Approximation Algorithms

Lecture 6:
k-Center via Parametric Pruning

Part I:
Metric *k*-Center

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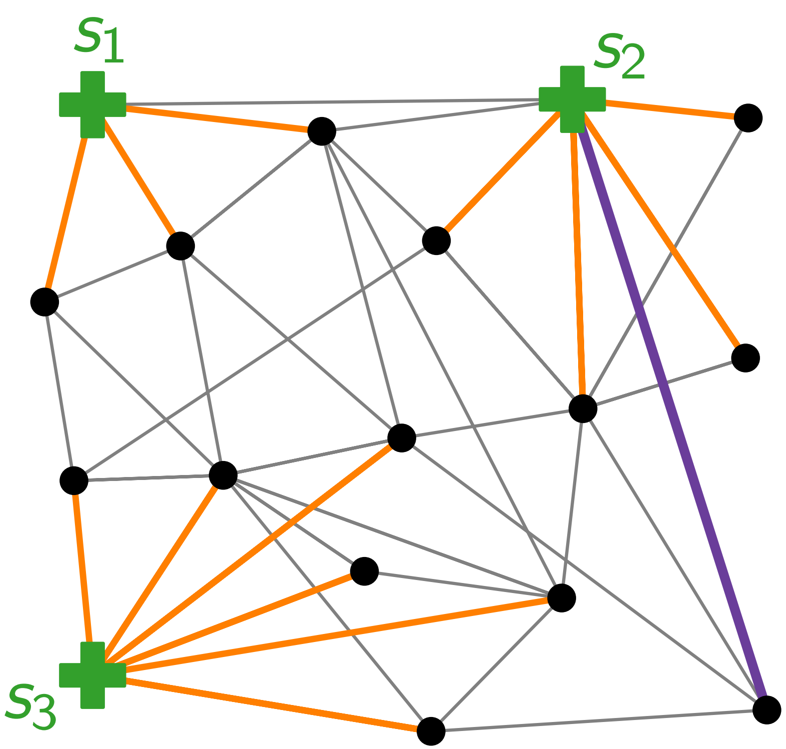
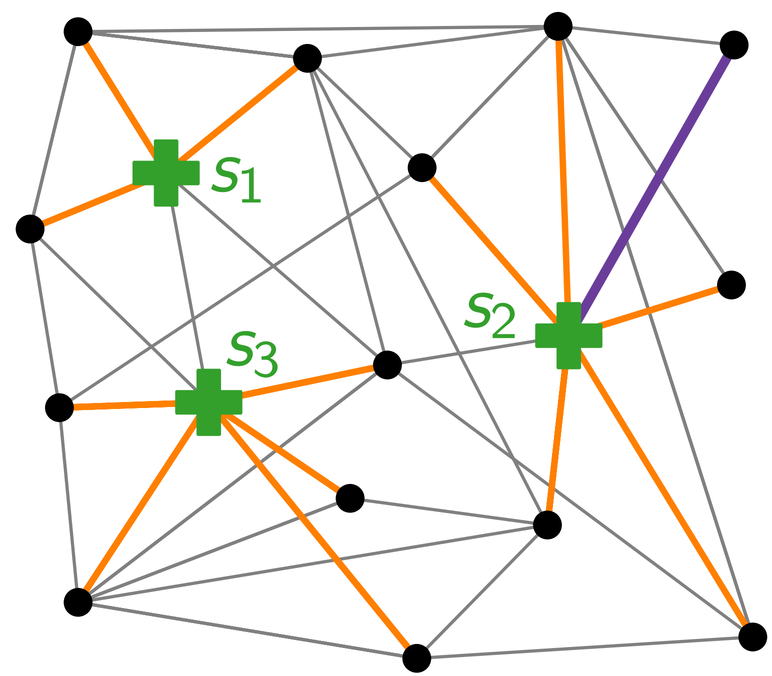
Metric k-Center

Given: A complete graph G with edge costs $c\colon E(G) \to \mathbb{Q}_{\geq 0}$ satisfying the triangle inequality and a natural number $k \leq |V(G)|$.

Given a set $S \subseteq V(G)$, $c(v, S)$ is the cost of the cheapest edge from v to a vertex in S.

Find: A k-element vertex set S that minimizes $\text{cost}(S) := \max_{v \in V} c(v, S)$.





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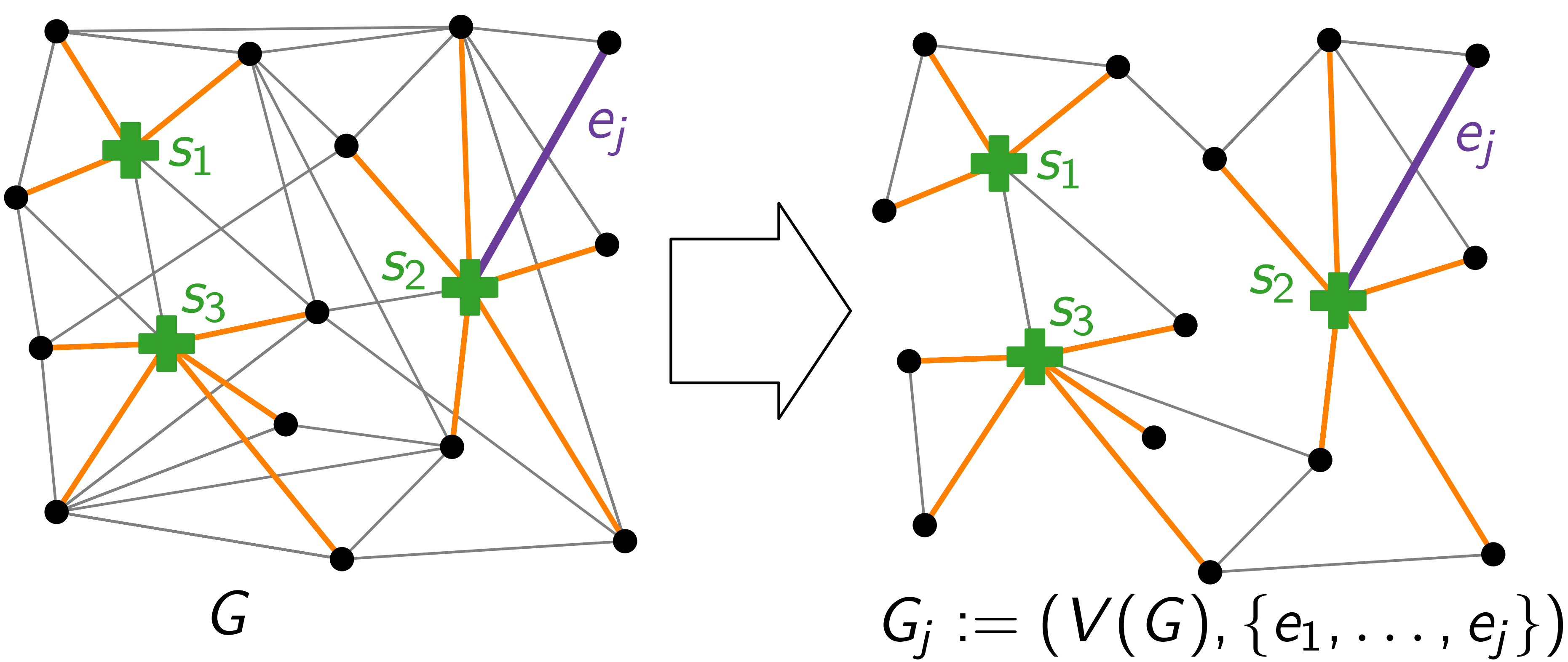
Part II:
Parametric Pruning

Parametric Pruning

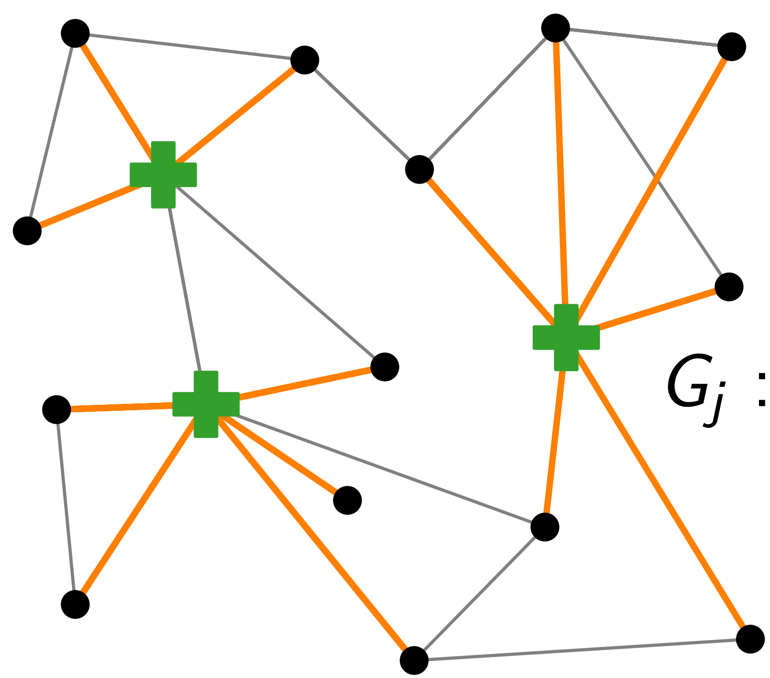
Let $E(G) = \{e_1, \ldots, e_m\}$ with $c(e_1) \leq \cdots \leq c(e_m)$.
Suppose that we know that $\text{OPT} = c(e_j)$.



... try each G_j.

. . . try each G_j.

Def. A vertex set D of a graph H is **dominating** if each vertex is either in D or adjacent to a vertex in D. The cardinality of a smallest dominating set in H is denoted by $\text{dom}(H)$.



$\text{dom}(G_j) = 3$

$G_j := (V(G), \{e_1, \ldots, e_j\})$

. . . but computing $\text{dom}(H)$ is NP-hard.

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Part III:
Square of a Graph

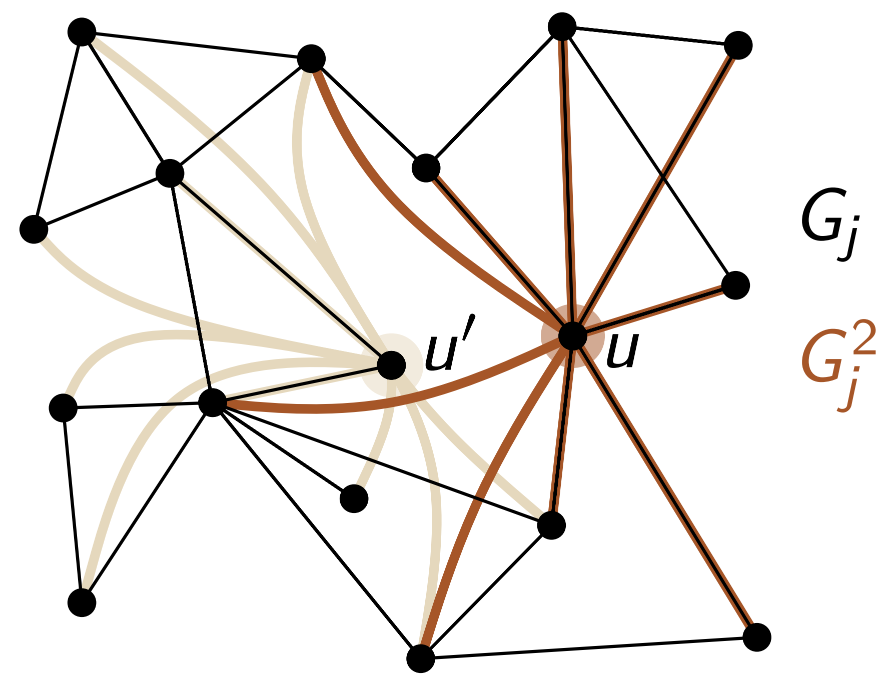
Square of a Graph

Idea: Find a small dominating set in a "coarsened" G_j.

Def. The **square** H^2 of a graph H has the same vertex set as H. Two vertices $u \neq v$ are adjacent in H^2 iff they are within distance at most **two** in H.

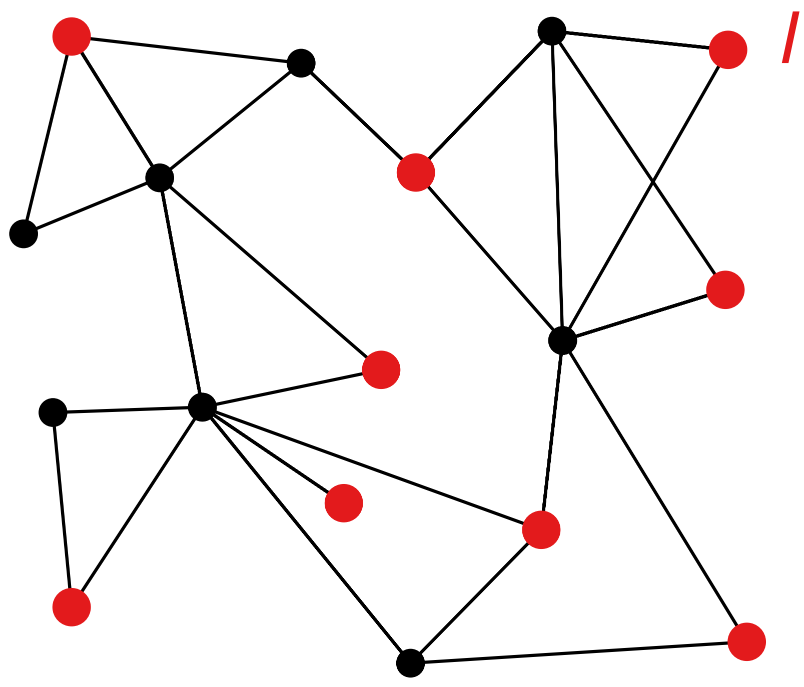
Obs. A dominating set of size at most k in G_j^2 is a 2-approximation for the metric k-Center of G.

Why? $\max_{e \in E(G_j)} c(e) = \mathrm{OPT}$ and edge costs are metric!



Independent Sets

Def. A vertex set I in a graph is called **independent** (or **stable**) if no pair of vertices in I forms an edge.
An independent set is called **maximal** if it does not have an independent superset.

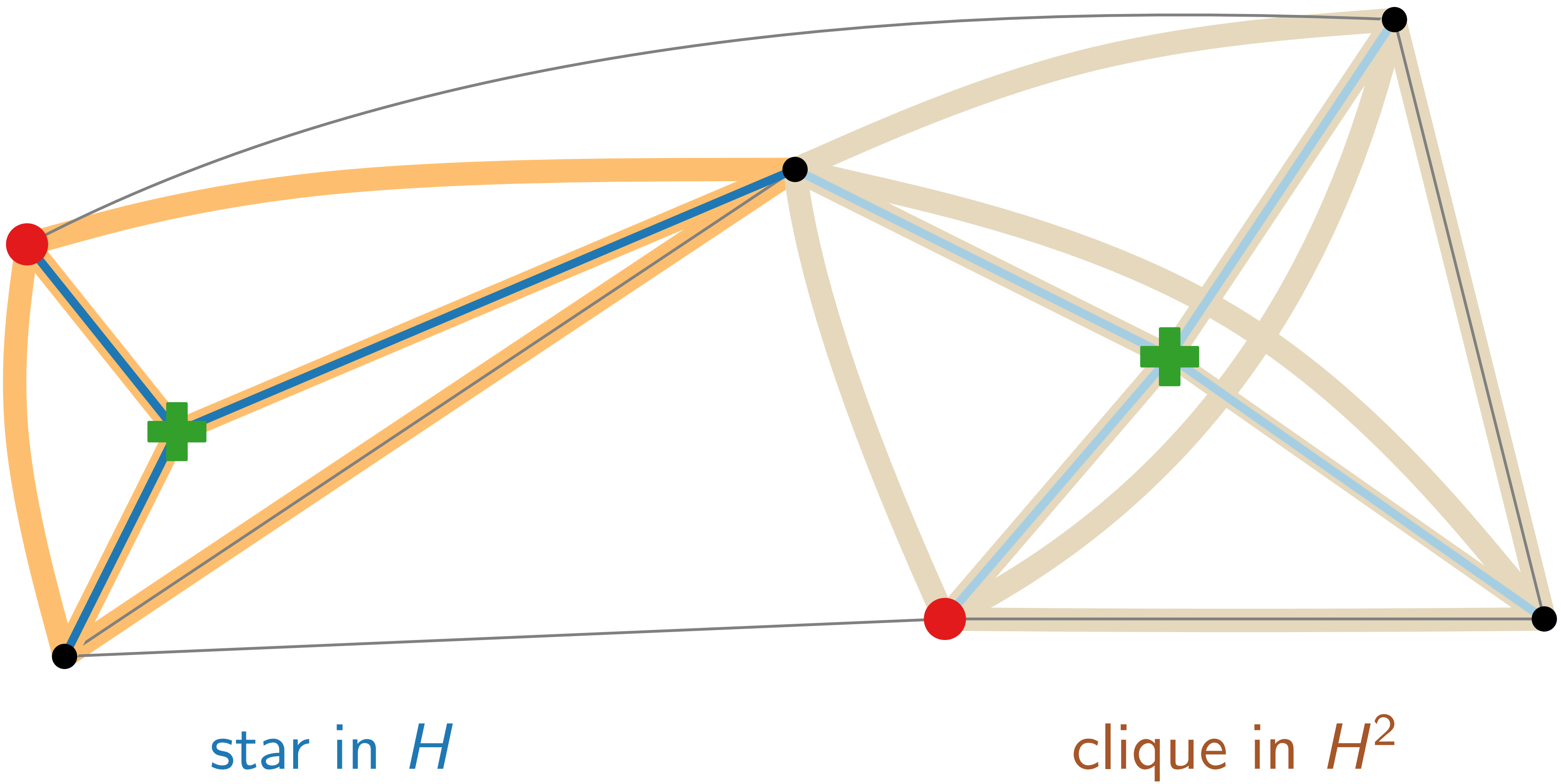


Obs. Maximal independent sets are dominating. :-)

Independent Sets in H^2

Lemma. For a graph H and an independent set I in H^2, $|I| \leq \text{dom}(H)$.

Proof. What does a dominating set of H look like in H^2?



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Part IV:
Factor-2 Approximation for Metric-k-Center

Factor-2 Approx. for Metric k-Center

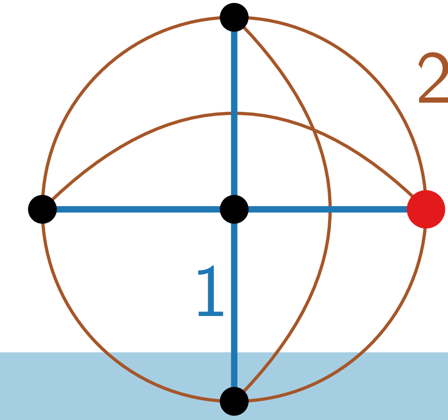
Metric-k-Center-Approx(G, c, k)

Sort the edges of G by cost: $c(e_1) \leq \cdots \leq c(e_m)$.
for $j = 1$ **to** m **do**
  Construct G_j^2.
  Find a maximal independent set I_j in G_j^2.
  if $|I_j| \leq k$ **then**
    return I_j

Lemma. For j provided by the algorithm, it holds that $c(e_j) \leq \mathrm{OPT}$.

Theorem. The above algorithm is a factor-2 approximation algorithm for the metric k-Center problem.

Can we do better . . . ?



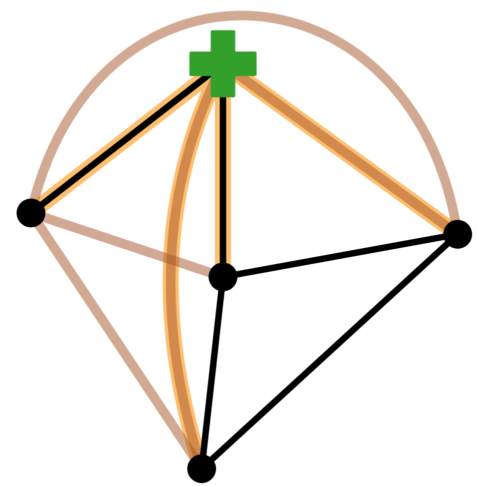
What about a tight example?

Theorem. Assuming P $\neq$ NP, for no $\varepsilon > 0$, there is a $(2-\varepsilon)$-approximation algorithm for the metric k-CENTER problem.

Proof. Reduce from DOMINATING SET to metric k-CENTER.
Given graph G and integer k, construct complete graph G' with $V(G') = V(G)$, $E(G') = E(G) \cup E'$.

with $c(e) = \begin{cases} 1, & \text{if } e \in E \\ 2, & \text{if } e \in E' \end{cases}$

← $\triangle$-inequality holds



Let S be a metric k-center of G'.
If $\mathrm{dom}(G) \leq k$, then $\mathrm{cost}(S) = 1$.
If $\mathrm{dom}(G) > k$, then $\mathrm{cost}(S) = 2$.

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Part V:
Metric-Weighted-Center

Metric-~~k~~-Center

Weighted

Given: A complete graph G with metric edge costs $c\colon E(G) \to \mathbb{Q}_{\geq 0}$ and ~~an integer $k \leq |V|$~~, vertex weights $w\colon V \to \mathbb{Q}_{\geq 0}$, and a budget $W \in \mathbb{Q}_{+}$

For $S \subseteq V(G)$,
$c(v, S)$ is the cost of the cheapest edge from v to a vertex in S.

vertex set S of weight at most W

Find: A ~~k-element vertex set S~~ such that $\text{cost}(S) := \max_{v \in V(G)} c(v, S)$ is minimized.

Algorithm for the Weighted Version

Algorithm Metric-Weighted-CENTER-Approx(G, c, w, W)

Sort the edges of G by cost: $c(e_1) \leq \cdots \leq c(e_m)$

for $j = 1$ **to** m **do**

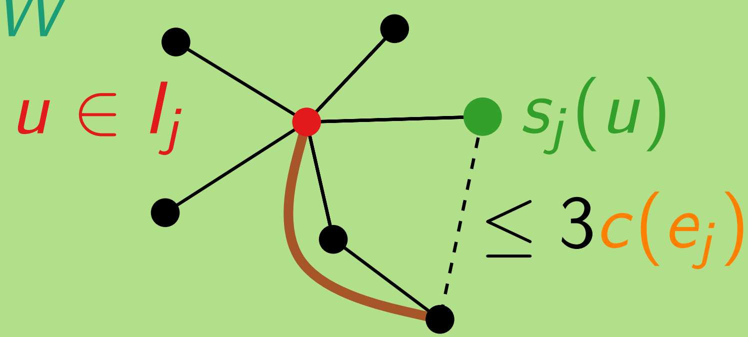
Construct G_j^2

Find a maximal independent set I_j in G_j^2

Compute $S_j := \{\, s_j(u) \mid u \in I_j \,\}$

if ~~$|I_j| \leq k$~~ $w(S_j) \leq W$ **then**

return ~~I_j~~ S_j

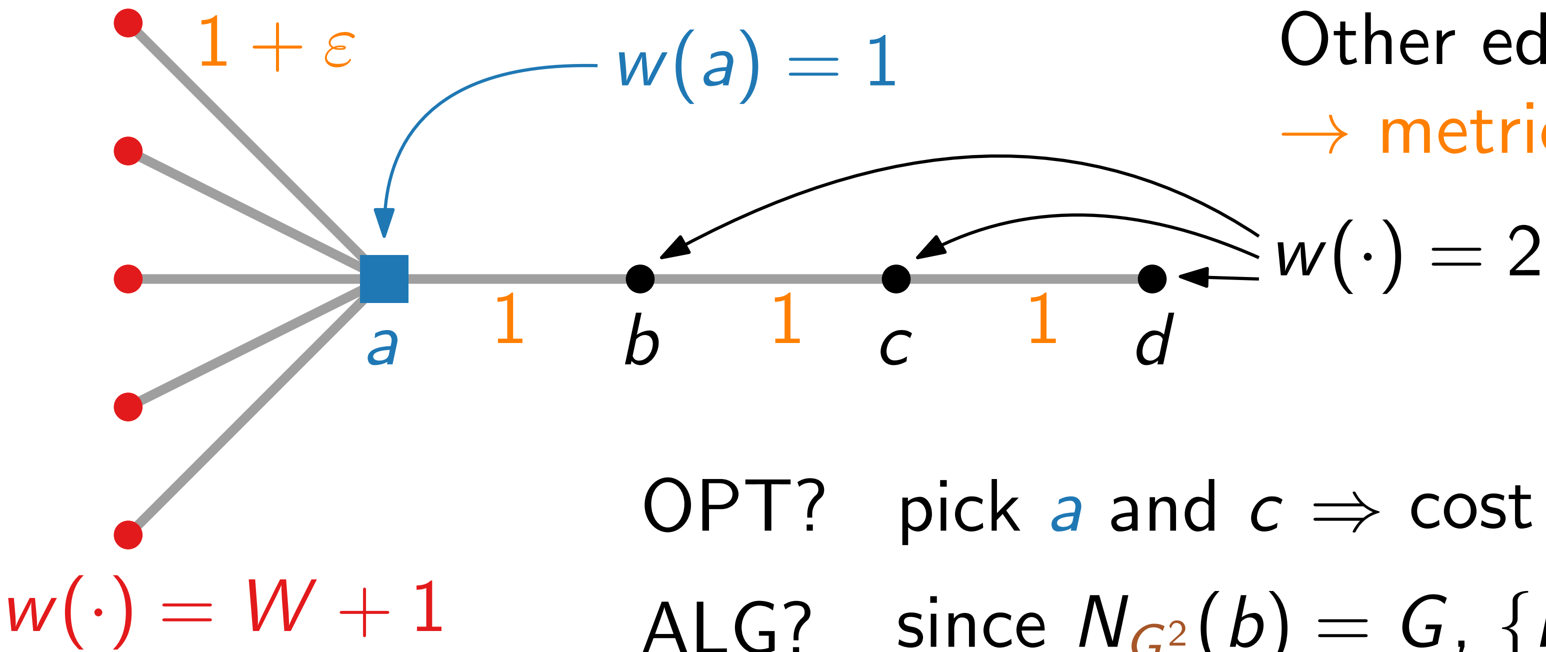


$s_j(u) :=$ lightest node in $N_{G_j}(u) \cup \{u\}$

Theorem. The above is a factor-3 approximation algorithm for METRIC-WEIGHTED-CENTER.

Tight Example. . . ?

Here, we need to have a budget W,
and edge costs satisfying the triangle inequality.



~~Consider $W = 3$.~~

Other edge costs?
$\rightarrow$ metric completion!

OPT? pick a and c $\Rightarrow$ cost $1+\varepsilon$.

ALG? since $N_{G^2}(b) = G$, $\{b\}$ is a maximal independent set in G^2
Thus, alg. picks only a $\Rightarrow$ cost 3.

How can we generalize this to larger W?