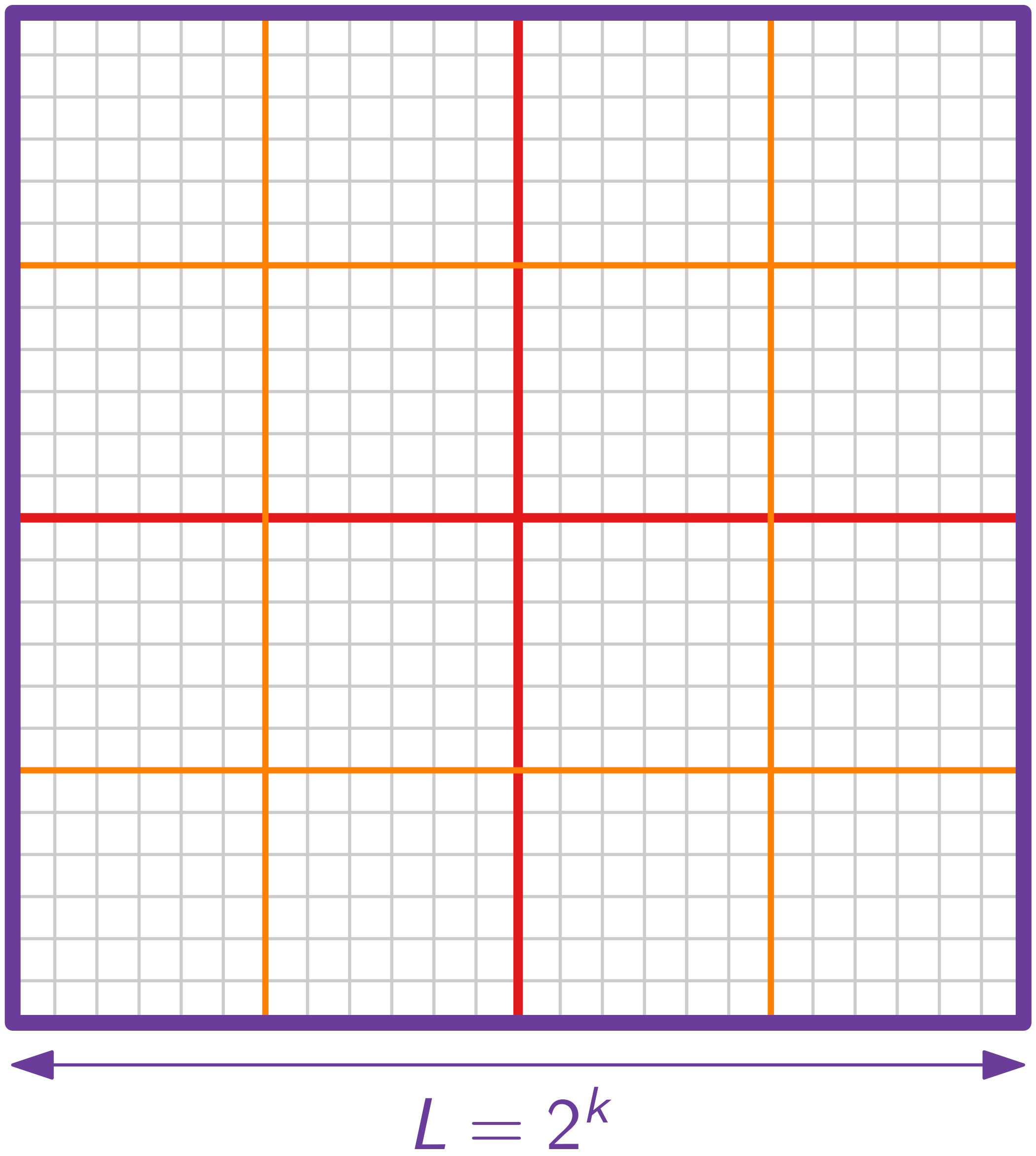

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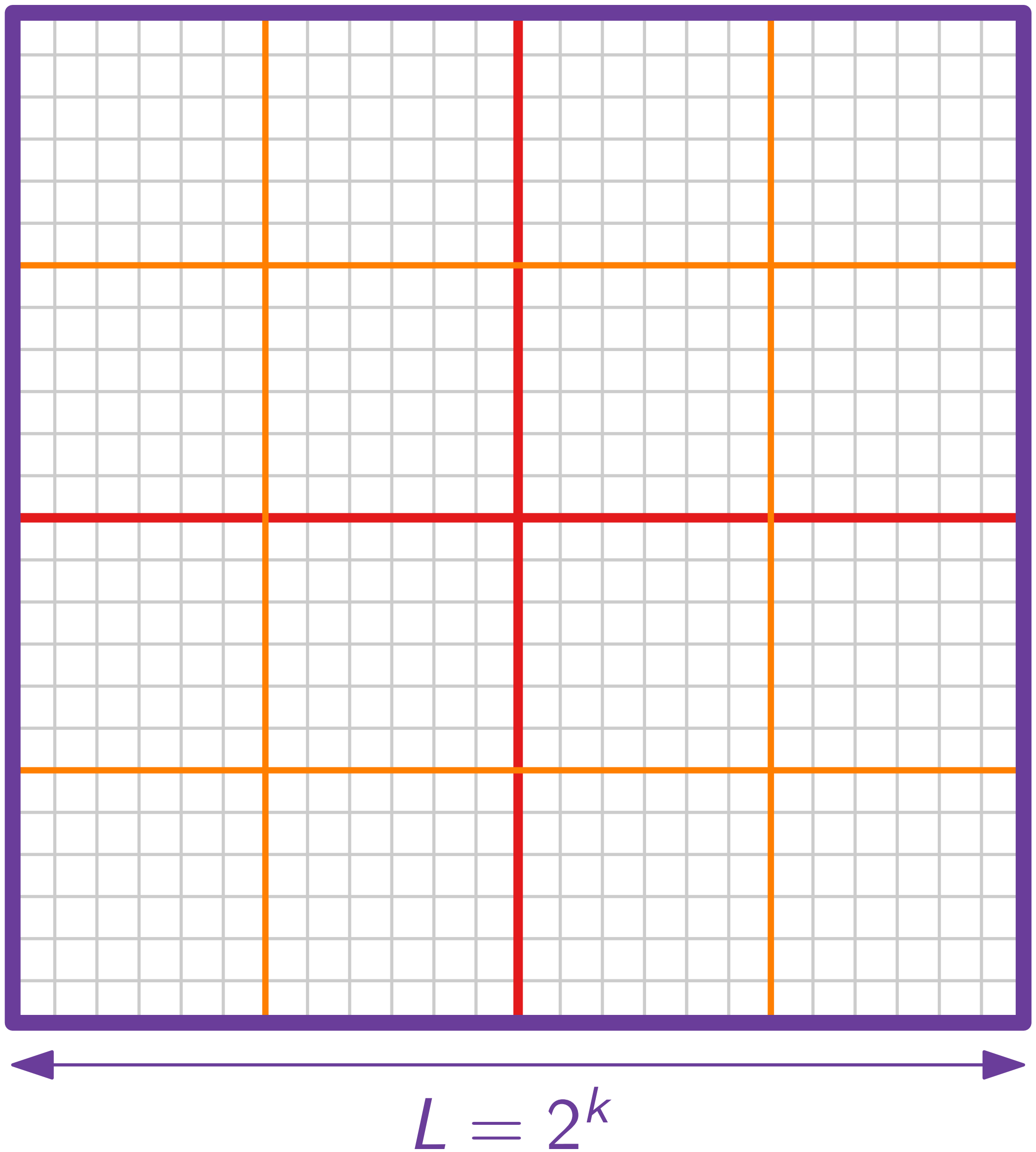

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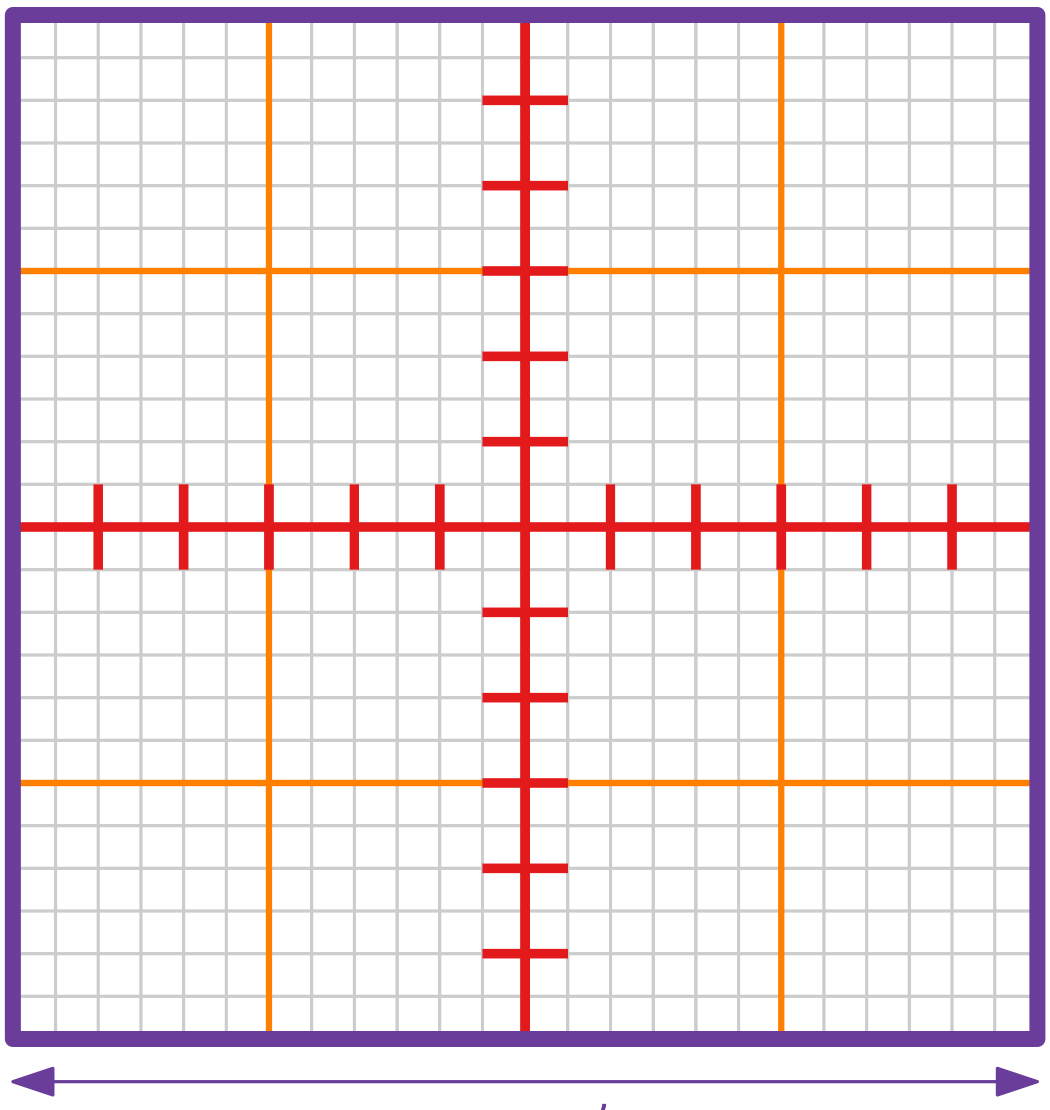

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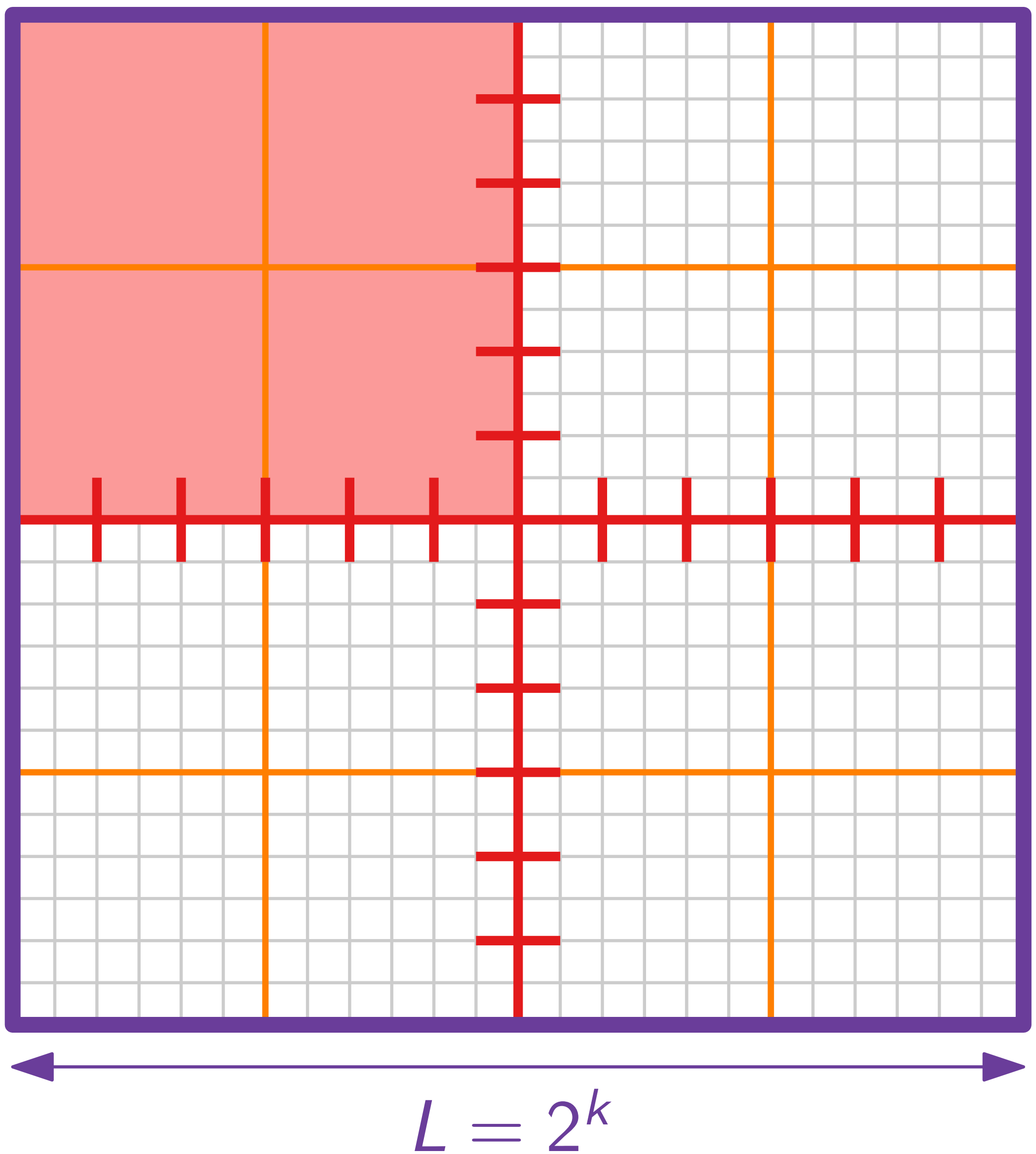

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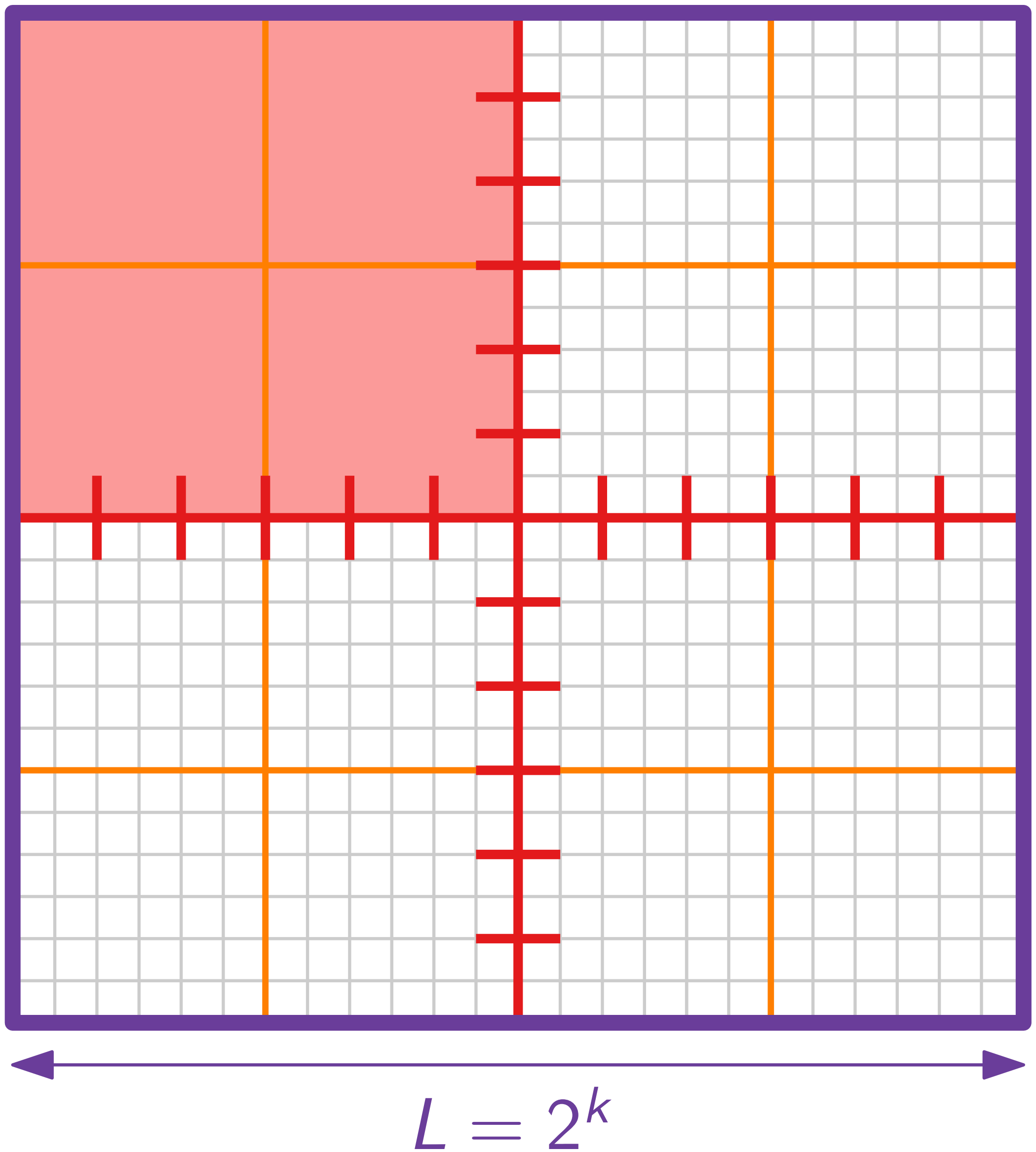

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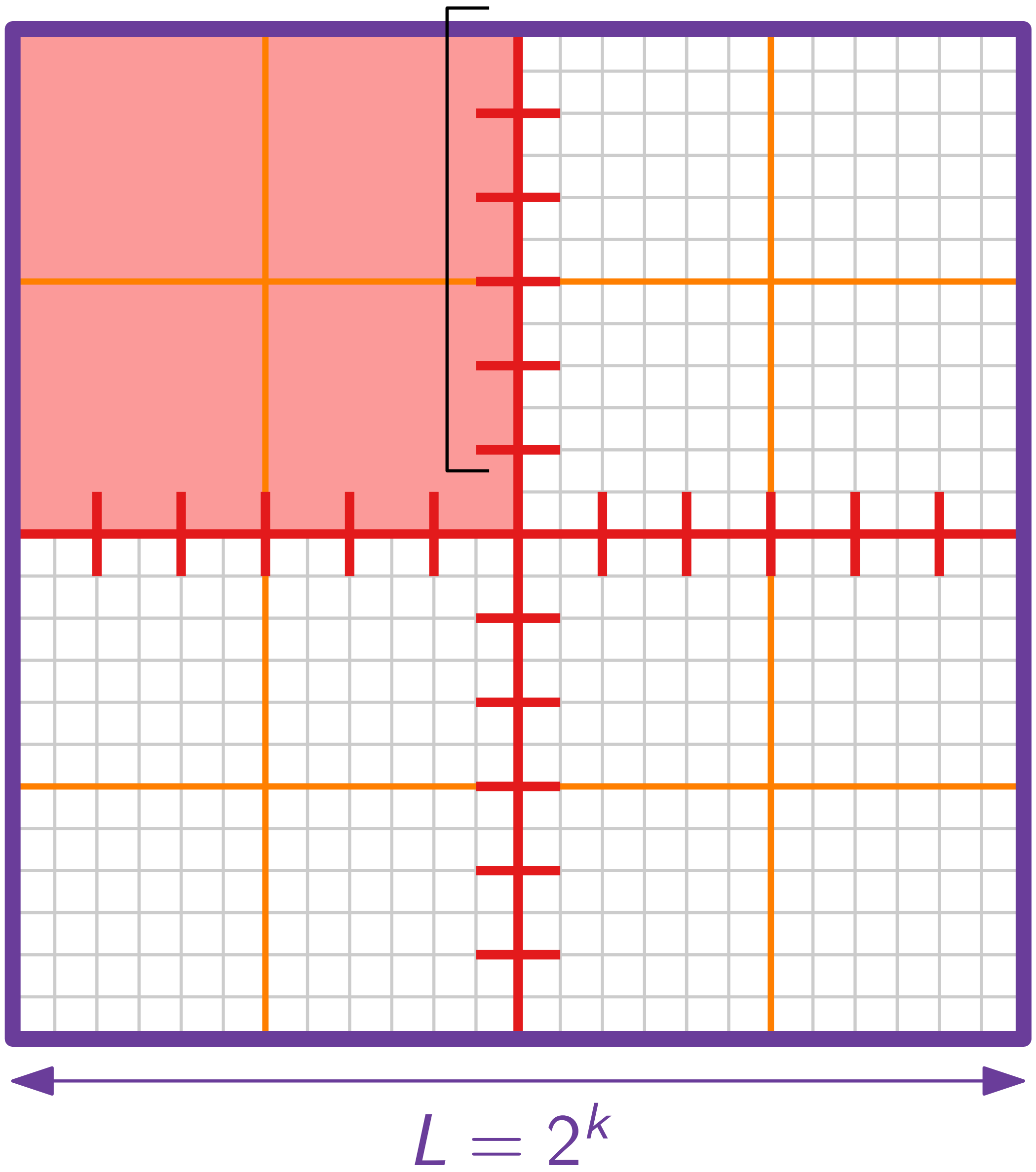

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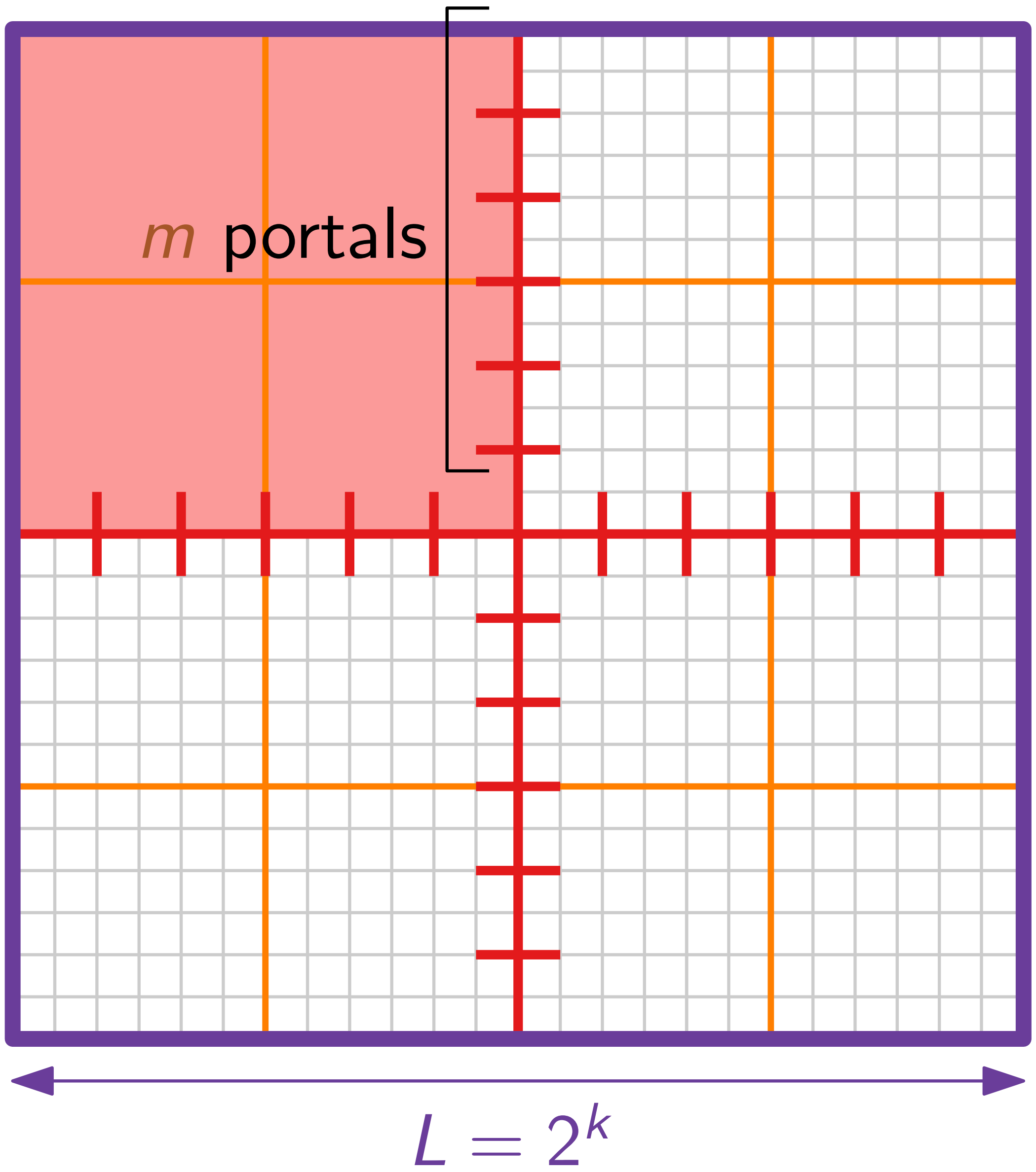

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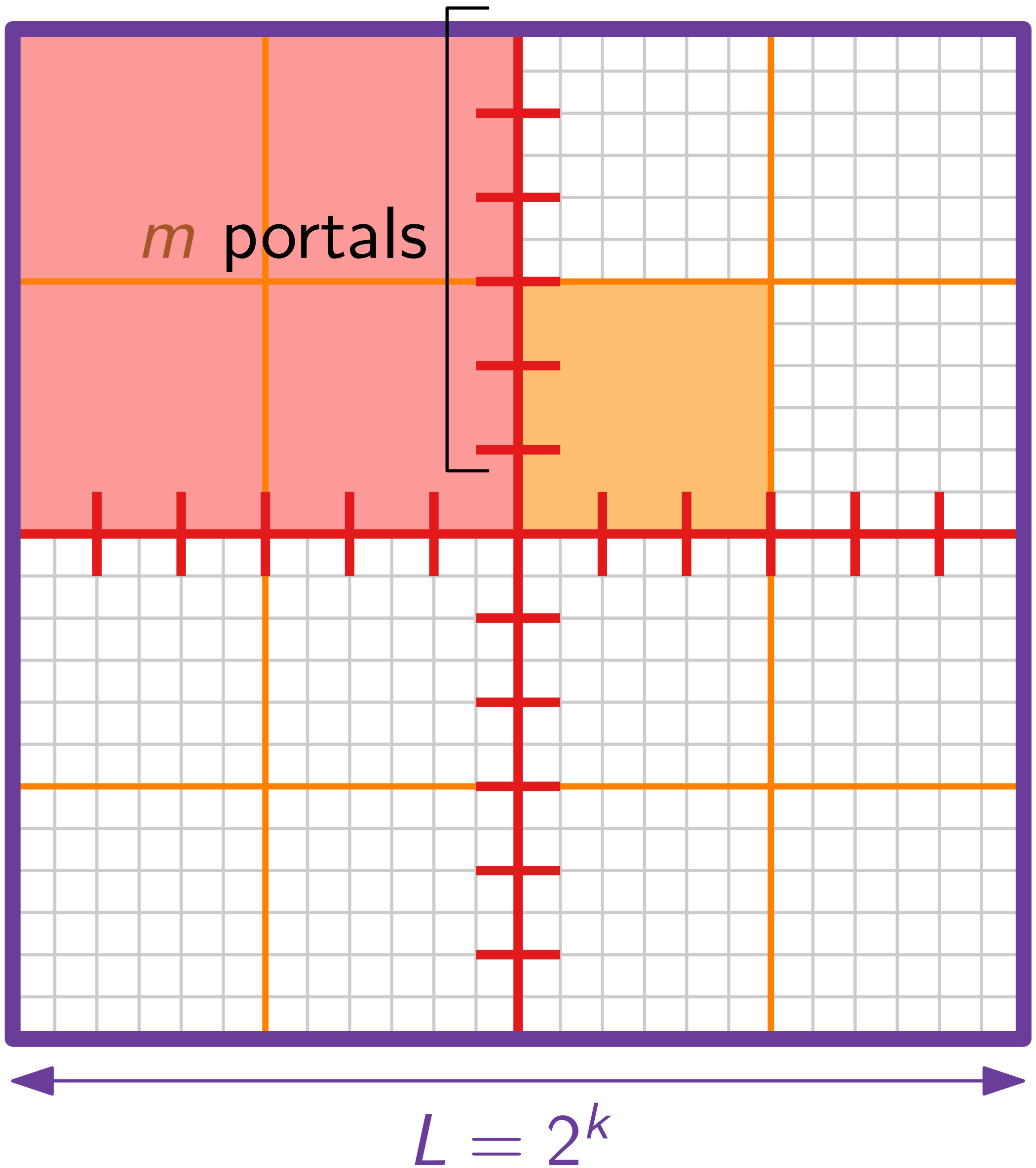

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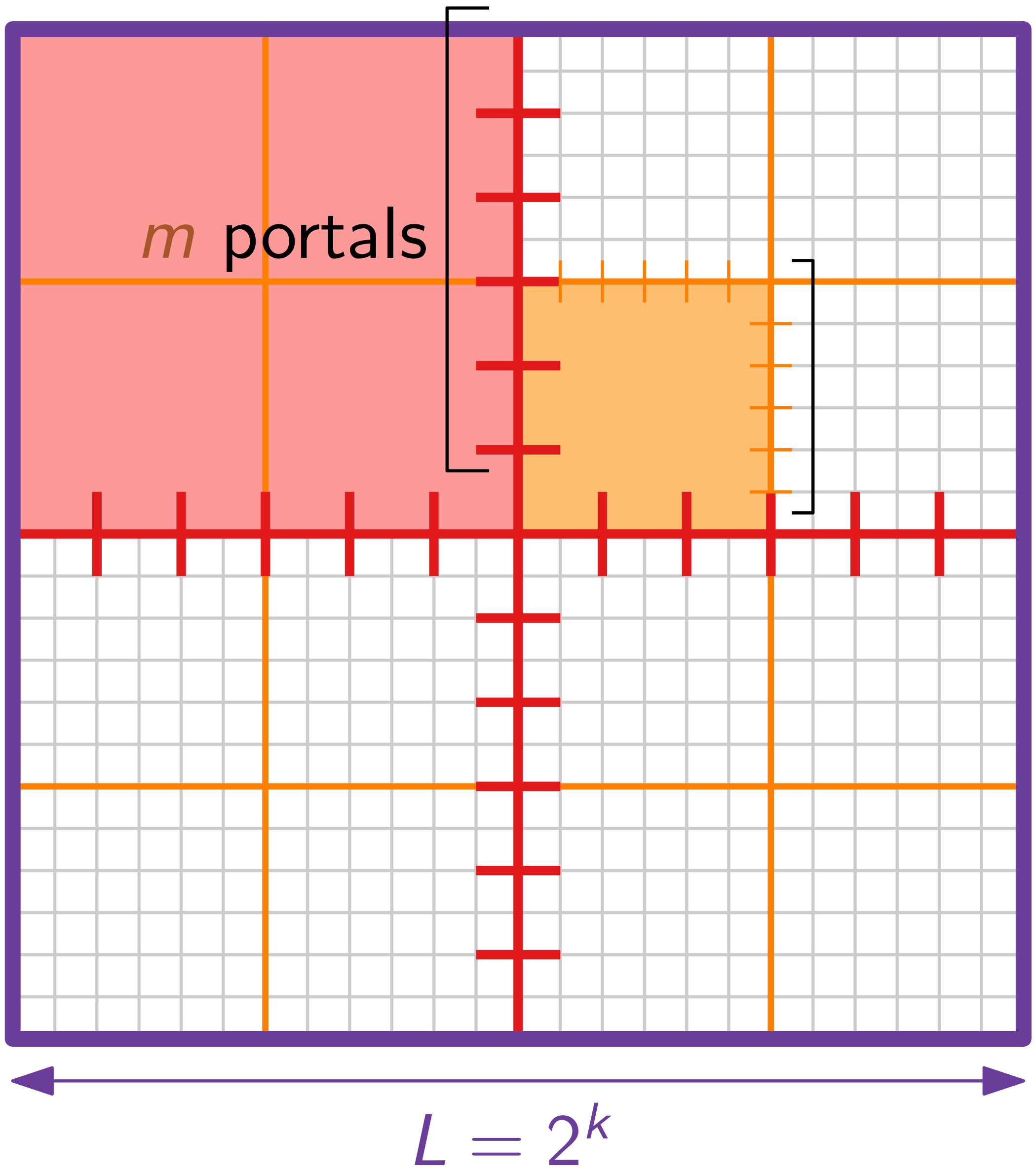

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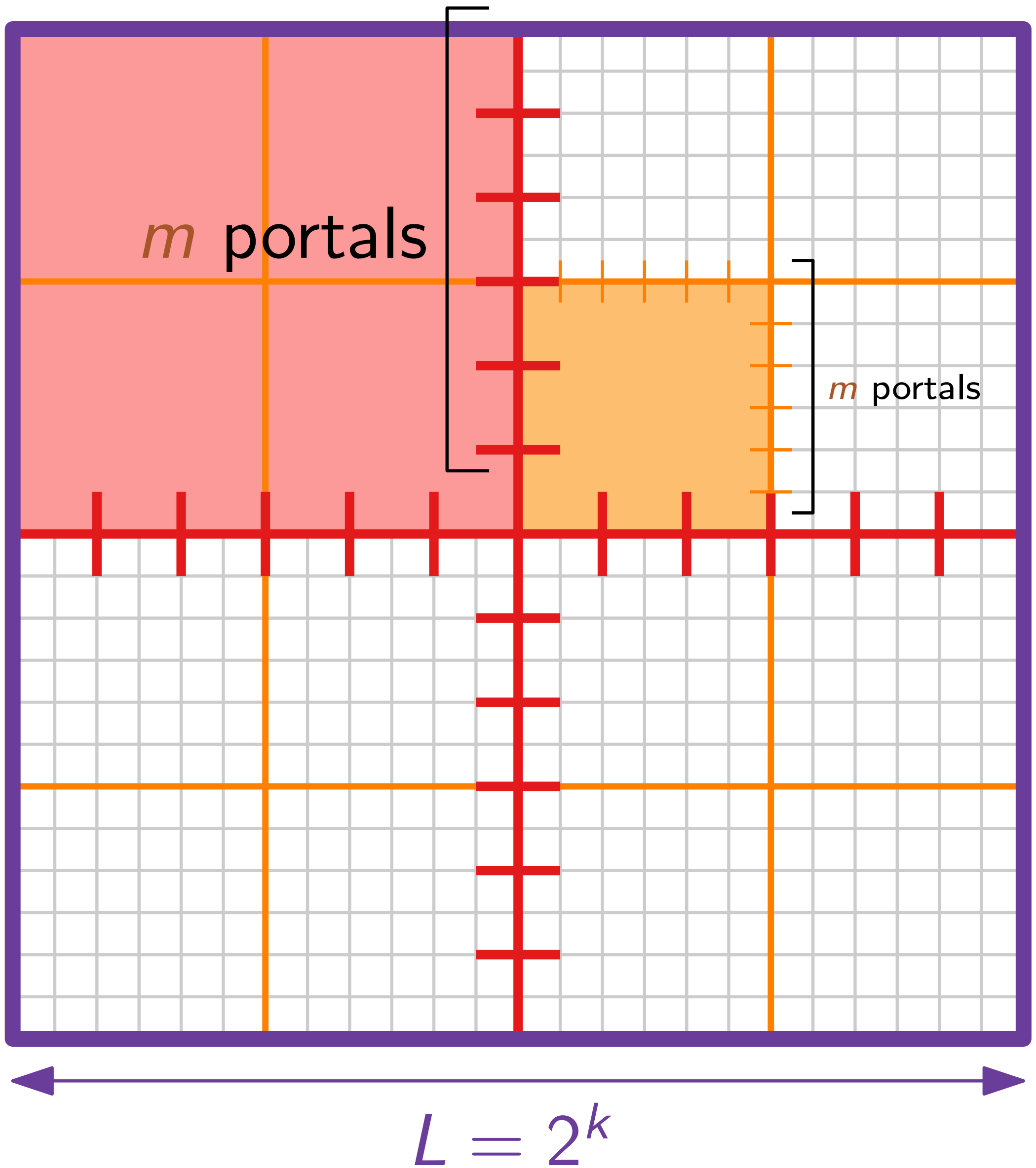

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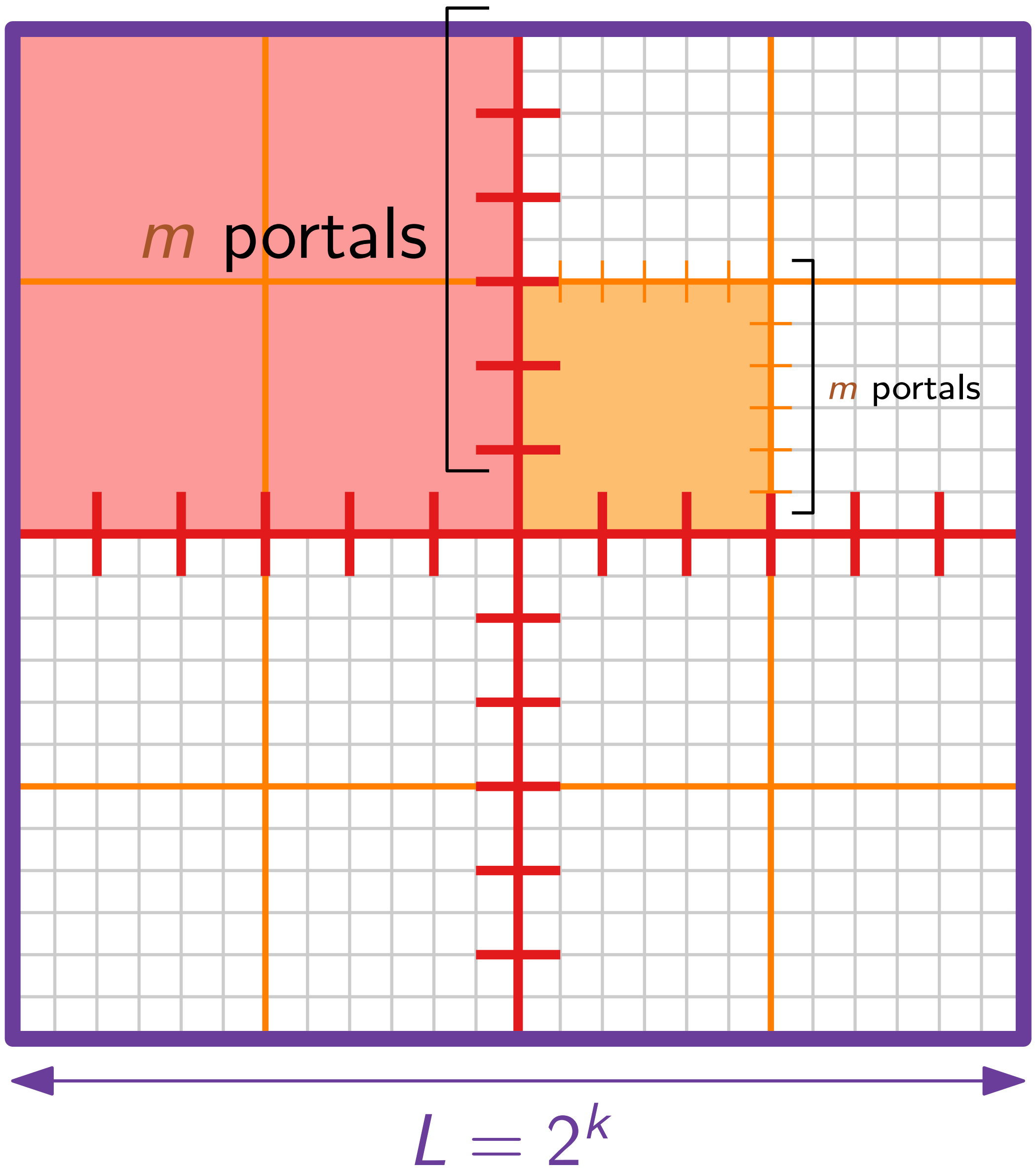

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- A level-i square has $\leq 4m$ portals on its boundary.

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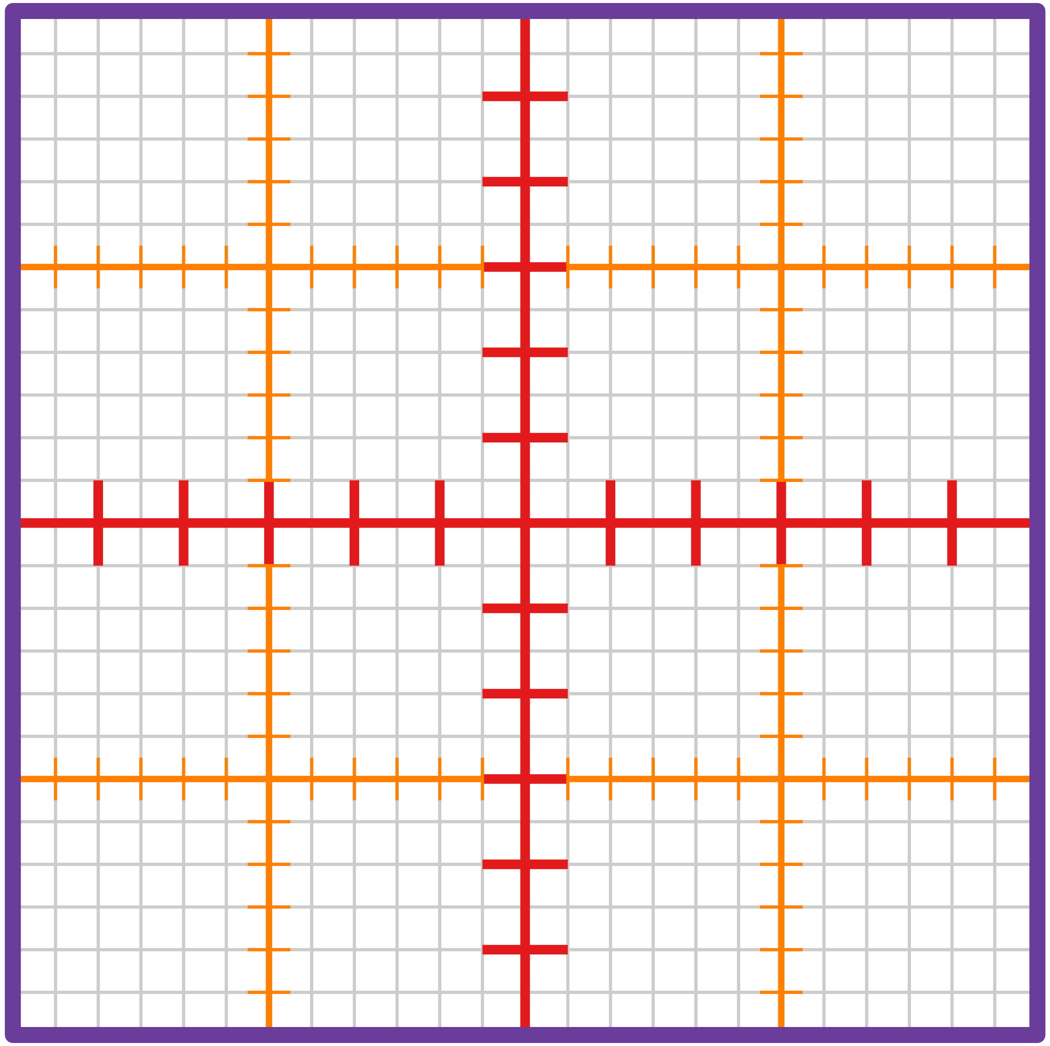
Part III:
Well-Behaved Tours

Well-Behaved Tours



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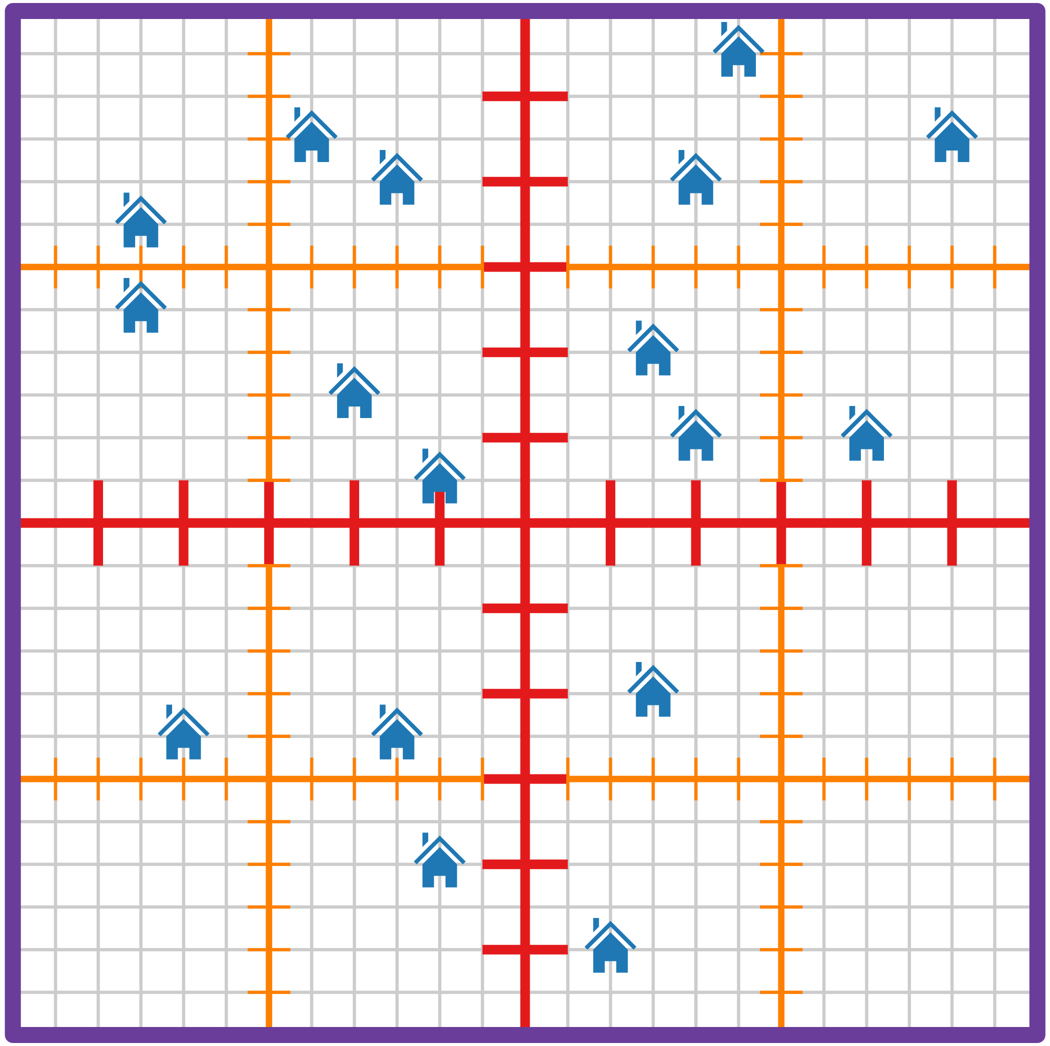

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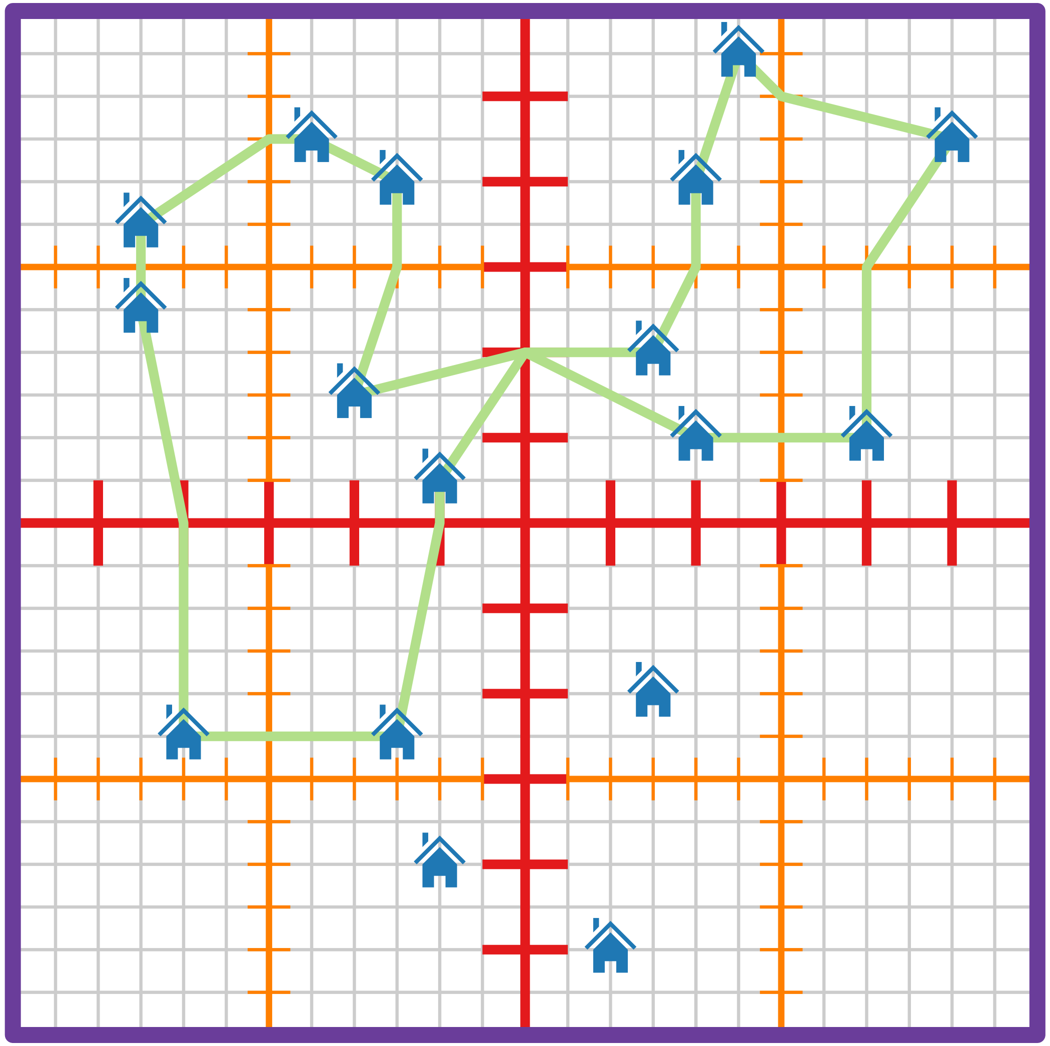

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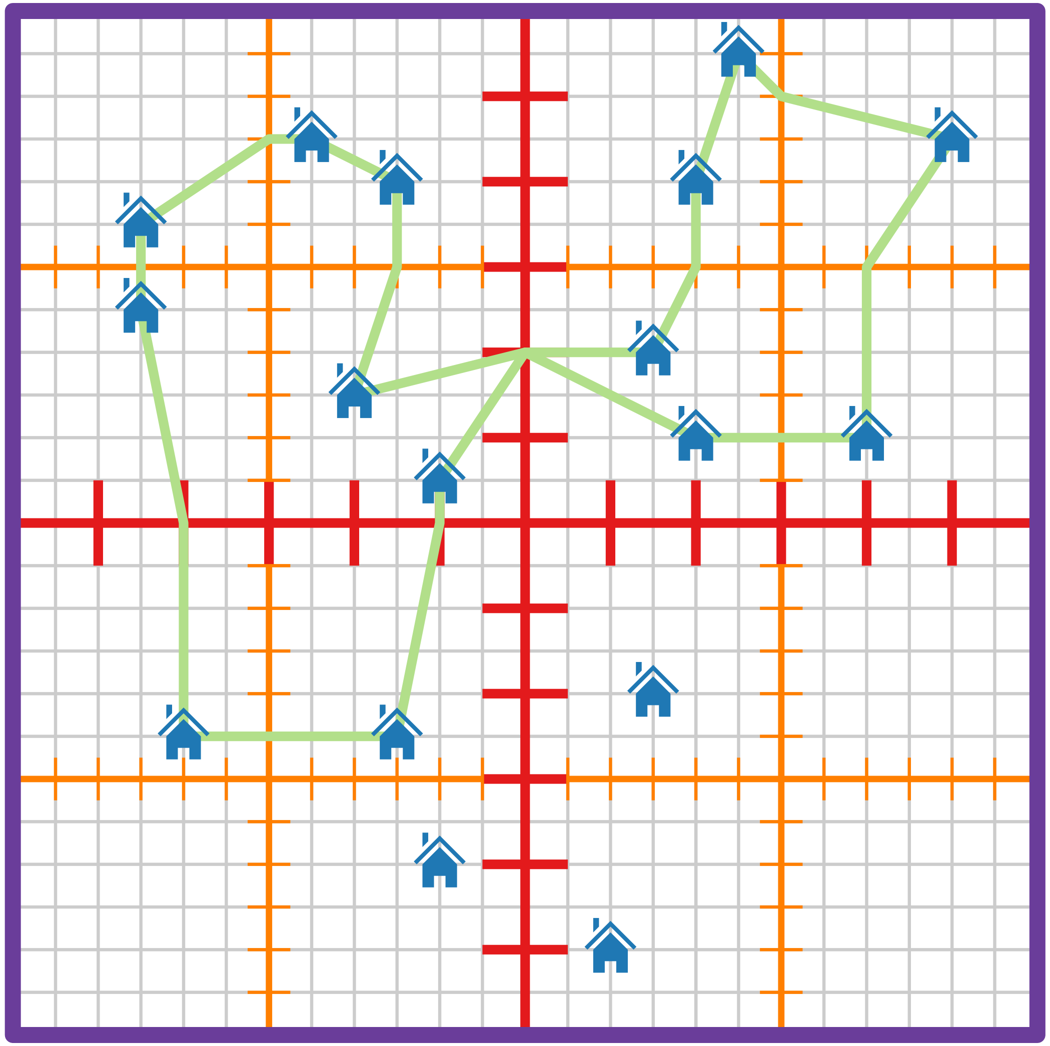

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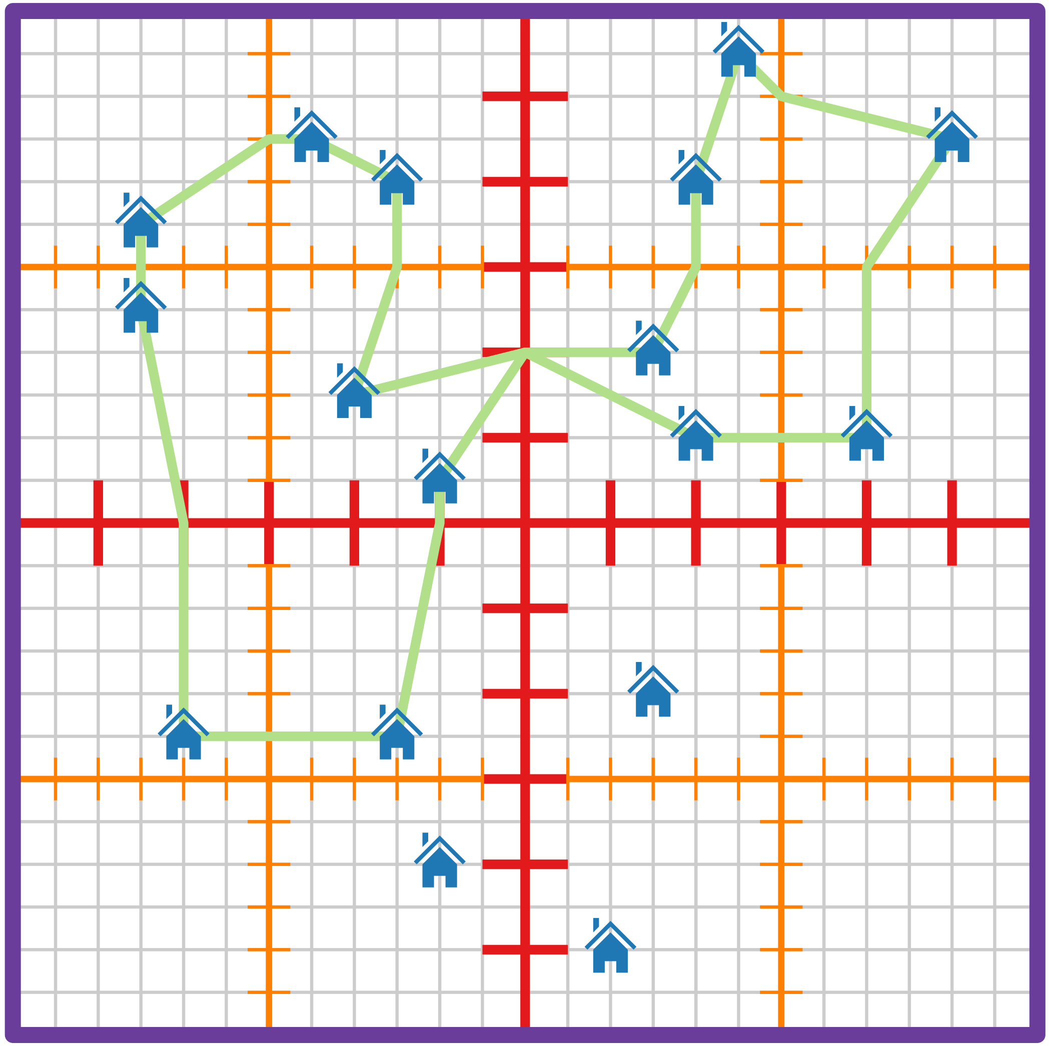

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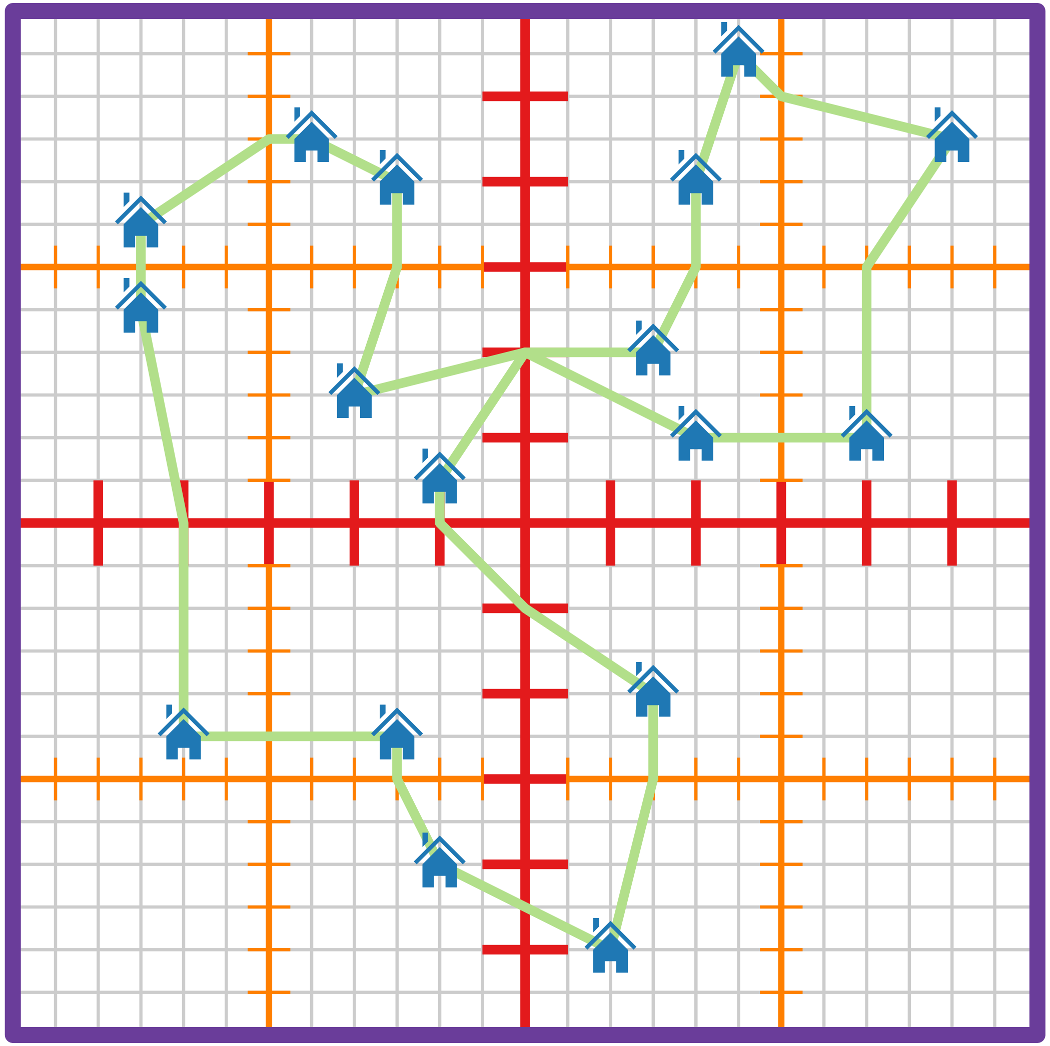

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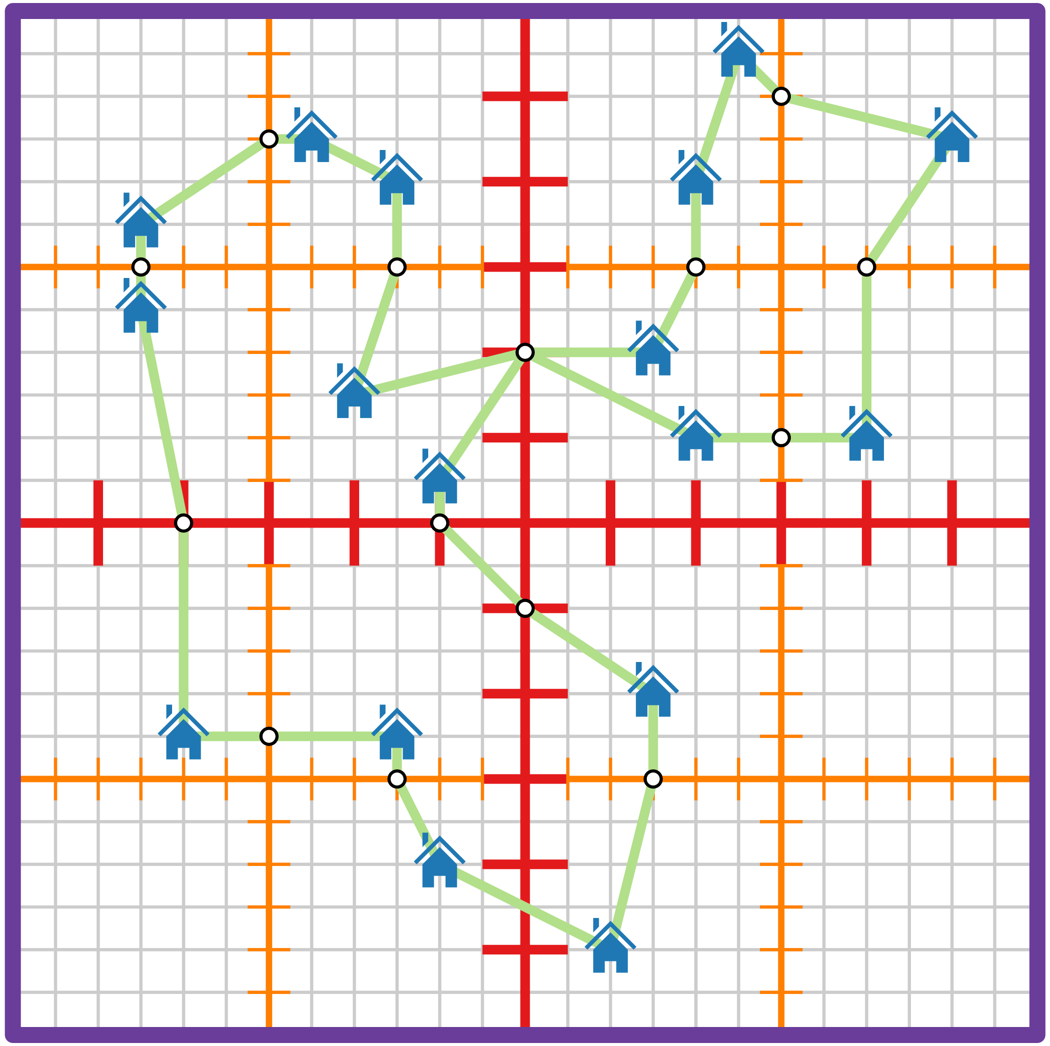
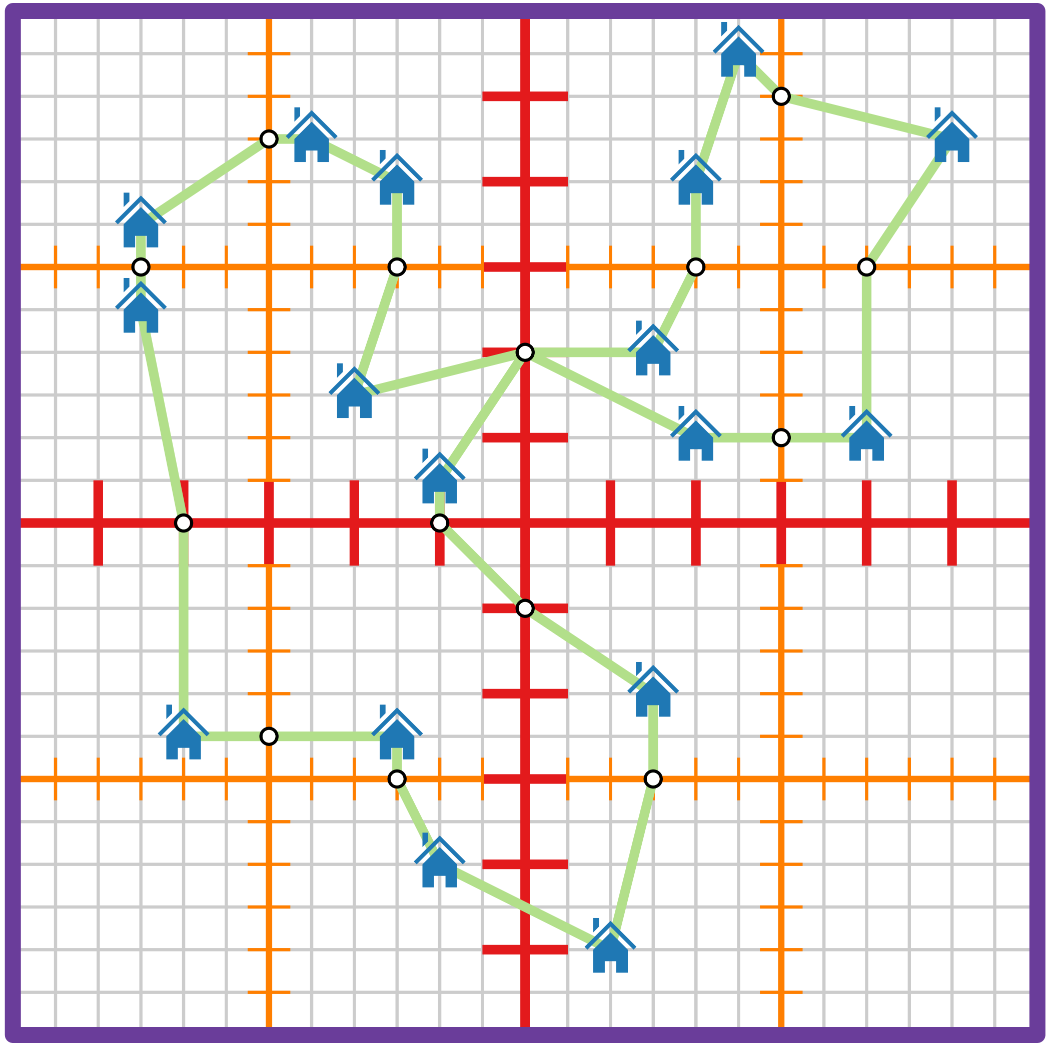
Well-Behaved Tours



A tour is *well-behaved* if

- it involves all houses and a subset of the portals,

Well-Behaved Tours



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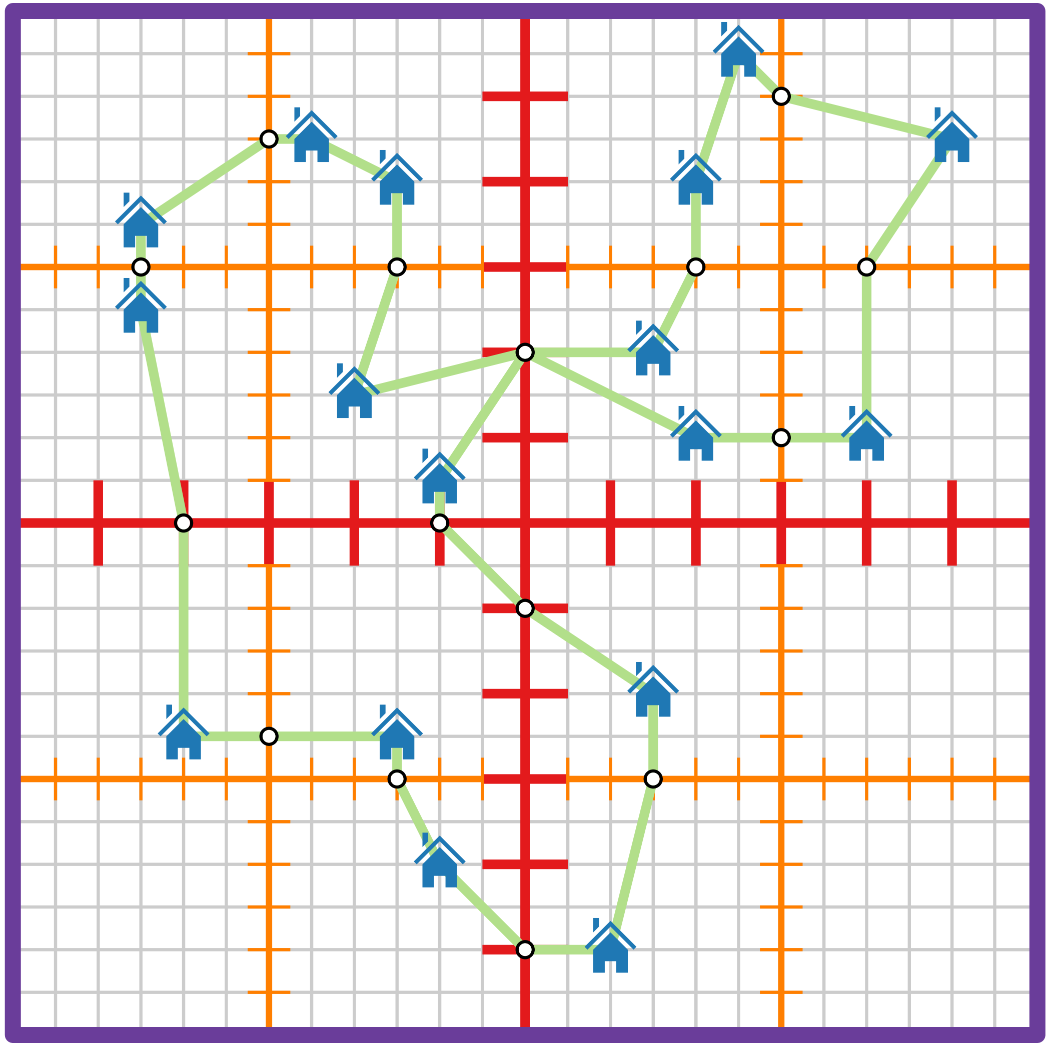

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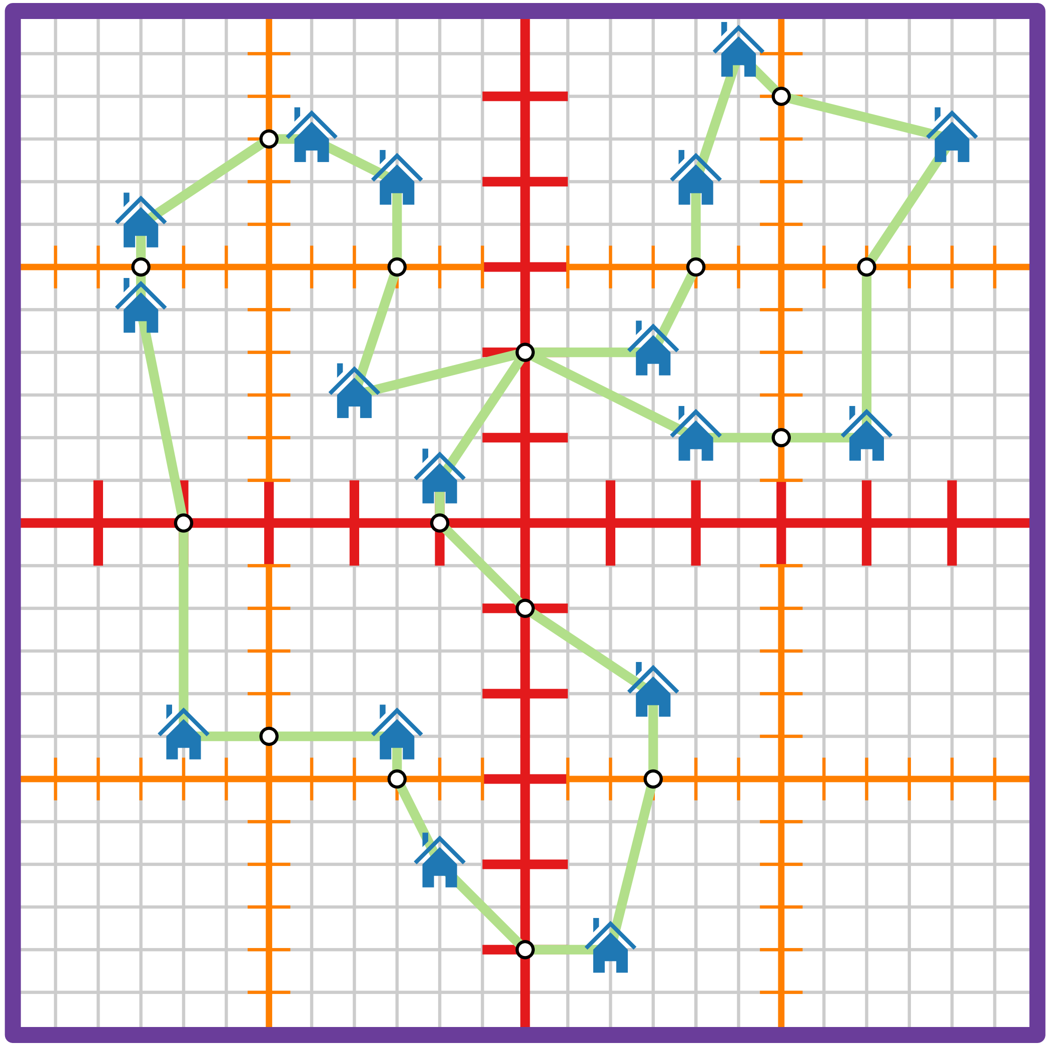

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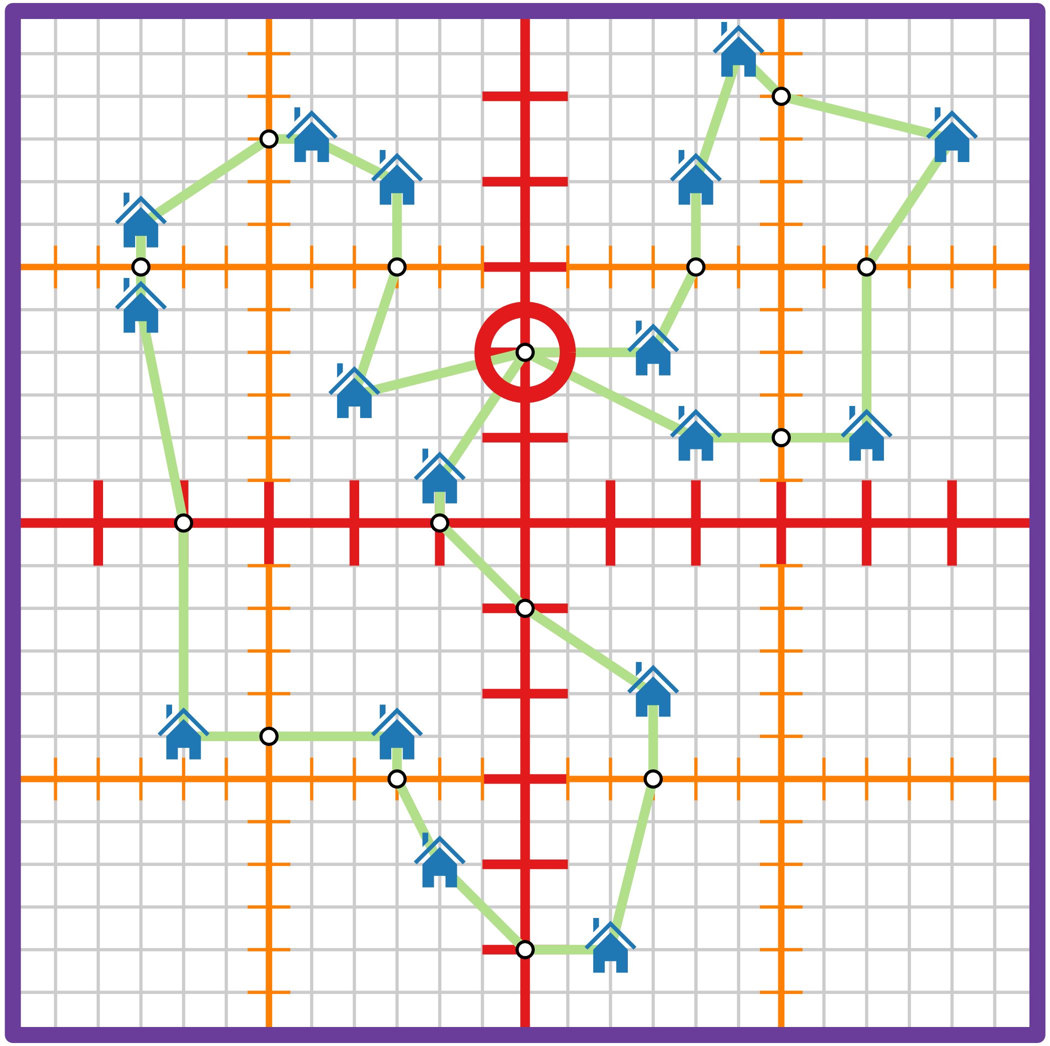

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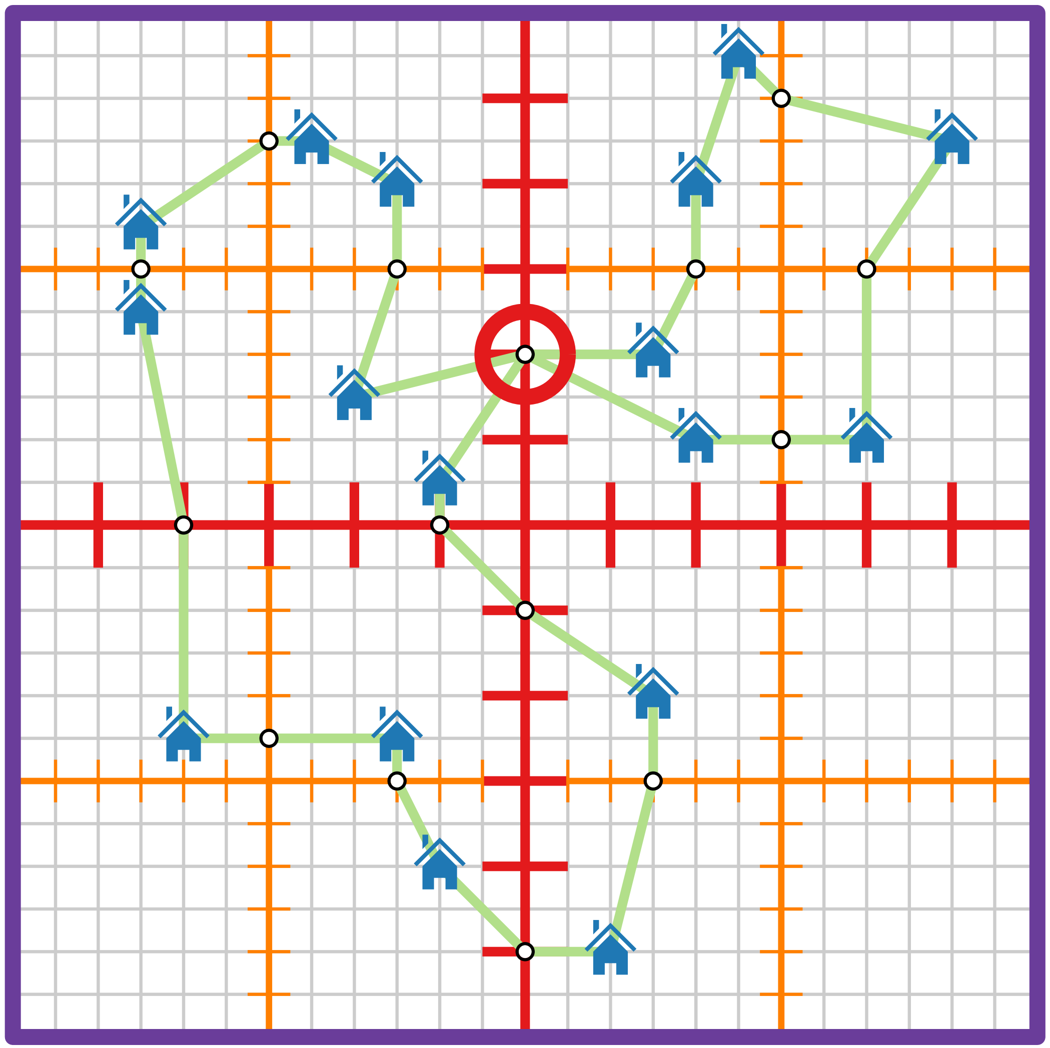

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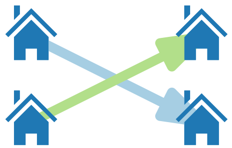
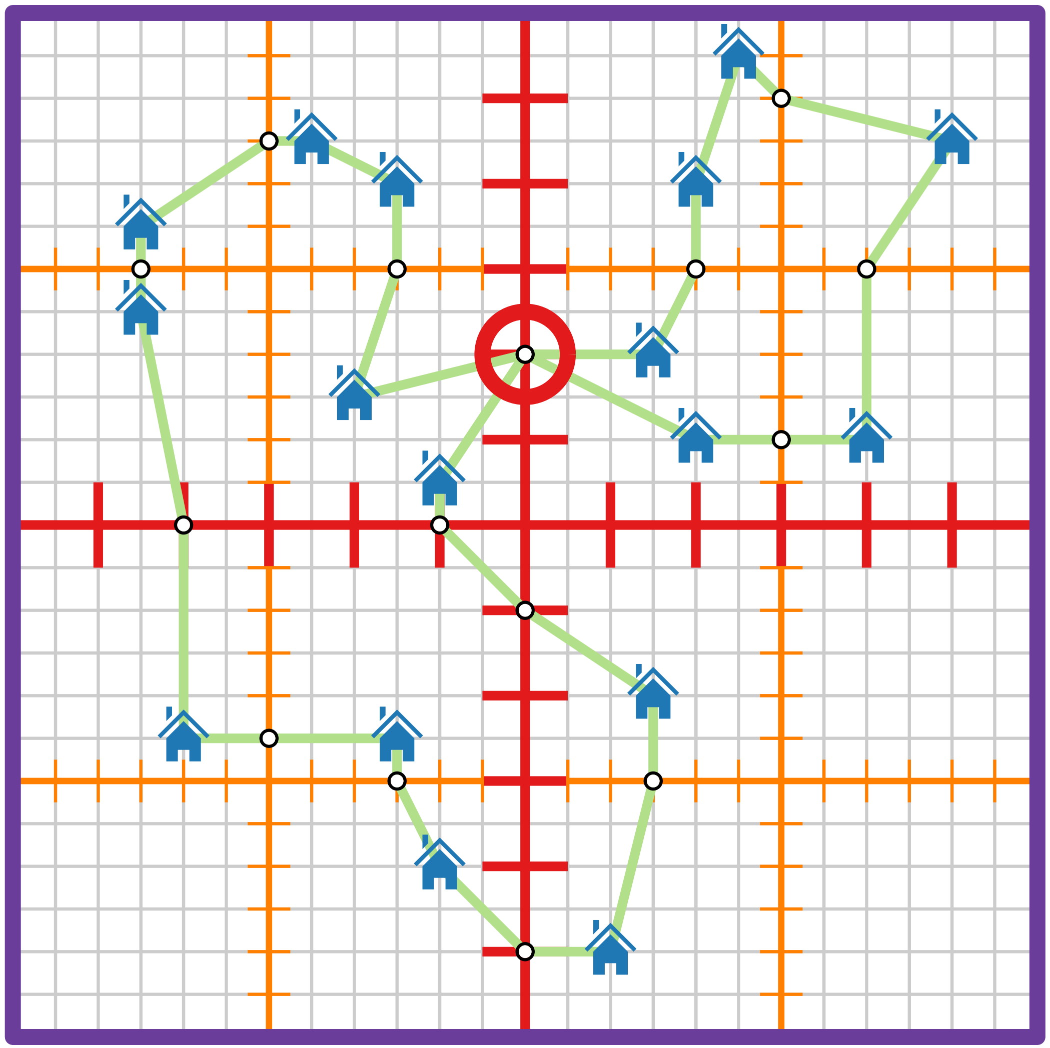
Crossing



Well-Behaved Tours



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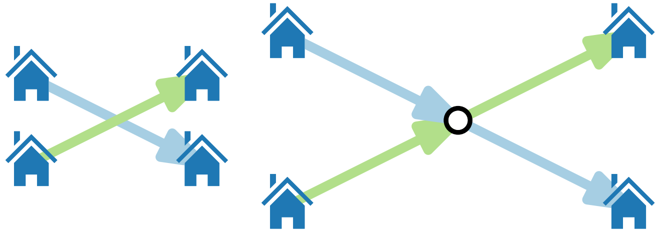

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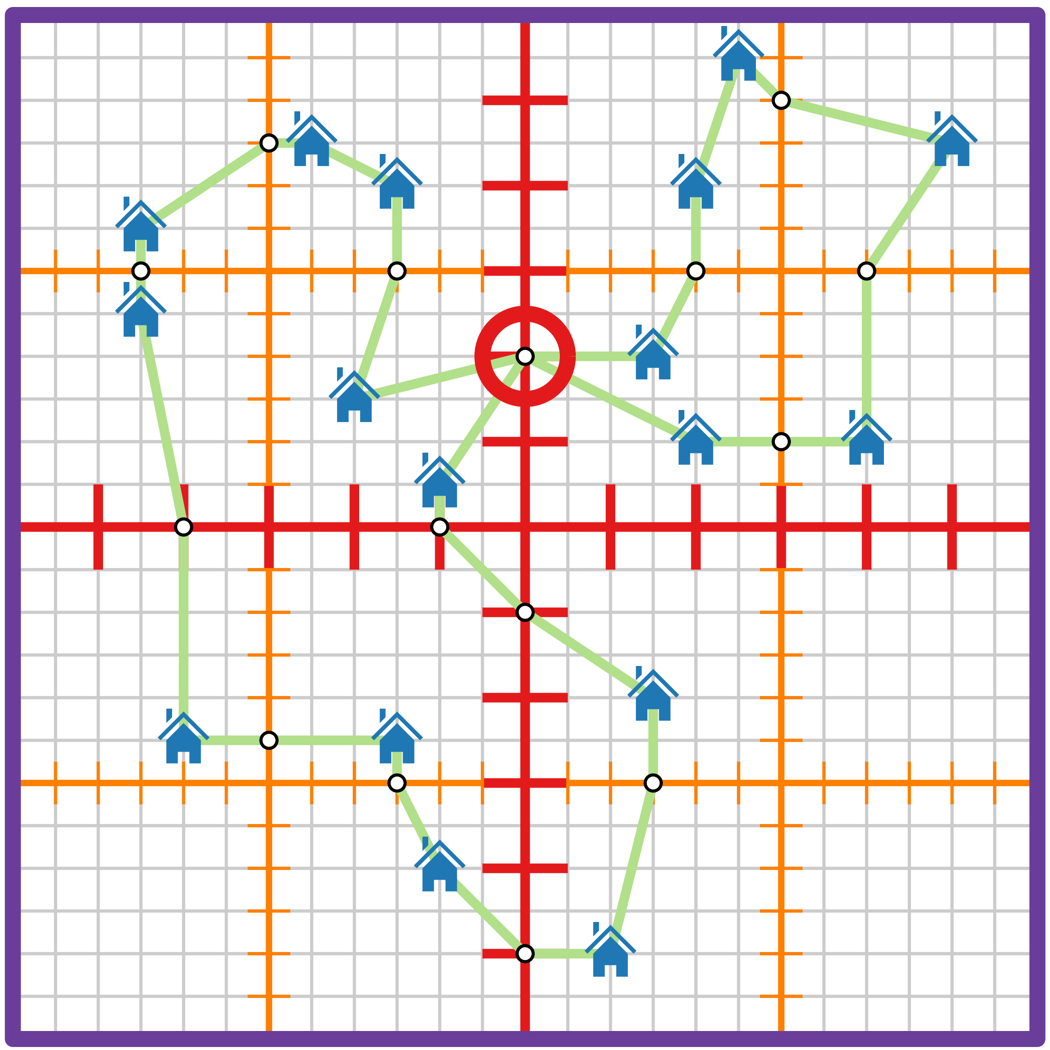

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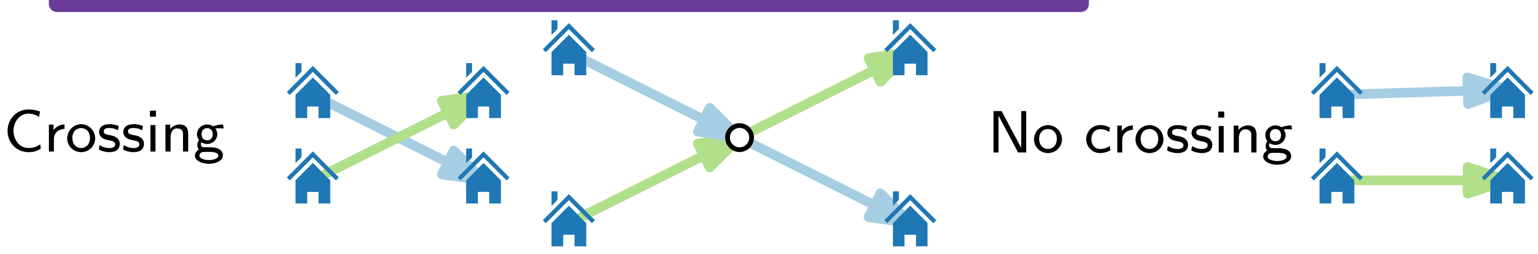
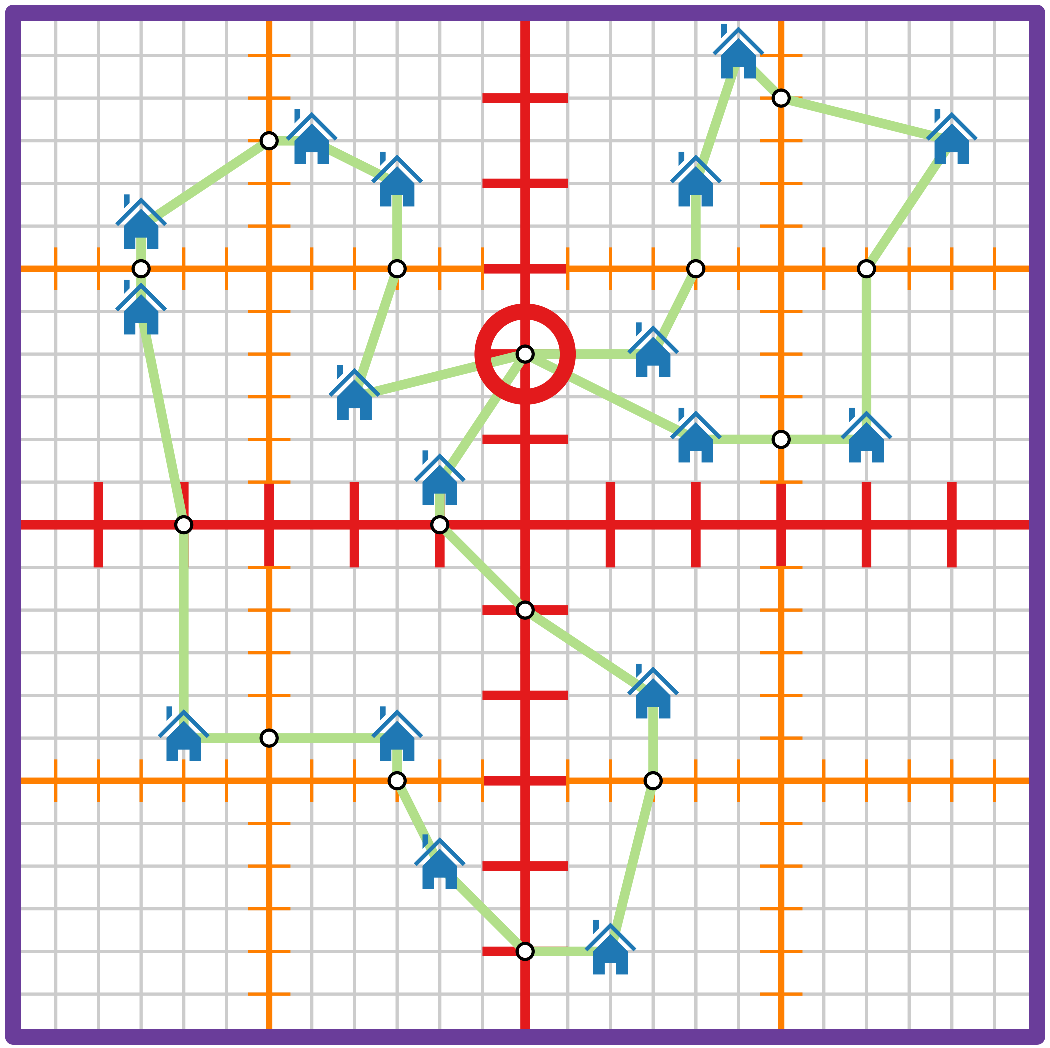


Well-Behaved Tours



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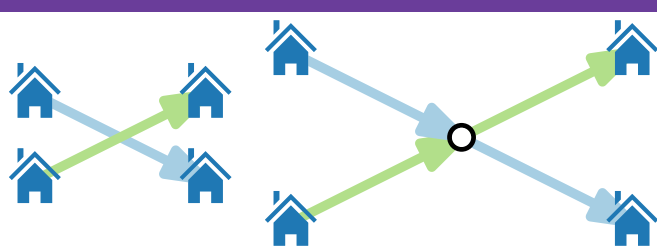
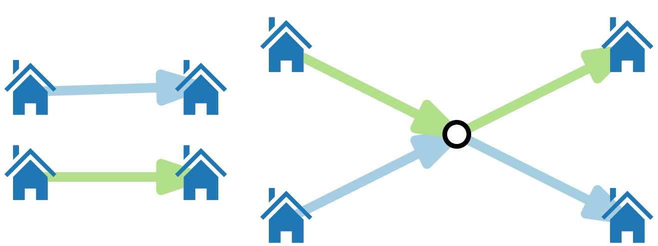


No crossing



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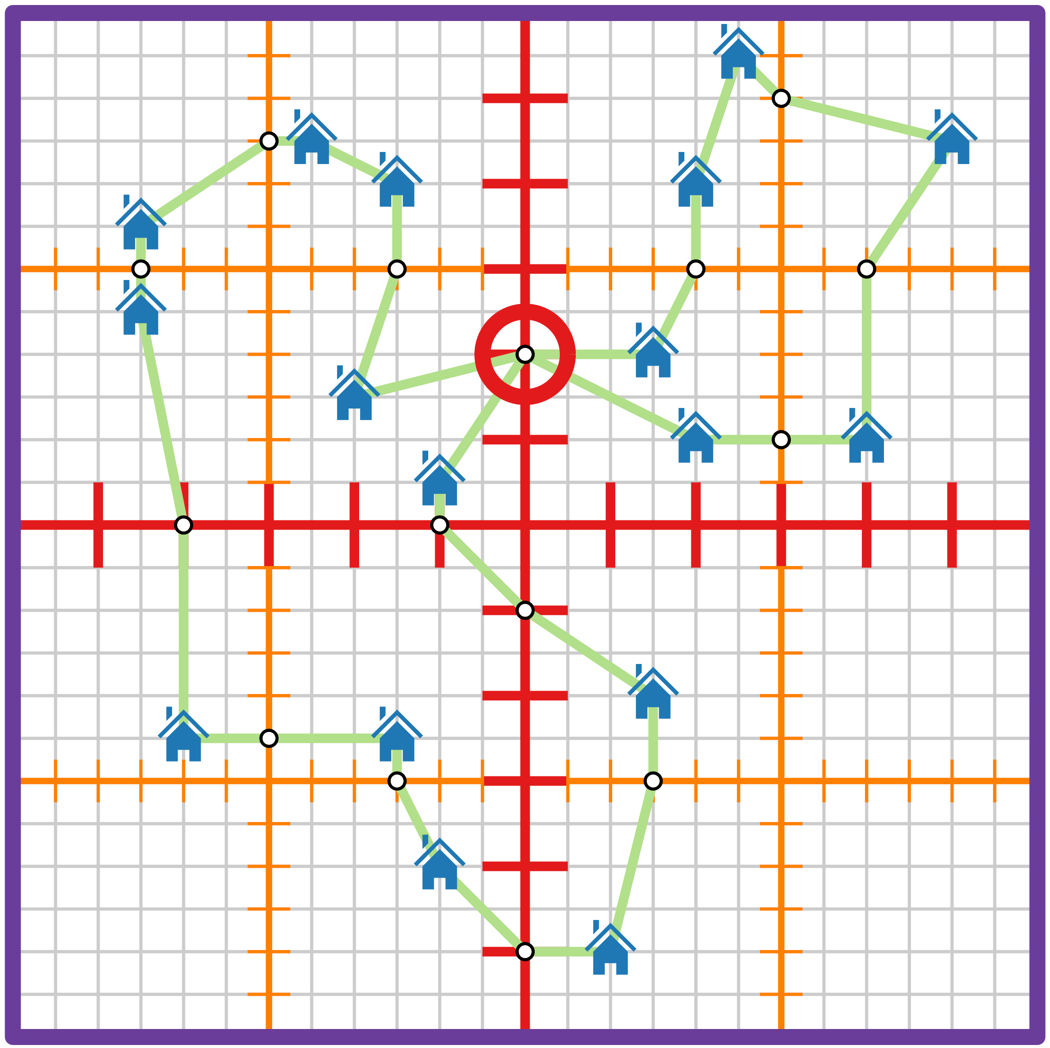

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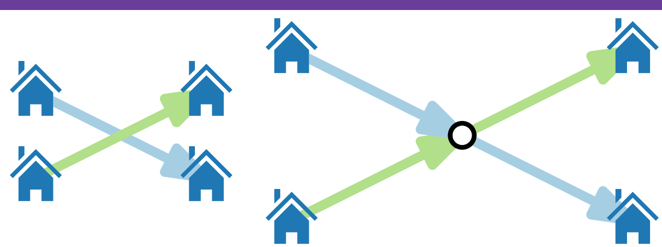
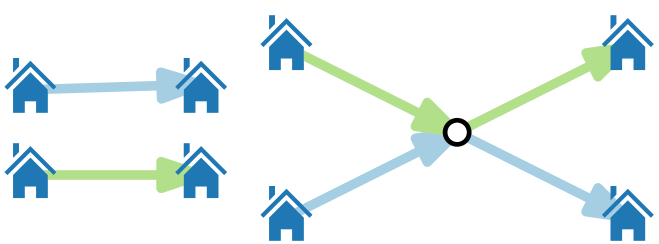
- it is crossing-free.

W.l.o.g. (homework):
No portal visited more than twice

Crossing



No crossing



Computing a Well-Behaved Tour

Computing a Well-Behaved Tour

Lemma. An optimal well-behaved tour can be computed in $2^{O(m)} = n^{O(1/\varepsilon)}$ time.

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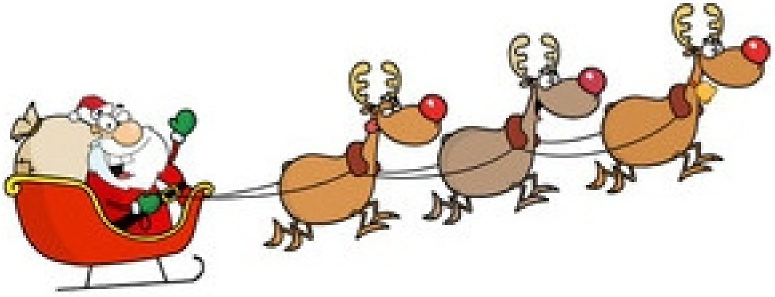

Computing a Well-Behaved Tour

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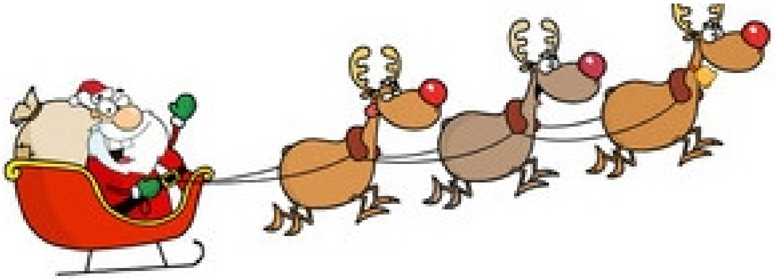
Sketch.

- Dynamic programming!



Computing a Well-Behaved Tour

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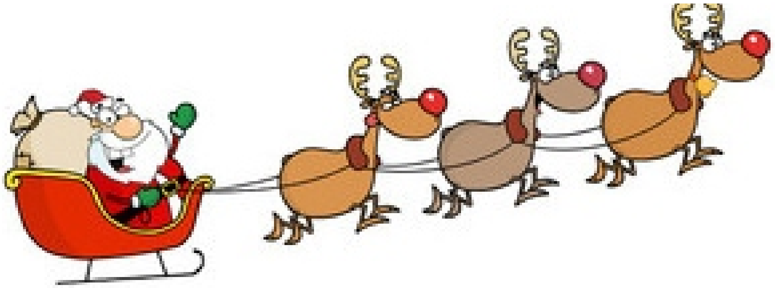
- Dynamic programming!
- Compute sub-structure of an optimal tour for each square in the dissection tree.

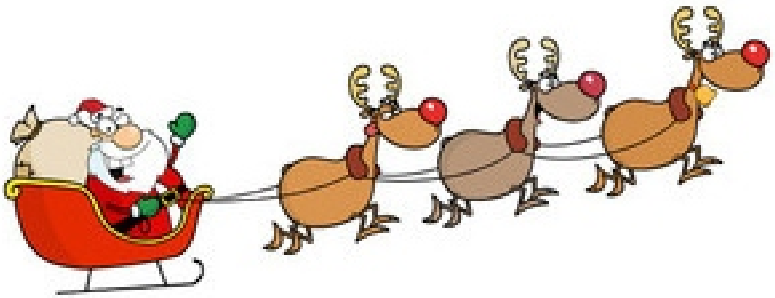


Computing a Well-Behaved Tour

Lemma. An optimal well-behaved tour can be computed in $2^{O(m)} = n^{O(1/\varepsilon)}$ time.

Sketch.

- Dynamic programming!
- Compute sub-structure of an optimal tour for each square in the dissection tree.
- These solutions can be efficiently propagated bottom-up through the dissection tree.

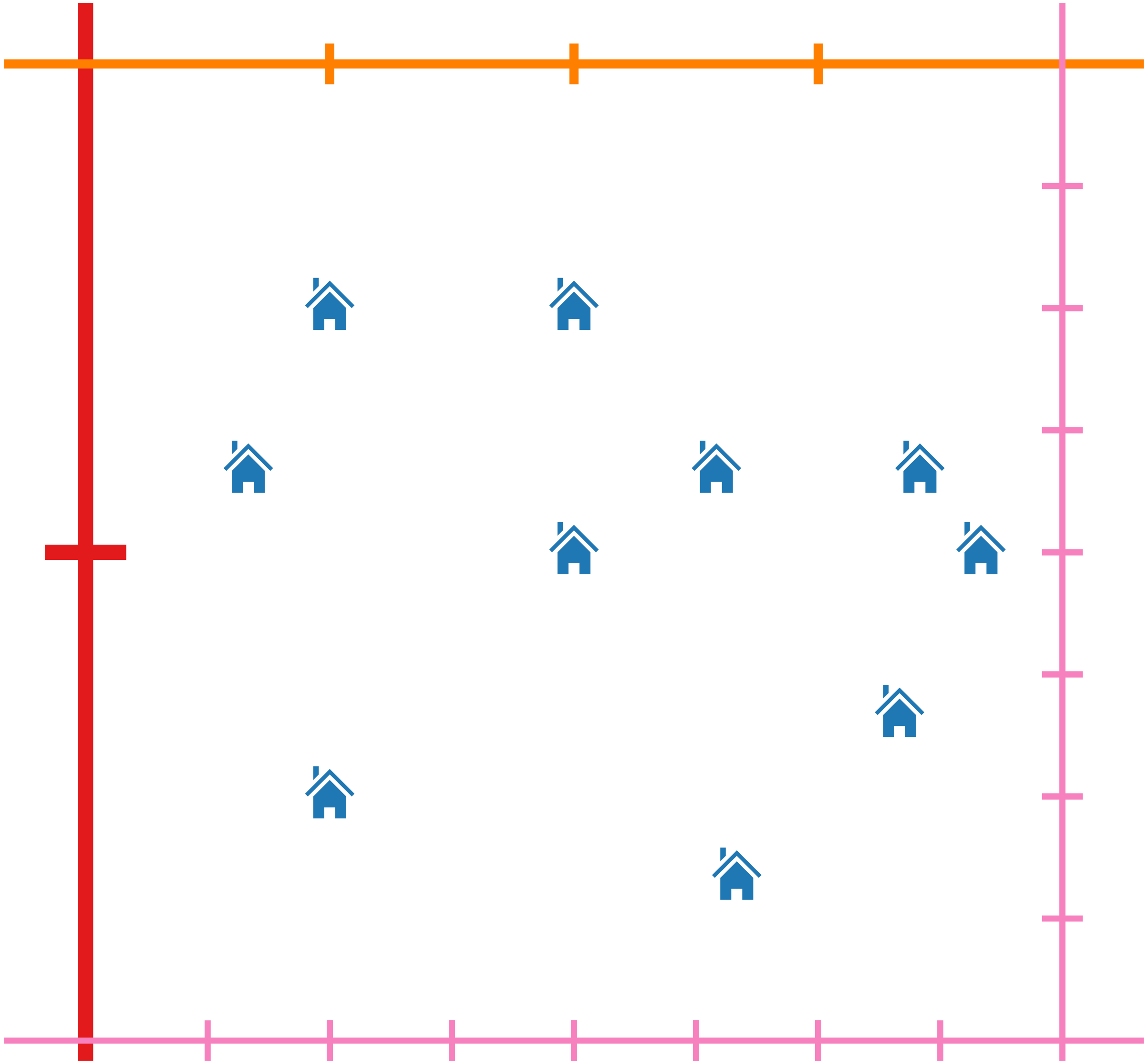


Approximation Algorithms

Lecture 9:
A PTAS for EUCLIDEAN TSP

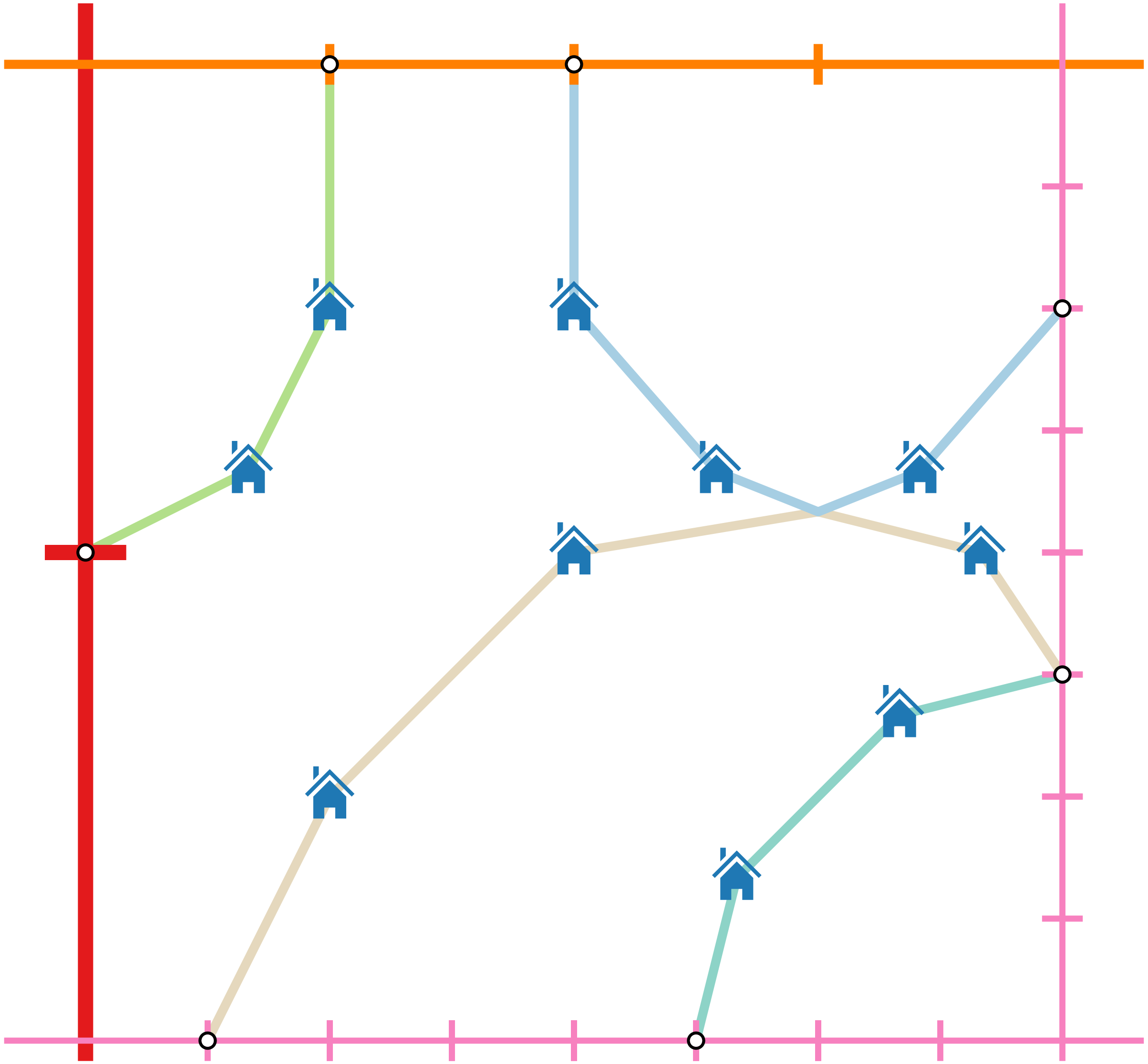
Part IV:
Dynamic Program

Dynamic Program (I)



Each well-behaved tour induces the following in each square Q of the dissection:

Dynamic Program (I)



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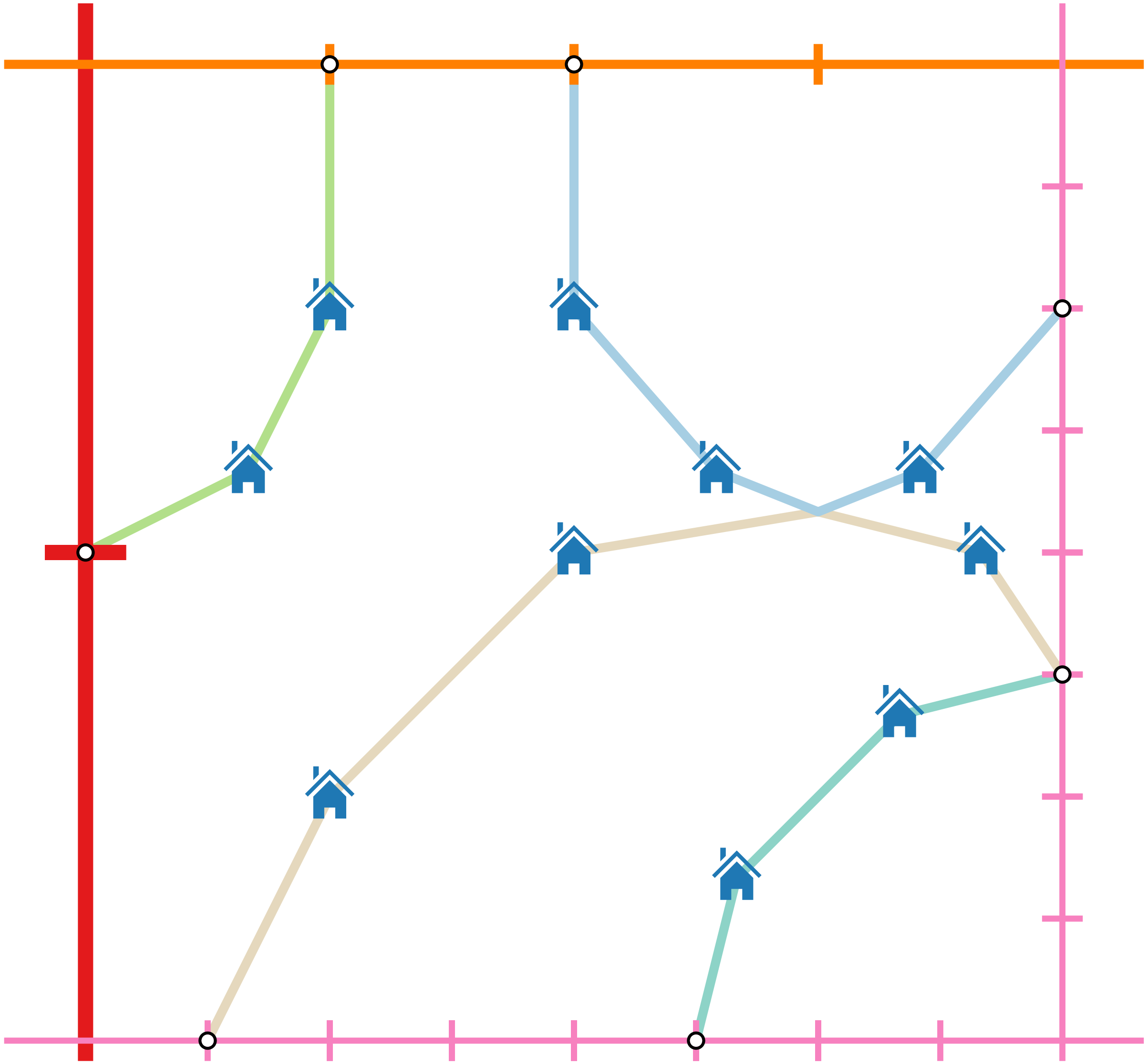

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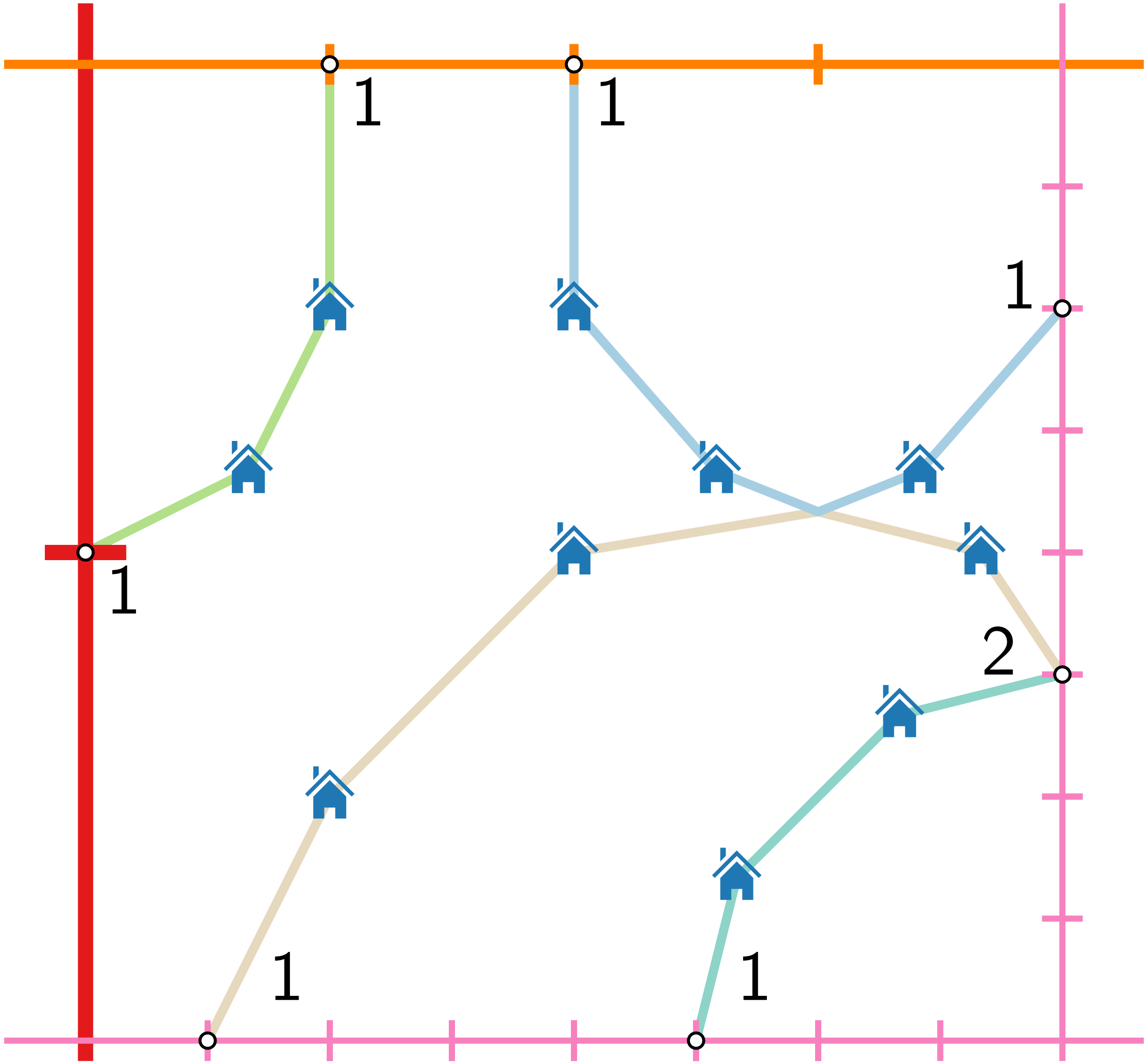

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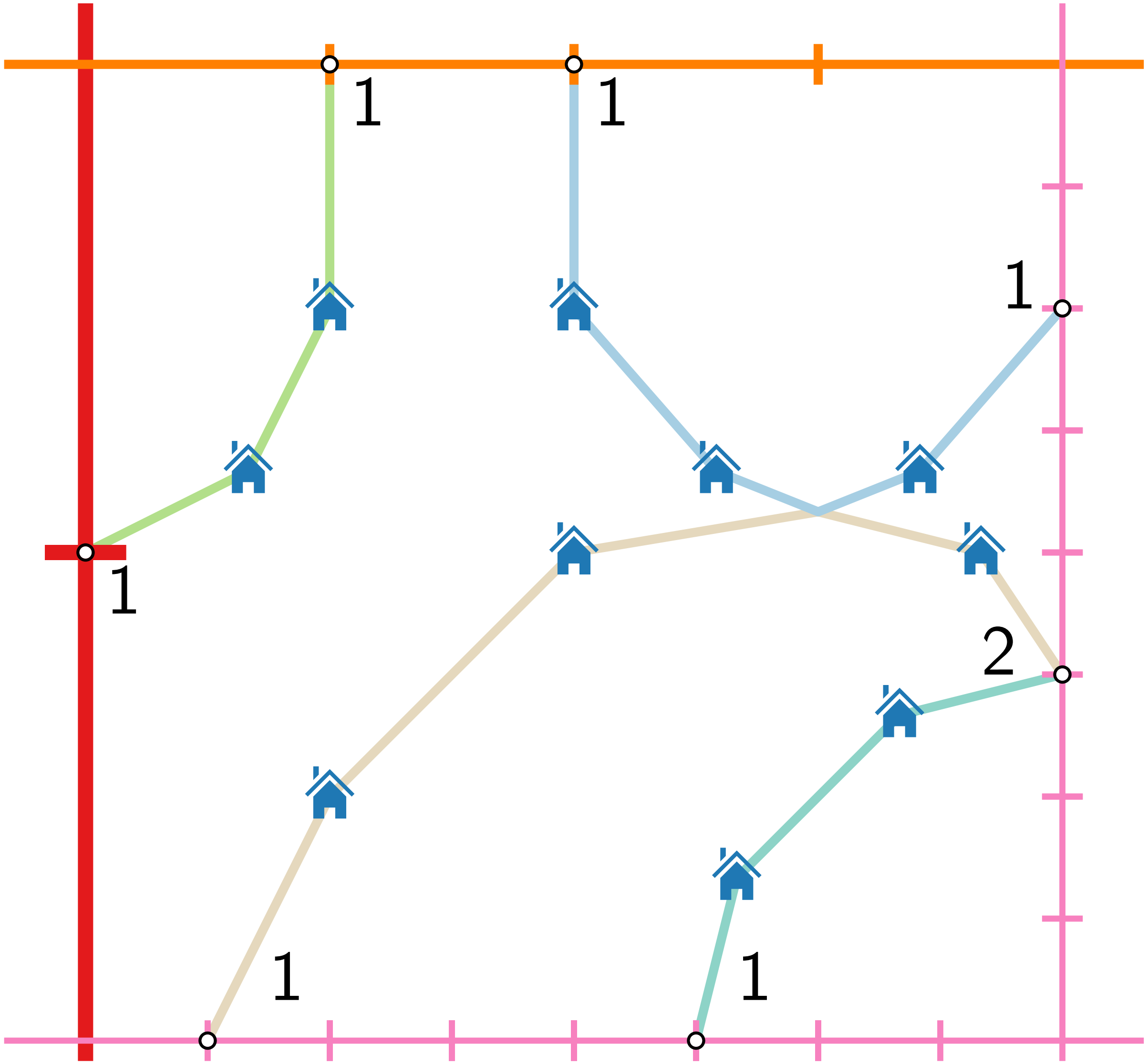

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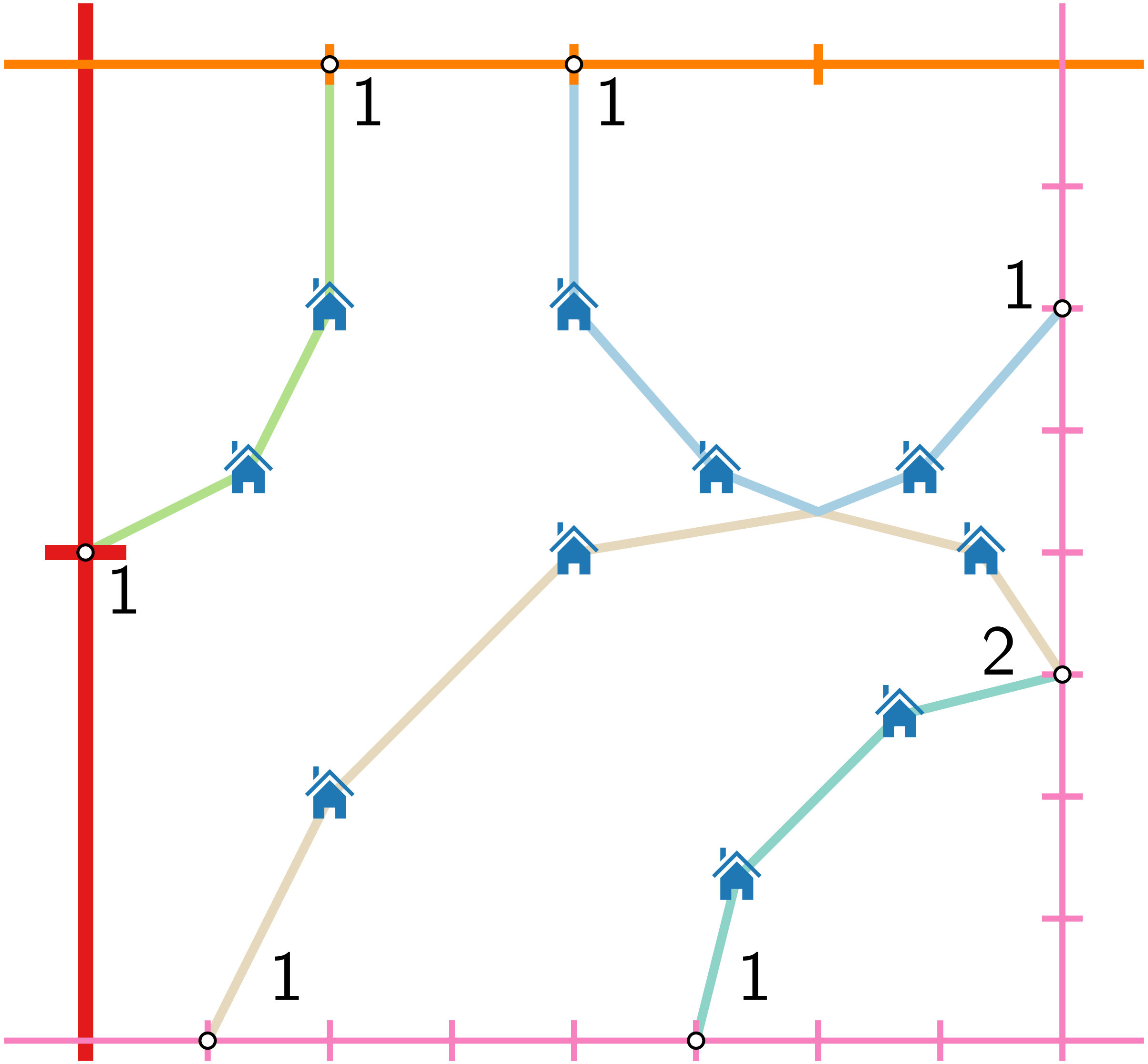
$\Rightarrow$ at most possibilities

Dynamic Program (I)



Each well-behaved tour induces the following in each square Q of the dissection:

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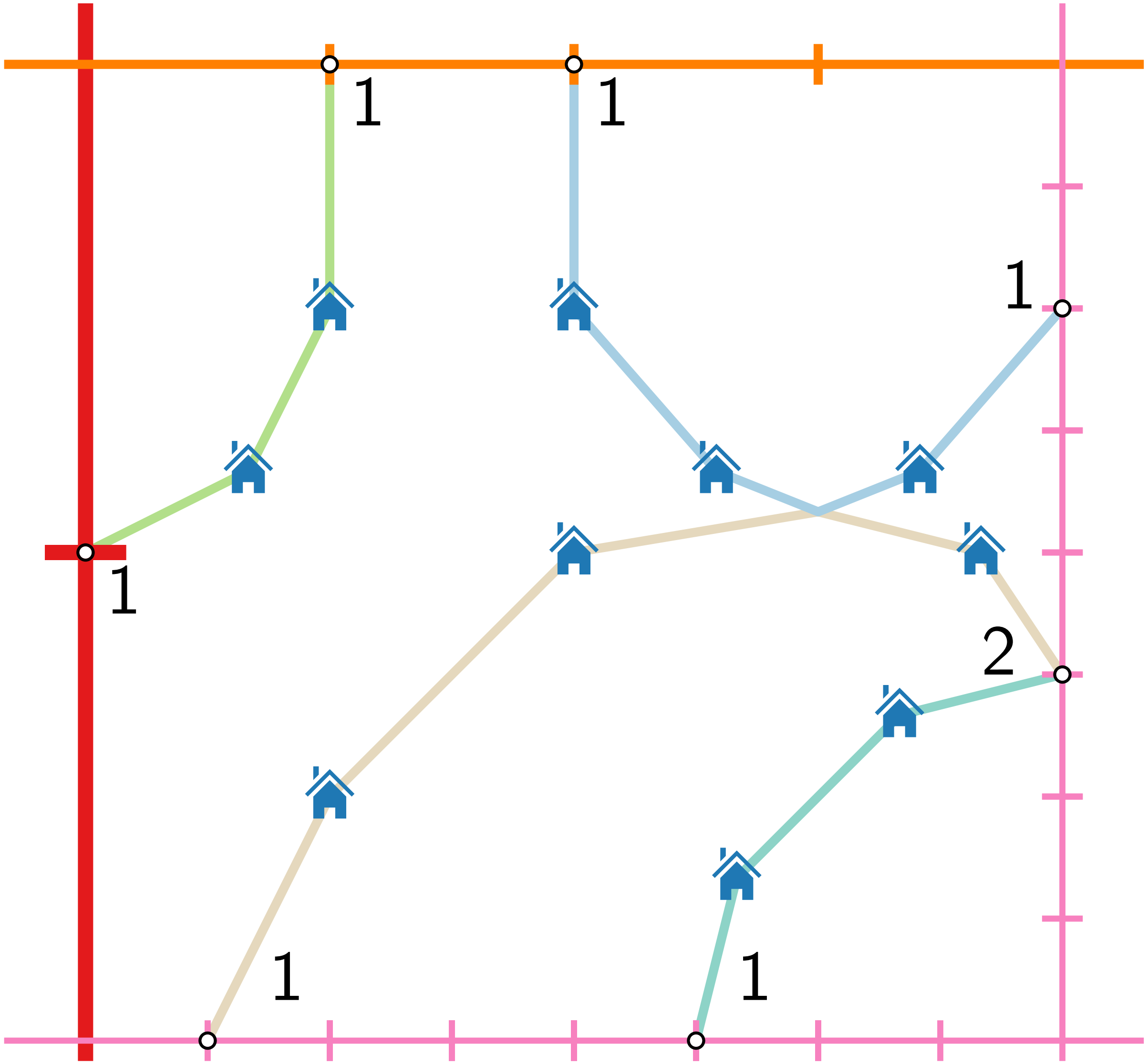

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$m = O((\log n)/\varepsilon)$

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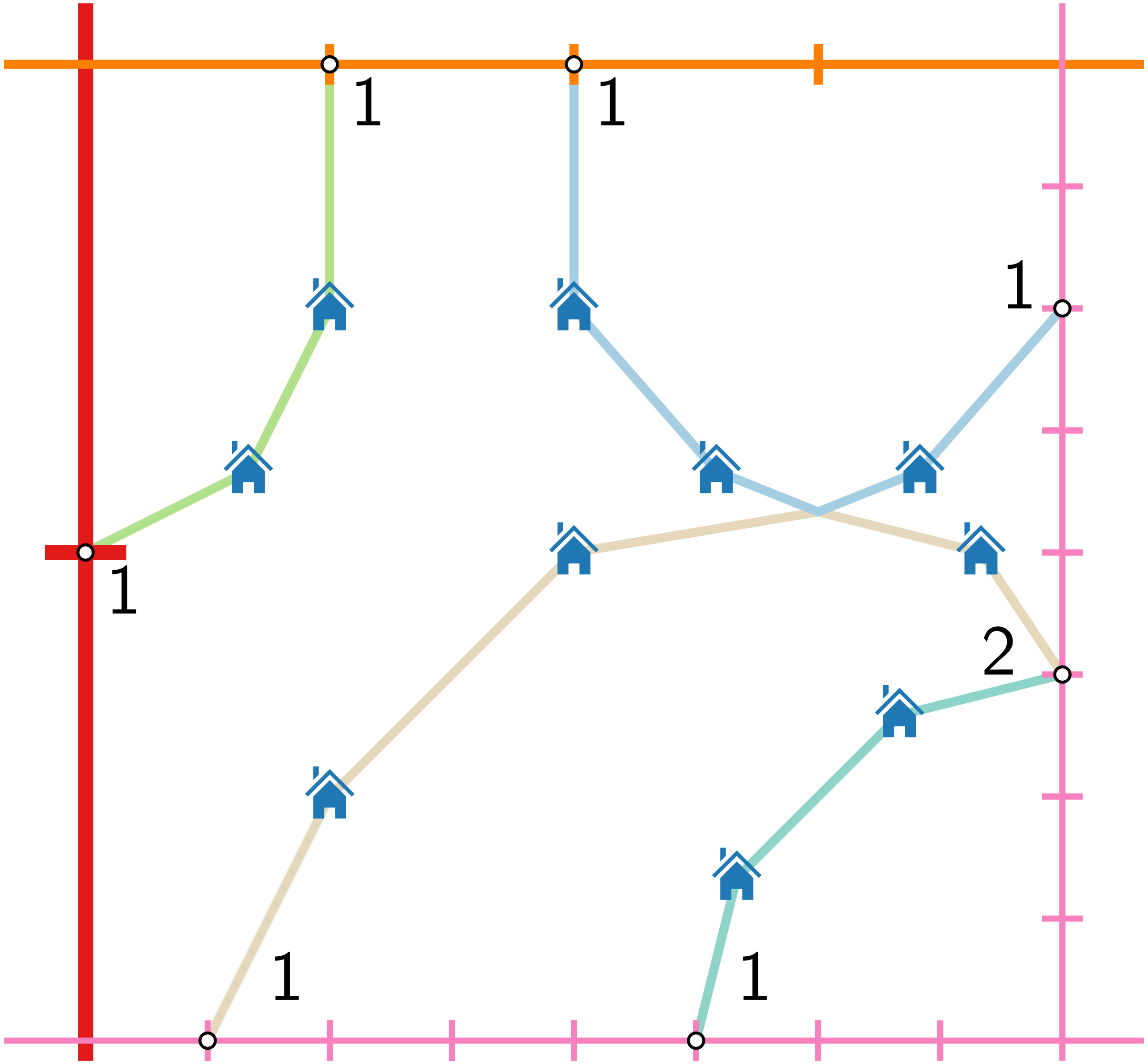

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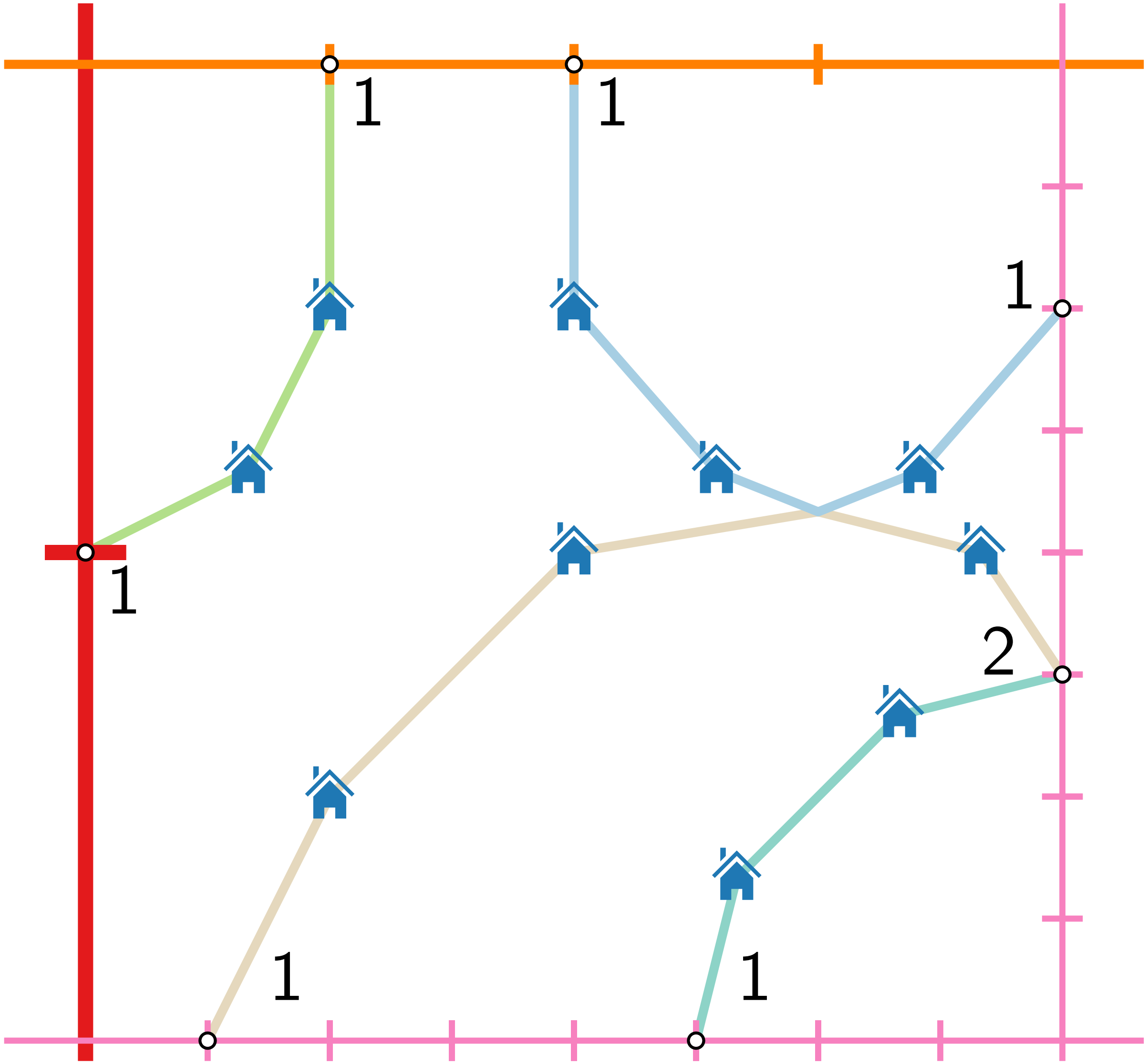

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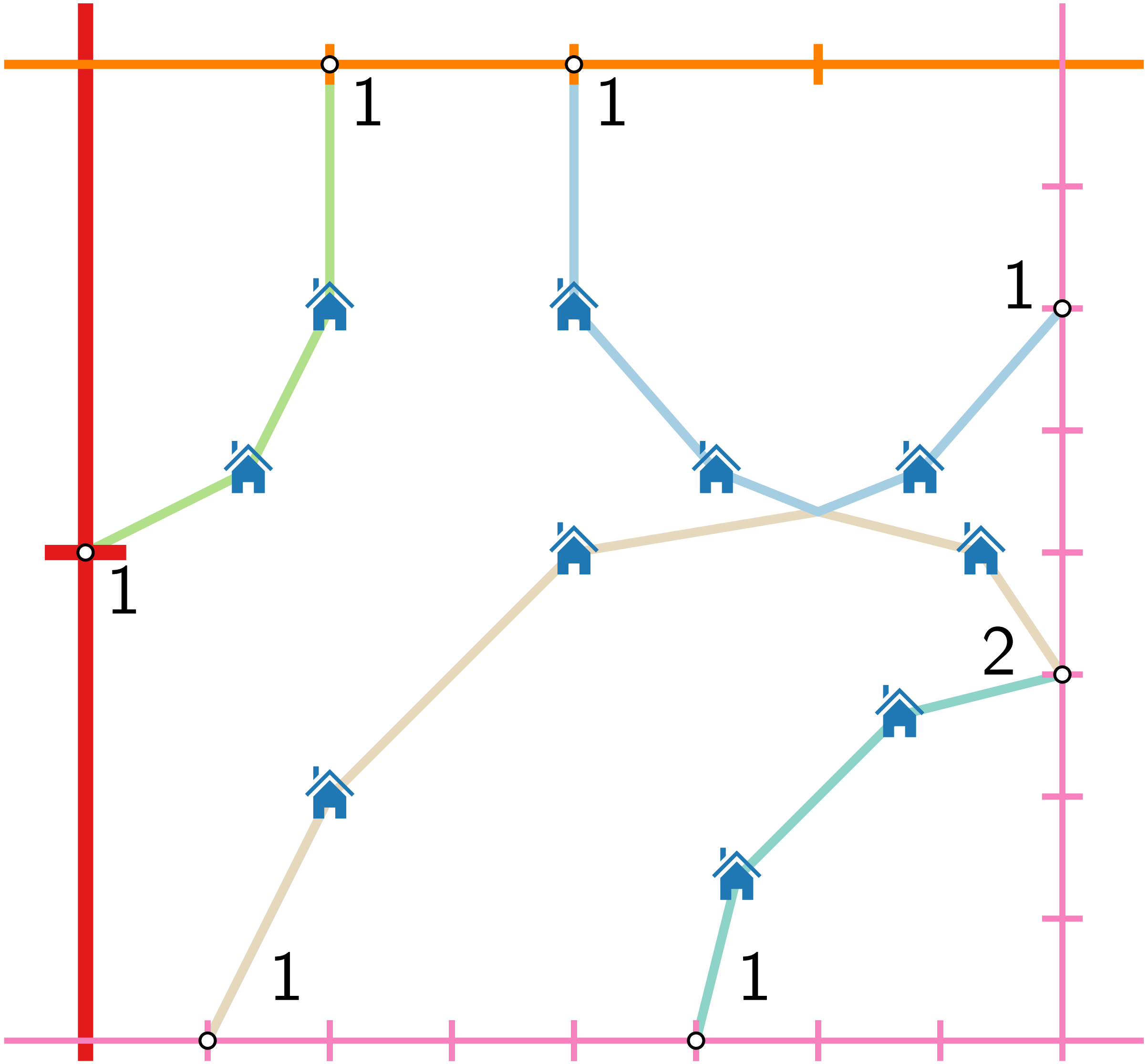

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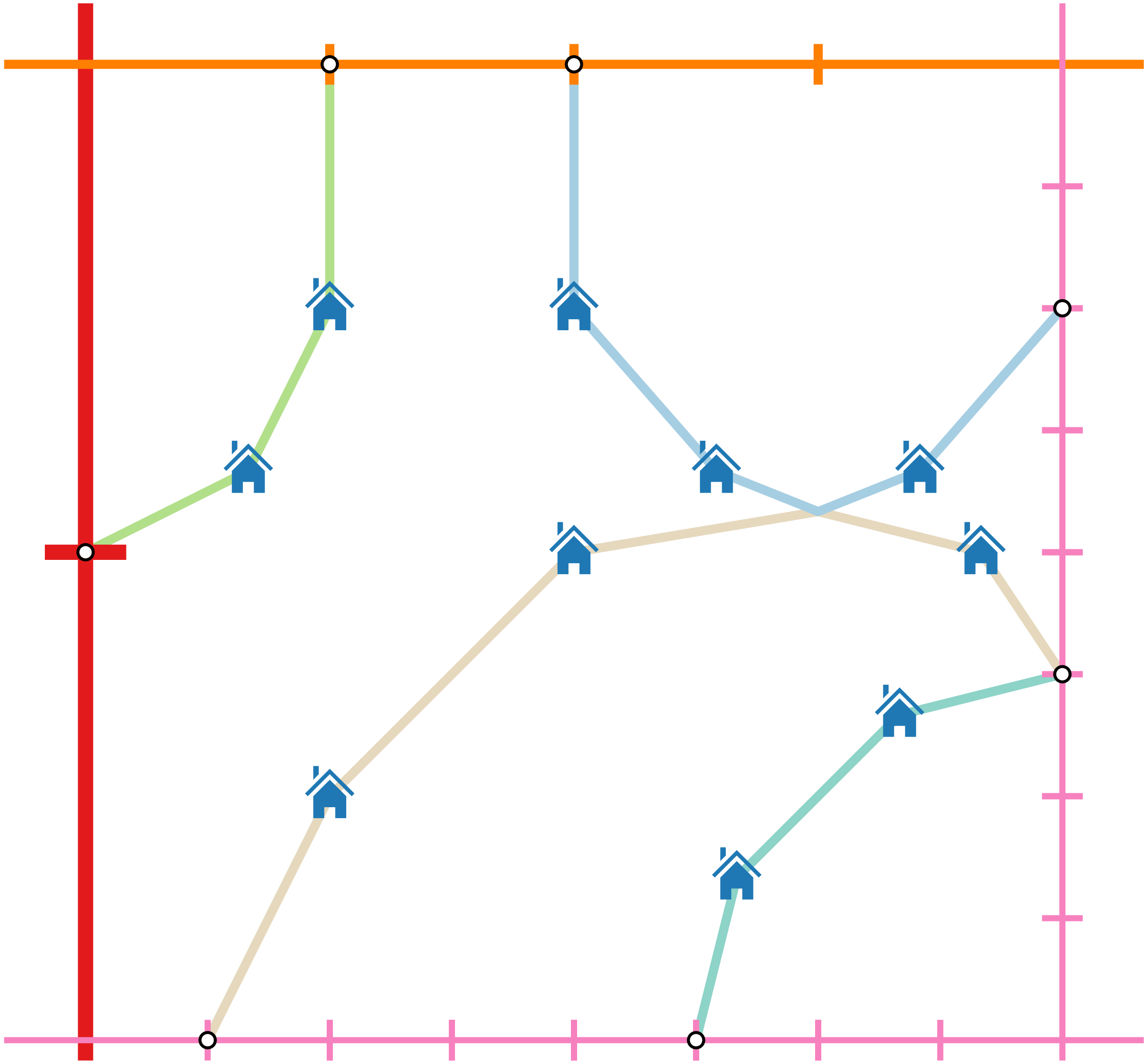
$\Rightarrow$ at most $3^{4m} \in 3^{O((\log n)/\varepsilon)} = n^{O(1/\varepsilon)}$ possibilities (*visit vectors*)

$m = O((\log n)/\varepsilon)$

Dynamic Program (I)



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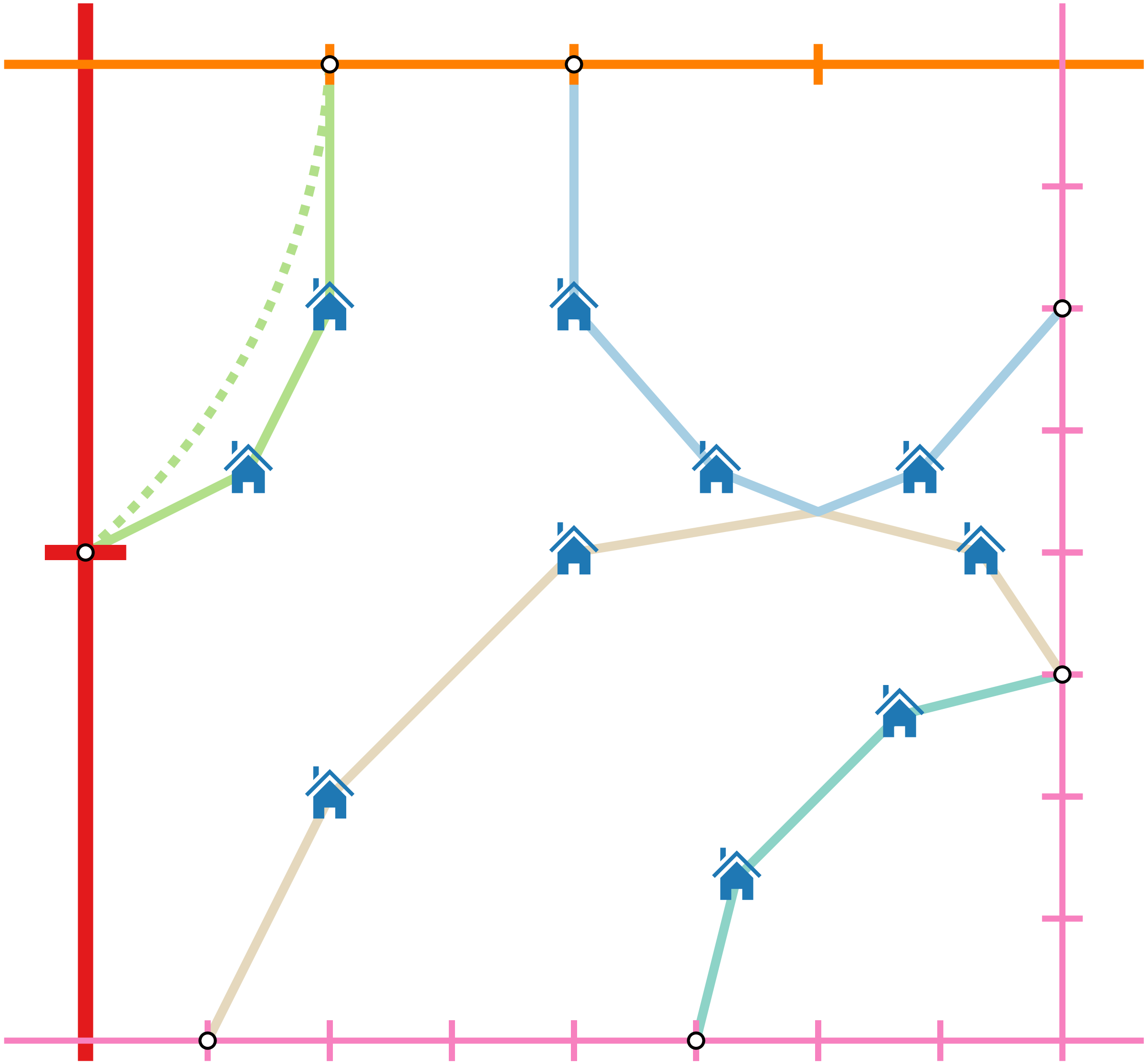

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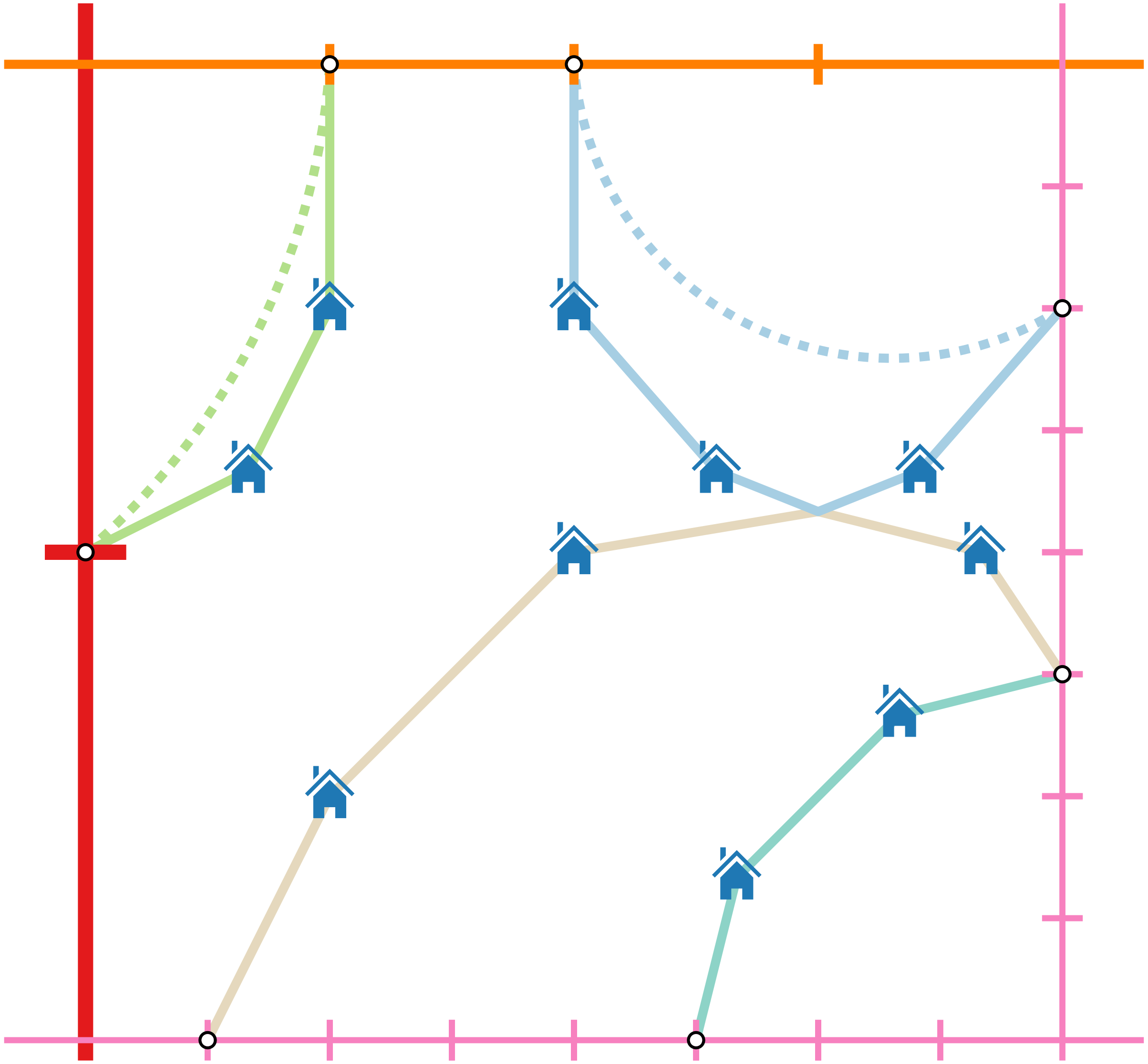

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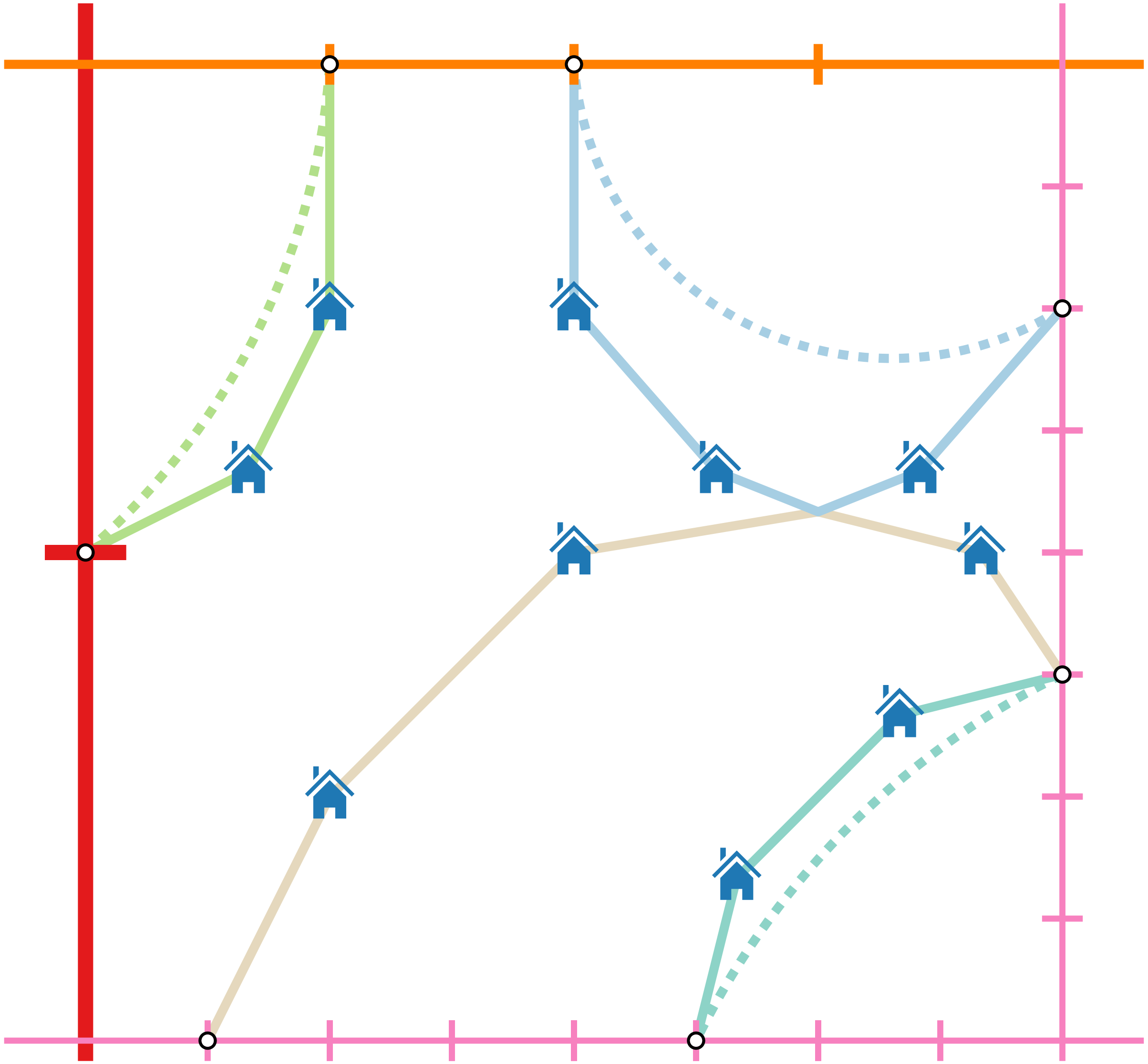

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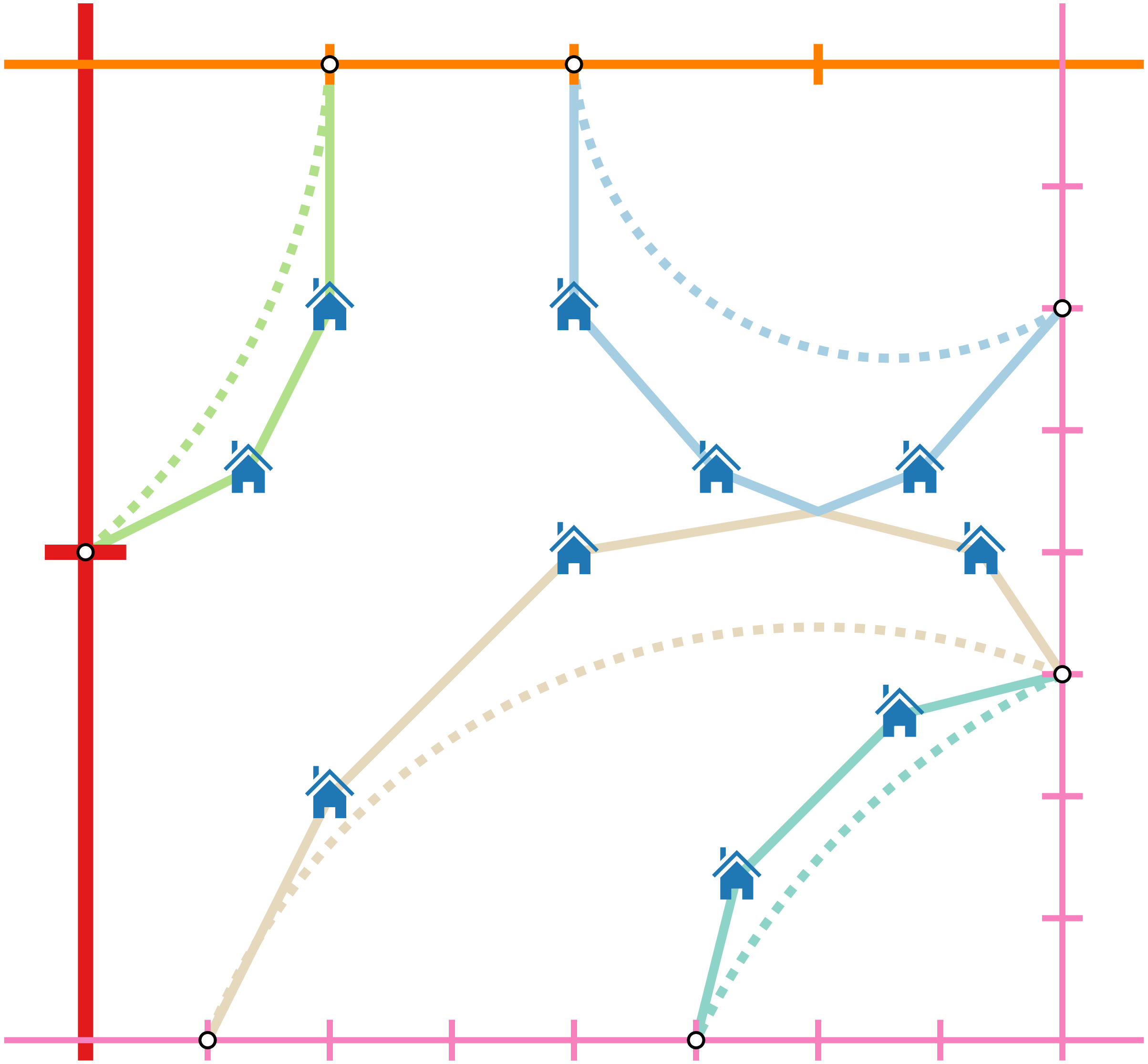

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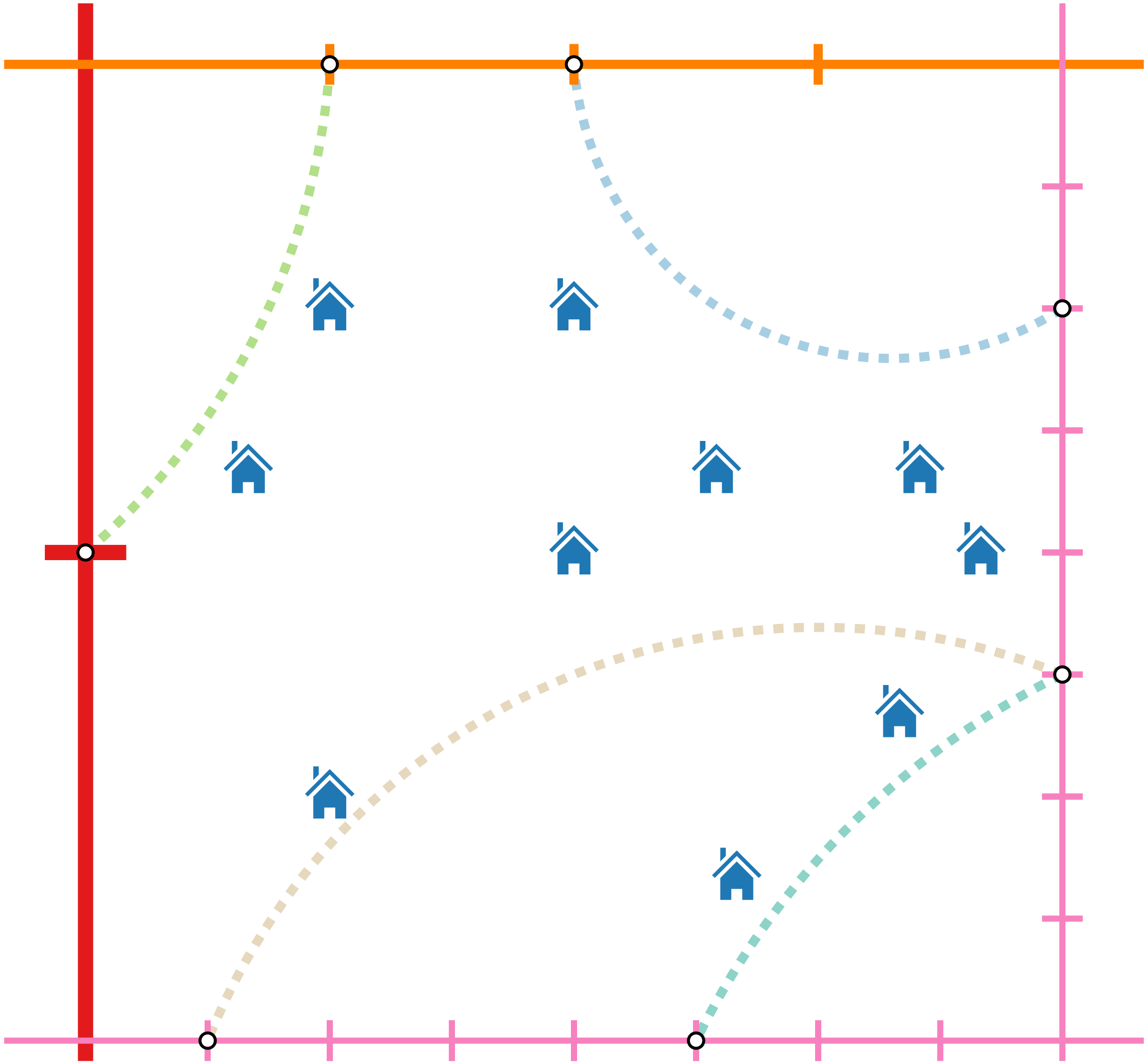

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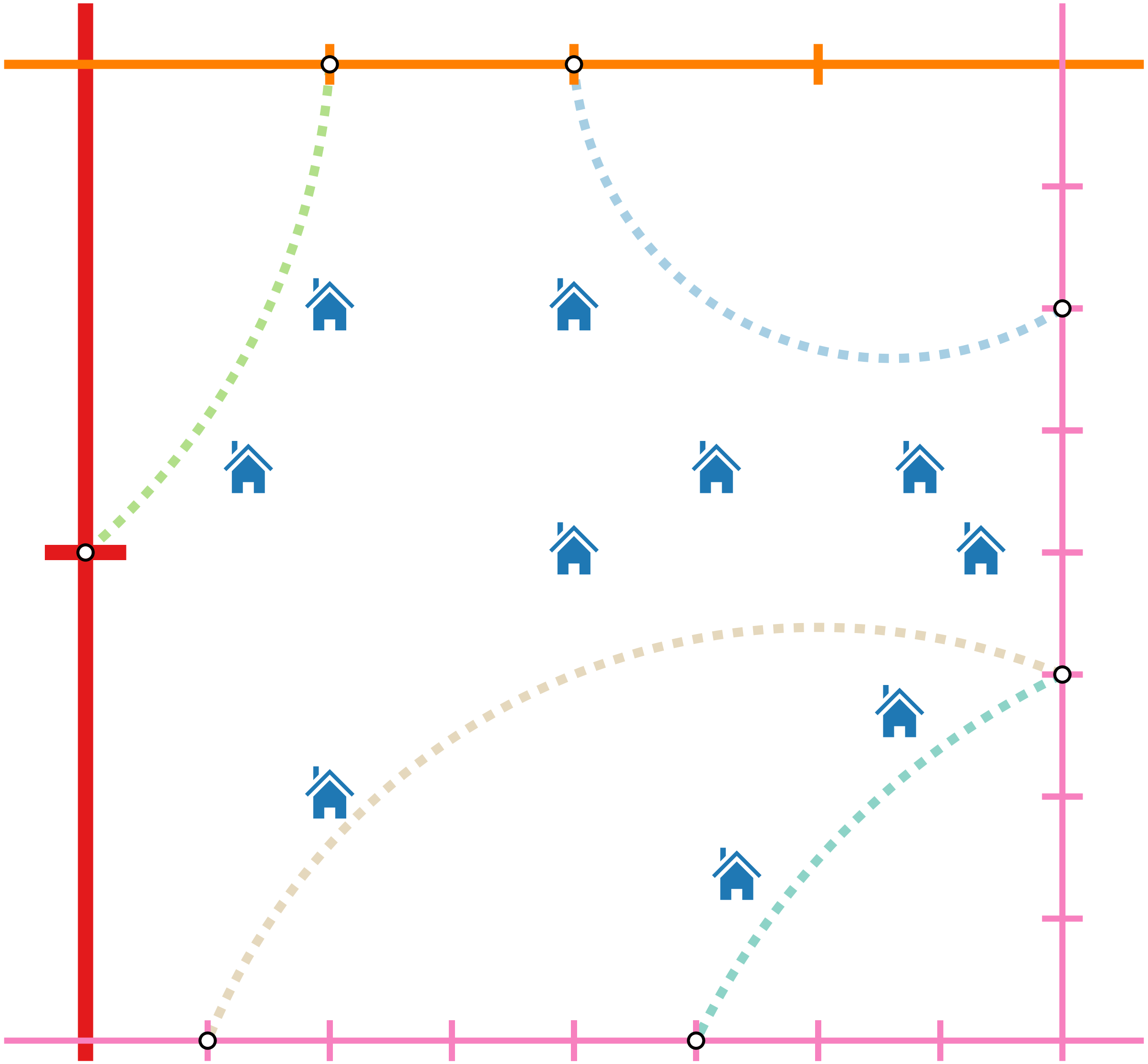

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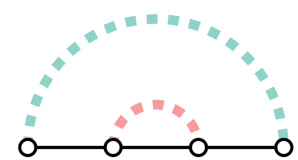

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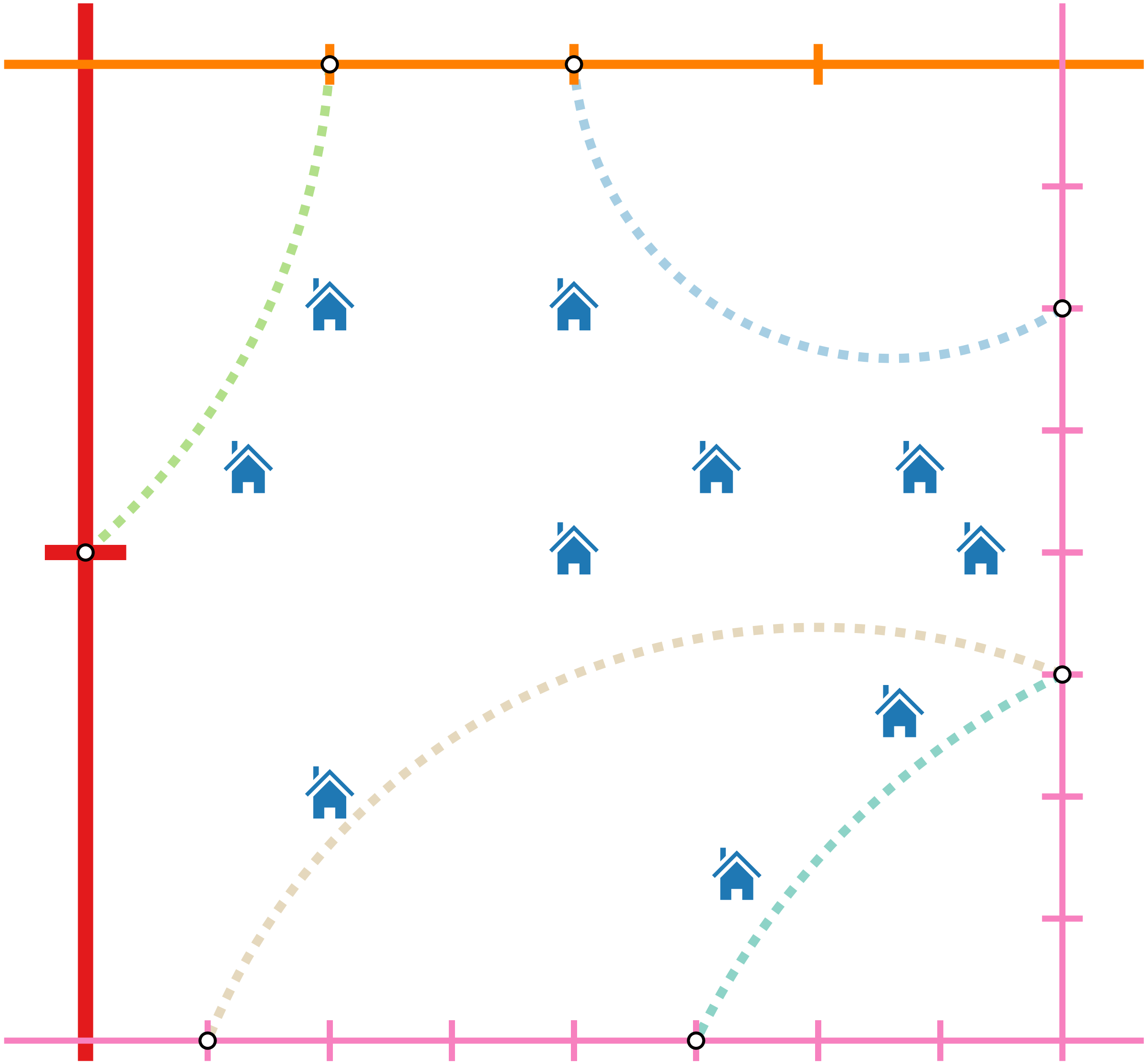

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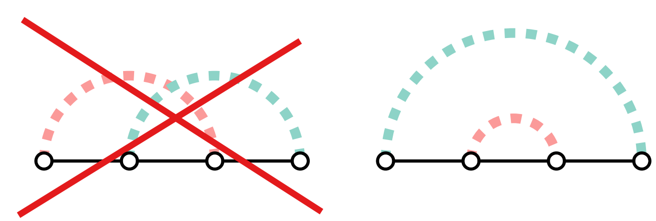

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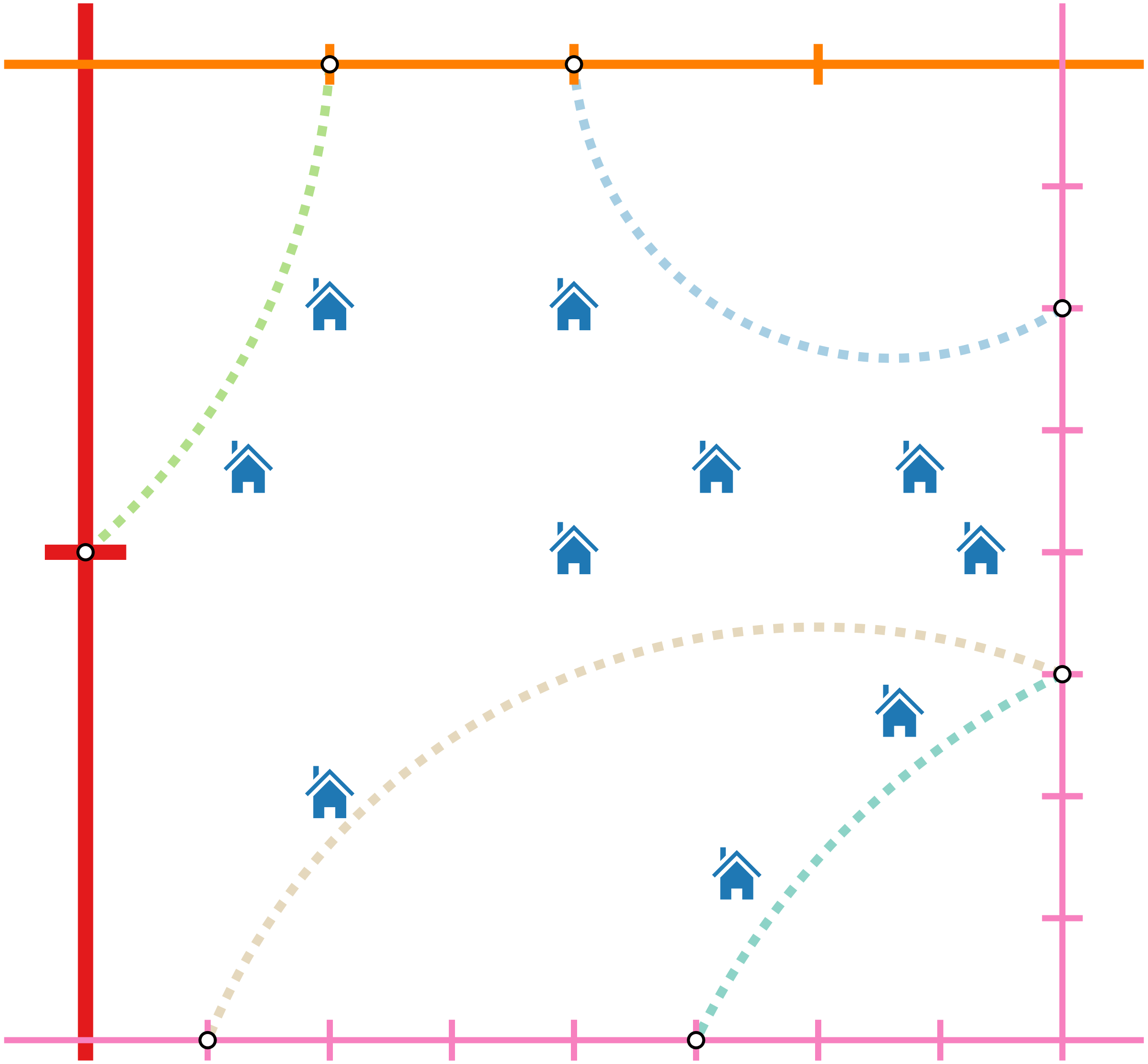

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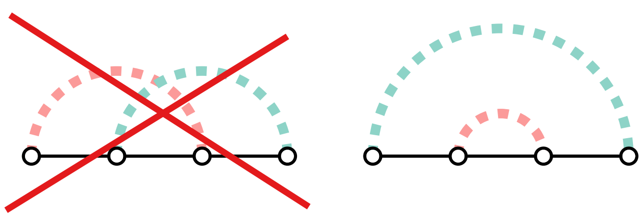
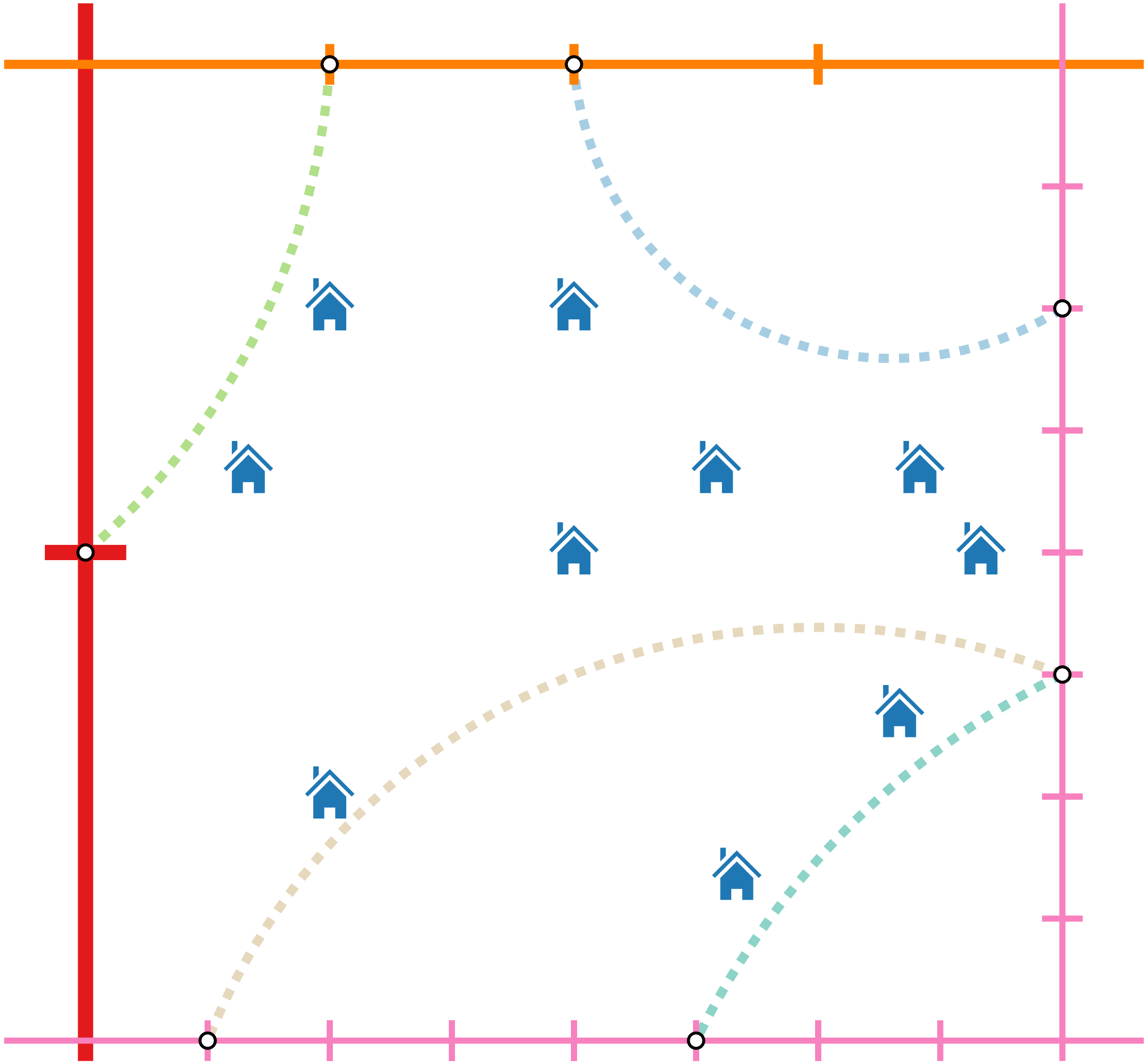


$\Rightarrow$ at most

crossing-free pairings

Dynamic Program (I)



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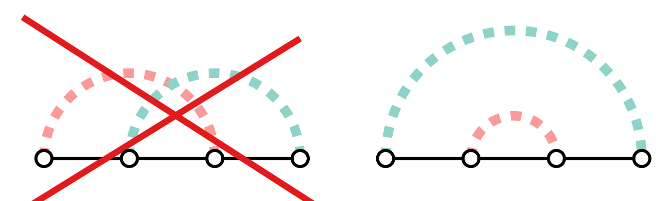
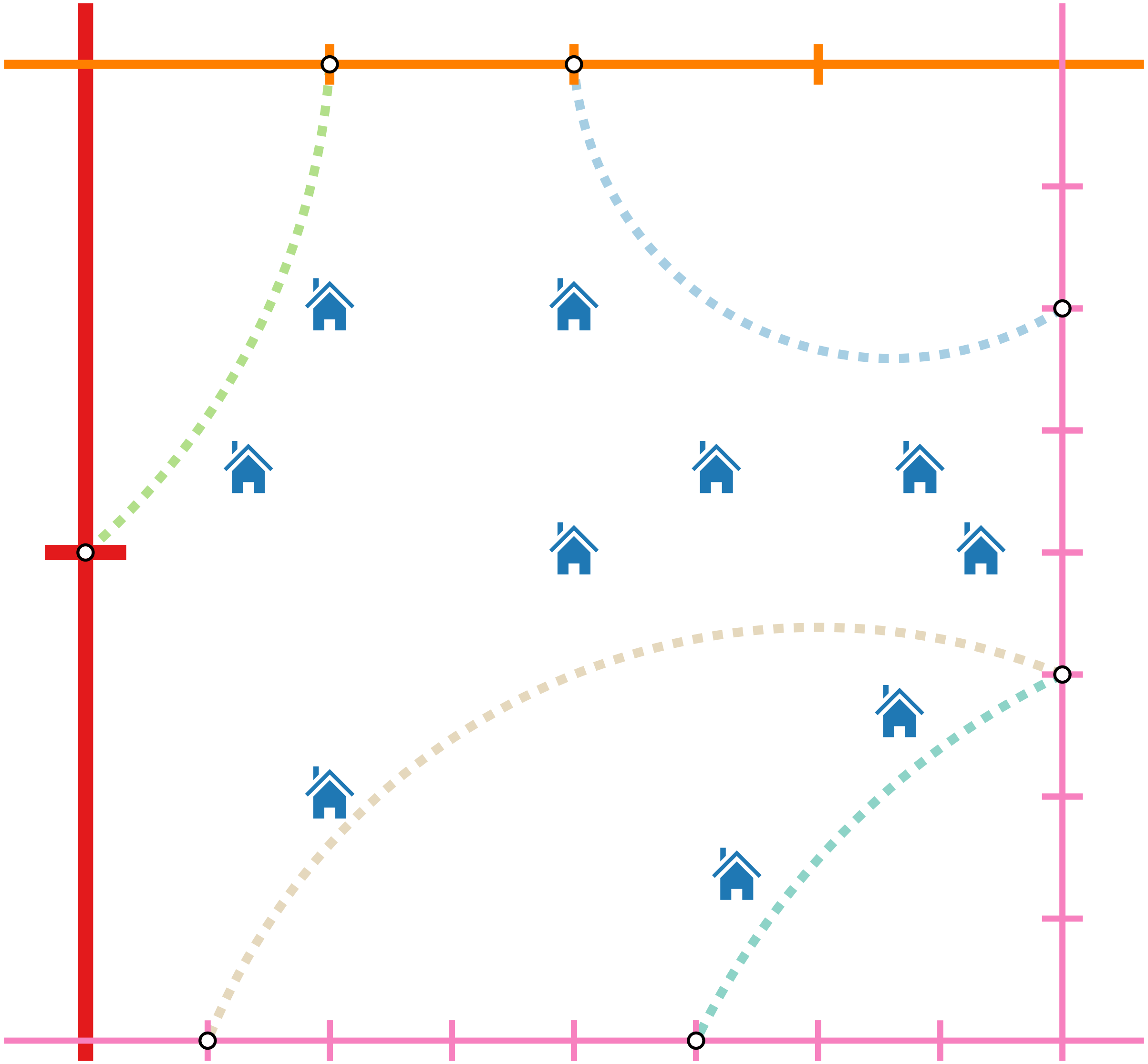


crossing-free pairings

$\Rightarrow$ at most $\underbrace{\qquad\qquad}_{\#\text{visit vectors}} \times \underbrace{\qquad\qquad}_{\#\text{realizable pairings}}$

Dynamic Program (I)



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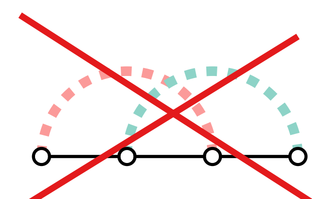
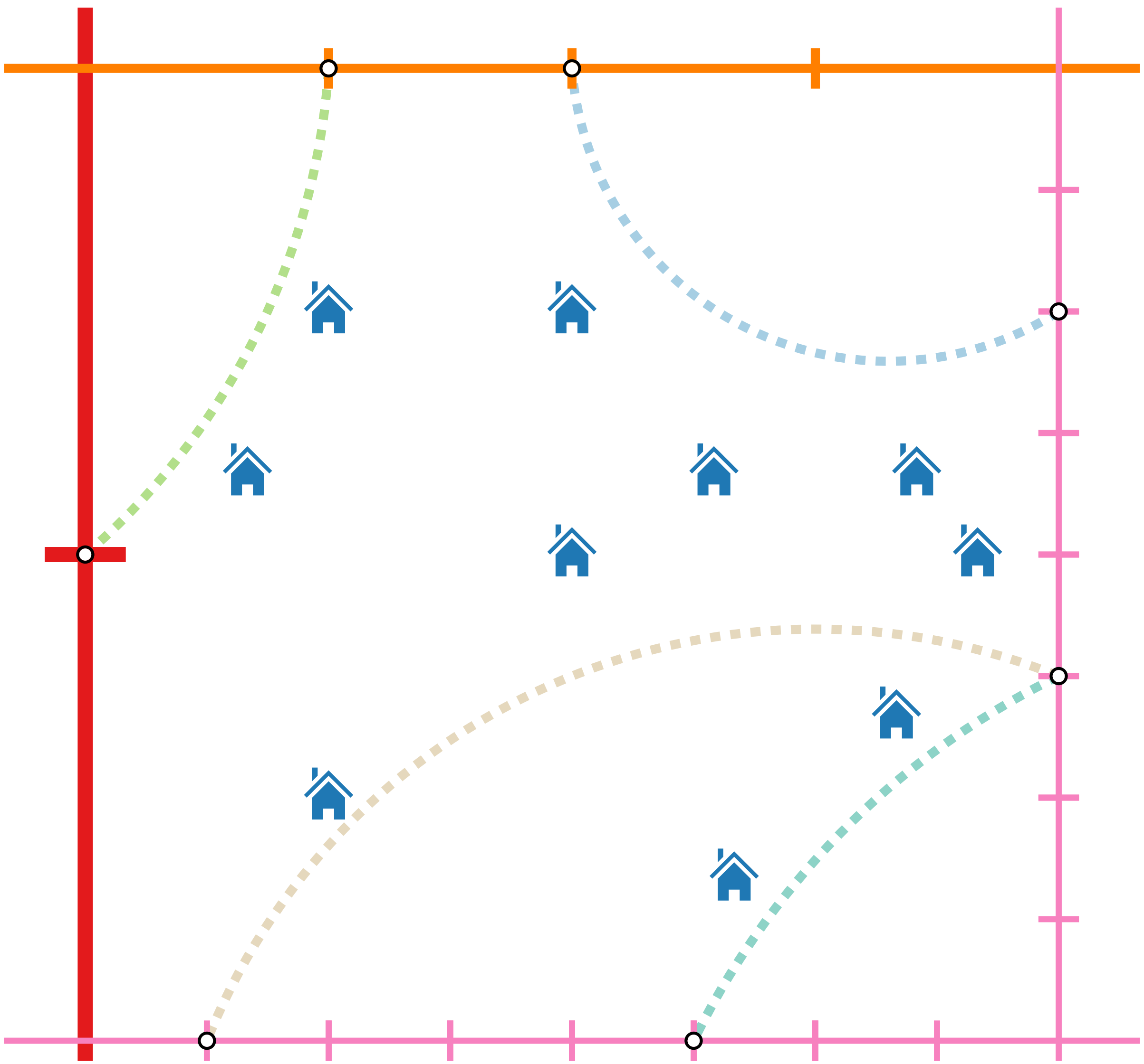


crossing-free pairings

$$\Rightarrow \text{at most } \underbrace{n^{O(1/\varepsilon)}}_{\#\text{visit vectors}} \times \underbrace{\qquad}_{\#\text{realizable pairings}}$$

Dynamic Program (I)



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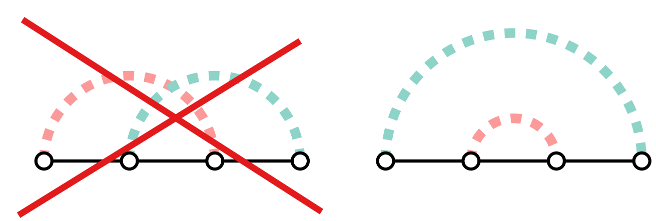
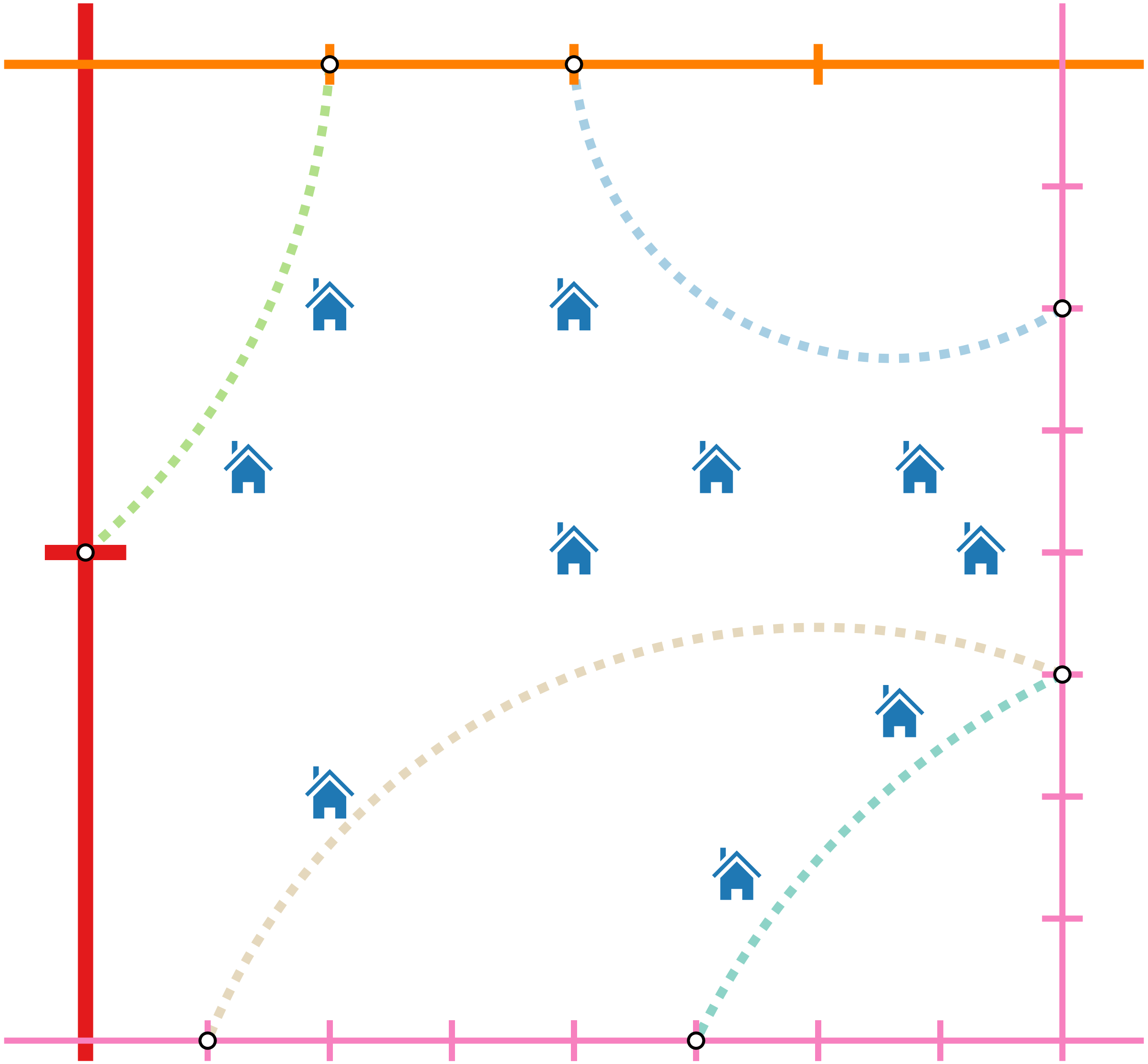


crossing-free pairings

$\Rightarrow$ at most $\underbrace{n^{O(1/\varepsilon)}}_{\#\text{visit vectors}} \times \underbrace{\qquad\qquad}_{\#\text{realizable pairings}}$ $= \#$ well-formed expr. with $\leq 4m$ pairs of parentheses

Dynamic Program (I)



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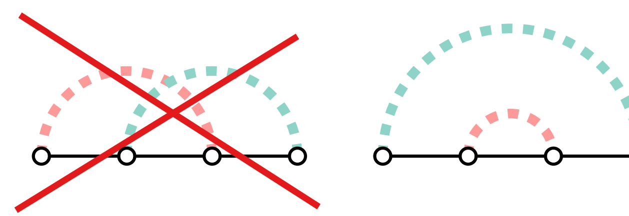
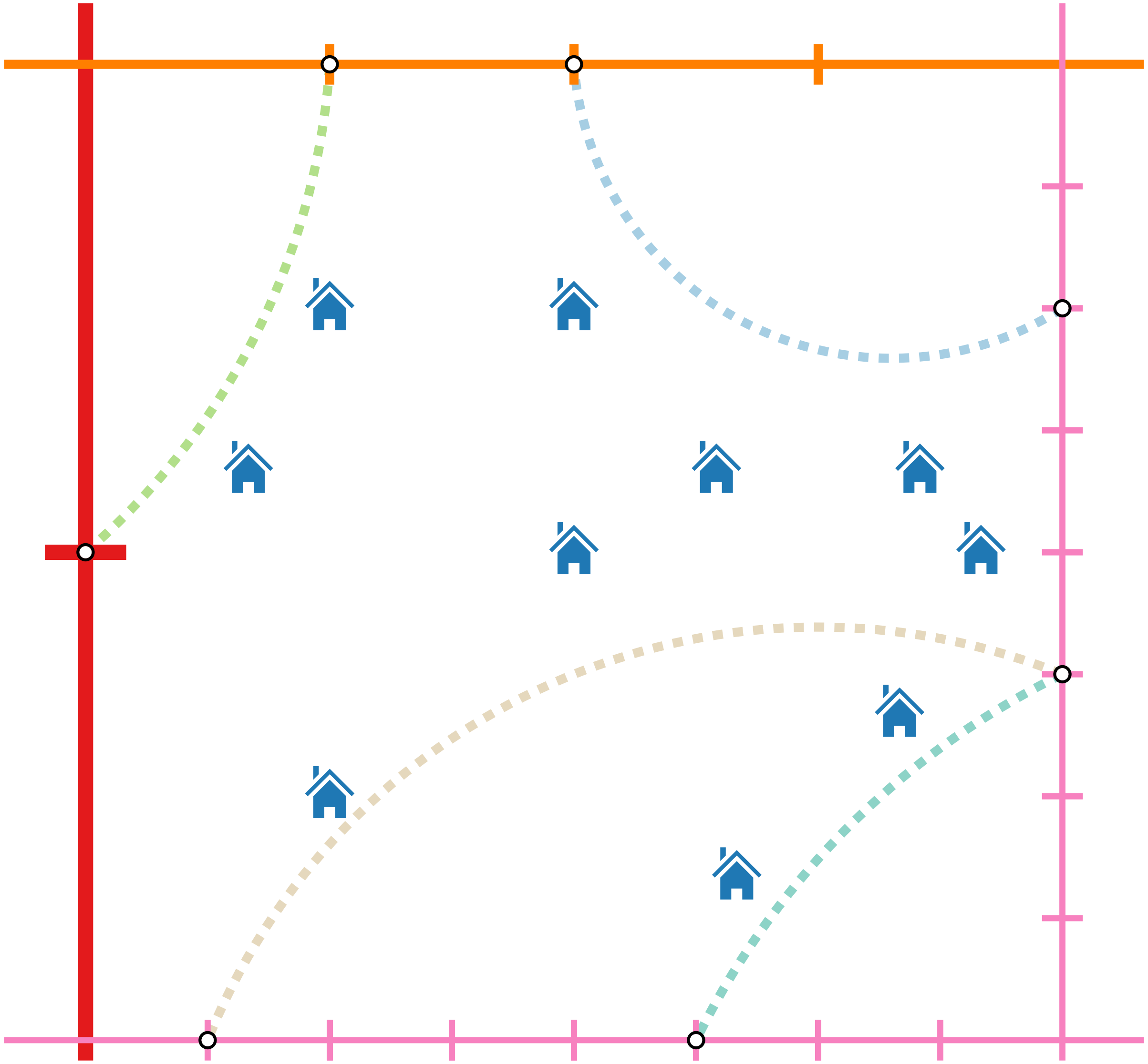


$\Rightarrow$ at most $\underbrace{n^{O(1/\varepsilon)}}_{\#\text{visit vectors}} \times \underbrace{2^{O(m)}}_{\#\text{realizable pairings}} =$ crossing-free pairings

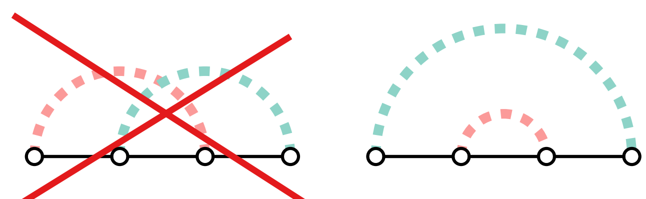
#realizable pairings $=$ # well-formed expr. with $\leq 4m$ pairs of parentheses

Dynamic Program (I)



Each well-behaved tour induces the following in each square Q of the dissection:

- a path cover of the houses in Q,
- ...such that each portal of Q is visited 0, 1 or 2 times,
- a **crossing-free pairing** of the visited portals.



$$\Rightarrow \text{at most } \underbrace{n^{O(1/\varepsilon)}}_{\#\text{visit vectors}} \times \underbrace{2^{O(m)}}_{\#\text{realizable pairings}} = n^{O(1/\varepsilon)} \text{ crossing-free pairings}$$

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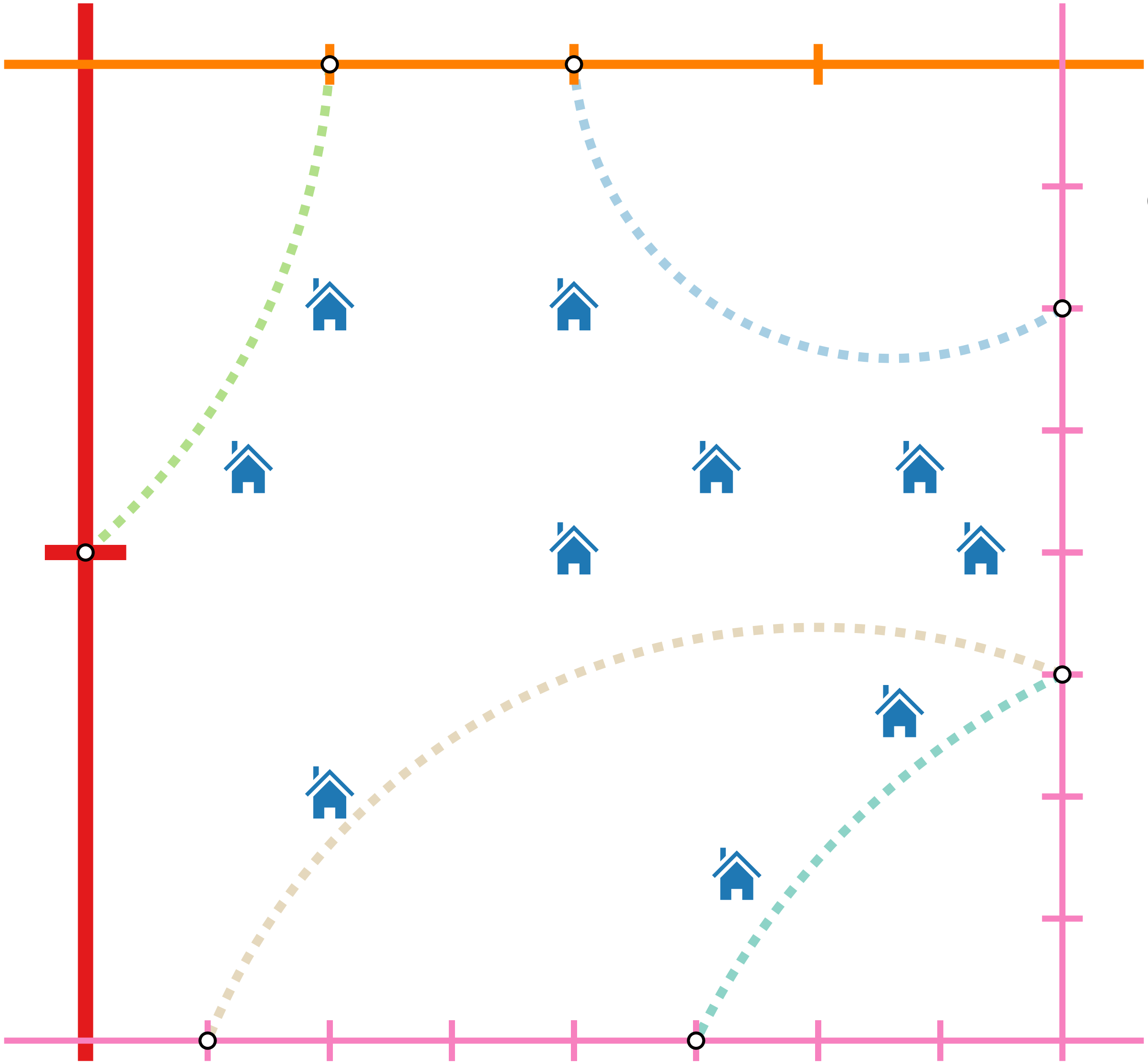
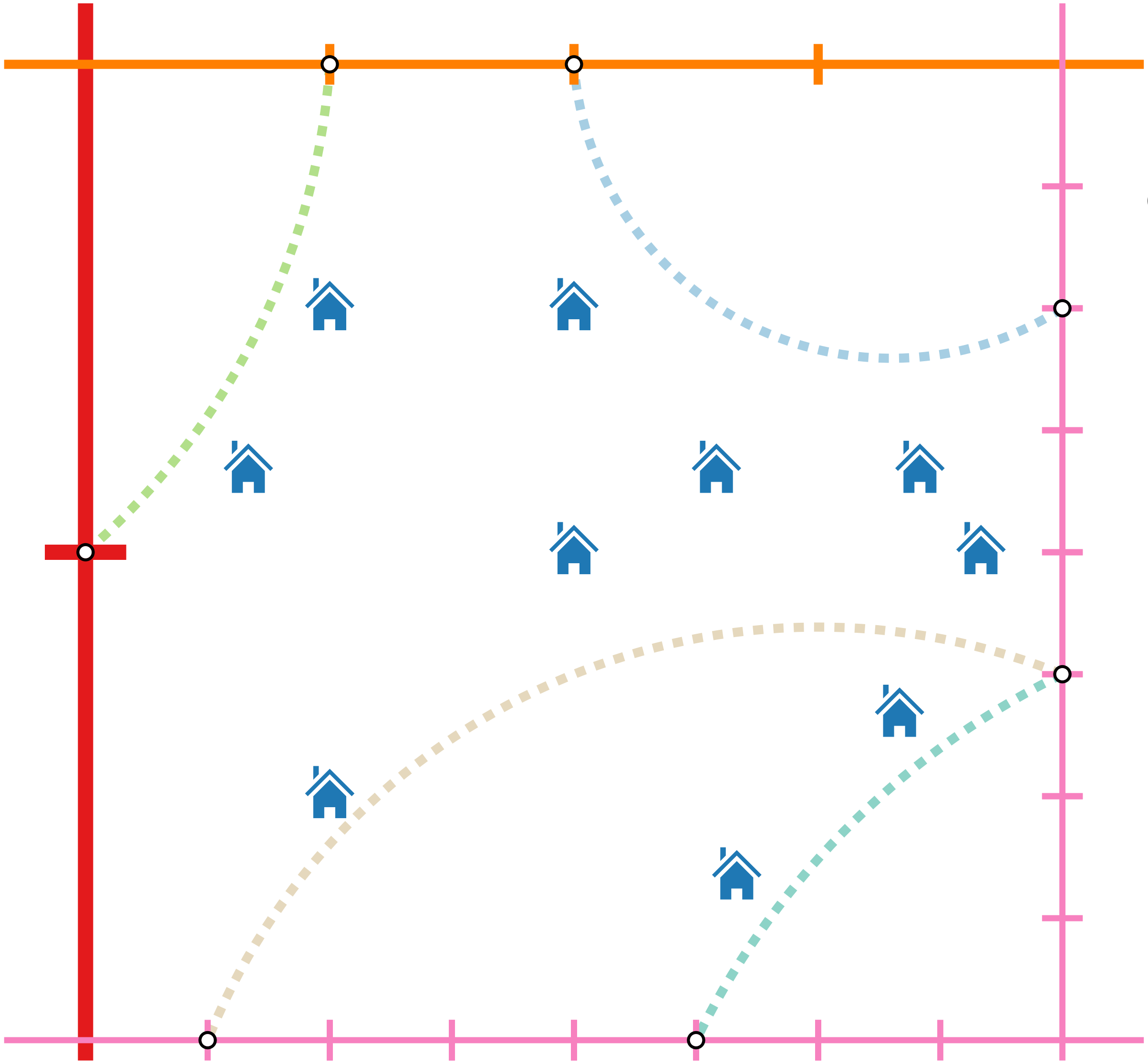
Dynamic Program (II)



Compute

Dynamic Program (II)



Compute

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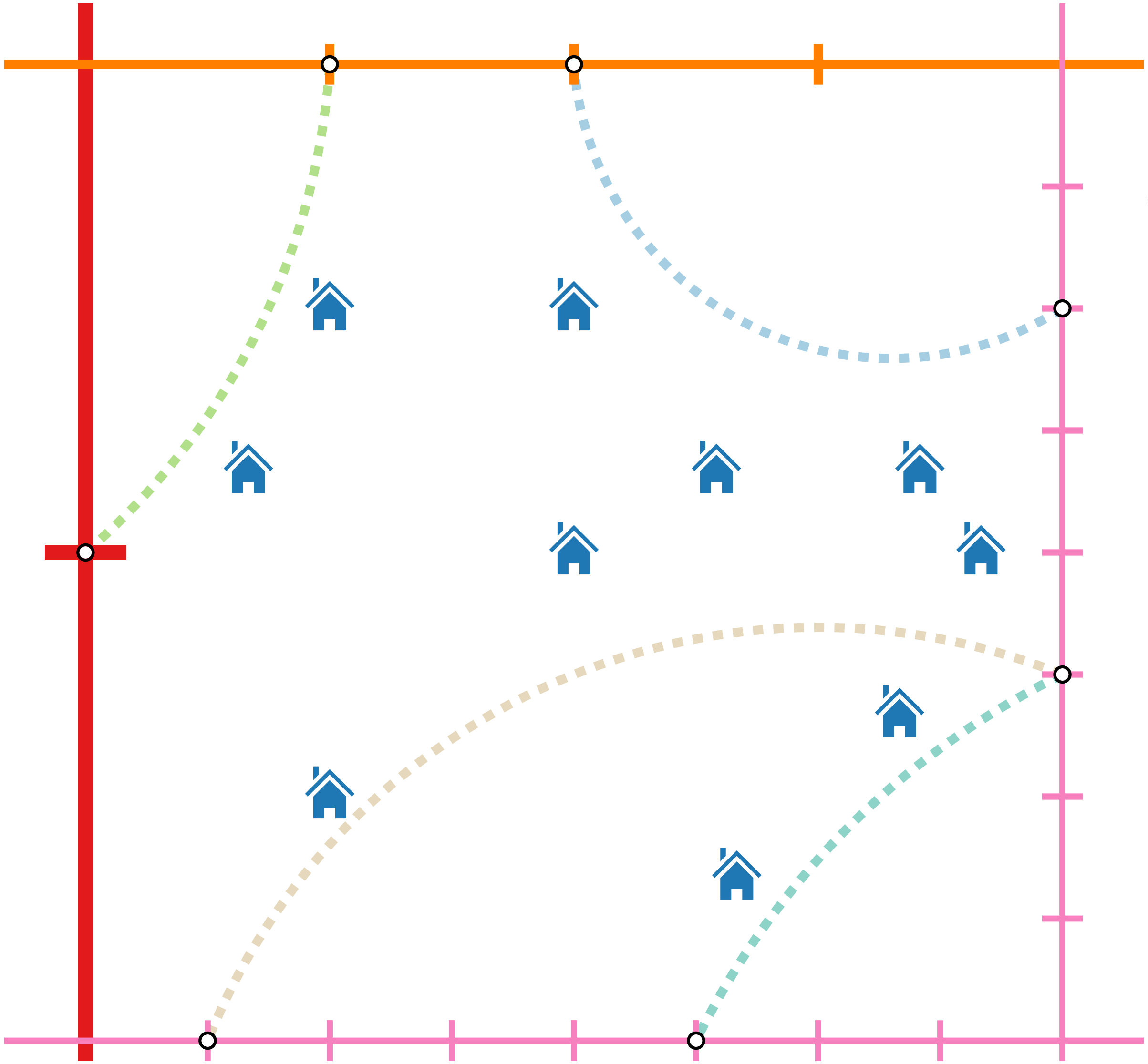
Dynamic Program (II)



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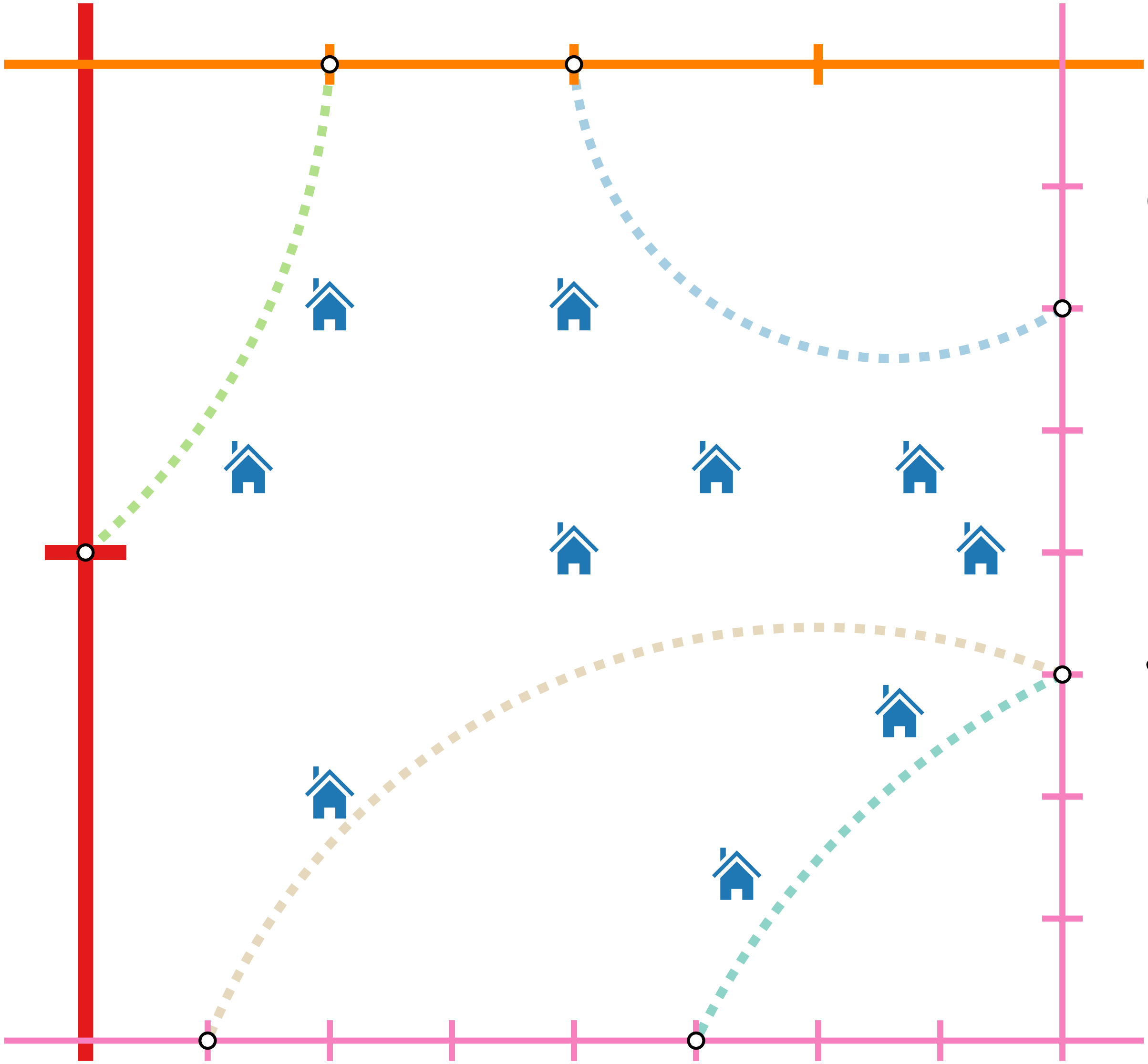

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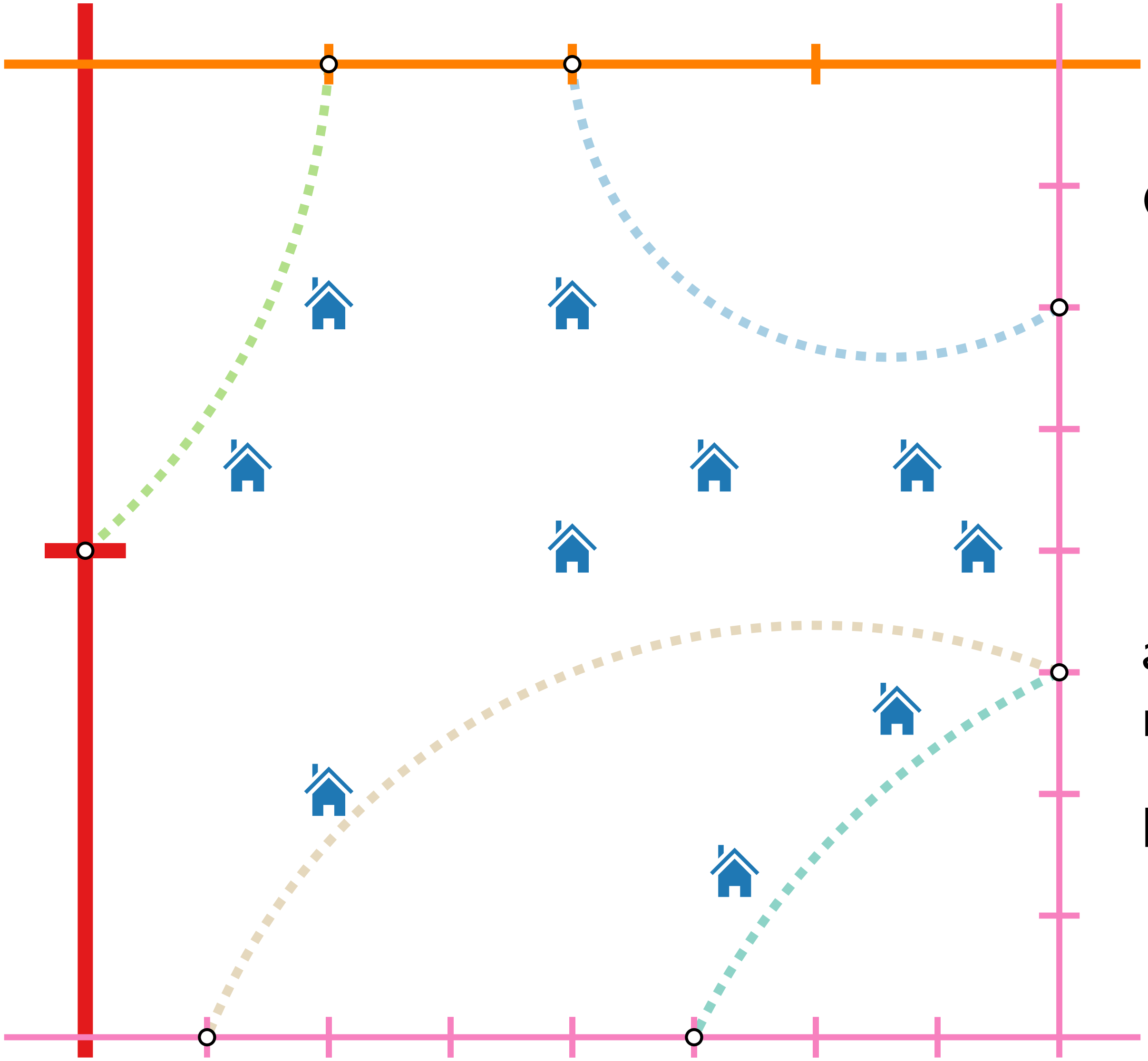

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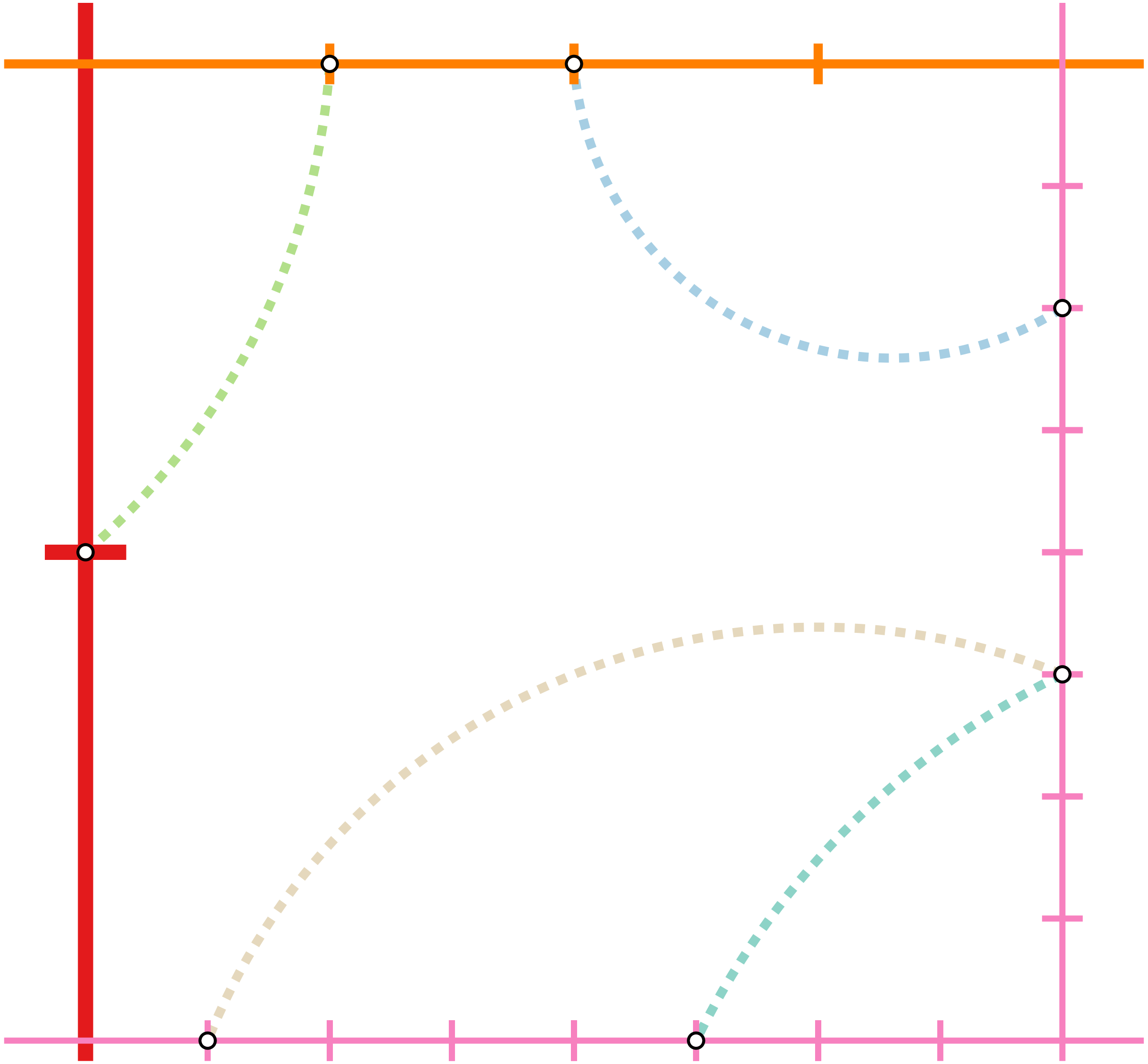
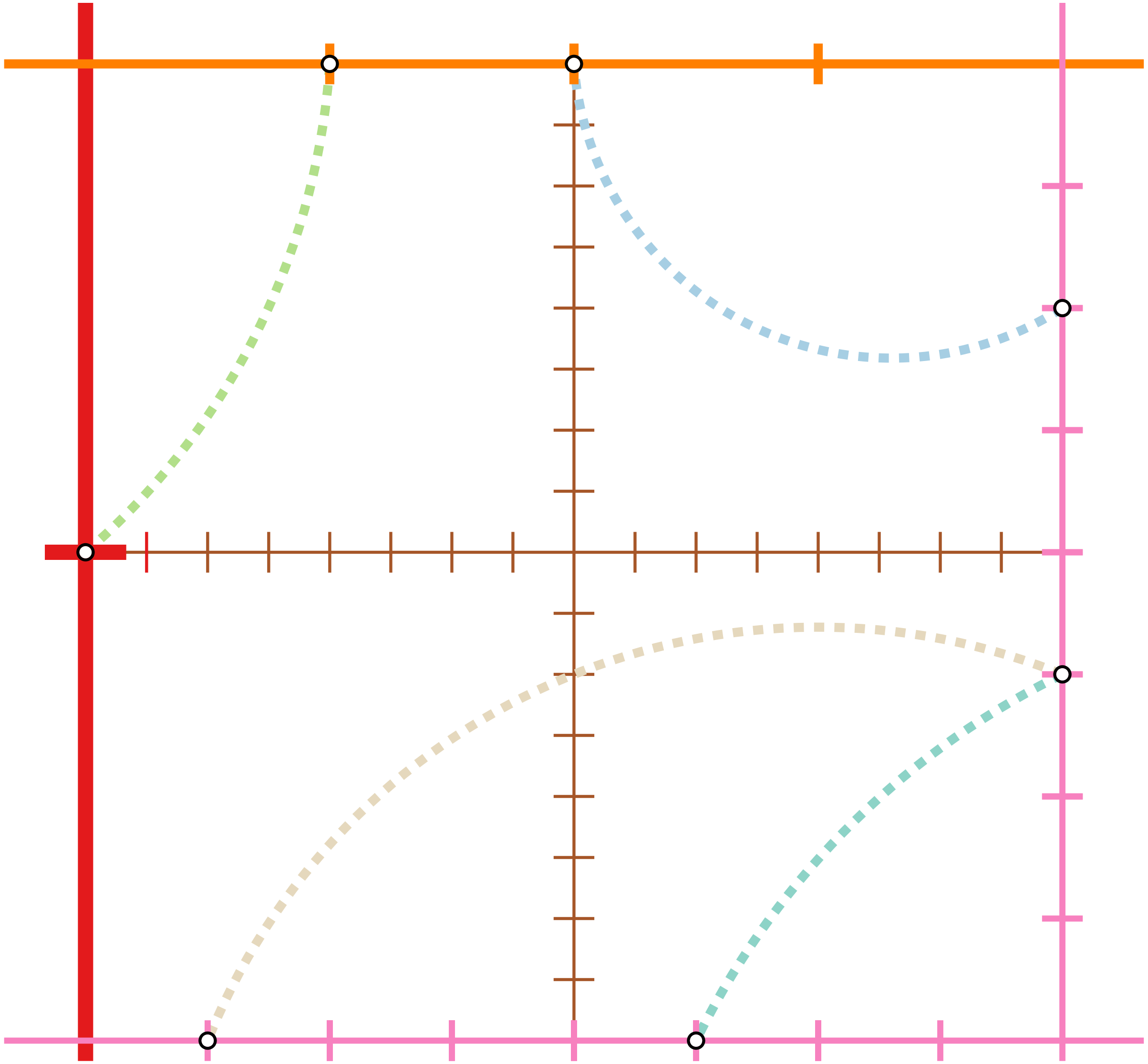
How?

Dynamic Program (III)



For a given square Q and pairing P:

Dynamic Program (III)



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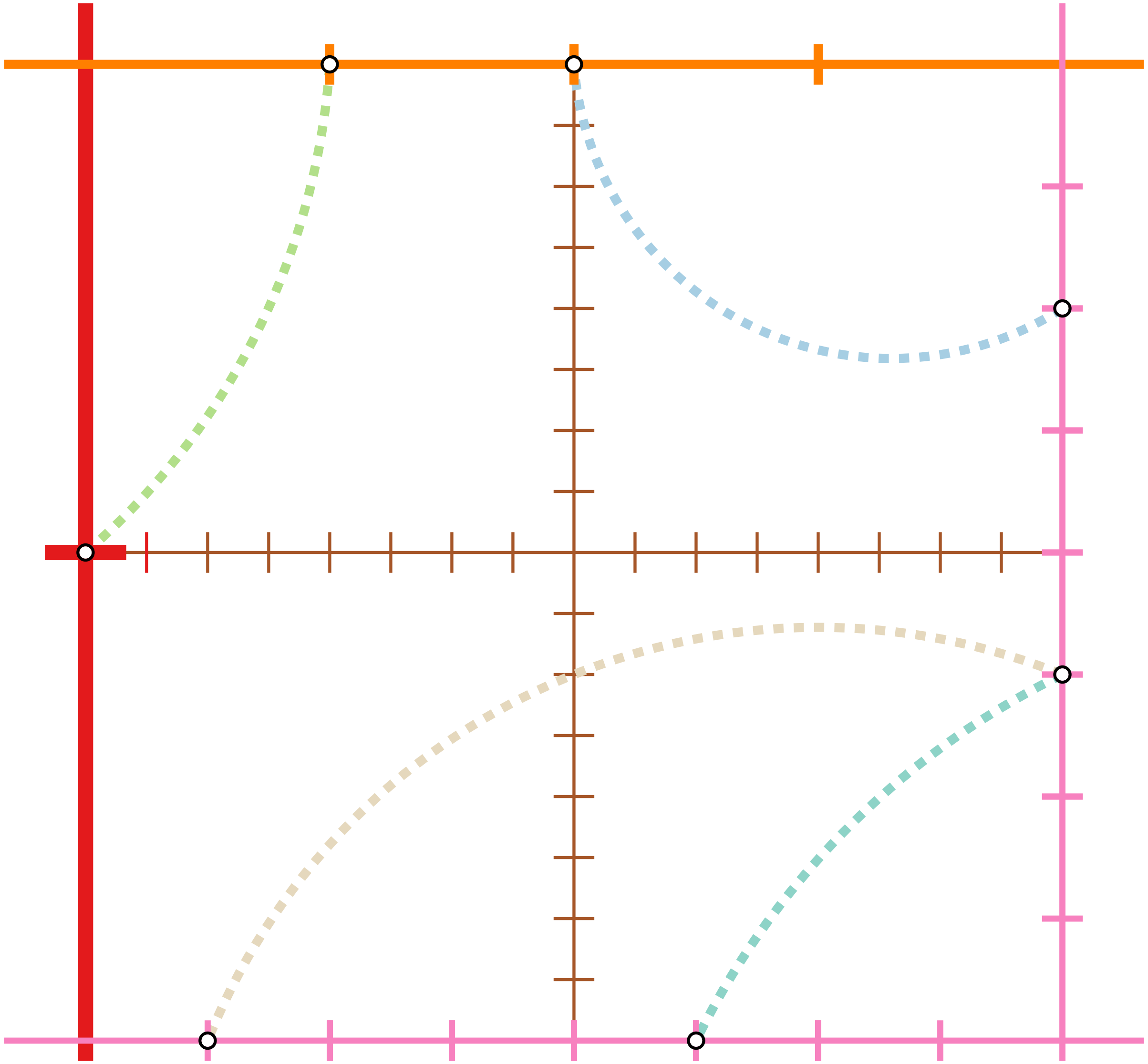

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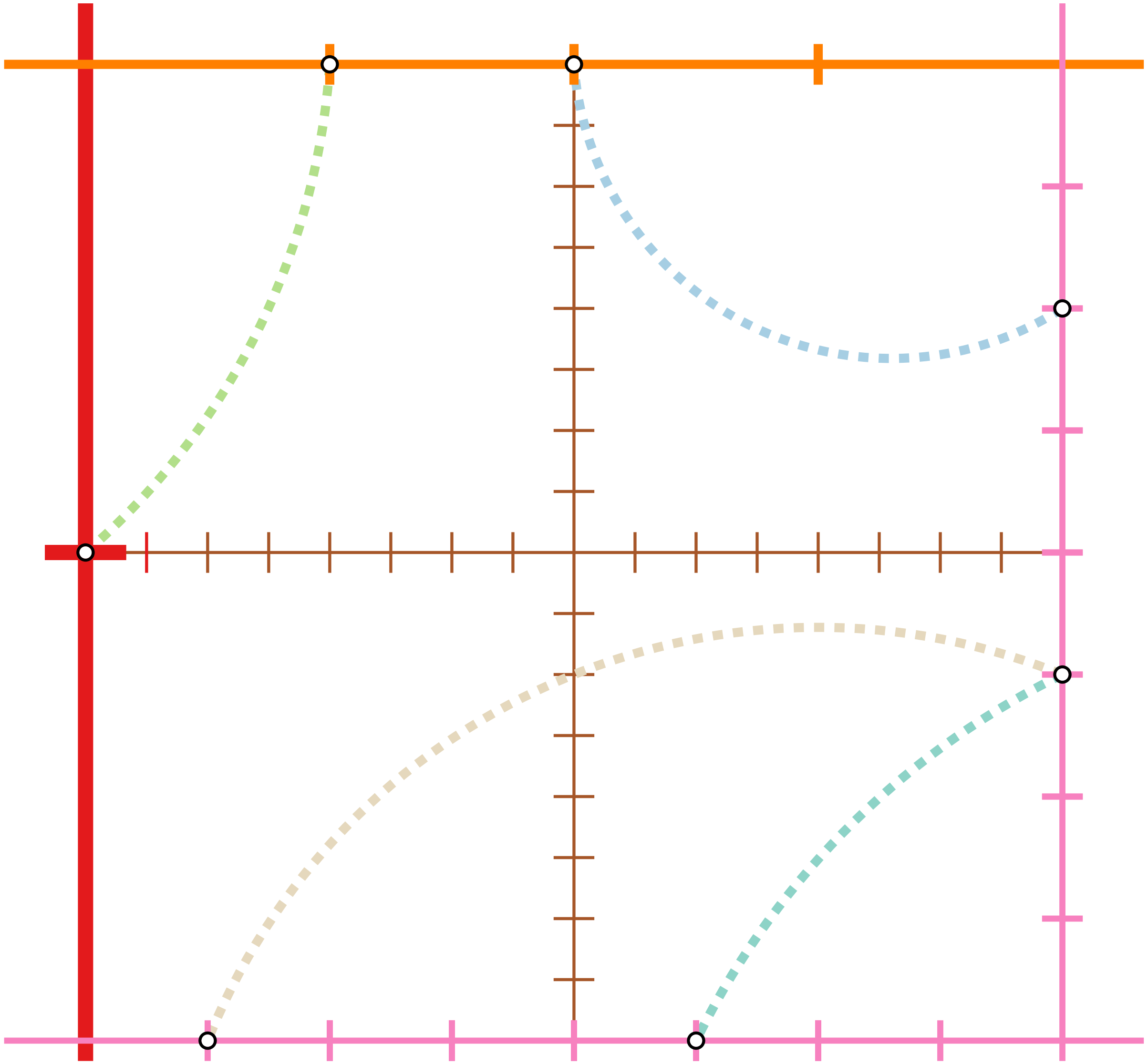
Dynamic Program (III)



For a given square Q and pairing P:

- Iterate over all $(n^{O(1/\varepsilon)})^4 =$ crossing-free pairings of the child squares.

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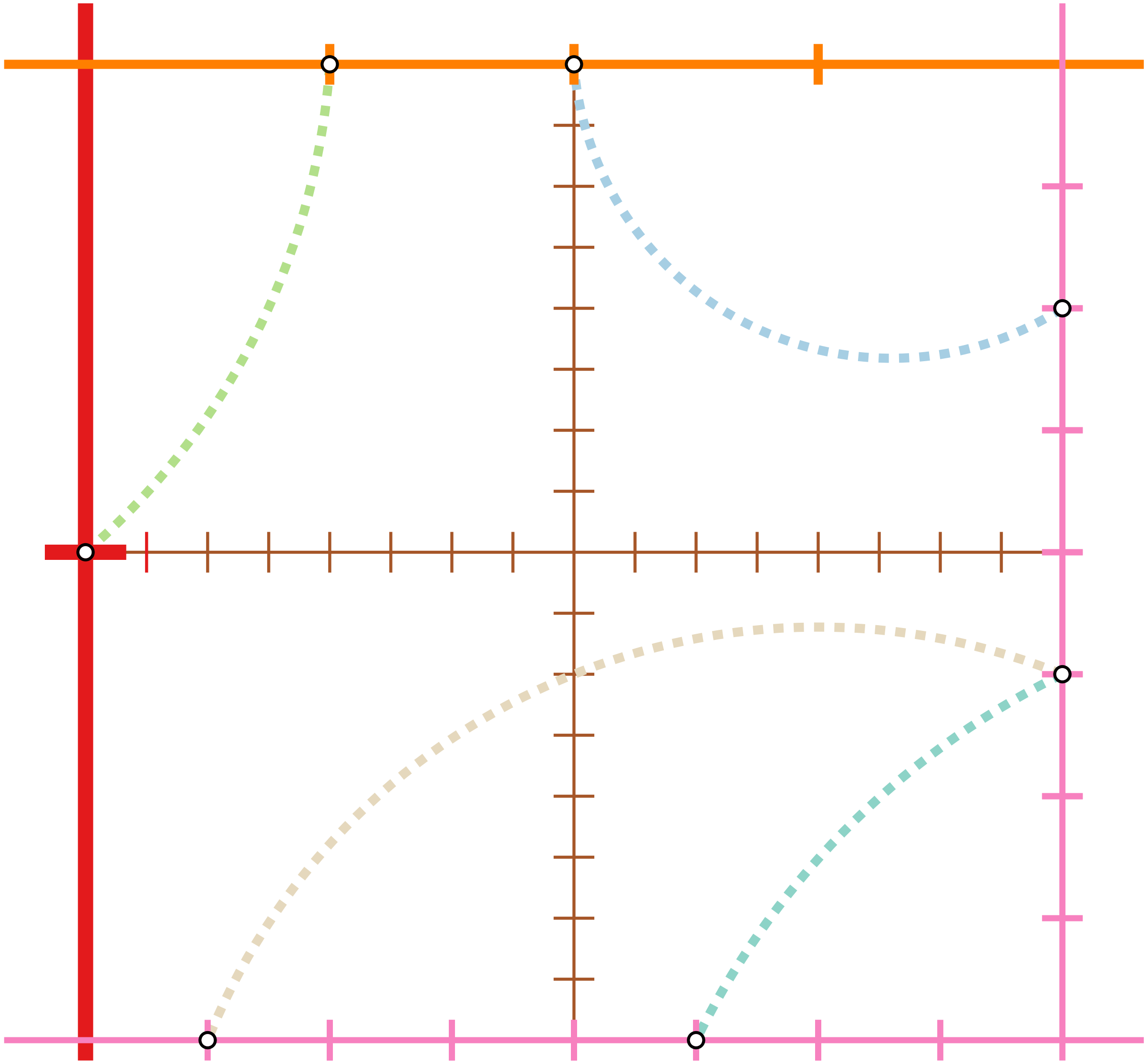

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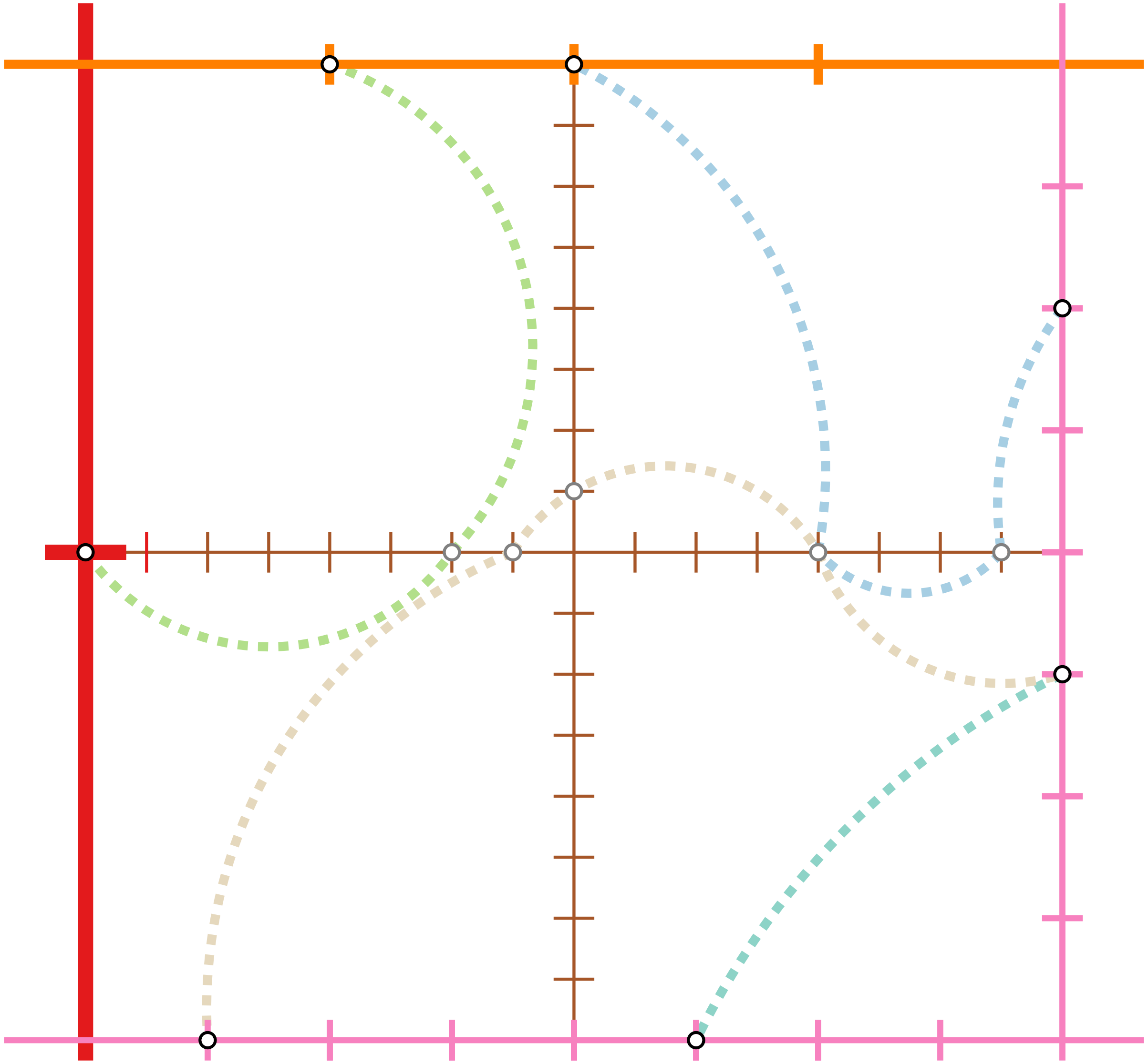

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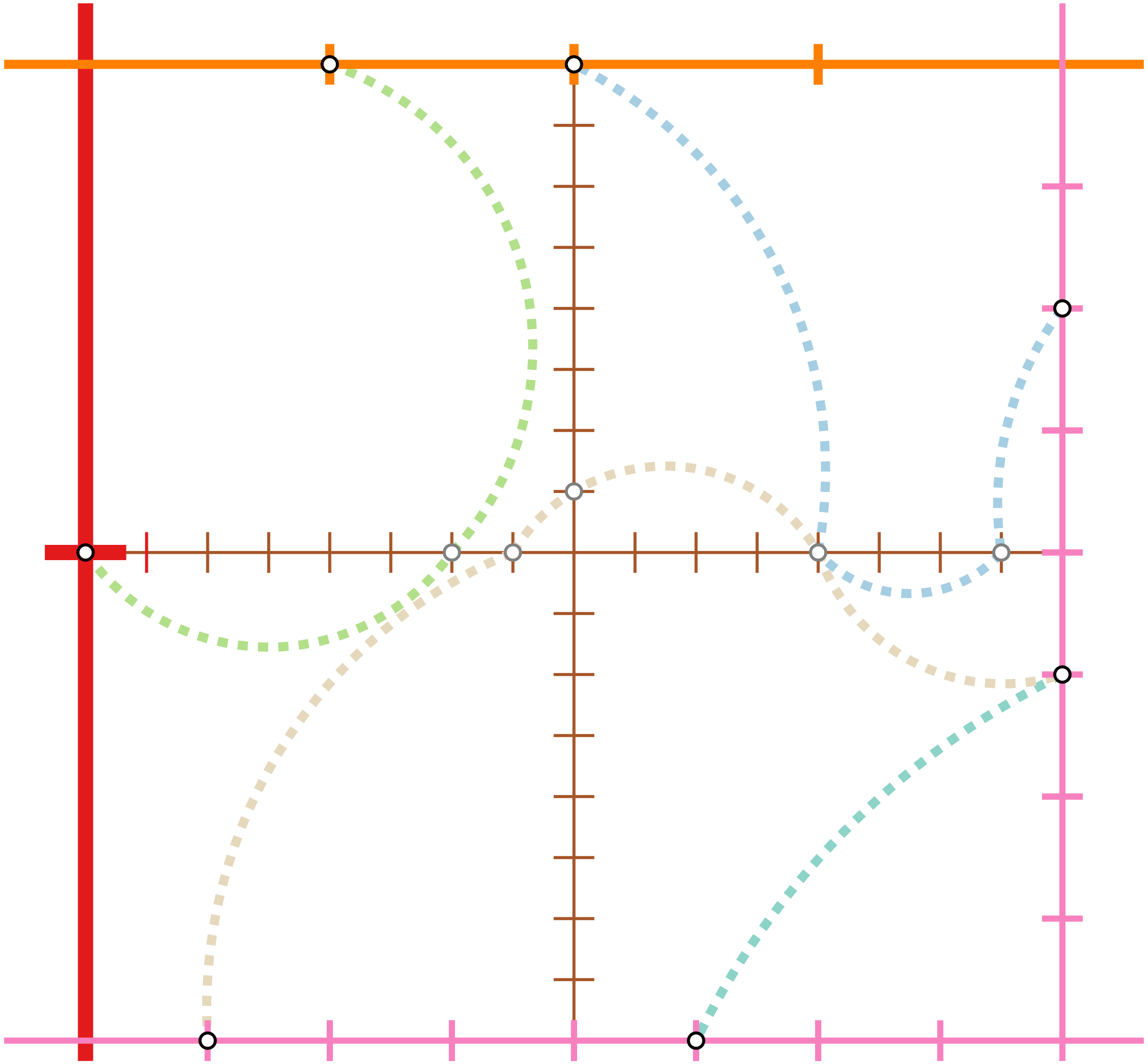

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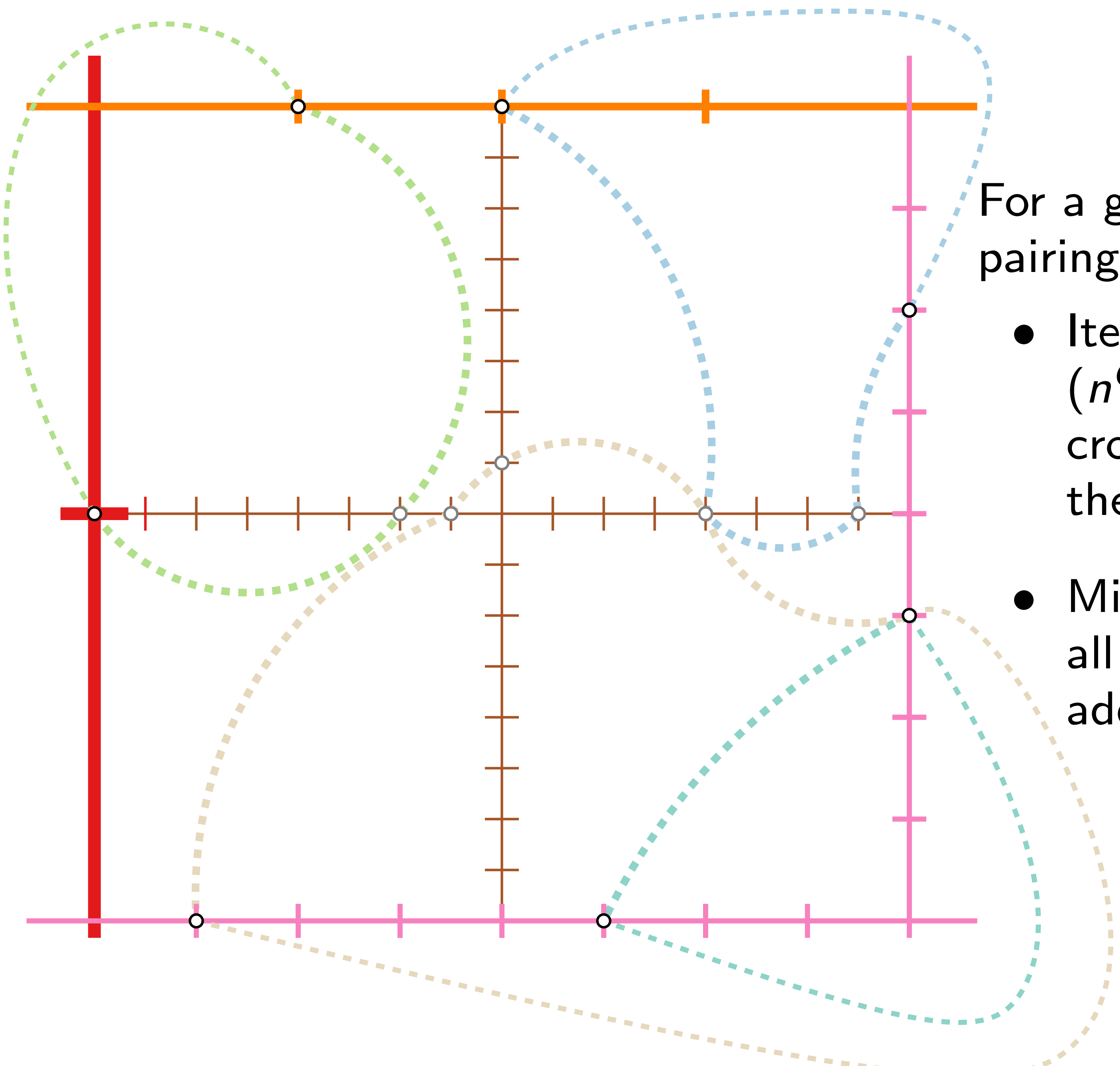

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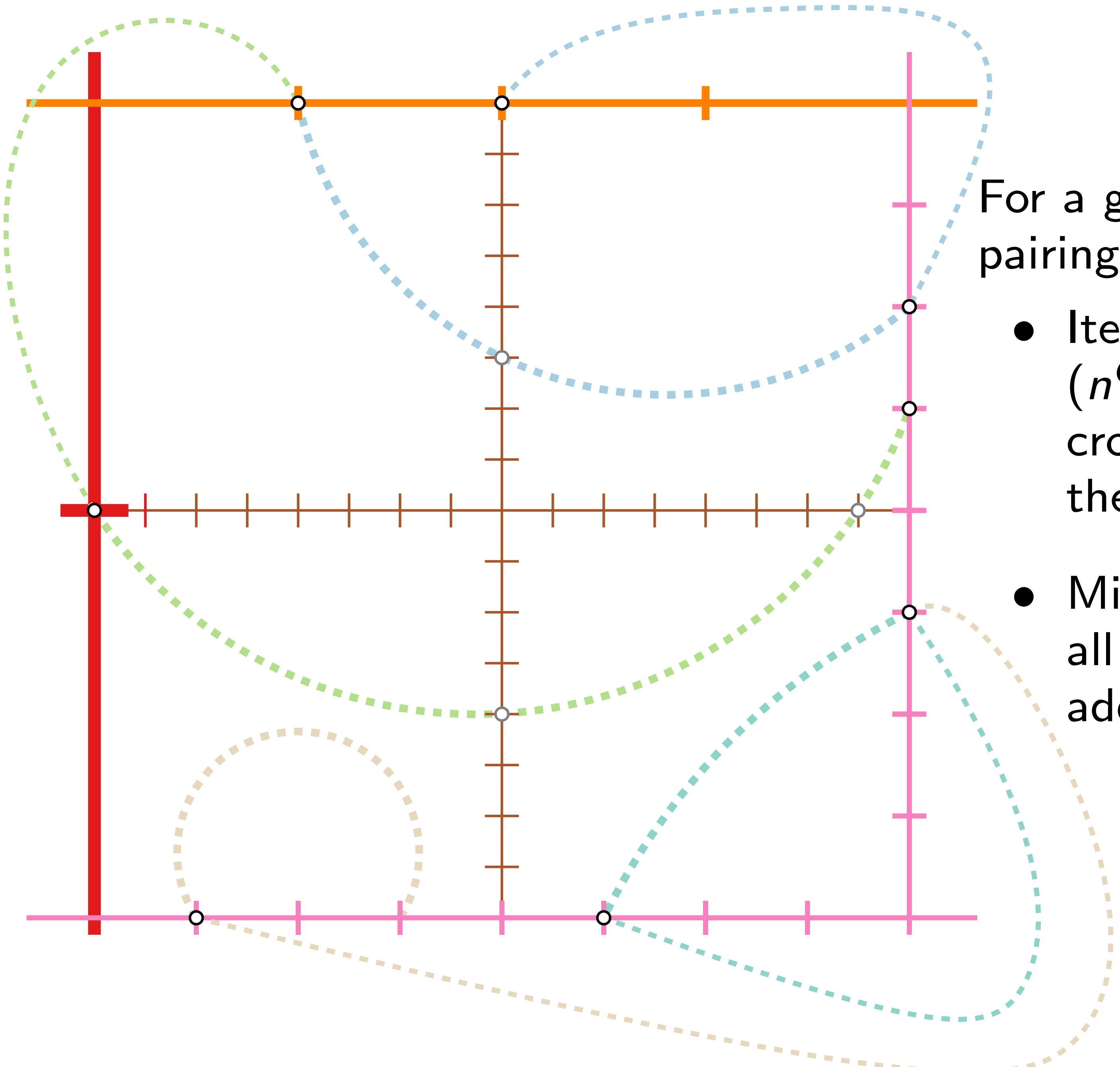

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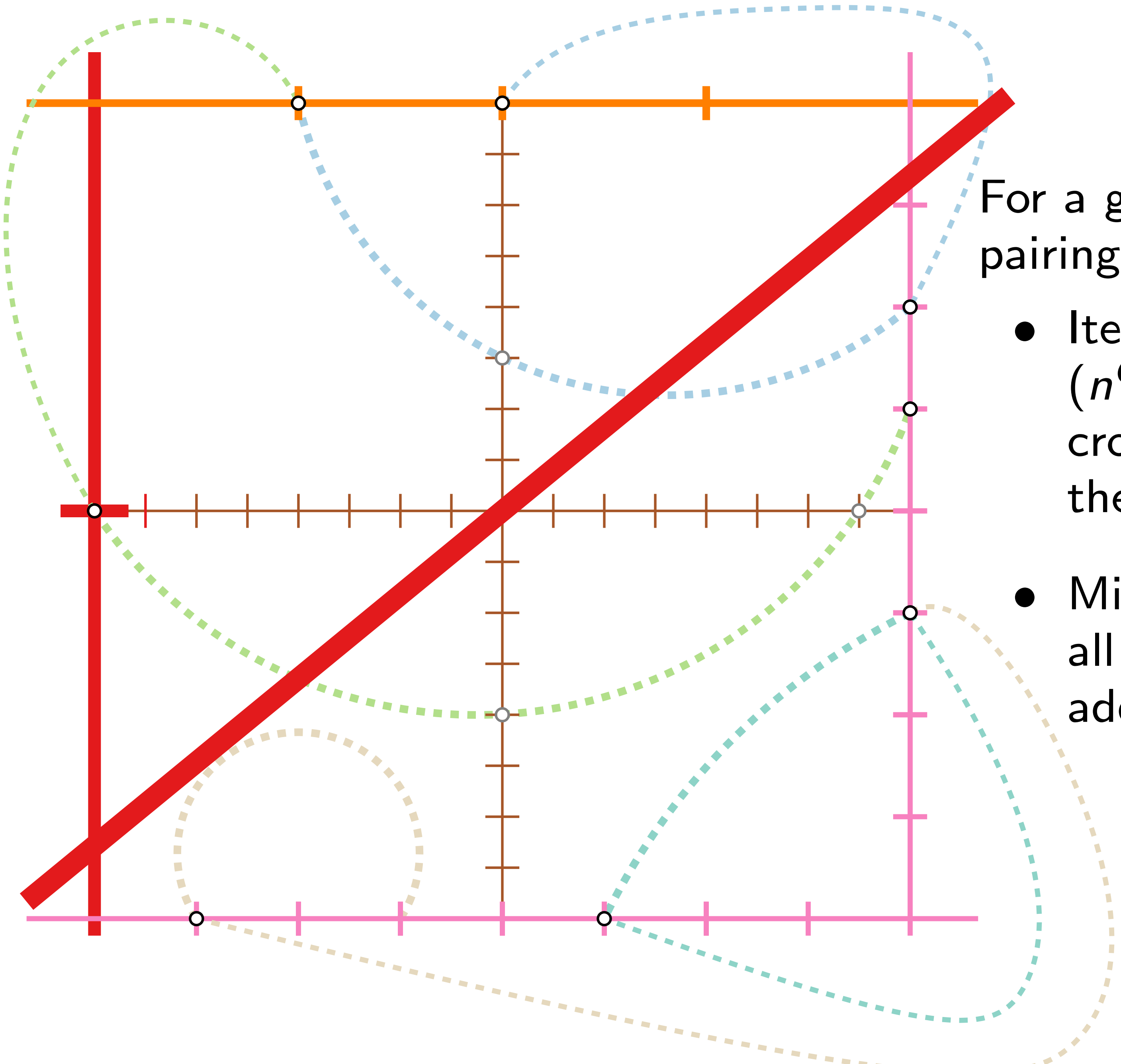

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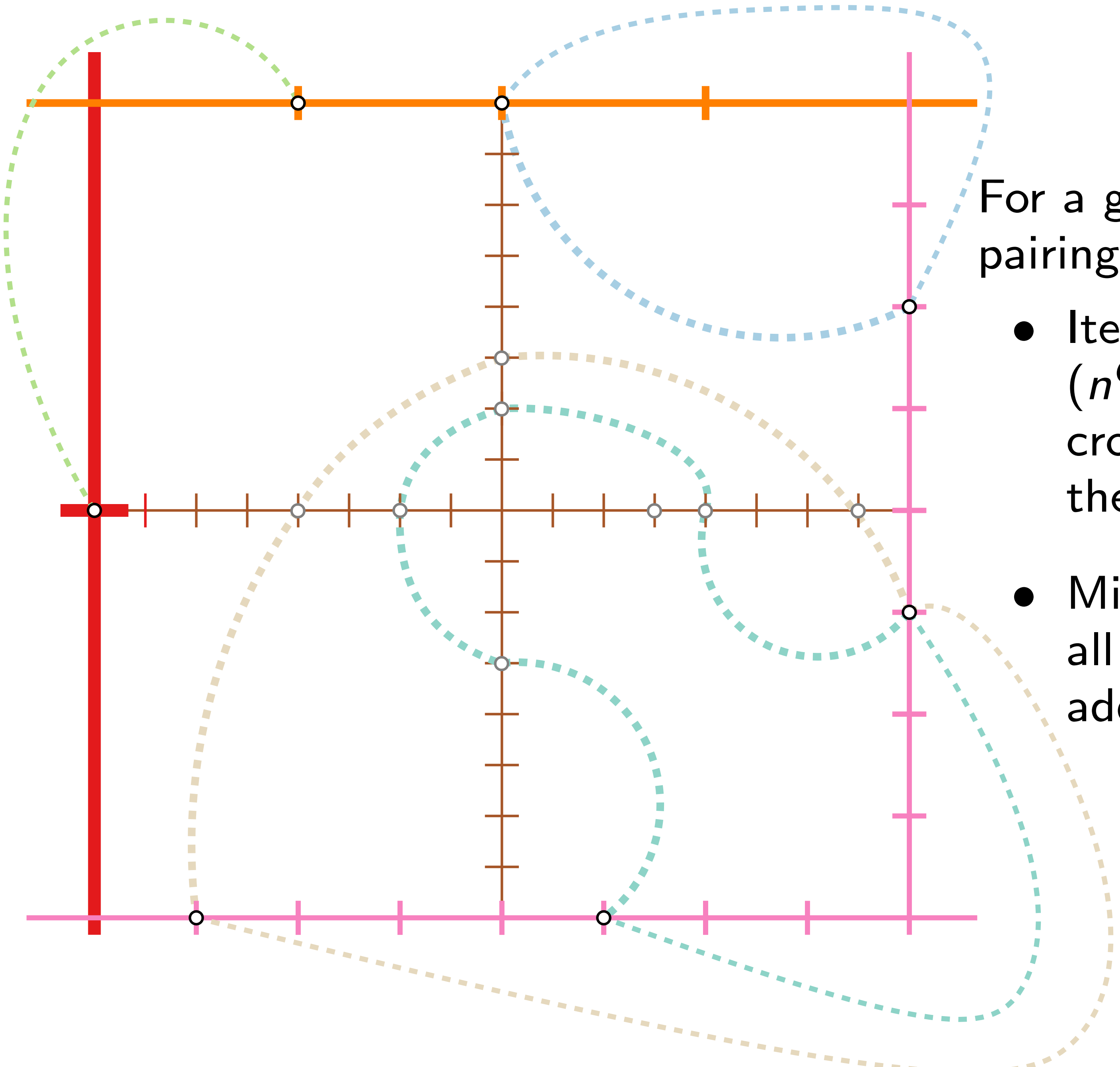

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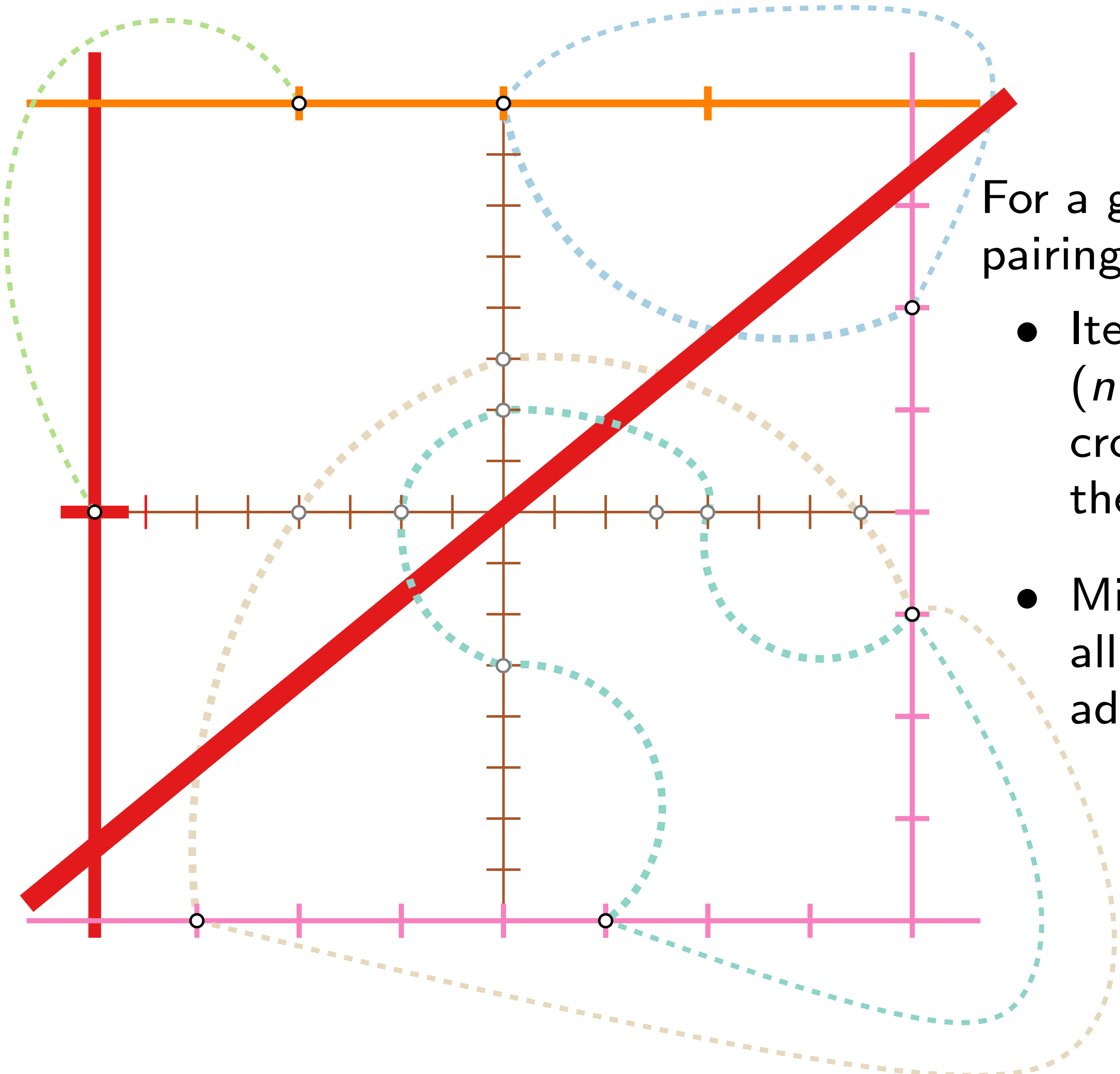

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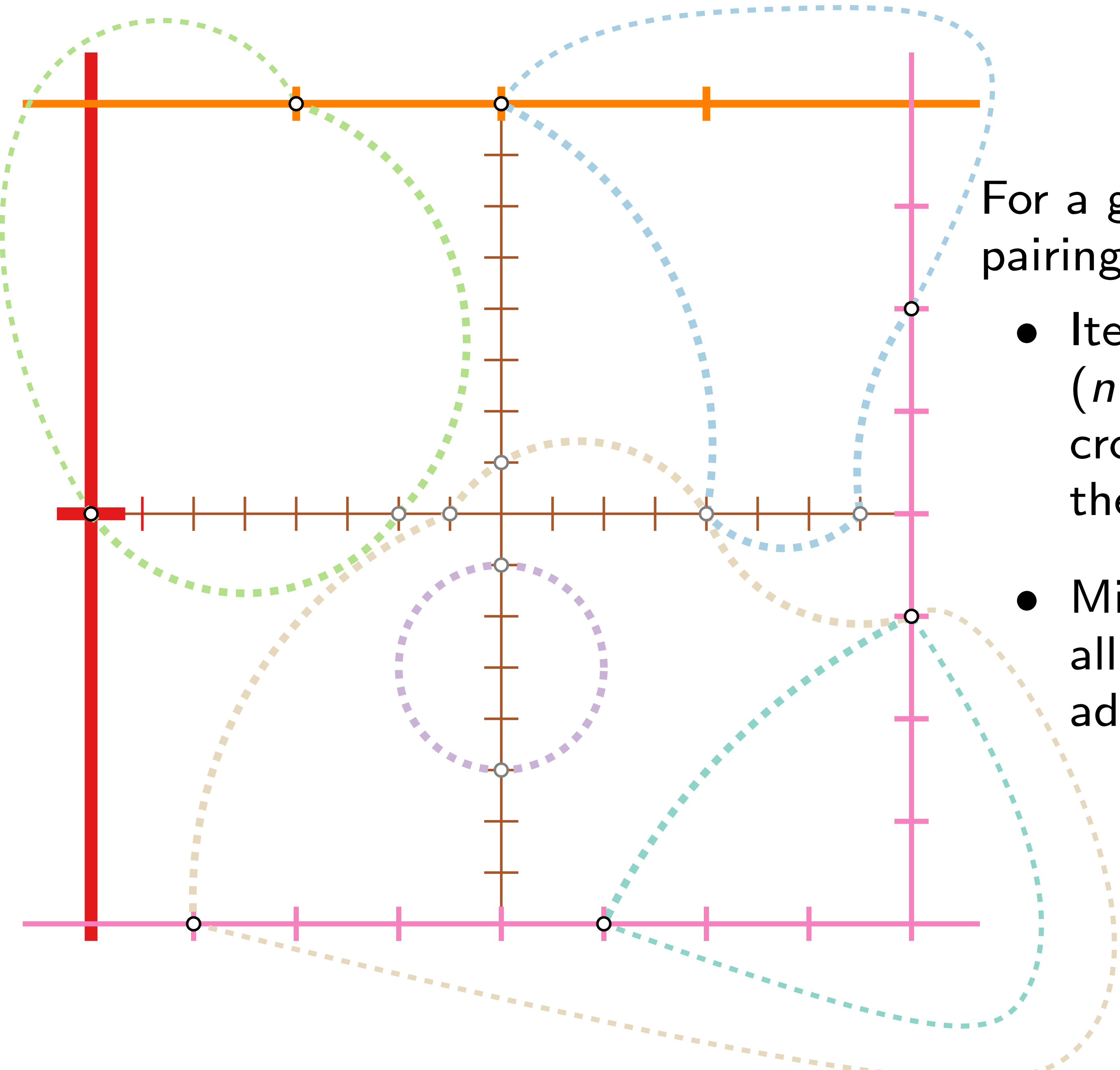

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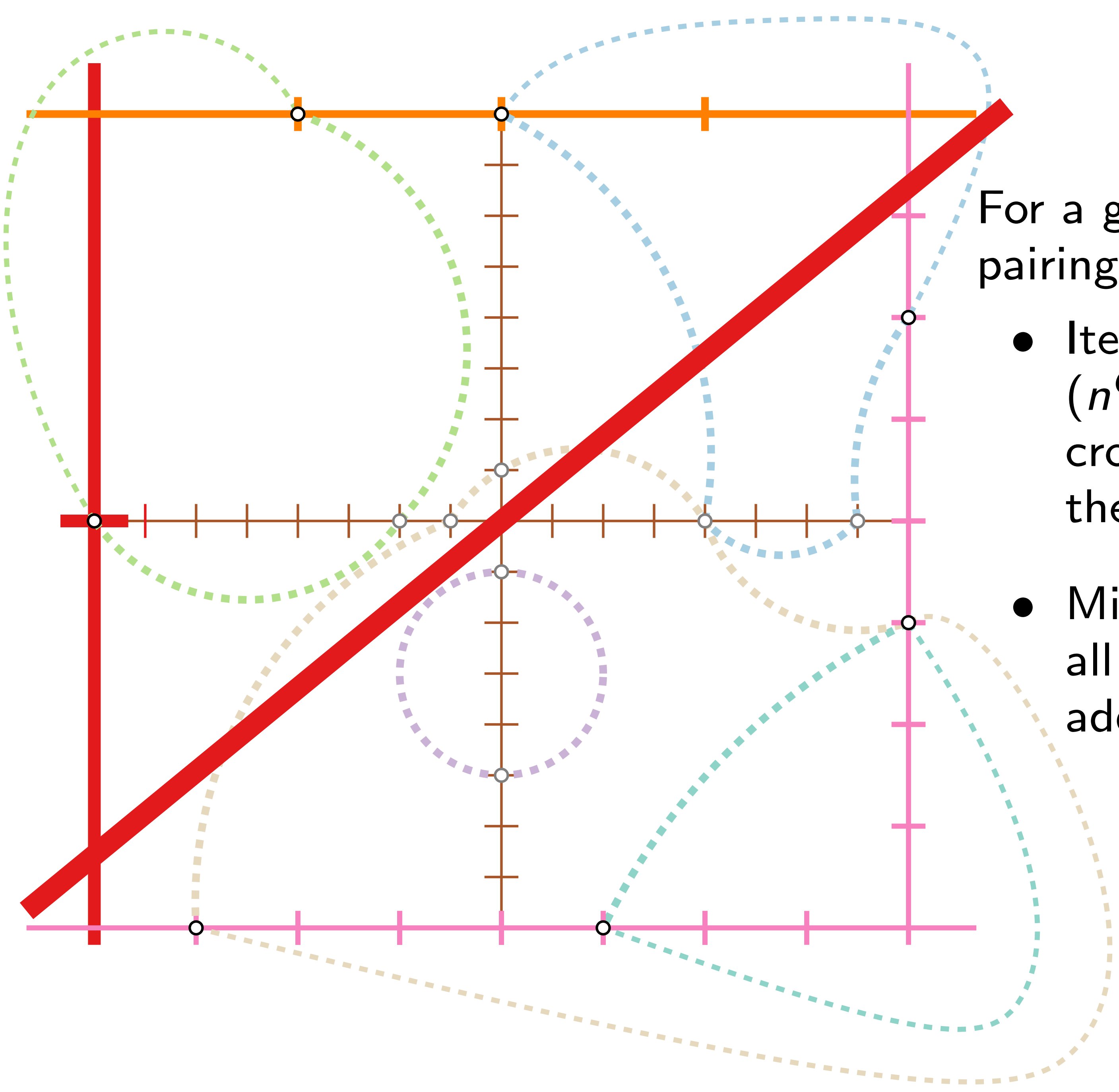

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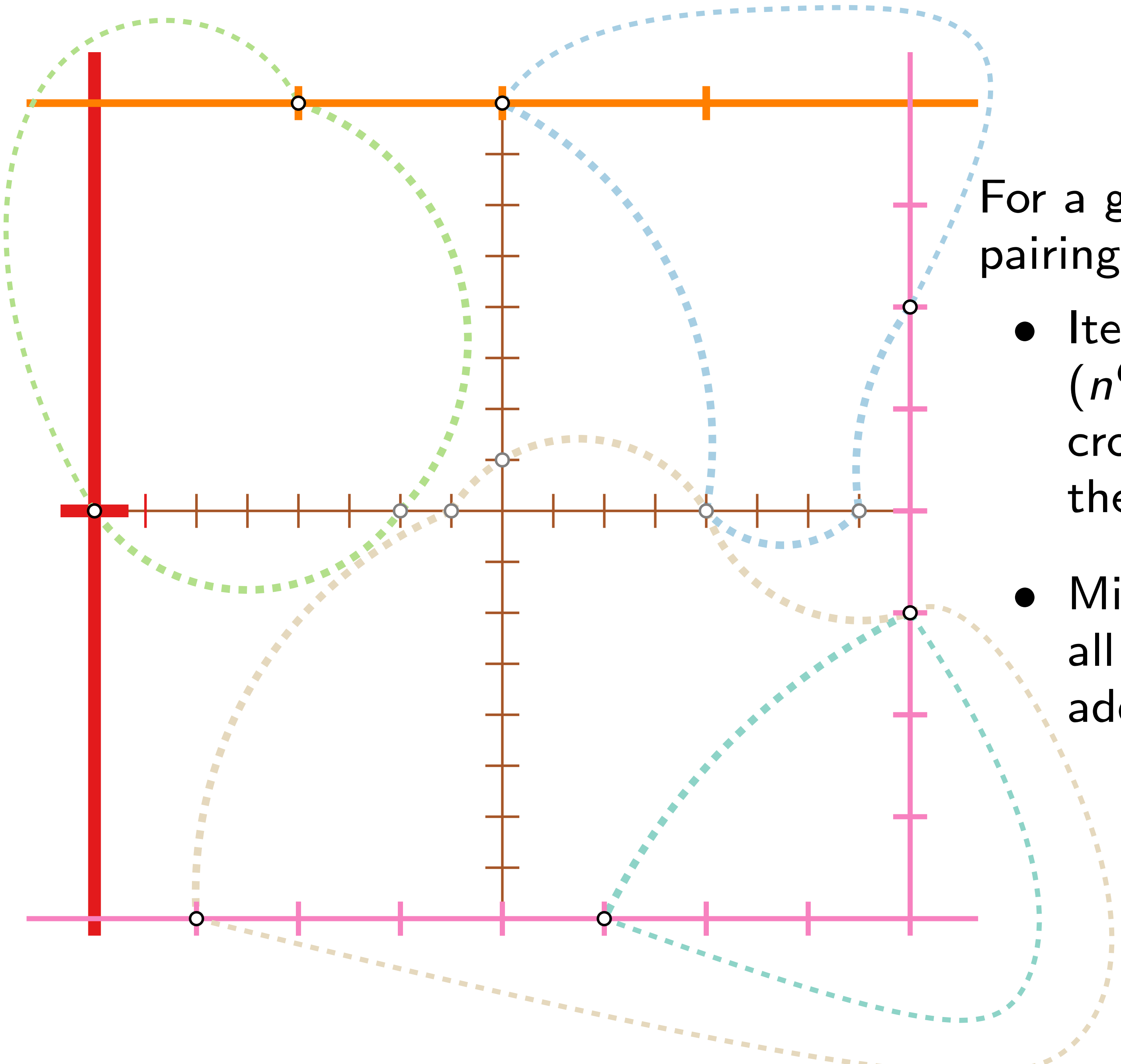

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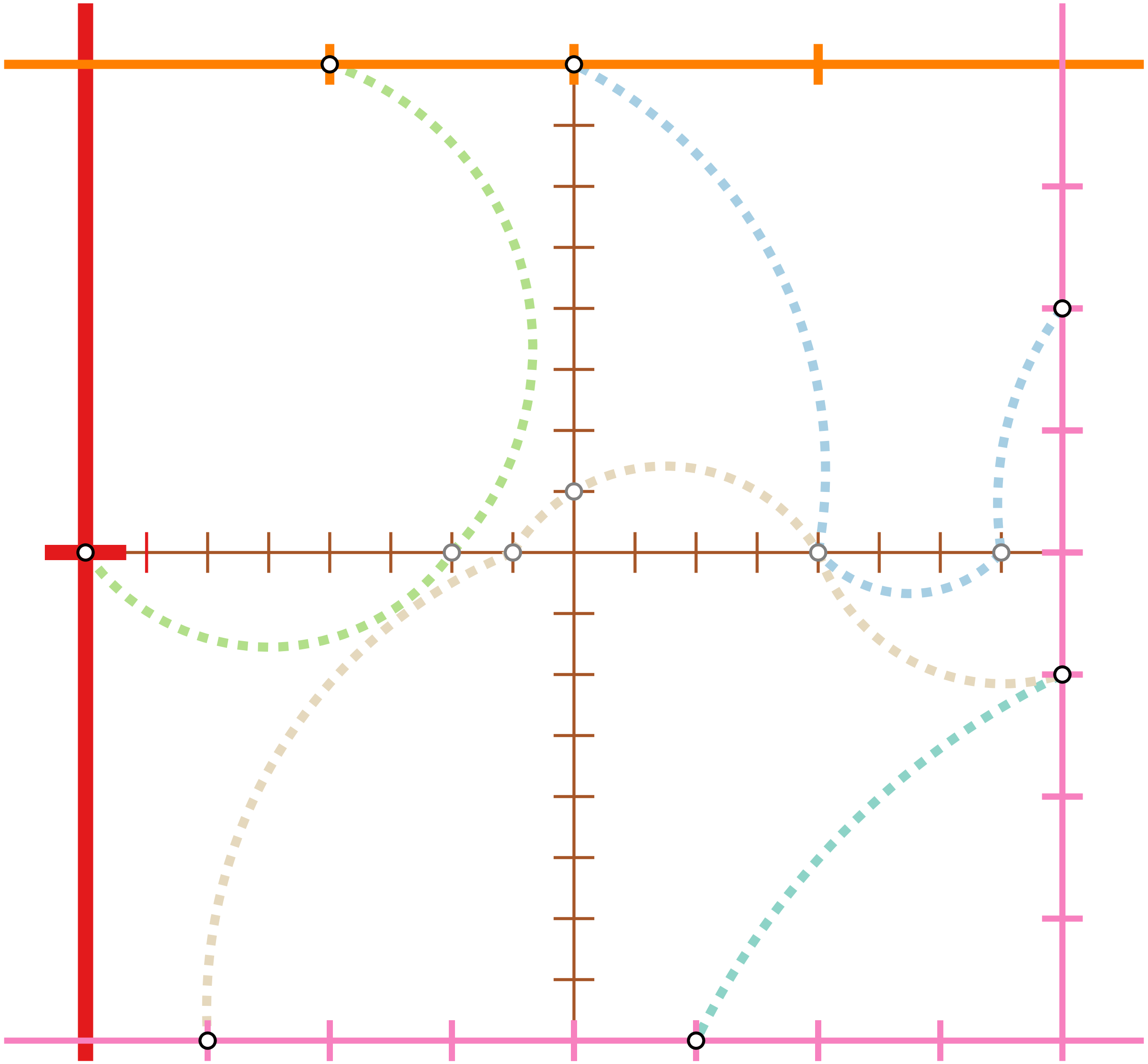

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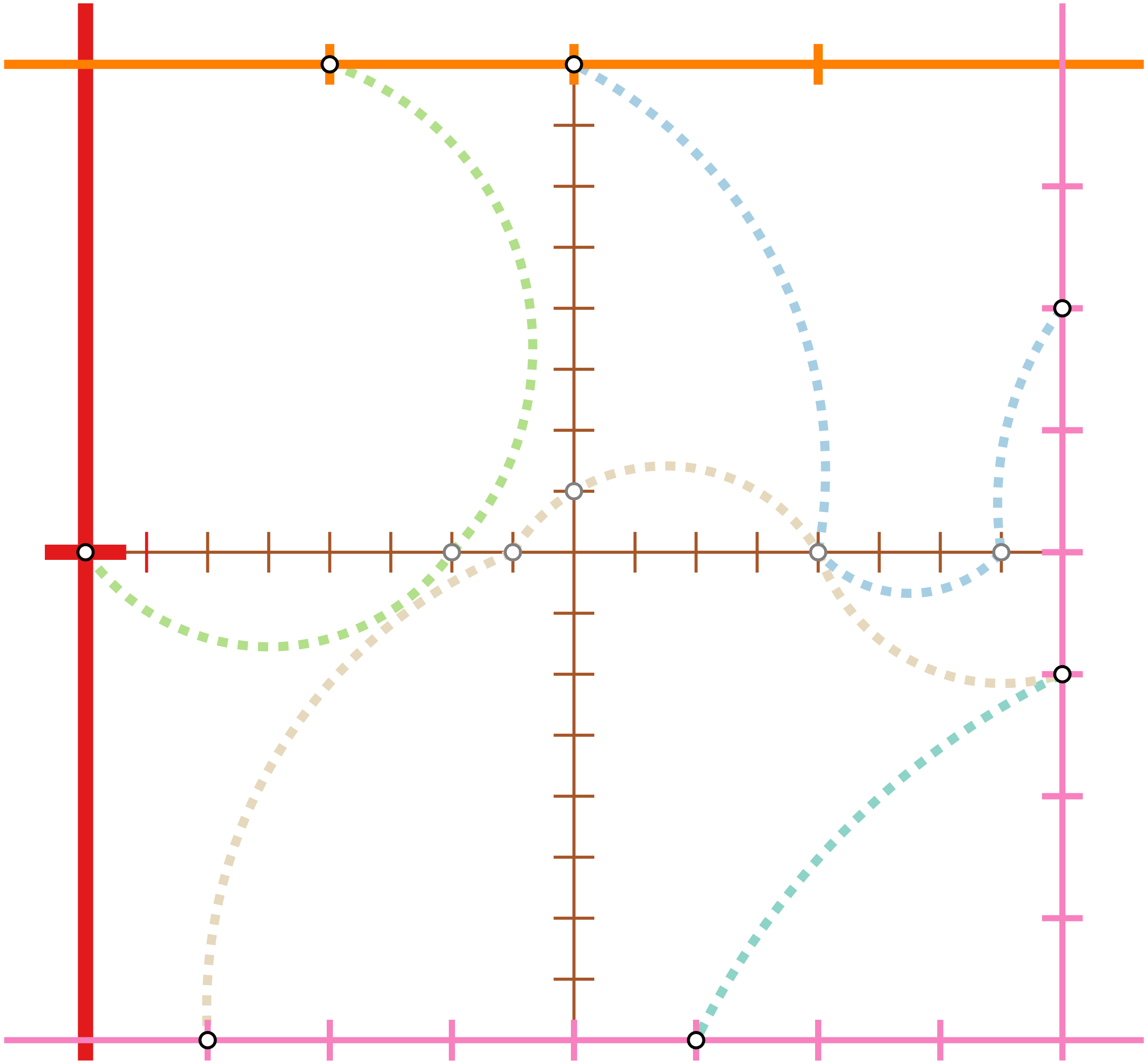

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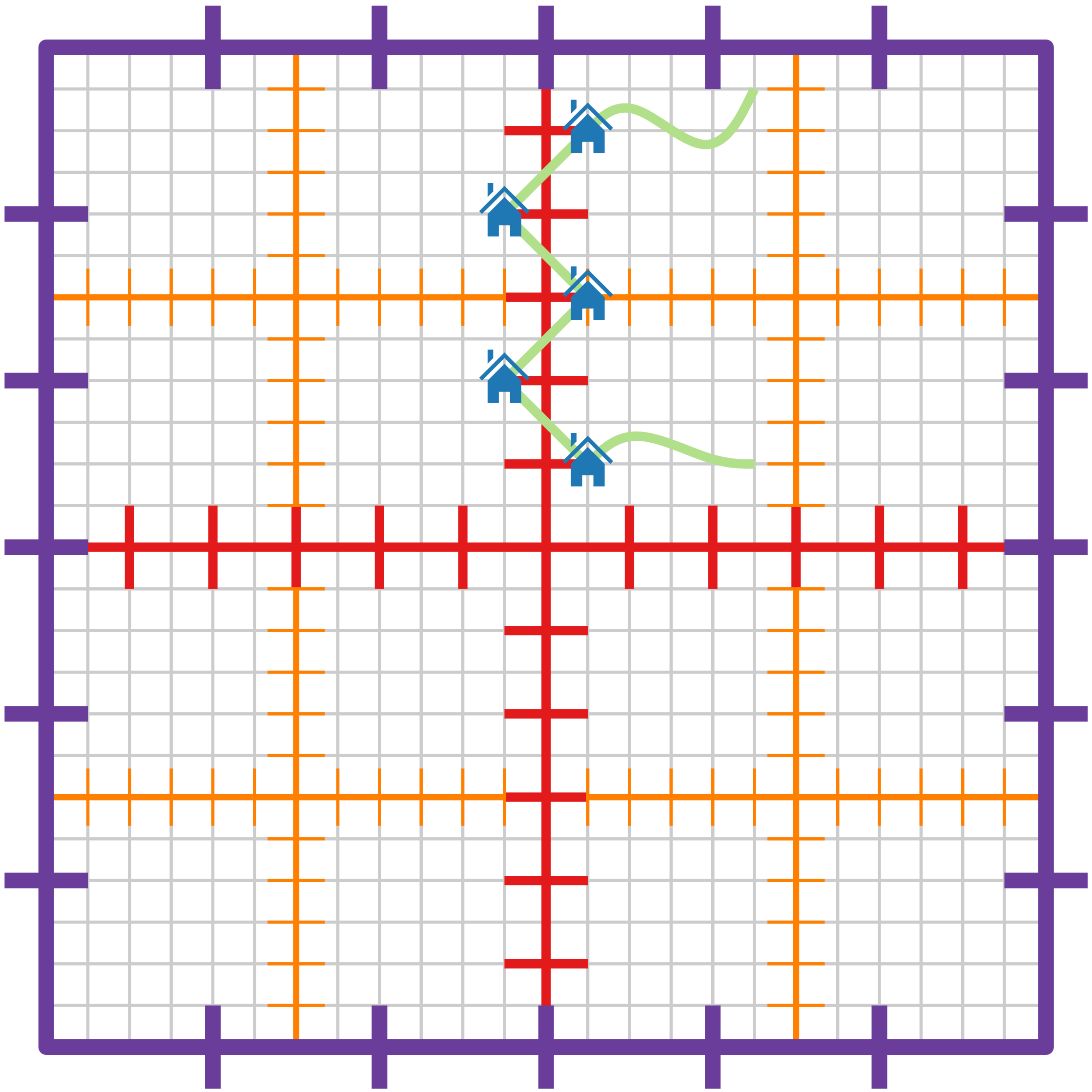
Lemma. An optimal well-behaved tour can be computed in $2^{O(m)} = n^{O(1/\varepsilon)}$ time.

Approximation Algorithms

Lecture 9:
A PTAS for EUCLIDEAN TSP

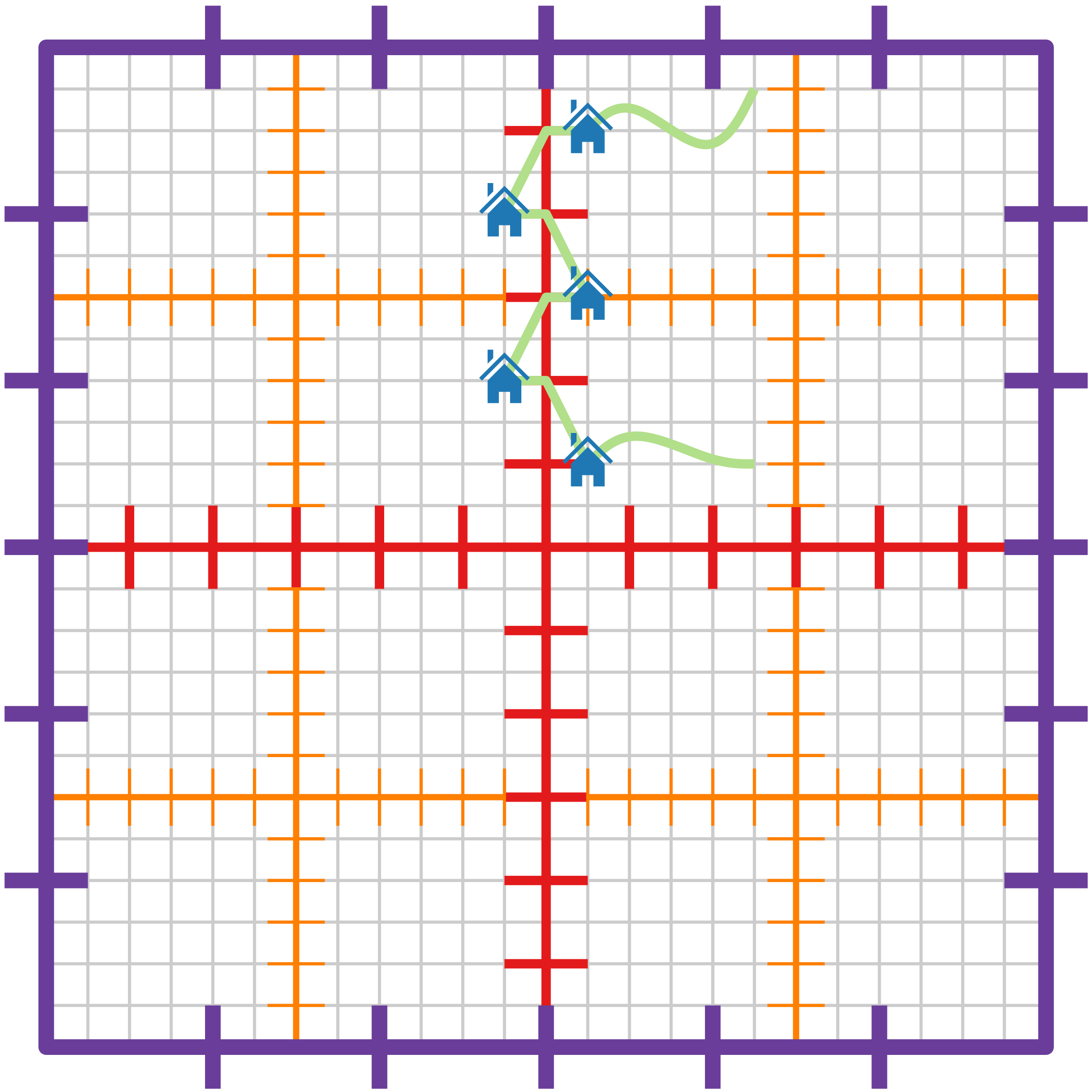
Part V:
Shifted Dissections

Shifted Dissections



- The best well-behaved tour can be a bad approximation.

Shifted Dissections



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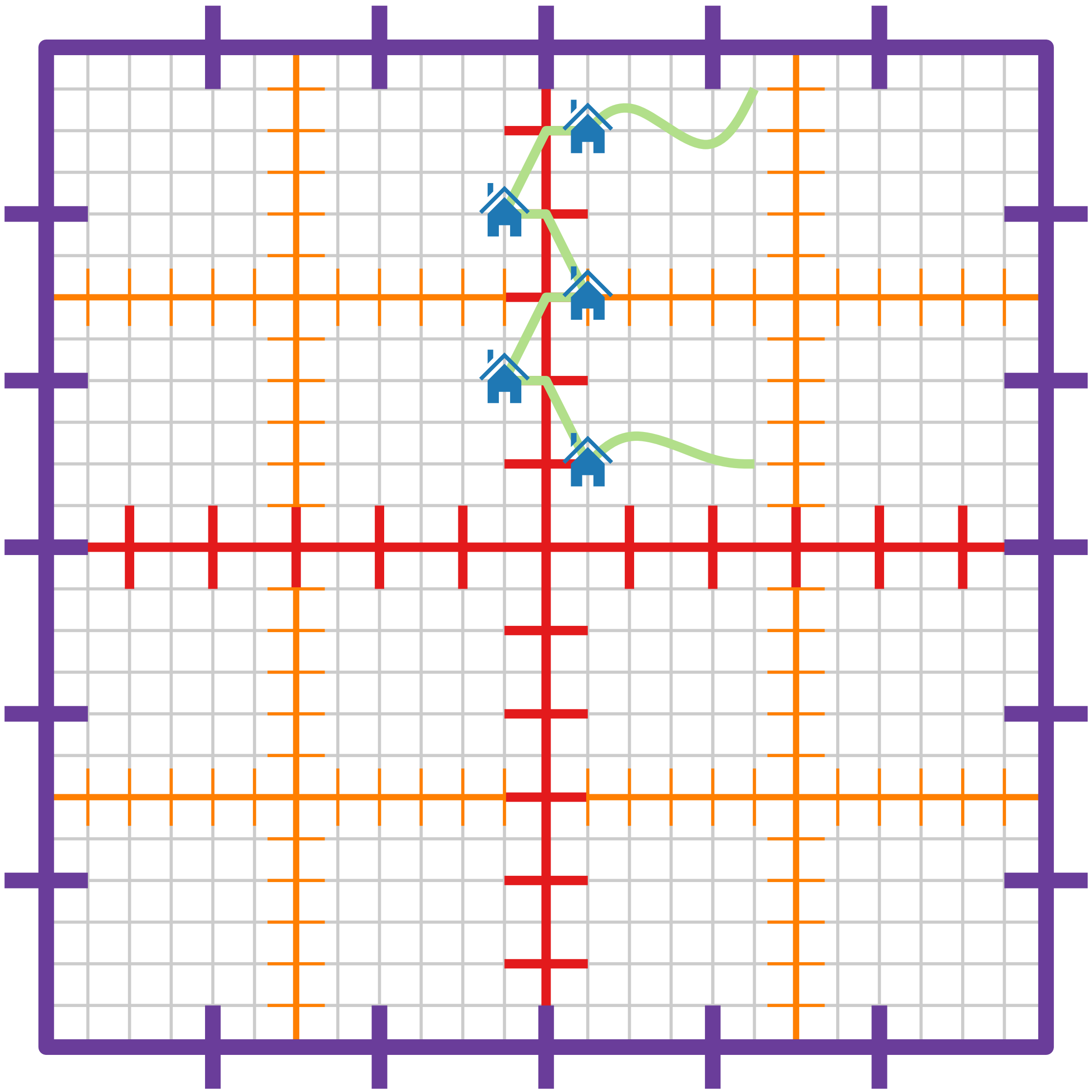

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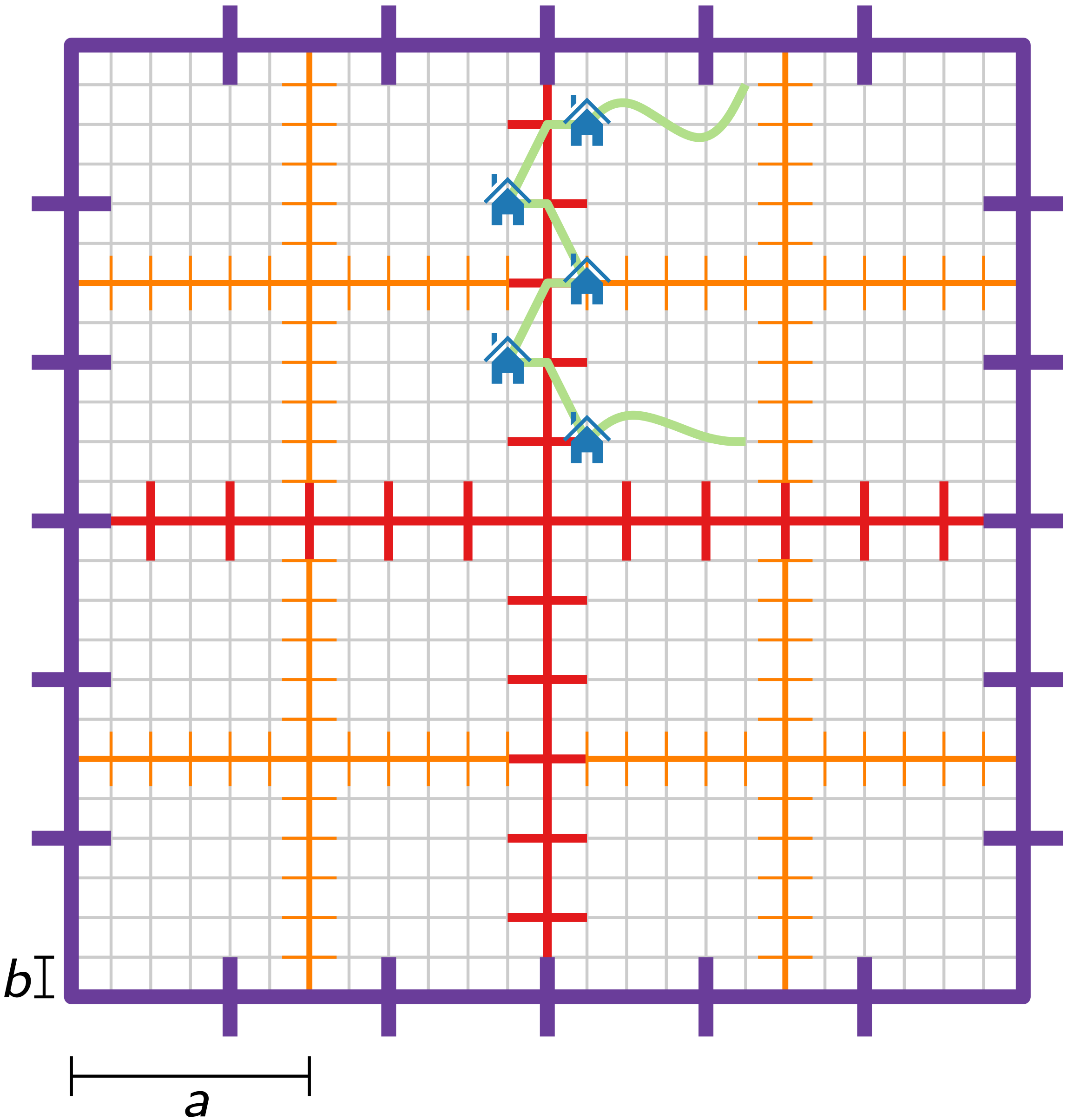
- Consider an (a, b)-shifted dissection:

$$x \mapsto (x + a) \bmod L$$
$$y \mapsto (y + b) \bmod L$$

Shifted Dissections



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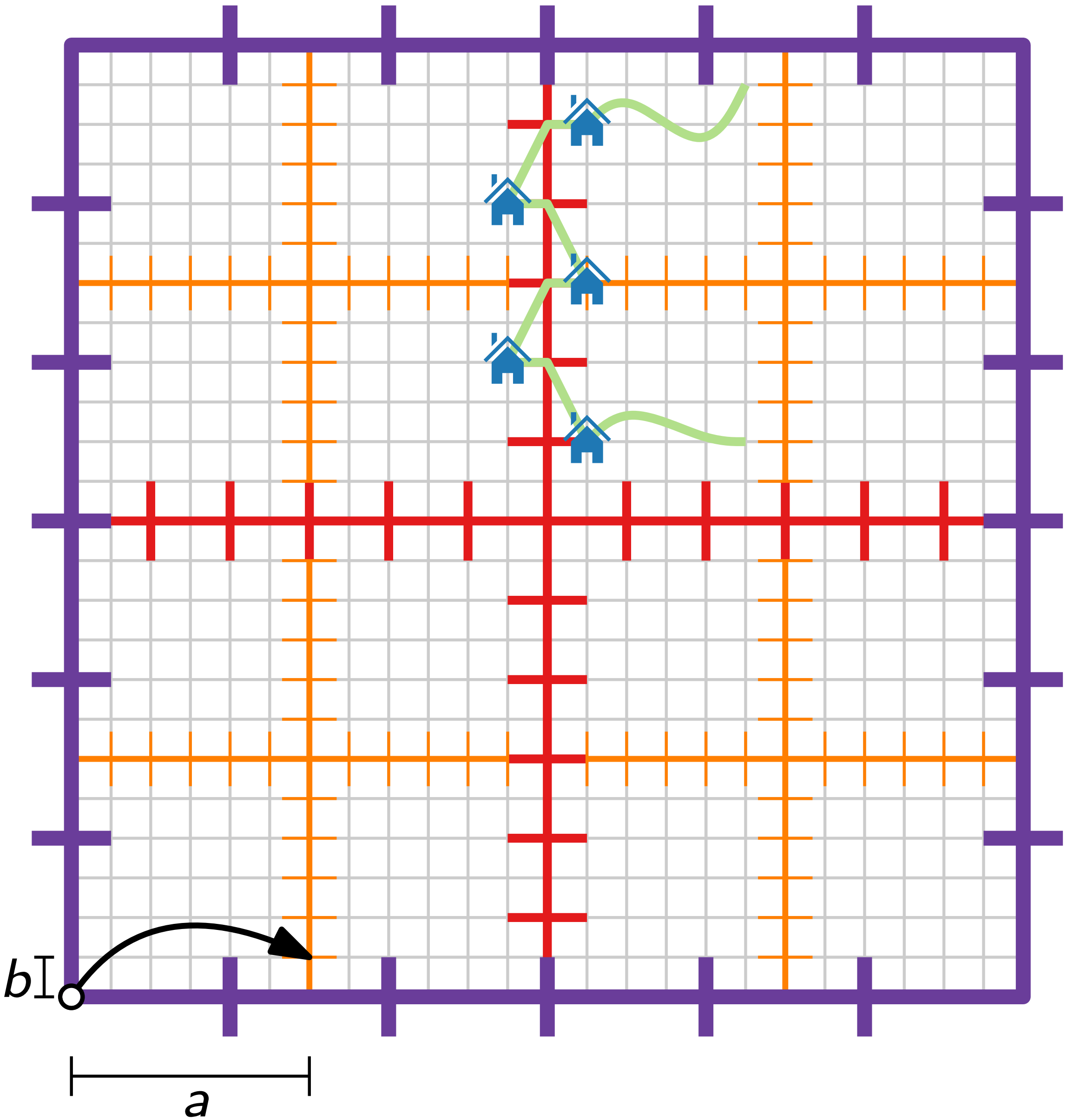

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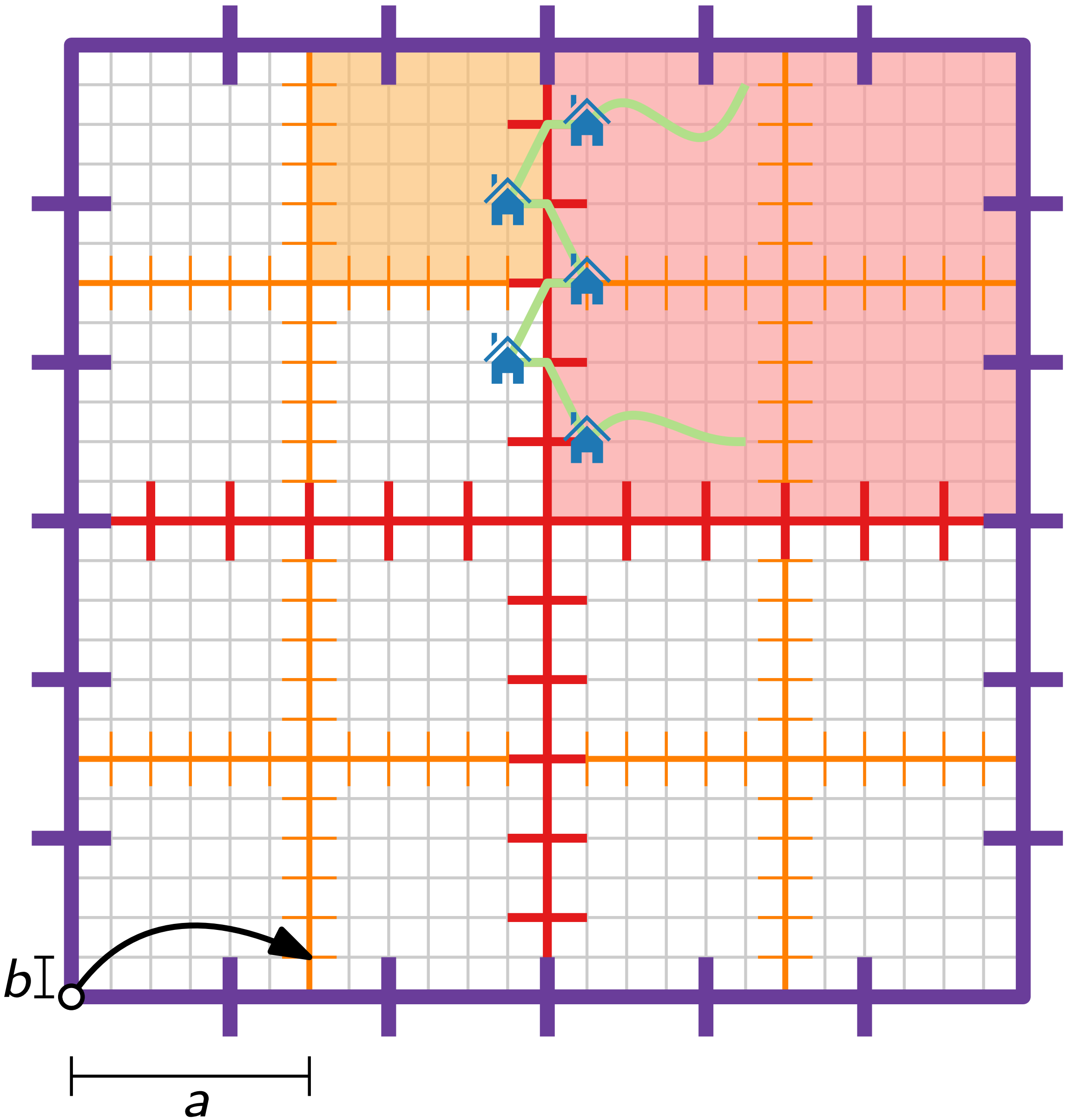

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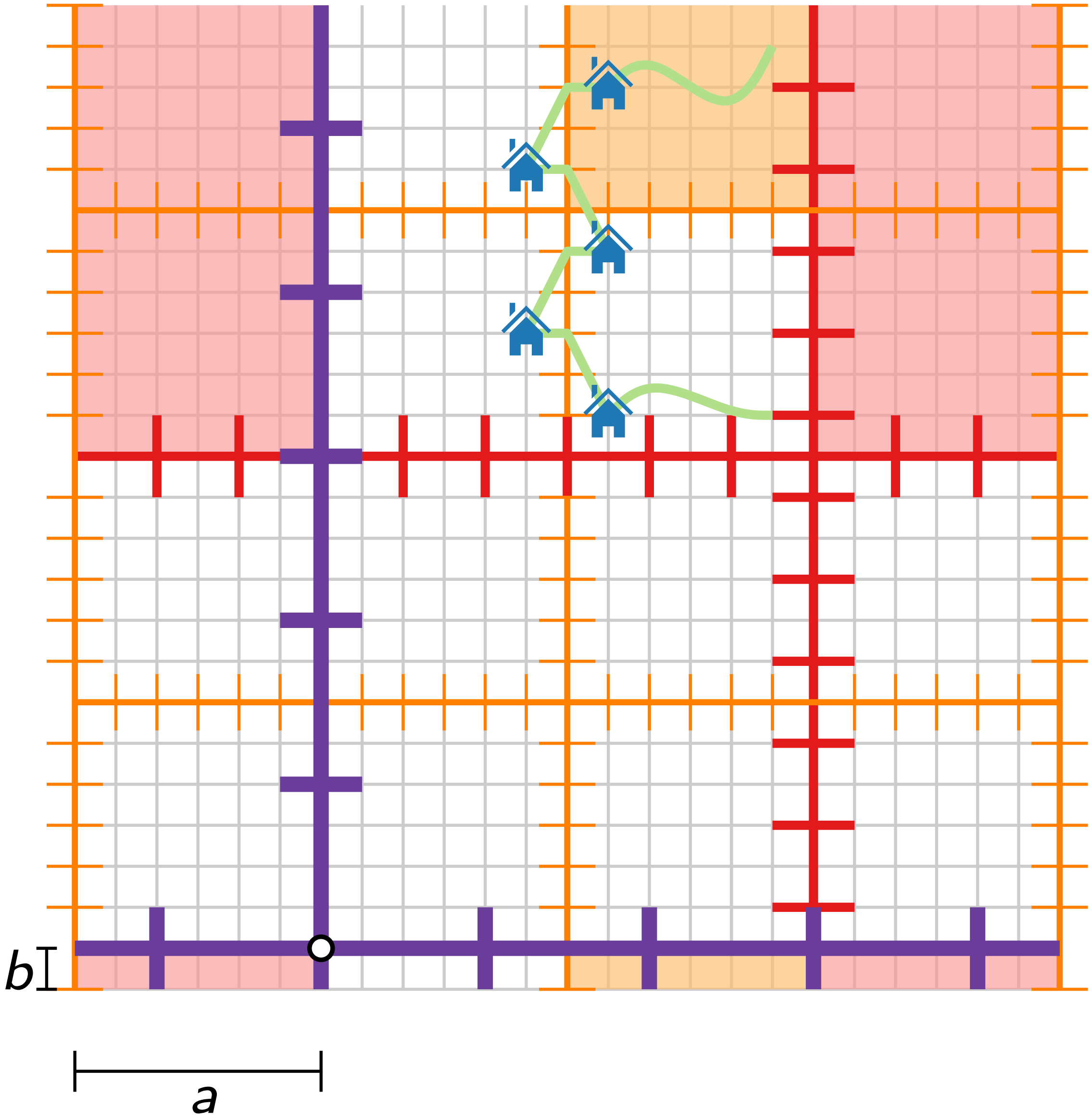

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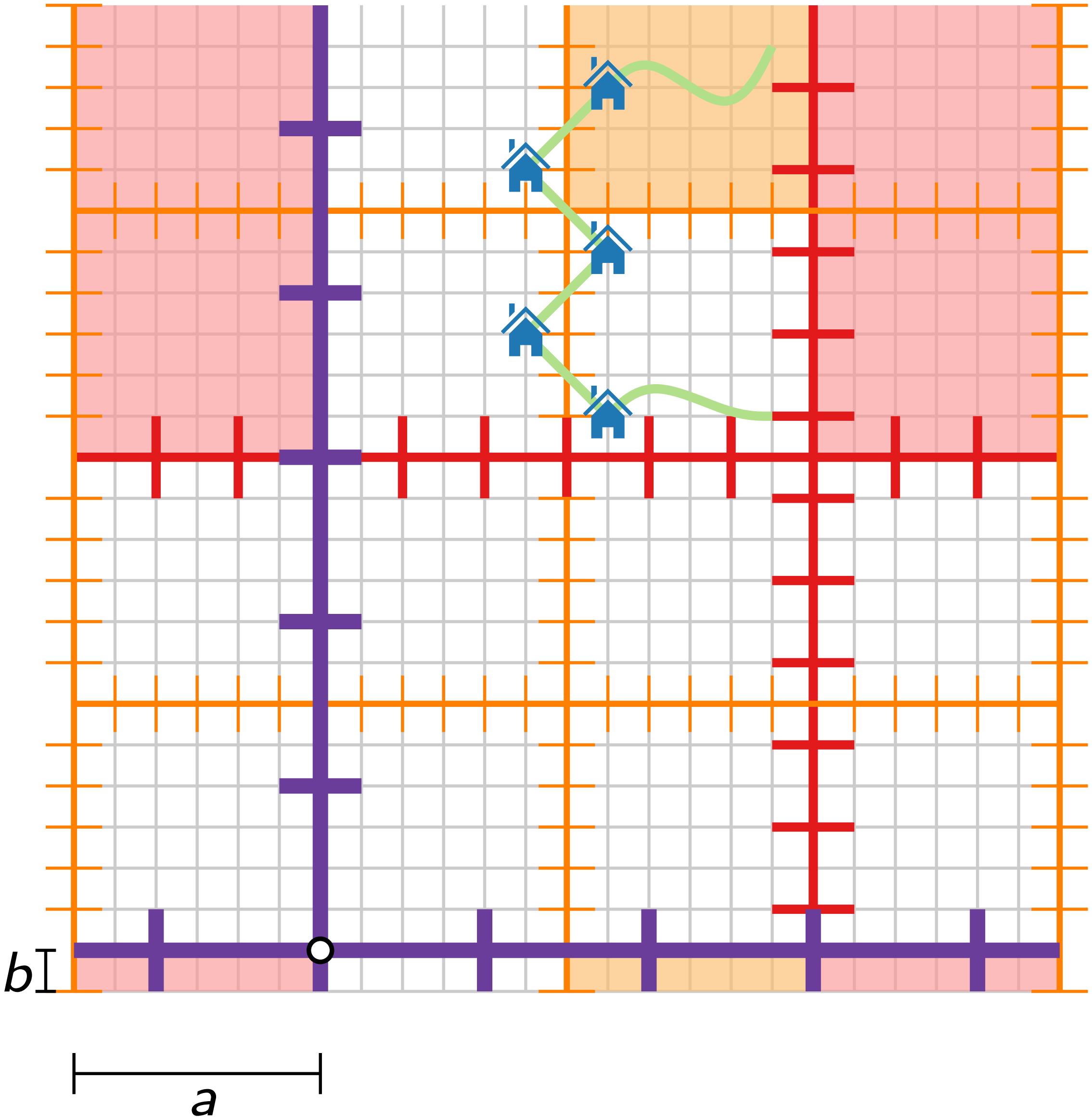

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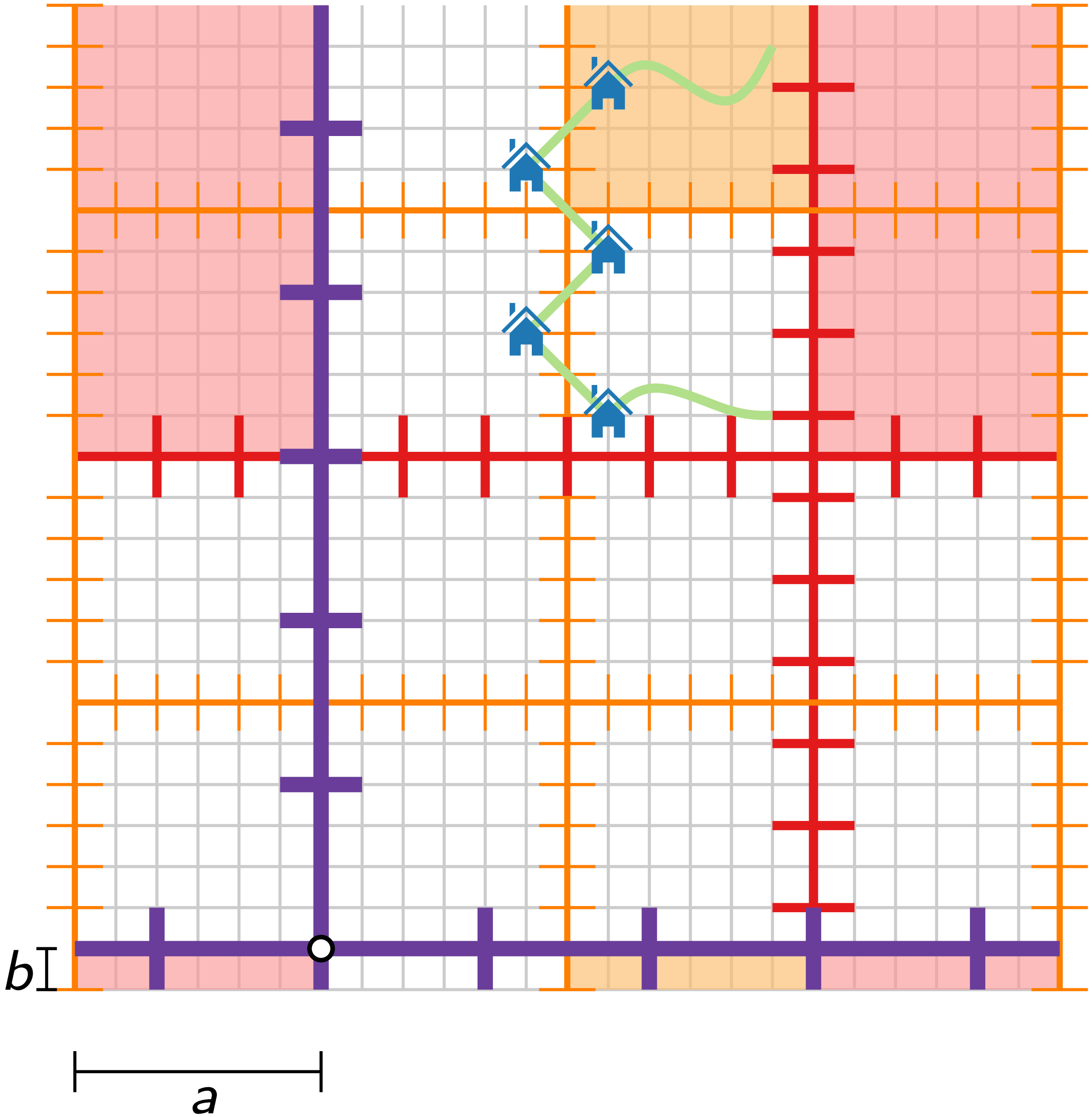

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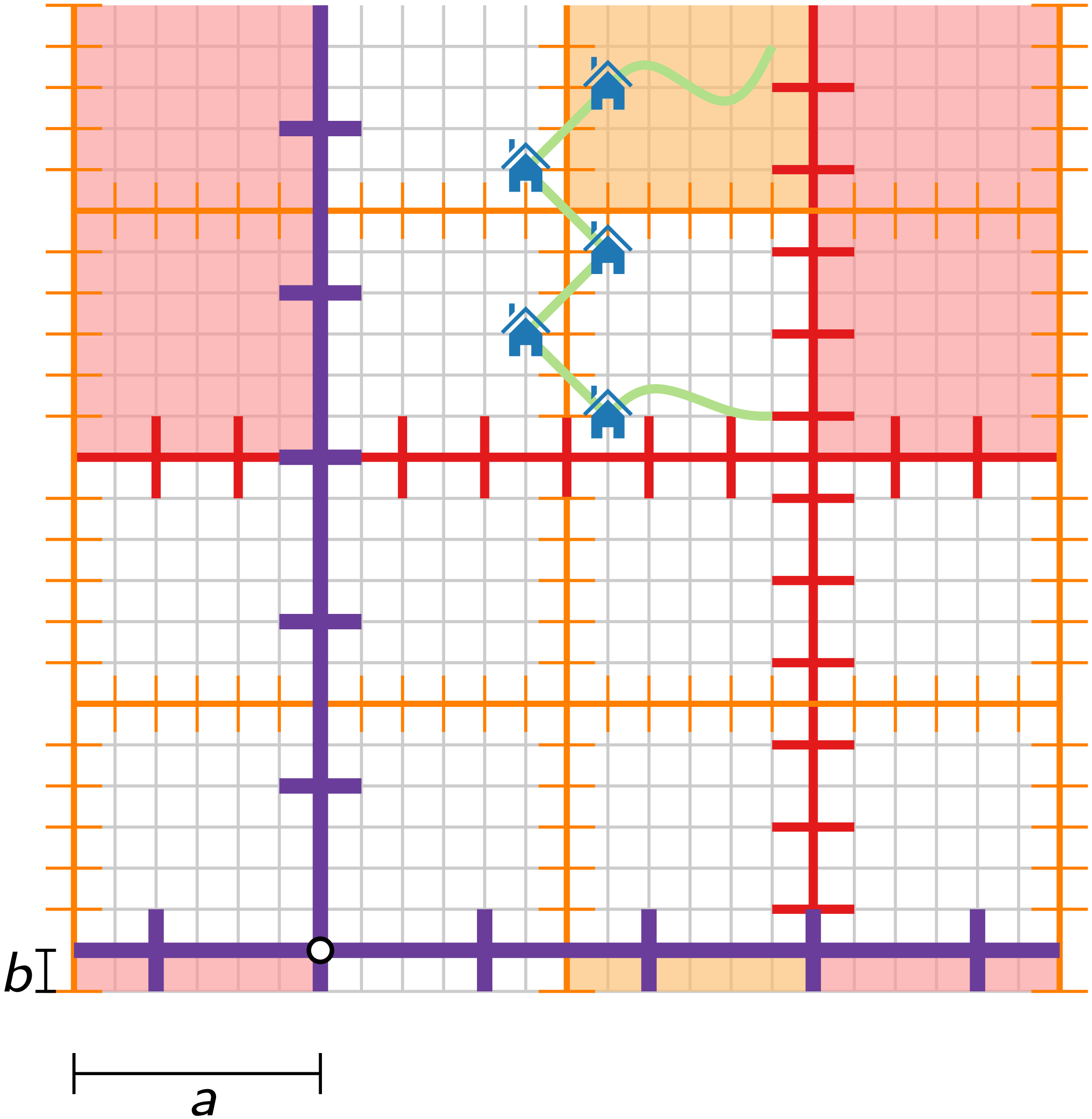
- Squares in the dissection tree are "wrapped around".

Shifted Dissections



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- Squares in the dissection tree are "wrapped around".
- Dynamic program must be modified accordingly.

Shifted Dissections (II)

Lemma. If π is an optimal tour and $N(\pi)$ is the number of crossings of π with the lines of the $(L \times L)$-grid, then $N(\pi) \leq$

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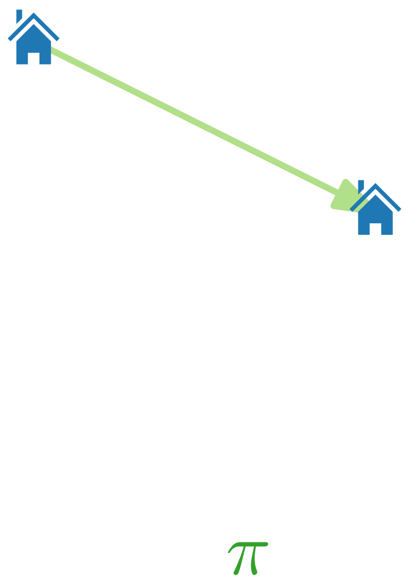

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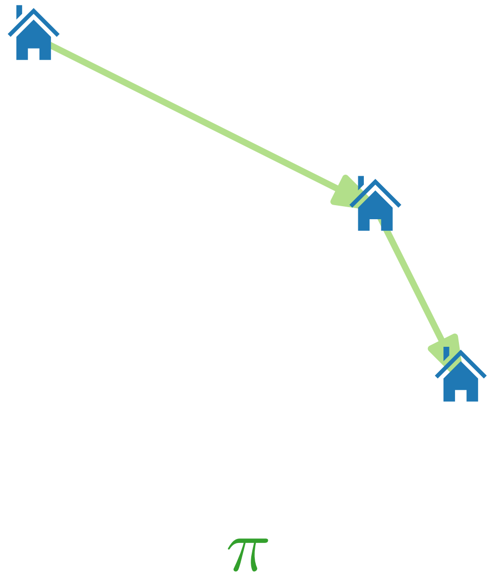

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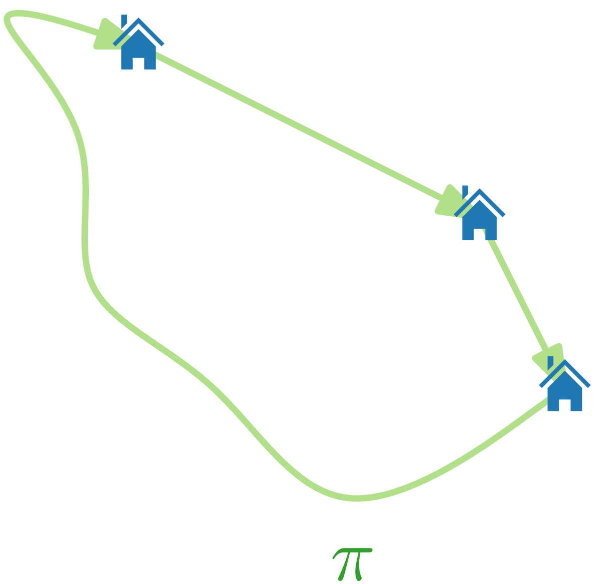

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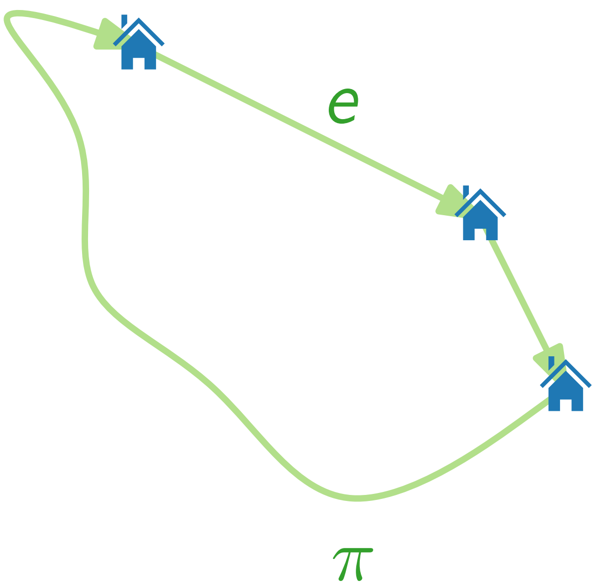

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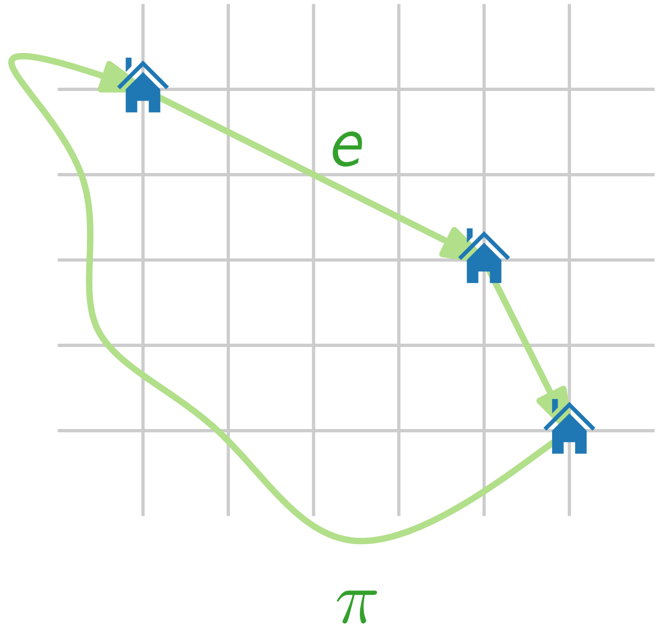

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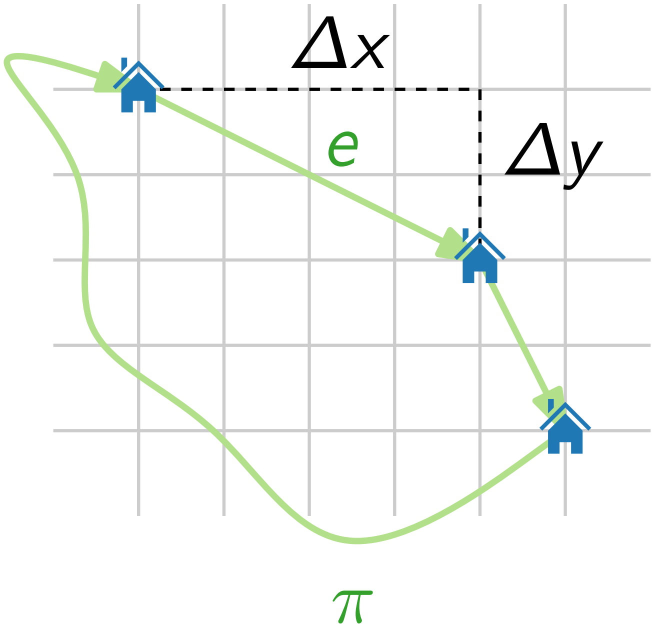

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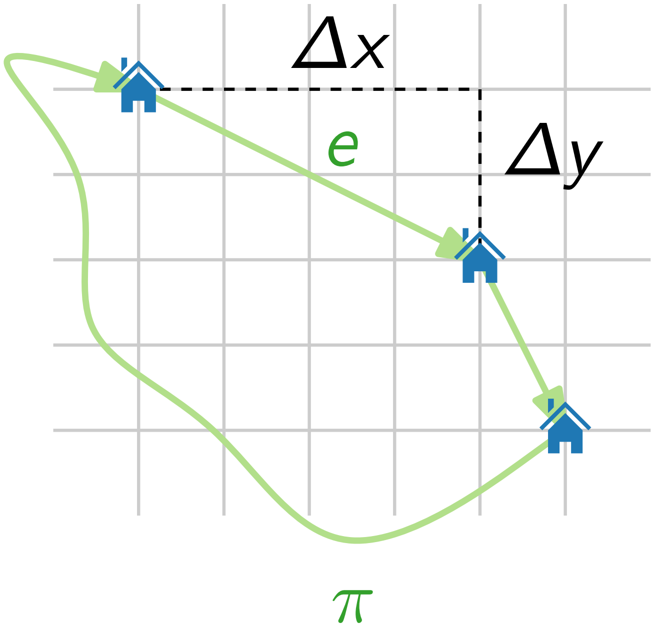
- Each edge e generates $N_e \leq \Delta x + \Delta y$ crossings.



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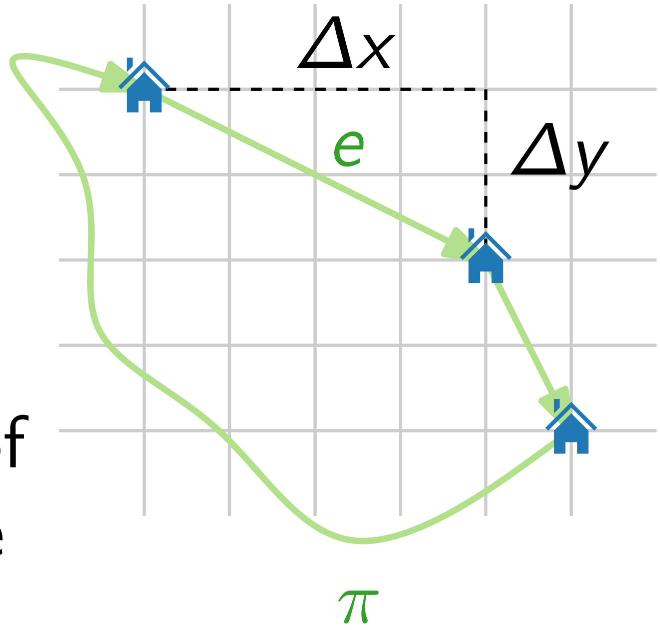

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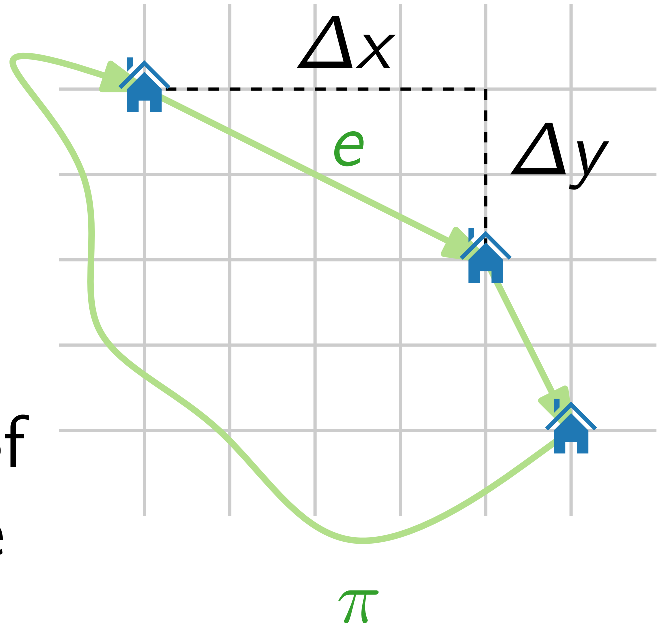
- $N_e{}^2 \leq$



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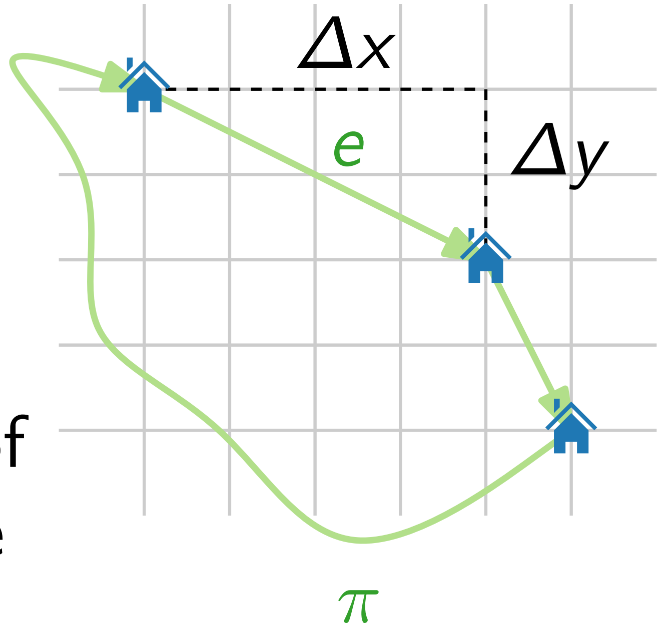

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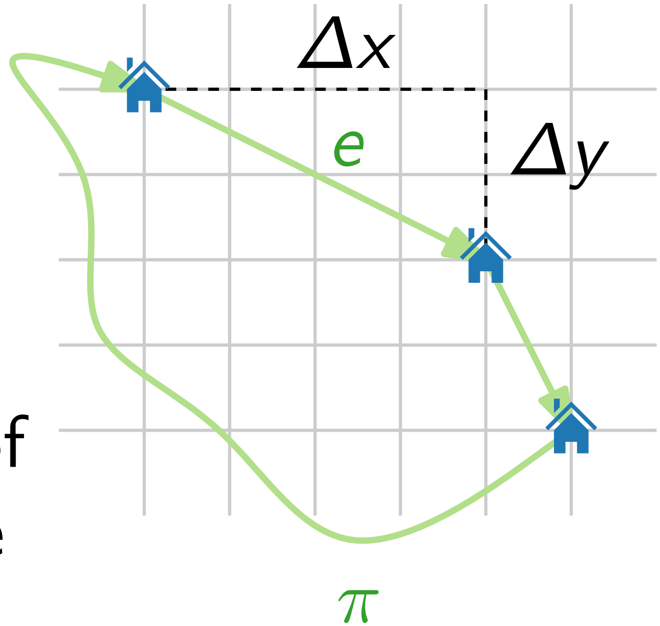

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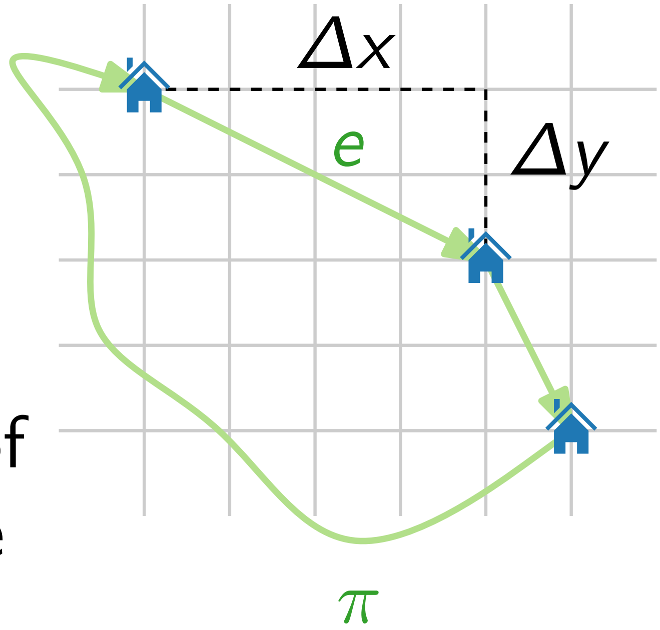

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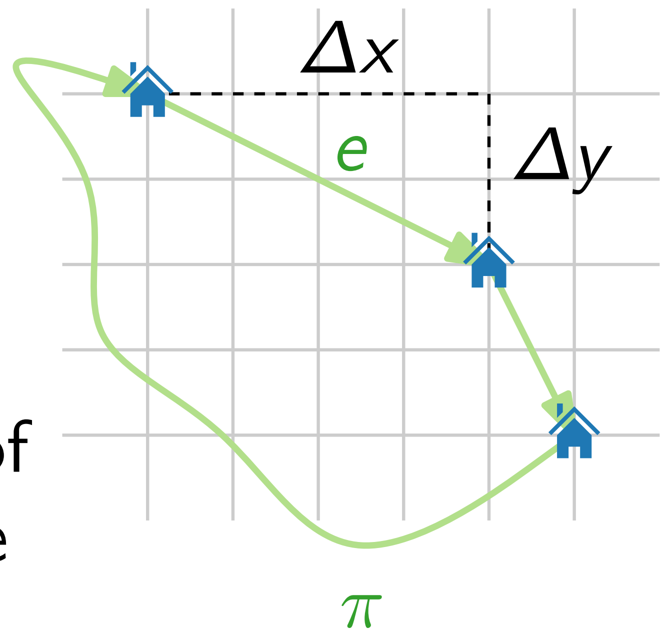

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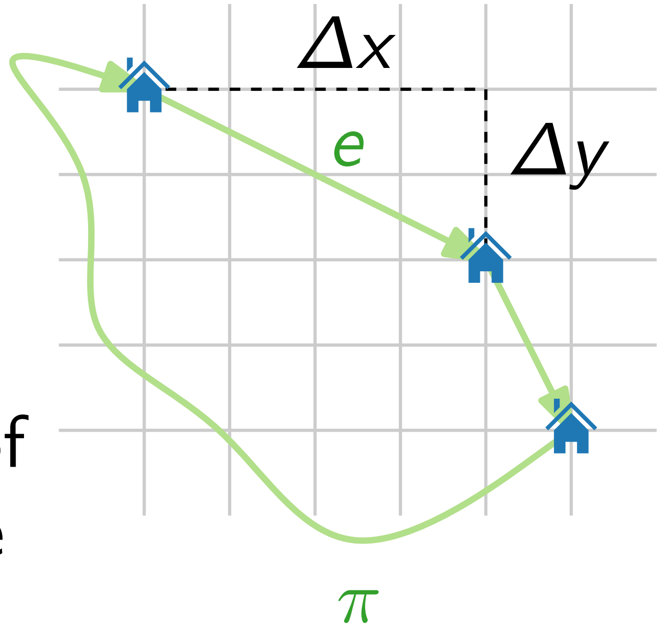

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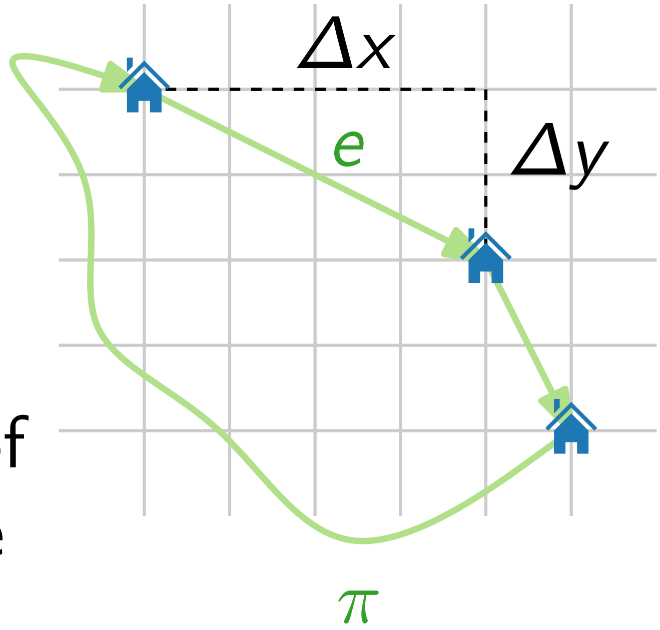

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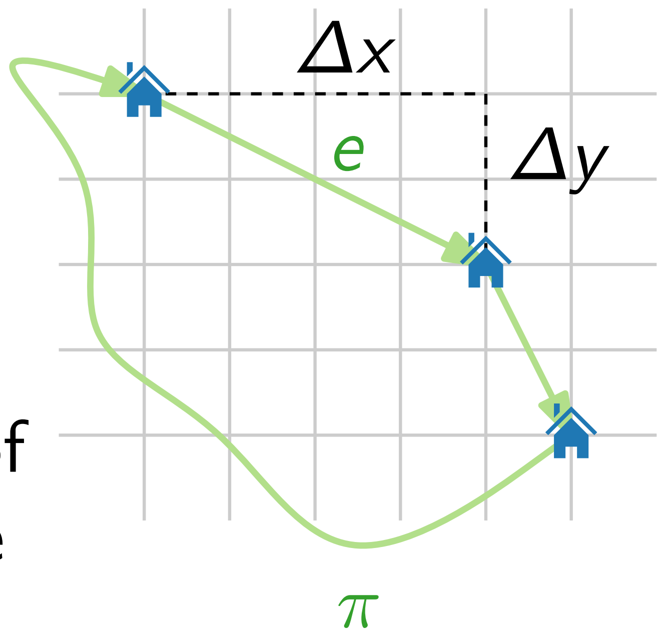

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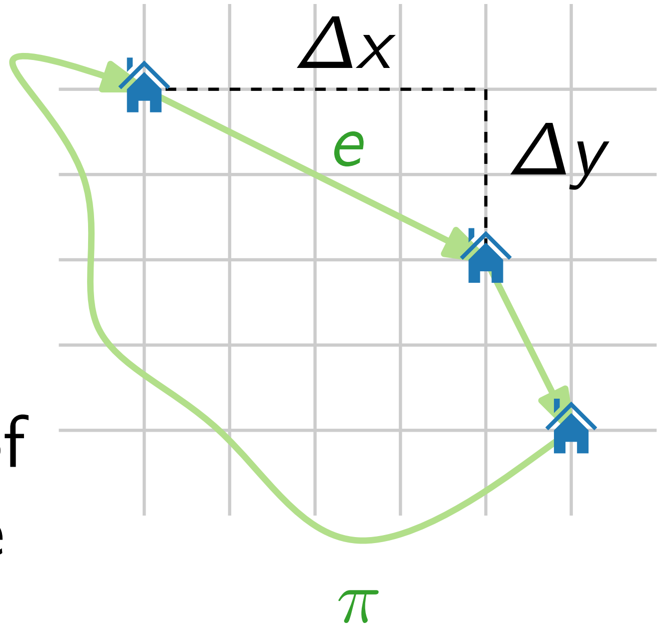

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Approximation Algorithms

Lecture 9:
A PTAS for EUCLIDEAN TSP

Part VI:
Approximation Factor

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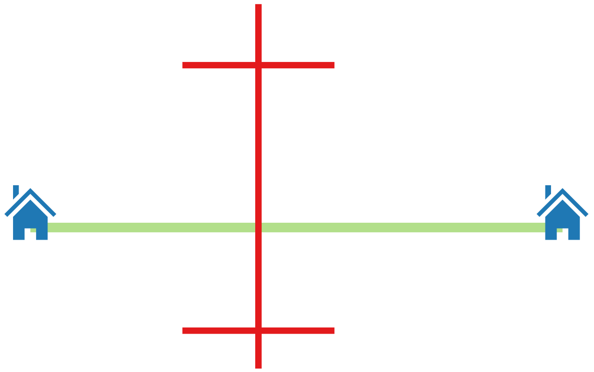

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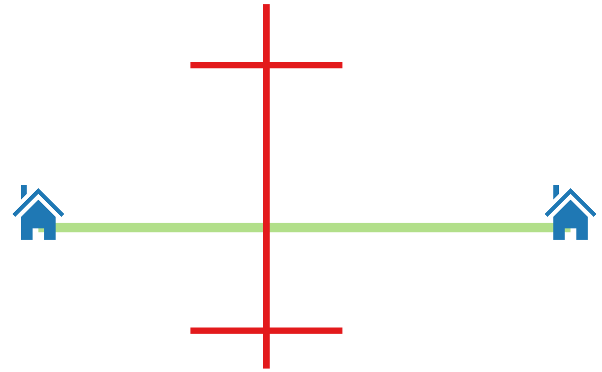

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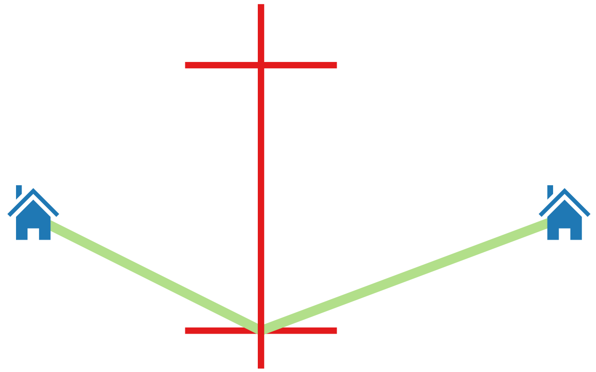

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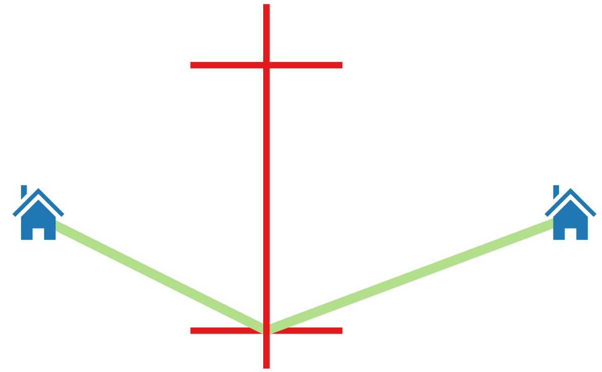

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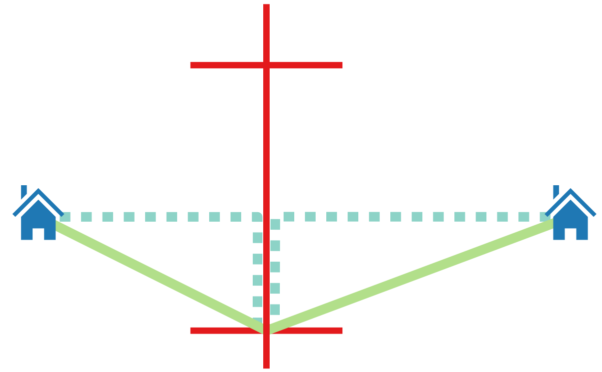

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- Summing over all $N(\pi) \leq \sqrt{2} \cdot \mathrm{OPT}$ intersection points and applying linearity of expectation yields the claim.

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Literature

- William J. Cook: Pursuit of the Traveling Salesman: Mathematics at the Limits of Computation.
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Randomized, $O\left(n(\log n)^{O(1/\varepsilon)}\right)$ time

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- Anna R. Karlin, Nathan Klein, Shayan Oveis Gharan: A (slightly) improved approximation algorithm for metric TSP. Proc. STOC, p. 32–45, 2021: approximation factor $1.5 - 10^{-36}$. Best paper award!