

Approximation Algorithms

Lecture 4: Linear Programming and LP-Duality

Part I: Introduction to Linear Programming

Maximizing Profits

You're the boss of a small company that produces two products P_1 and P_2 . For the production of x_1 units of P_1 and x_2 units of P_2 , your profit in € is:

$$G(x_1, x_2) = 30x_1 + 50x_2$$

Three machines M_A , M_B and M_C produce the required components A , B and C for the products. The components are used in different quantities for the products, and each machine requires some time for the production.

$$M_A: \quad 4x_1 + 11x_2 \leq 880$$

$$M_B: \quad x_1 + x_2 \leq 150$$

$$M_C: \quad x_2 \leq 60$$

Which choice of (x_1, x_2) maximizes the profit?

Solution

Linear constraints:

$$M_A: 4x_1 + 11x_2 \leq 880$$

$$M_B: x_1 + x_2 \leq 150$$

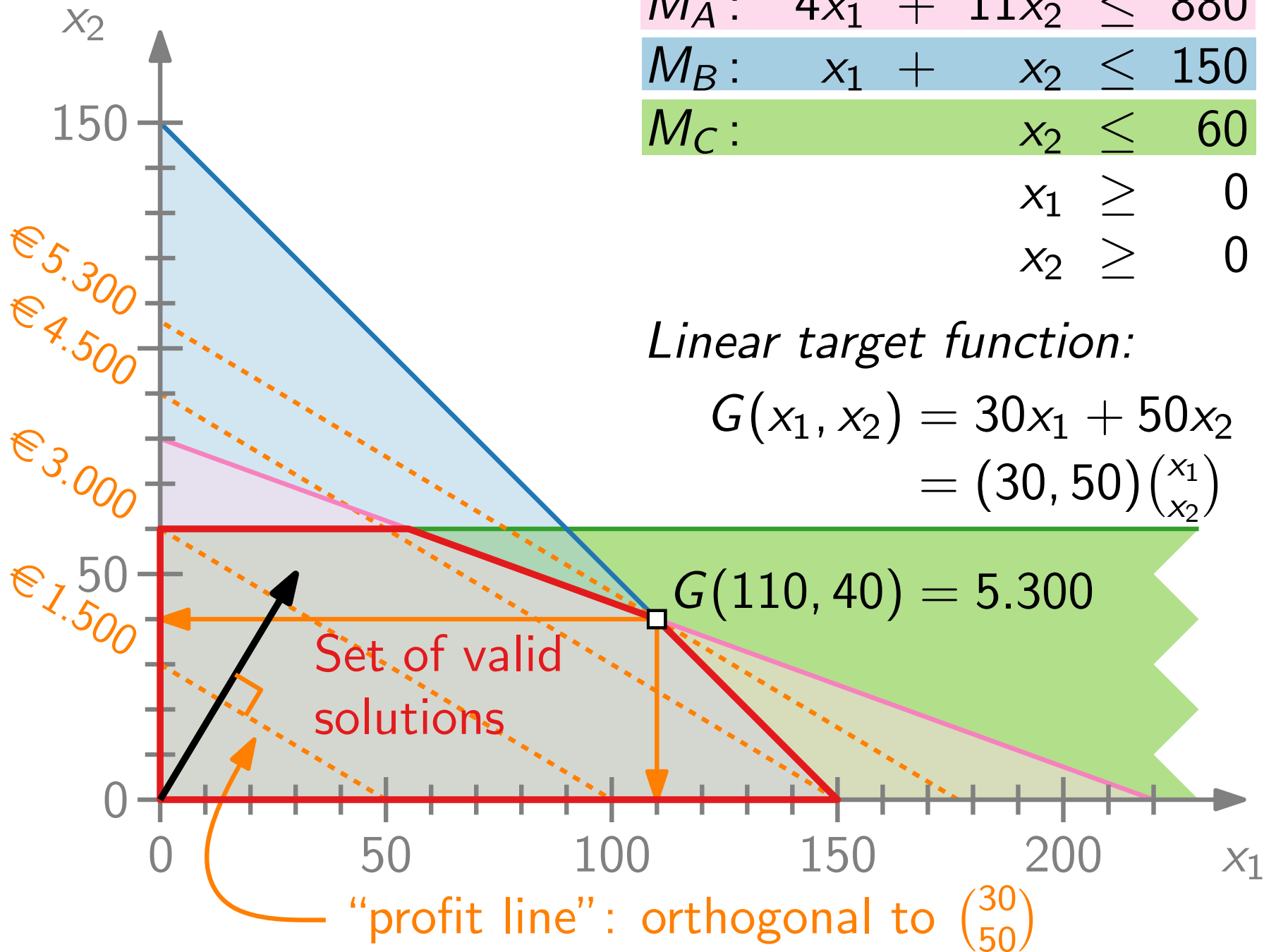
$$M_C: x_2 \leq 60$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

Linear target function:

$$\begin{aligned} G(x_1, x_2) &= 30x_1 + 50x_2 \\ &= (30, 50) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$



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Part II: Upper Bounds for LPs

Motivation: Upper and Lower Bounds

Consider an NP-hard minimization problem.

Decision Problem:

Is a given U an **upper bound** on OPT ?

A feasible sol. S provides efficiently verifiable “yes”-certificate.

Lower bounds / “no”-certificates?

\leadsto probably not! (conjecture: $NP \neq coNP$)

For an approximation algorithm, we need a lower bound

$L \geq OPT/\alpha$ (i.e., an approximate “no”-certificate)!

Examples:

- Vertex Cover: lower bound by maximal matchings
- TSP: lower bound by MSTs or by minimum cycle covers

Linear Programming

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array} \quad \text{standard form}$$

Example. $c = \begin{pmatrix} 7 \\ 1 \\ 5 \end{pmatrix} \quad A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & 2 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 10 \\ 6 \end{pmatrix}$

$$\begin{array}{ll} \text{minimize} & 7x_1 + x_2 + 5x_3 \\ \text{subject to} & x_1 - x_2 + 3x_3 \geq 10 \\ & 5x_1 + 2x_2 - x_3 \geq 6 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	14	$+$	x_2	1	$+$	$5x_3$	15	$=$	30
subject to	x_1	2	$-$	x_2	1	$+$	$3x_3$	9	\geq	10
	$5x_1$	10	$+$	$2x_2$	2	$-$	x_3	3	\geq	6
									\geq	0
	x_1, x_2, x_3									

Valid solution?

$$x = (2, 1, 3)$$

$\Rightarrow \text{obj}(x) = 30$ is upper bound for OPT

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Part III: Lower Bounds for LPs

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	+	x_2	+	$5x_3$	
subject to	$\overset{\text{IV}}{\underset{+}{2 \cdot x_1}}$	–	$\overset{\text{IV}}{\underset{+}{2 \cdot x_2}}$	+	$\overset{\text{IV}}{\underset{+}{2 \cdot 3x_3}}$	$\geq 2 \cdot 10$
	$5x_1$	+	$2x_2$	–	x_3	≥ 6
					x_1, x_2, x_3	≥ 0

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \text{OPT} \geq 10$$

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\ &\geq 10 + 6 \Rightarrow \text{OPT} \geq 16 \end{aligned}$$

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\ &\geq 2 \cdot 10 + 6 \Rightarrow \text{OPT} \geq 26 \end{aligned}$$

Linear Programming – Lower Bounds

$$\begin{array}{ll}
 \text{minimize} & 7x_1 + x_2 + 5x_3 \\
 \text{subject to} & y_1(x_1 - x_2 + 3x_3) \geq 10y_1 \\
 & y_2(5x_1 + 2x_2 - x_3) \geq 6y_2 \\
 & x_1, x_2, x_3 \geq 0
 \end{array}$$

$$\begin{aligned}
 7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
 &\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2 \\
 10y_1 + 6y_2 &\text{ is lower bound for OPT if ...}
 \end{aligned}$$

Linear Programming – Lower Bounds

$$\begin{array}{ll}
 \text{minimize} & 7x_1 + x_2 + 5x_3 \\
 \text{subject to} & y_1 \left(\overset{\text{IV}}{+}x_1 - \overset{\text{IV}}{+}x_2 + \overset{\text{IV}}{+}3x_3 \right) \geq 10 y_1 \\
 & y_2 \left(\overset{+}{+}5x_1 + \overset{+}{+}2x_2 - \overset{+}{+}x_3 \right) \geq 6 y_2 \\
 & x_1, x_2, x_3 \geq 0
 \end{array}$$

$$\begin{aligned}
 7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
 &\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
 \end{aligned}$$

maximize

$$\begin{array}{llll}
 y_1 & + & 5y_2 & \leq 7 \\
 -y_1 & + & 2y_2 & \leq 1 \\
 3y_1 & - & y_2 & \leq 5 \\
 & & y_1, y_2 & \geq 0
 \end{array}$$

Primal–Dual

primal program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

dual program

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \\ & y \geq 0 \end{array}$$

dual of the dual program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

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Part IV: LP-Duality and Complementary Slackness

LP-Duality

minimize	$c^T x$	Primal
subject to	$Ax \geq b$	
	$x \geq 0$	

maximize	$b^T y$	Dual
subject to	$A^T y \leq c$	
	$y \geq 0$	

Theorem. The primal program has a finite optimum \Leftrightarrow the dual program has a finite optimum. Moreover, if $x^* = (x_1^*, \dots, x_n^*)$ and $y^* = (y_1^*, \dots, y_m^*)$ are *optimal* solutions for the primal and dual program, respectively, then

$$\sum_{j=1}^n c_j x_j^* = \sum_{i=1}^m b_i y_i^* .$$

Weak LP-Duality

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \\ & y \geq 0 \end{array}$$

Theorem. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ are valid solutions for the primal and dual program, resp., then

$$\sum_{j=1}^n c_j x_j \geq \sum_{i=1}^m b_i y_i .$$

Proof.

$$\sum_{j=1}^n c_j x_j \geq \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} y_i \right) x_j = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j \right) y_i \geq \sum_{i=1}^m b_i y_i .$$

Complementary Slackness

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \\ & y \geq 0 \end{array}$$

Theorem. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ be valid solutions for the primal and dual program, respectively. Then x and y are optimal if and only if the following conditions are met:

Primal CS:

For each $j = 1, \dots, n$: $x_j = 0$ or $\sum_{i=1}^m a_{ij} y_i = c_j$

Dual CS:

For each $i = 1, \dots, m$: $y_i = 0$ or $\sum_{j=1}^n a_{ij} x_j = b_i$

Proof. Follows from LP-duality: For every summand...

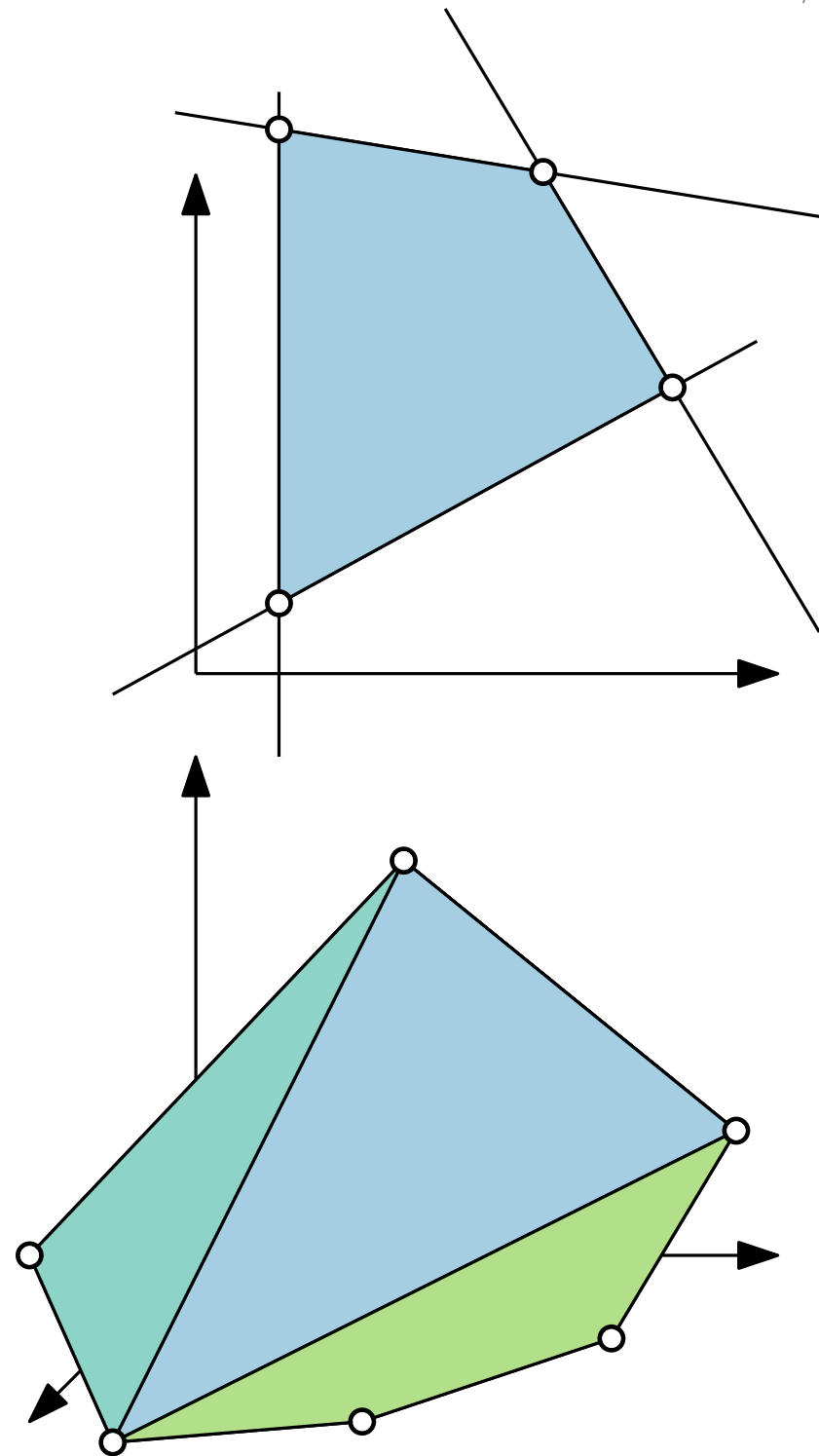
$$\sum_{j=1}^n c_j x_j \geq \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} y_i \right) x_j = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j \right) y_i \geq \sum_{i=1}^m b_i y_i.$$

LPs and Convex Polytopes

The feasible solutions of an LP with n variables form a **convex polytope** in \mathbb{R}^n (intersection of halfspaces).

Corners of the polytope are called **extreme point solutions** \Leftrightarrow n linearly independent inequalities (constraints) are satisfied with equality.

If an optimal solution exists, some extreme point is also optimal.



Integer Linear Programs (ILPs)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \in \mathbb{N} \end{array}$$

Many NP-optimization problems can be formulated as ILPs; thus ILPs are NP-hard to solve.

LP-relaxation provides a lower bound: $\text{OPT}_{\text{LP}} \leq \text{OPT}_{\text{ILP}}$

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Part V: Min–Max Relationships

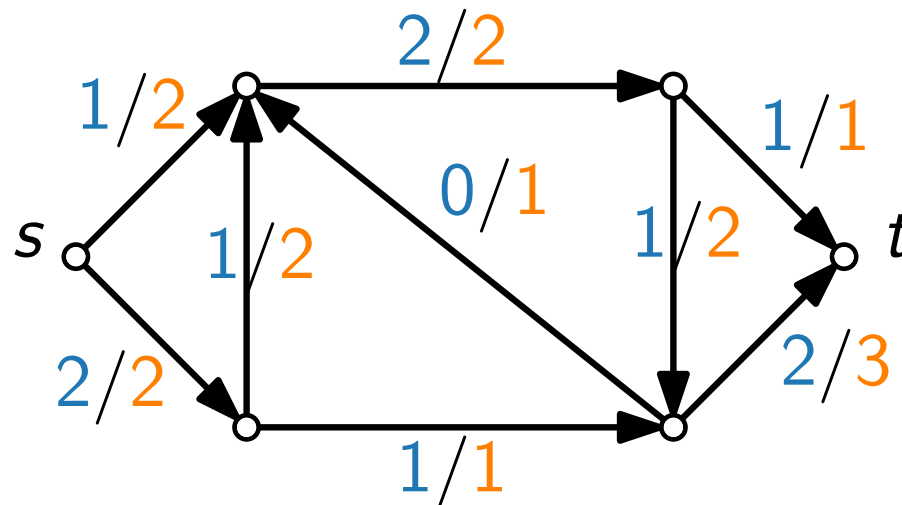
Max-Flow Problem

Given: A directed graph G with edge capacities $c: E(G) \rightarrow \mathbb{Q}_+$ and two special vertices: the source s and sink t .

Find: A **maximum s – t flow** (i.e., non-negative edge weights f) such that

- $f(u, v) \leq c(u, v)$ for each edge $(u, v) \in E(G)$,
- $\sum_{u: (u,v) \in E} f(u, v) = \sum_{z: (v,z) \in E} f(v, z)$ for each $v \in V(G) \setminus \{s, t\}$.

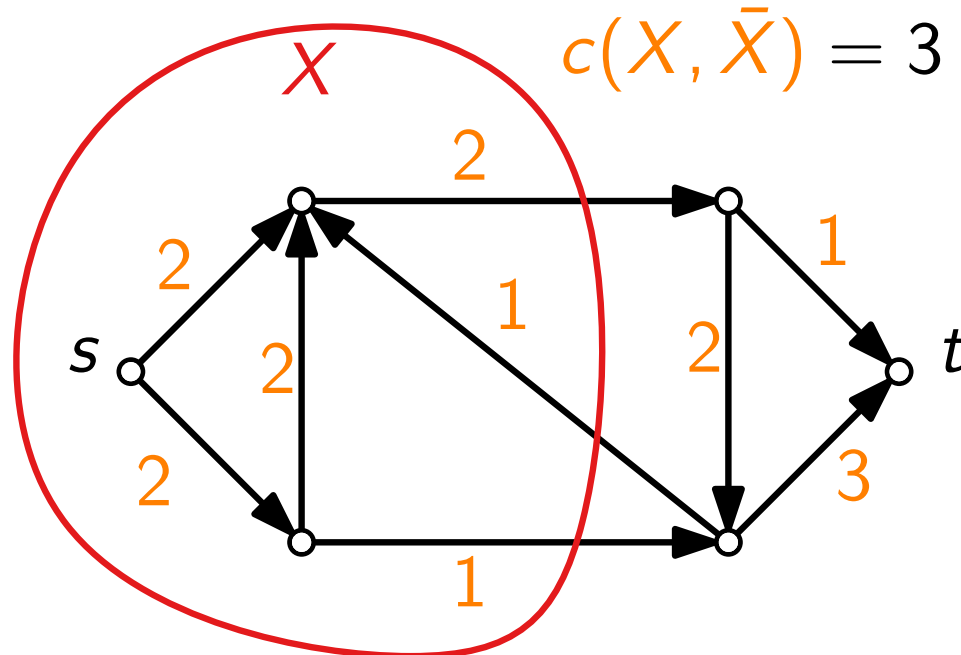
The **flow value** is the inflow to t minus the outflow from t .



Min-Cut Problem

Given: A directed graph G with edge capacities $c: E(G) \rightarrow \mathbb{Q}_+$ and two special vertices: the source s and sink t .

Find: An s - t cut, i.e., a vertex set X with $s \in X$ and $t \in \bar{X}$ such that the total weight $c(X, \bar{X})$ of the edges from X to \bar{X} is minimum.



Max-Flow-Min-Cut Theorem

Theorem. The value of a **maximum s – t flow** and the weight of a **minimum s – t cut** are the same.

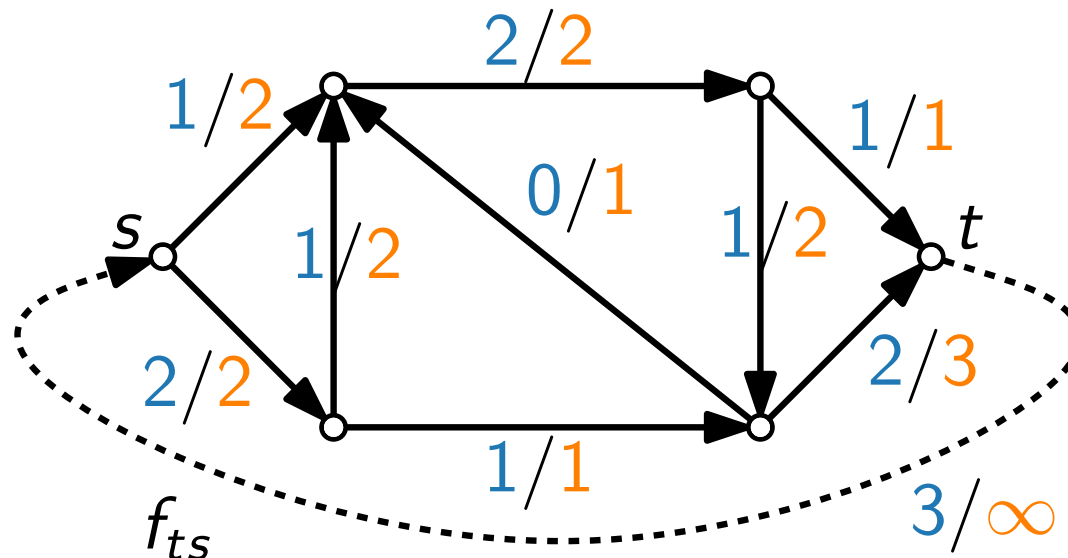
Proof. Special case of LP-Duality ...

maximize f_{ts}
 subject to

$$\sum_{u: (u,v) \in E(G)} f_{uv} - \sum_{z: (v,z) \in E(G)} f_{vz} \leq 0 \quad \forall v \in V(G)$$

$$f_{uv} \leq c_{uv} \quad \forall (u,v) \neq (t,s)$$

$$f_{uv} \geq 0 \quad \forall (u,v) \in E(G)$$



Max-Flow-Min-Cut Theorem

Theorem. The value of a **maximum s – t flow** and the weight of a **minimum s – t cut** are the same.

Proof. Special case of LP-Duality ...

$$\begin{array}{ll}
 \text{maximize} & f_{ts} \\
 \text{subject to} & \\
 & \sum_{u: (u,v) \in E(G)} f_{uv} - \sum_{z: (v,z) \in E(G)} f_{vz} \leq 0 \quad \forall v \in V(G) \\
 & f_{uv} \leq c_{uv} \quad \forall (u,v) \neq (t,s) \\
 & f_{uv} \geq 0 \quad \forall (u,v) \in E(G)
 \end{array}$$

d_{uv}
 p_v

$$\text{maximize } c^T x = \sum_{(u,v) \in E(G)} (0 \cdot f_{uv}) + 1 \cdot f_{ts} \Rightarrow c^T = (0, \dots, 0, 1)$$

Which constraints contain f_{uv} for $(u,v) \neq (t,s)$? d_{uv}, p_u, p_v

$$\Rightarrow d_{uv} - p_u + p_v \geq 0$$

Which constraints contain f_{ts} ? p_s, p_t

$$\Rightarrow p_s - p_t \geq 1$$

Max-Flow-Min-Cut Theorem

Theorem. The value of a **maximum s – t flow** and the weight of a **minimum s – t cut** are the same.

Proof. Special case of LP-Duality ...

$$\begin{array}{ll}
 \text{maximize} & f_{ts} \\
 \text{subject to} & \\
 & \sum_{u: (u,v) \in E(G)} f_{uv} - \sum_{z: (v,z) \in E(G)} f_{vz} \leq 0 \quad \forall v \in V(G) \\
 & f_{uv} \leq c_{uv} \quad \forall (u,v) \neq (t,s) \\
 & f_{uv} \geq 0 \quad \forall (u,v) \in E(G)
 \end{array}$$

d_{uv}
 p_v

$$\begin{array}{ll}
 \text{minimize} & \sum_{(u,v) \in E(G) \setminus \{(t,s)\}} c_{uv} \cdot d_{uv} \\
 \text{subject to} & \\
 & d_{uv} - p_u + p_v \geq 0 \quad \forall (u,v) \in E(G) \setminus \{(t,s)\} \\
 & p_s - p_t \geq 1 \\
 & d_{uv} \geq 0 \quad \forall (u,v) \in E(G) \\
 & p_u \geq 0 \quad \forall u \in V(G)
 \end{array}$$

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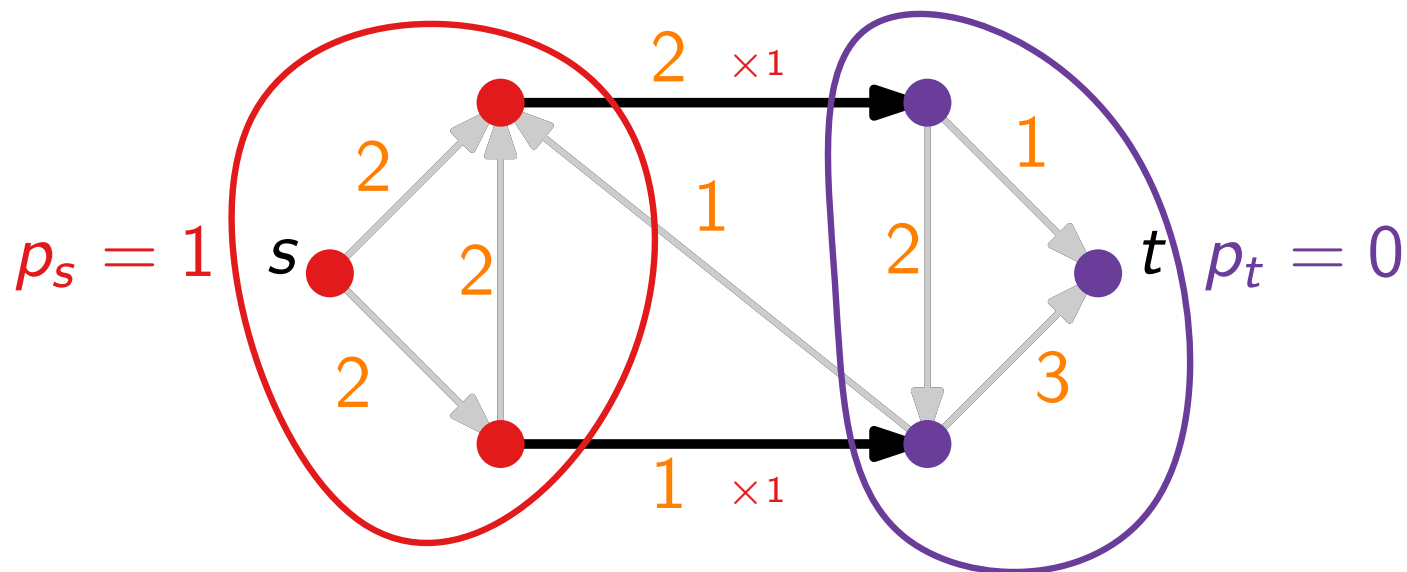
Lecture 4: Linear Programming and LP-Duality

Part VI: Dual LP of Max Flow

Dual LP – Interpretation as ILP

$$\begin{array}{ll}
 \text{minimize} & \sum_{(u,v) \in E(G) \setminus \{(t,s)\}} c_{uv} \cdot d_{uv} \\
 \text{subject to} & d_{uv} - p_u + p_v \geq 0 \quad \forall (u,v) \neq (t,s) \\
 & p_s - p_t \geq 1 \\
 & d_{uv} \geq 0 \in \{0, 1\} \quad \forall (u,v) \in E(G) \\
 & p_u \geq 0 \in \{0, 1\} \quad \forall u \in V(G)
 \end{array}$$

equivalent to Min-Cut!



Dual LP – Fractional Cuts

minimize $\sum_{(u,v) \in E(G) \setminus \{(t,s)\}} c_{uv} \cdot d_{uv}$

subject to

$$d_{uv} - p_u + p_v \geq 0 \quad \forall (u, v) \in E(G) \setminus \{(t, s)\}$$

$$p_s - p_t \geq 1$$

$$d_{uv} \geq 0 \quad \forall (u, v) \in E(G)$$

$$p_u \geq 0 \quad \forall u \in V(G)$$

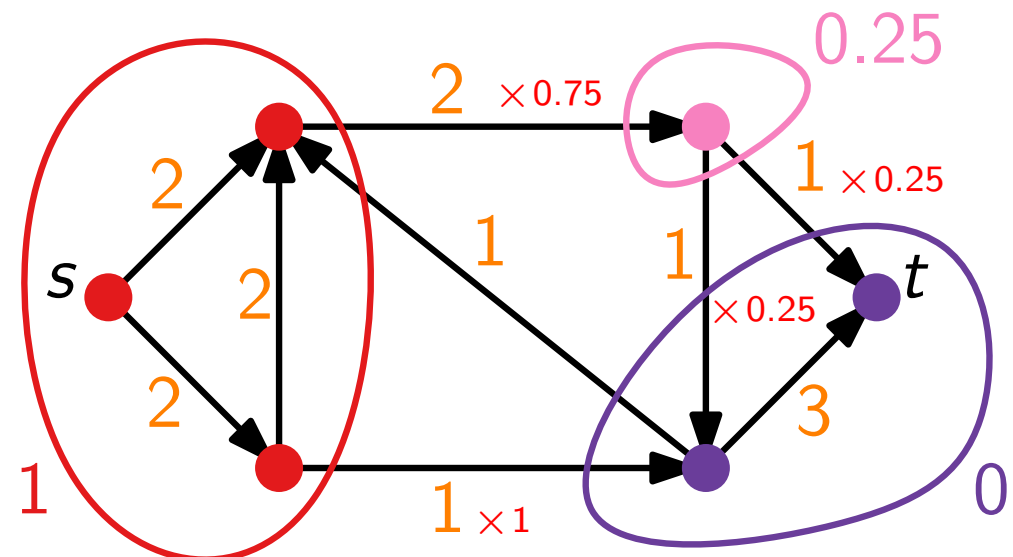
LP-relaxation of the ILP

Moreover, all extreme-point solutions of this polytope are **integral!** (HW)

Note that every s – t path $s = v_0, \dots, v_k = t$ has length ≥ 1 w.r.t. d :

$$\sum_{i=0}^{k-1} d_{i,i+1} \geq \sum_{i=0}^{k-1} (p_i - p_{i+1})$$

$$= p_s - p_t \geq 1$$



Dual LP – Complementary Slackness

$$\begin{array}{ll}
 \text{maximize} & f_{ts} \\
 \text{subject to} & \sum_{u: (u,v) \in E(G)} f_{uv} - \sum_{z: (v,z) \in E(G)} f_{vz} \leq 0 \quad \forall v \in V(G) \\
 & f_{uv} \leq c_{uv} \quad \forall (u,v) \neq (t,s) \\
 & f_{uv} \geq 0 \quad \forall (u,v) \in E(G)
 \end{array}$$

$$\begin{array}{ll}
 \text{minimize} & \sum_{(u,v) \in E(G) \setminus \{(t,s)\}} c_{uv} \cdot d_{uv} \\
 \text{subject to} & d_{uv} - p_u + p_v \geq 0 \\
 & p_s - p_t \geq 1 \\
 & d_{uv} \geq 0 \\
 & p_u \geq 0
 \end{array}$$

Primal CS:

$$\forall j: \quad x_j = 0 \quad \text{or} \quad \sum_{i=1}^m a_{ij} y_i = c_j$$

Dual CS:

$$\forall i: \quad y_i = 0 \quad \text{or} \quad \sum_{j=1}^n a_{ij} x_j = b_i$$

For a max flow and min cut:

- For each forward edge (u, v) of the cut: $f_{uv} = c_{uv}$.
($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)
- For each backward edge (u, v) of the cut: $f_{uv} = 0$.
(Otherwise, by primal CS: $d_{uv} - 0 + 1 = 0$.)

