Approximation Algorithms

Lecture 8: Approximation Schemes and the KNAPSACK Problem

Part I: KNAPSACK

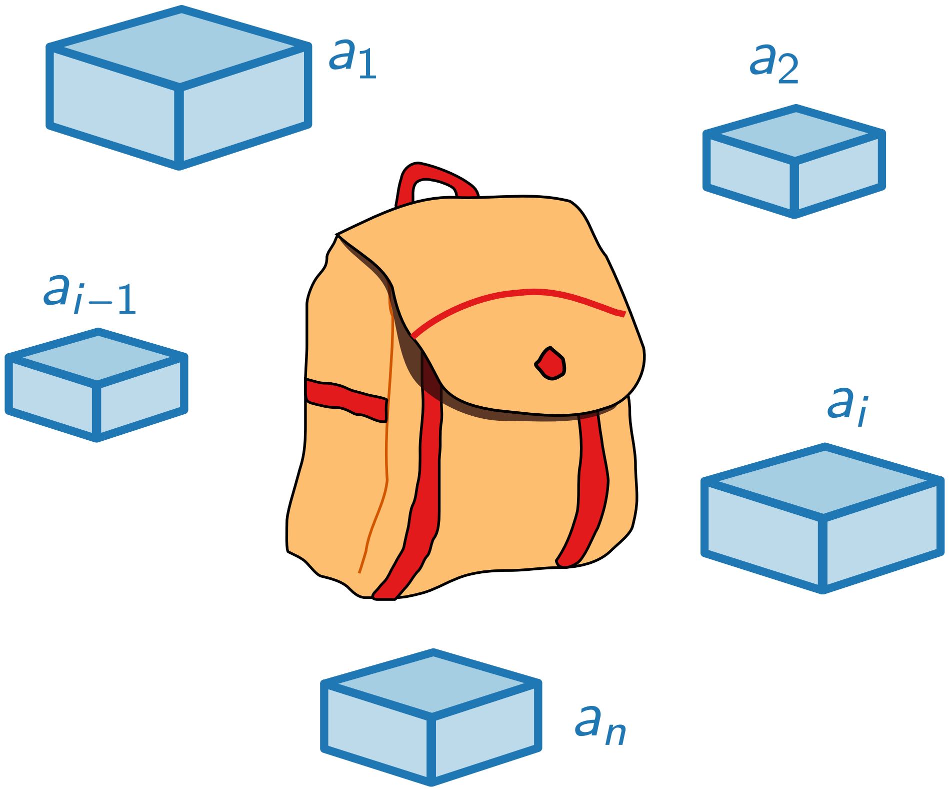
Alexander Wolff

Winter term 2025

Knapsack

Given:

- A set $S = \{a_1, \ldots, a_n\}$ of **objects**.



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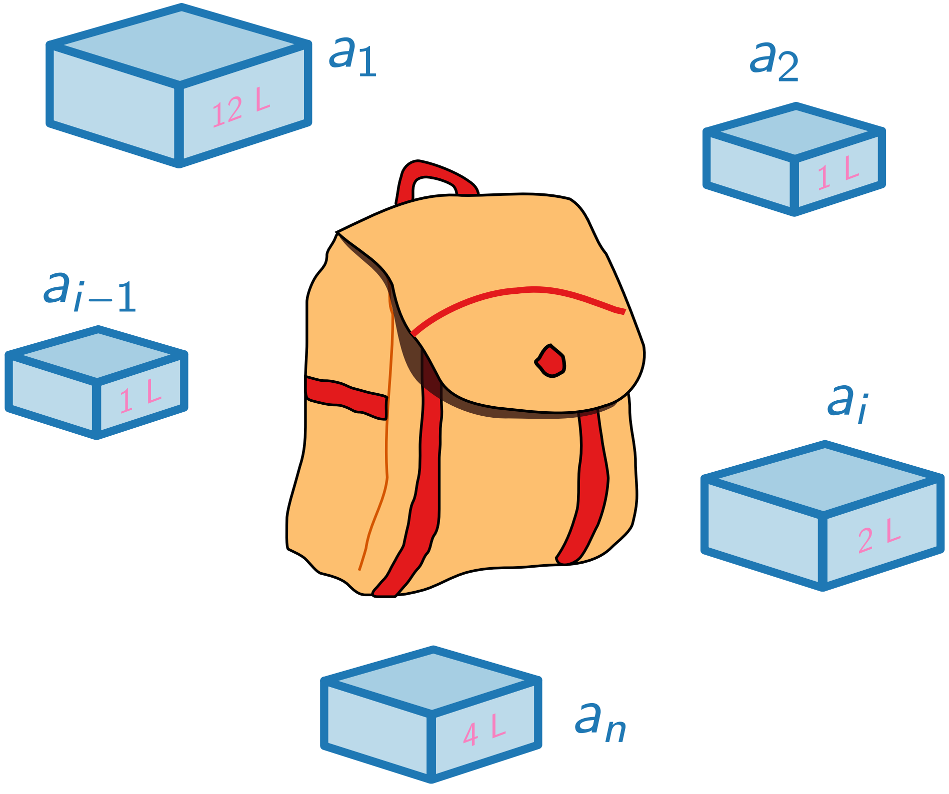

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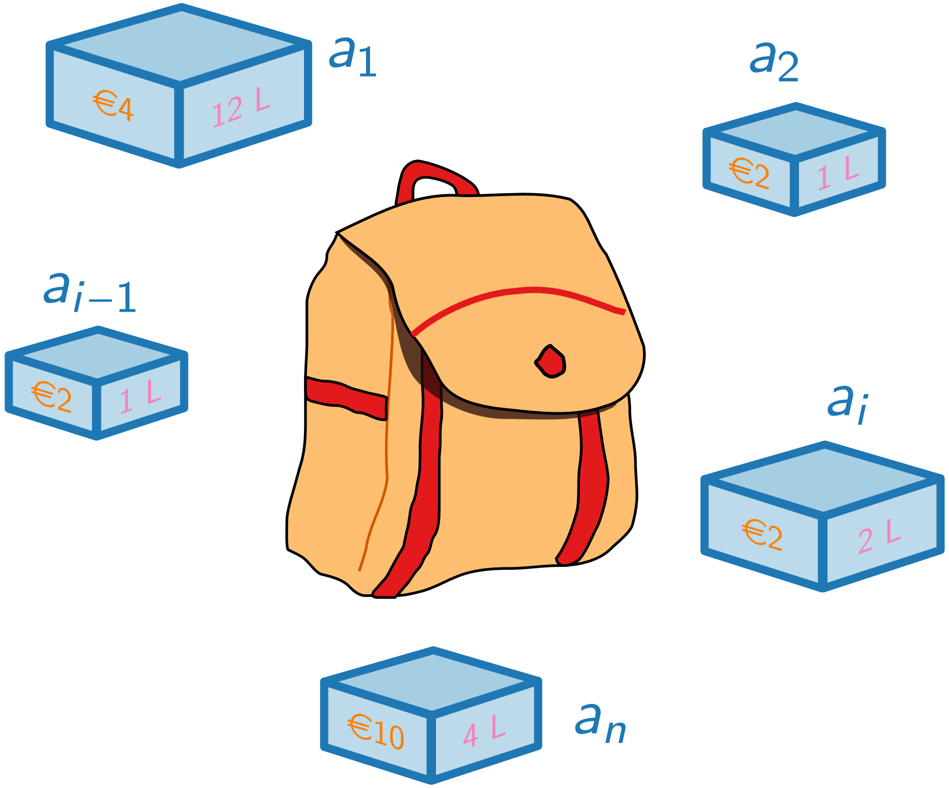

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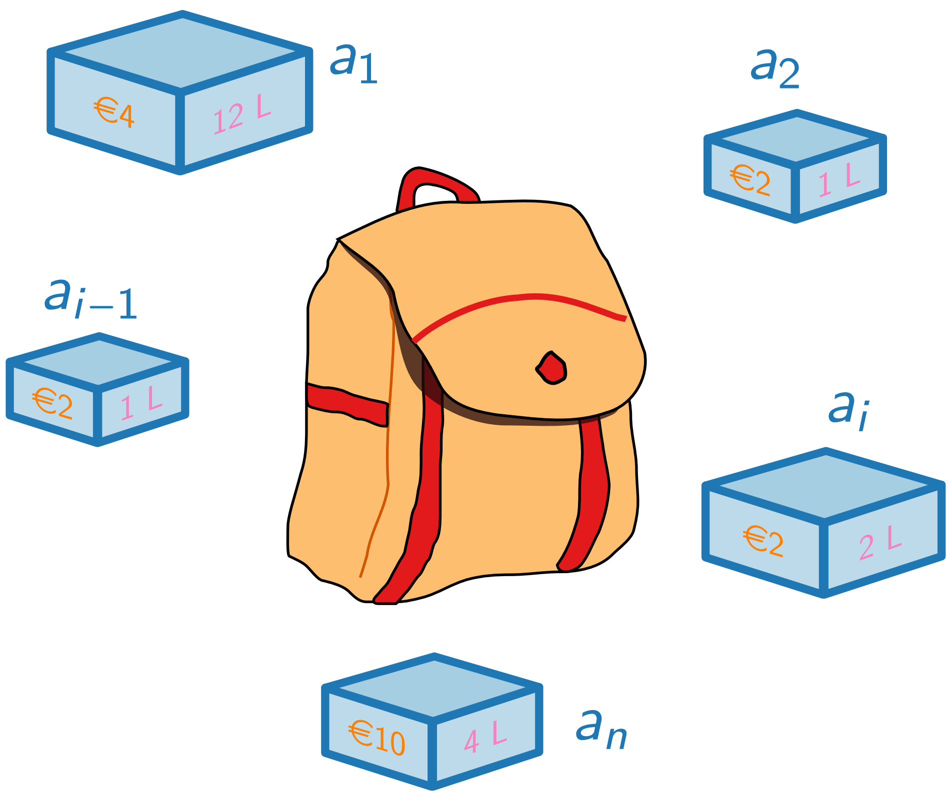

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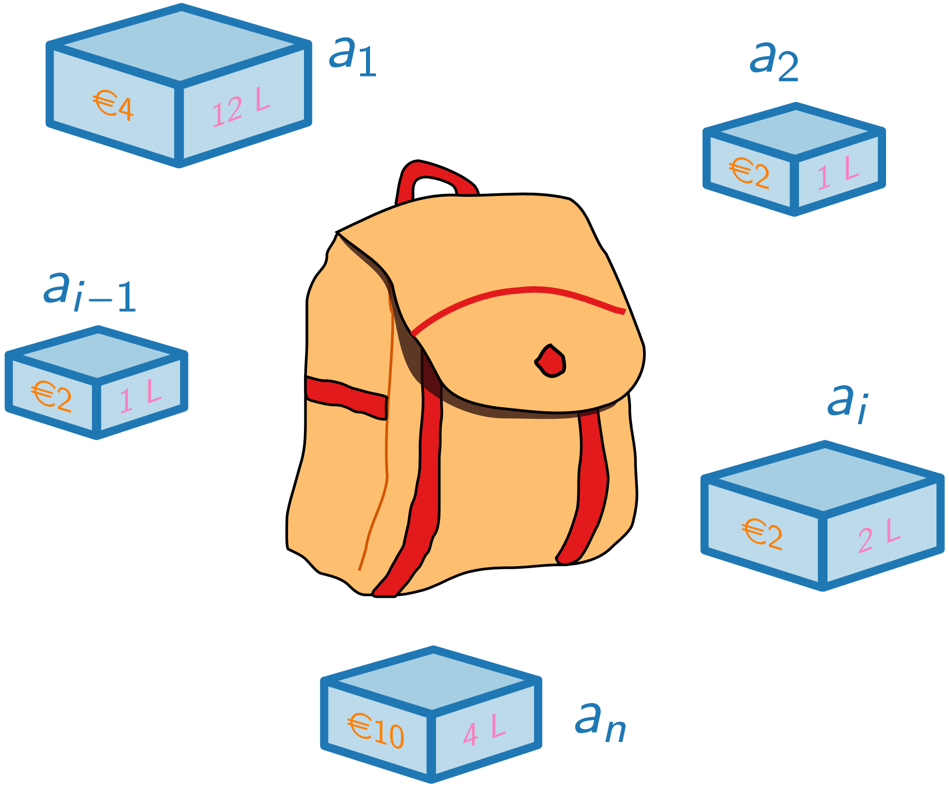
Task: Find a subset of objects whose **total size** is at most B and whose **total profit** is maximum.



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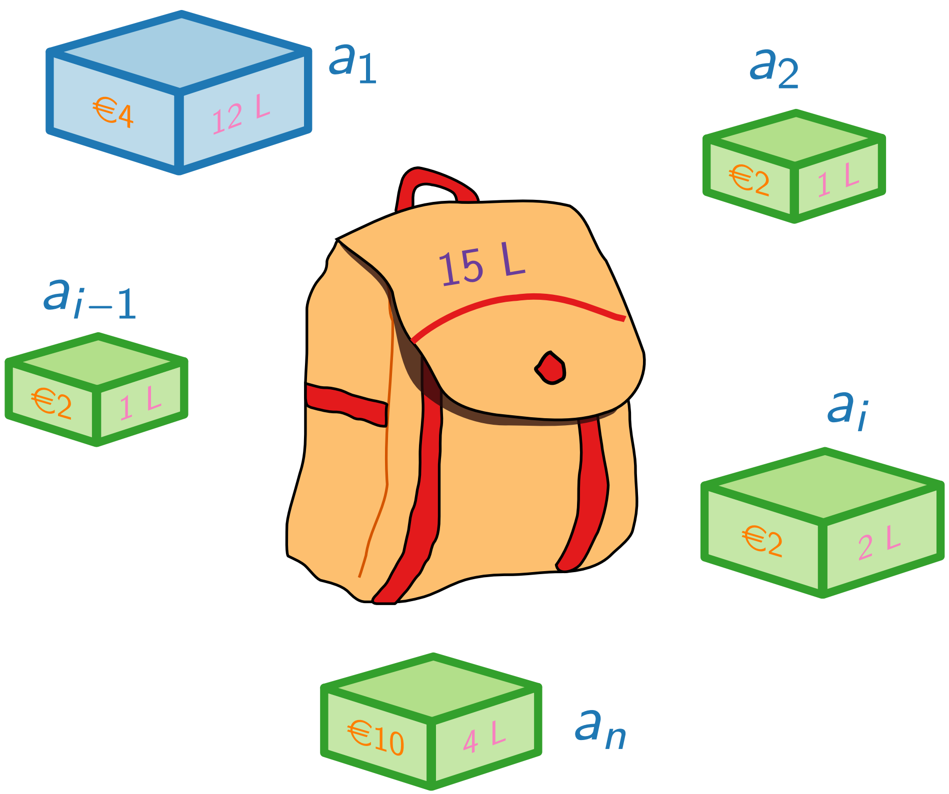

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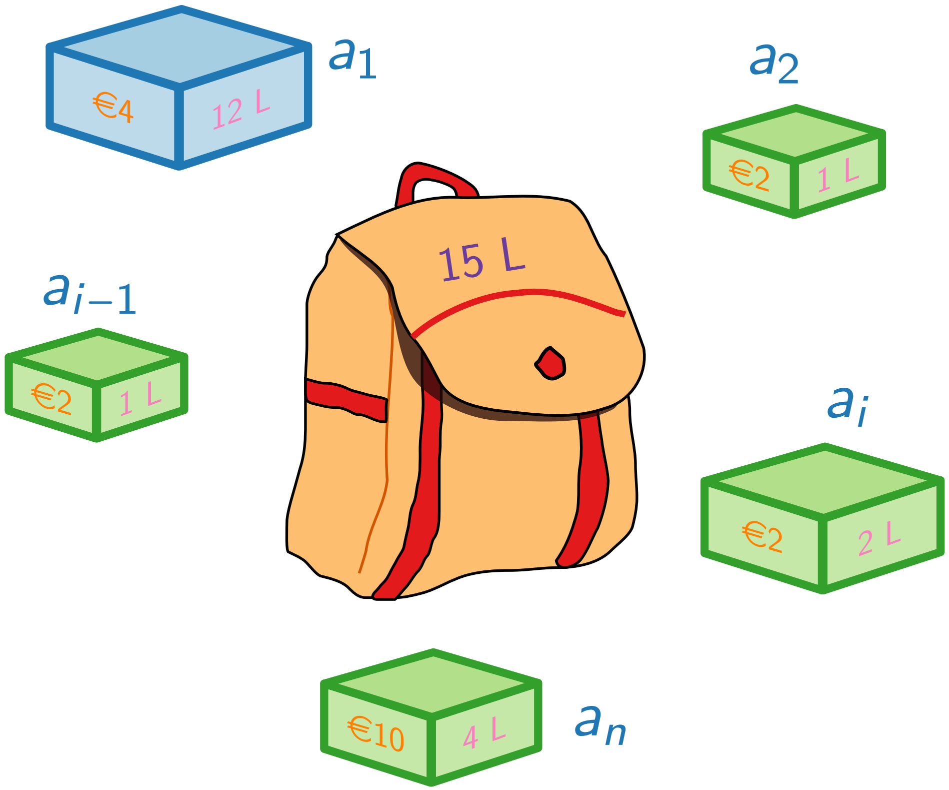
NP-hard



Approximation Algorithms

Lecture 8:
Approximation Schemes and the KNAPSACK Problem

Part II:
Pseudo-Polynomial Algorithms and Strong NP-Hardness

Pseudo-Polynomial Algorithms

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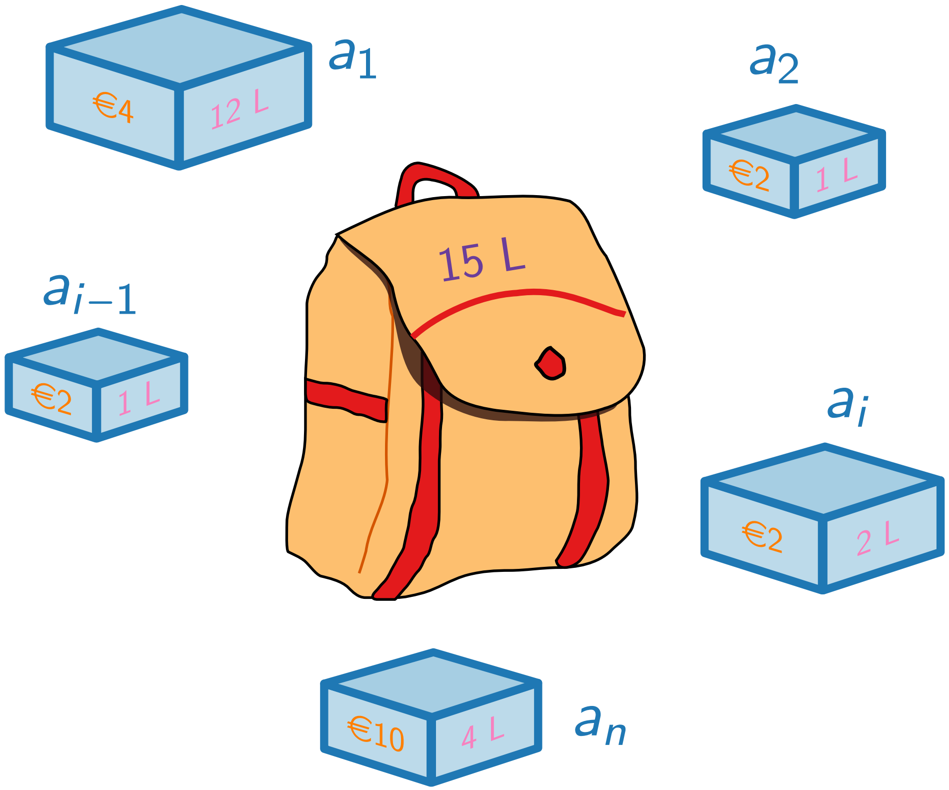

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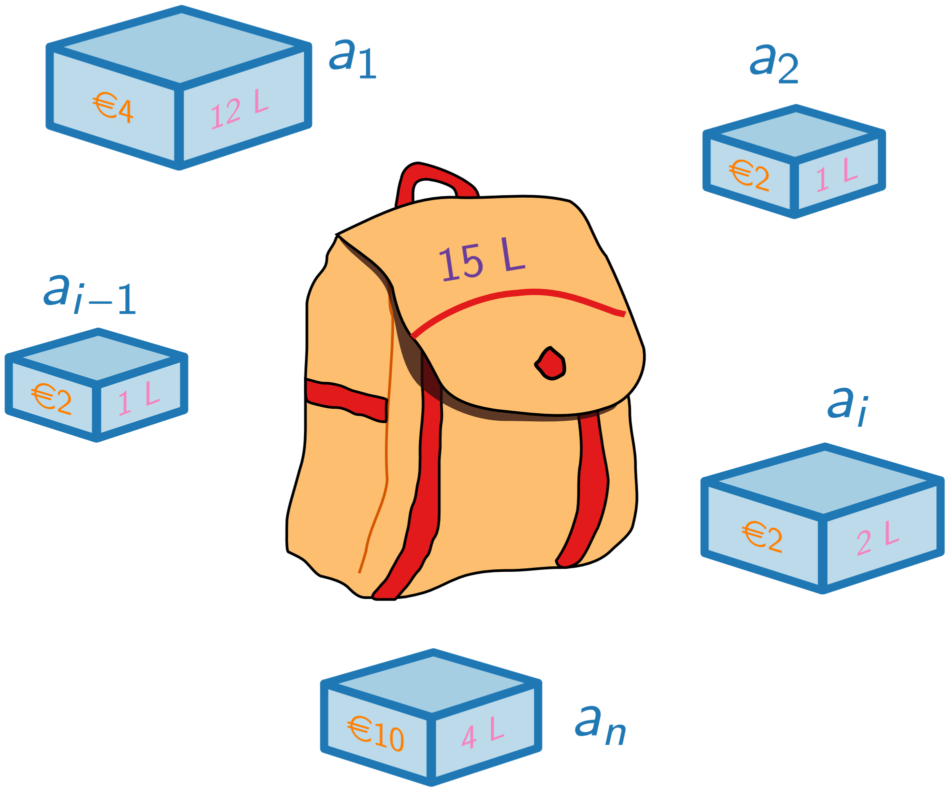

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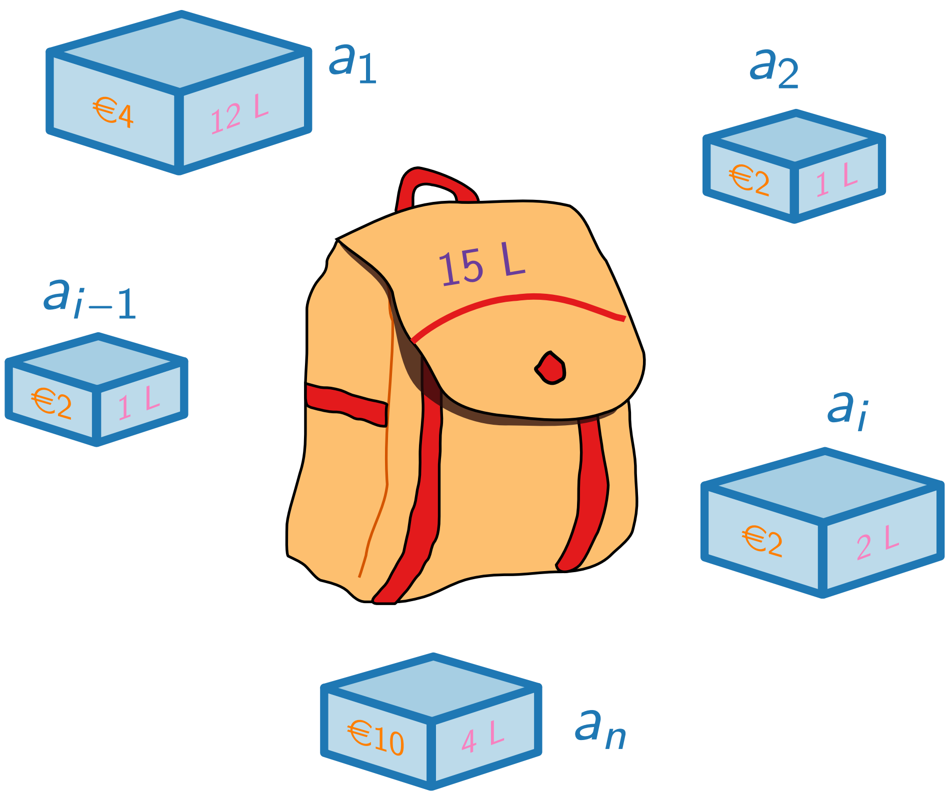

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The running time of a polynomial algorithm for Π is polynomial in $|I|$.

The running time of a **pseudo-polynomial algorithm** is polynomial in $|I|_u$.

The running time of a pseudo-polynomial algorithm may not be polynomial in $|I|$.

Strong NP-Hardness

An optimization problem is called **strongly NP-hard** if it remains NP-hard under unary encoding.

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Theorem. A strongly NP-hard problem has no pseudo-polynomial algorithm unless P = NP.

Approximation Algorithms

Lecture 8: Approximation Schemes and the KNAPSACK Problem

Part III: Pseudo-Polynomial Algorithm for KNAPSACK

Pseudo-Polynomial Alg. for KNAPSACK

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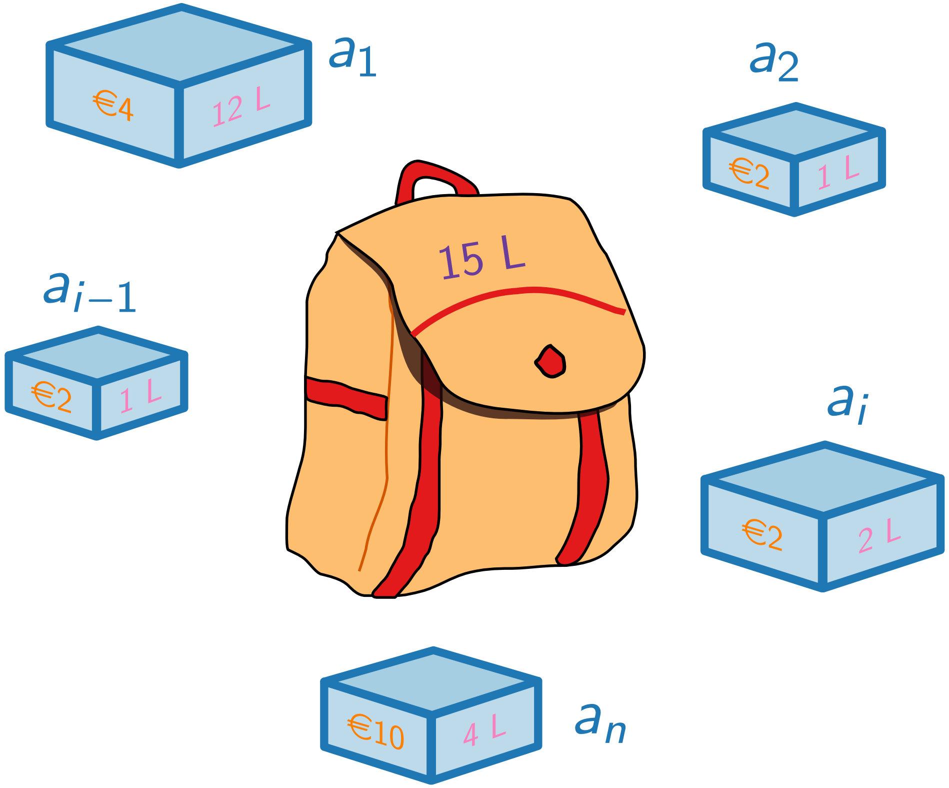

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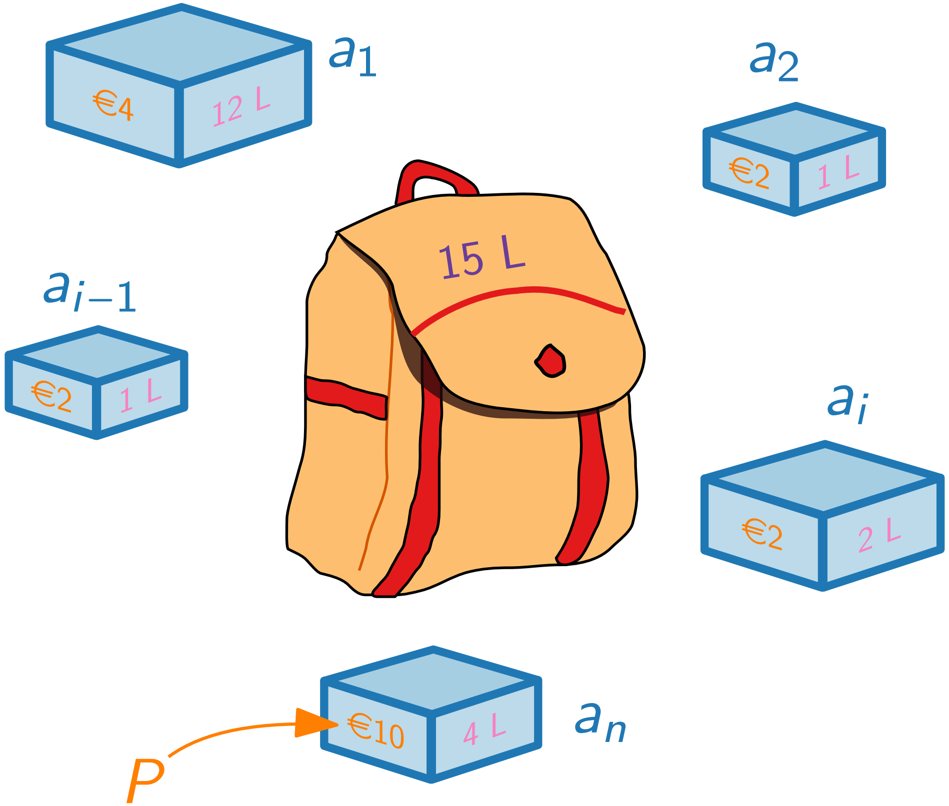

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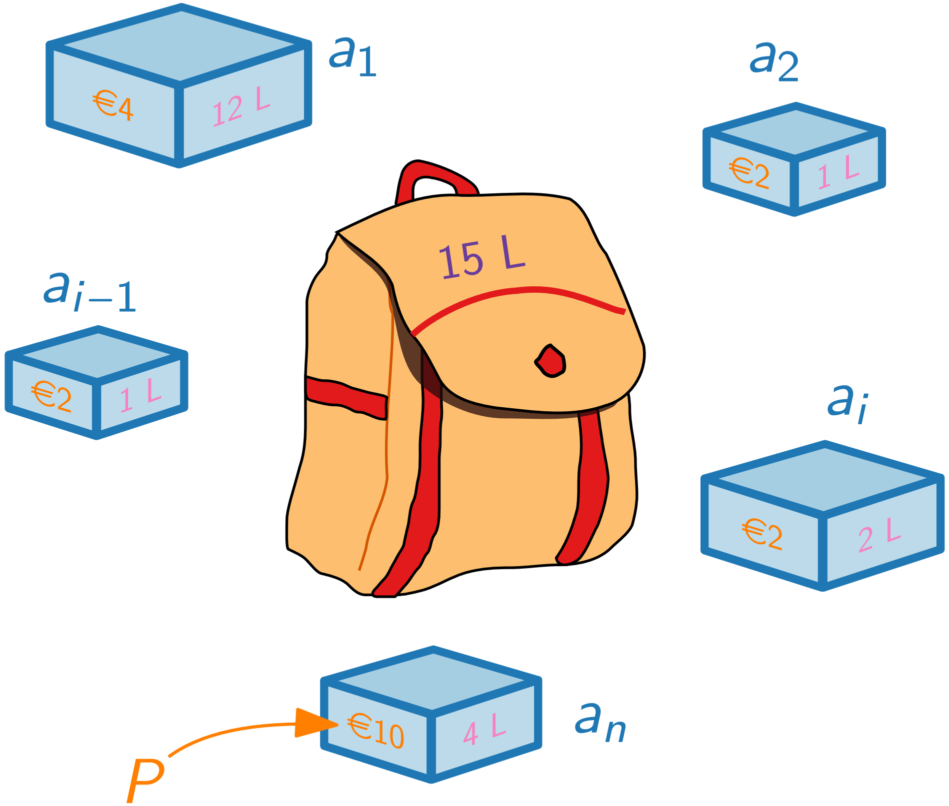

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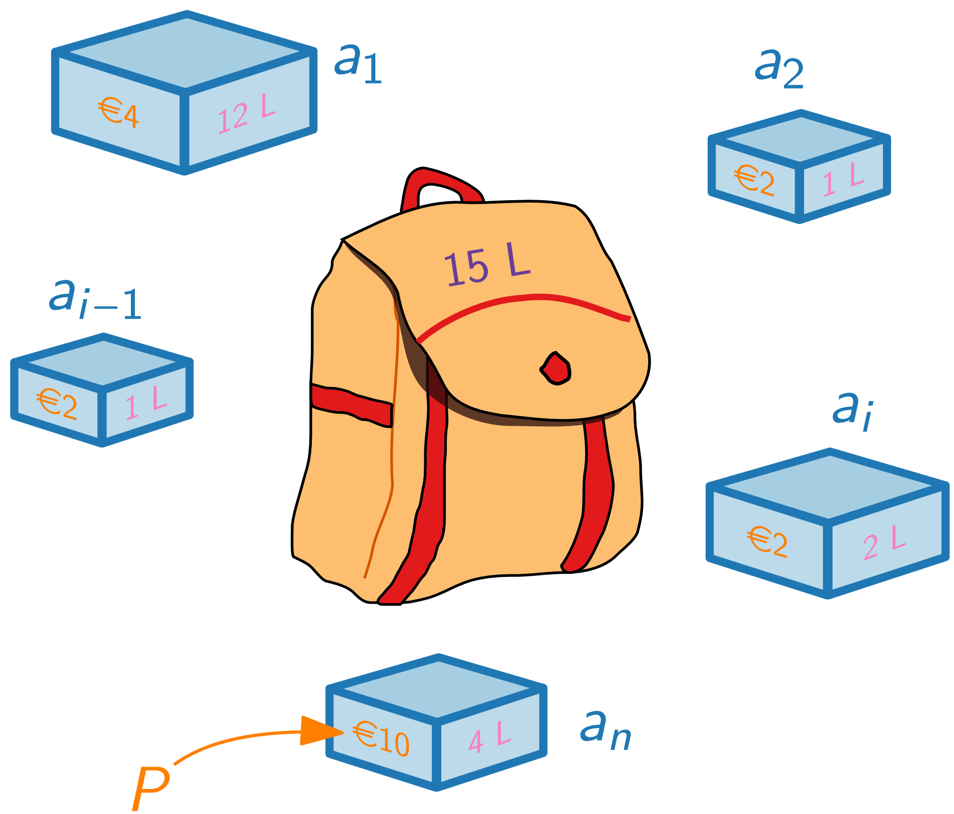

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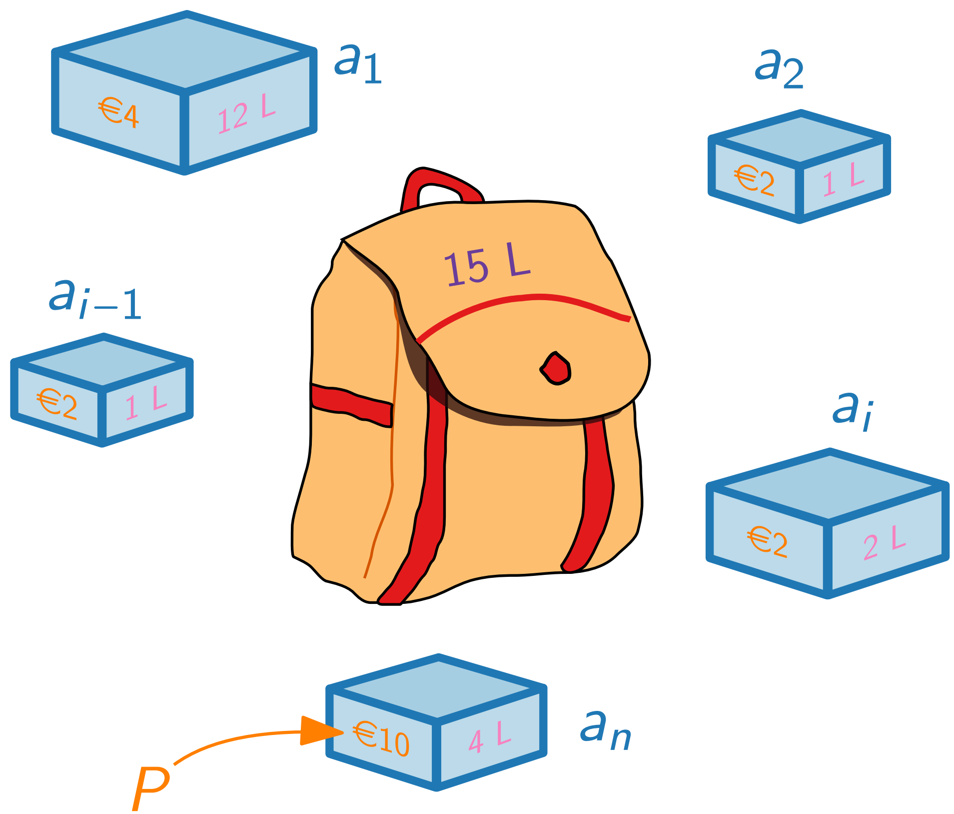

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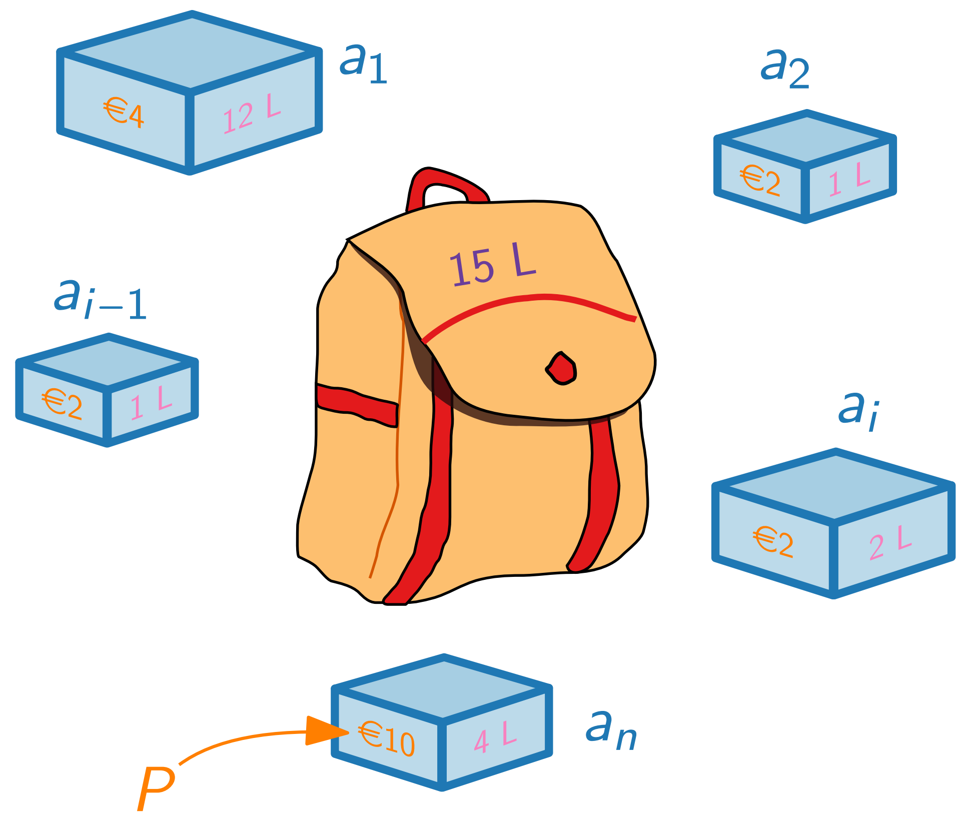

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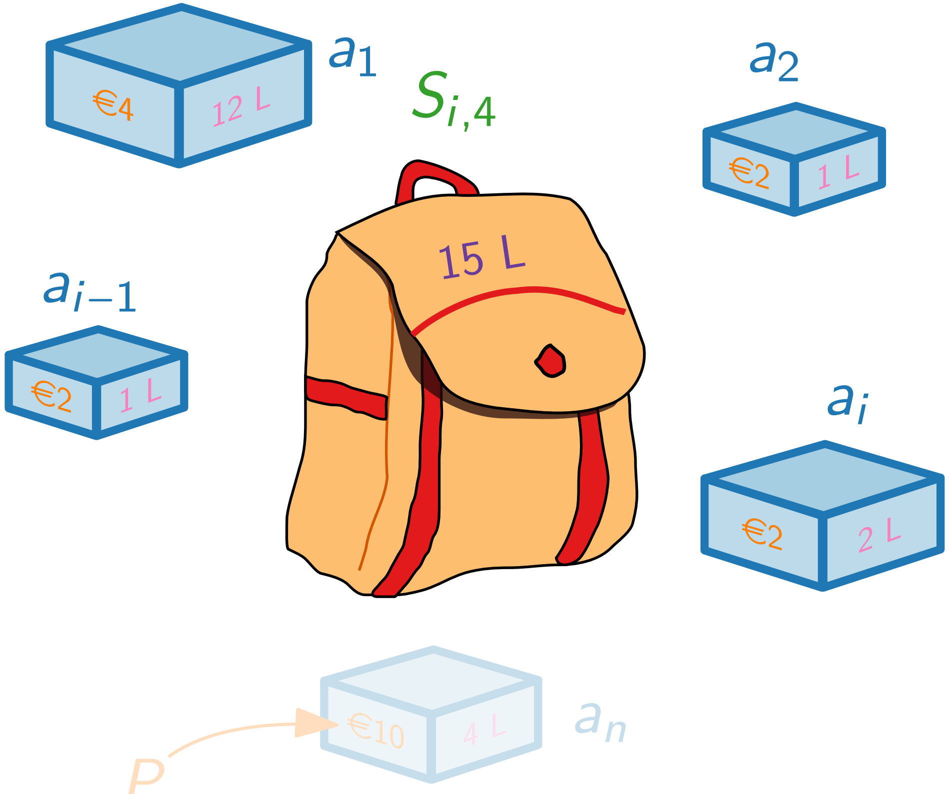

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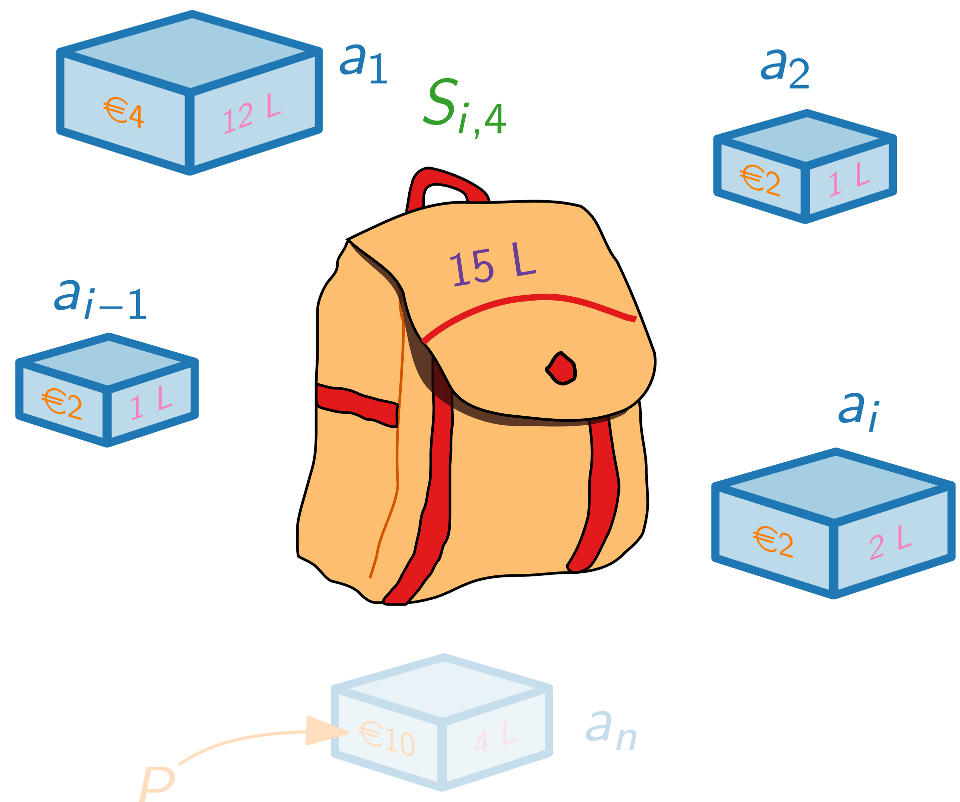

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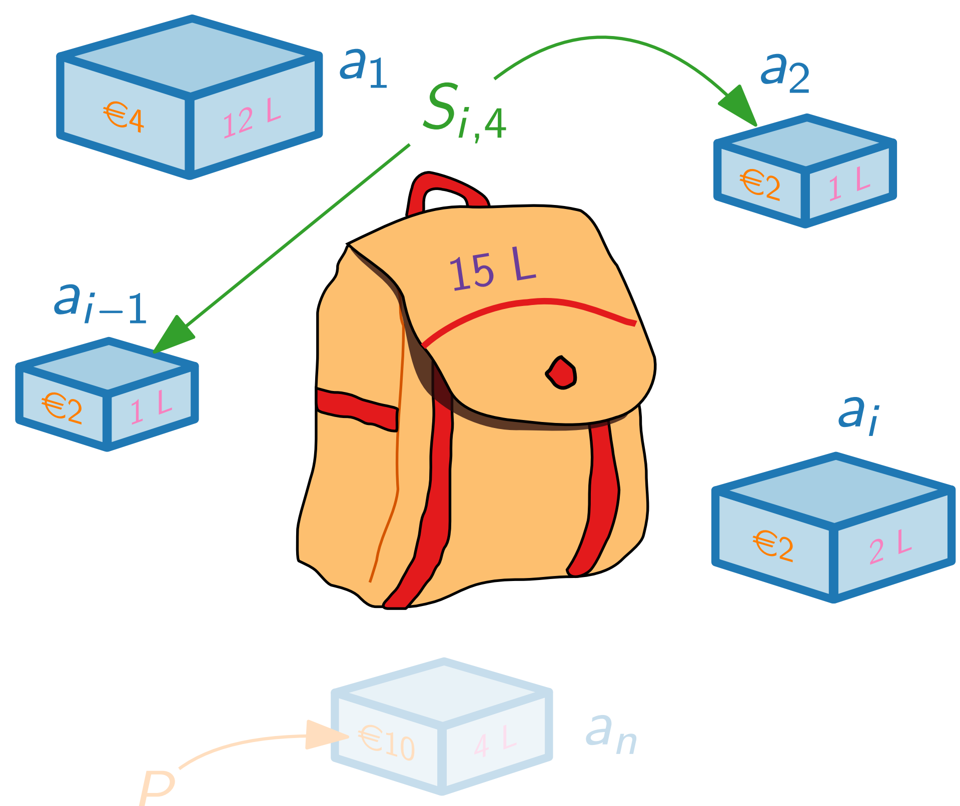

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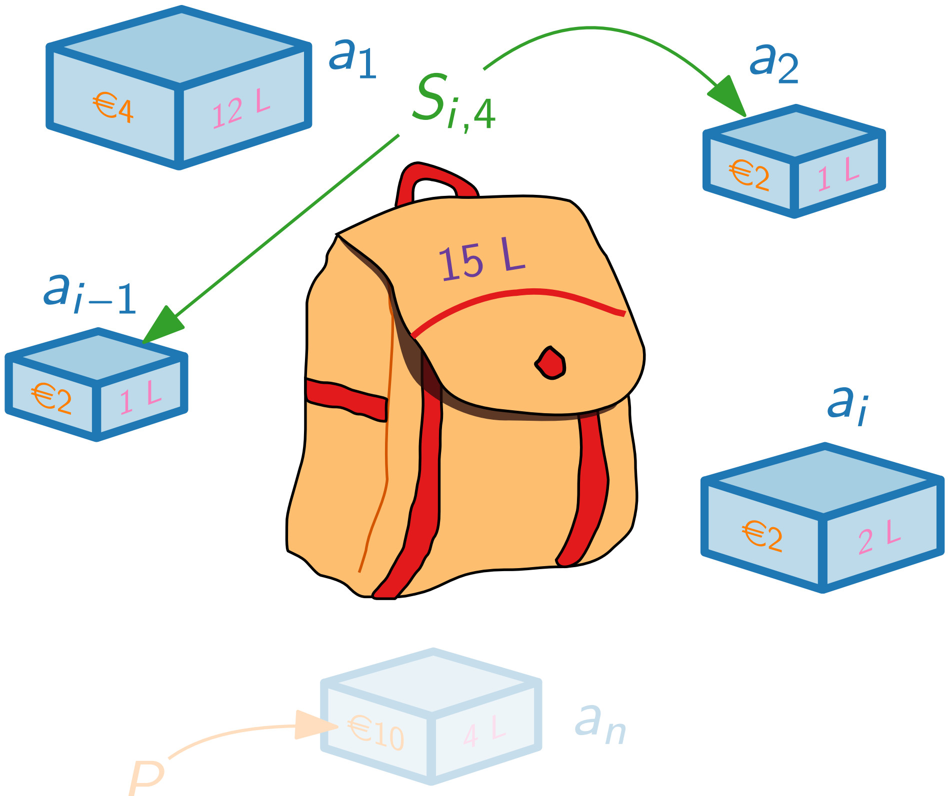

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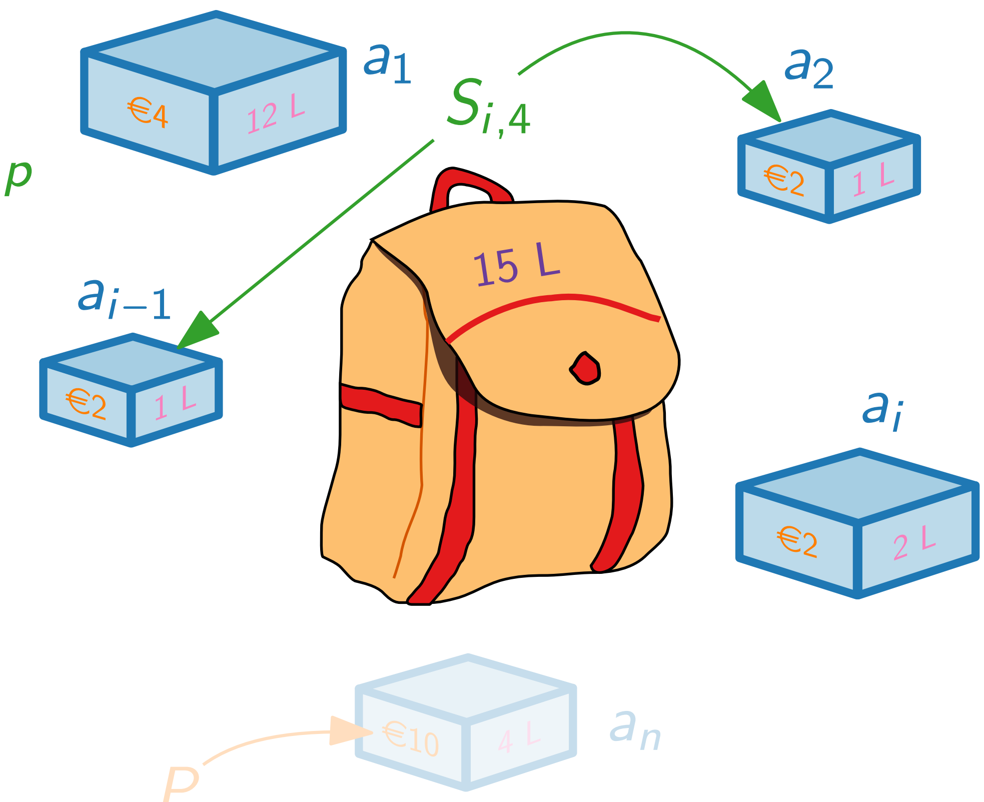

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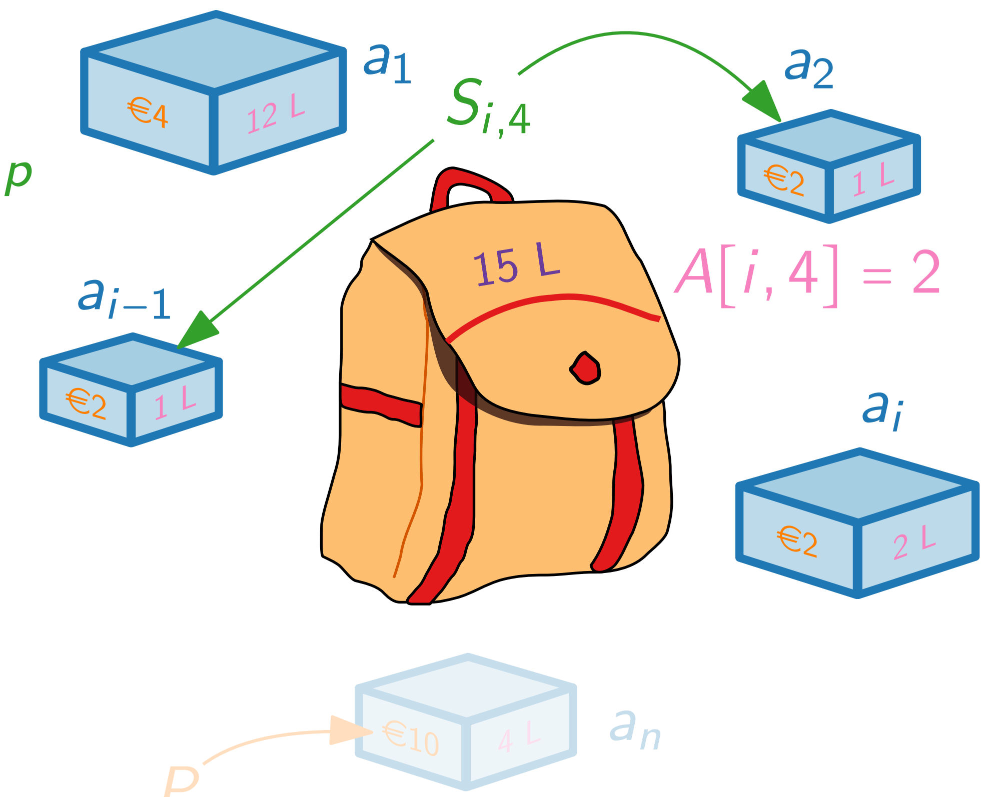

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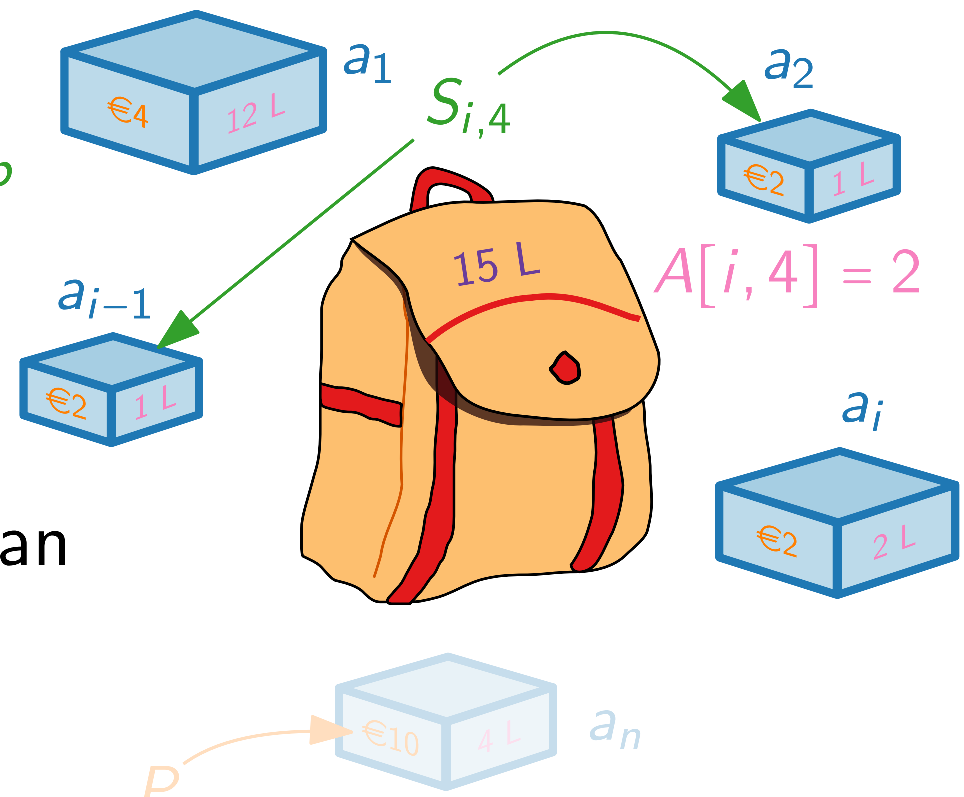

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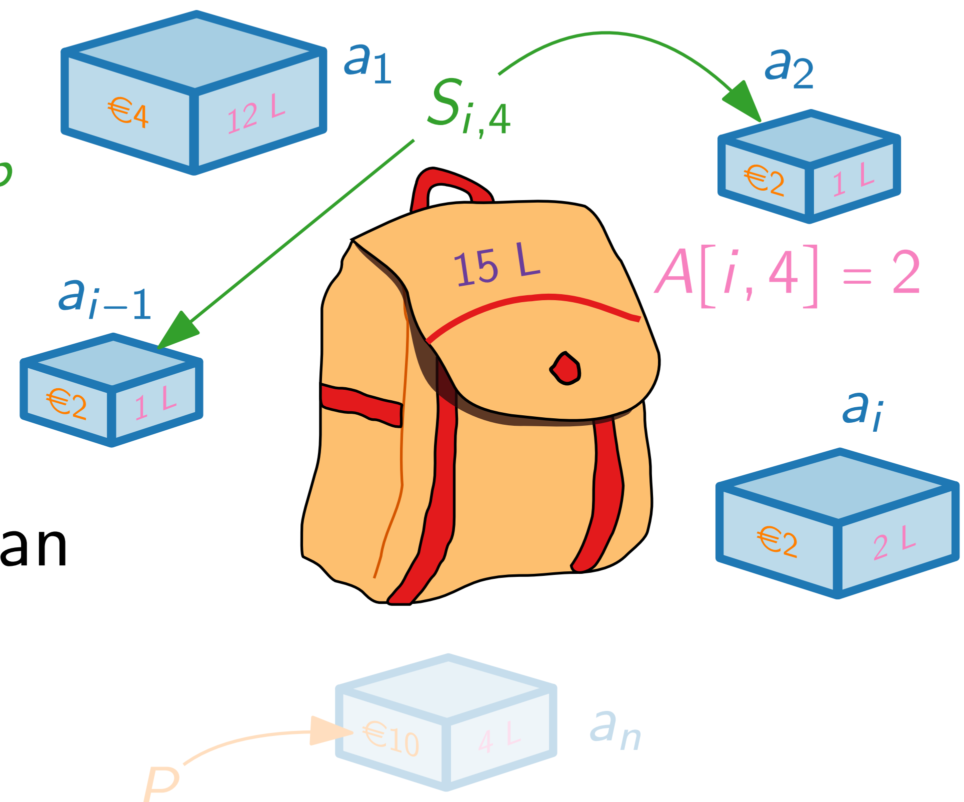
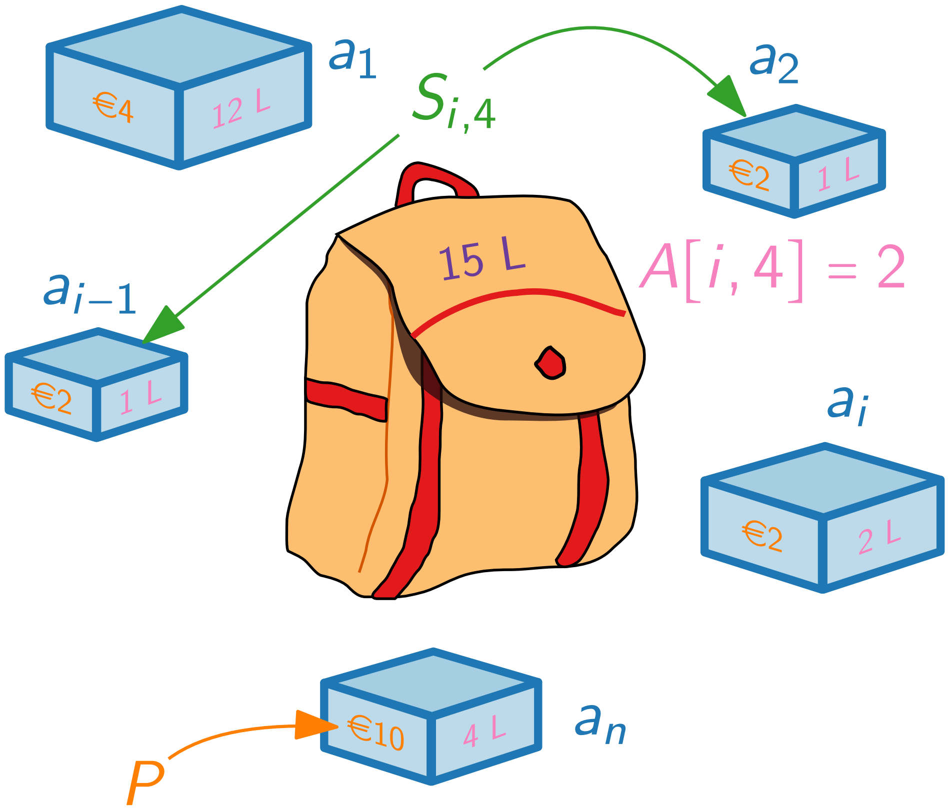
If all $A[i, p]$ are known, then we can compute $\text{OPT} = \max\{\, p \mid A[n, p] \leq B \,\}$.



Pseudo-Polynomial Alg. for KNAPSACK

$A[1,p]$ can be computed for every $p \in \{0,\ldots,nP\}$.



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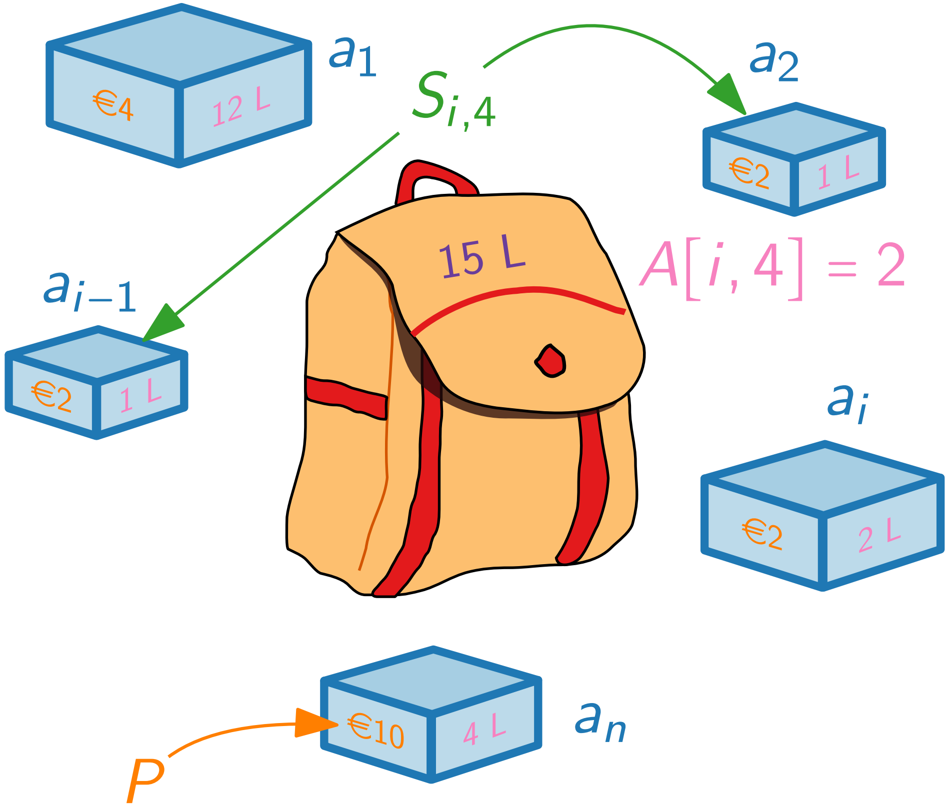

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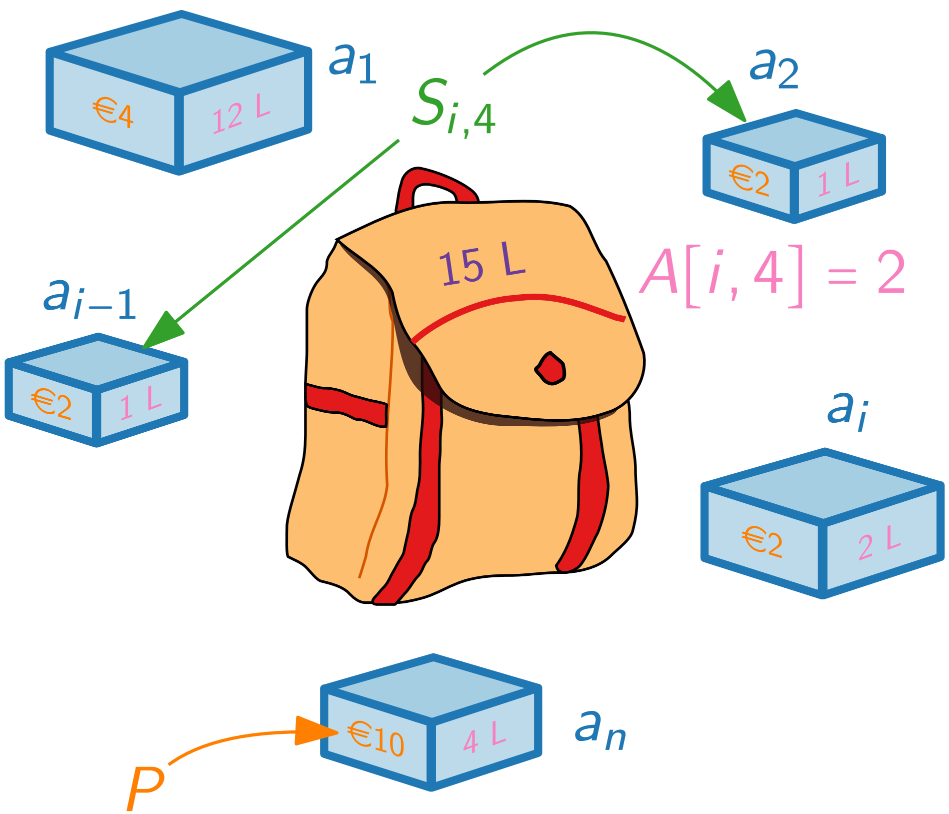

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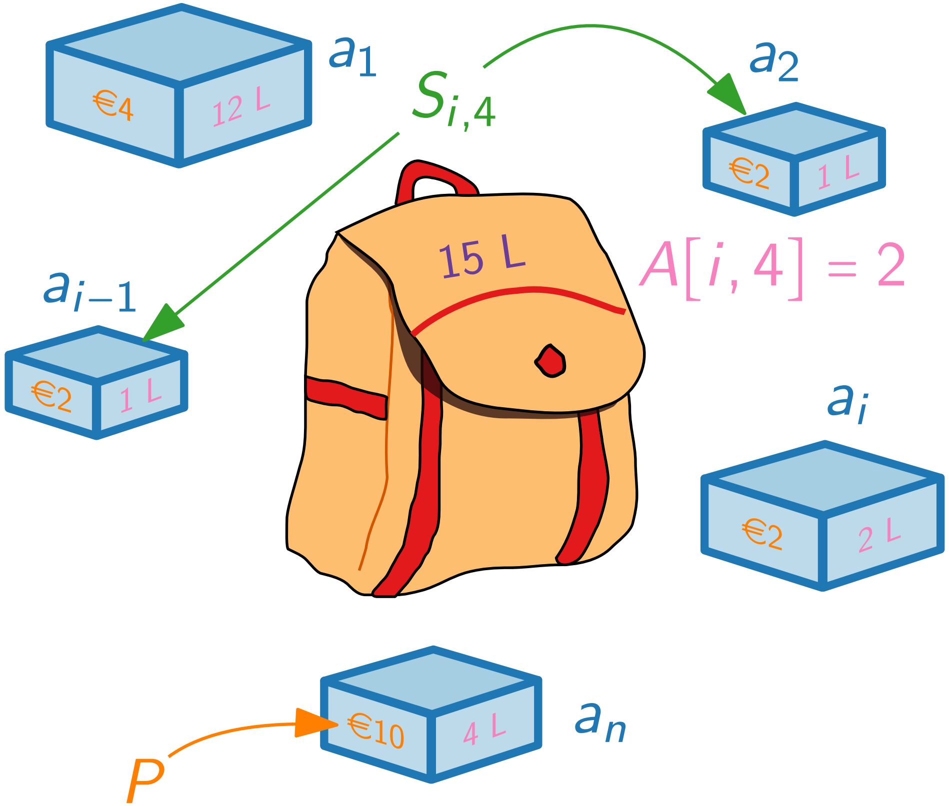

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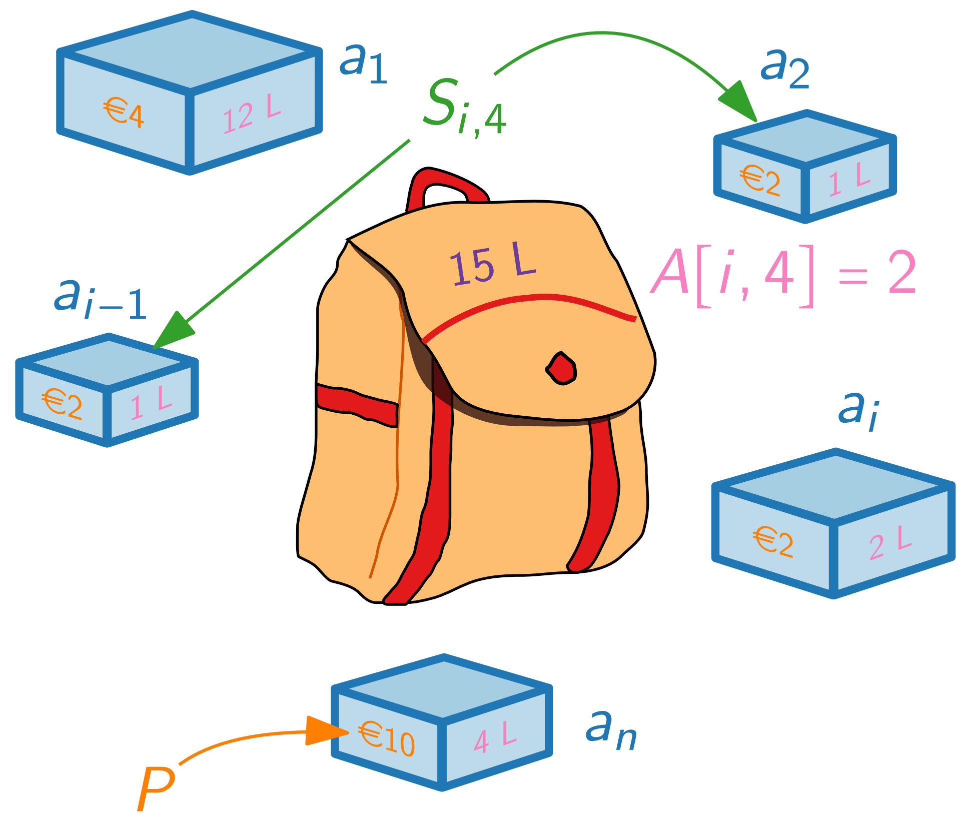

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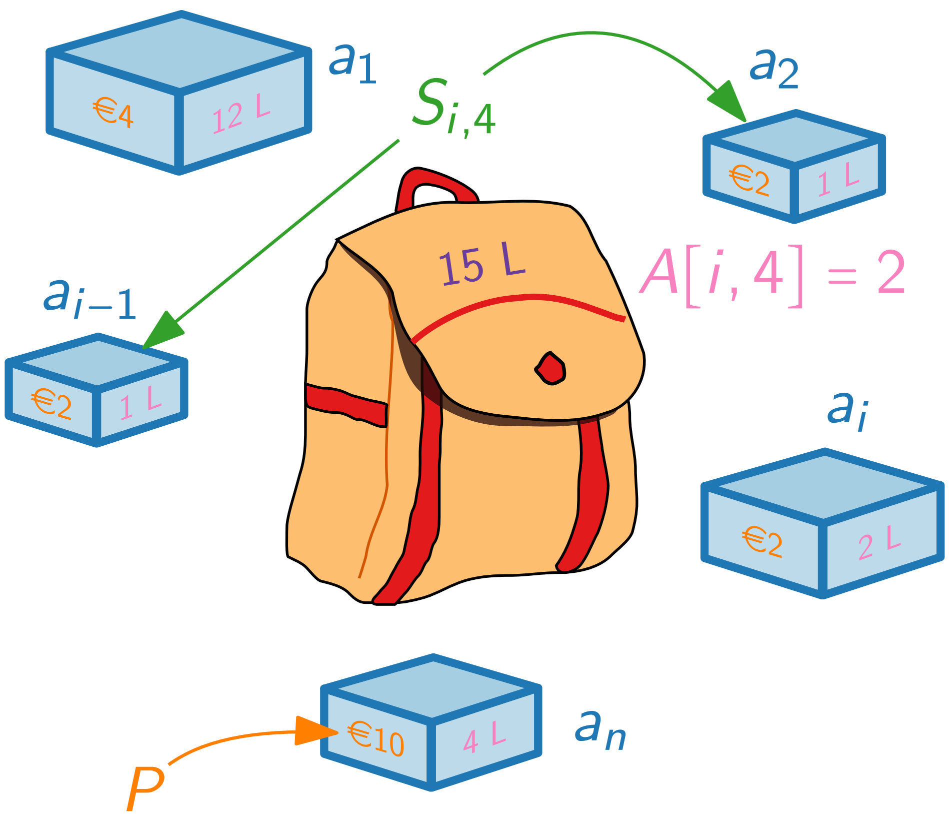

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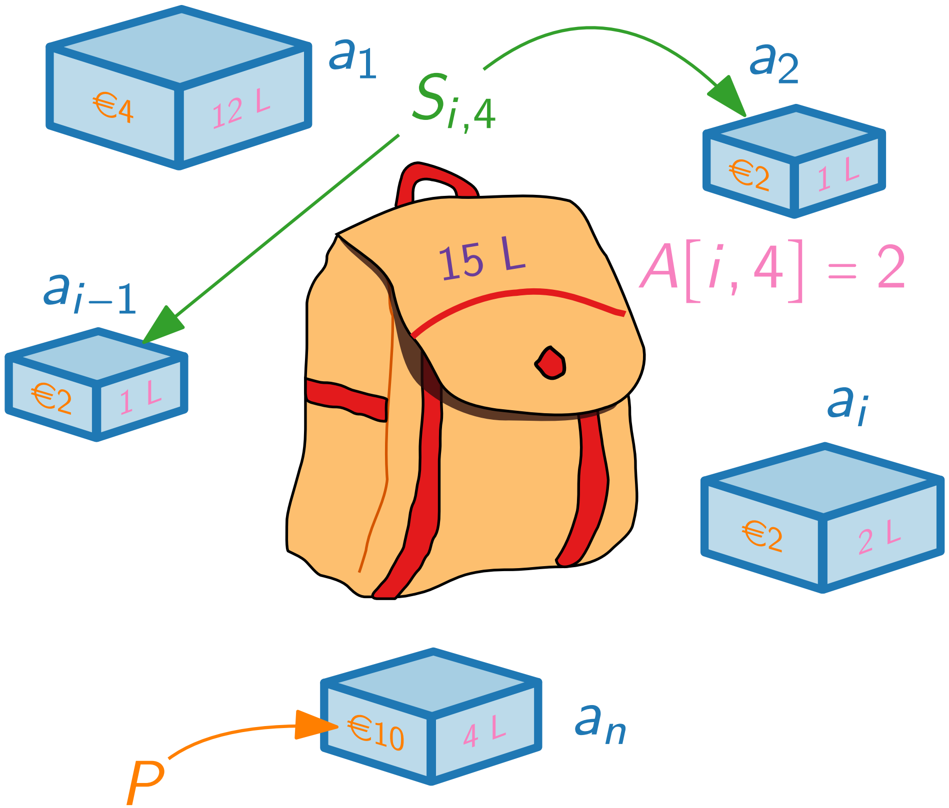

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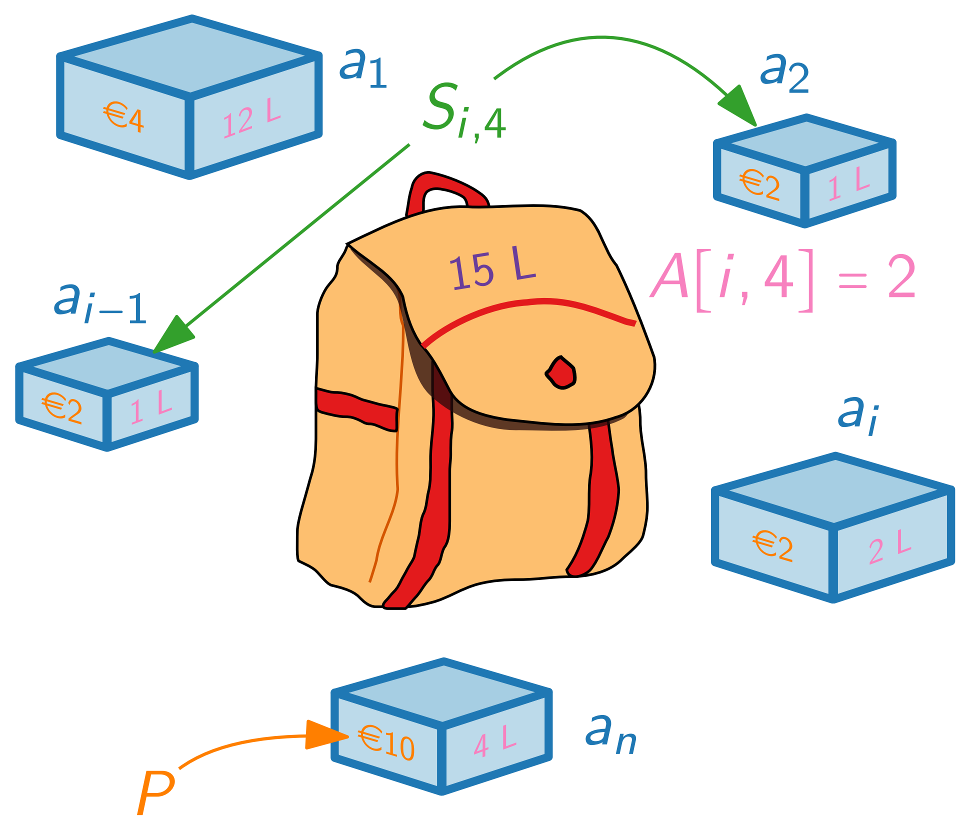

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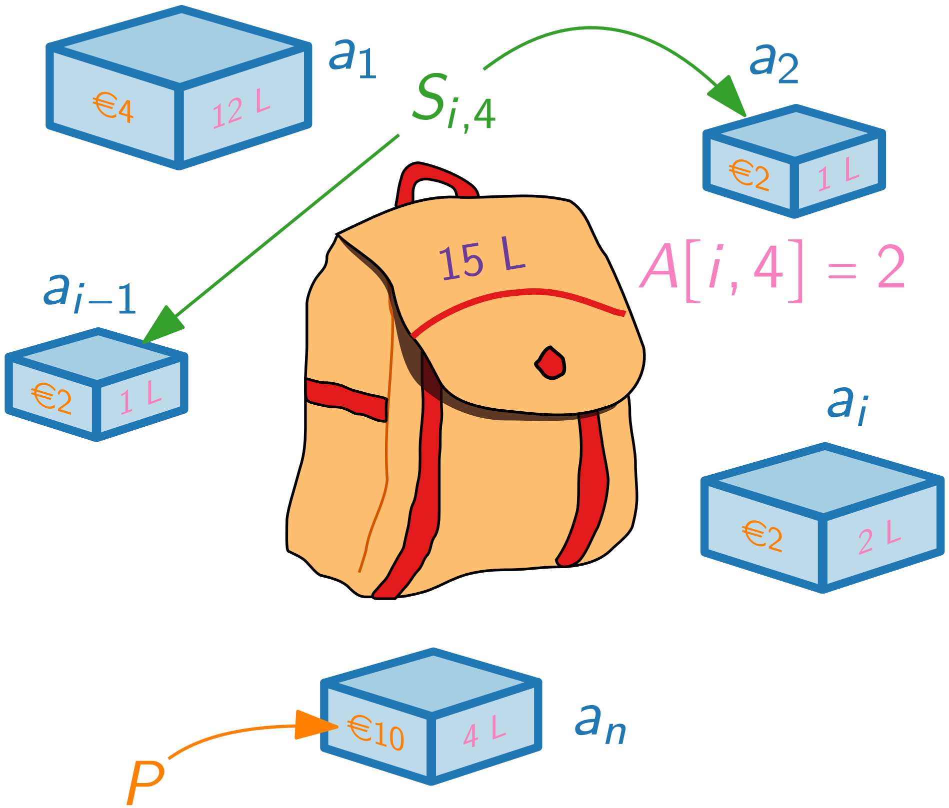
$\Rightarrow$ All values $A[i, p]$ can be computed in total time $O(n^2 P)$.



Pseudo-Polynomial Alg. for KNAPSACK

$A[1,p]$ can be computed for every $p \in \{0,\ldots,nP\}$.

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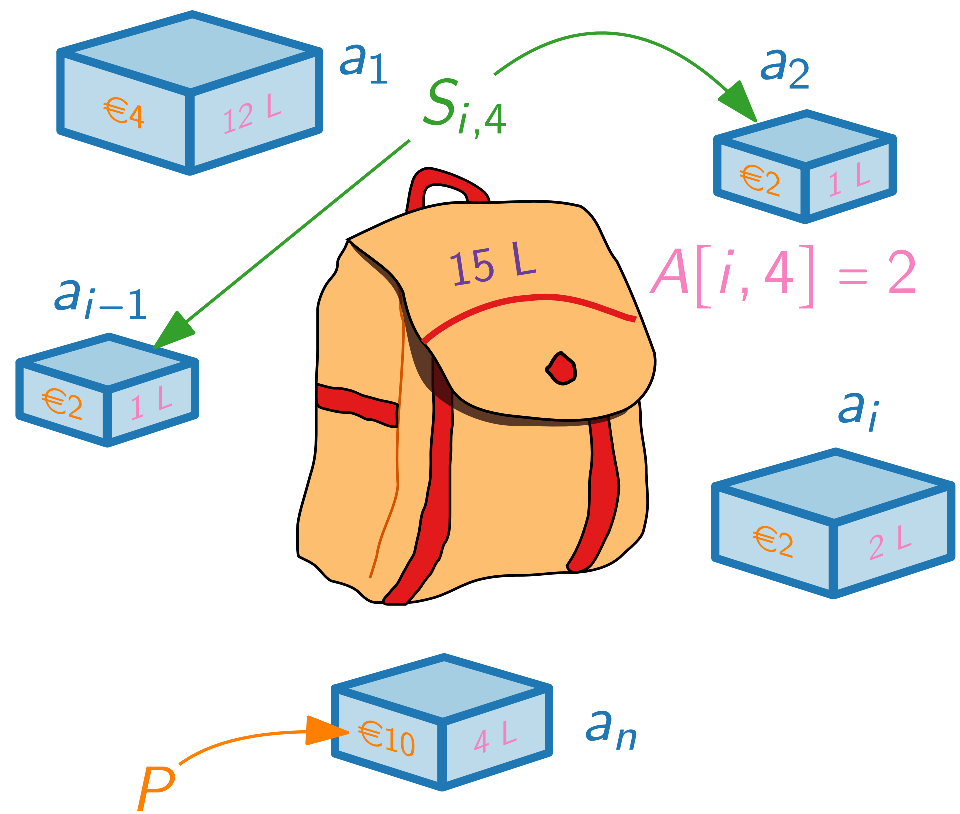

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Observe. The running time $O(n^2 P)$ is polynomial in n if P is polynomial in n.

Approximation Algorithms

Lecture 8:
Approximation Schemes and the Knapsack Problem

Part IV:
Approximation Schemes

Approximation Schemes

Let Π be an optimization problem.

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Approximation Algorithms

Lecture 8:
Approximation Schemes and the Knapsack Problem

Part V:
FPTAS for Knapsack

An FPTAS for KNAPSACK via Scaling

FPTAS idea: **Scale** profits to polynomial size (as required by the error parameter ε)...

An FPTAS for Knapsack via Scaling

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Lemma. $\text{profit}(S') \geq (1-\varepsilon) \cdot \text{OPT}$.

Proof. Let $\text{OPT} = \{o_1, \dots, o_\ell\}$.

Obs. 1. For $i = 1, \dots, \ell$: $\text{profit}(o_i) - K \leq K \cdot \text{profit}'(o_i) \leq \text{profit}(o_i)$

$\Rightarrow K \cdot \sum_i \text{profit}'(o_i) \geq \text{OPT} - \ell K \geq \text{OPT} - nK = \text{OPT} - \varepsilon P$.

Obs. 2. $\text{profit}(S') \geq K \cdot \text{profit}'(S') \geq K \cdot \sum_i \text{profit}'(o_i) \geq \text{OPT} - \varepsilon P$

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Theorem. KnapsackScaling is an FPTAS for Knapsack with running time $O(n^3/\varepsilon)$

An FPTAS for KNAPSACK via Scaling

KnapsackScaling (I, ε)
  $K = \varepsilon P / n$ // scaling factor
  for $i = 1$ **to** n **do** $\text{profit}'(a_i) = \lfloor \text{profit}(a_i)/K \rfloor$
  Compute optimal solution S' for I w.r.t. $\text{profit}'(\cdot)$.
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Theorem. KnapsackScaling is an FPTAS for KNAPSACK with running time $O(n^3/\varepsilon) = O\left(n^2 \cdot \frac{P}{\varepsilon P/n}\right)$.

Approximation Algorithms

Lecture 8:
Approximation Schemes and the KNAPSACK Problem

Part VI:
Connections Between the Concepts

FPTAS and Pseudo-Polynomial Algorithms

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Corollary. Let Π be an NP-hard optimization problem that fulfills the restrictions above.
If Π is strongly NP-hard, then there is no FPTAS for Π (unless P = NP).