Approximation Algorithms

Lecture 7: Scheduling Jobs on Parallel Machines

Part I: ILP & Parametric Pruning

Alexander Wolff

Winter term 2025

Scheduling on Parallel Machines

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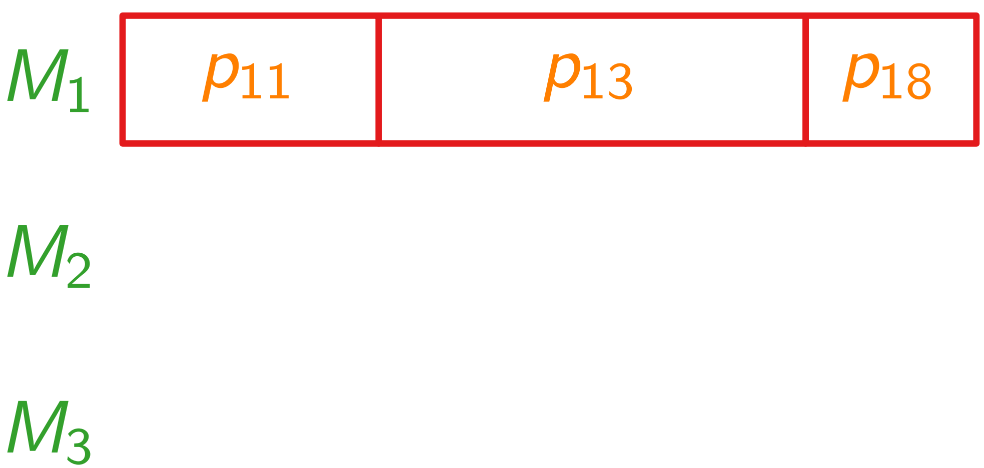

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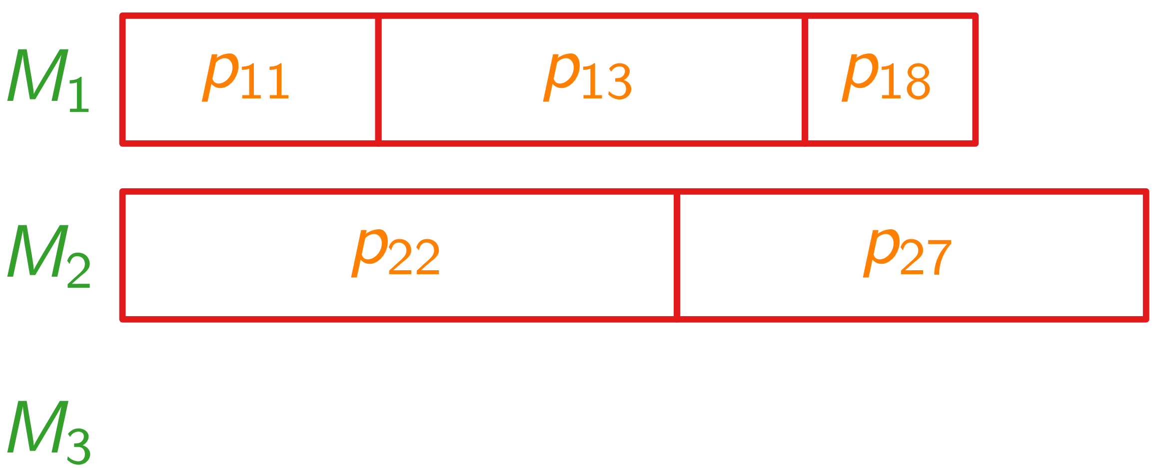

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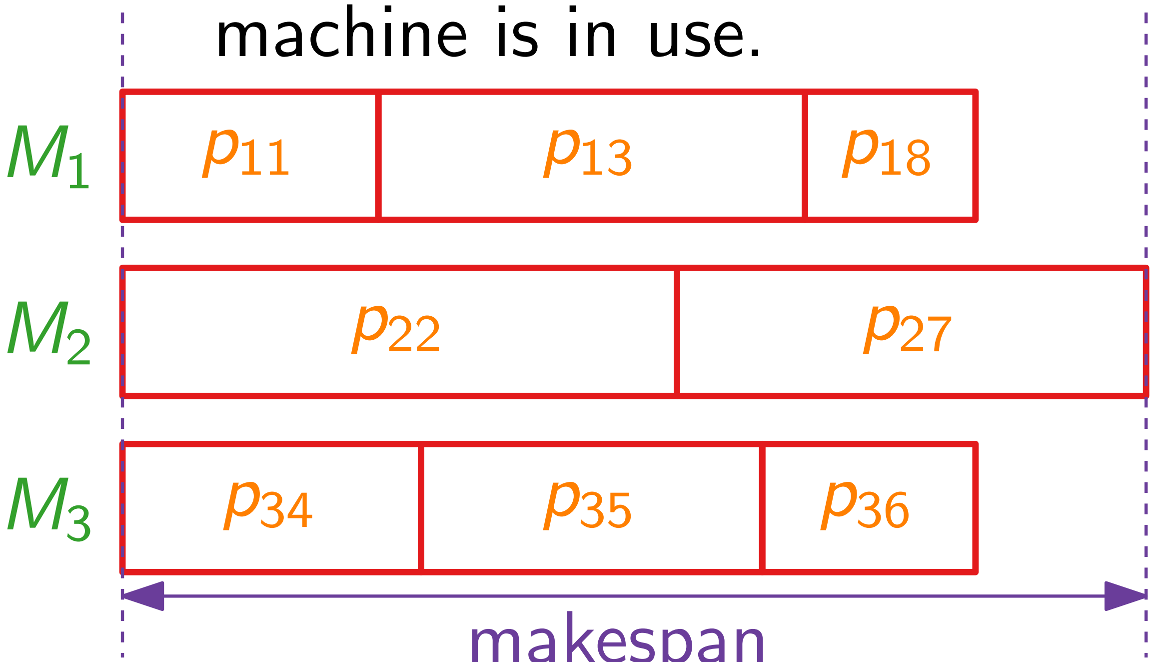

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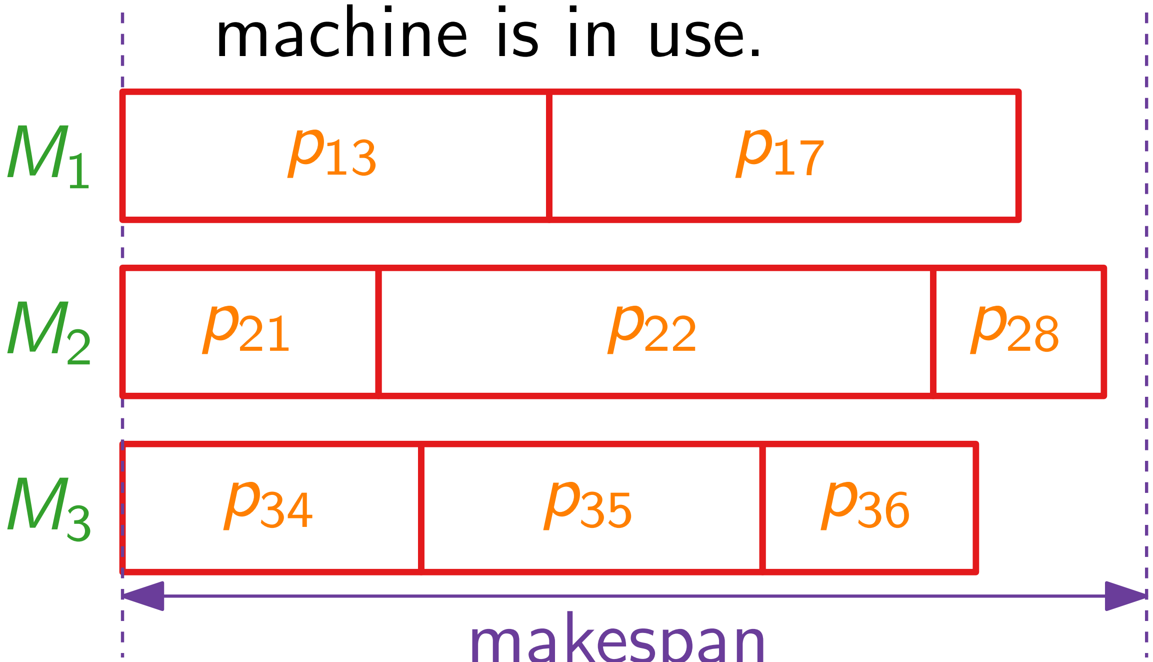

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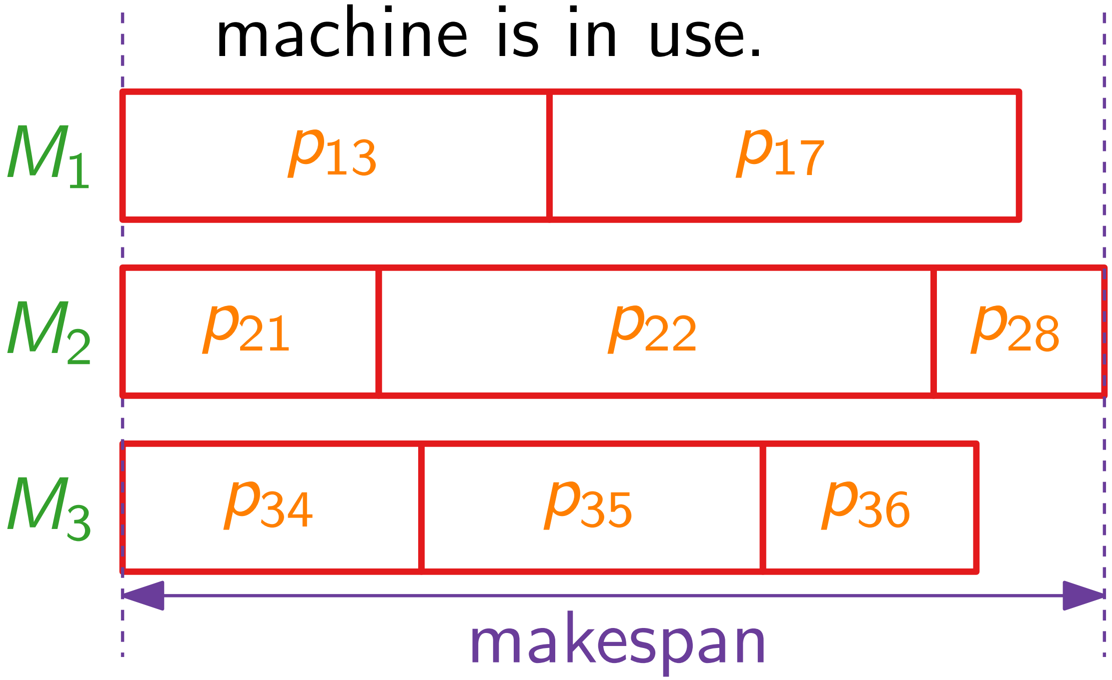

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Solution: m machines and one job with processing time m
$\Rightarrow$ OPT $= m$ and $\text{OPT}_{\text{frac}} = 1$.

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Strengthen the ILP $\rightarrow$ implicit (non-linear) constraint:

If $p_{ij} > t$, then set $x_{ij} = 0$.

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But why does this LP give a good integrality gap?

Approximation Algorithms

Lecture 7:
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Part II:
Properties of Extreme-Point Solutions

Properties of Extreme Point Solutions

Use binary search to find the smallest T so that LP(T) has a solution.

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Lemma 1.
Every extreme-point solution of LP(T) has at most $|\mathcal{J}| + |\mathcal{M}|$ positive variables.

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$$x_{ij} \geq 0, \qquad (i,j) \in S_T$$

Lemma 1.
Every extreme-point solution of LP(T) has at most $|\mathcal{J}| + |\mathcal{M}|$ positive variables.

Lemma 2.
Every extreme-point solution of LP(T) sets at least $|\mathcal{J}| - |\mathcal{M}|$ jobs integrally.

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$\Rightarrow \beta \le |\mathcal{M}|$ and $\alpha \ge |\mathcal{J}| - |\mathcal{M}|$ $\square$

Evaluation of the Lecture and the Tutorial

- Please participate **now**! Check your email.
- It takes only a few minutes.
- We get the results only if at least 6 students participate.
- Please evaluate Alexander Wolff (for the lecture) and Antonio Lauerbach (for the tutorial).

Approximation Algorithms

Lecture 7:
Scheduling Jobs on Parallel Machines

Part III:
An Algorithm

Extreme Point Solutions of LP(T)

Definition: Bipartite graph $G = (\mathcal{M} \cup \mathcal{J}, E)$ with $(i, j) \in E \Leftrightarrow x_{ij} \neq 0$ (in extreme-point sol.).

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And why is this useful . . . ?

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Theorem. This is a factor-2 approximation algorithm (assuming that we have an F-perfect matching).

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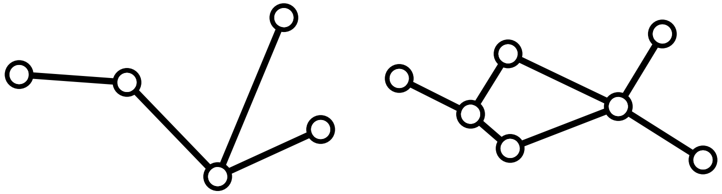
Approximation Algorithms

Lecture 7:
Scheduling Jobs on Parallel Machines

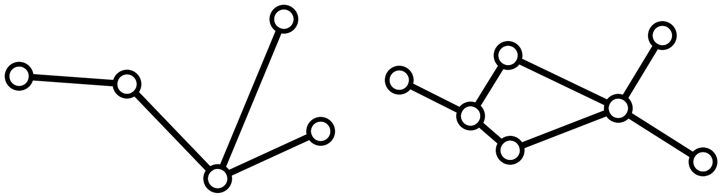
Part IV:
Pseudo-Trees and -Forests

Pseudo-Trees and -Forests



Pseudo-tree: a connected graph with at most as many edges as vertices.

Pseudo-Trees and -Forests



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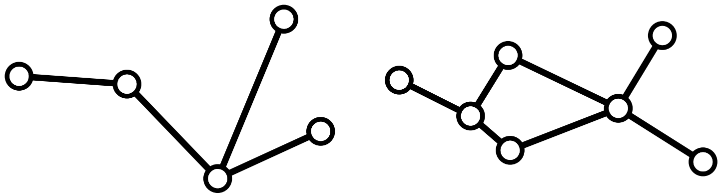

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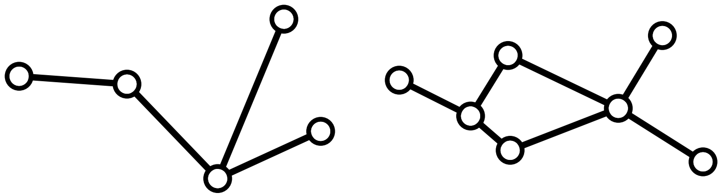
Pseudo-Trees and -Forests



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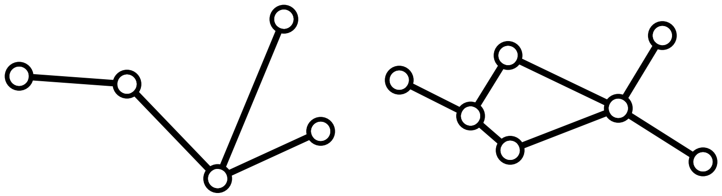

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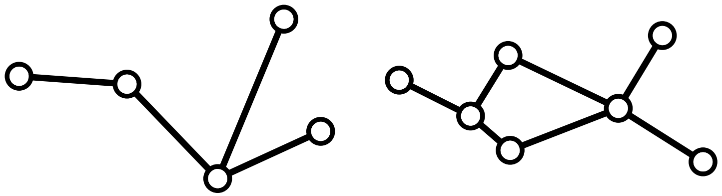

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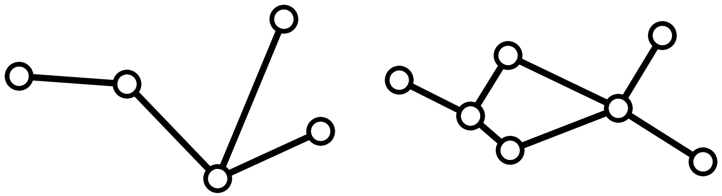

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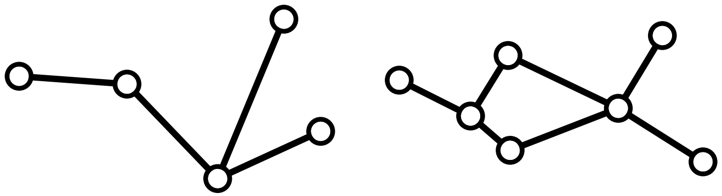

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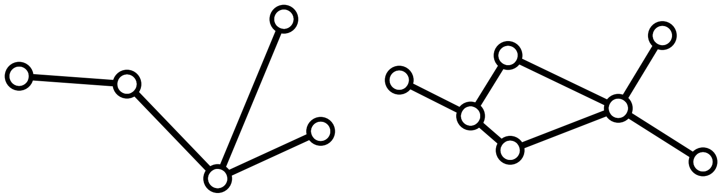

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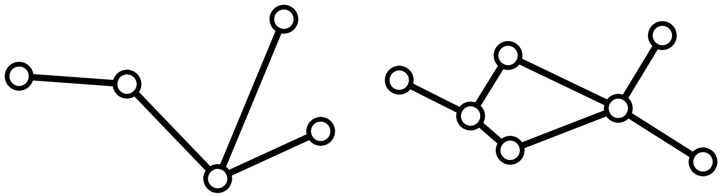

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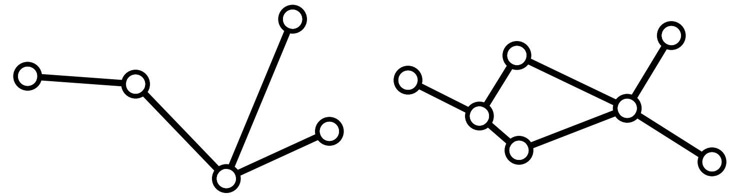

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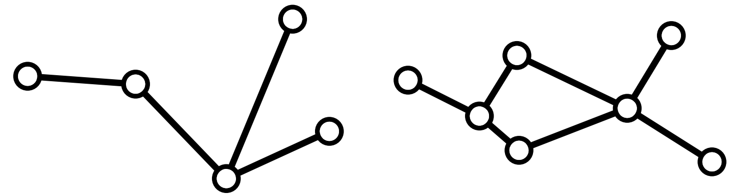

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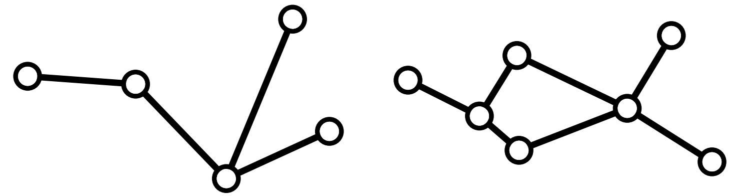

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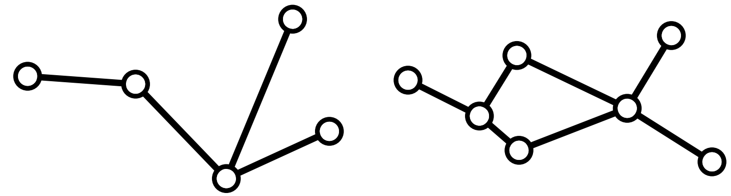

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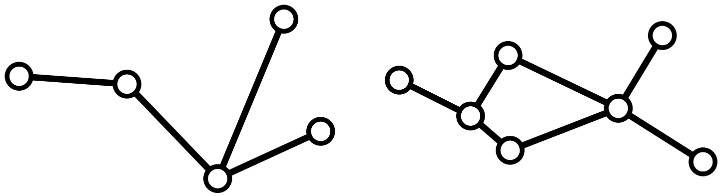

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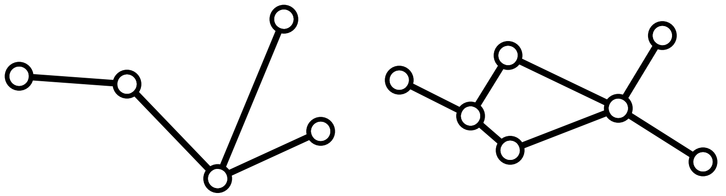

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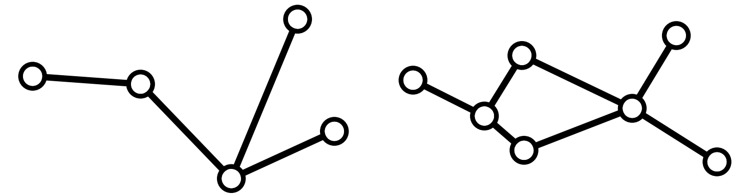

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