

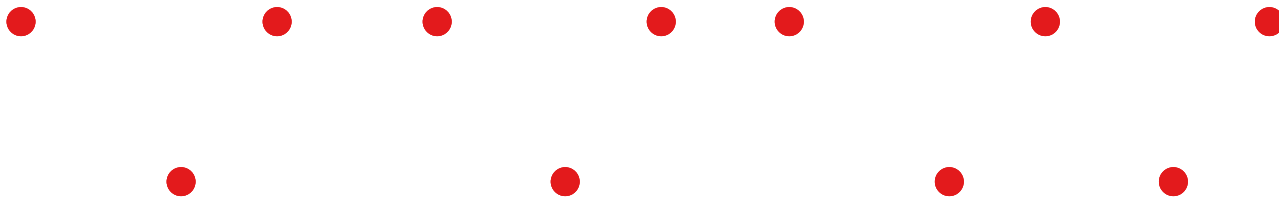
Approximation Algorithms

Lecture 5: LP-based Approximation Algorithms for SETCOVER

Part I: SETCOVER as an ILP

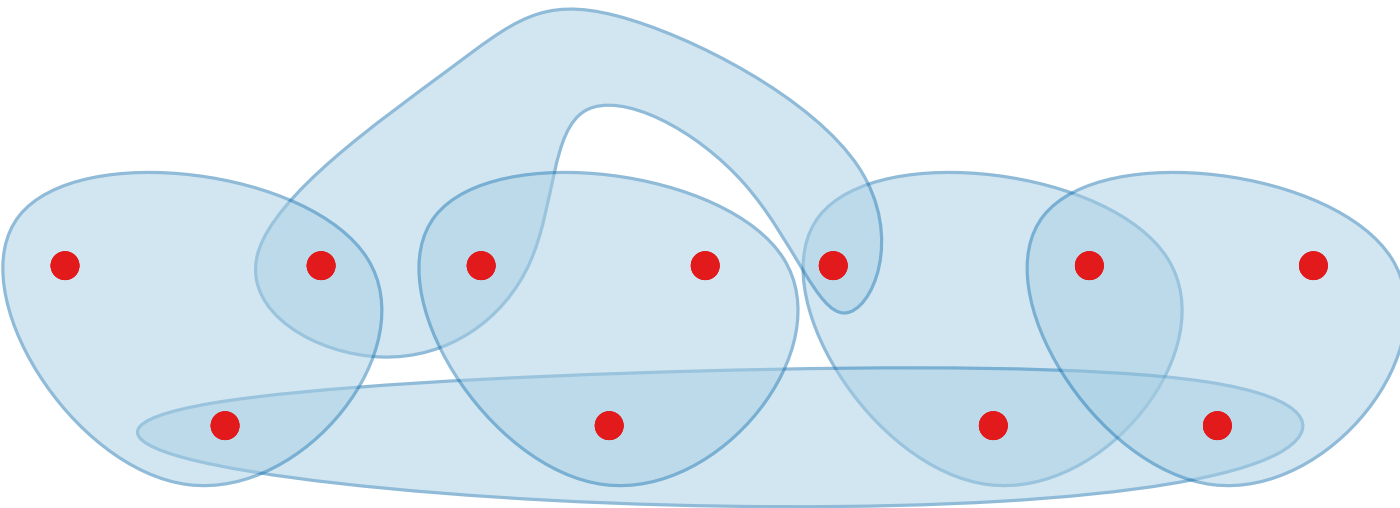
SETCOVER as an ILP

Ground set U



SETCOVER as an ILP

Ground set U
Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$

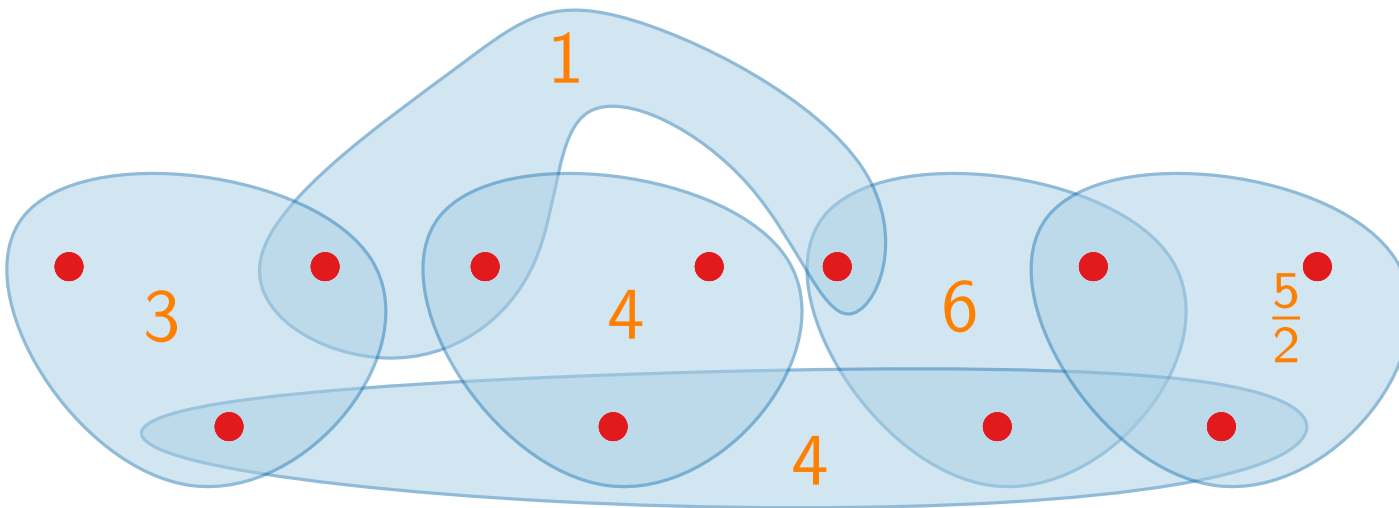


SETCOVER as an ILP

Ground set U

Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$

Costs $c: \mathcal{S} \rightarrow \mathbb{Q}^+$



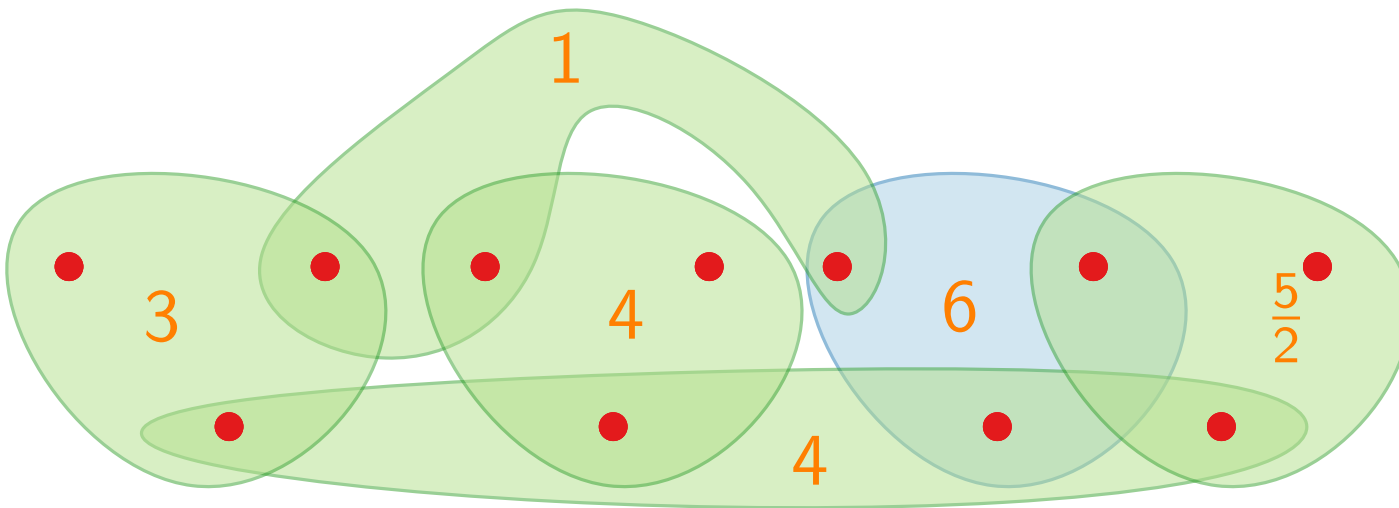
SETCOVER as an ILP

Ground set U

Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$

Costs $c: \mathcal{S} \rightarrow \mathbb{Q}^+$

Find cover $\mathcal{S}' \subseteq \mathcal{S}$
of U with
minimum cost.



SETCOVER as an ILP

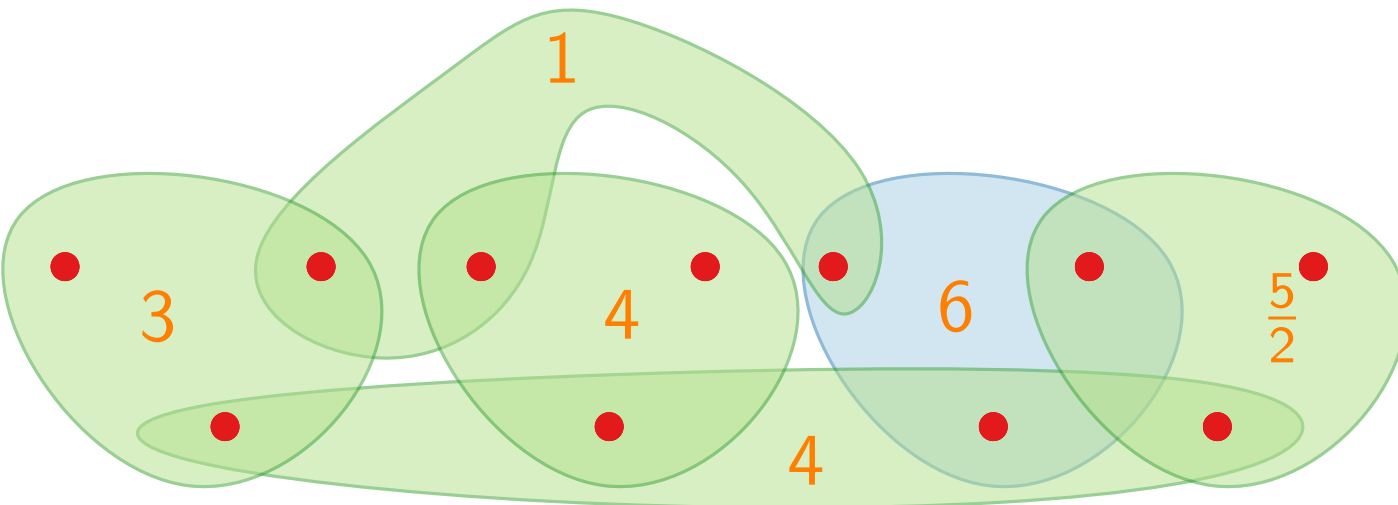
minimize

subject to

Ground set U

Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$

Costs $c: \mathcal{S} \rightarrow \mathbb{Q}^+$



Find cover $\mathcal{S}' \subseteq \mathcal{S}$
of U with
minimum cost.

SETCOVER as an ILP

minimize

subject to

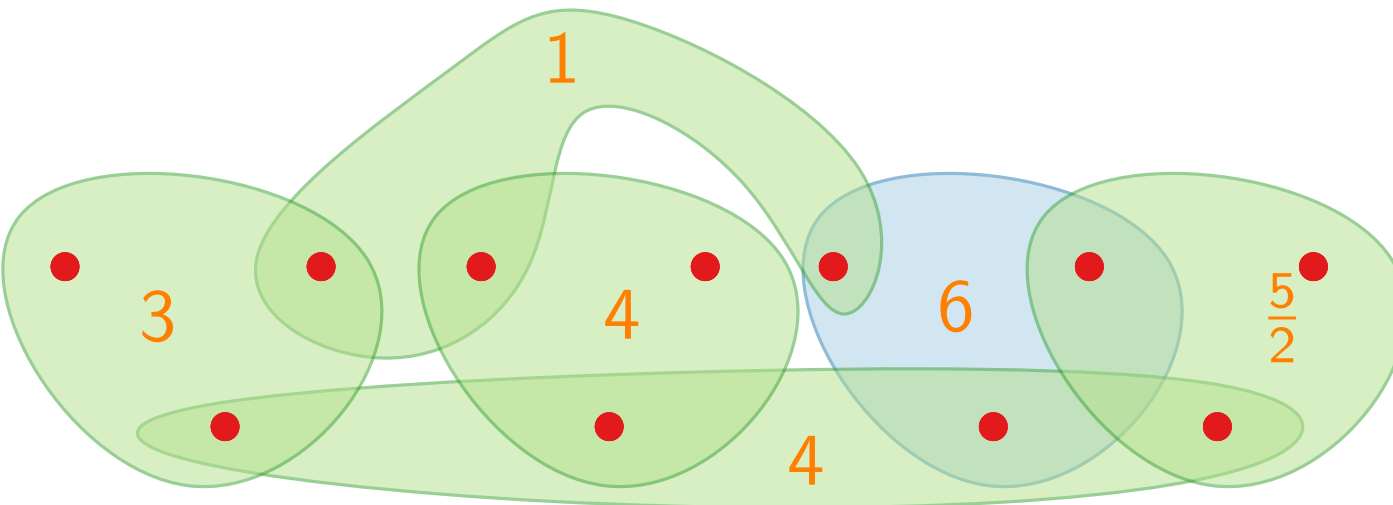
$$x_S$$

$$\forall S \in \mathcal{S}$$

Ground set U

Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$

Costs $c: \mathcal{S} \rightarrow \mathbb{Q}^+$



Find cover $\mathcal{S}' \subseteq \mathcal{S}$
of U with
minimum cost.

SETCOVER as an ILP

minimize

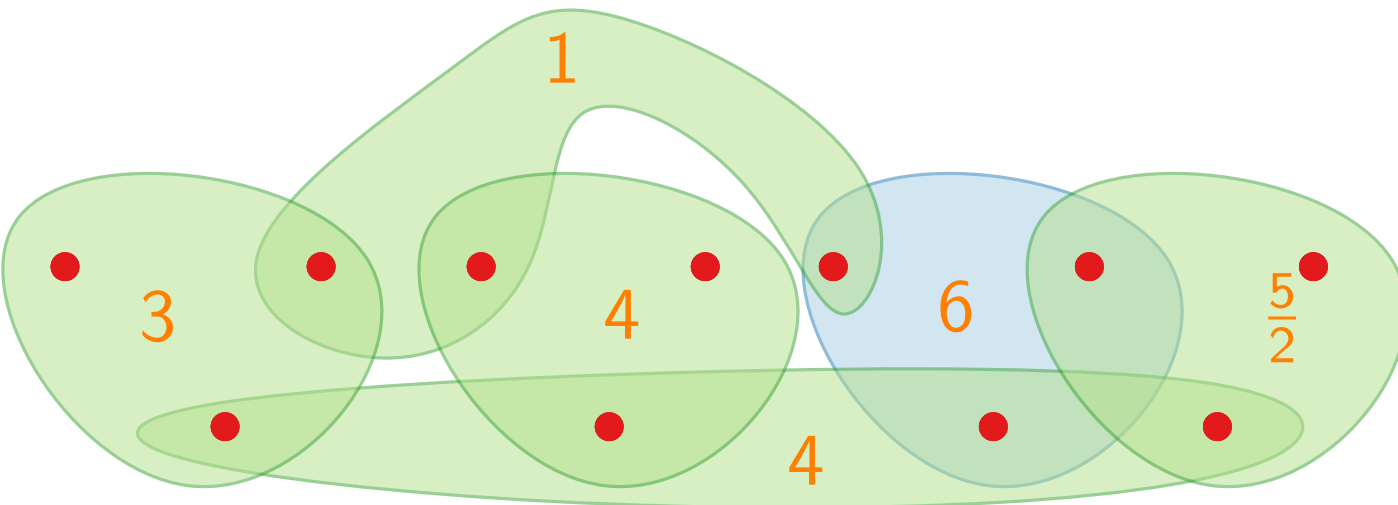
subject to

$$x_S \in \{0, 1\} \quad \forall S \in \mathcal{S}$$

Ground set U

Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$

Costs $c: \mathcal{S} \rightarrow \mathbb{Q}^+$



Find cover $\mathcal{S}' \subseteq \mathcal{S}$
of U with
minimum cost.

SETCOVER as an ILP

$$\text{minimize} \quad \sum_{S \in \mathcal{S}} c_S x_S$$

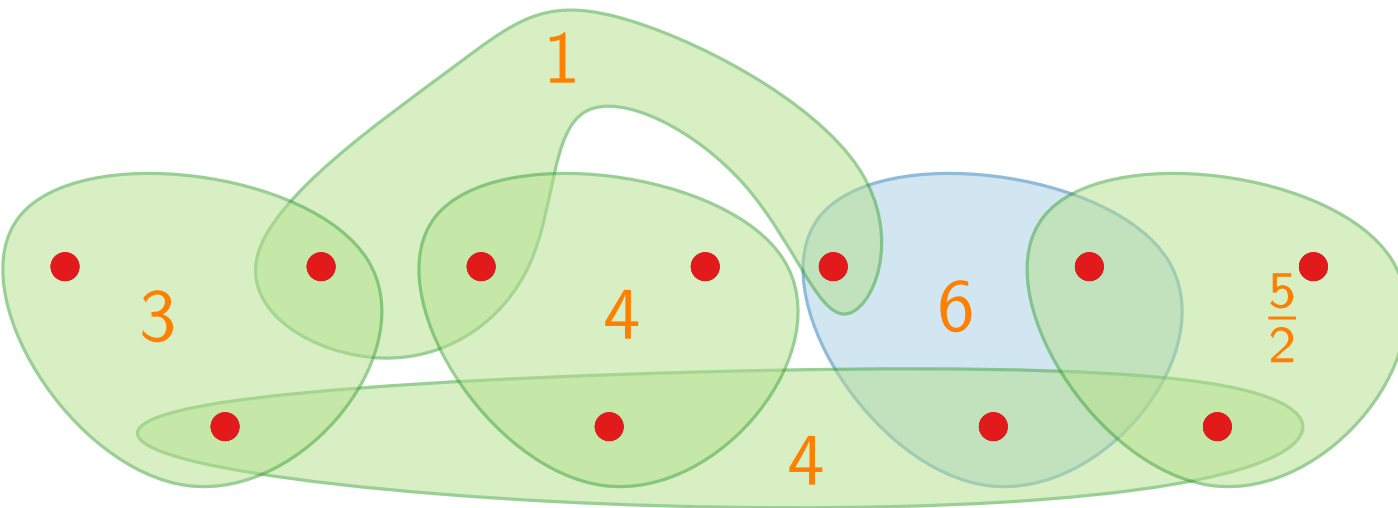
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Ground set U

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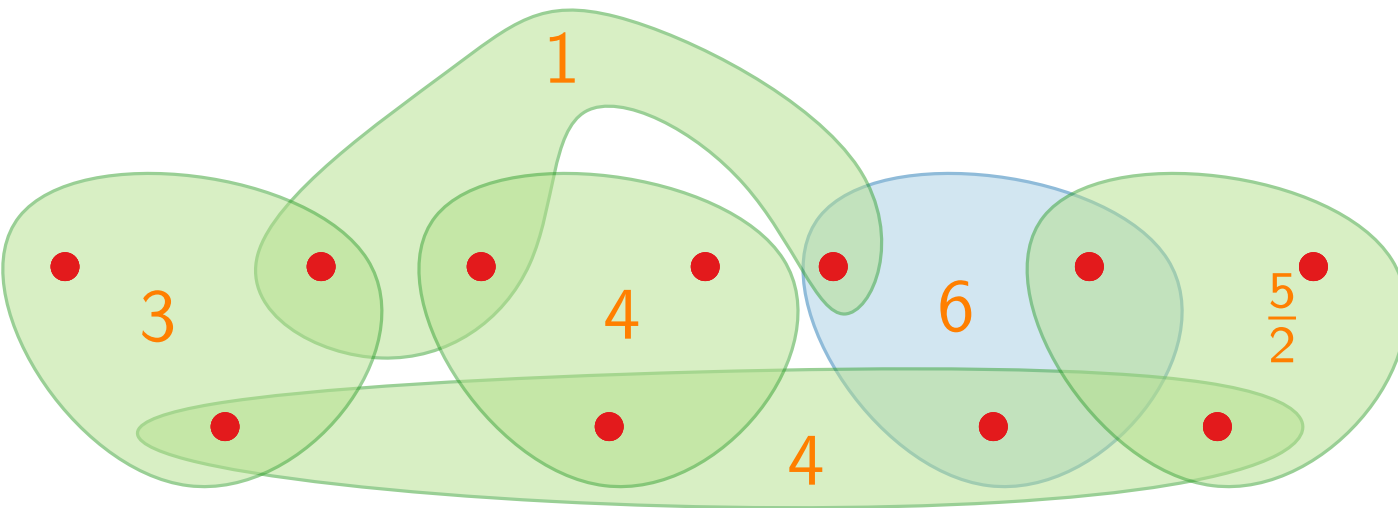
SETCOVER as an ILP

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Ground set U

Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$

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Find cover $\mathcal{S}' \subseteq \mathcal{S}$
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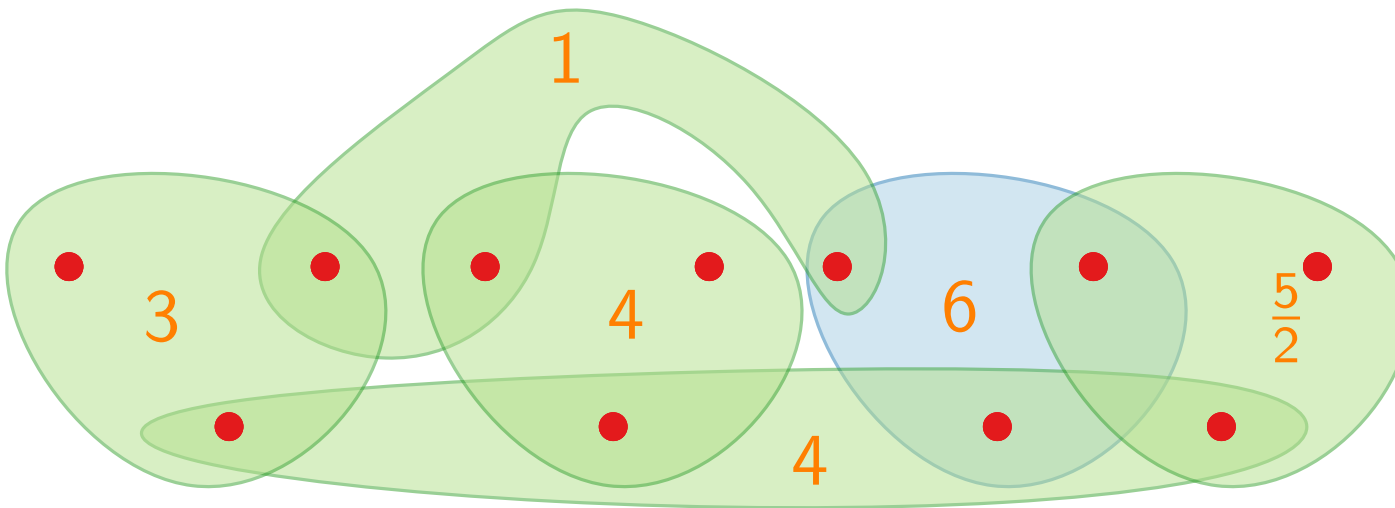
SETCOVER as an ILP

$$\begin{array}{ll}\text{minimize} & \sum_{S \in \mathcal{S}} c_S x_S \\ \text{subject to} & \sum_{S \ni u} x_S \geq 1 \quad \forall u \in U \\ & x_S \in \{0, 1\} \quad \forall S \in \mathcal{S}\end{array}$$

Ground set U

Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$

Costs $c: \mathcal{S} \rightarrow \mathbb{Q}^+$



Find cover $\mathcal{S}' \subseteq \mathcal{S}$
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Approximation Algorithms

Lecture 5: LP-based Approximation Algorithms for SETCOVER

Part II: LP-Rounding

Technique I) LP-Rounding



Consider a minimization problem Π in ILP form.

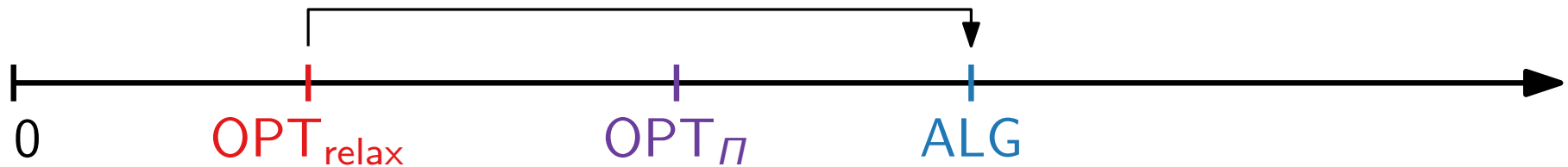
Technique I) LP-Rounding



Consider a minimization problem Π in ILP form.

Compute a solution for the LP-relaxation.

Technique I) LP-Rounding



Consider a minimization problem Π in ILP form.

Compute a solution for the **LP-relaxation**.

Round to obtain an **integer solution** for Π .

Technique I) LP-Rounding



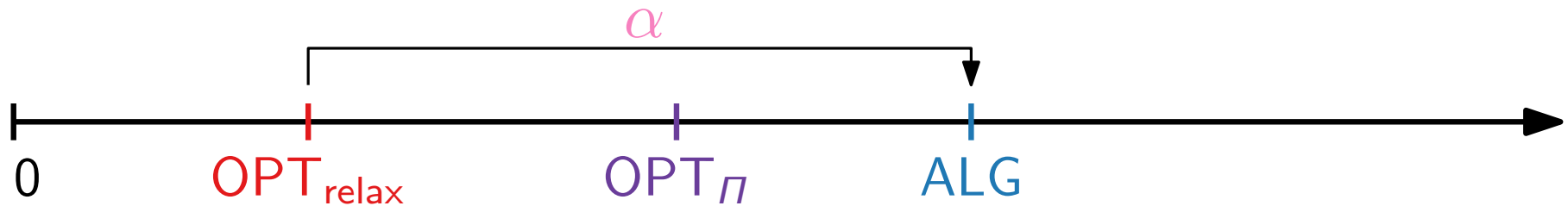
Consider a minimization problem Π in ILP form.

Compute a solution for the **LP-relaxation**.

Round to obtain an **integer solution** for Π .

Difficulty: Ensure the **feasibility** of the solution.

Technique I) LP-Rounding



Consider a minimization problem Π in ILP form.

Compute a solution for the **LP-relaxation**.

Round to obtain an **integer solution** for Π .

Difficulty: Ensure the **feasibility** of the solution.

Approximation factor: $ALG / OPT_{\Pi} \leq ALG / OPT_{relax}$.

SETCOVER – LP-Relaxation

$$\begin{array}{ll}\text{minimize} & \sum_{S \in \mathcal{S}} c_S x_S \\ \text{subject to} & \sum_{S \ni u} x_S \geq 1 \quad \forall u \in U \\ & x_S \geq 0 \quad \forall S \in \mathcal{S}\end{array}$$

SETCOVER – LP-Relaxation

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Optimal?

SETCOVER – LP-Relaxation

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Optimal?



SETCOVER – LP-Relaxation

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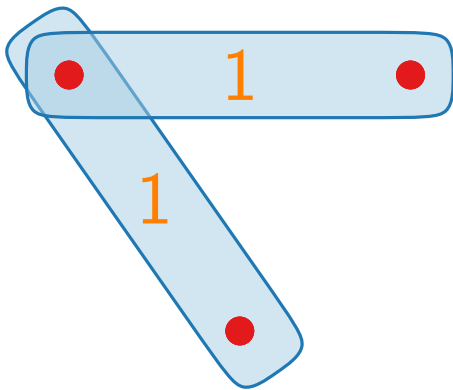
Optimal?



SETCOVER – LP-Relaxation

$$\begin{array}{ll}\text{minimize} & \sum_{S \in \mathcal{S}} c_S x_S \\ \text{subject to} & \sum_{S \ni u} x_S \geq 1 \quad \forall u \in U \\ & x_S \geq 0 \quad \forall S \in \mathcal{S}\end{array}$$

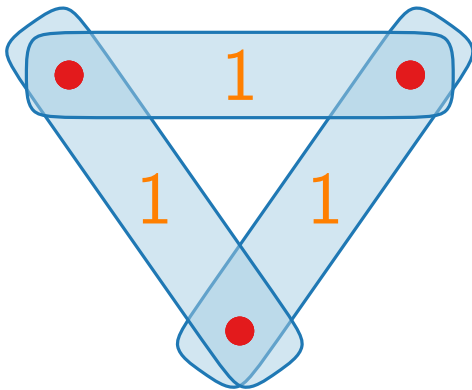
Optimal?



SETCOVER – LP-Relaxation

$$\begin{array}{ll}\text{minimize} & \sum_{S \in \mathcal{S}} c_S x_S \\ \text{subject to} & \sum_{S \ni u} x_S \geq 1 \quad \forall u \in U \\ & x_S \geq 0 \quad \forall S \in \mathcal{S}\end{array}$$

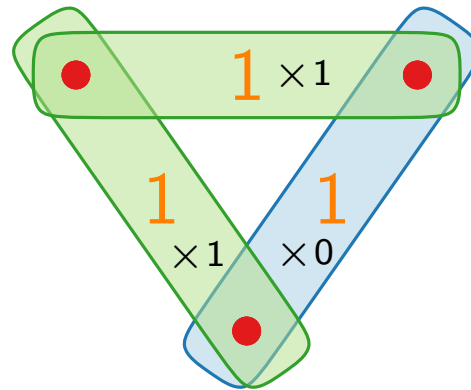
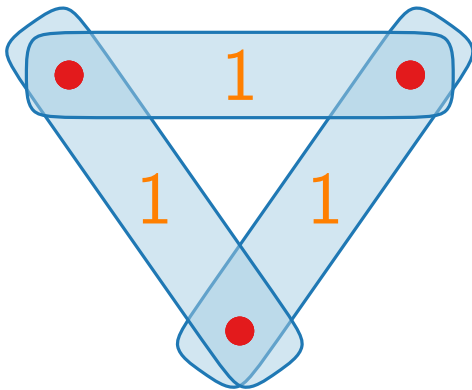
Optimal?



SETCOVER – LP-Relaxation

$$\begin{array}{ll}\text{minimize} & \sum_{S \in \mathcal{S}} c_S x_S \\ \text{subject to} & \sum_{S \ni u} x_S \geq 1 \quad \forall u \in U \\ & x_S \geq 0 \quad \forall S \in \mathcal{S}\end{array}$$

Optimal?

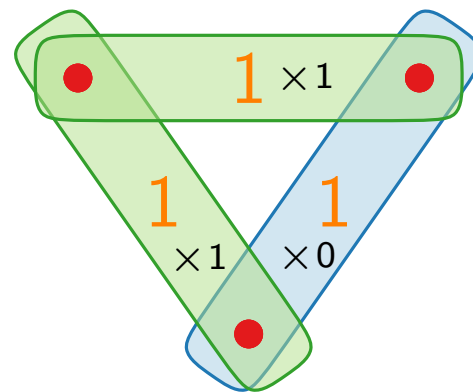
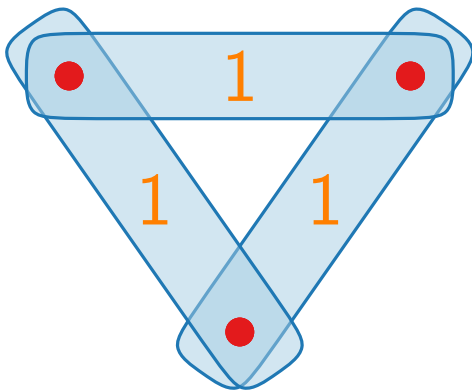


integer: 2

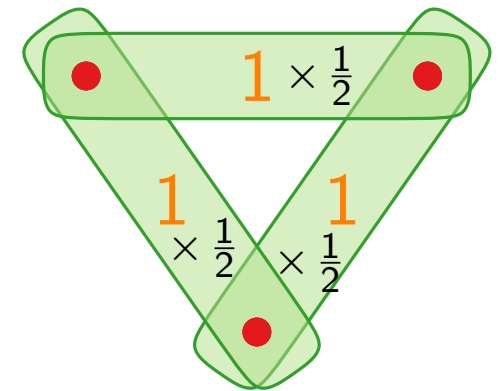
SETCOVER – LP-Relaxation

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Optimal?



integer: 2



fractional: $\frac{3}{2}$

LP-Rounding: Approach I

$$\begin{array}{ll}\text{minimize} & \sum_{S \in \mathcal{S}} c_S x_S \\ \text{subject to} & \sum_{S \ni u} x_S \geq 1 \quad \forall u \in U \\ & x_S \geq 0 \quad \forall S \in \mathcal{S}\end{array}$$

LP-Rounding-One(U, \mathcal{S}, c)

LP-Rounding: Approach I

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LP-Rounding-One(U, \mathcal{S}, c)

Compute optimal solution x for the LP-relaxation.

Round each x_S with $x_S > 0$ to 1.

LP-Rounding: Approach I

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LP-Rounding-One(U, \mathcal{S}, c)

Compute optimal solution x for the LP-relaxation.

Round each x_S with $x_S > 0$ to 1.

- Generates a feasible solution.
- Scaling factor arbitrarily large.

LP-Rounding: Approach I

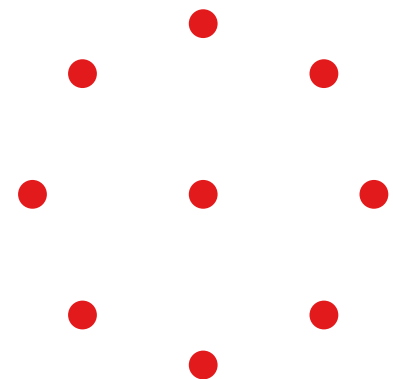
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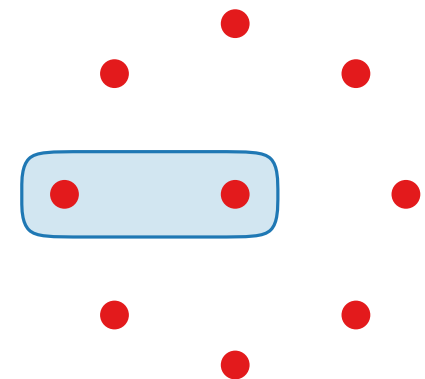
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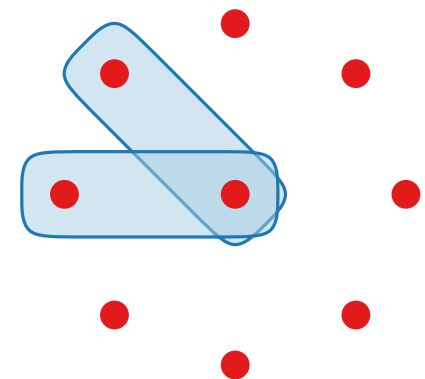
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LP-Rounding: Approach I

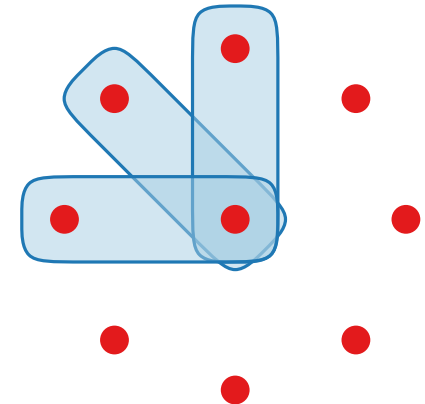
$$\begin{aligned} &\text{minimize} && \sum_{S \in \mathcal{S}} c_S x_S \\ &\text{subject to} && \sum_{S \ni u} x_S \geq 1 \quad \forall u \in U \\ &&& x_S \geq 0 \quad \forall S \in \mathcal{S} \end{aligned}$$

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LP-Rounding: Approach I

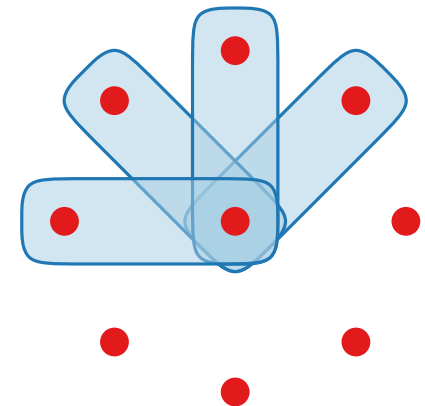
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LP-Rounding: Approach I

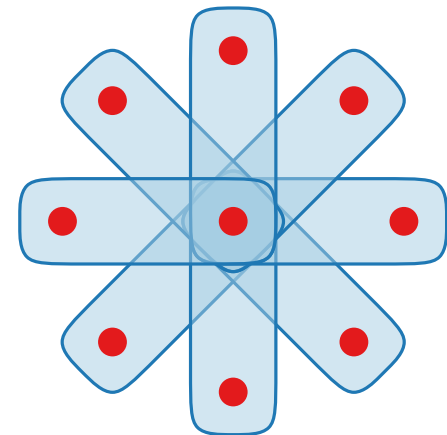
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LP-Rounding: Approach I

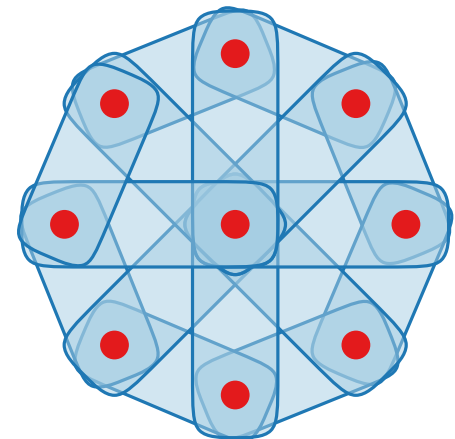
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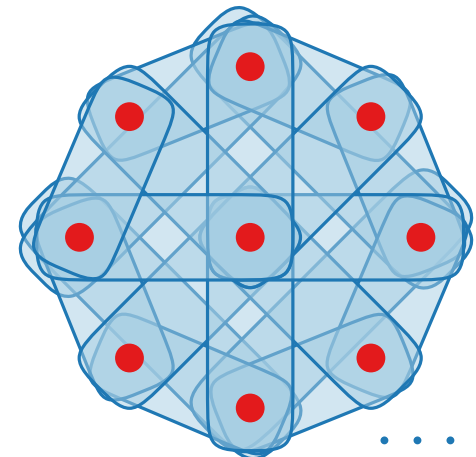
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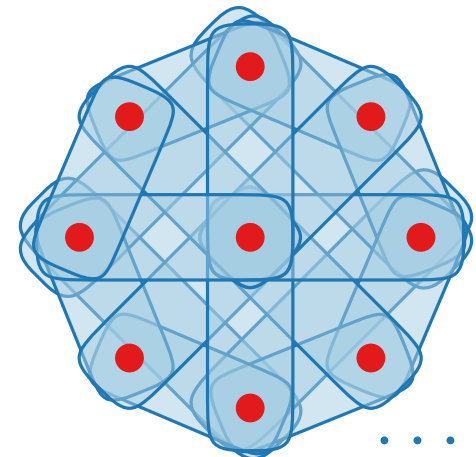
LP-Rounding-One(U, \mathcal{S}, c)

Compute optimal solution x for the LP-relaxation.

Round each x_S with $x_S > 0$ to 1.

- Generates a feasible solution.
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Use frequency f



LP-Rounding: Approach II

$$\begin{array}{ll}\text{minimize} & \sum_{S \in \mathcal{S}} c_S x_S \\ \text{subject to} & \sum_{S \ni u} x_S \geq 1 \quad \forall u \in U \\ & x_S \geq 0 \quad \forall S \in \mathcal{S}\end{array}$$

LP-Rounding-Two(U, \mathcal{S}, c)

Compute optimal solution x for LP-relaxation.

Round each x_S with $x_S \geq \frac{1}{f}$ to 1; remaining to 0.

Let f be the frequency of (i.e., the number of sets containing) the most frequent element.

LP-Rounding: Approach II

$$\begin{array}{ll}\text{minimize} & \sum_{S \in \mathcal{S}} c_S x_S \\ \text{subject to} & \sum_{S \ni u} x_S \geq 1 \quad \forall u \in U \\ & x_S \geq 0 \quad \forall S \in \mathcal{S}\end{array}$$

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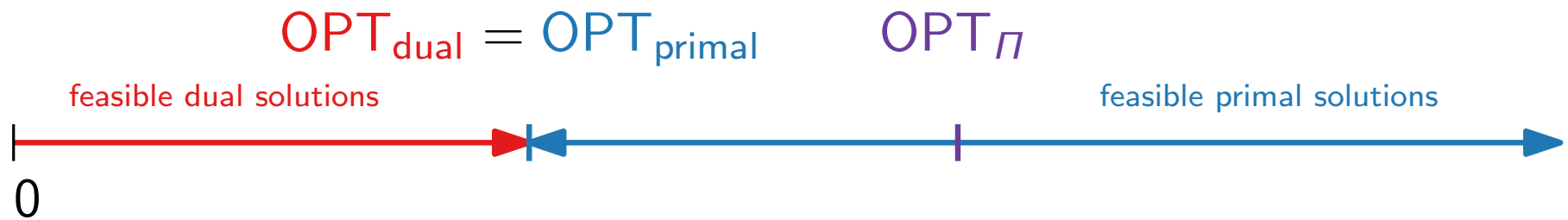
Theorem. LP-Rounding-Two is a factor- f approximation algorithm for SETCOVER.

Approximation Algorithms

Lecture 5: LP-based Approximation Algorithms for SETCOVER

Part III: The Primal-Dual Schema

Technique II) Primal–Dual Approach



Consider a minimization problem Π in ILP form.

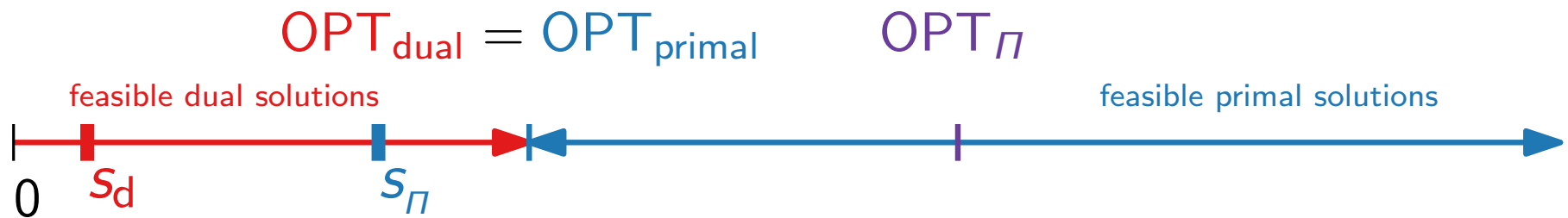
Technique II) Primal–Dual Approach



Consider a minimization problem Π in ILP form.

- Start with (trivial) **feasible dual solution** and **infeasible primal solution** (e.g., all variables = 0).

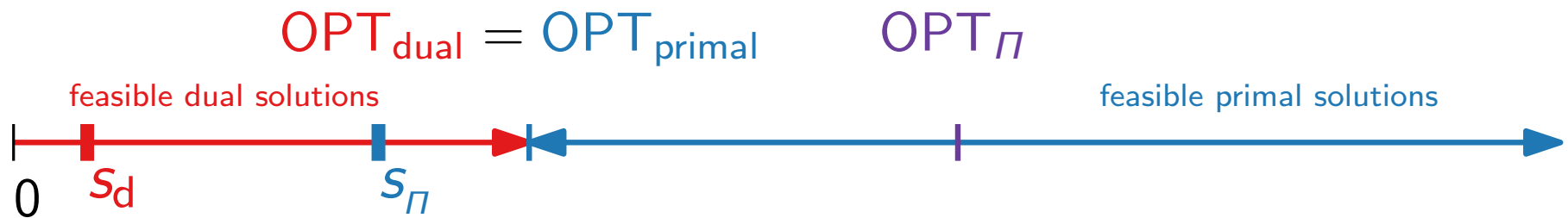
Technique II) Primal–Dual Approach



Consider a minimization problem π in ILP form.

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Technique II) Primal–Dual Approach



Consider a minimization problem π in ILP form.

- Start with (trivial) **feasible dual solution** and **infeasible primal solution** (e.g., all variables = 0).
- Compute **dual** solution s_d and **integral primal** solution s_π for π iteratively:
Increase s_d according to CS and make s_π “more feasible”.

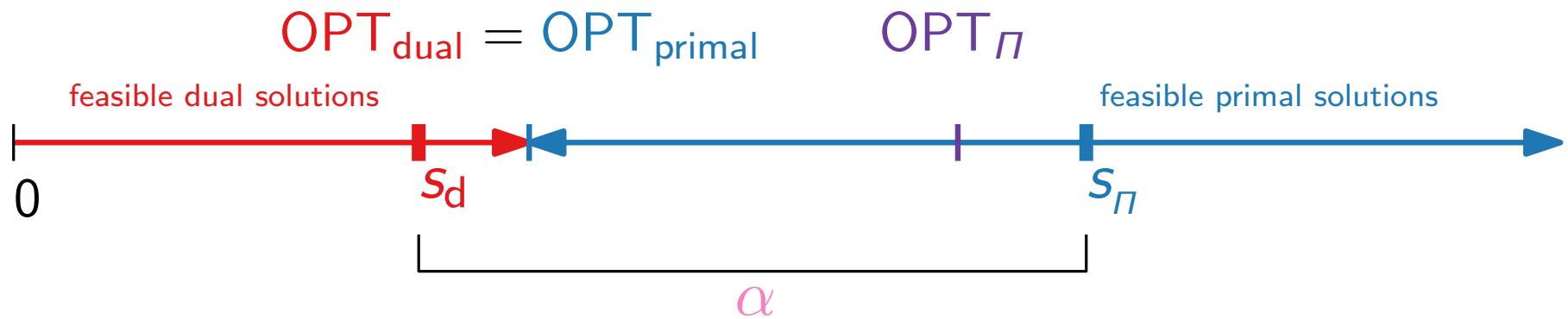
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Technique II) Primal–Dual Approach

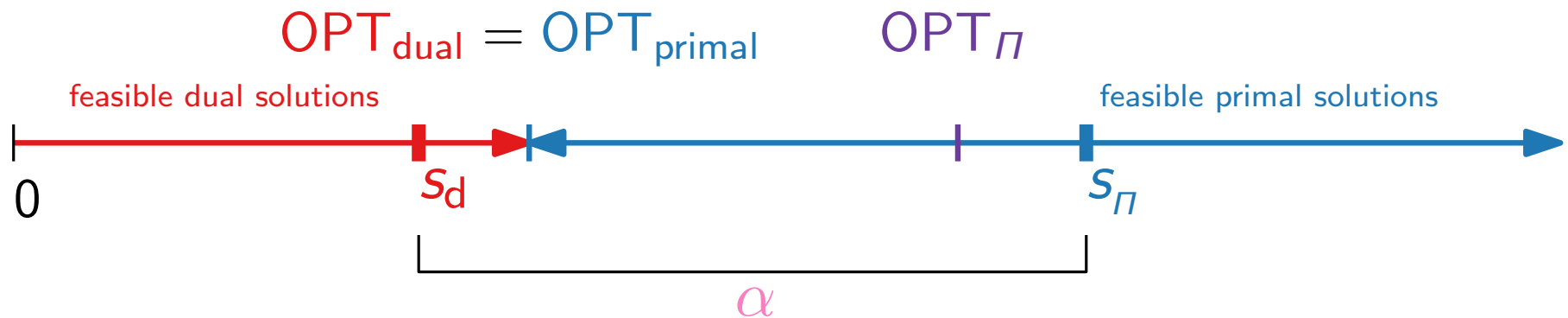


Consider a minimization problem Π in ILP form.

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Increase s_d according to CS and make s_Π “more feasible”.

Approximation factor $\leq \text{obj}(s_\Pi) / \text{obj}(s_d)$

Technique II) Primal–Dual Approach



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- Start with (trivial) **feasible dual solution** and **infeasible primal solution** (e.g., all variables = 0).
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Increase s_d according to CS and make s_Π “more feasible”.

Approximation factor $\leq \text{obj}(s_\Pi) / \text{obj}(s_d)$

Advantage: Don't need LP-“machinery”; possibly faster, more flexible.

SETCOVER – Dual LP

$$\begin{array}{ll}\text{minimize} & \sum_{S \in \mathcal{S}} c_S x_S \\ \text{subject to} & \sum_{S \ni u} x_S \geq 1 \quad \forall u \in U \\ & x_S \geq 0 \quad \forall S \in \mathcal{S}\end{array}$$

SETCOVER – Dual LP

$$\begin{array}{ll}\text{minimize} & \sum_{S \in \mathcal{S}} c_S x_S \\ \text{subject to} & \sum_{S \ni u} x_S \geq 1 \quad \forall u \in U \\ & x_S \geq 0 \quad \forall S \in \mathcal{S}\end{array}$$

maximize

subject to

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maximize

subject to

$$y_u \geq 0 \quad \forall u \in U$$

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SETCOVER – Dual LP

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$$\begin{array}{ll}\text{maximize} & \sum_{u \in U} y_u \\ \text{subject to} & \sum_{u \in S} y_u \leq c_S \quad \forall S \in \mathcal{S} \\ & y_u \geq 0 \quad \forall u \in U\end{array}$$

Complementary Slackness

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \\ & y \geq 0 \end{array}$$

Theorem. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ be valid solutions for the primal and dual program, respectively.

Then x and y are optimal \Leftrightarrow following conditions are met:

Primal CS:

For each $j = 1, \dots, n$: $x_j = 0$ or $\sum_{i=1}^m a_{ij} y_i = c_j$

Dual CS:

For each $i = 1, \dots, m$: $y_i = 0$ or $\sum_{j=1}^n a_{ij} x_j = b_i$

Relaxing Complementary Slackness

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

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$$\Leftrightarrow \sum_{j=1}^n c_j x_j = \sum_{i=1}^m b_i y_i$$

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~~Primal CS:~~ Relaxed Primal CS

For each $j = 1, \dots, n$: $x_j = 0$ or ~~$\sum_{i=1}^m a_{ij} y_i = c_j$~~
 $c_j / \alpha \leq \sum_{i=1}^m a_{ij} y_i \leq c_j$

Dual CS:

For each $i = 1, \dots, m$: $y_i = 0$ or $\sum_{j=1}^n a_{ij} x_j = b_i$

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~~Dual CS:~~ Relaxed Dual CS

For each $i = 1, \dots, m$: $y_i = 0$ or $\sum_{j=1}^n a_{ij} x_j = b_i$
 $b_i \leq \sum_{j=1}^n a_{ij} x_j \leq \beta \cdot b_i$

$$\Leftrightarrow \sum_{j=1}^n c_j x_j = \sum_{i=1}^m b_i y_i$$

Relaxing Complementary Slackness

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

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 $b_i \leq \sum_{j=1}^n a_{ij} x_j \leq \beta \cdot b_i$

$$\Leftrightarrow \sum_{j=1}^n c_j x_j = \sum_{i=1}^m b_i y_i \Rightarrow \sum_{j=1}^n c_j x_j \leq \alpha \beta \sum_{i=1}^m b_i y_i \leq \alpha \beta \cdot \text{OPT}_{\text{LP}}$$

Primal–Dual Schema

Start with a feasible **dual** and infeasible **primal** solution (often trivial).

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Do so until the relaxed CS conditions are met.

Maintain that the **primal** solution is integer-valued.

The feasibility of the **primal** solution and the relaxed CS conditions provide an approximation ratio.

Relaxed CS for SETCOVER

$$\begin{array}{ll}\text{minimize} & \sum_{S \in \mathcal{S}} c_S x_S \\ \text{subject to} & \sum_{S \ni u} x_S \geq 1 \quad \forall u \in U \\ & x_S \geq 0 \quad \forall S \in \mathcal{S}\end{array}$$

$$\begin{array}{ll}\text{maximize} & \sum_{u \in U} y_u \\ \text{subject to} & \sum_{u \in S} y_u \leq c_S \quad \forall S \in \mathcal{S} \\ & y_u \geq 0 \quad \forall u \in U\end{array}$$

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(Unrelaxed) primal CS:

Relaxed CS for SETCOVER

$$\begin{array}{ll}\text{minimize} & \sum_{S \in \mathcal{S}} c_S x_S \\ \text{subject to} & \sum_{S \ni u} x_S \geq 1 \quad \forall u \in U \\ & x_S \geq 0 \quad \forall S \in \mathcal{S}\end{array}$$

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Relaxed CS for SETCOVER

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(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

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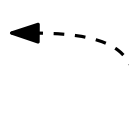
critical set 


Relaxed CS for SETCOVER

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critical set 

 only chooses critical sets

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(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

critical set \leftarrow

\rightarrow only chooses critical sets

Relaxed dual CS:

Relaxed CS for SETCOVER

$$\begin{array}{ll}\text{minimize} & \sum_{S \in \mathcal{S}} c_S x_S \\ \text{subject to} & \sum_{S \ni u} x_S \geq 1 \quad \forall u \in U \\ & x_S \geq 0 \quad \forall S \in \mathcal{S}\end{array}$$

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(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

critical set \leftarrow

\rightarrow only chooses critical sets

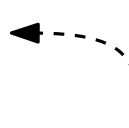
Relaxed dual CS: $y_u \neq 0 \Rightarrow$


Relaxed CS for SETCOVER

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Relaxed dual CS: $y_u \neq 0 \Rightarrow 1 \leq \sum_{S \ni u} x_S \leq f$

Relaxed CS for SETCOVER

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Relaxed dual CS: $y_u \neq 0 \Rightarrow 1 \leq \sum_{S \ni u} x_S \leq f \cdot 1$

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(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

critical set \leftarrow

\rightarrow only chooses critical sets

trivial for binary x \leftarrow

Relaxed dual CS: $y_u \neq 0 \Rightarrow 1 \leq \sum_{S \ni u} x_S \leq f \cdot 1$

Primal–Dual Schema for SETCOVER

PrimalDualSetCover(U, \mathcal{S}, c)

$x \leftarrow 0, y \leftarrow 0$

repeat

|

until all elements are covered.

return x

Primal–Dual Schema for SETCOVER

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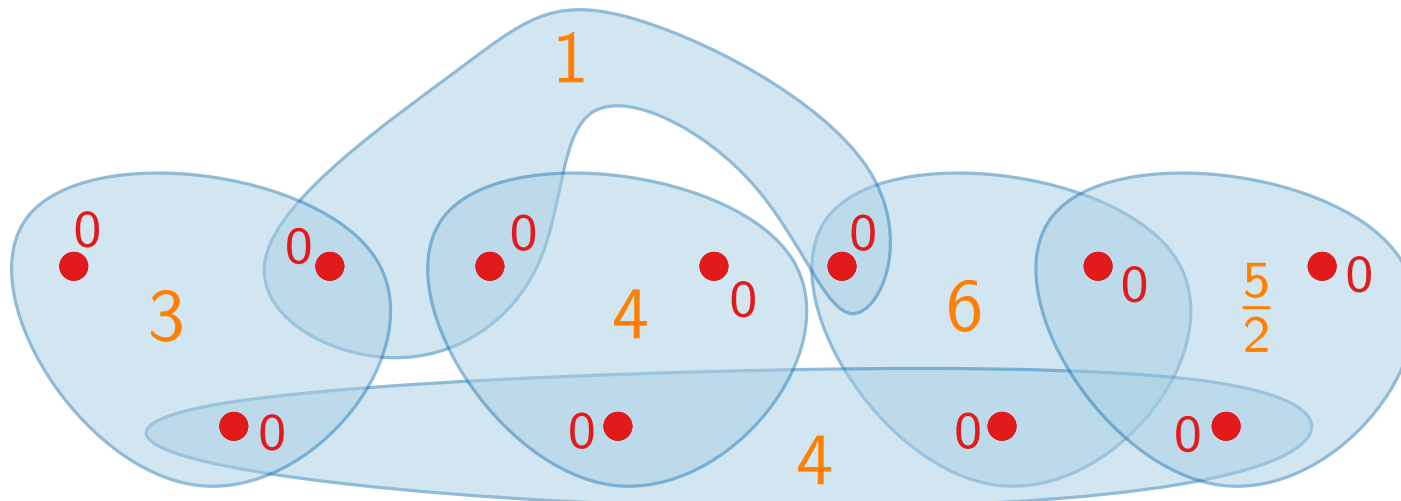
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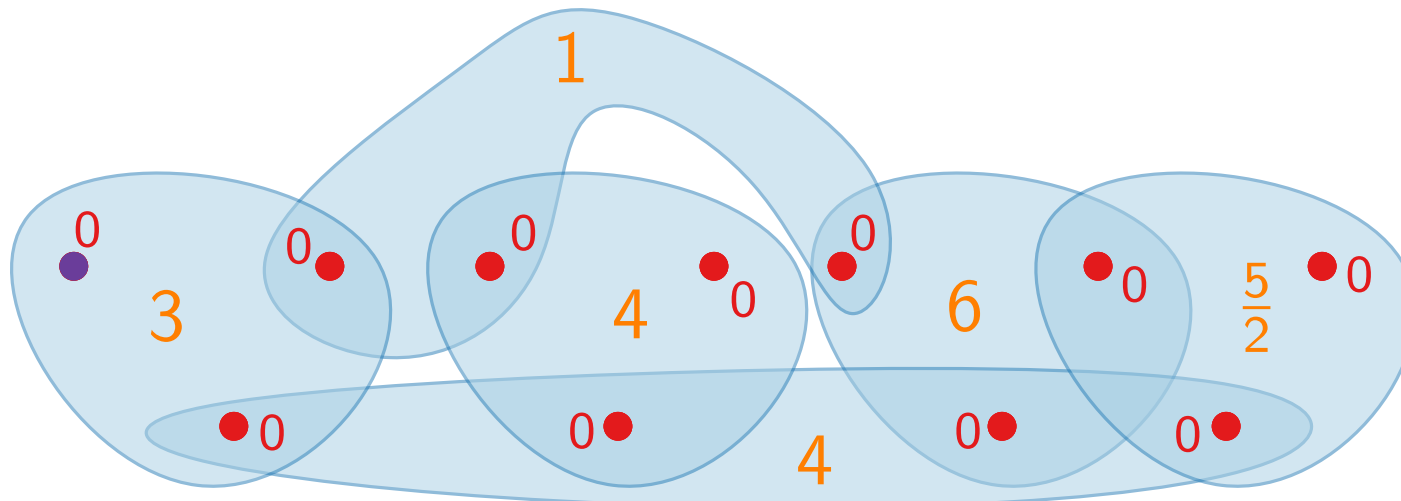
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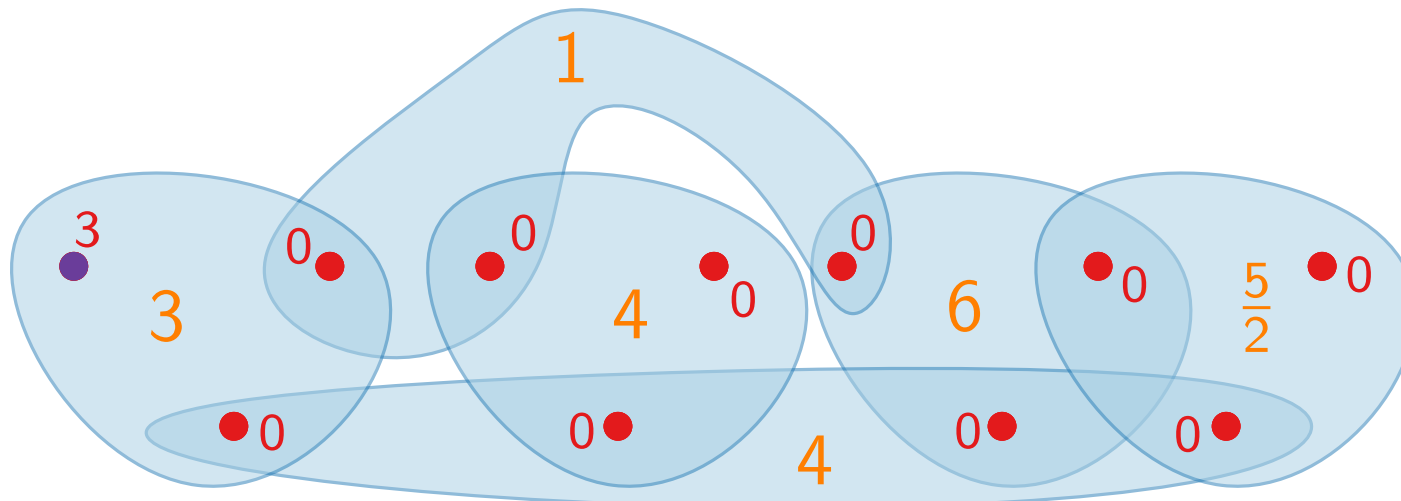
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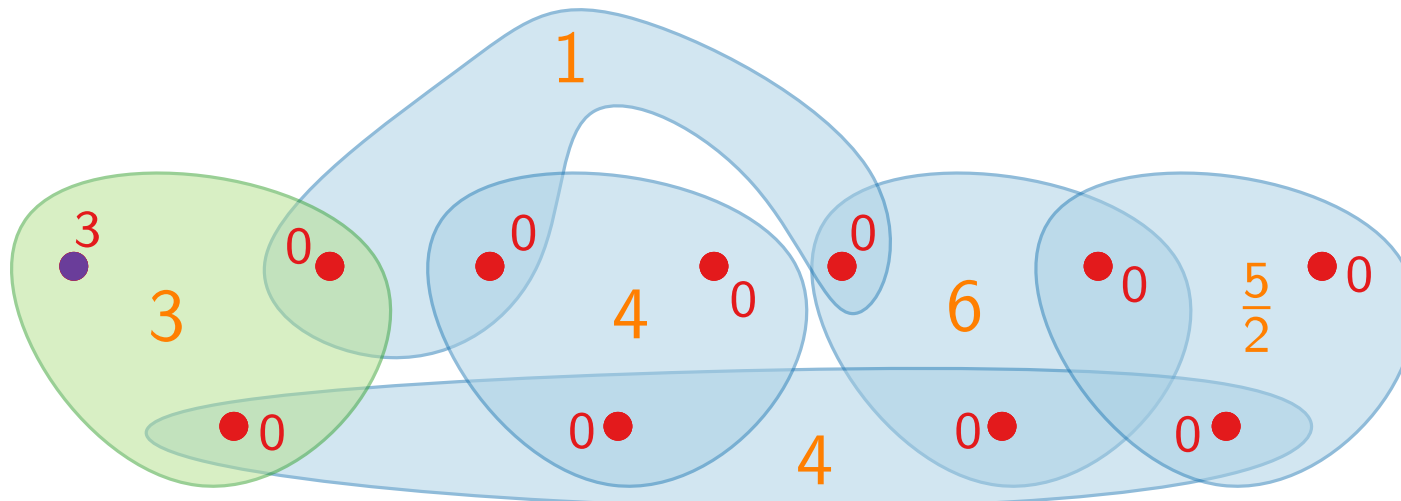
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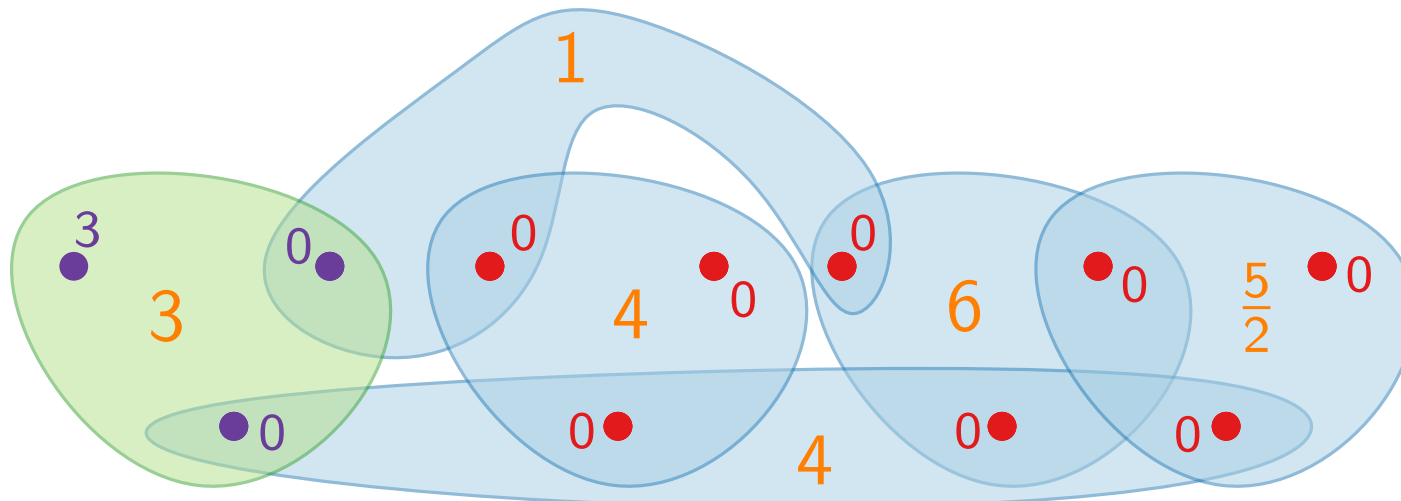
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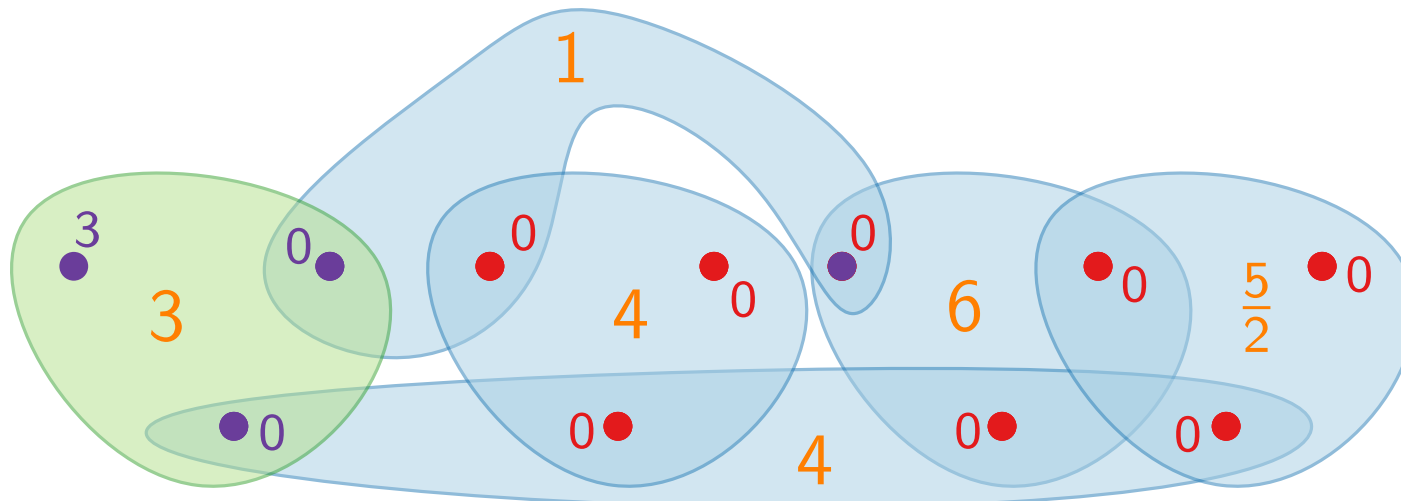
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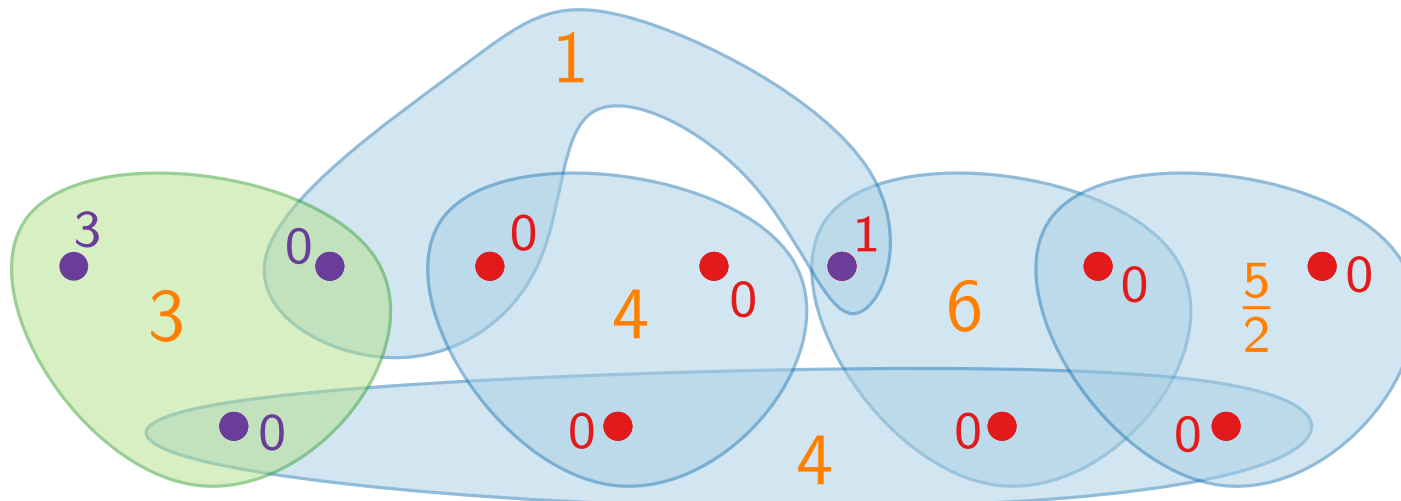
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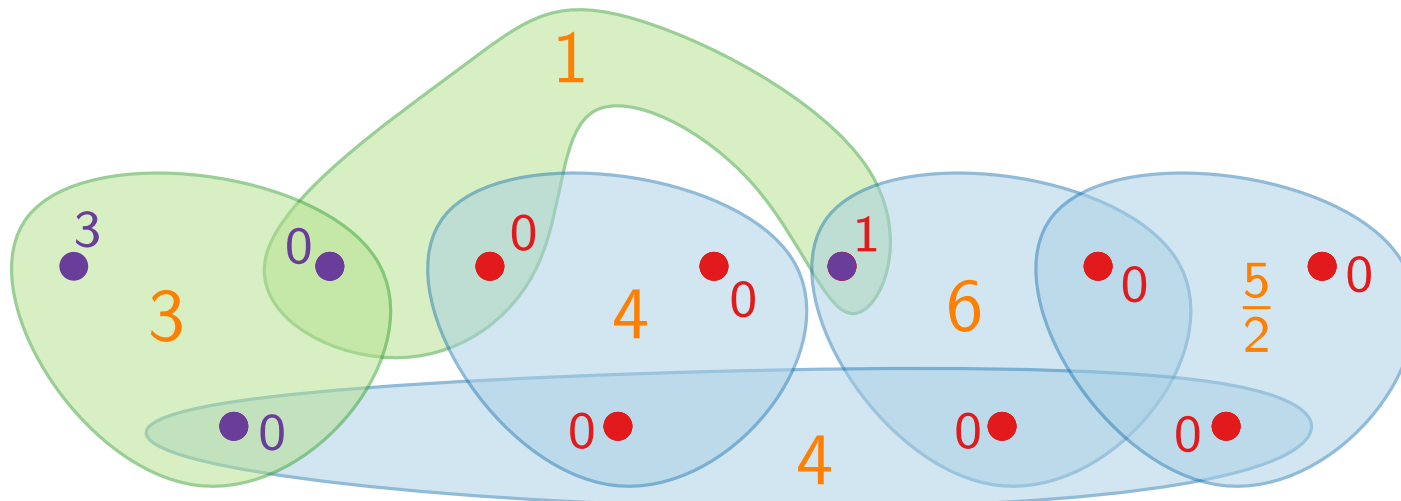
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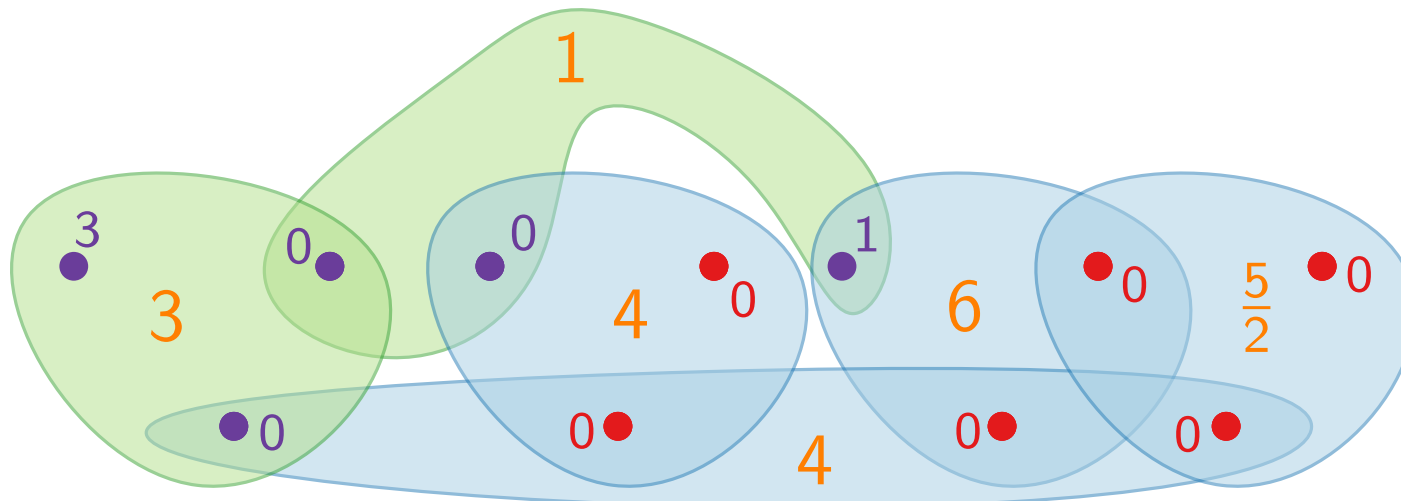
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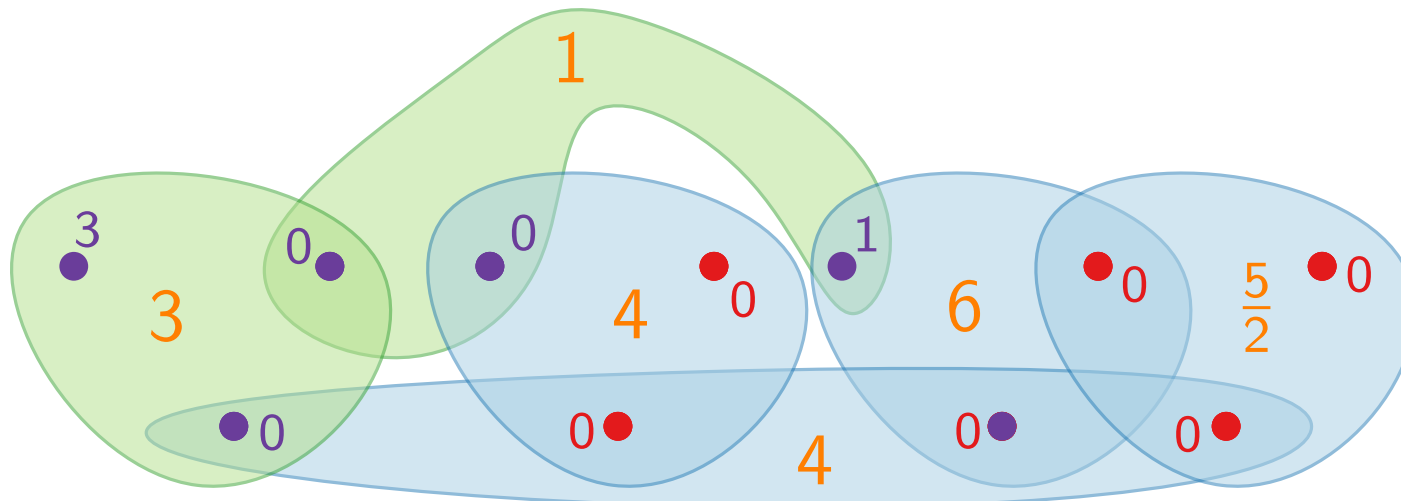
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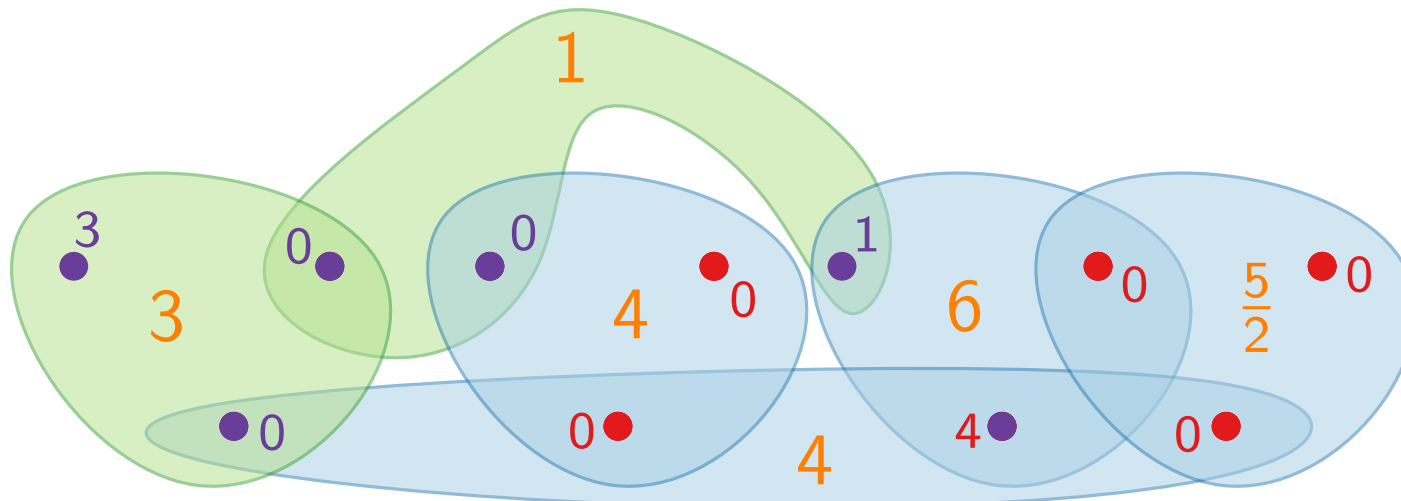
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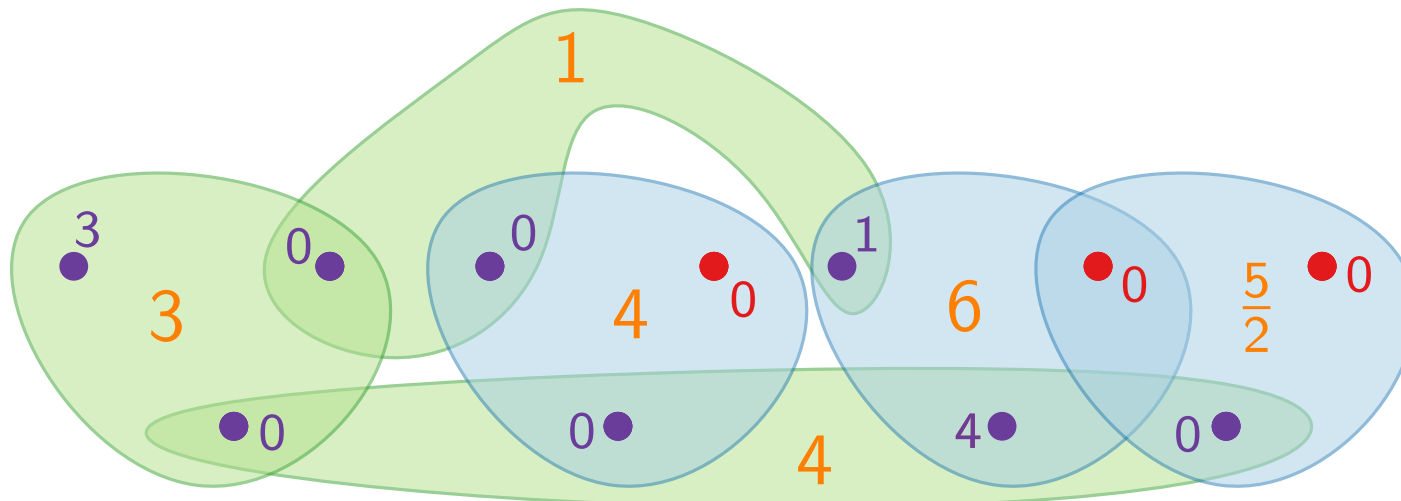
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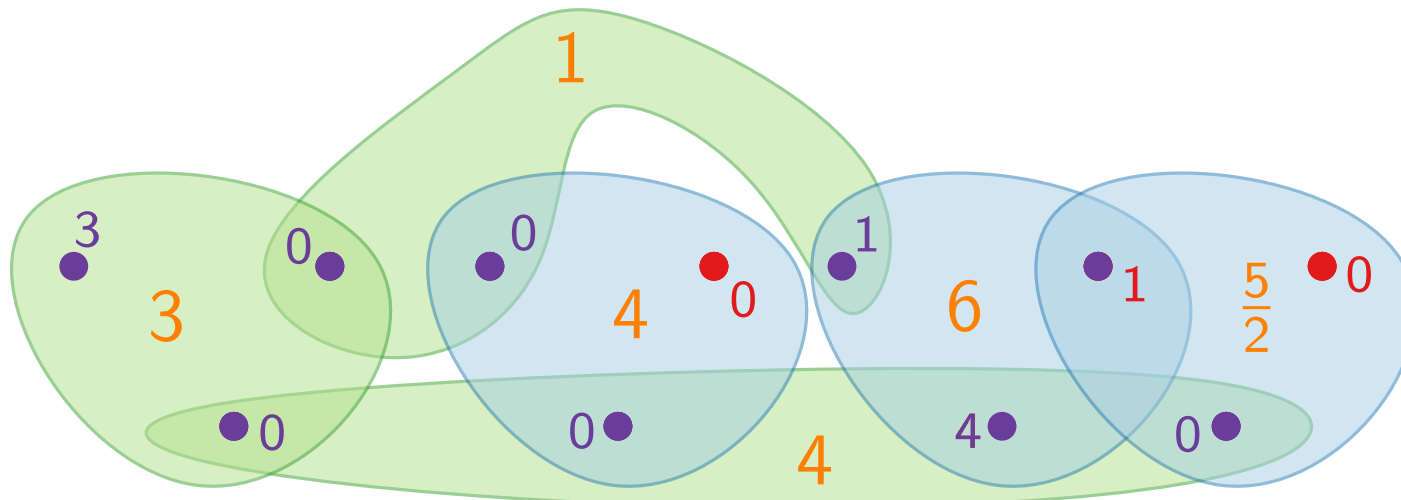
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Primal–Dual Schema for SETCOVER

PrimalDualSetCover(U, \mathcal{S}, c)

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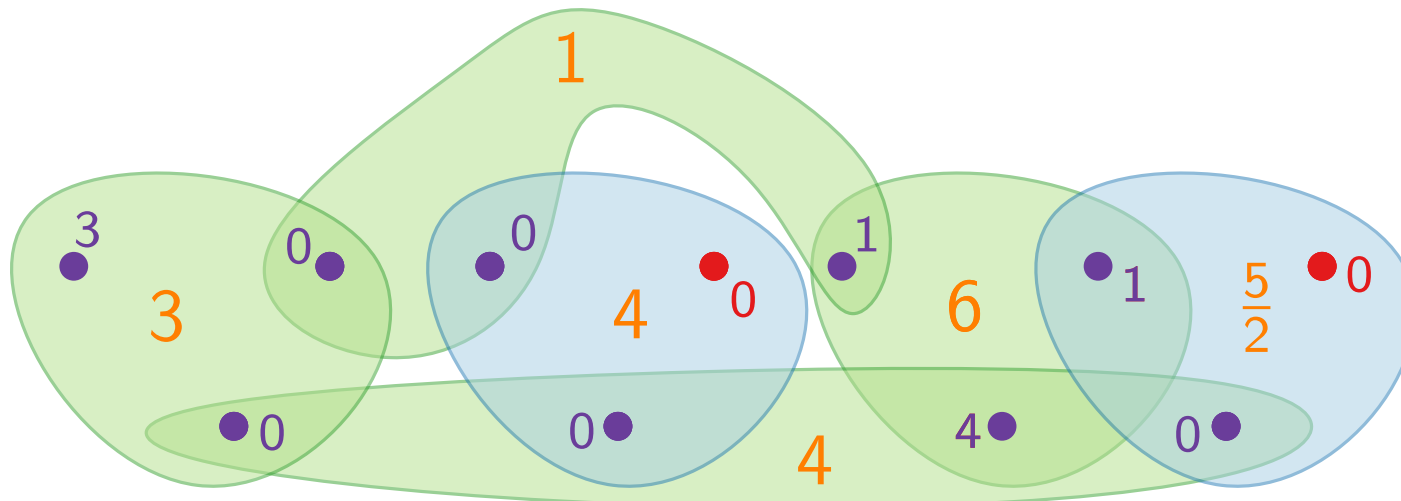
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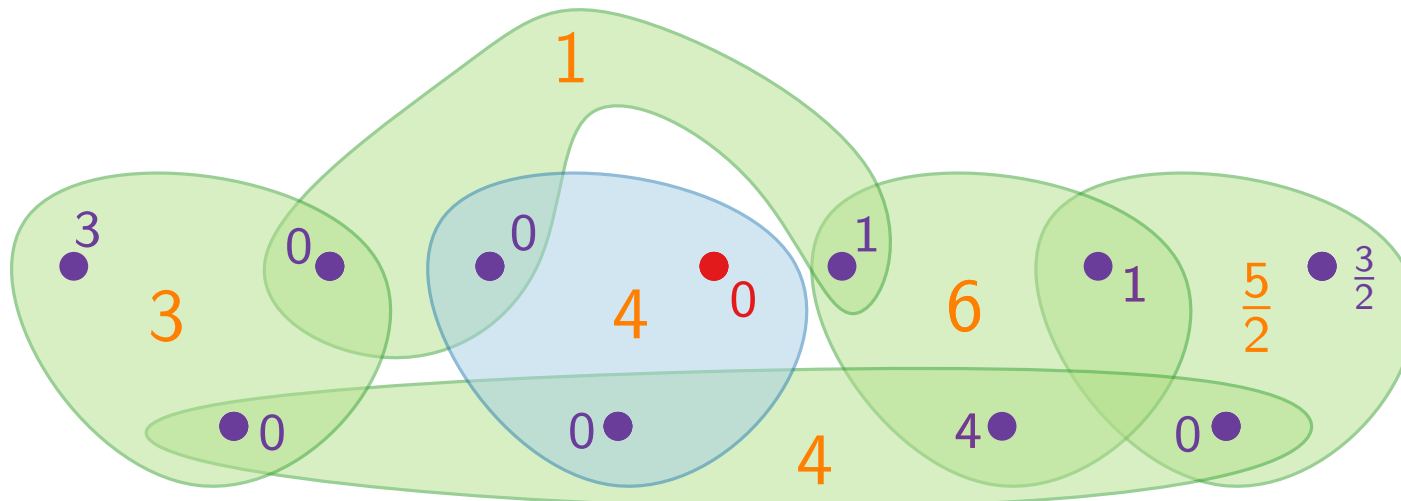
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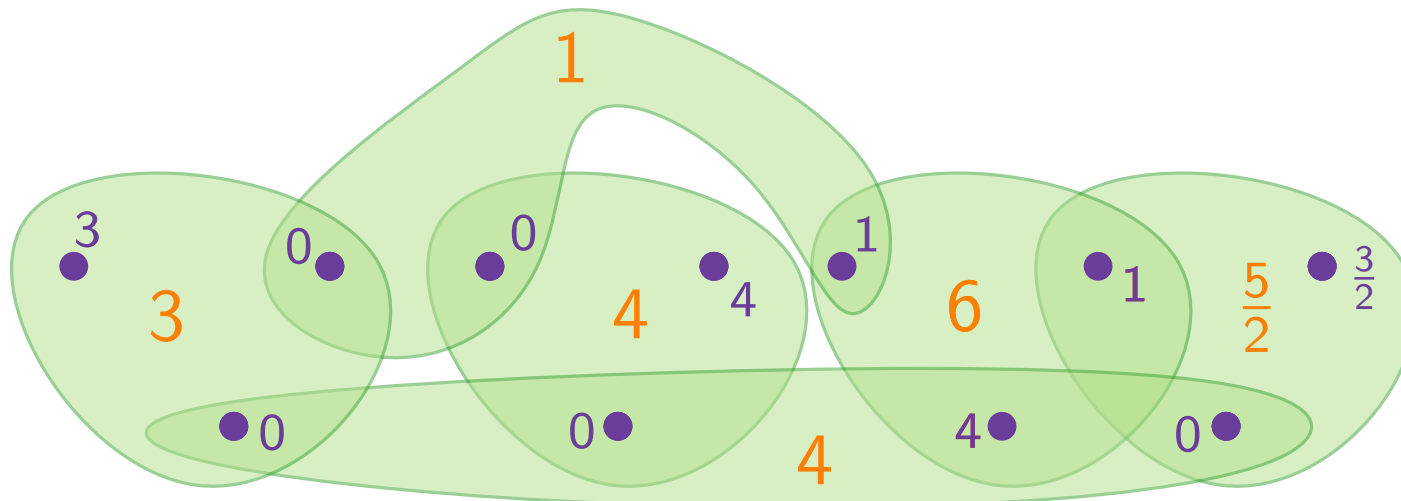
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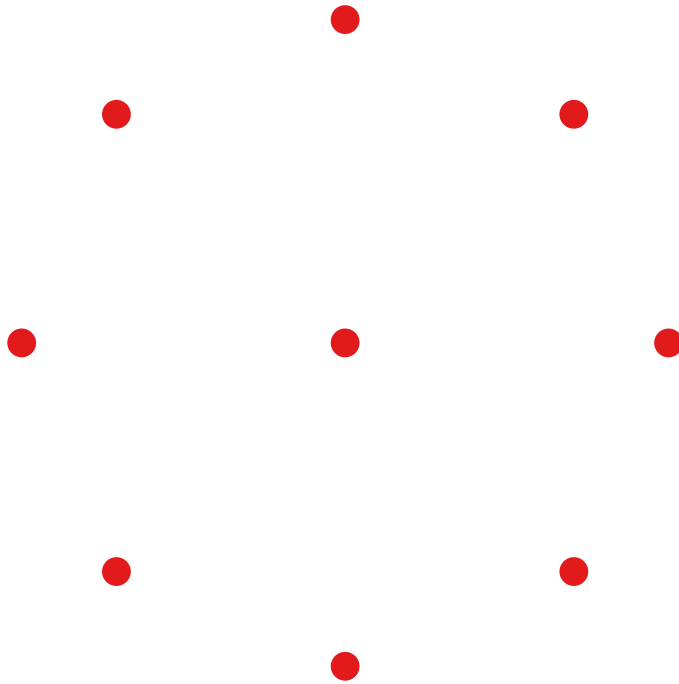
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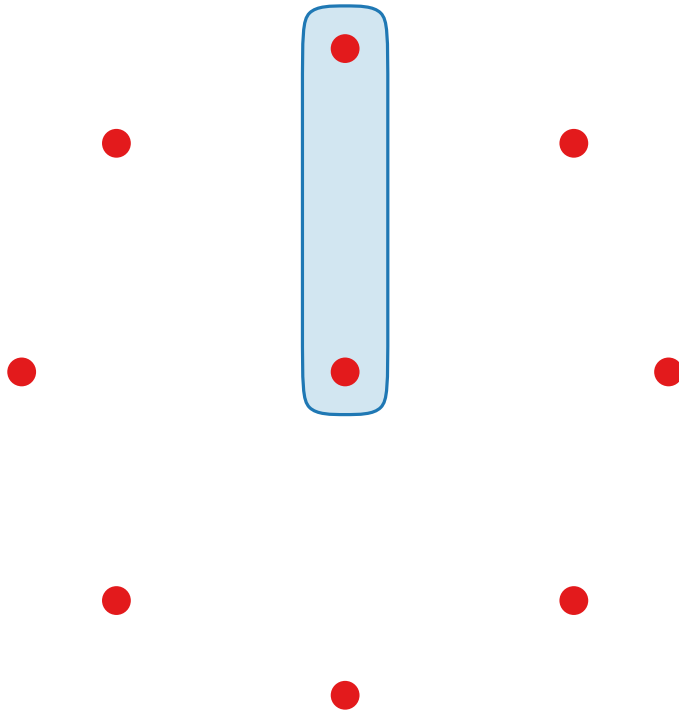


Theorem. PrimalDualSetCover is a factor- f approximation algorithm for SETCOVER. This bound is tight.

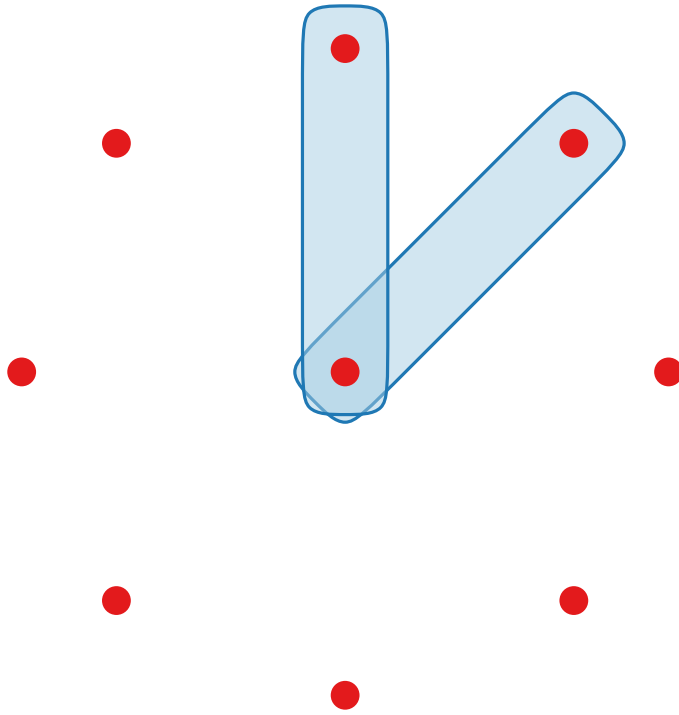
Tight Example



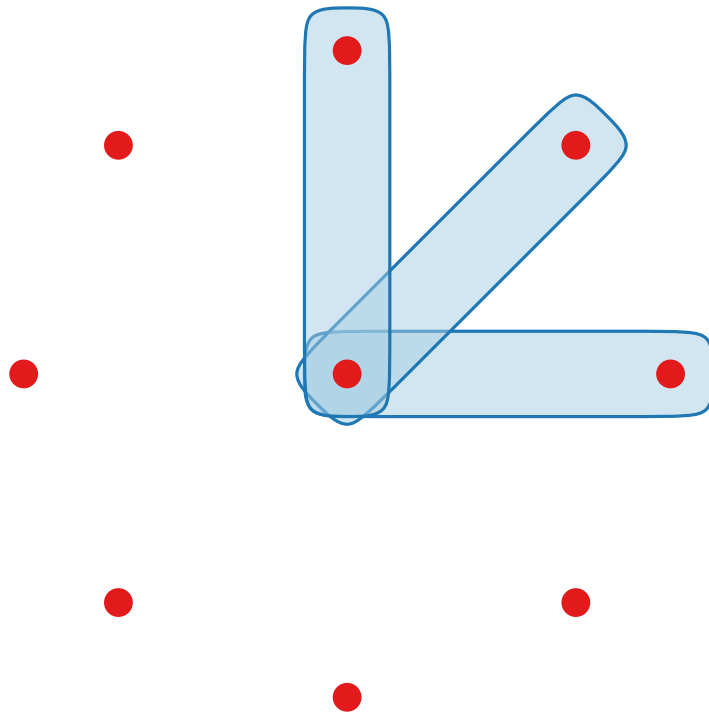
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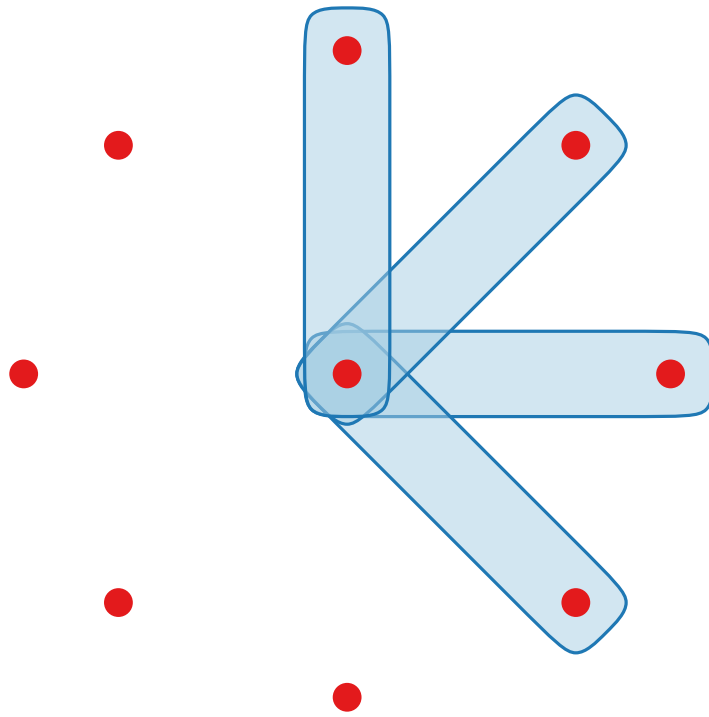
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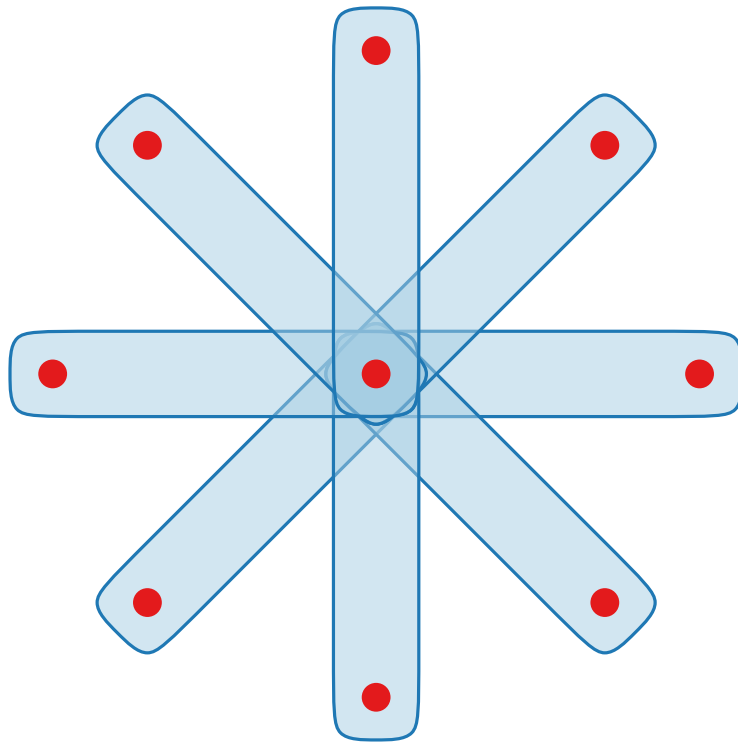
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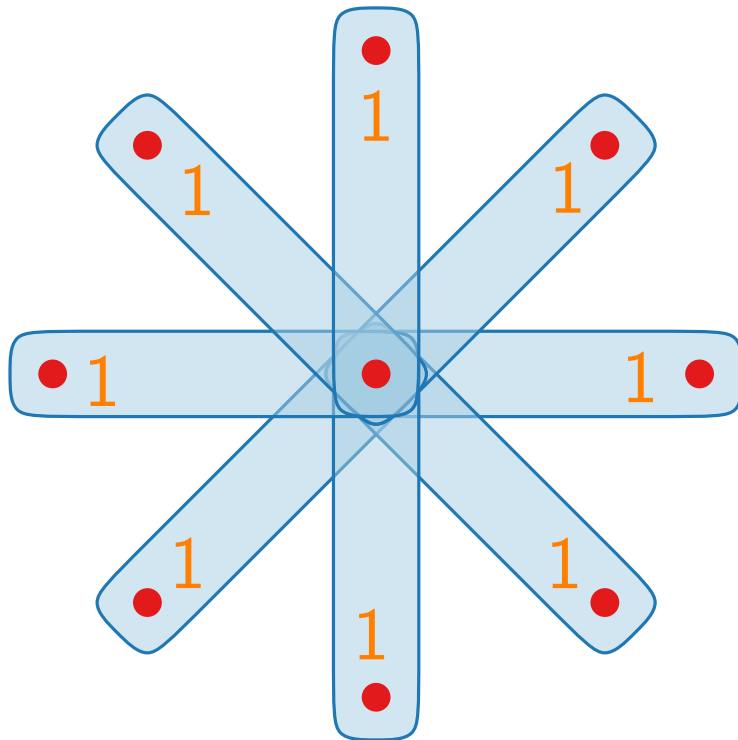
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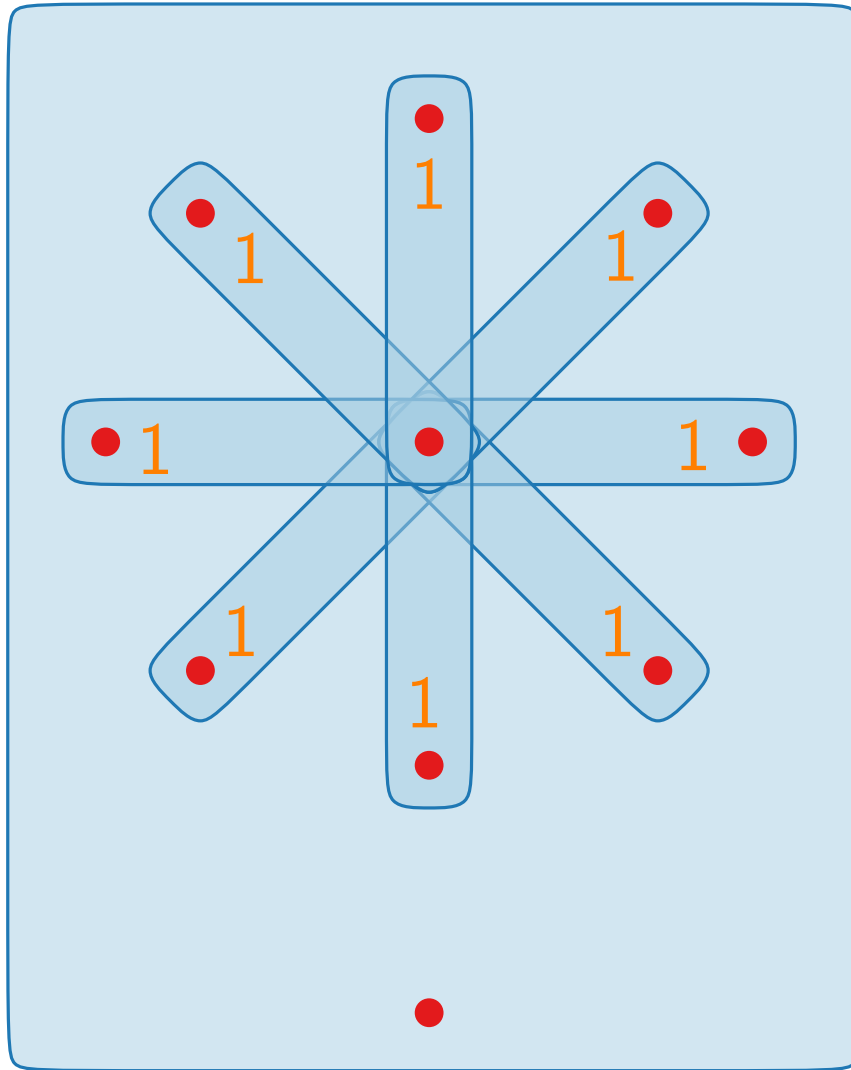
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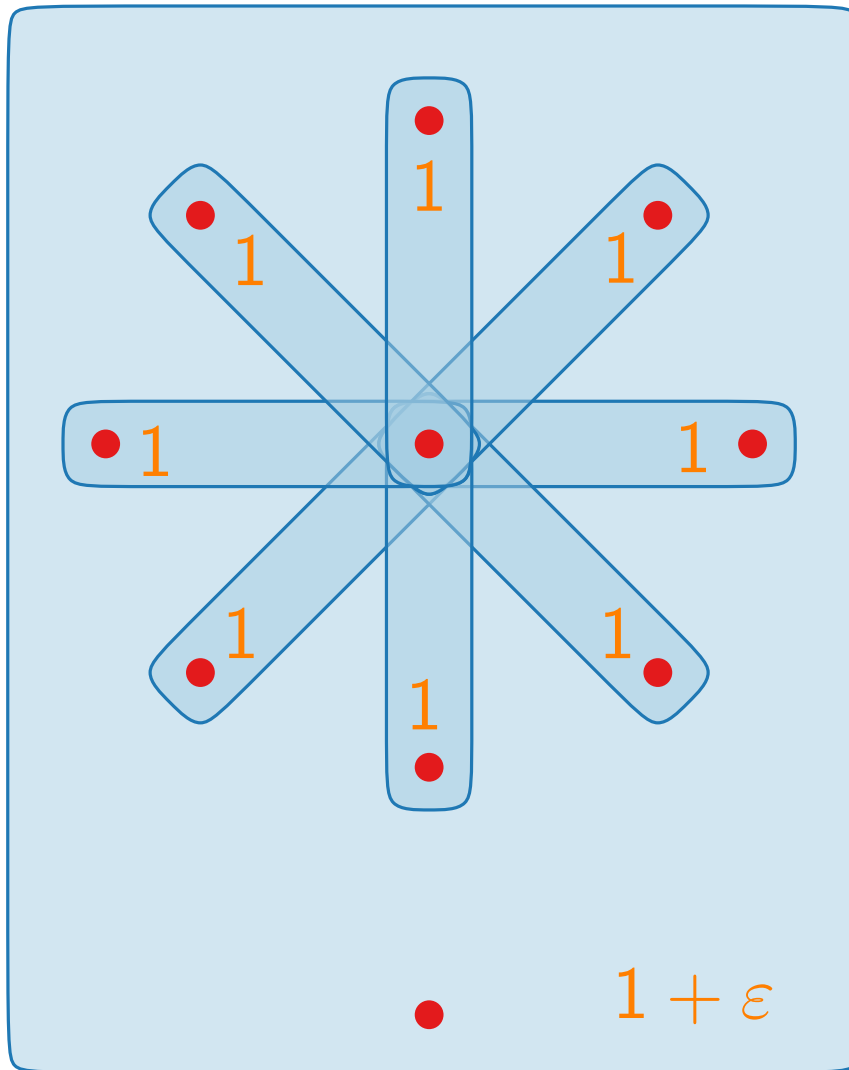
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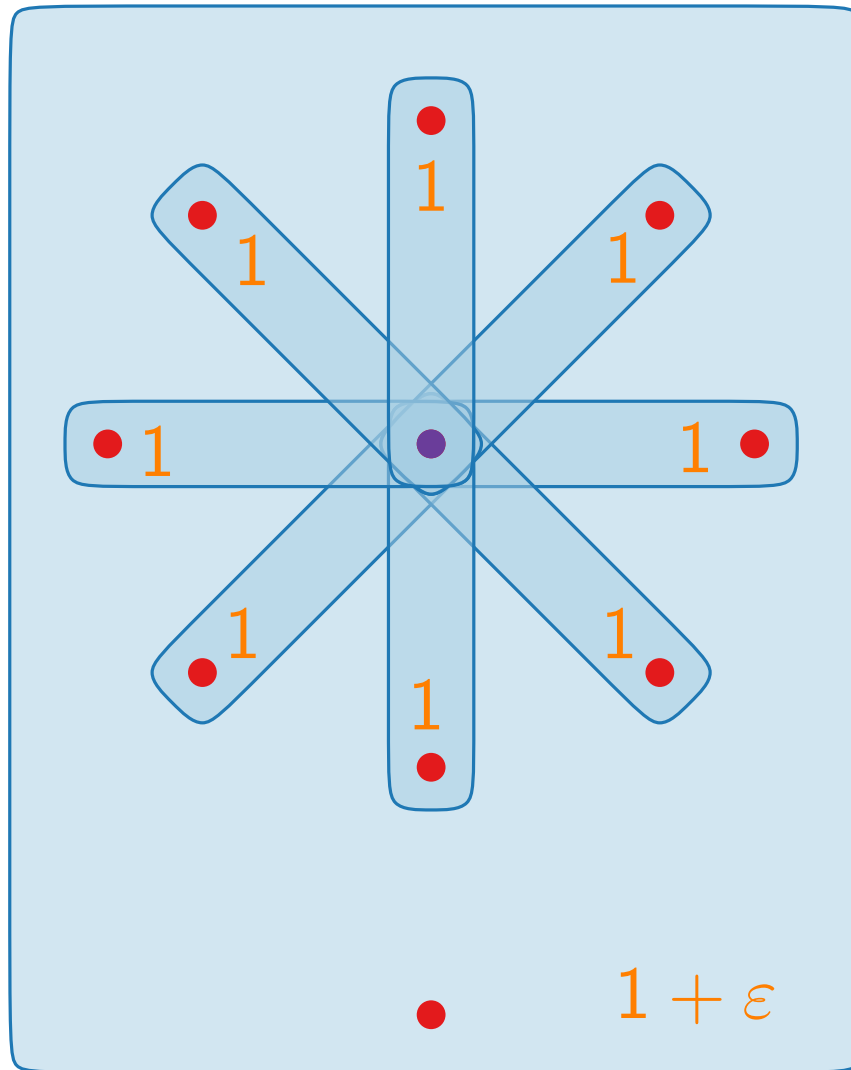
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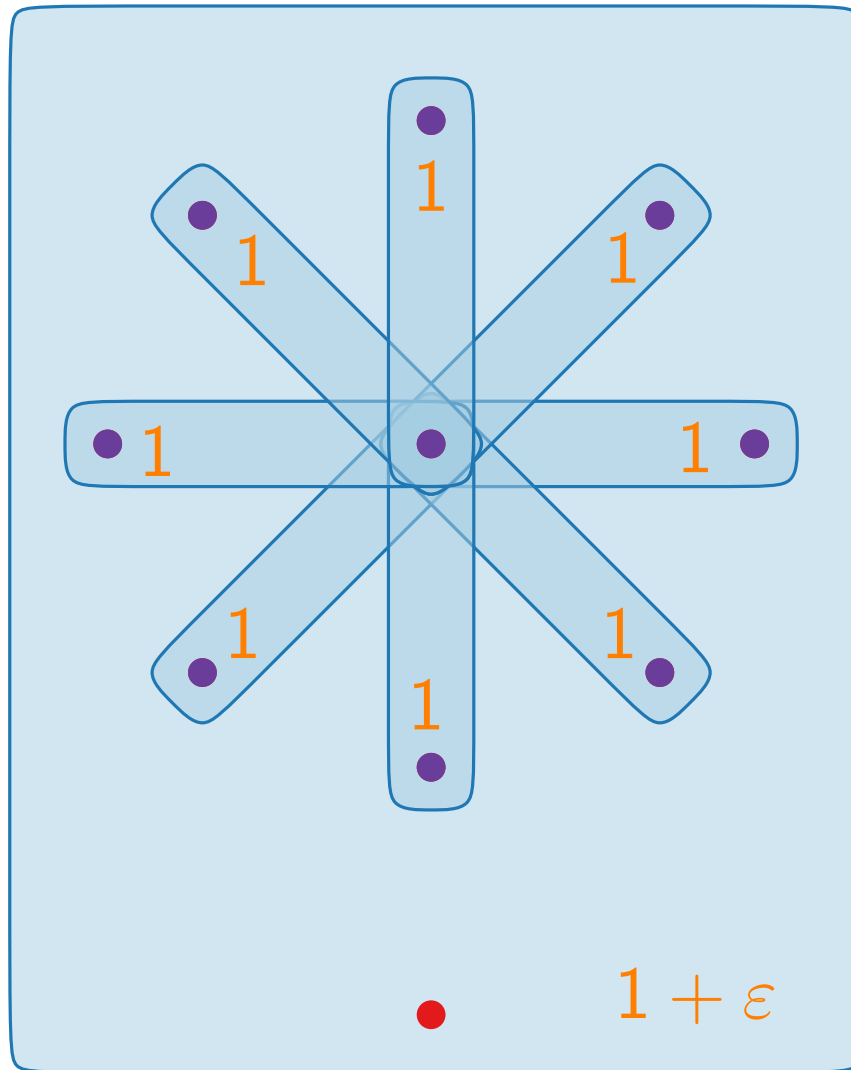
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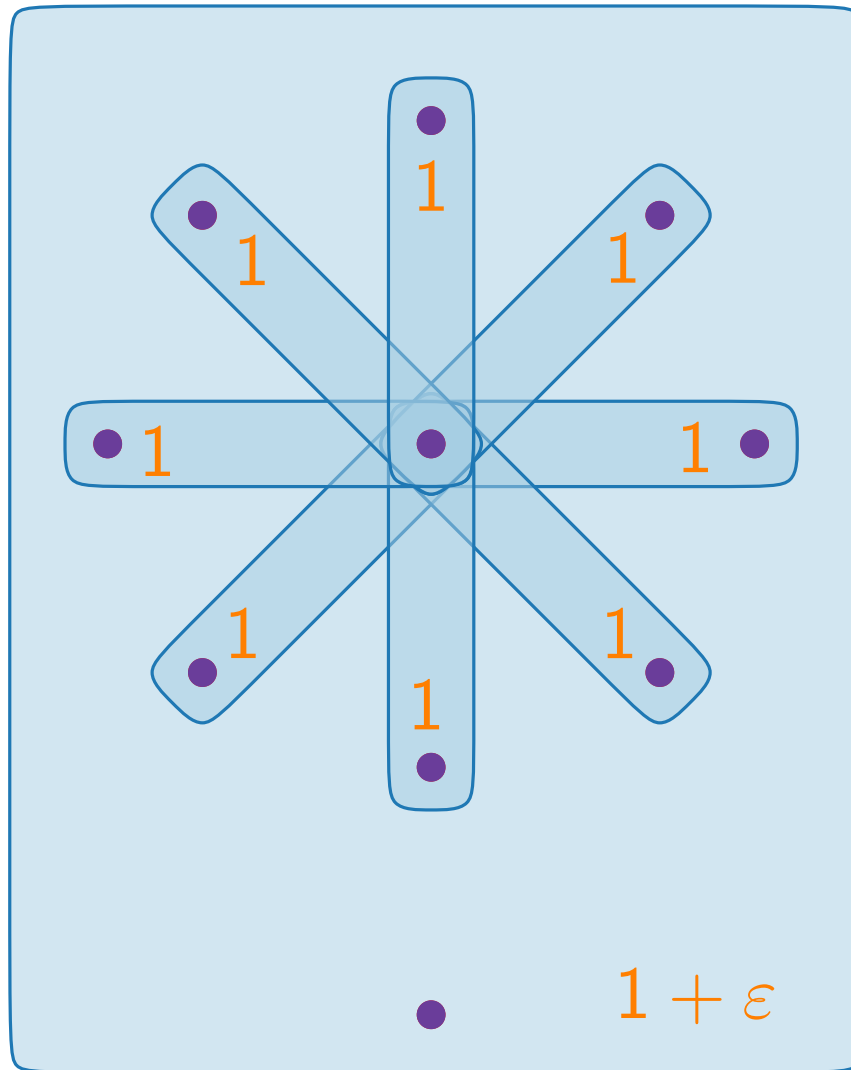
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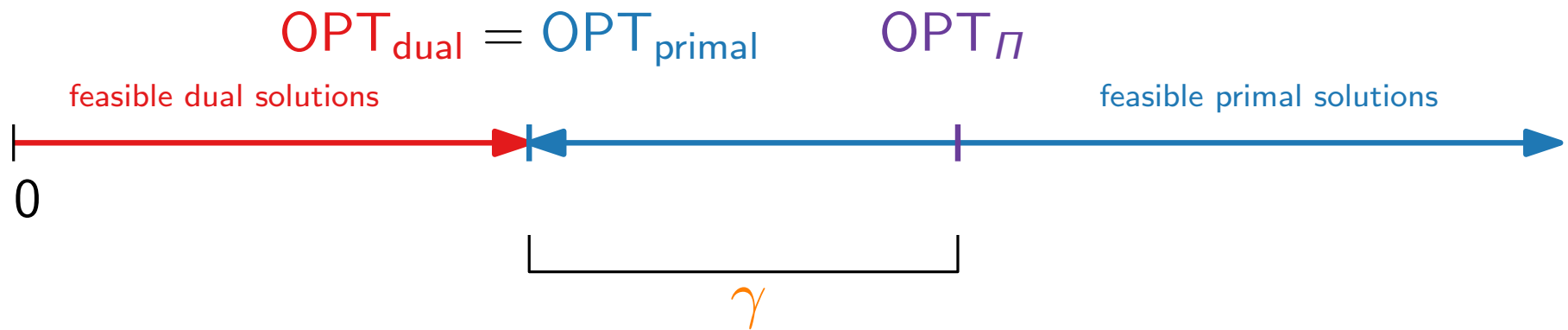
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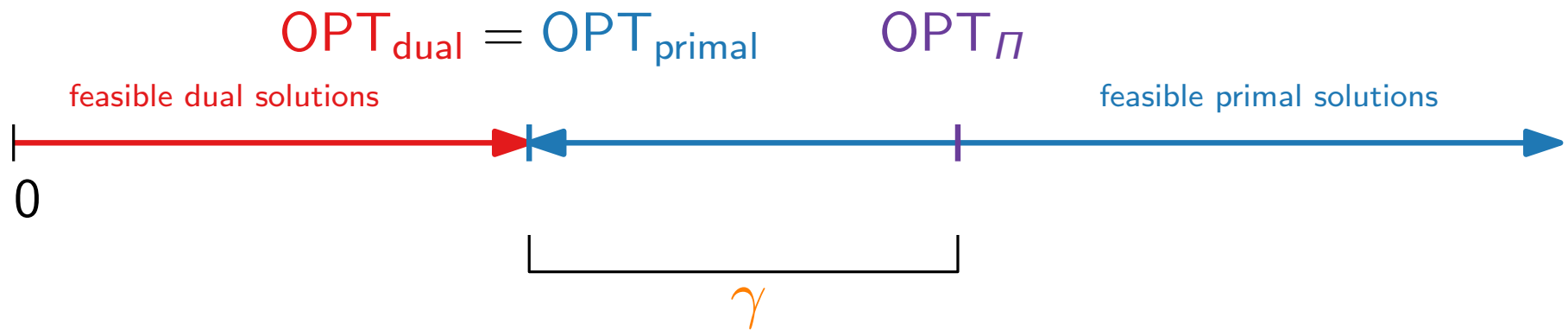


Integrality Gap



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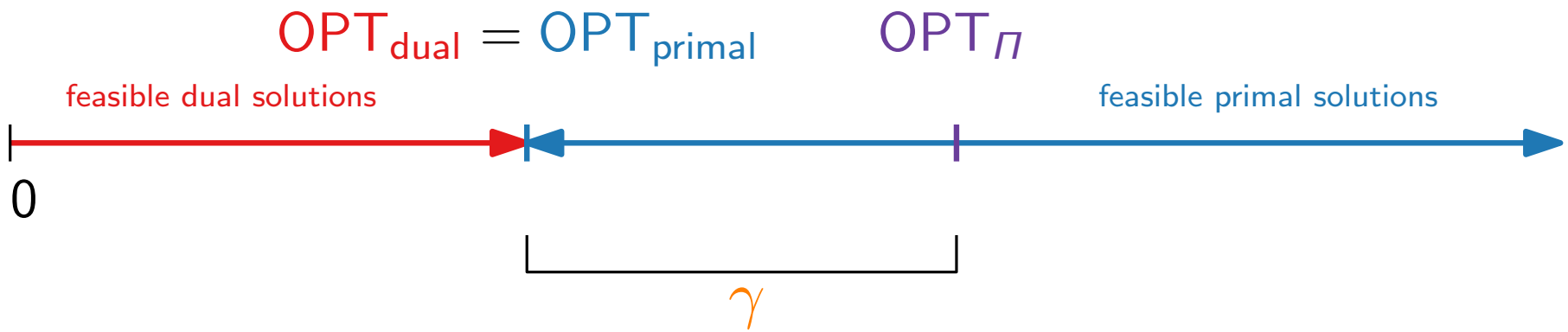
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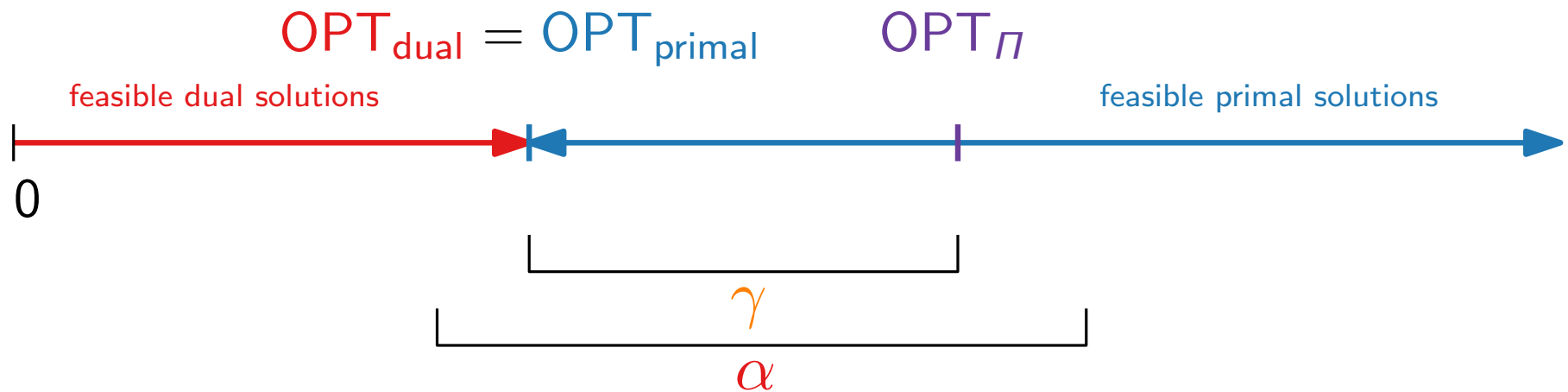


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Approximation Algorithms

Lecture 5: LP-based Approximation Algorithms for SETCOVER

Part IV: Dual Fitting

Technique III) Dual Fitting



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Combinatorial algorithm (e.g., greedy) computes feasible primal solution s_{Π} and infeasible dual solution s_d that completely “pays” for s_{Π} ,

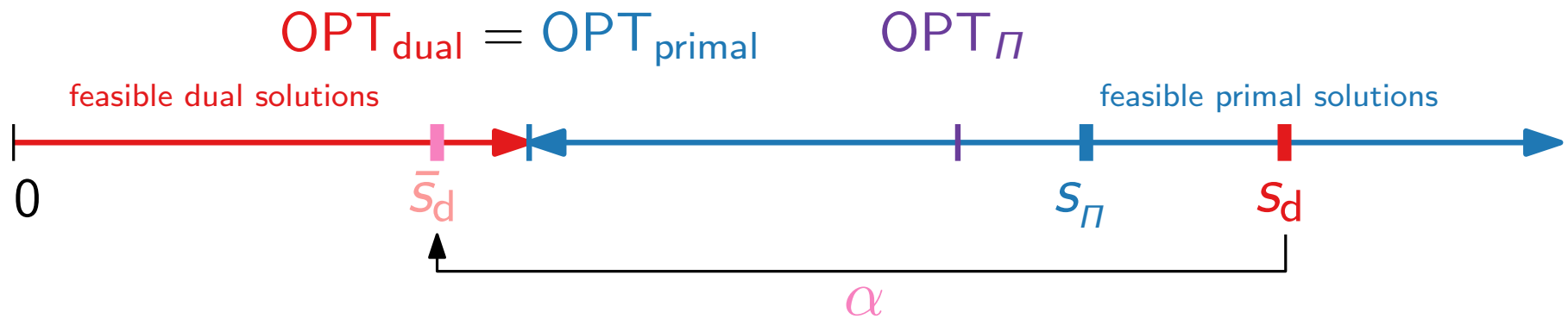
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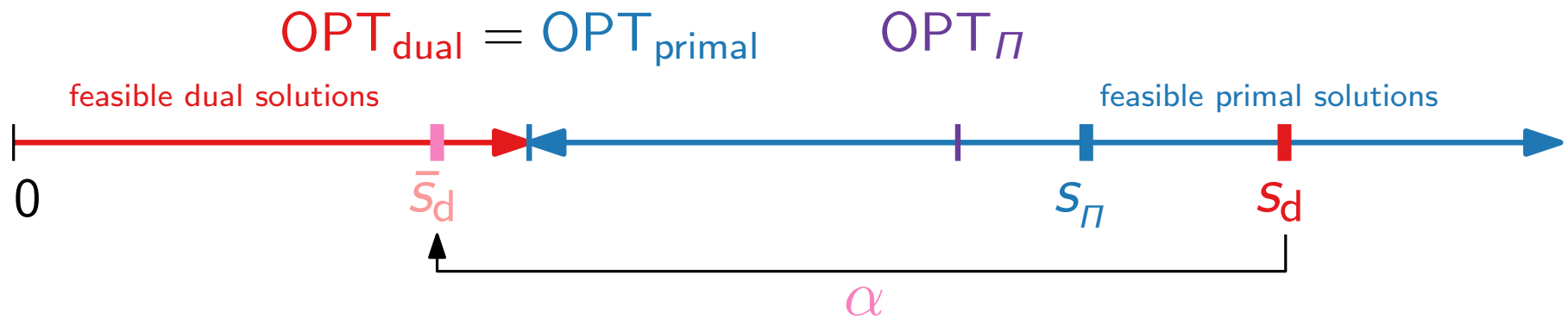


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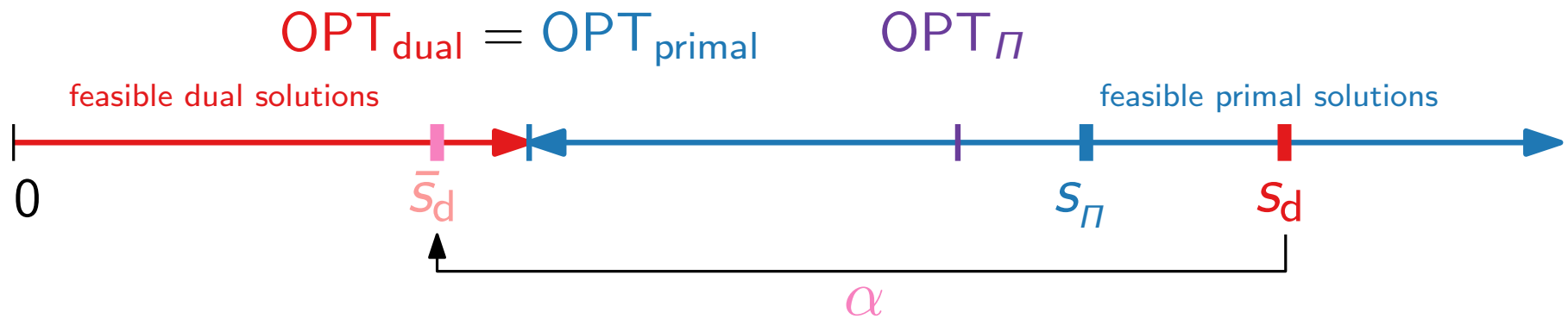
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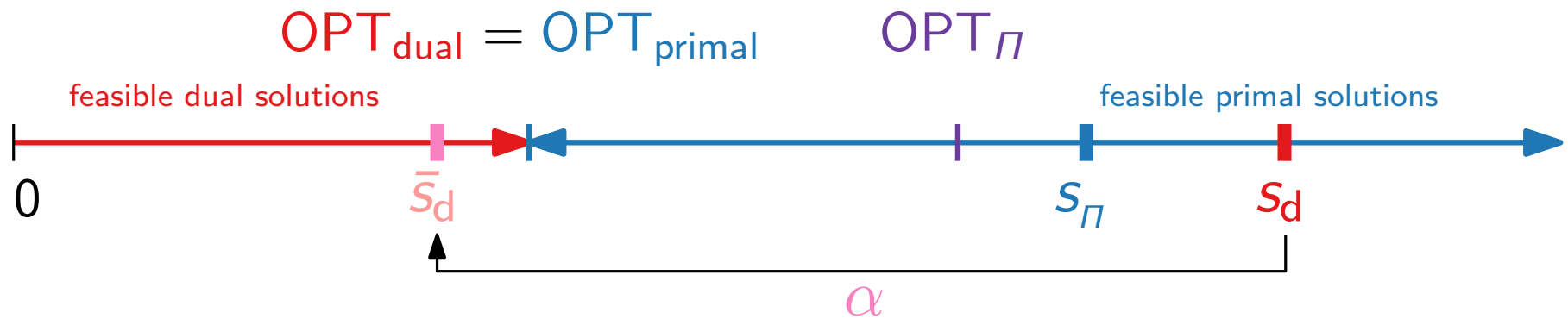
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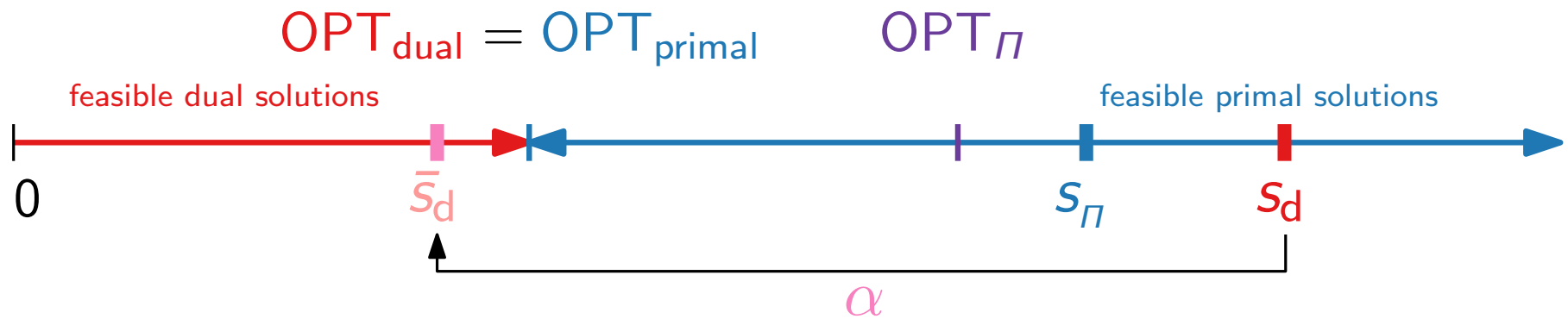
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\Rightarrow Scaling factor α is approximation factor :-)

Dual Fitting for SETCOVER

Combinatorial (greedy) algorithm (see Lecture #2):

GreedySetCover(universe U , $\mathcal{S} \subseteq 2^U$, costs $c: \mathcal{S} \rightarrow \mathbb{Q}_{\geq 0}$)

$C \leftarrow \emptyset$

$\mathcal{S}' \leftarrow \emptyset$

while $C \neq U$ **do**

$S \leftarrow$ set from \mathcal{S} that minimizes $\frac{c(S)}{|S \setminus C|}$

foreach $u \in S \setminus C$ **do**

$\text{price}(u) \leftarrow \frac{c(S)}{|S \setminus C|}$

$C \leftarrow C \cup S$

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Reminder: $\sum_{u \in U} \text{price}(u) \dots$

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Reminder: $\sum_{u \in U} \text{price}(u)$ completely pays for \mathcal{S}' .

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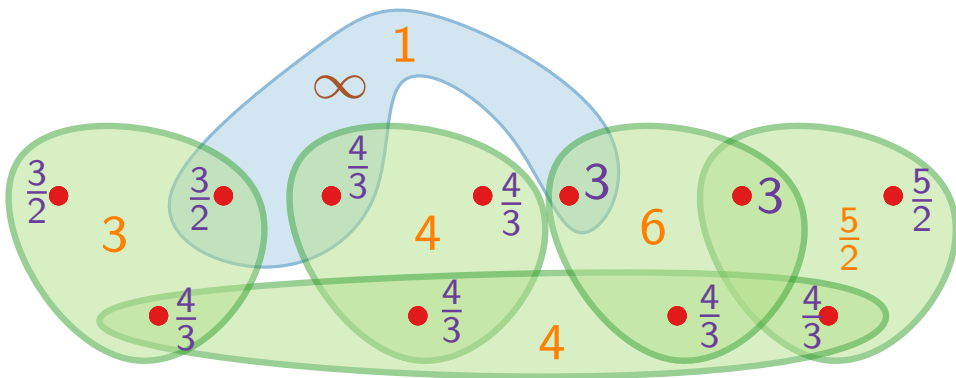
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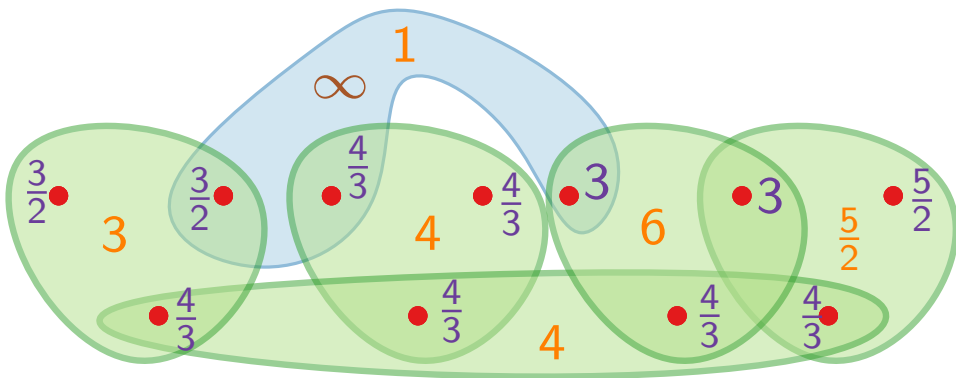
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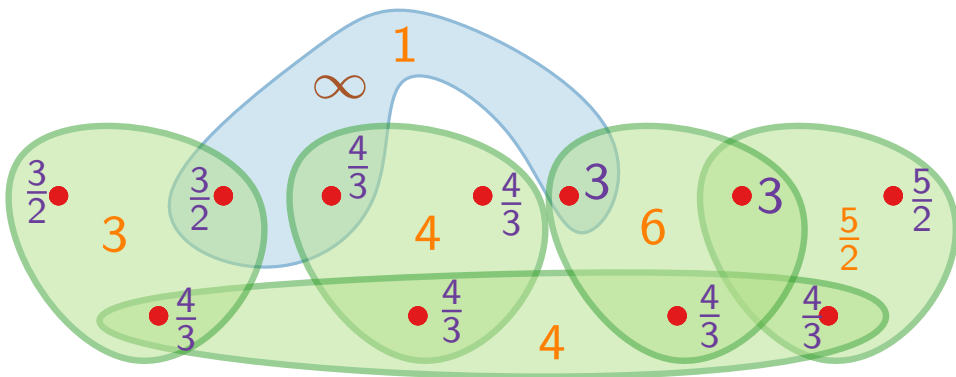
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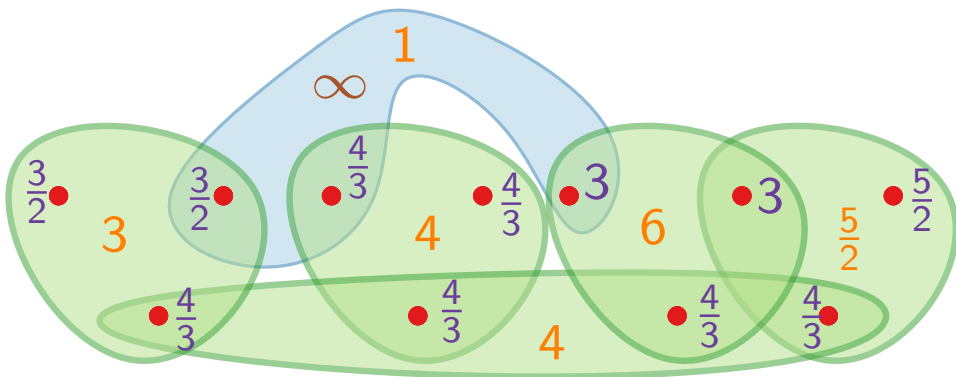
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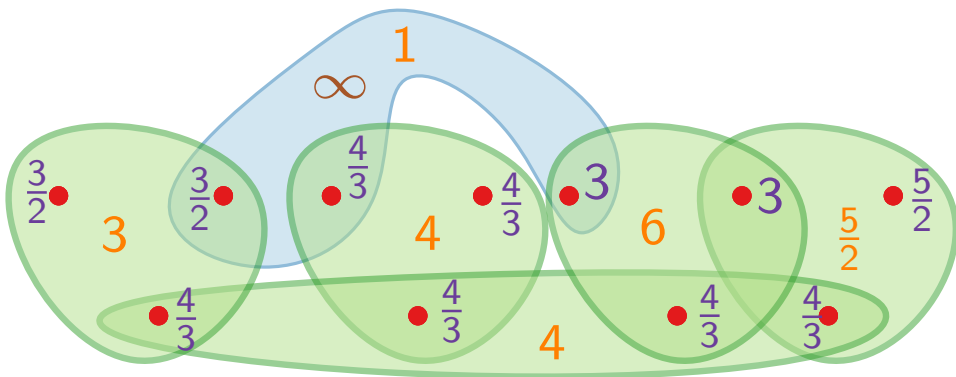
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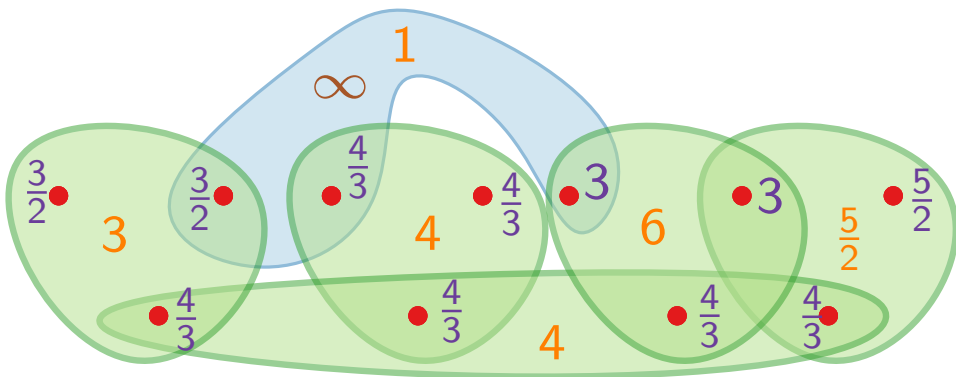
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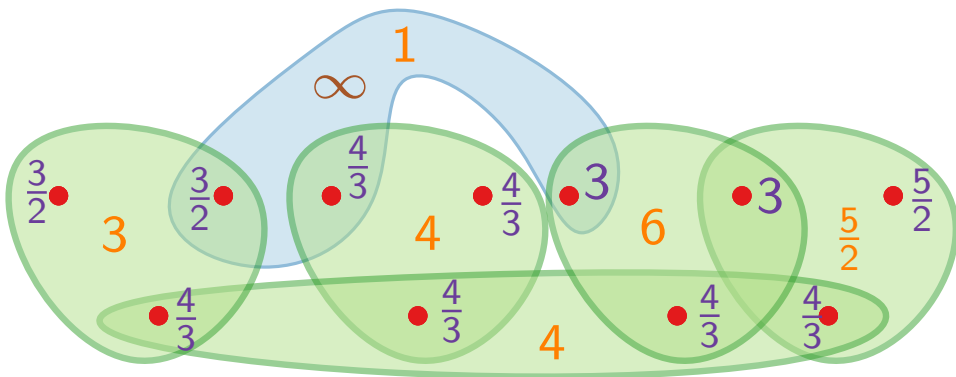
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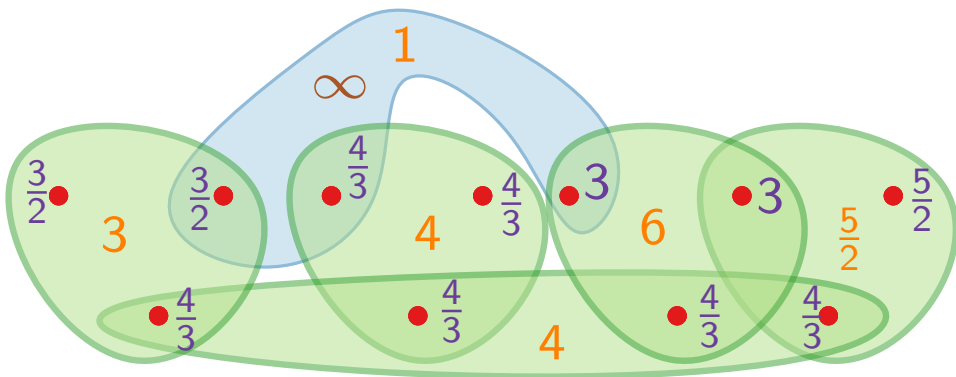
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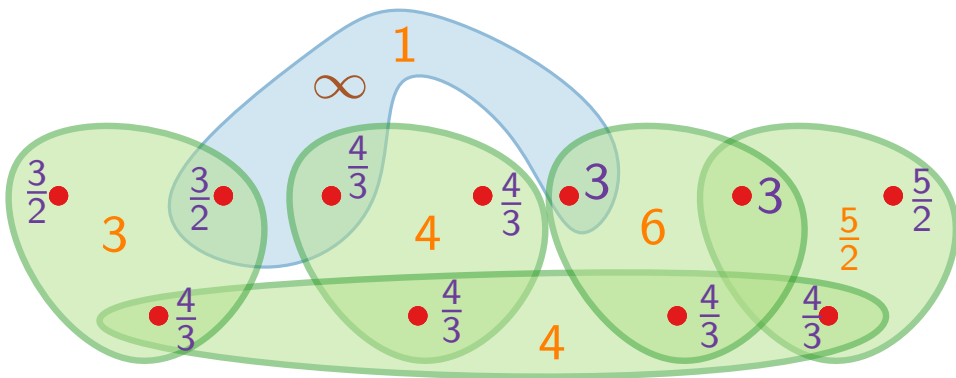
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