Approximation Algorithms

Lecture 6:
k-Center via Parametric Pruning

Part I:
Metric k-Center

Alexander Wolff

Winter term 2025

Metric k-Center

Given: A graph G

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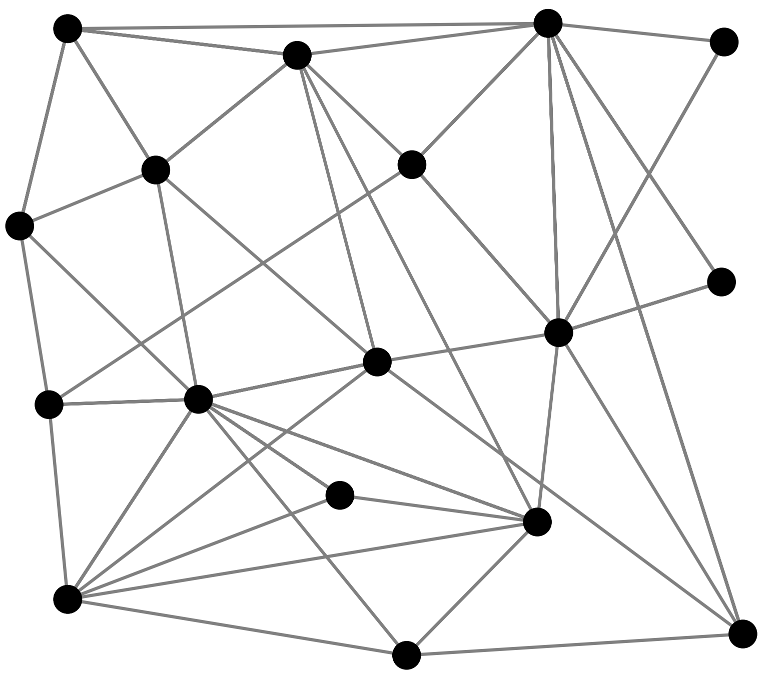
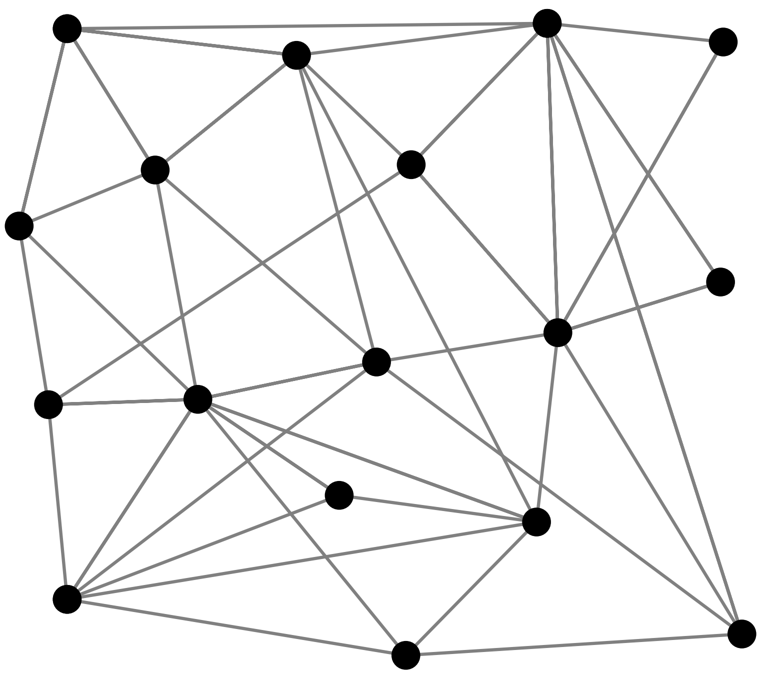


Metric k-Center

Given: A complete graph G with edge costs $c\colon E(G) \to \mathbb{Q}_{\geq 0}$ satisfying the triangle inequality



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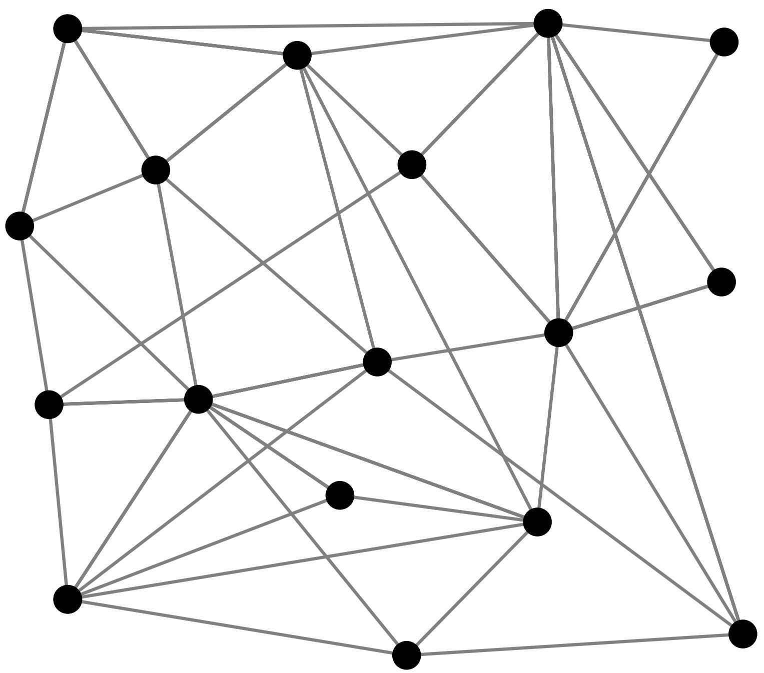

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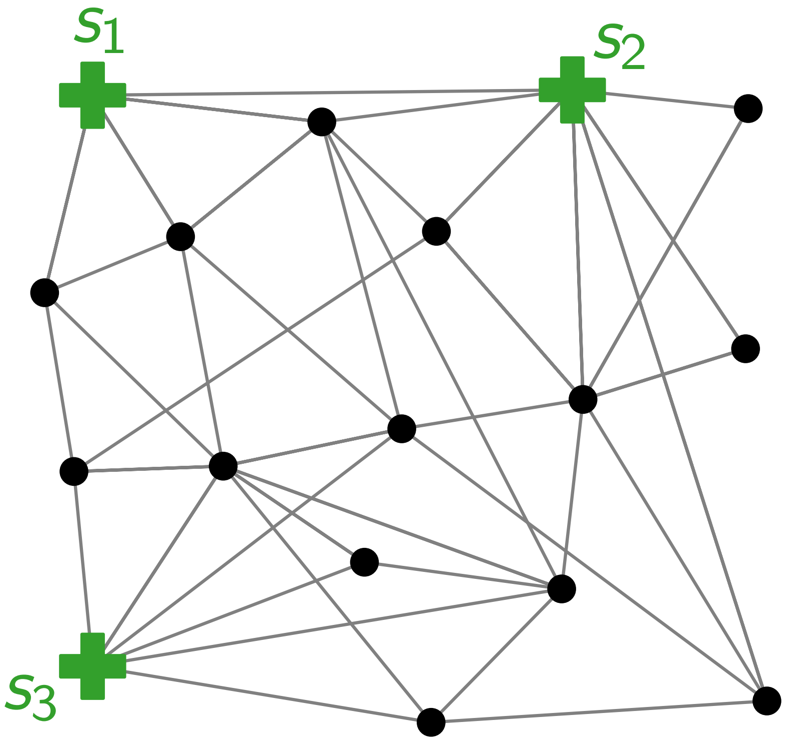

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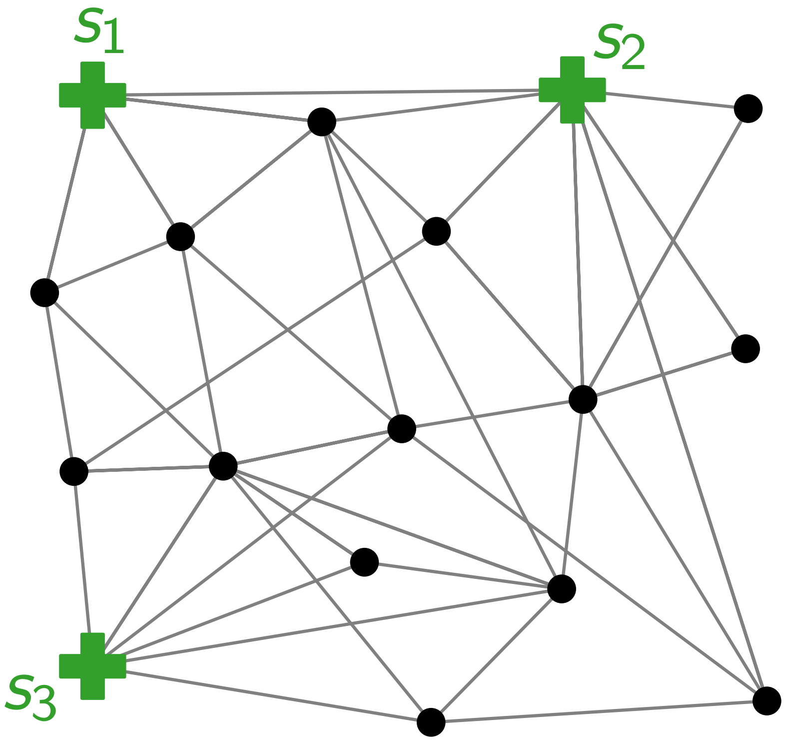
Given a set $S \subseteq V(G)$, $c(v, S)$ is the cost of the cheapest edge from v to a vertex in S.



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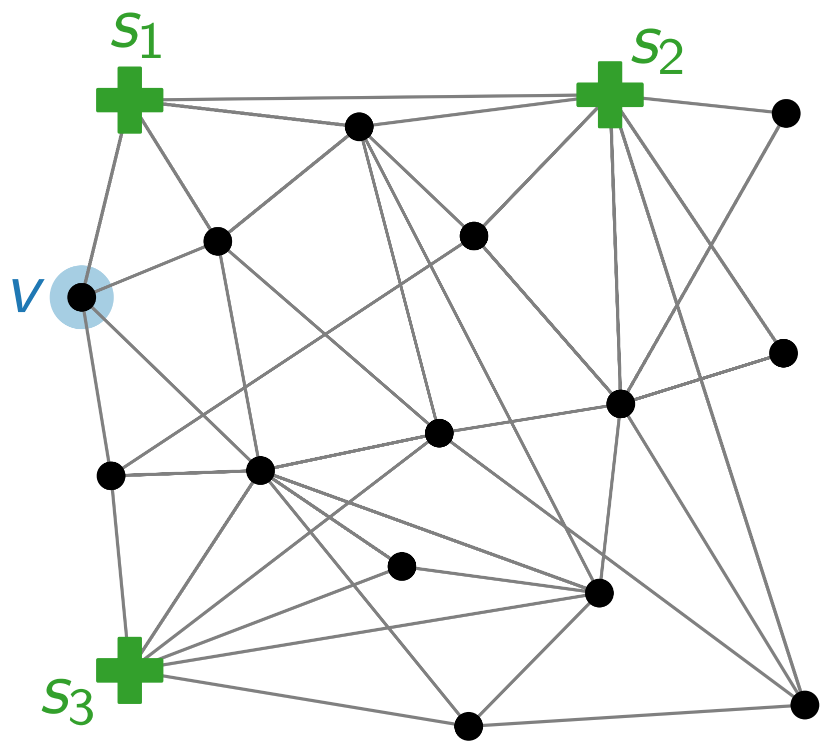

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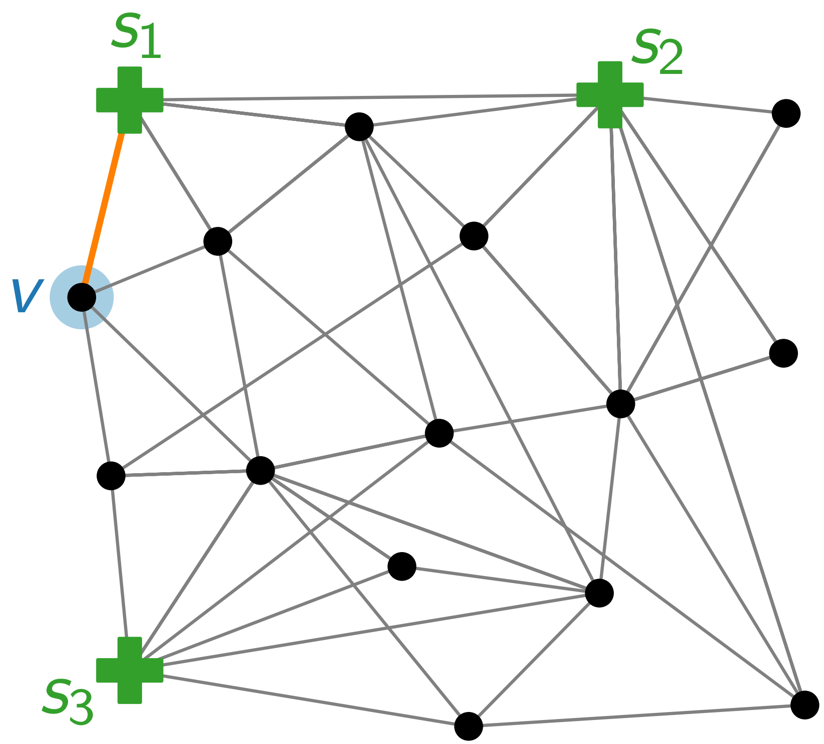

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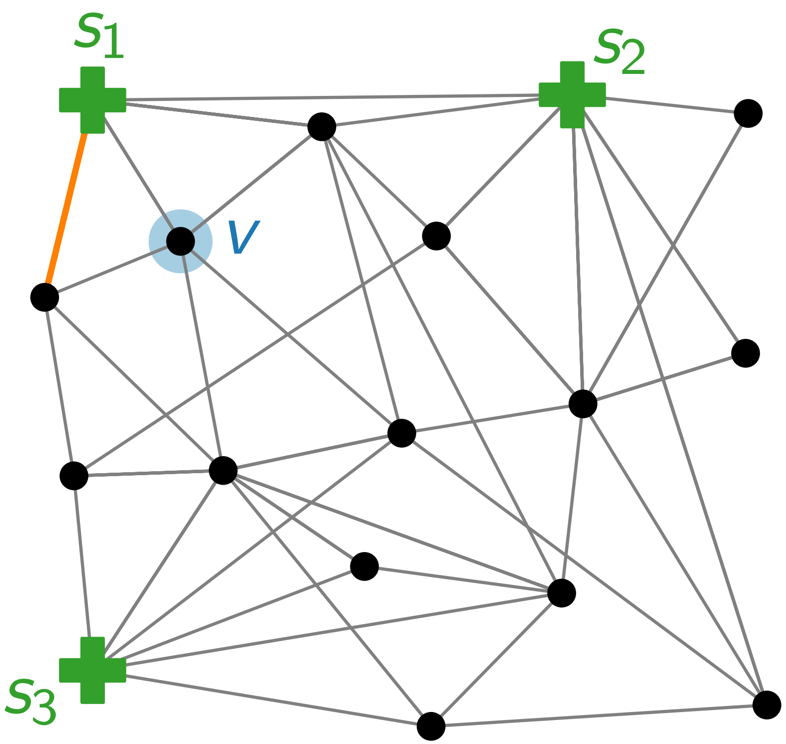

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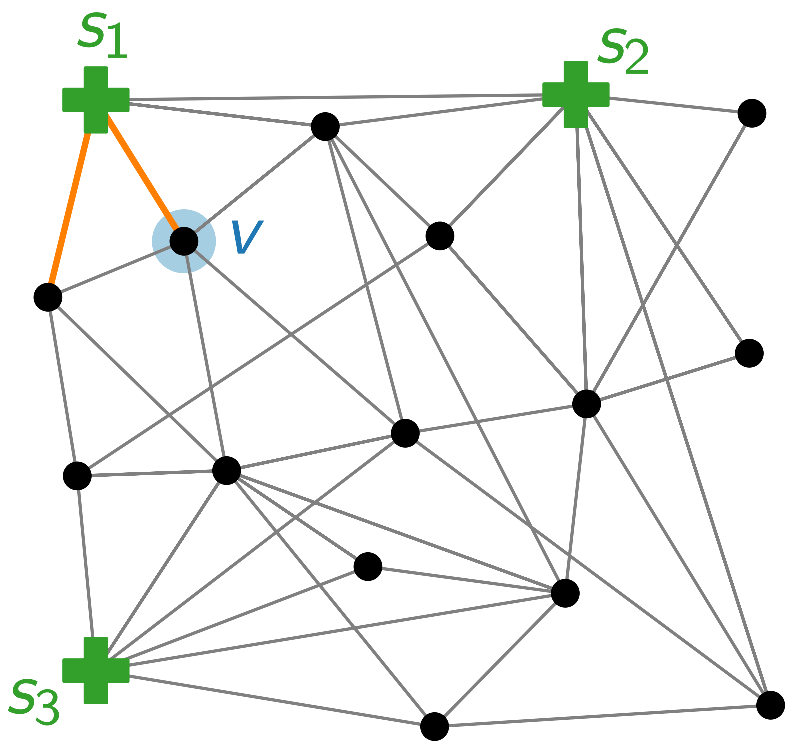

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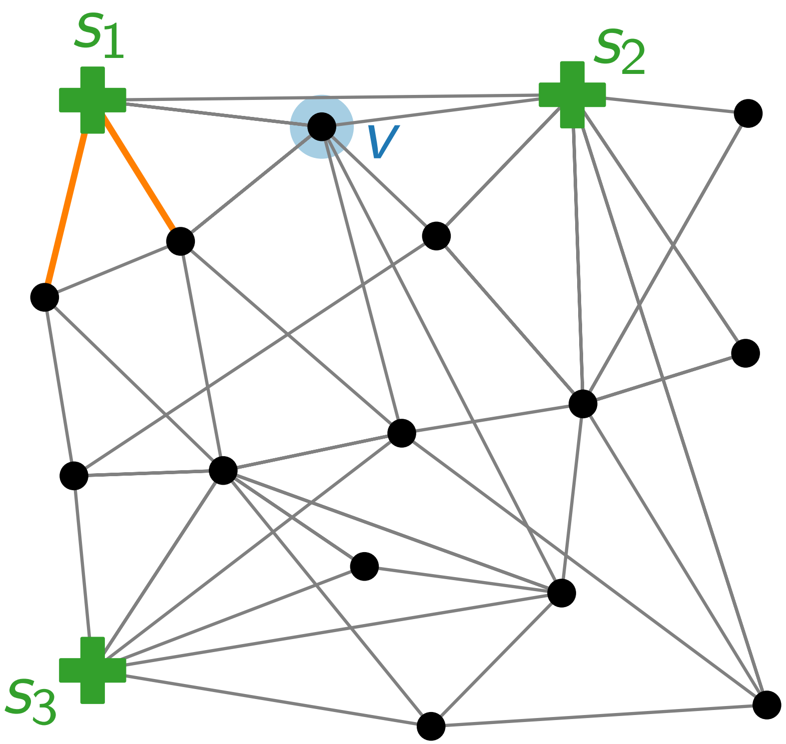

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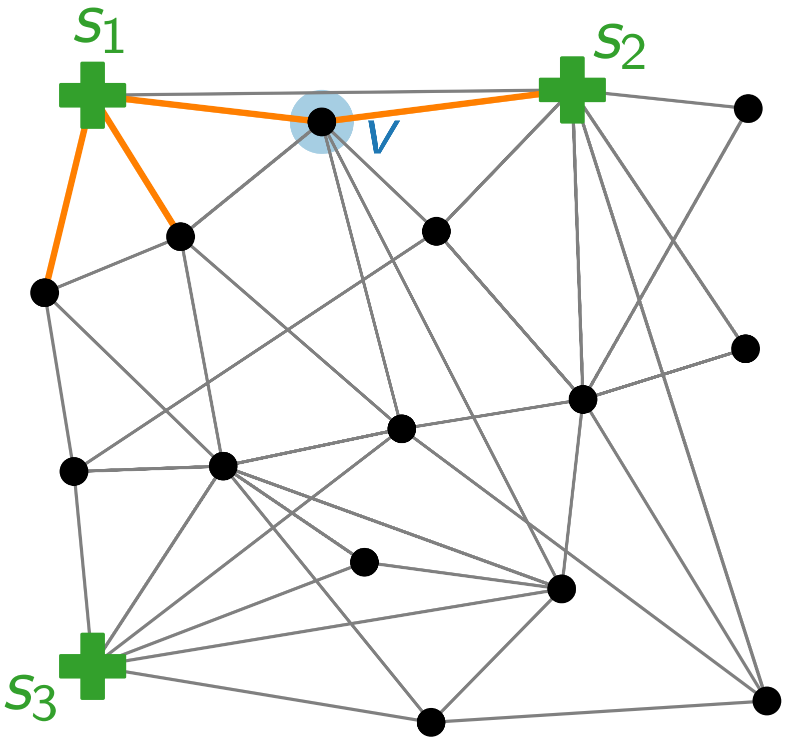

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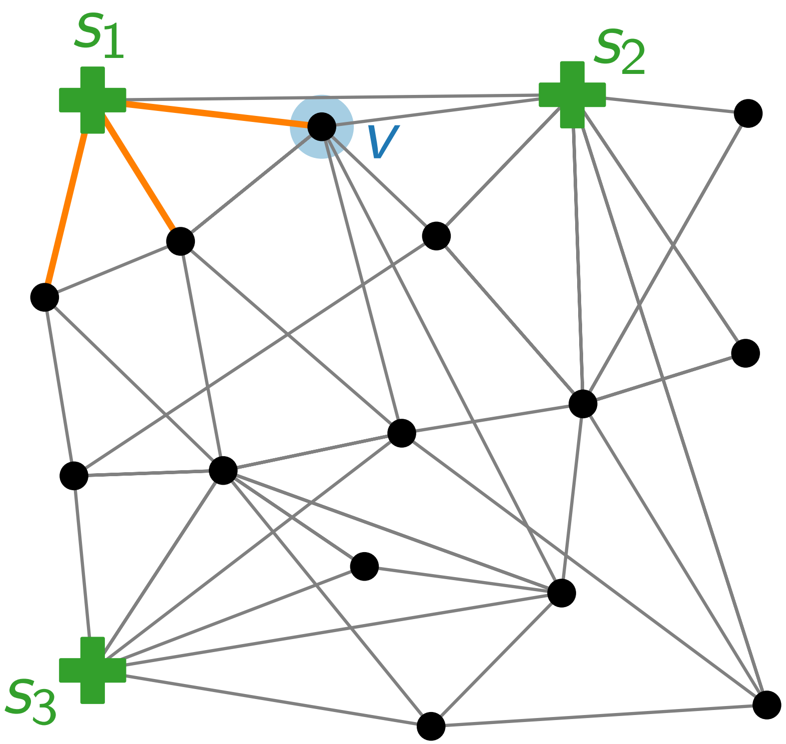

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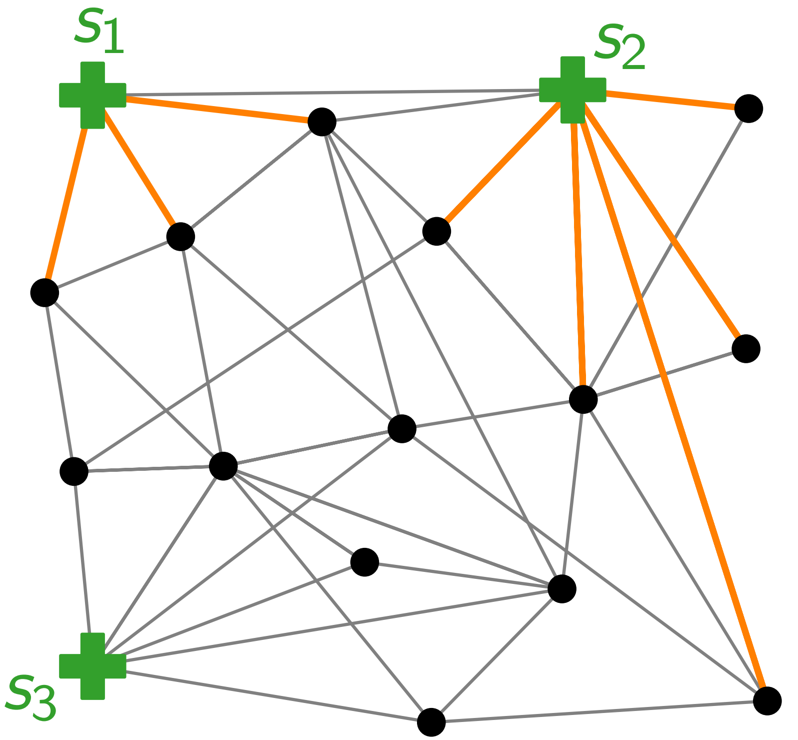

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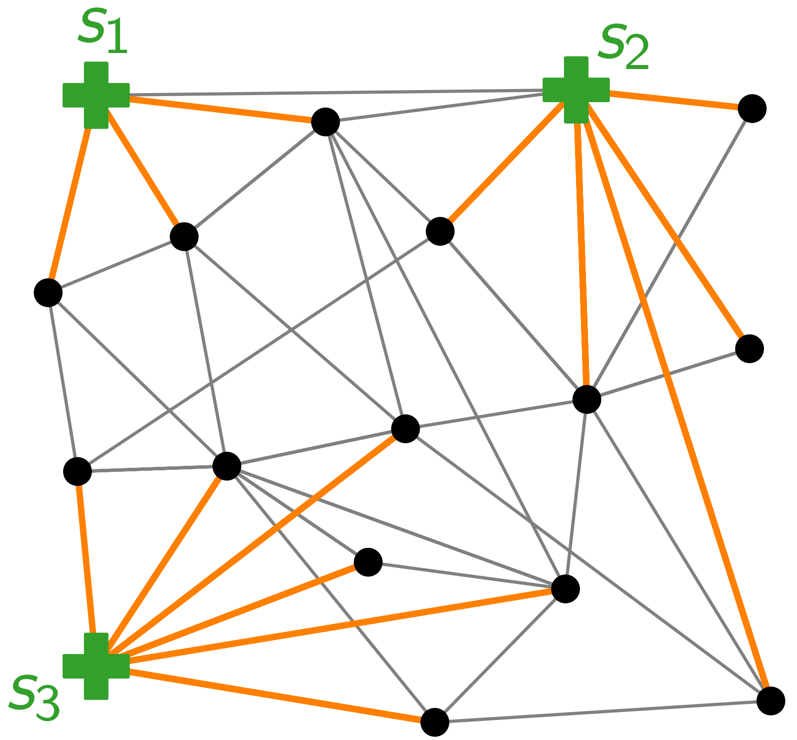

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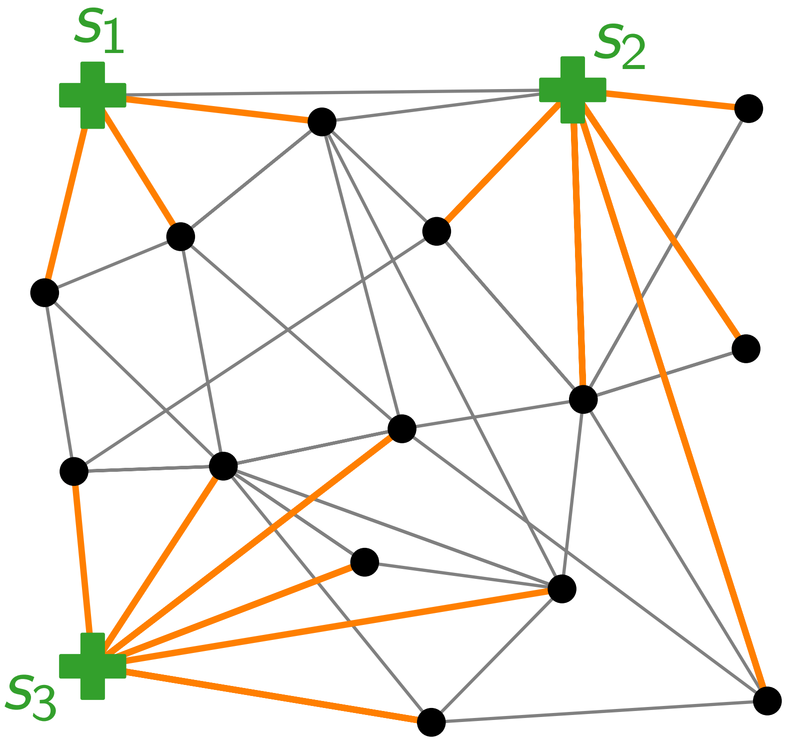

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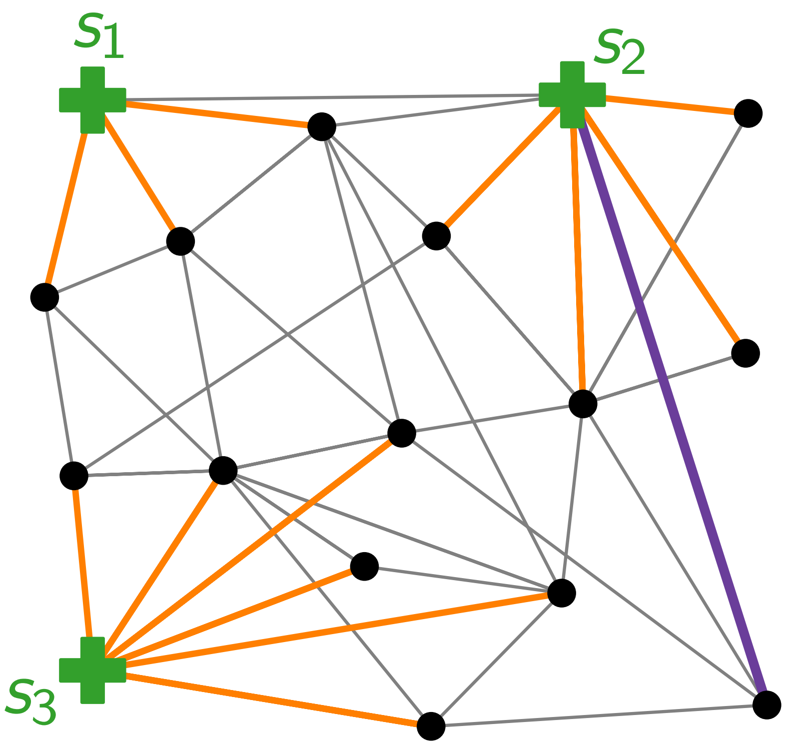

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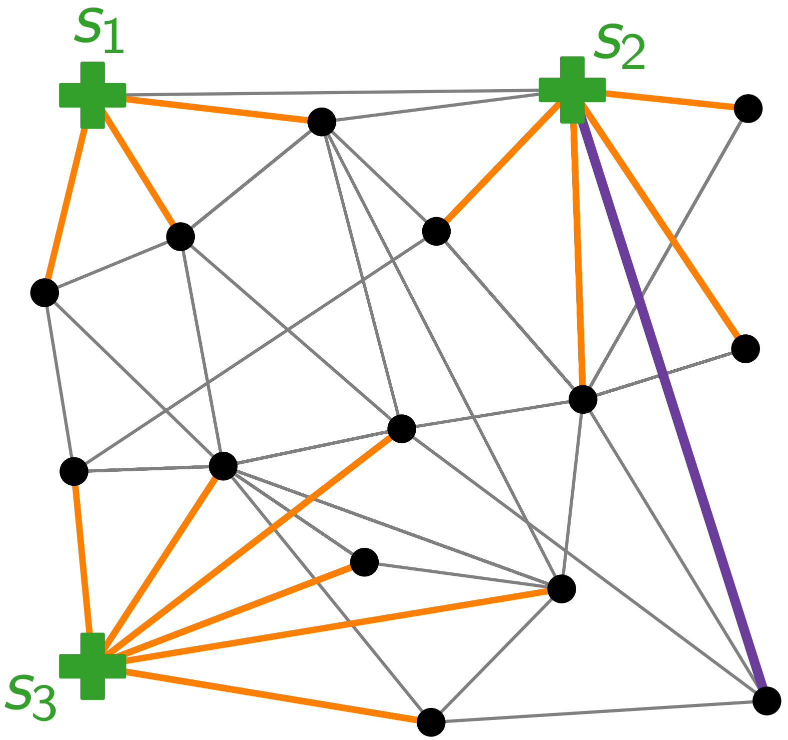
Find: A vertex set S that minimizes $\text{cost}(S) := \max_{v \in V} c(v, S)$.



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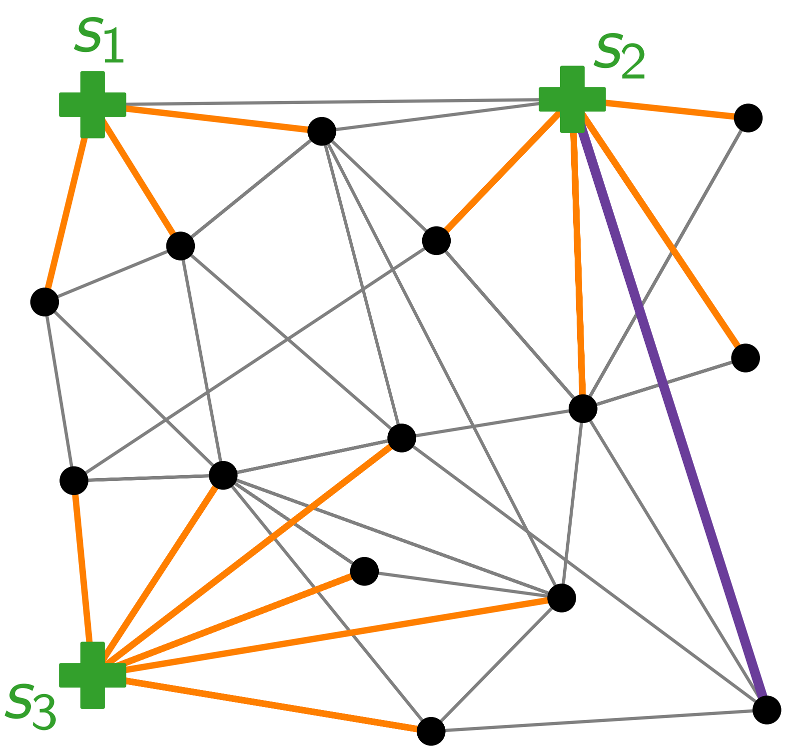

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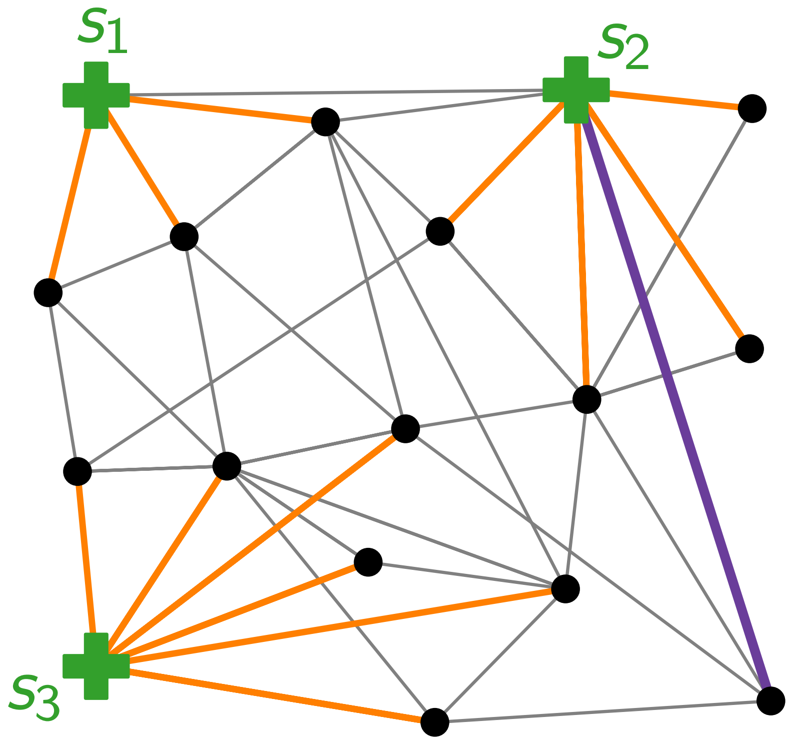

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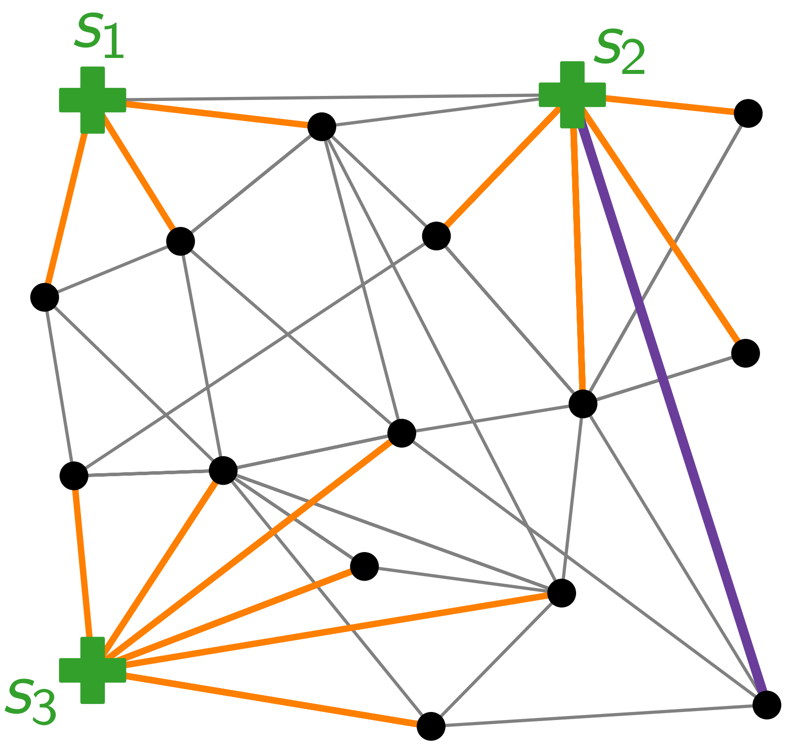

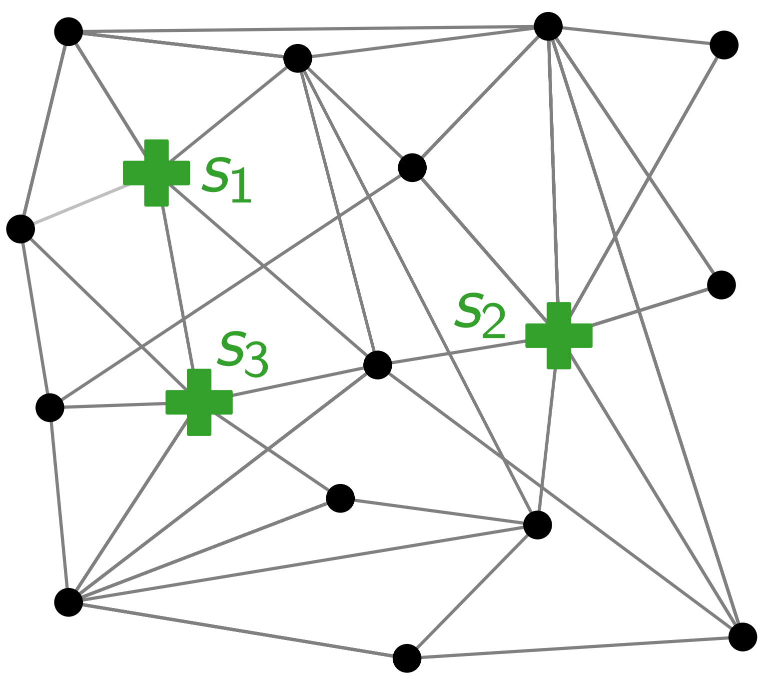

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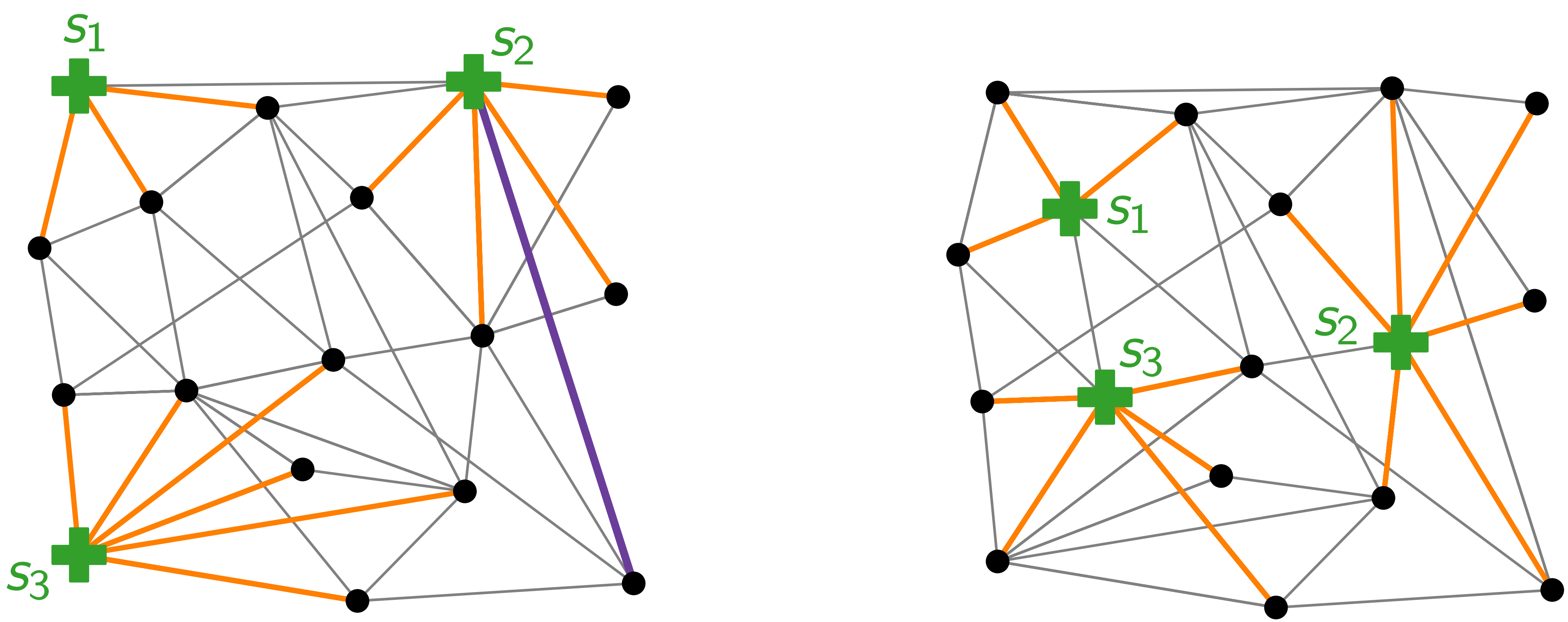

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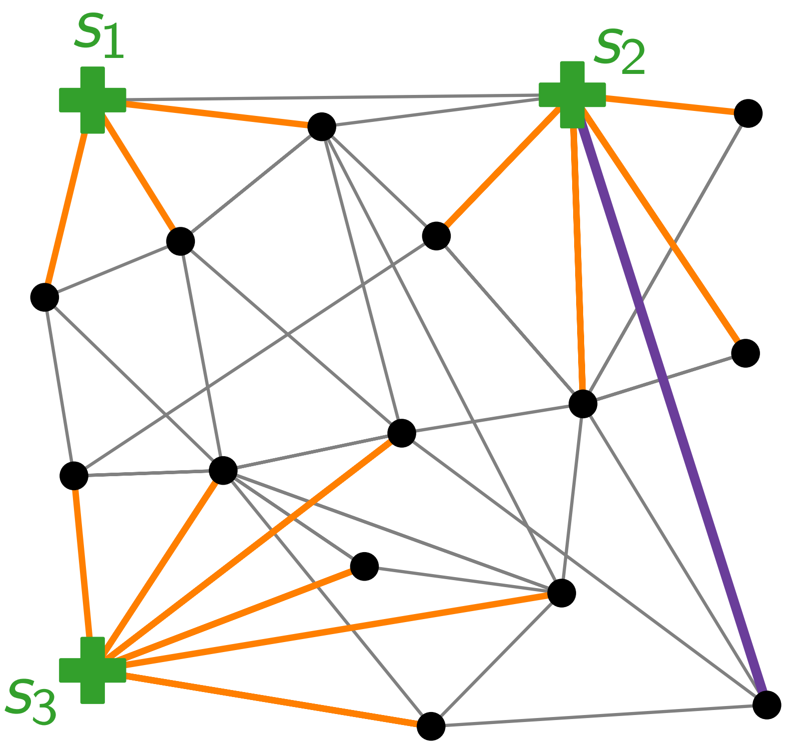
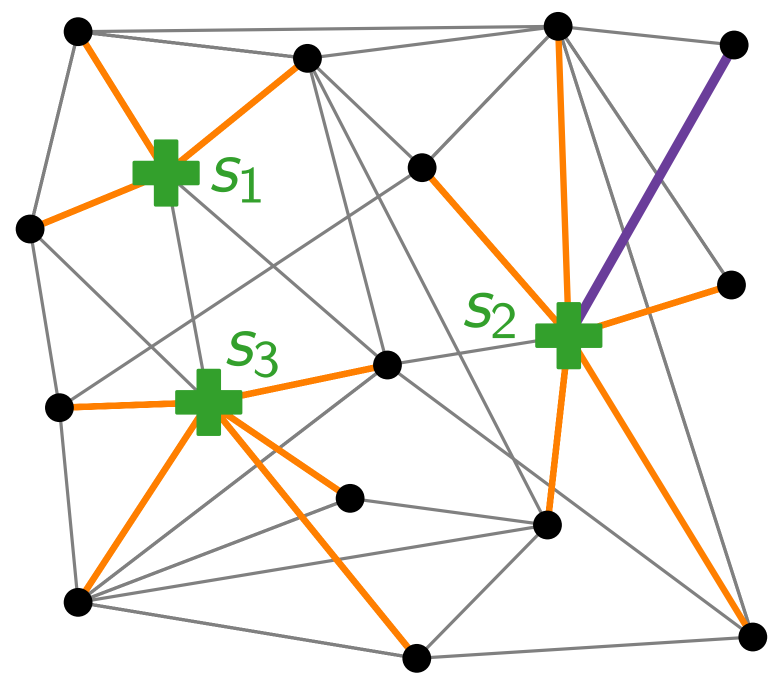


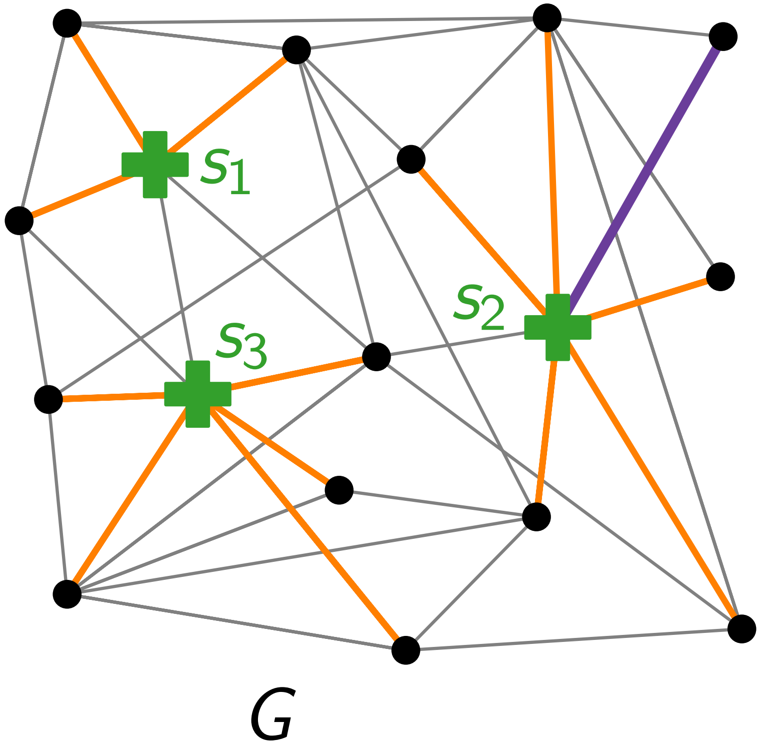


Approximation Algorithms

Lecture 6:
k-CENTER via Parametric Pruning

Part II:
Parametric Pruning

Parametric Pruning



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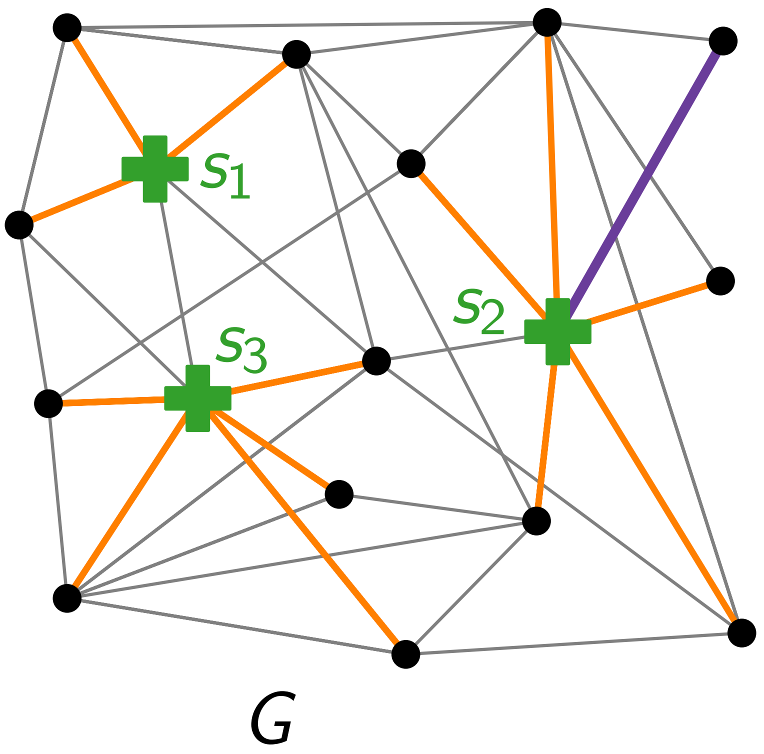
Let $E(G) = \{e_1, \ldots, e_m\}$ with $c(e_1) \leq \cdots \leq c(e_m)$.



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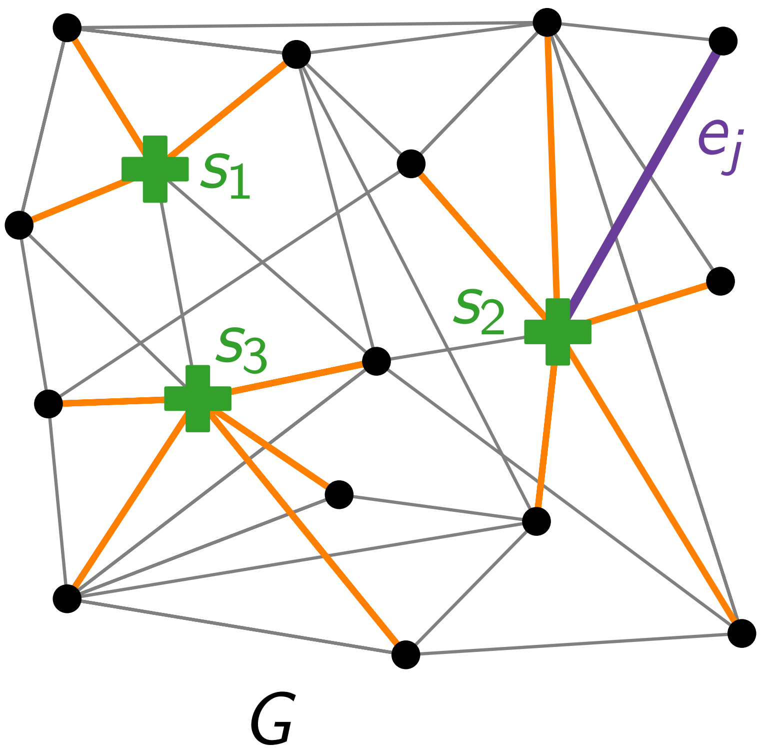

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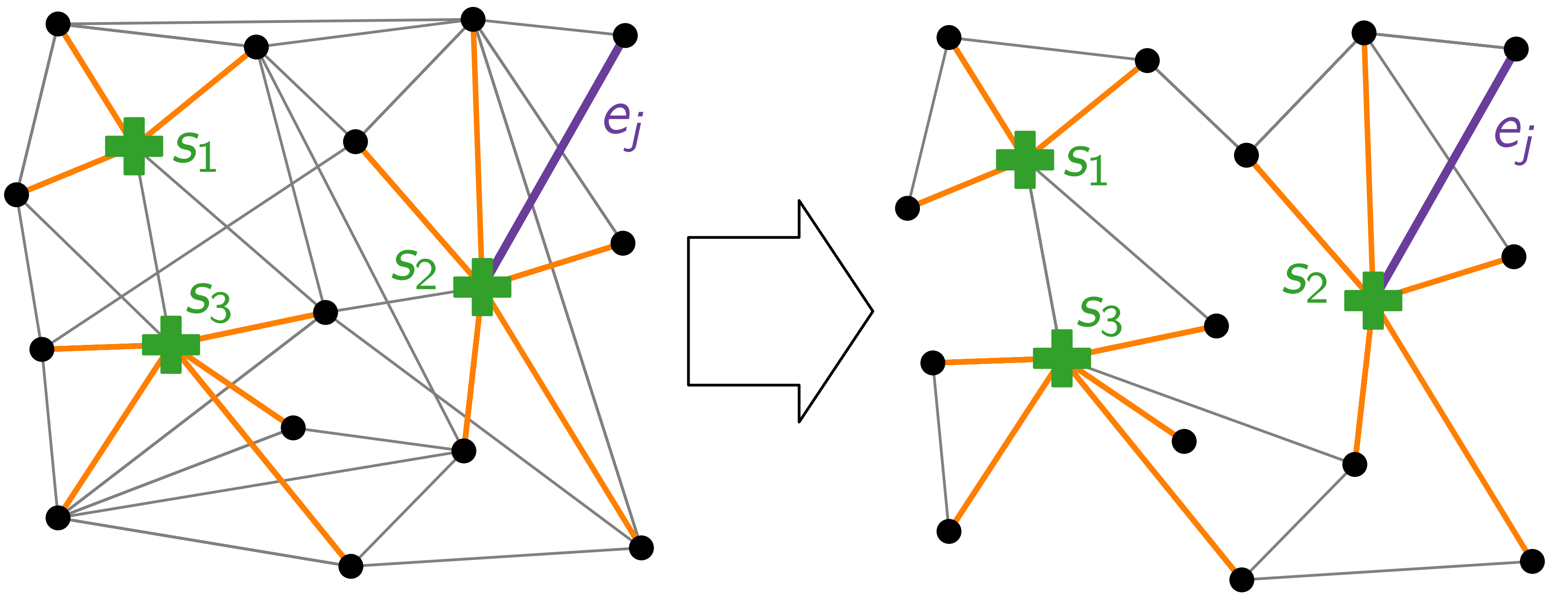

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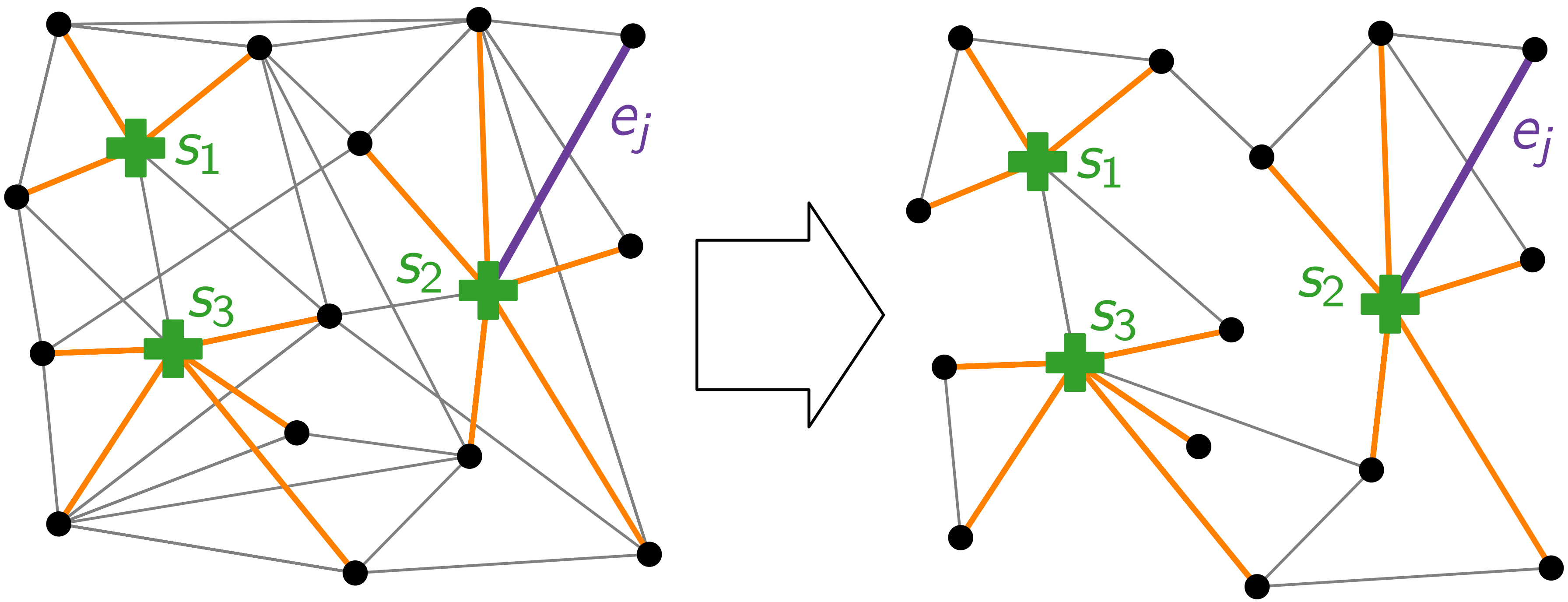

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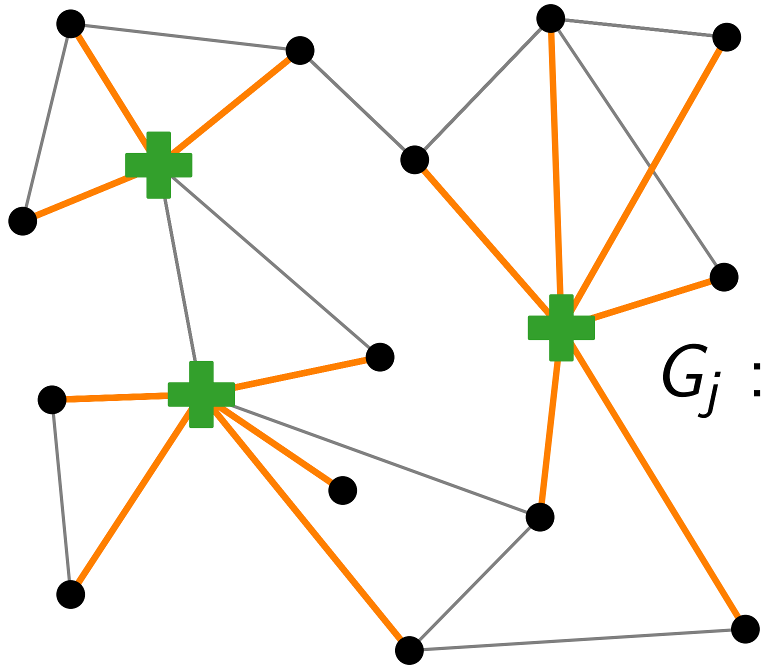
Def.



$$G_j := (V(G), \{e_1, \dots, e_j\})$$

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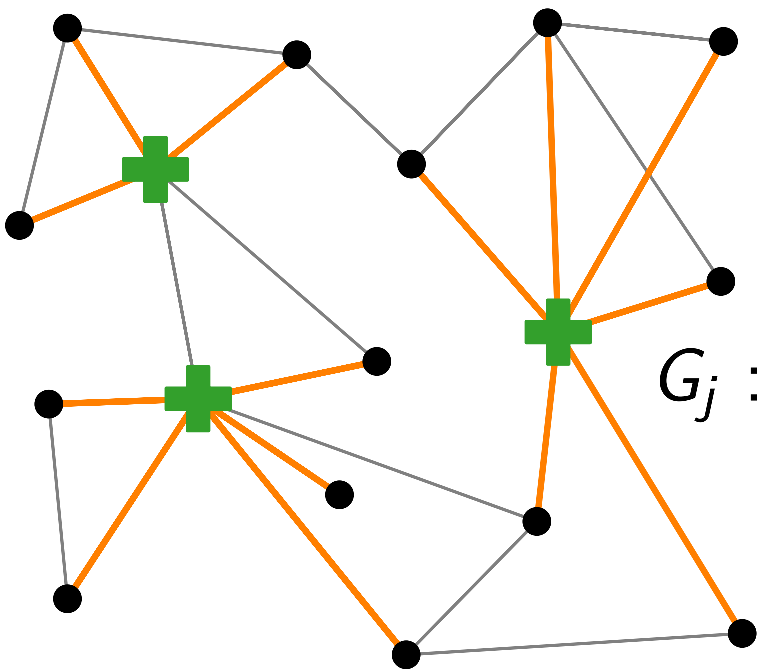
Def. A vertex set D of a graph H is **dominating** if each vertex is either in D or adjacent to a vertex in D.



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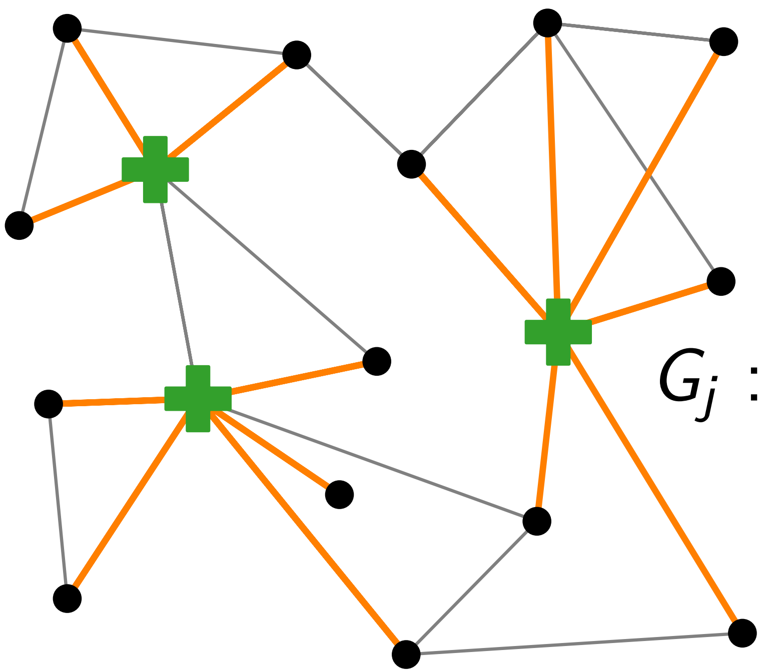

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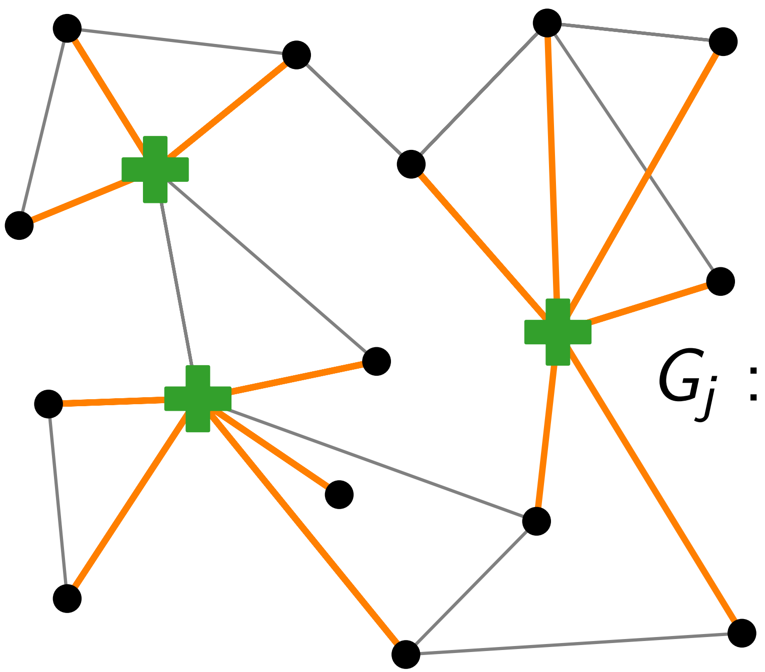

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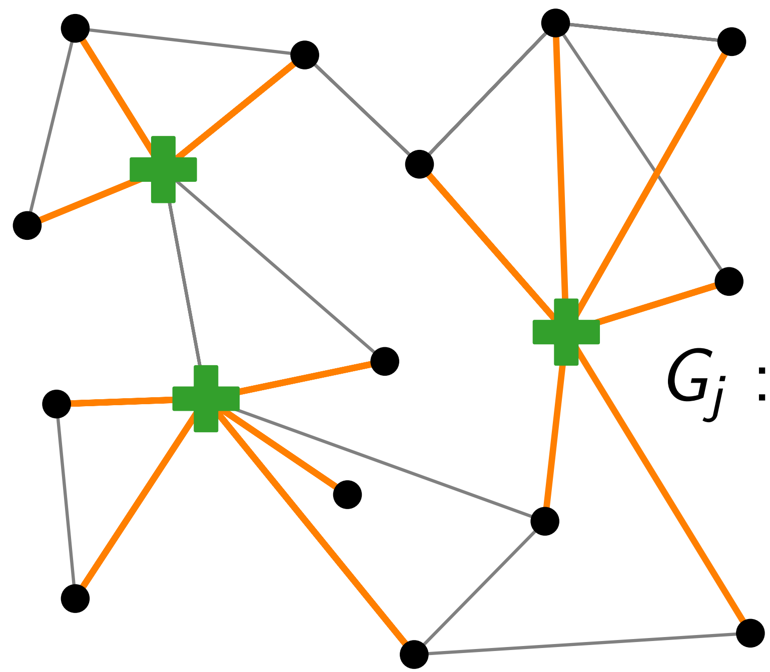

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. . . but computing $\mathrm{dom}(H)$ is NP-hard.

Approximation Algorithms

Lecture 6:
k-Center via Parametric Pruning

Part III:
Square of a Graph

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Idea: Find a small dominating set in a "coarsened" G_j.

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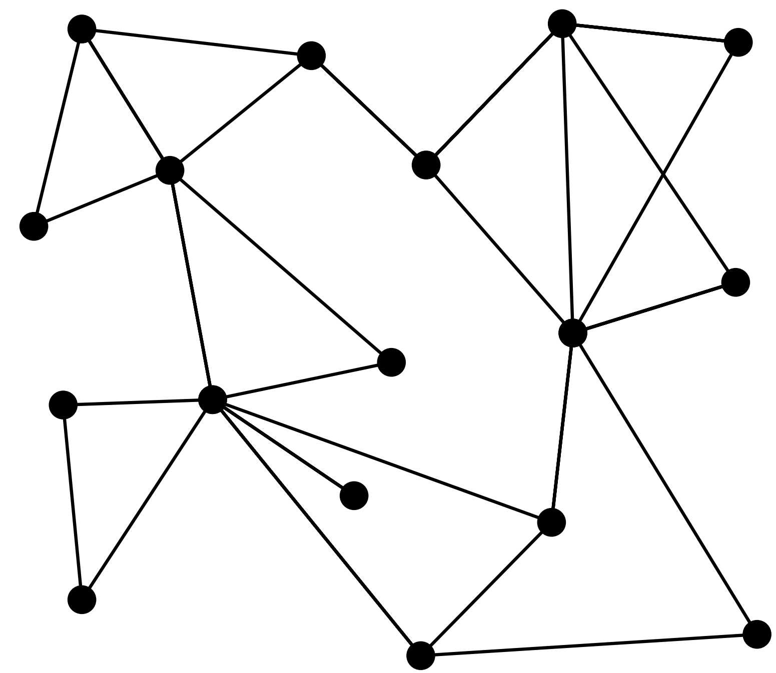

G_j

Square of a Graph

Idea: Find a small dominating set in a "coarsened" G_j.

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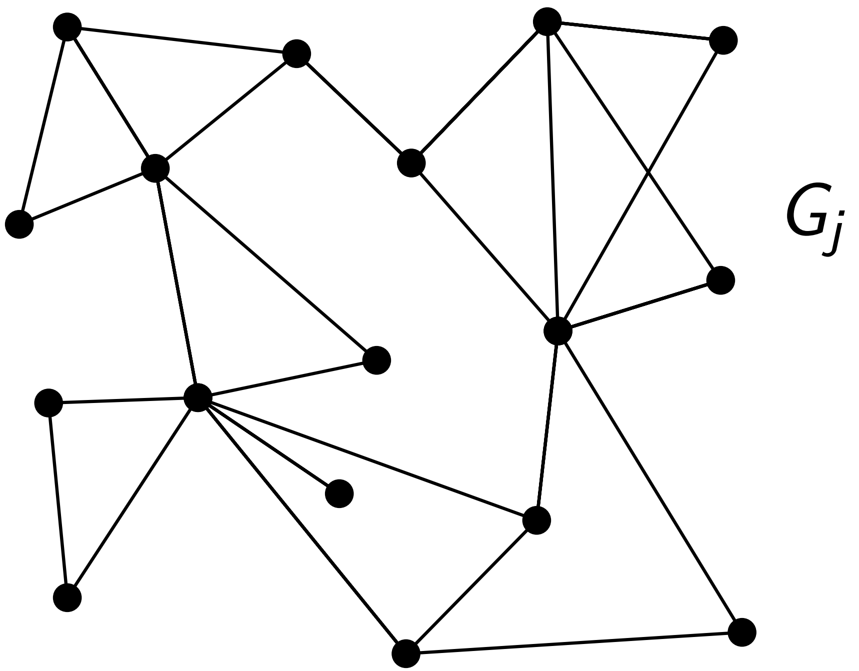

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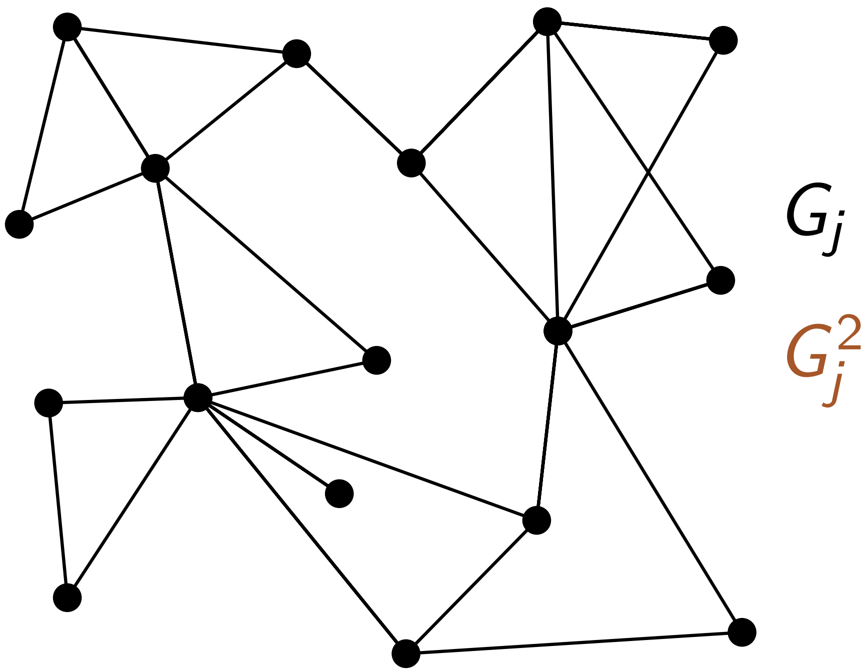

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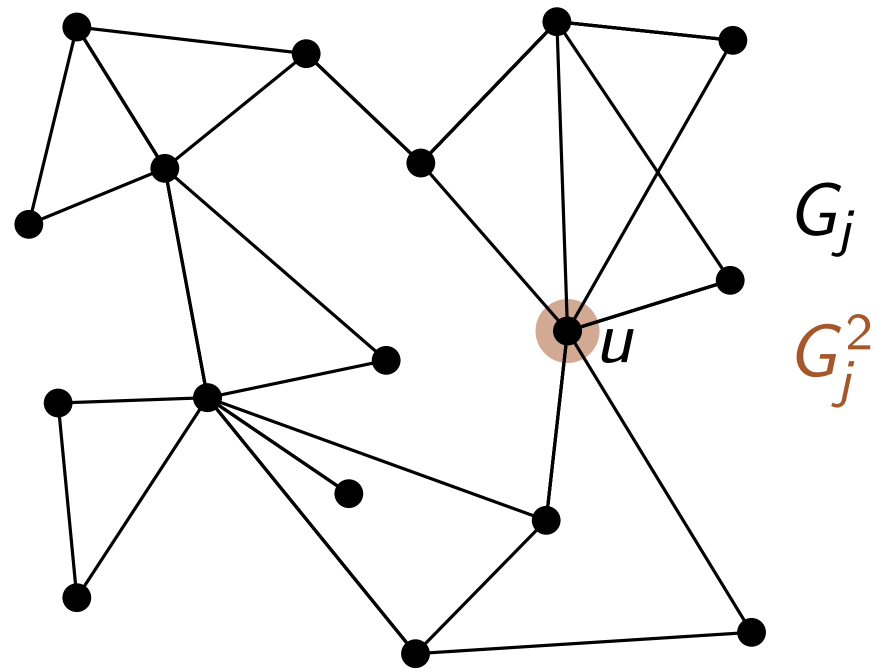

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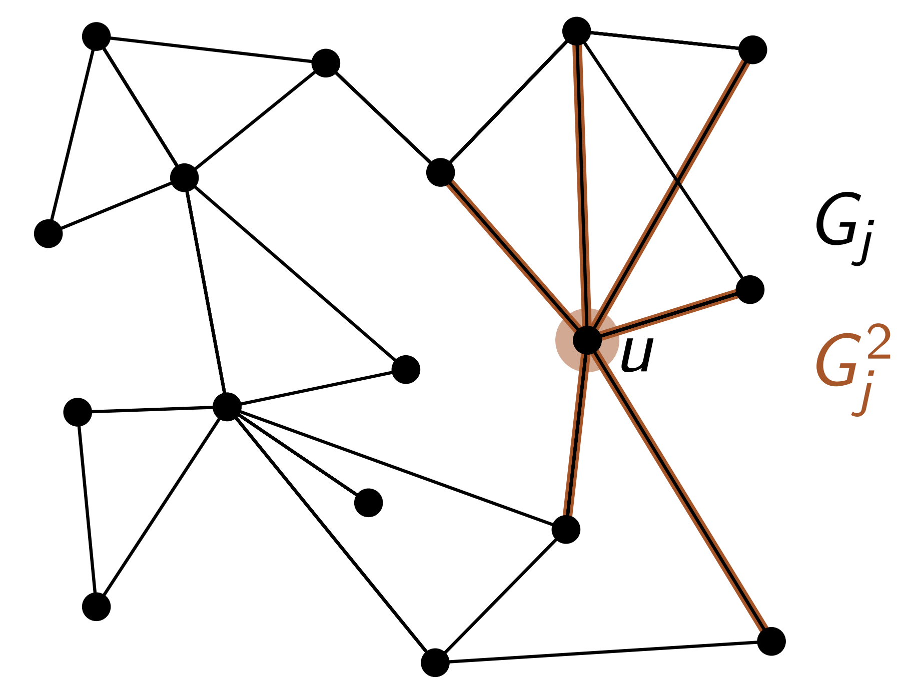

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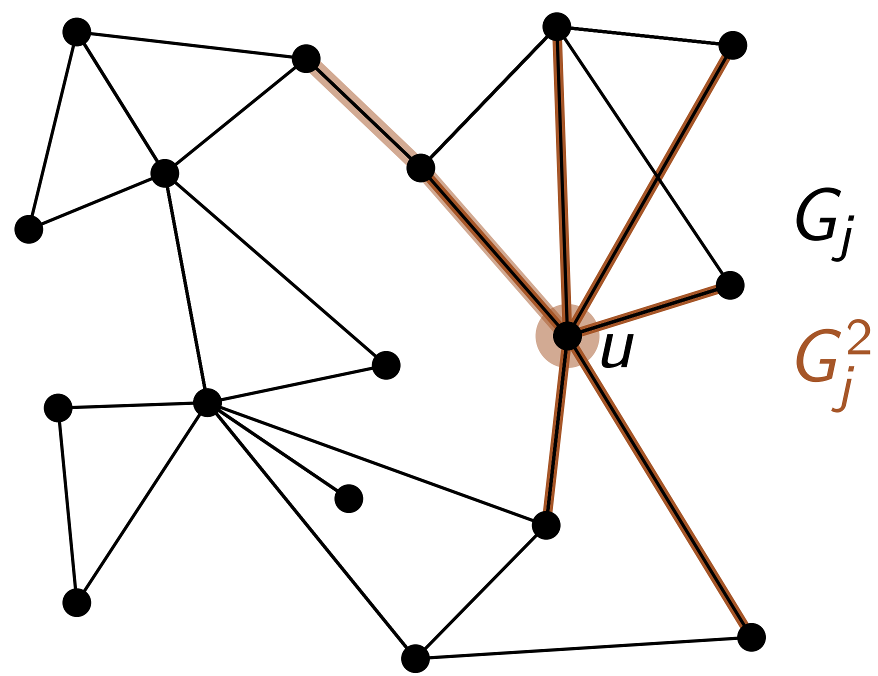

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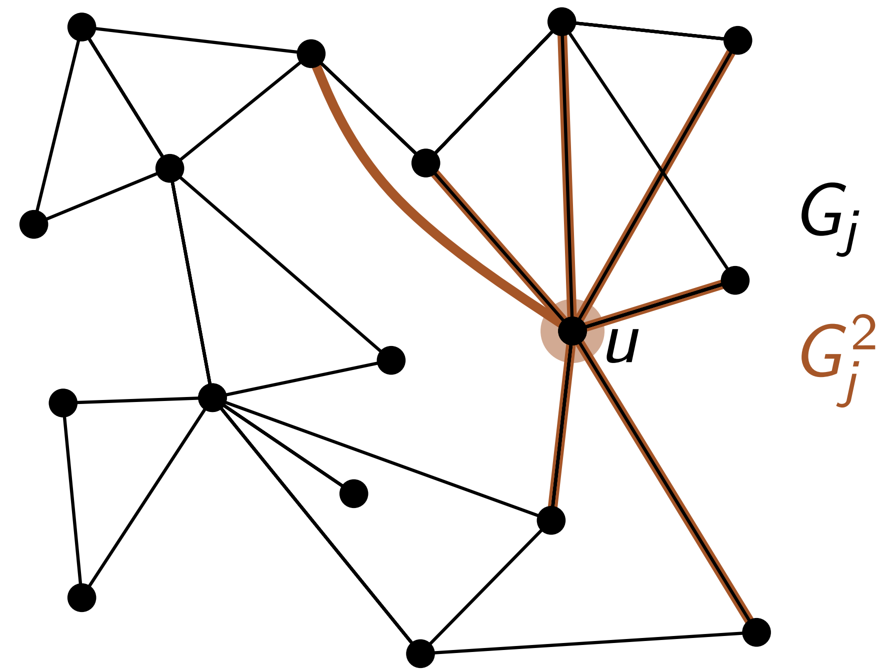

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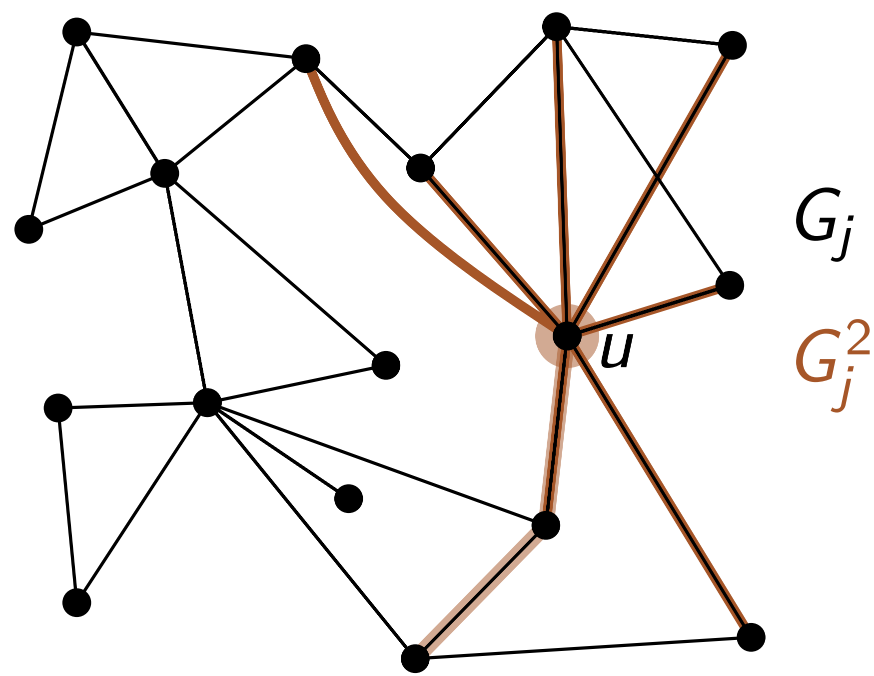

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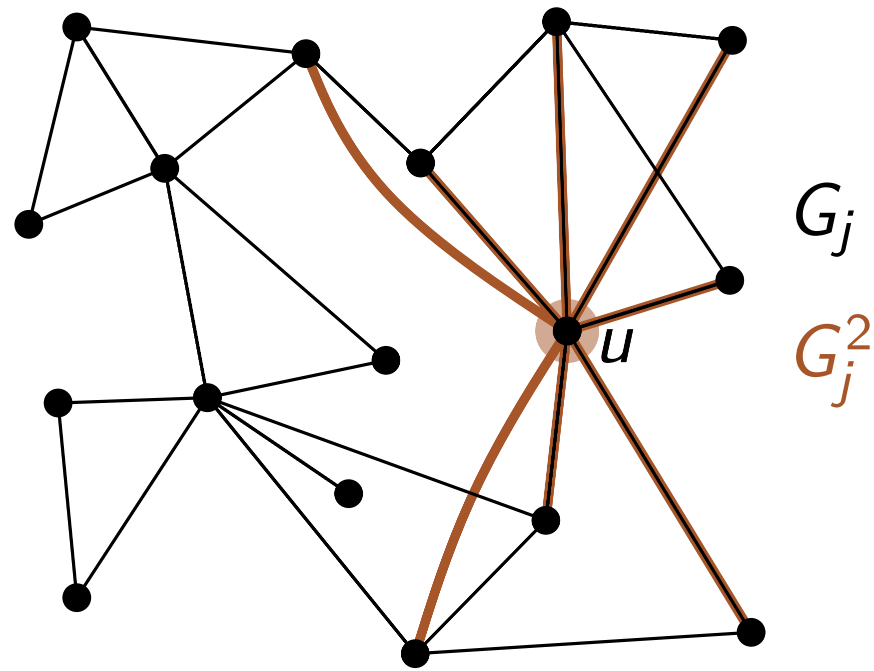

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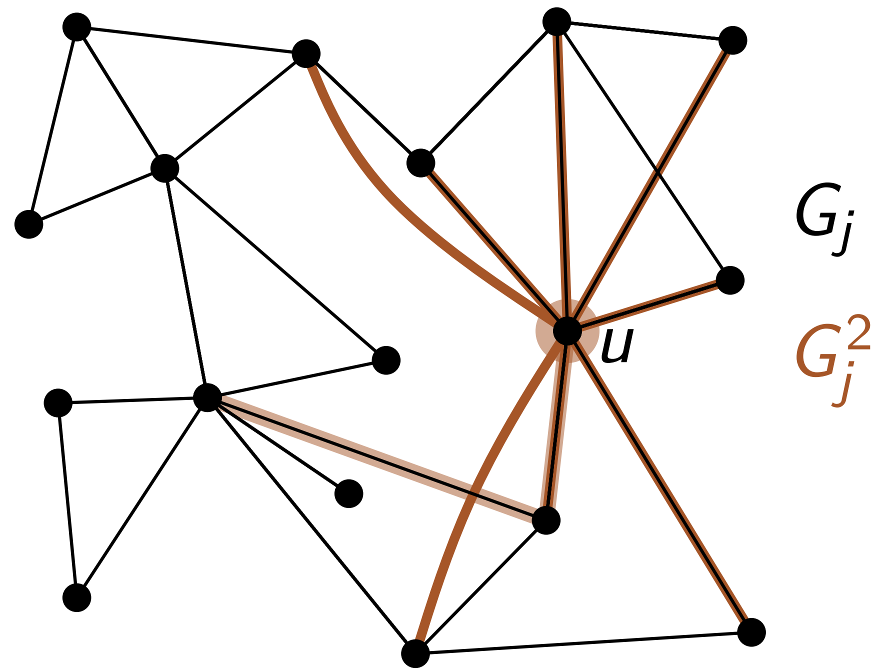

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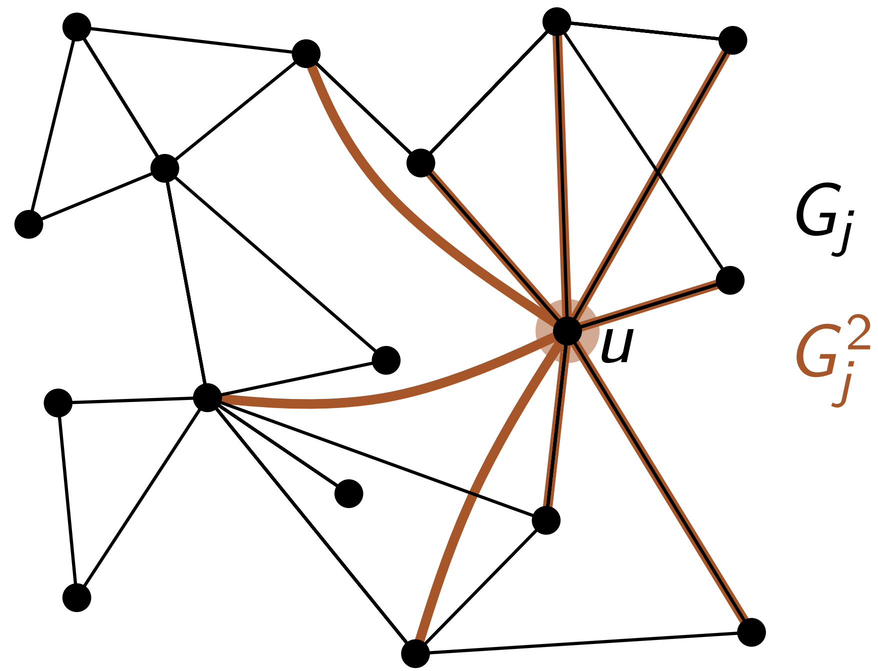

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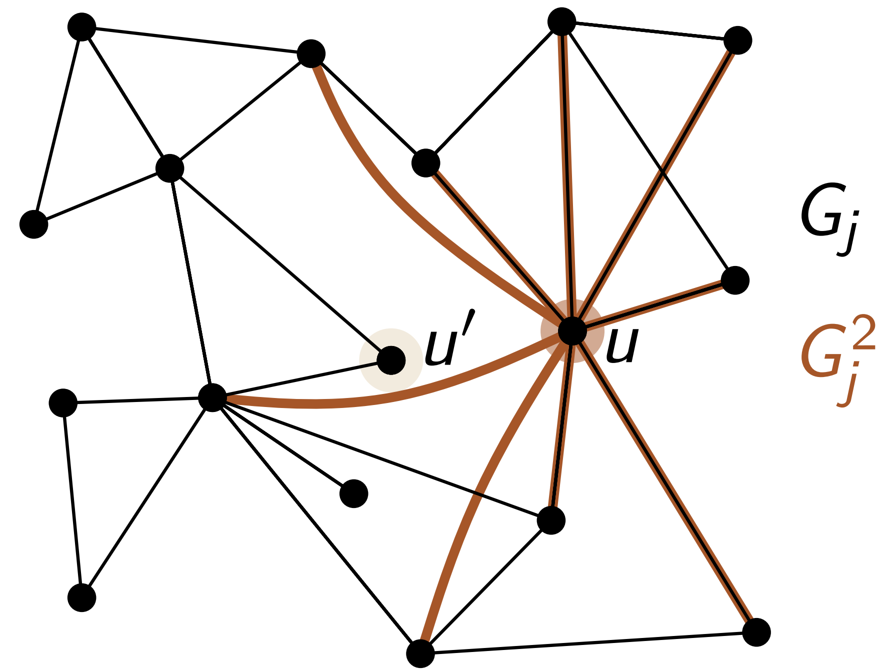

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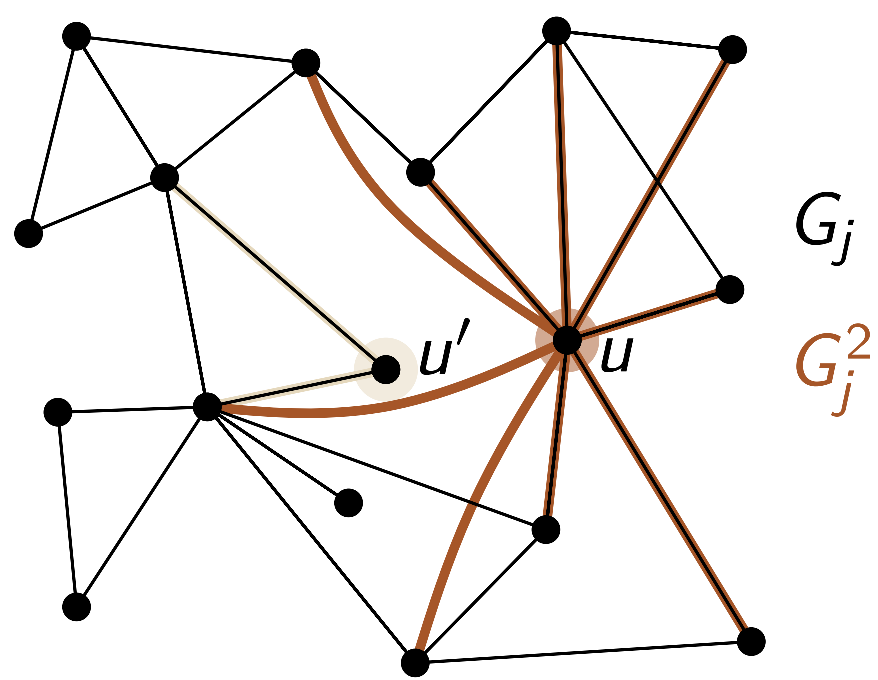

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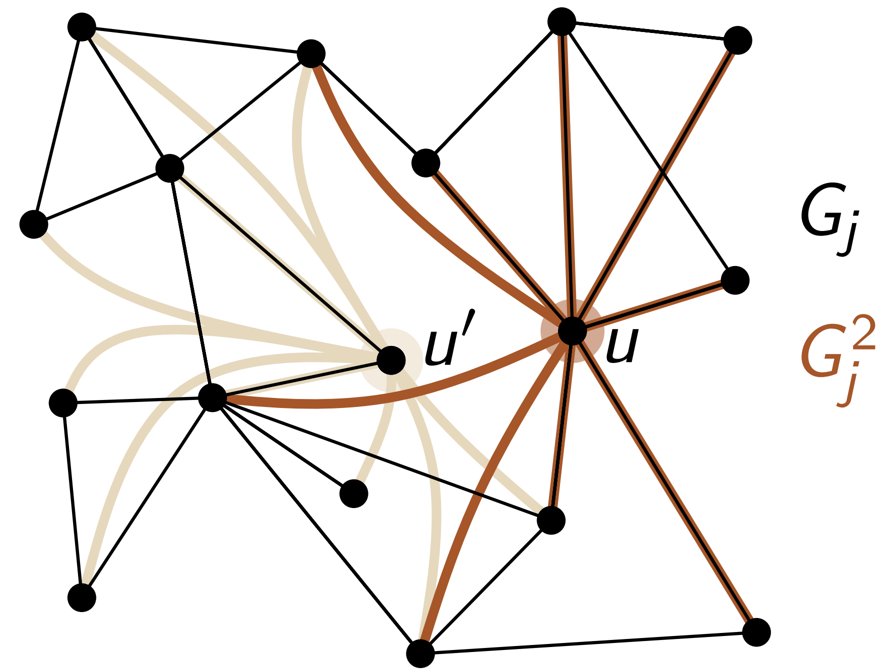

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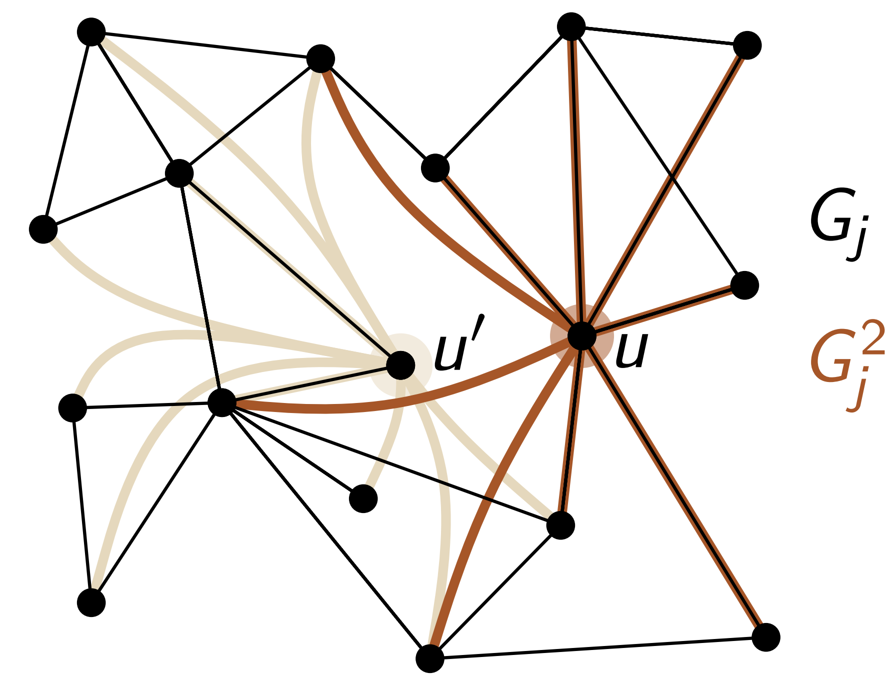

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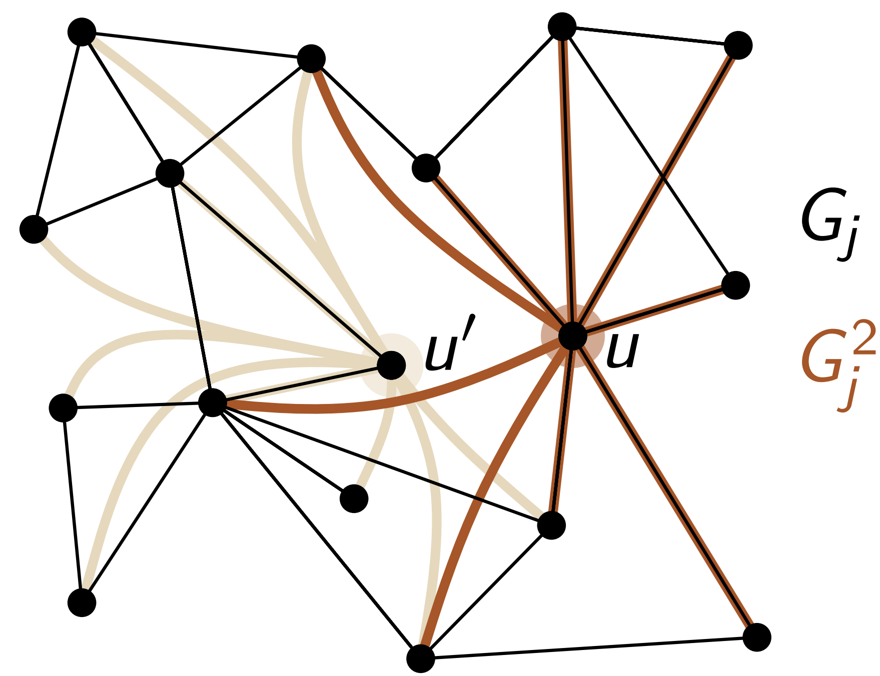

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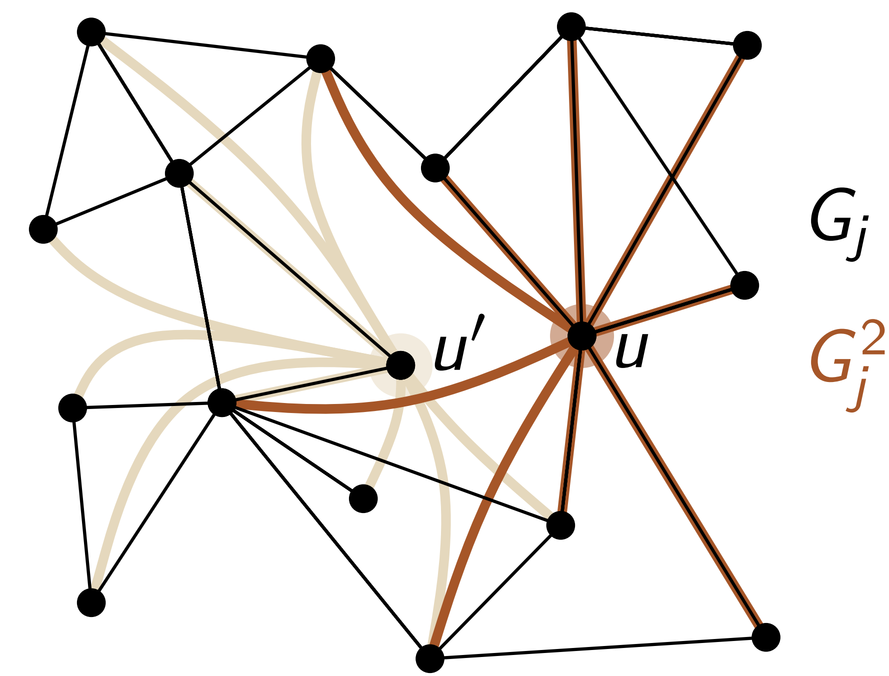

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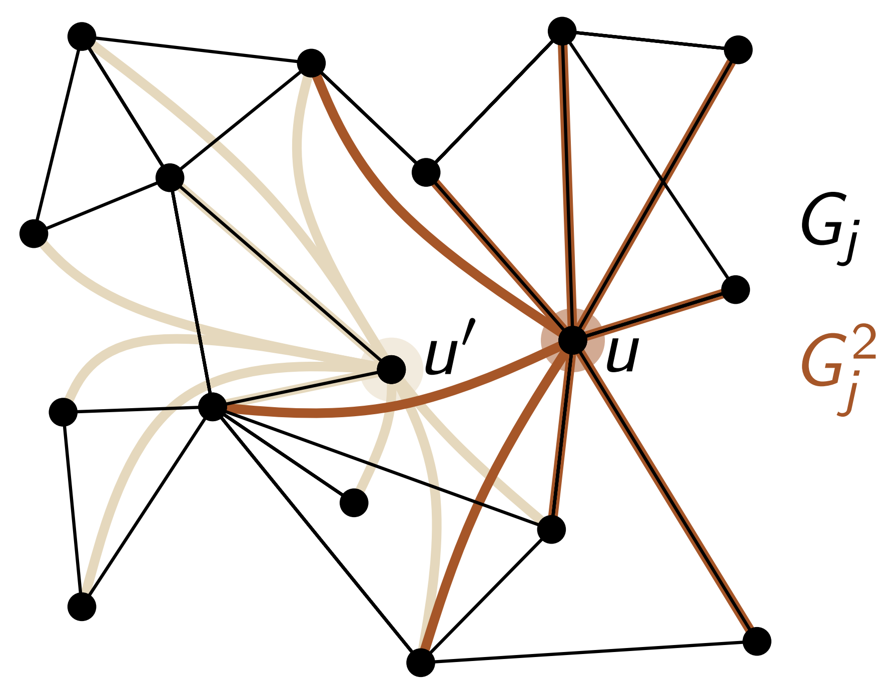

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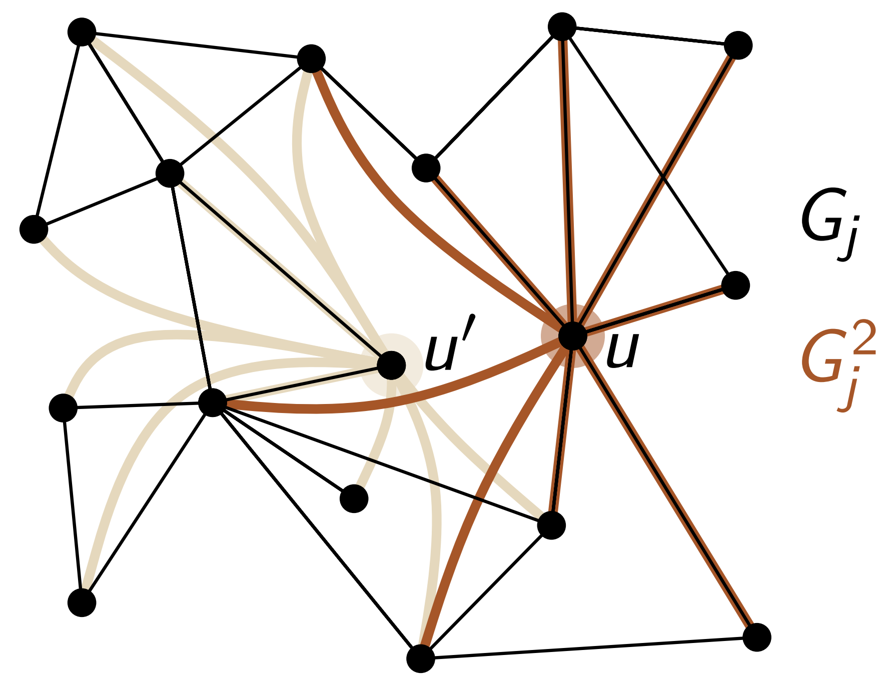

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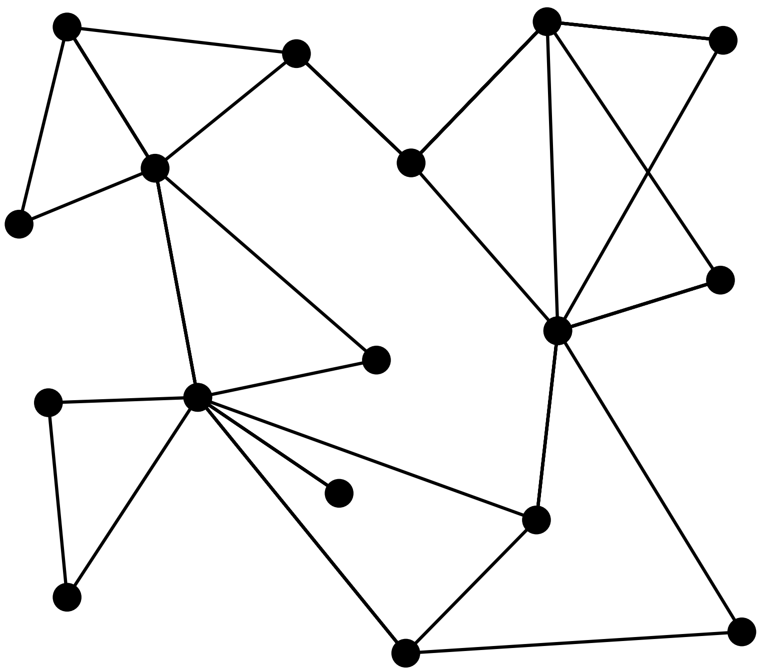

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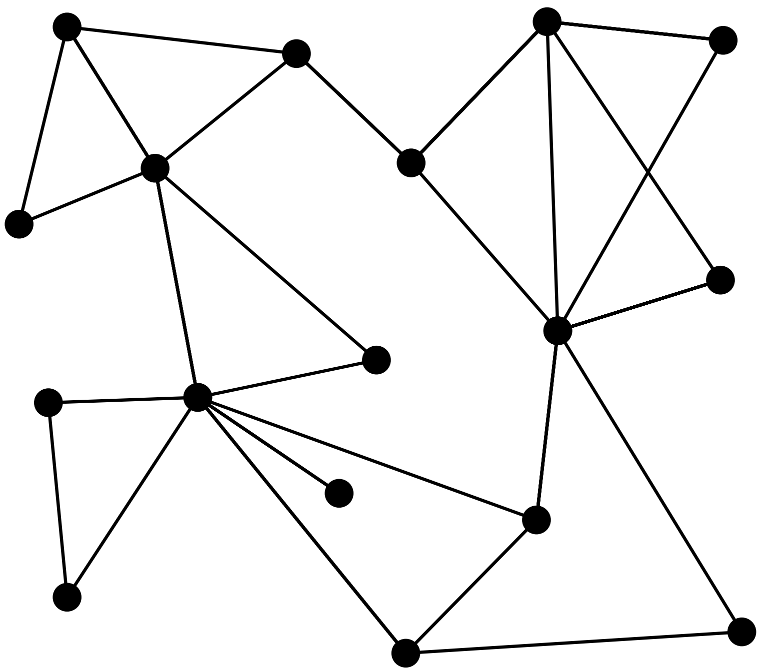

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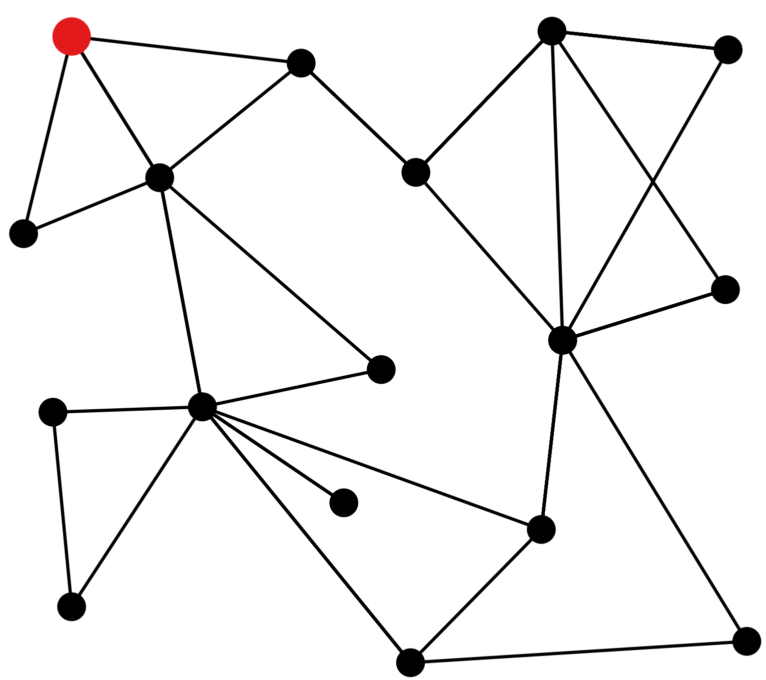

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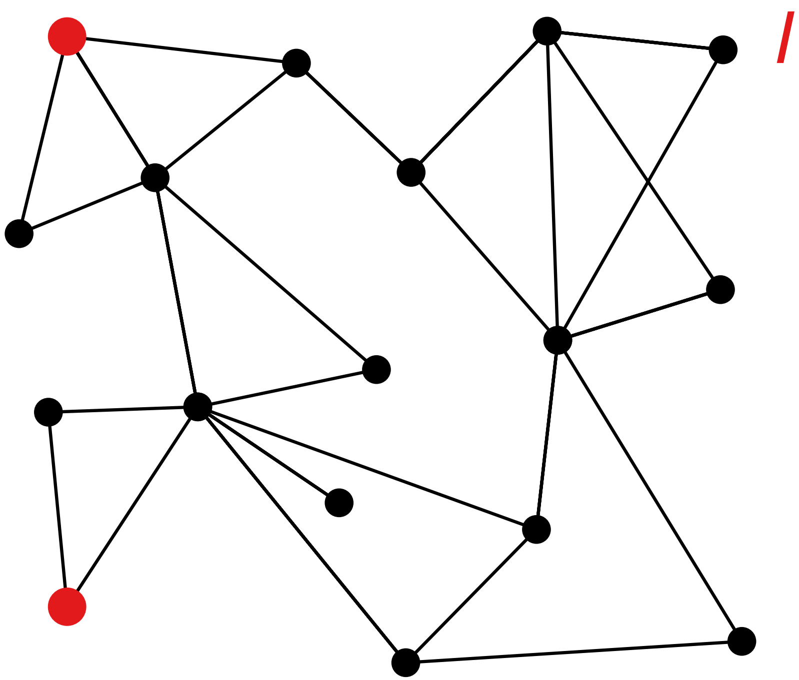

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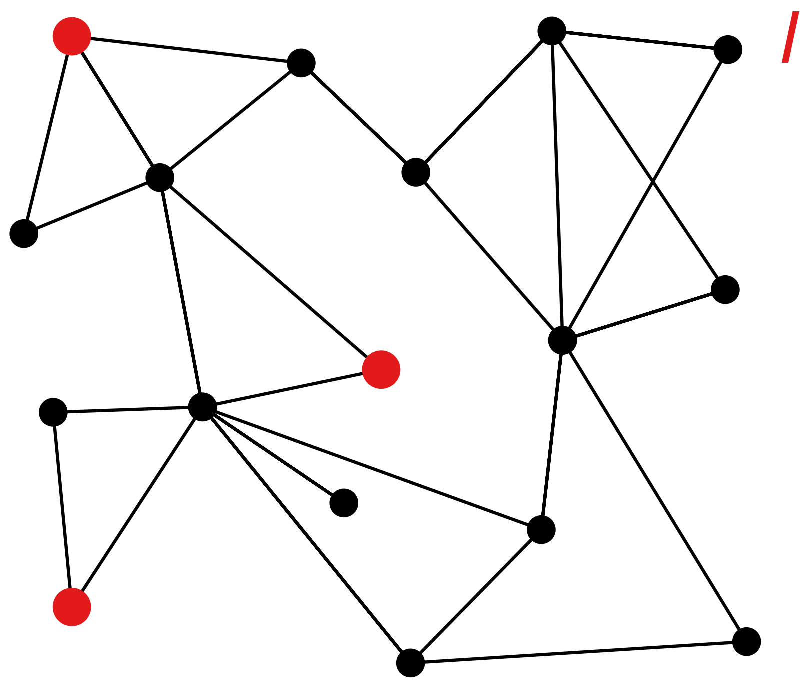

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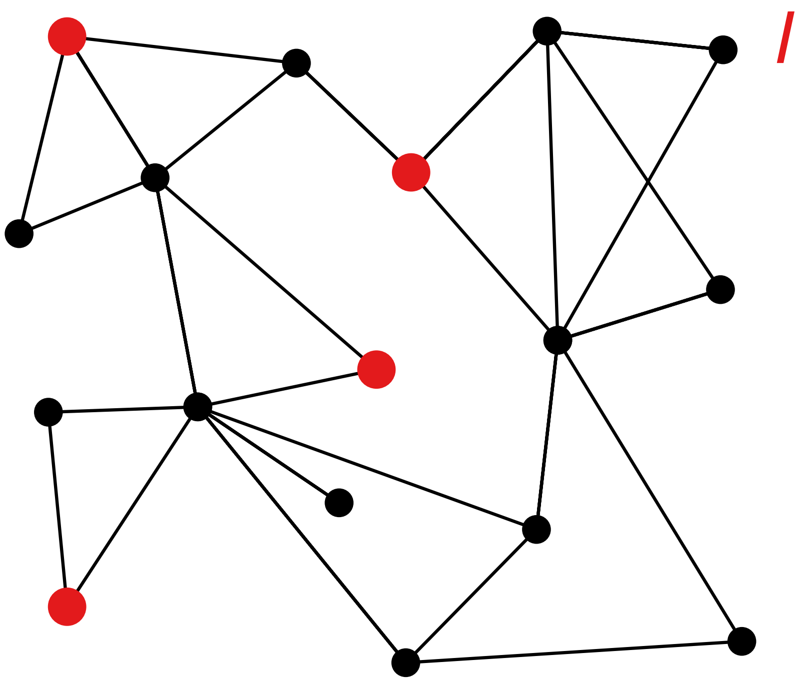

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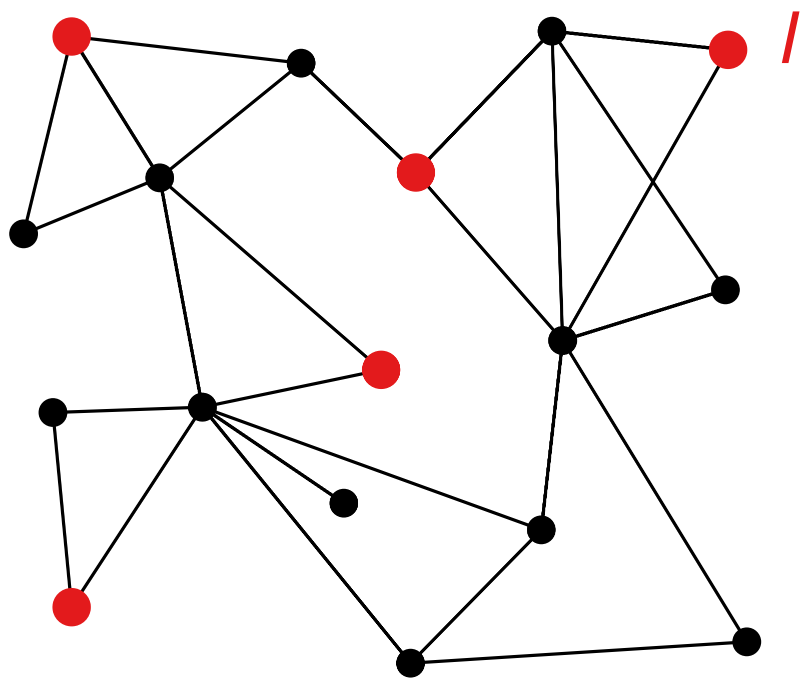

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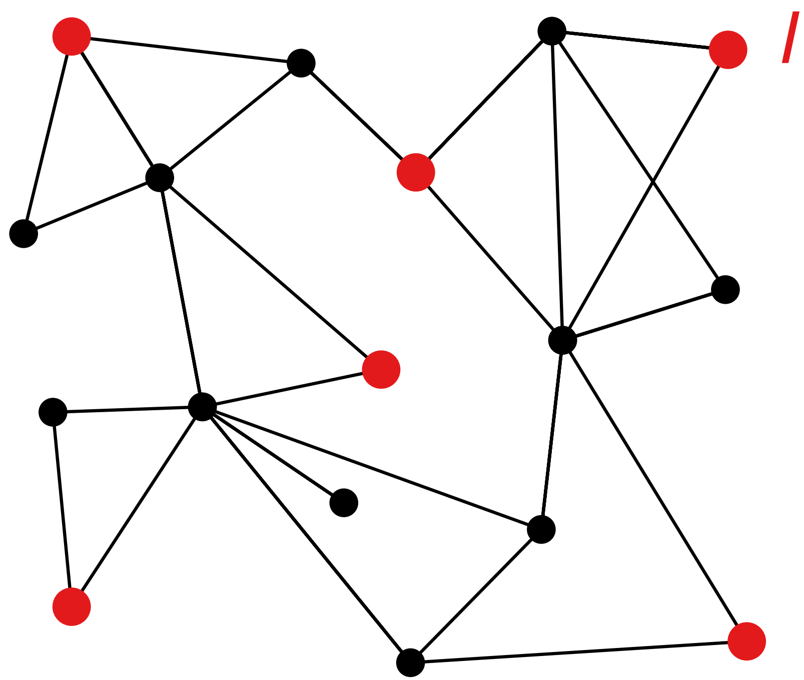

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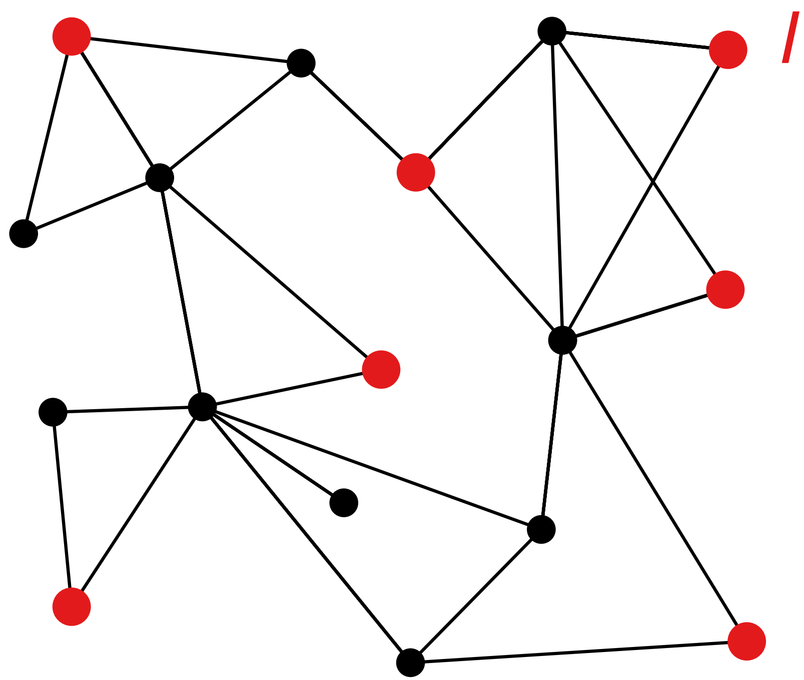

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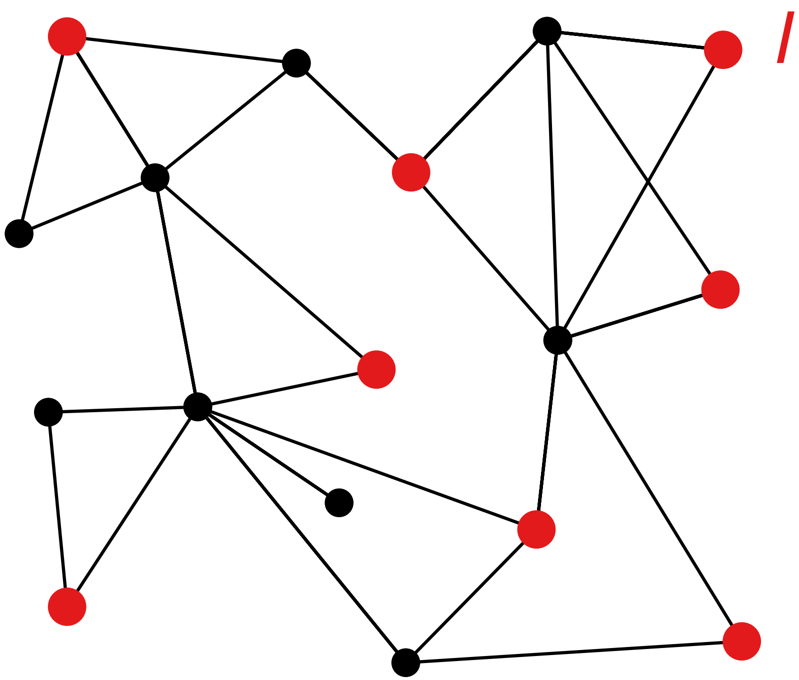

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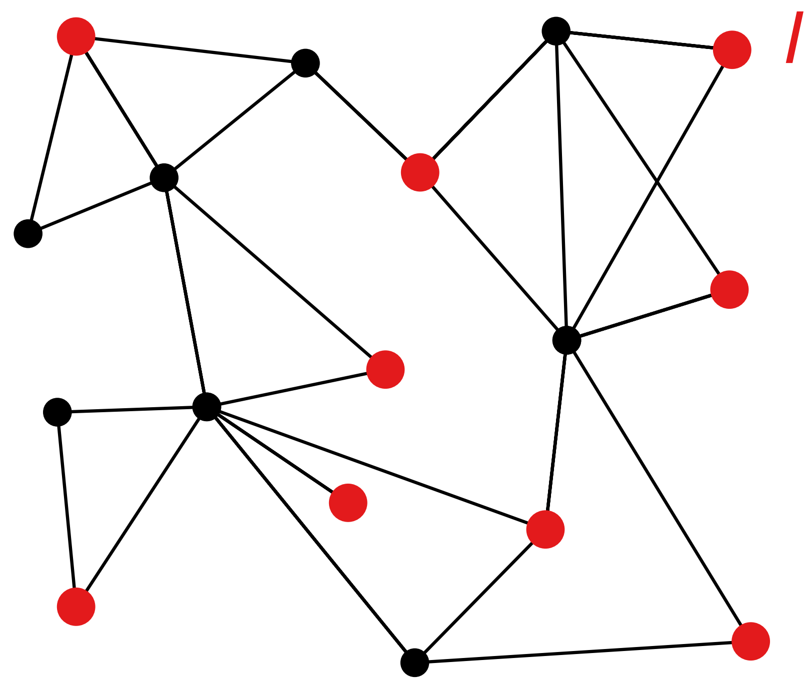

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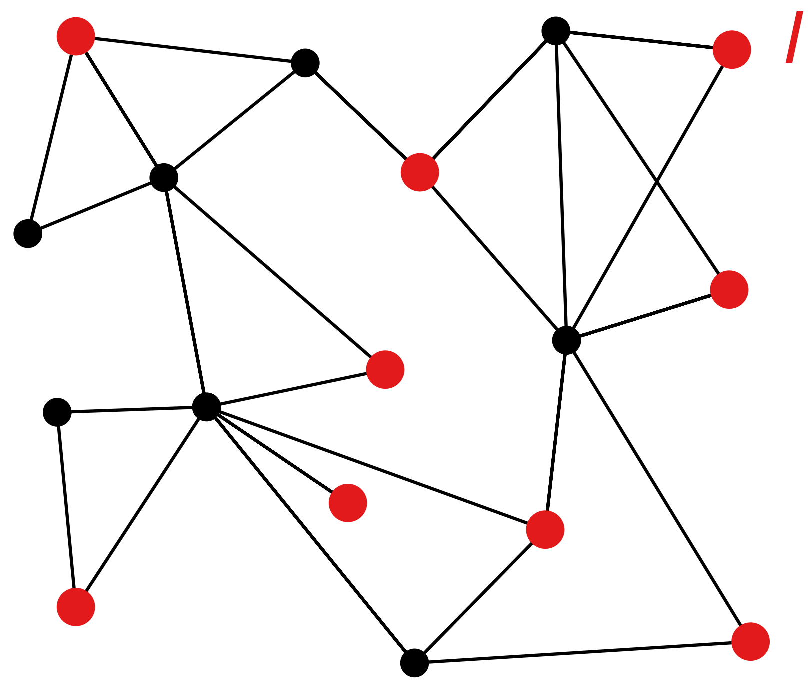

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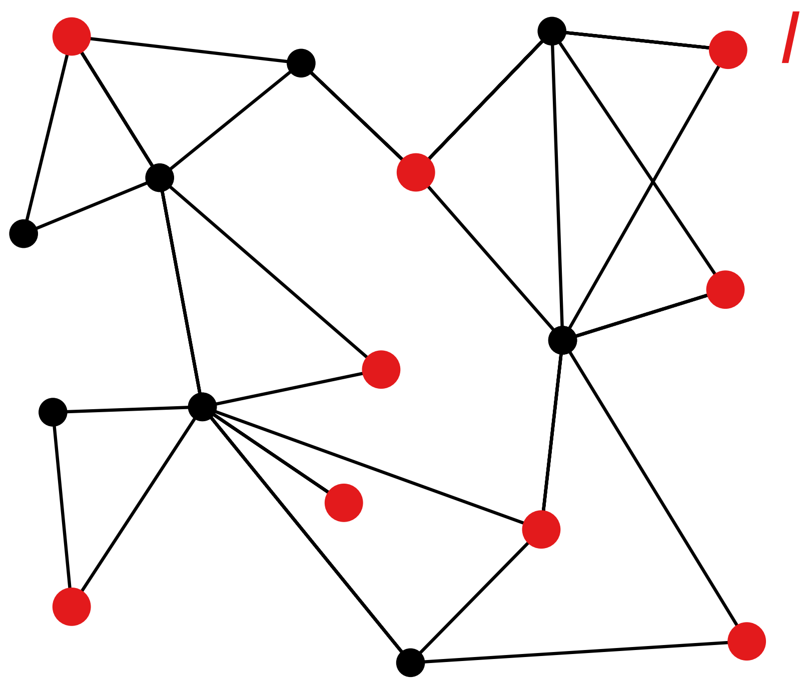

Obs. Maximal independent sets are dominating. :-)

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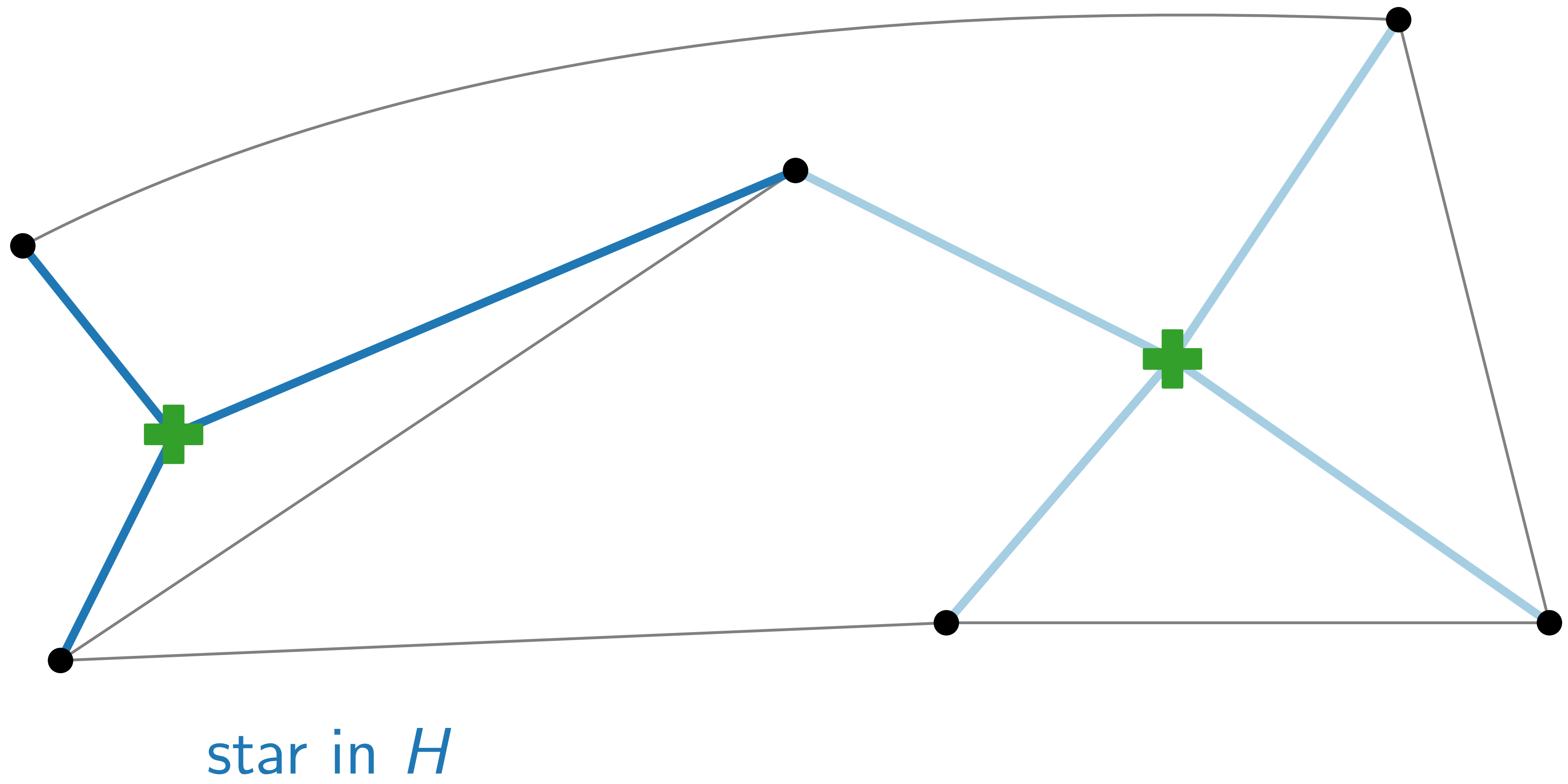

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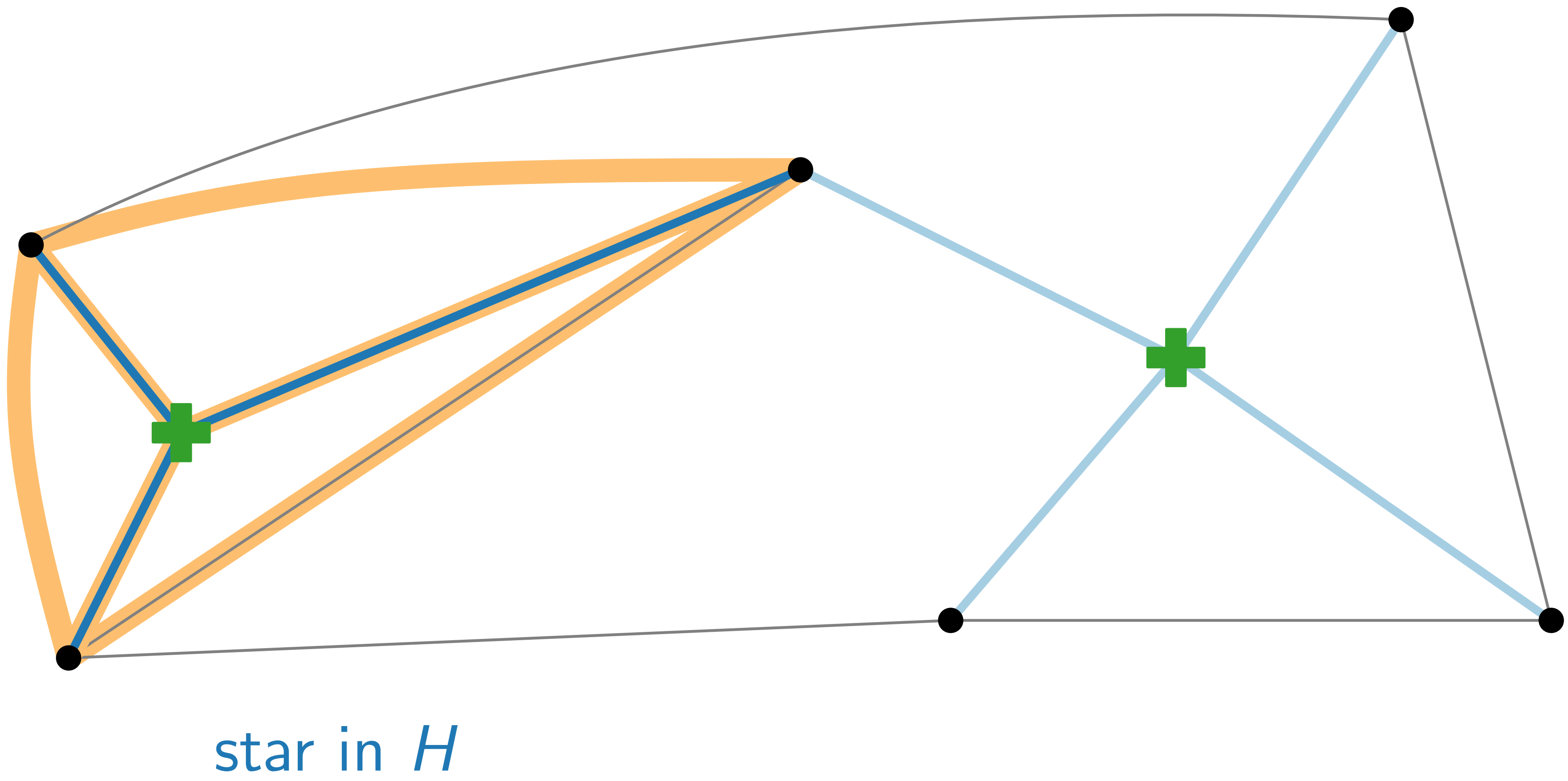

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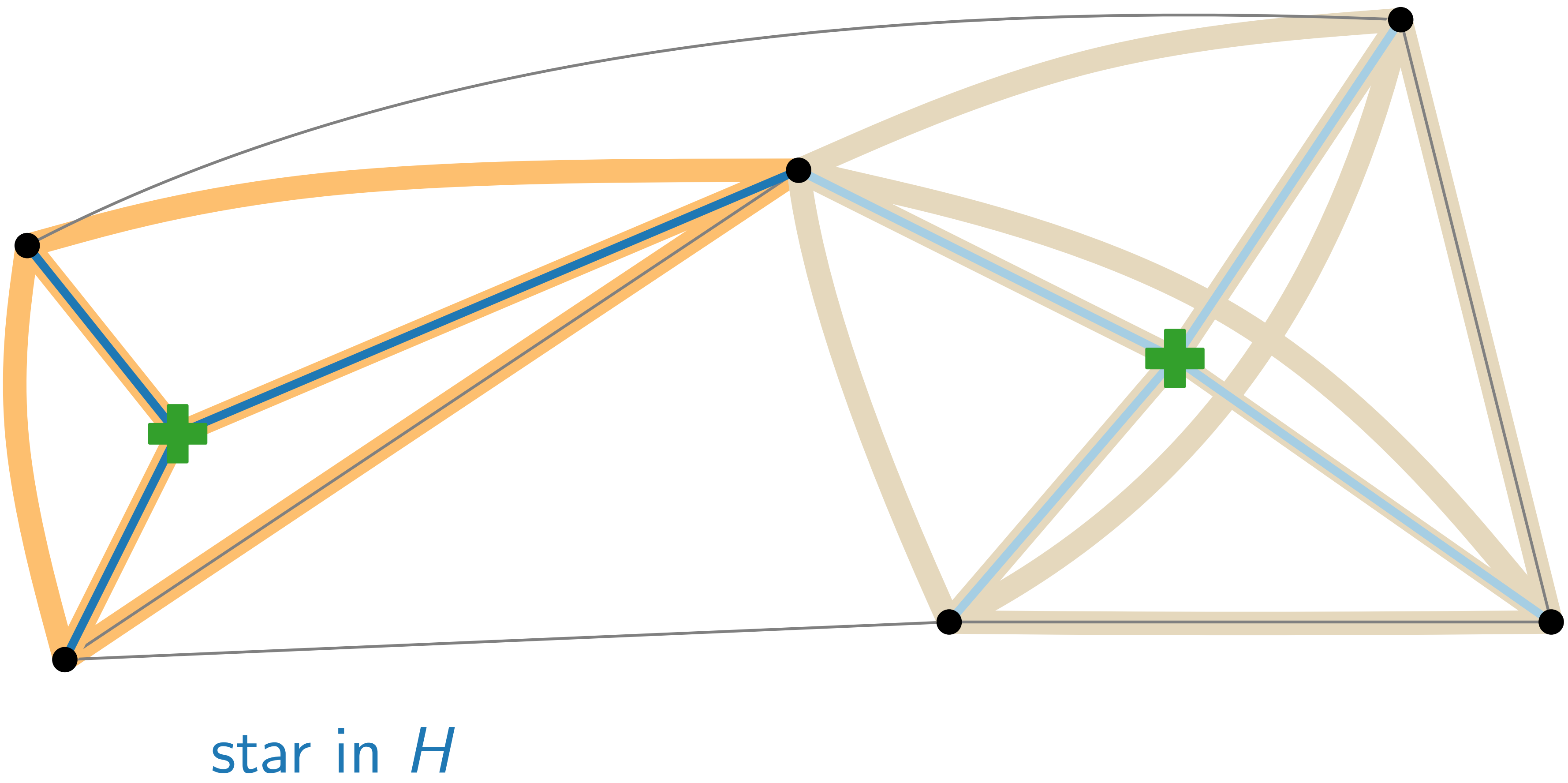

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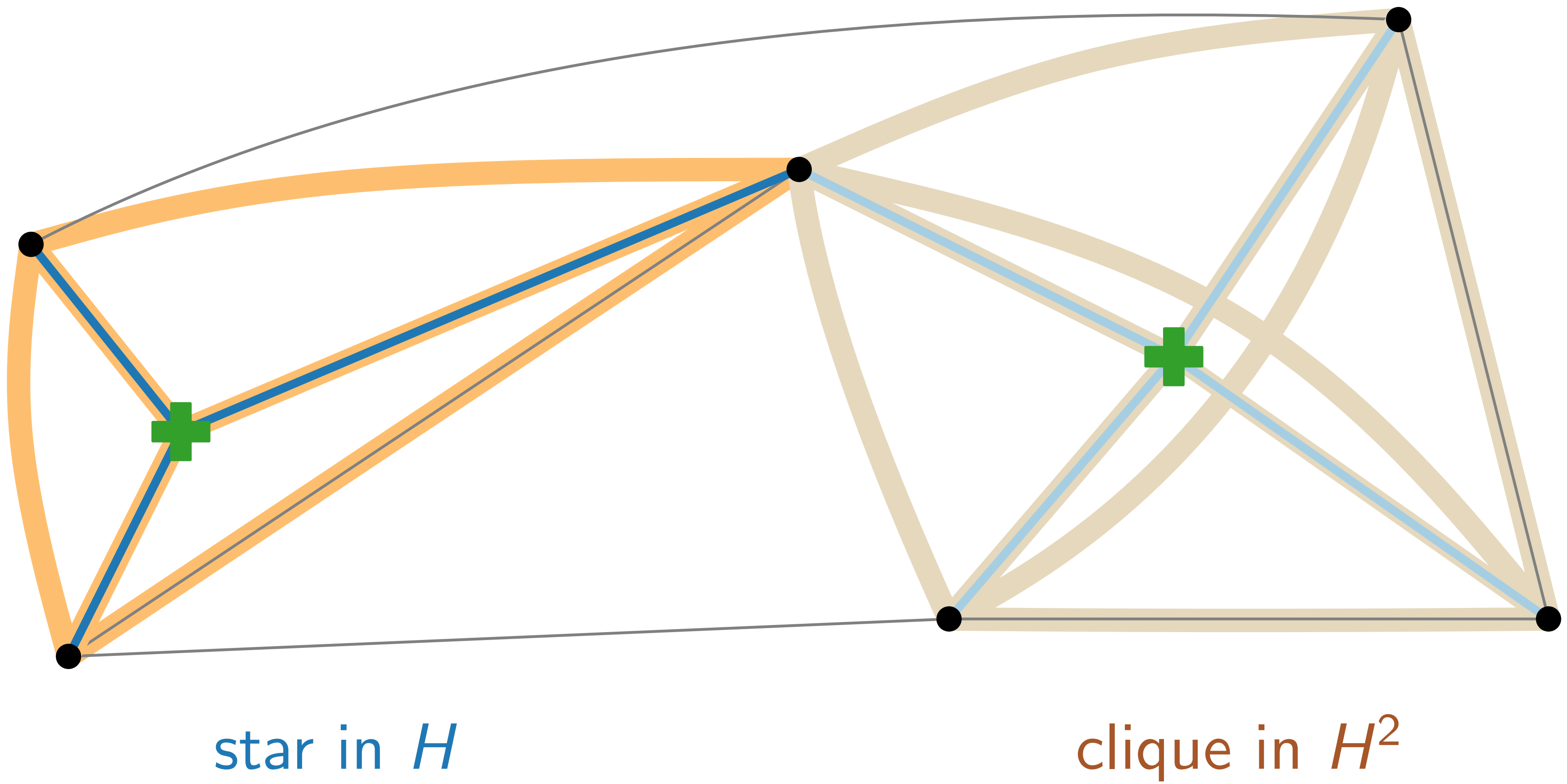

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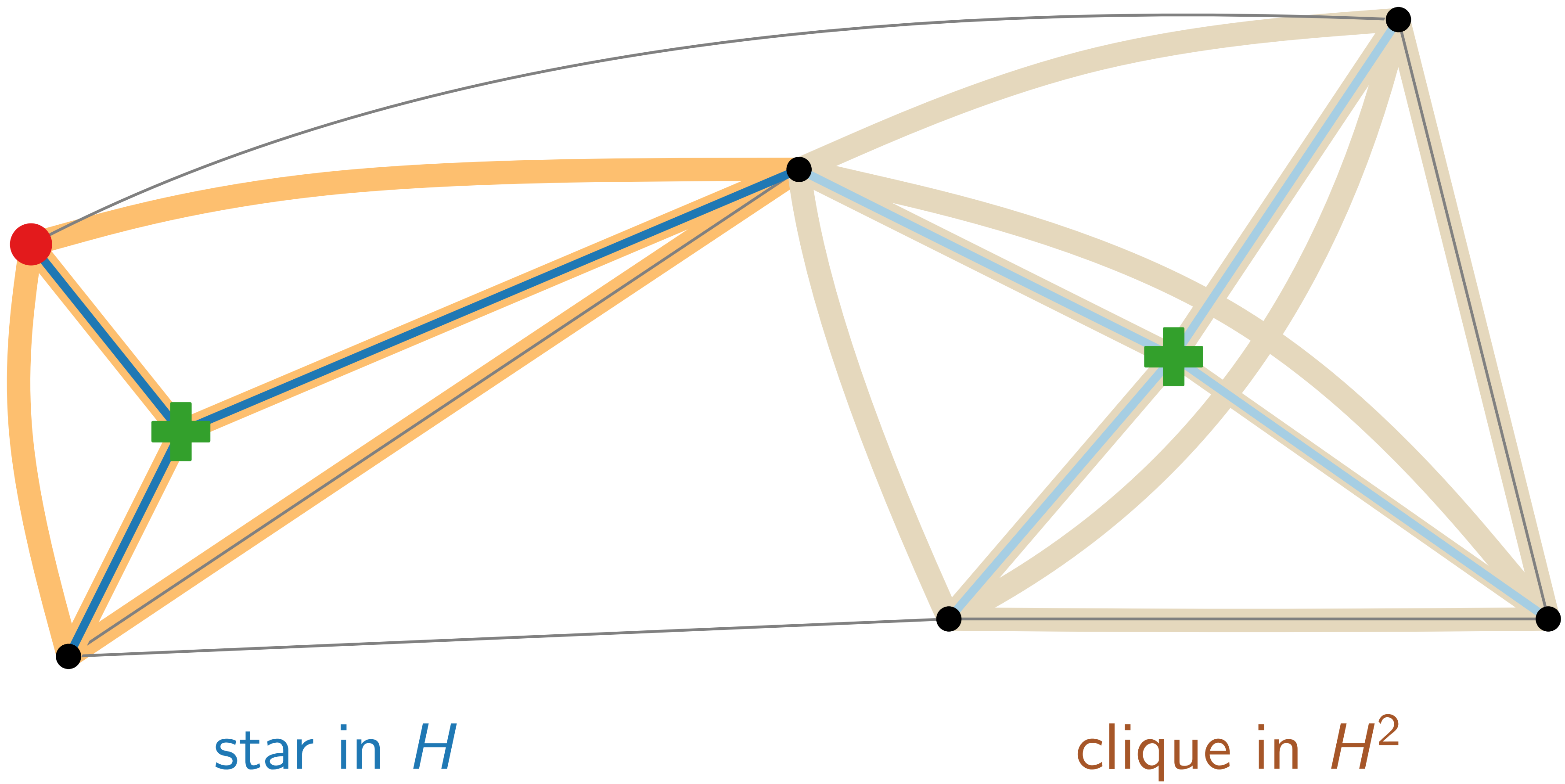

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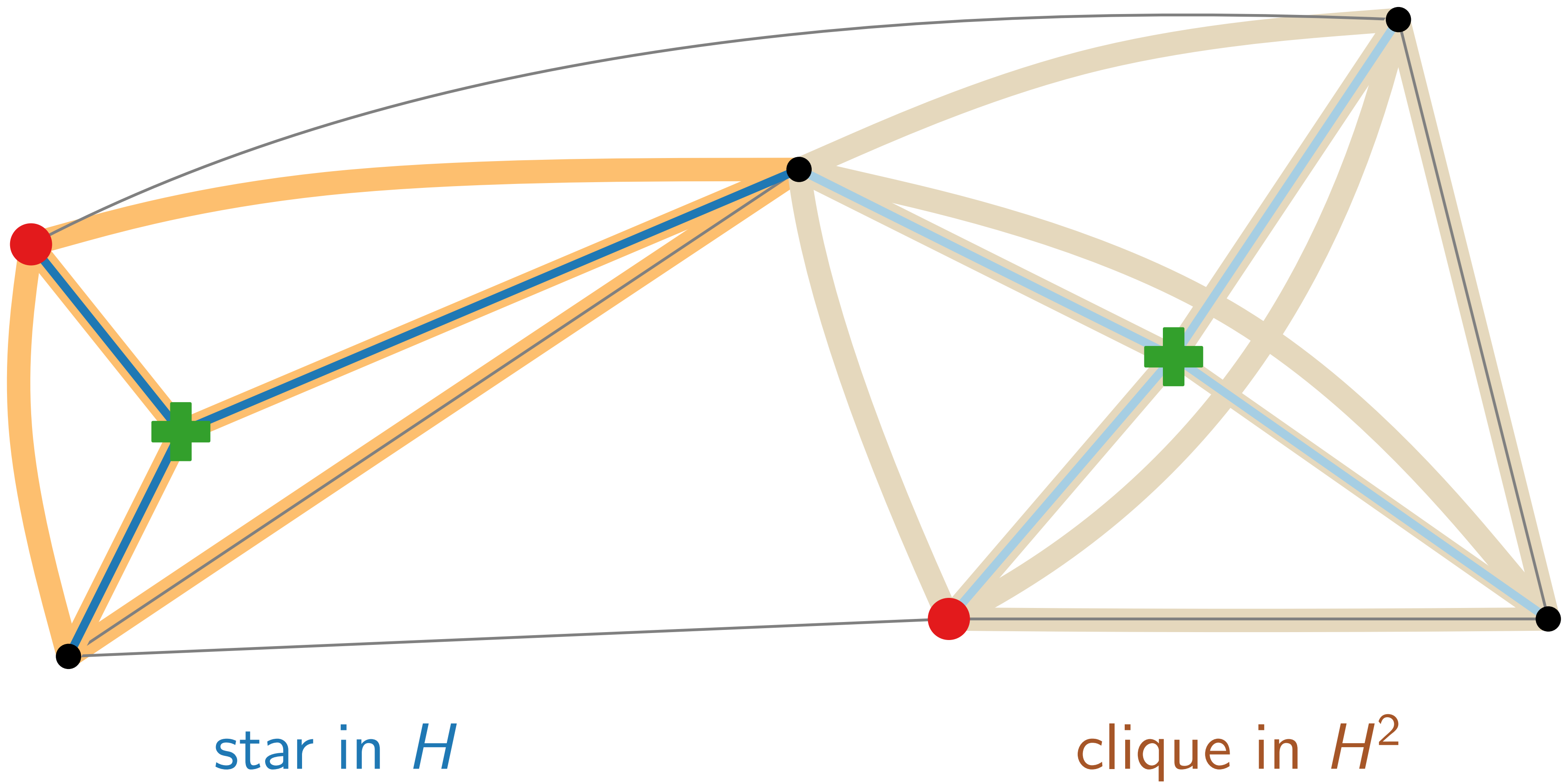

Approximation Algorithms

Lecture 6:
k-Center via Parametric Pruning

Part IV:
Factor-2 Approximation for Metric-k-Center

Factor-2 Approx. for Metric k-Center

Metric-k-Center-Approx(G, c, k)

Sort the edges of G by cost: $c(e_1) \leq \cdots \leq c(e_m)$.

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Lemma. For j provided by the algorithm, it holds that $c(e_j) \leq \mathrm{OPT}$.

Theorem. The above algorithm is a factor-2 approximation algorithm for the metric k-Center problem.

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What about a tight example?

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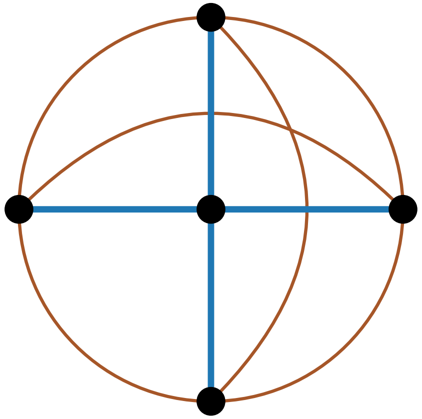

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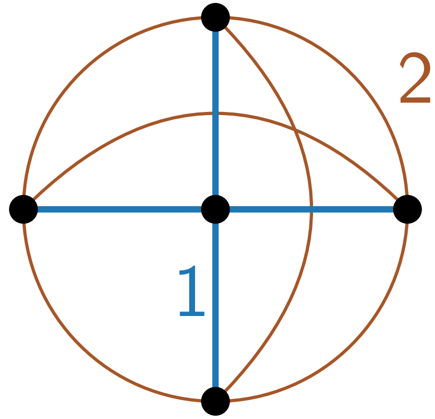

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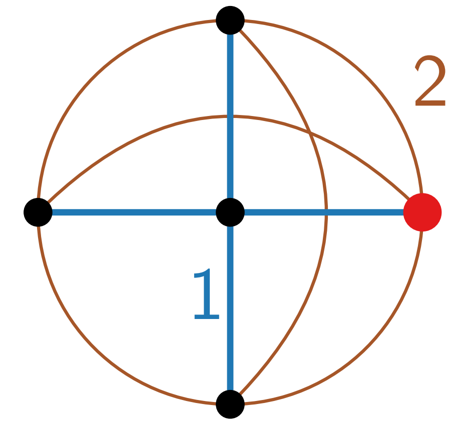

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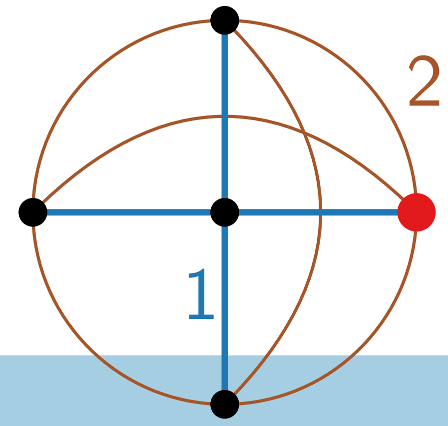

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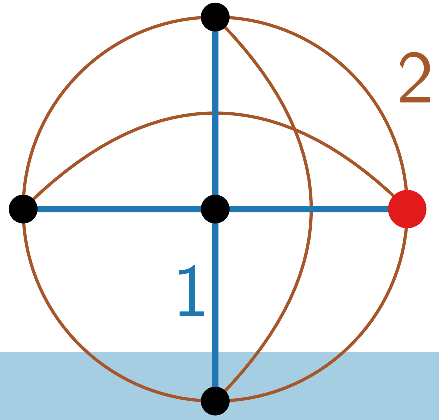

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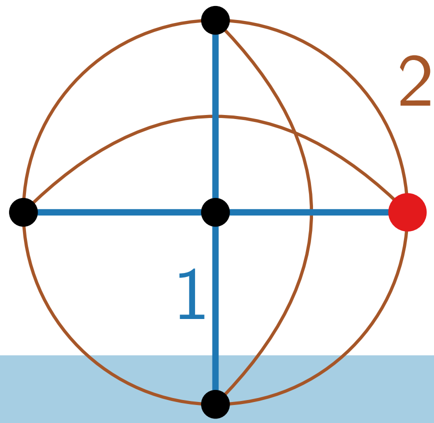

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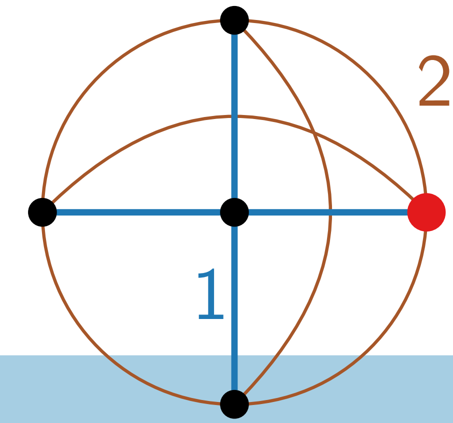

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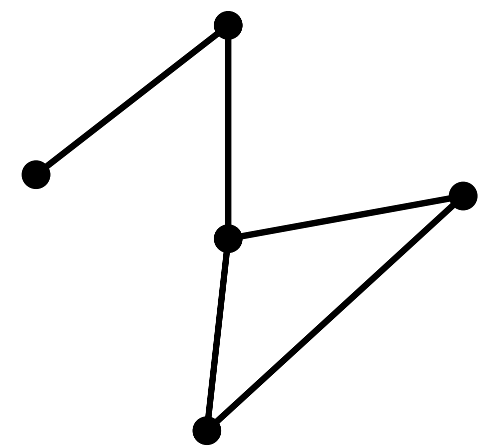

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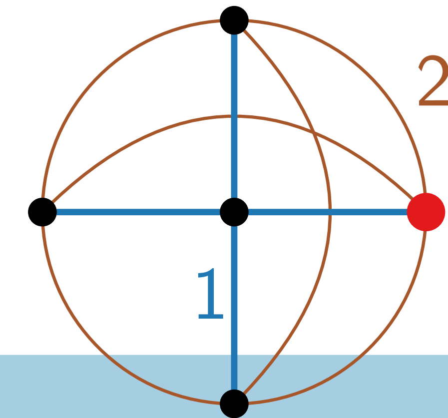

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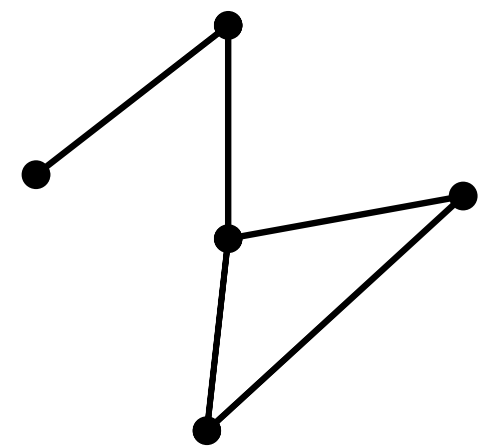
Proof. Reduce from DOMINATING SET to metric k-CENTER. Given graph G and integer k, construct complete graph G' with $V(G') = V(G)$, $E(G') = E(G) \cup E'$.



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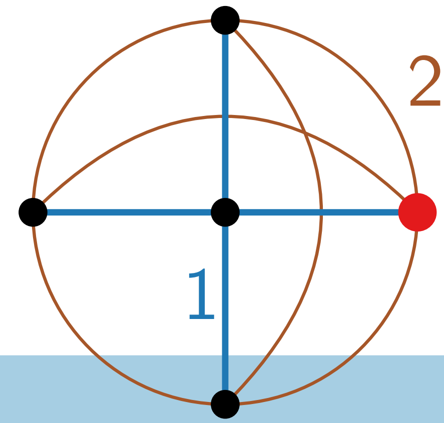

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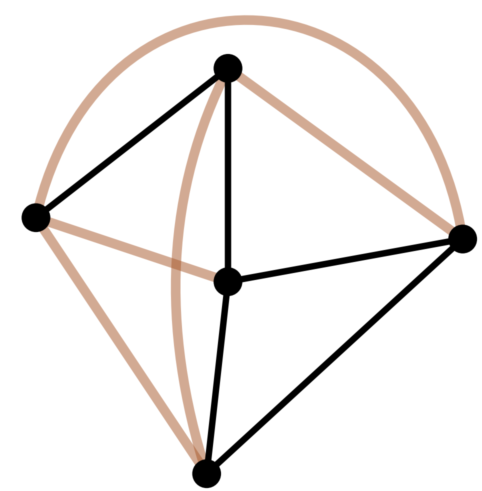

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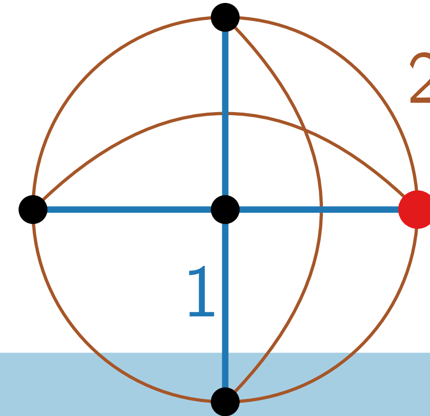

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Proof. Reduce from DOMINATING SET to metric k-CENTER.
Given graph G and integer k, construct complete graph G' with $V(G') = V(G)$, $E(G') = E(G) \cup E'$.
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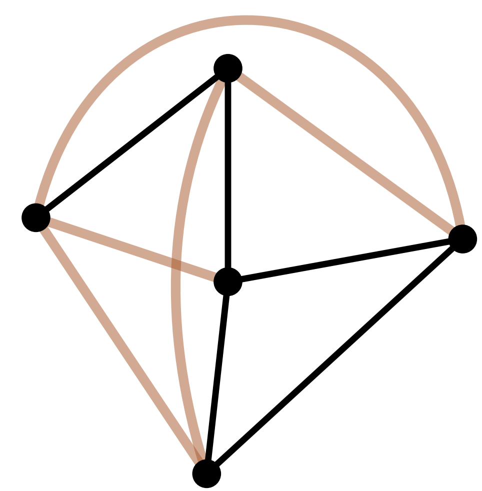

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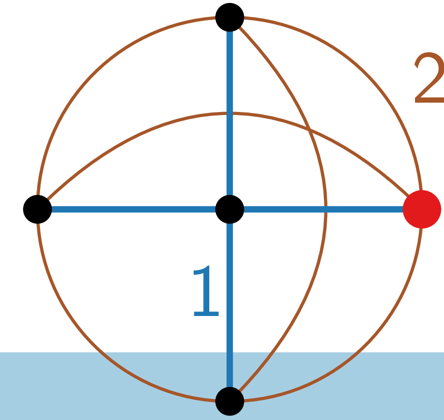

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Theorem. Assuming P $\neq$ NP, for no $\varepsilon > 0$, there is a $(2 - \varepsilon)$-approximation algorithm for the metric k-Center problem.

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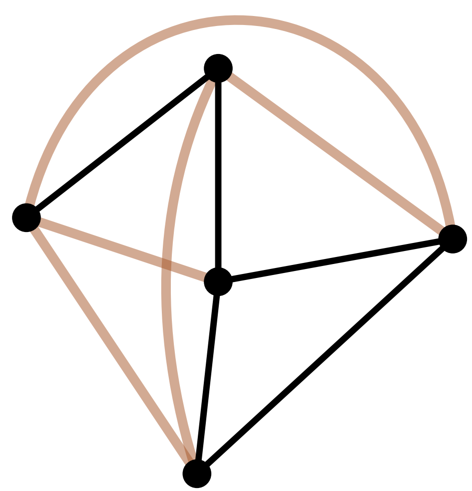

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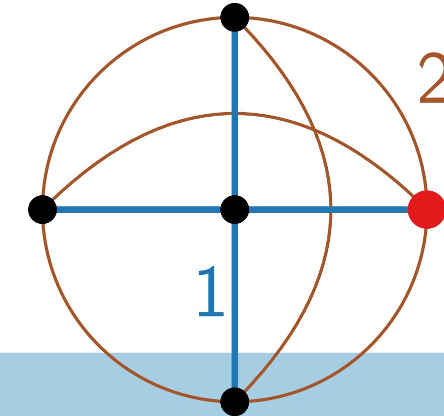

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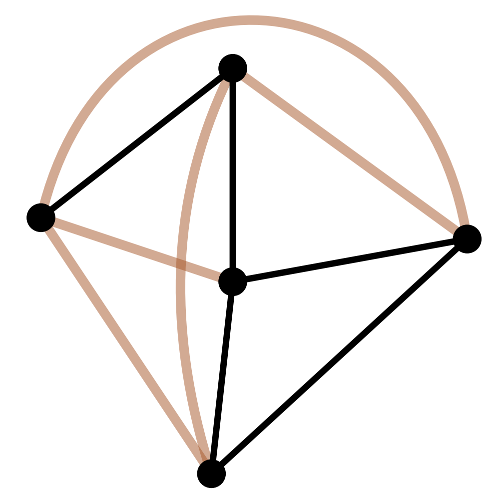

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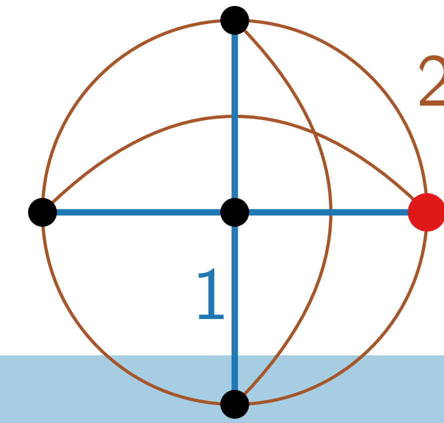

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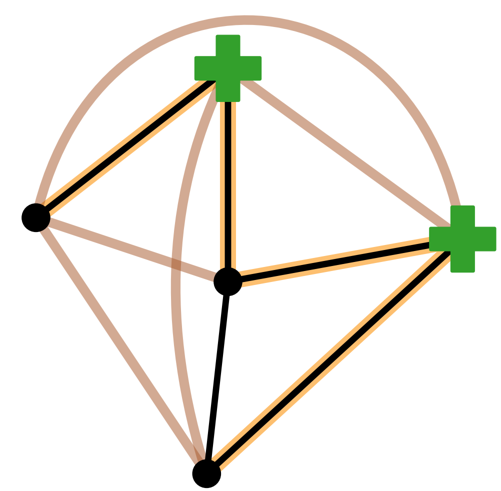

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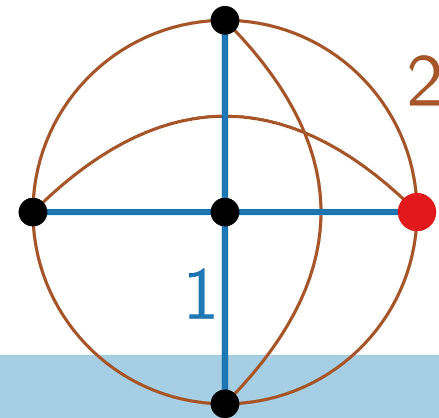

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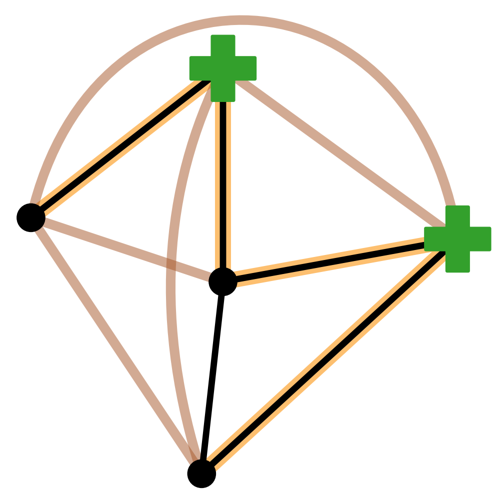

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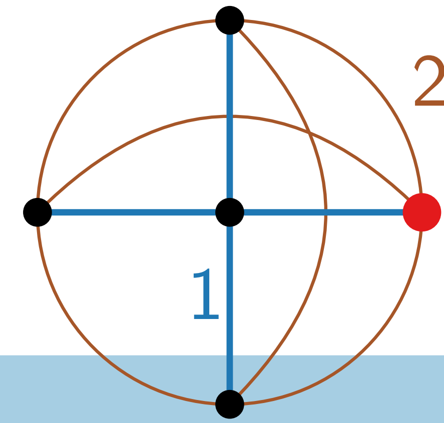

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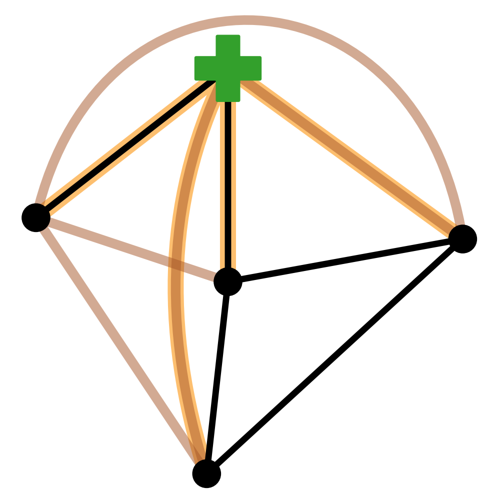

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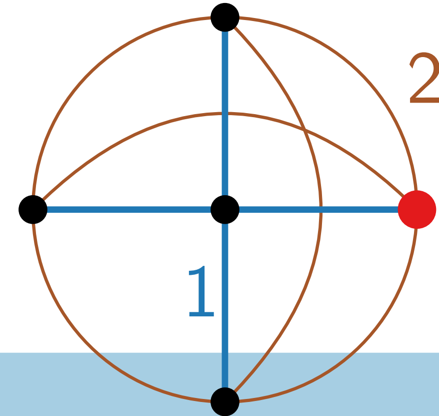

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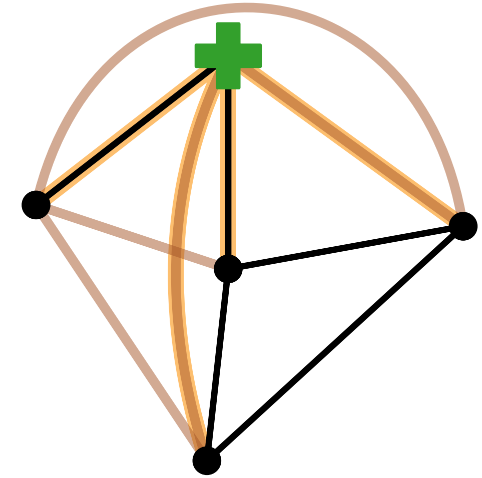

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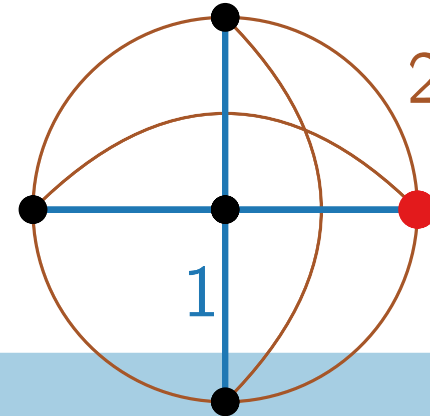

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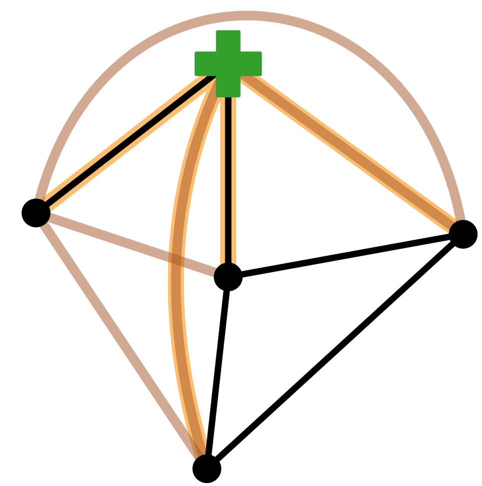

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Approximation Algorithms

Lecture 6:
k-Center via Parametric Pruning

Part V:
Metric-Weighted-Center

Metric-k-Center

Given: A complete graph G with metric edge costs $c\colon E(G) \to \mathbb{Q}_{\geq 0}$ and an integer $k \leq |V|$.

For $S \subseteq V(G)$,
$c(v, S)$ is the cost of the cheapest edge from v to a vertex in S.

Find: A k-element vertex set S such that $\text{cost}(S) := \max_{v \in V(G)} c(v, S)$ is minimized.

Metric-k-Center

Weighted

Given: A complete graph G with metric edge costs $c \colon E(G) \to \mathbb{Q}_{\geq 0}$ and an integer $k \leq |V|$.

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vertex set S of weight at most W

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Algorithm for the Weighted Version

Algorithm Metric-k CENTER-Approx(G, c, k)

Sort the edges of G by cost: $c(e_1) \leq \cdots \leq c(e_m)$

for $j = 1$ **to** m **do**

  Construct G_j^2

  Find a maximal independent set I_j in G_j^2

  if $|I_j| \leq k$ **then**

    return I_j

Algorithm for the Weighted Version

Algorithm Metric-Weighted-CENTER-Approx(G, c, w, W)

Sort the edges of G by cost: $c(e_1) \leq \cdots \leq c(e_m)$

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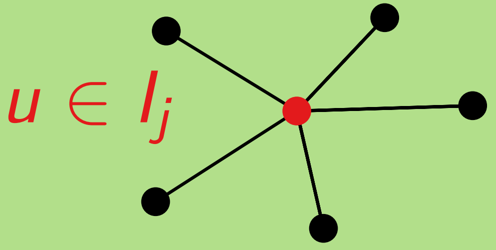

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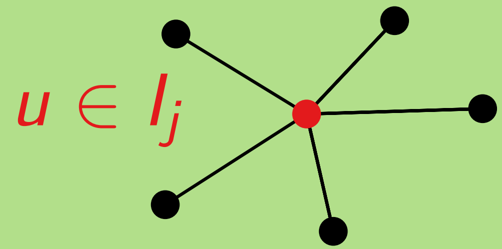

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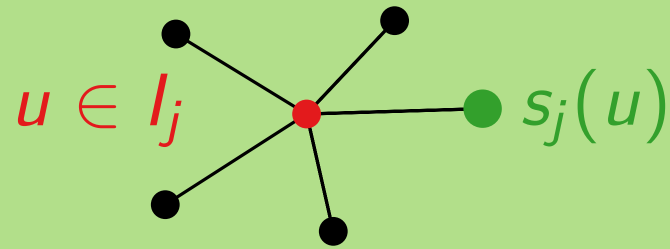

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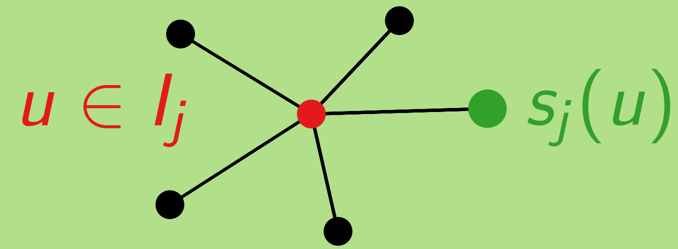

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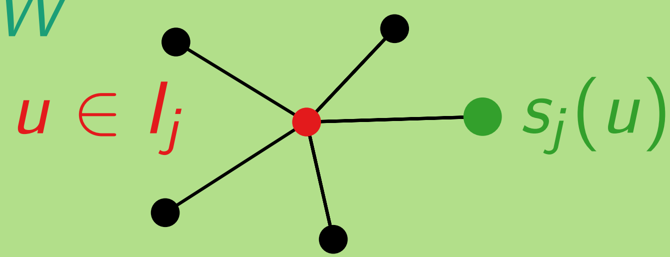

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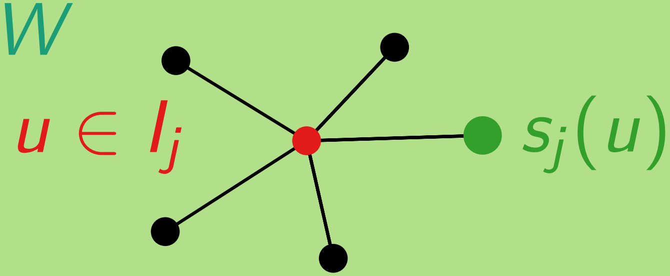

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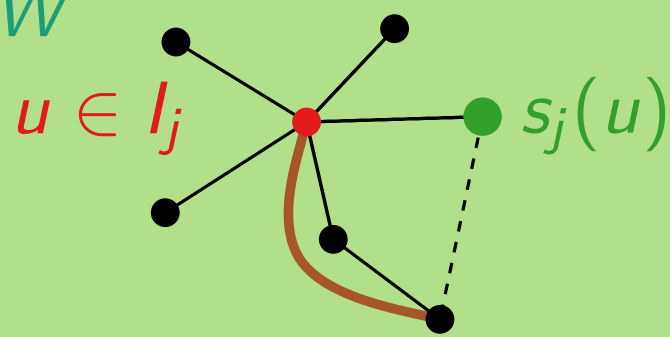

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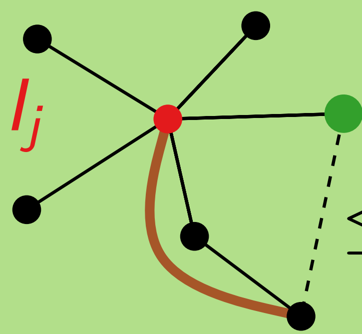

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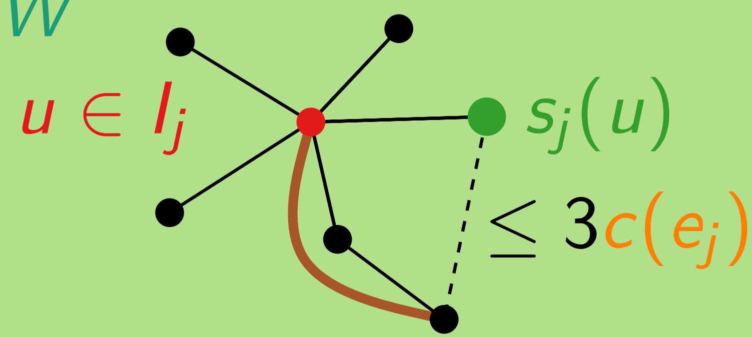

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Theorem. The above is a factor-3 approximation algorithm for METRIC-WEIGHTED-CENTER.

Tight Example. . . ?

Here, we need to have a budget W,
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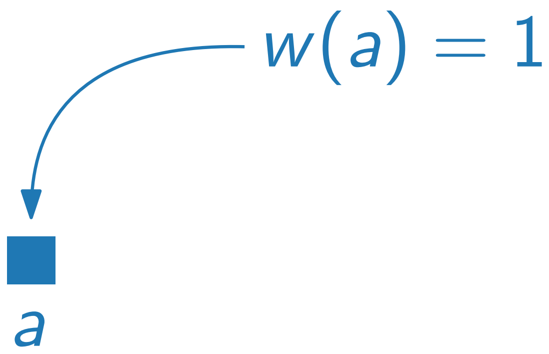

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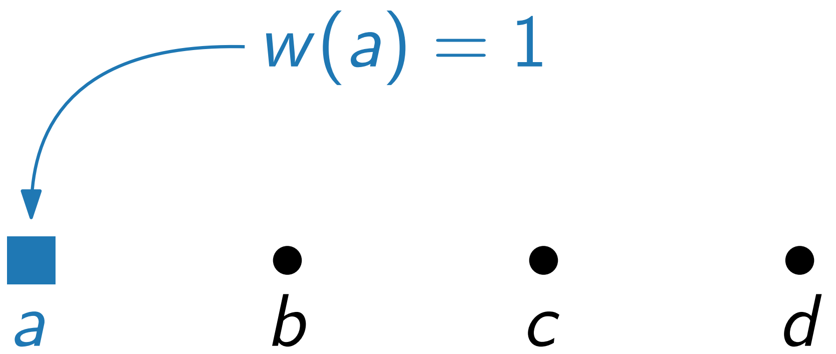

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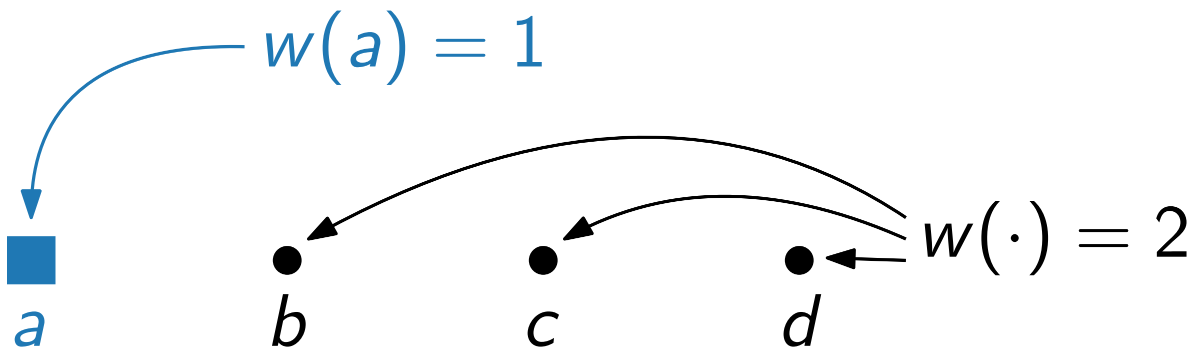

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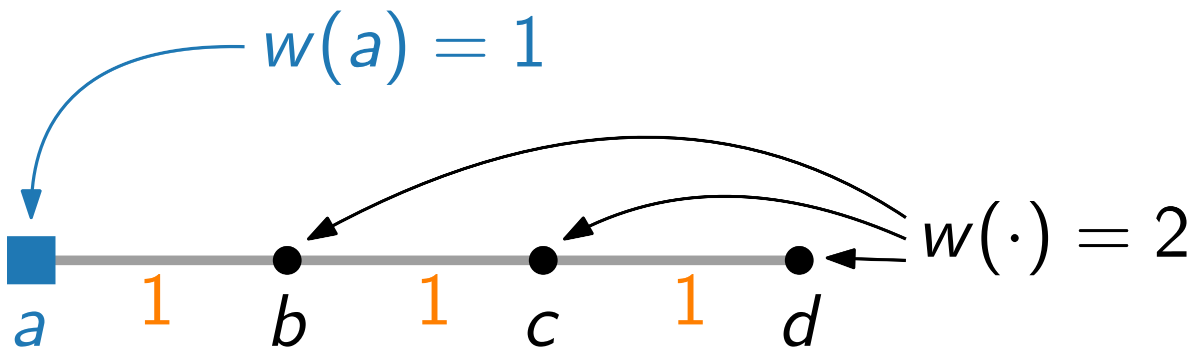

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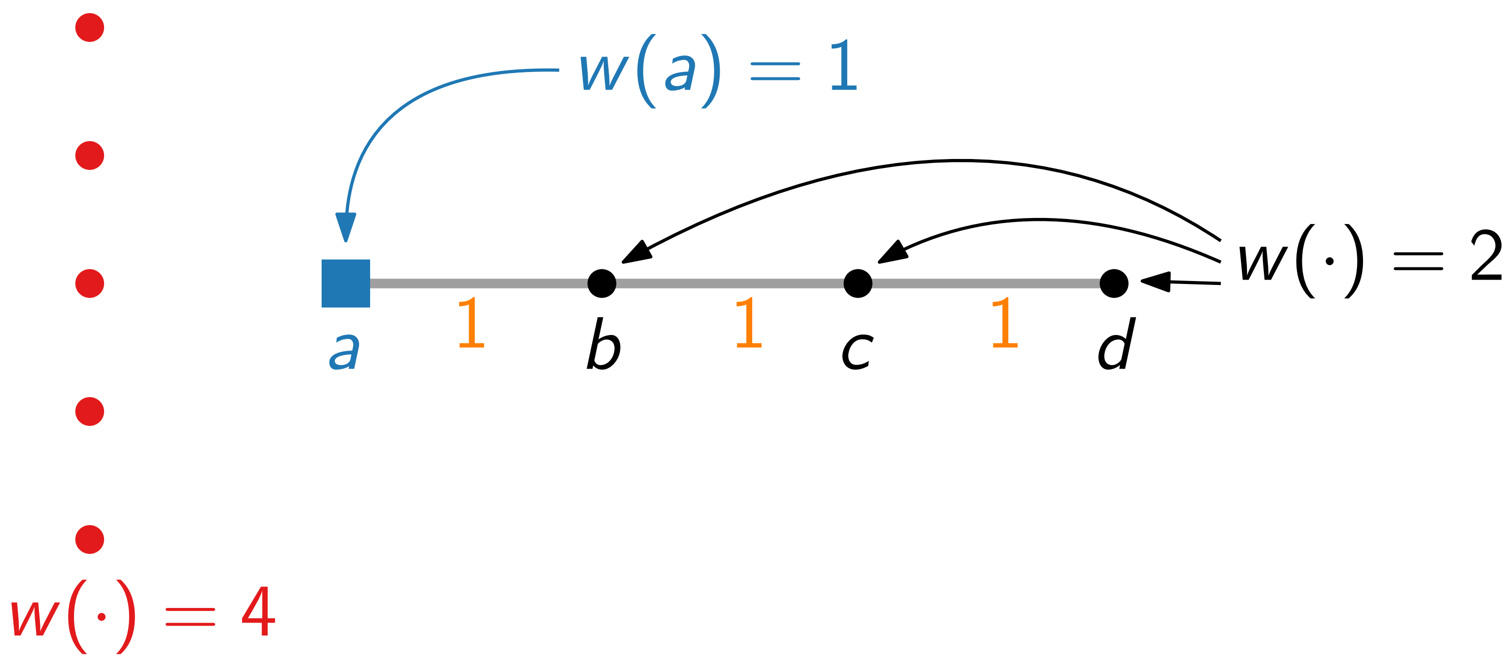

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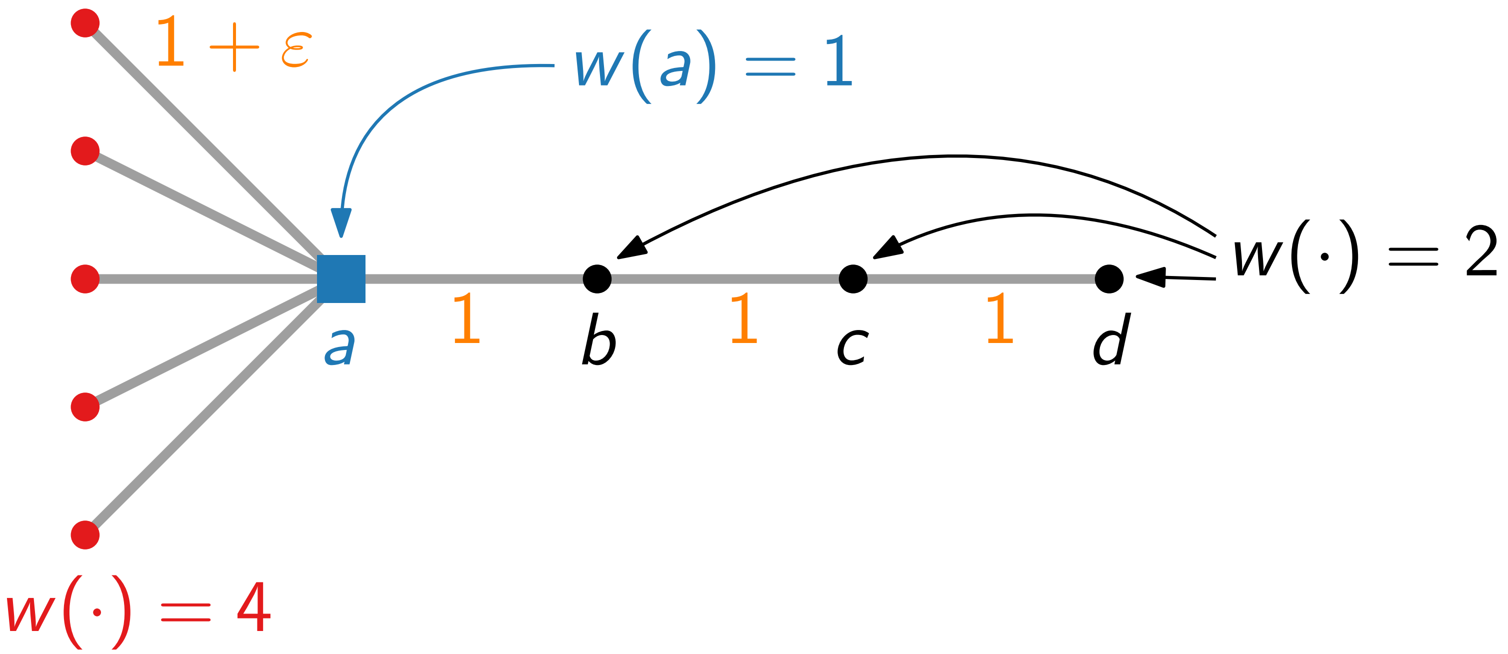

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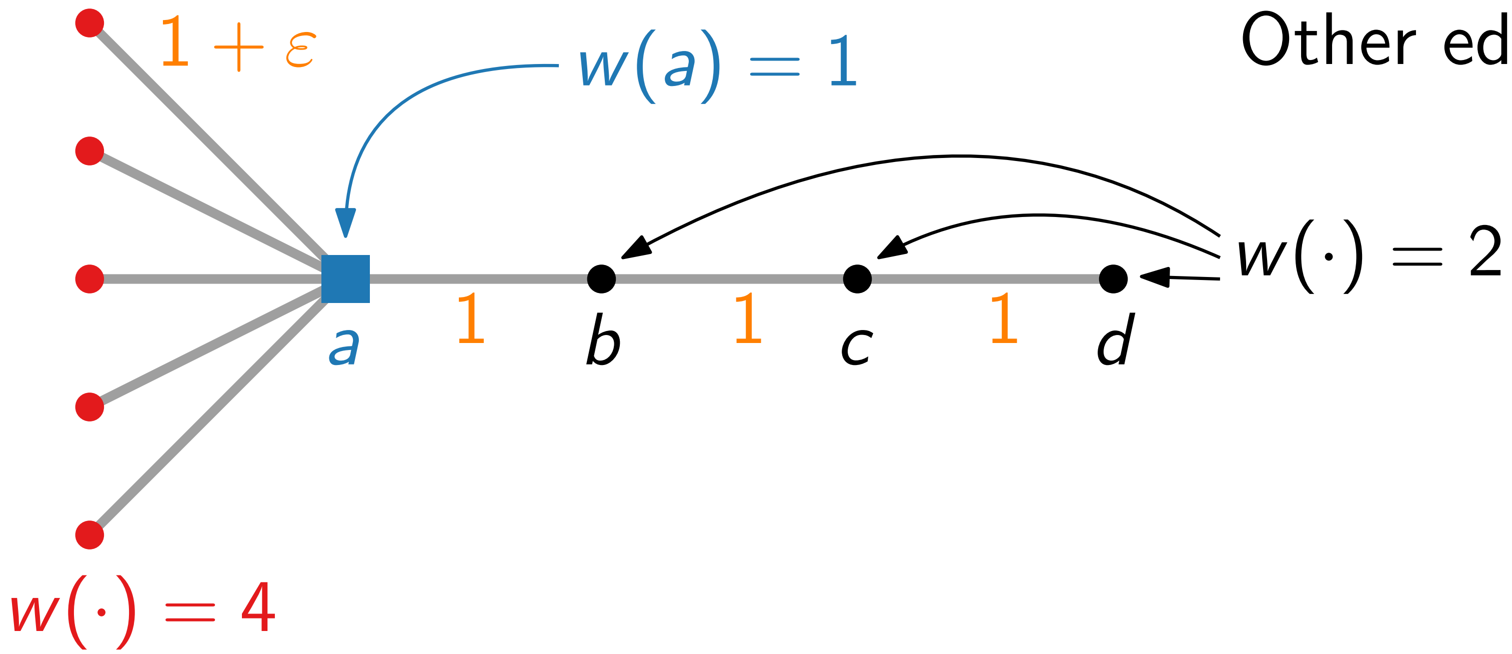

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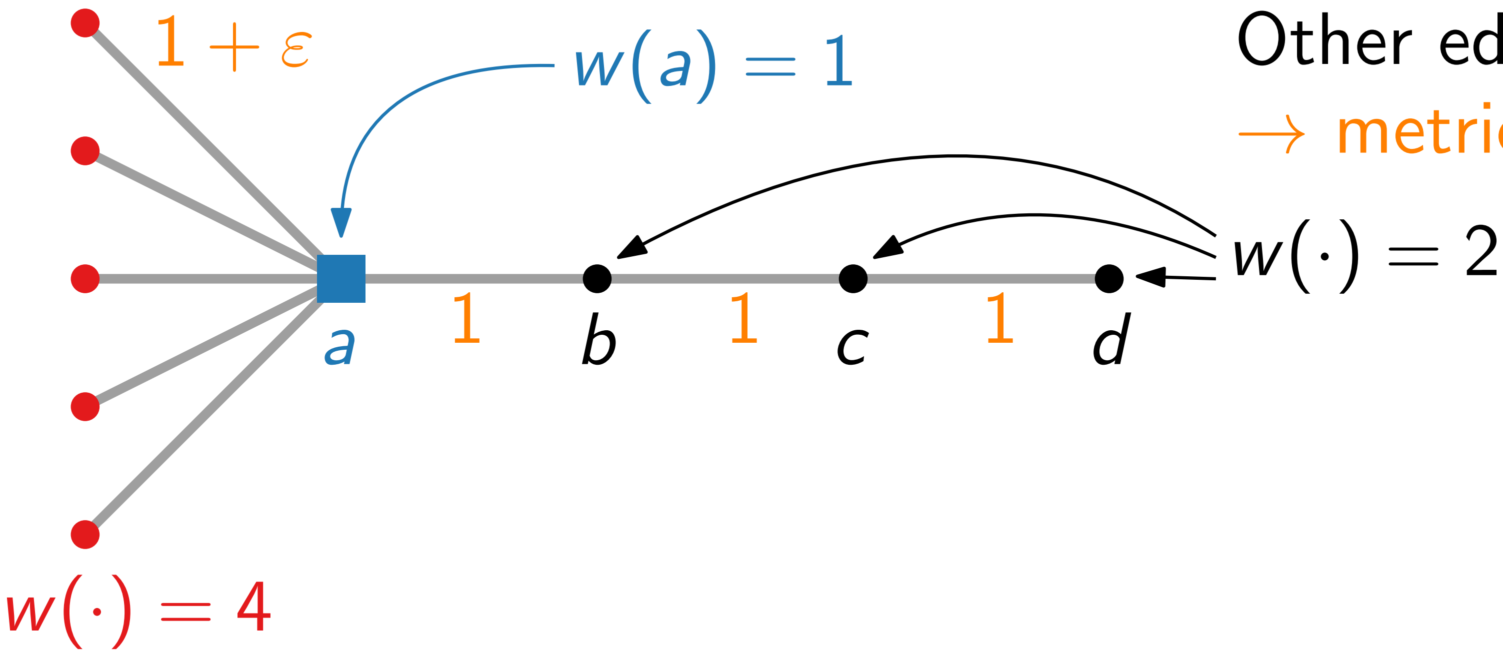
$\rightarrow$ metric completion!



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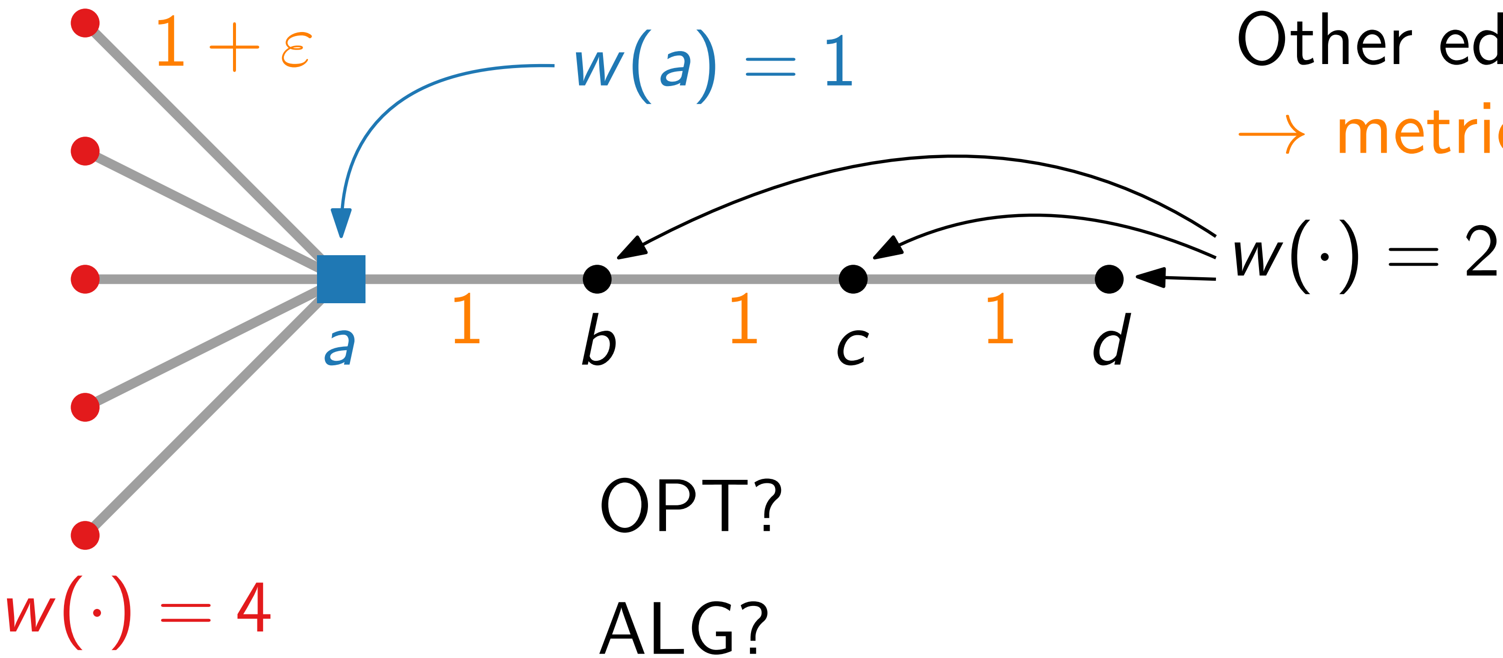

OPT?

ALG?

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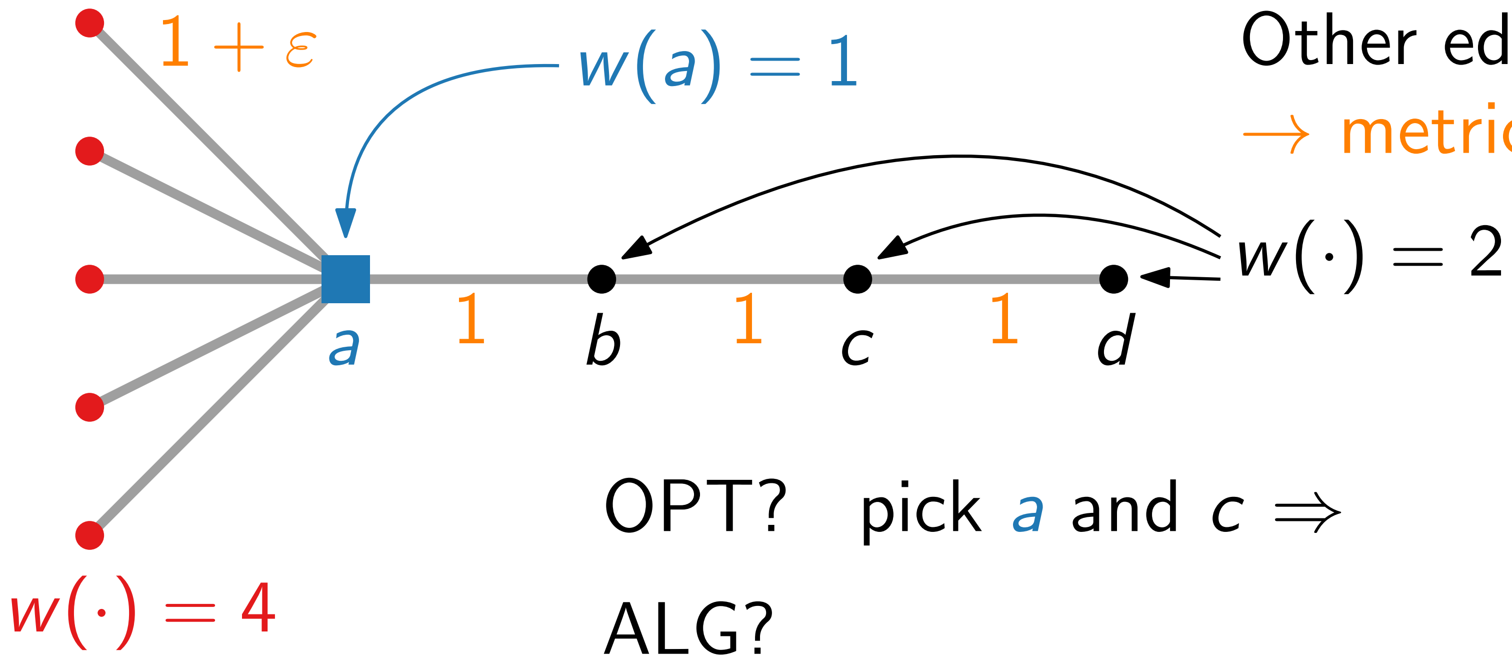

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Other edge costs?

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OPT? pick a and c $\Rightarrow$

ALG?

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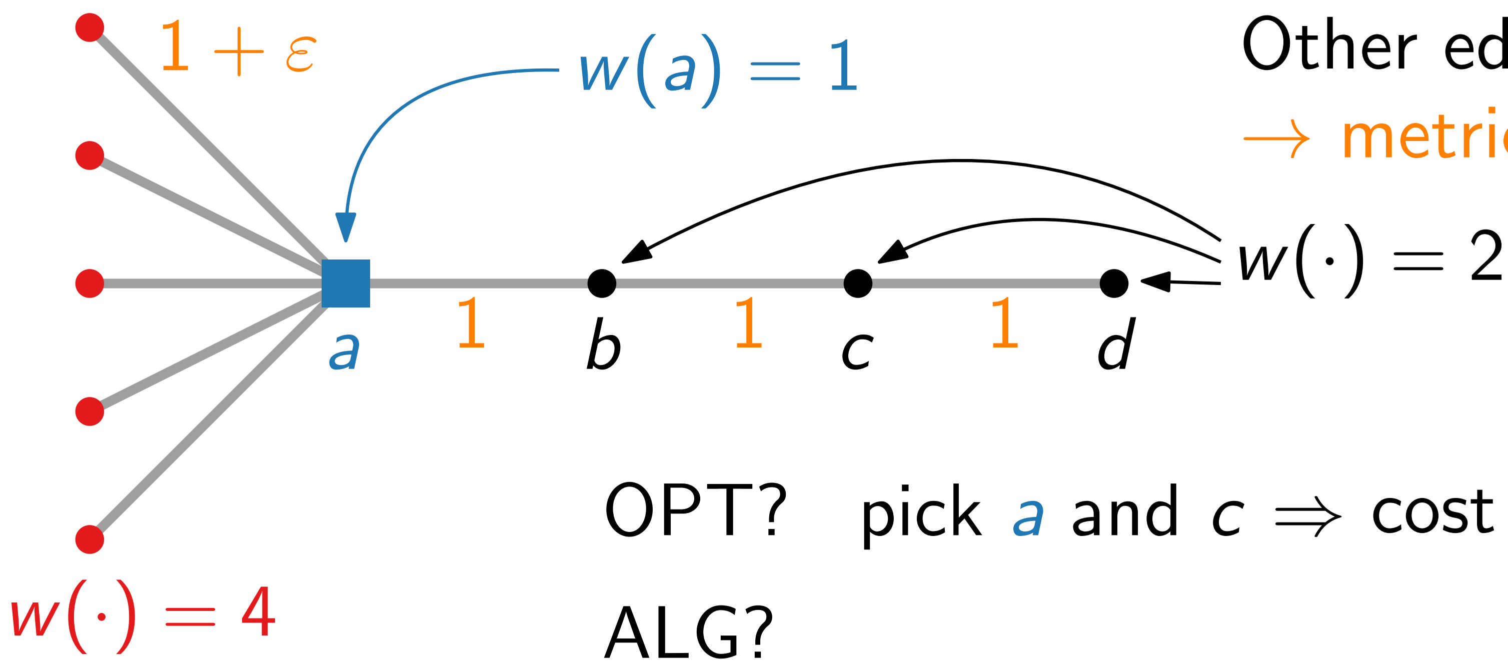

OPT? pick a and c $\Rightarrow$ cost $1 + \varepsilon$.

ALG?

Tight Example. . . ?

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Consider $W = 3$.

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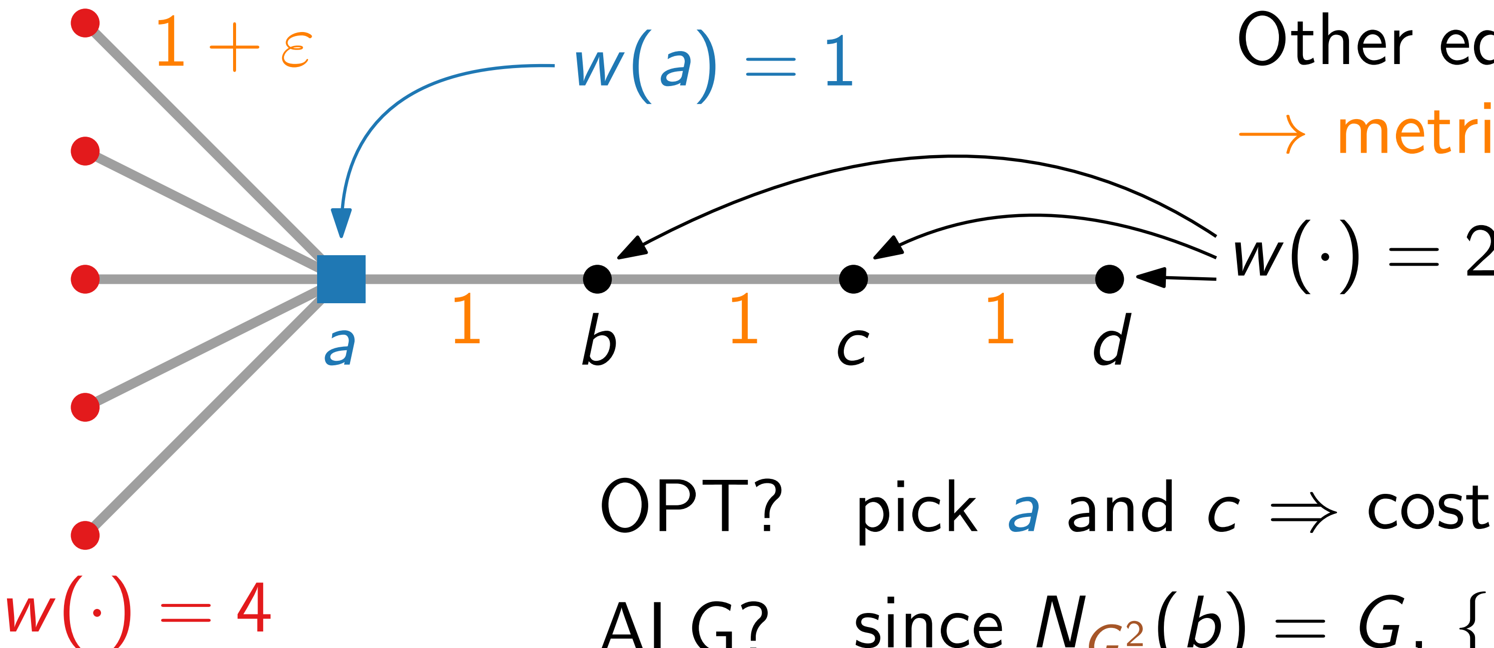

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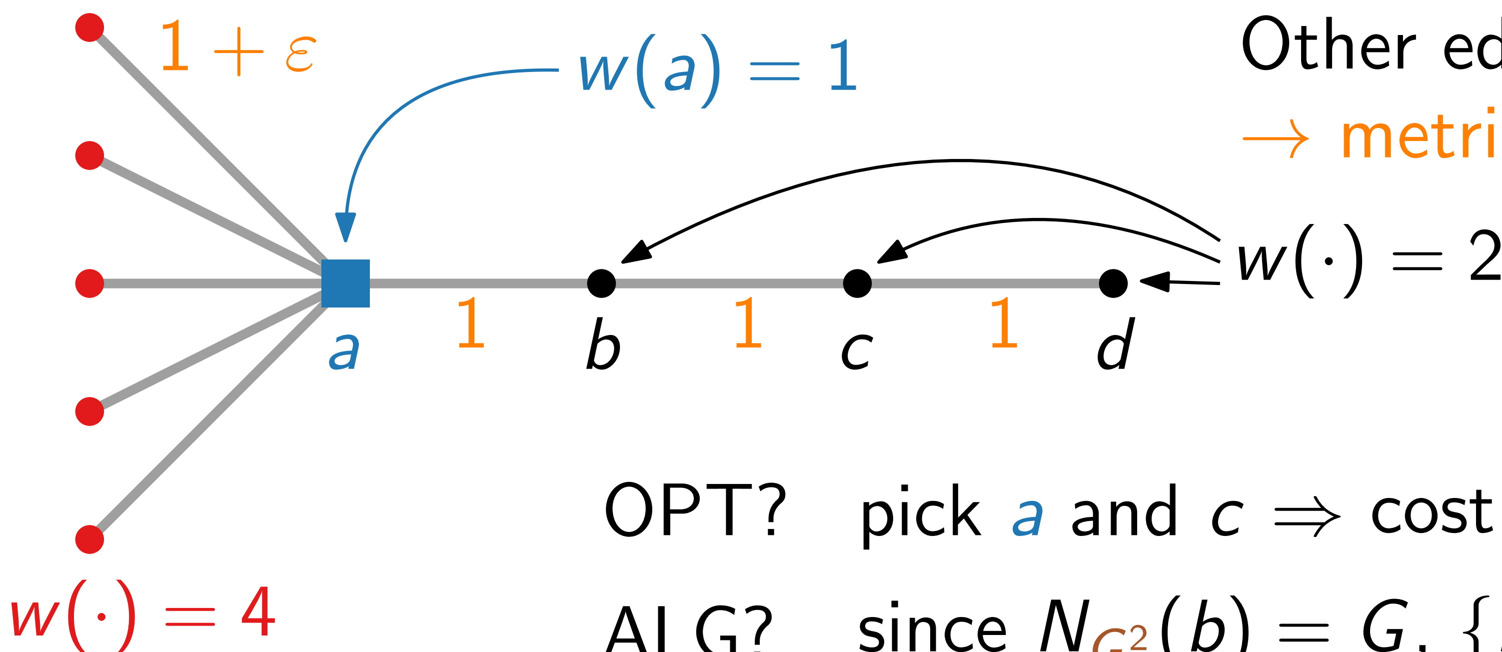

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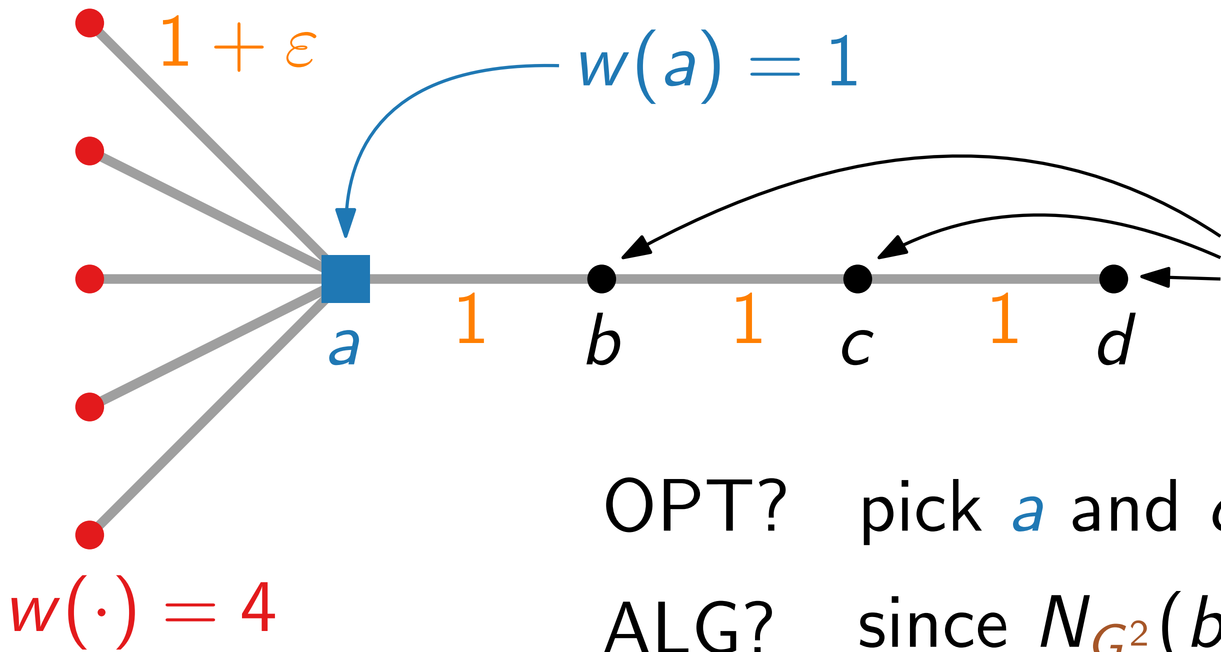

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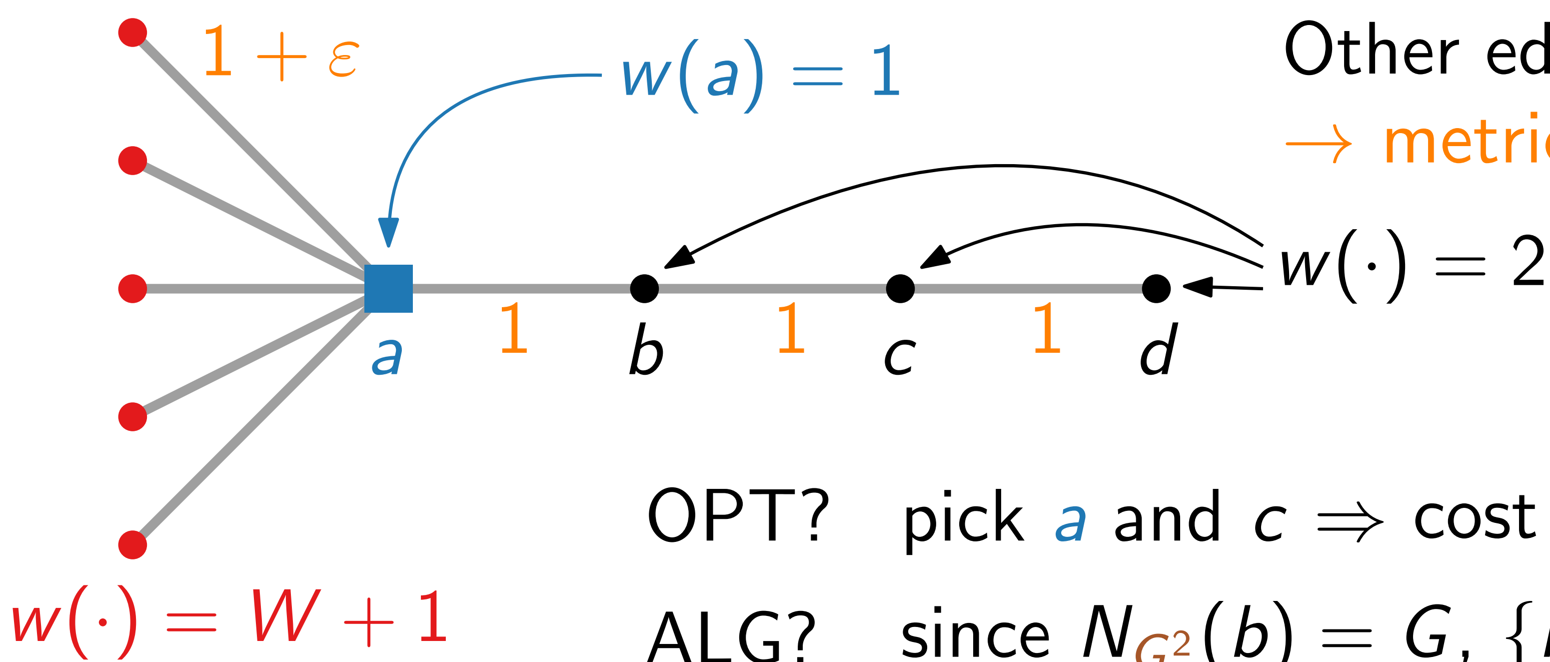

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