Approximation Algorithms

Lecture 3: SteinerTree and MultiwayCut

Part I: SteinerTree

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SteinerTree

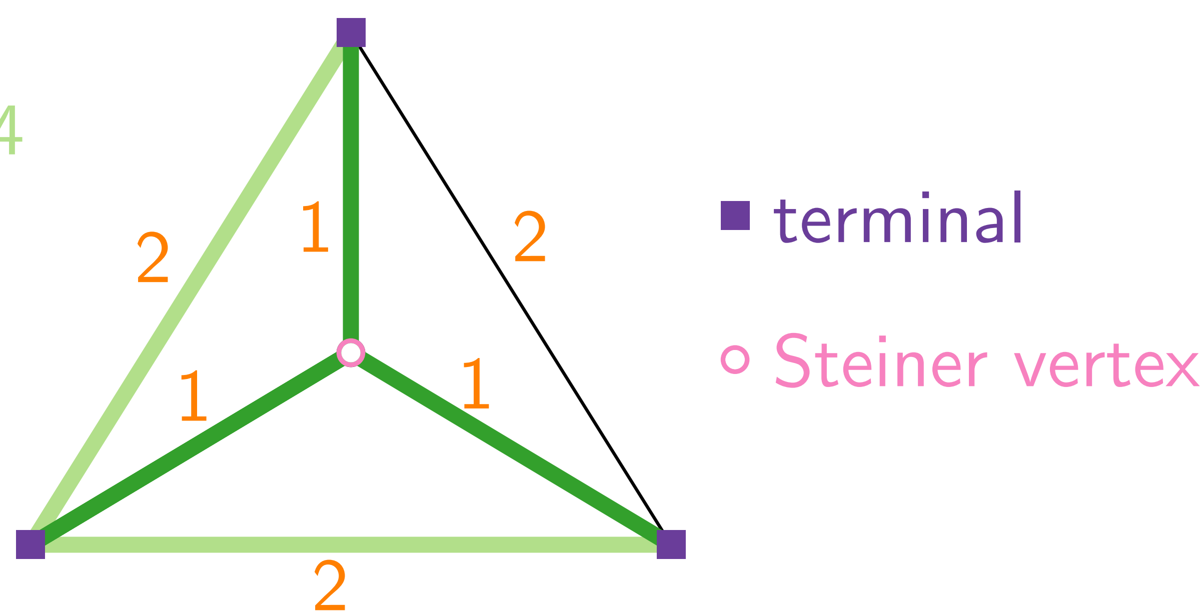
Given: A graph G with edge weights $c\colon E(G) \to \mathbb{Q}^+$
and a partition of $V(G)$ into a set T of **terminals**
and a set S of **Steiner vertices**.

Find: A subtree B of G that

- contains all terminals (i.e., $T \subseteq V(B)$) and
- has minimum cost $c(B) := \sum_{e \in E(B)} c(e)$ among all subtrees with this property.

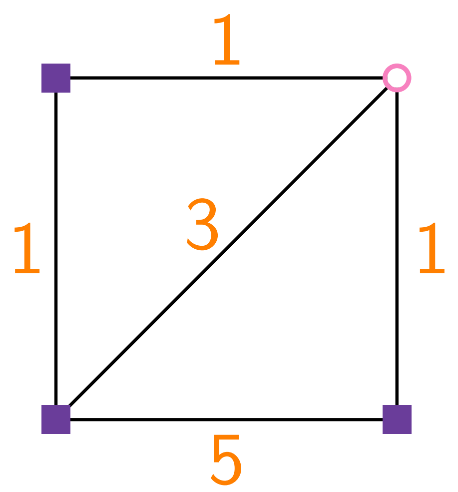
valid solution with cost 4

optimum solution with cost 3

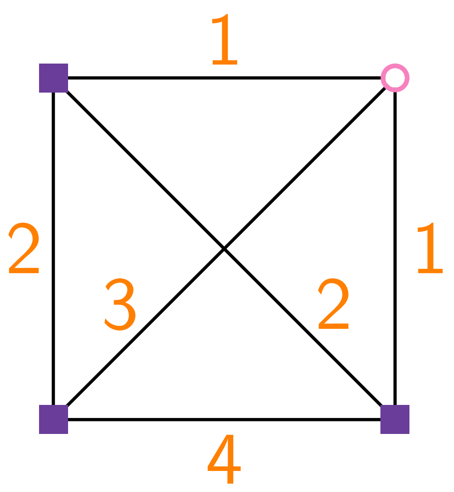


METRICSTEINERTREE

Restriction of STEINERTREE where the graph G is complete and the cost function is **metric**, i.e., for every triple u, v, w of vertices, we have $c(u, w) \leq c(u, v) + c(v, w)$.



– not complete
– not metric



– complete
– metric

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Part II: Approximation-Preserving Reduction

Approximation-Preserving Reduction

Let Π_1, Π_2 be minimization problems. An **approximation-preserving reduction** from Π_1 to Π_2 ist a tuple (f, g) of poly-time computable functions with the following properties.

- For each instance I_1 of Π_1,
  $I_2 = f(I_1)$ is an instance of Π_2 with $\mathrm{OPT}_{\Pi_2}(I_2) \leq \mathrm{OPT}_{\Pi_1}(I_1)$.
- For each feasible solution t of I_2,
  $s = g(I_1, t)$ is a feasible sol. of I_1 with $\mathrm{obj}_{\Pi_1}(I_1, s) \leq \mathrm{obj}_{\Pi_2}(I_2, t)$.

$$
\begin{tikzcd}
\text{problems} & \Pi_1 & & \Pi_2 \\
\text{instances} & I_1 \arrow[d, dashed] \arrow[rr, "f"] & & I_2 \arrow[d] \\
\text{solutions} & s & & t \arrow[ll, "g"]
\end{tikzcd}
$$

Approximation-Preserving Reduction

Theorem. Let Π_1, Π_2 be minimization problems with an approximation-preserving reduction (f, g) from Π_1 to Π_2. If there is a factor-α approximation algorithm for Π_2, then there is a factor-α approximation algorithm for Π_1.

$$\begin{tikzcd}
\Pi_1 & \Pi_2 \\
I_1 \arrow[r, "f"] \arrow[d, dashed] & I_2 \arrow[d, "A"] \\
s & t \arrow[l, "g"]
\end{tikzcd}$$

Proof.

Let A be a factor-α approx. alg. for Π_2.

Let I_1 be an instance of Π_1.

Set $I_2 := f(I_1)$, $t := A(I_2)$ and $s := g(I_1, t)$.

Then:

$$\mathrm{obj}_{\Pi_1}(I_1, s) \leq \mathrm{obj}_{\Pi_2}(I_2, t) \leq \alpha \cdot \mathrm{OPT}_{\Pi_2}(I_2) \leq \alpha \cdot \mathrm{OPT}_{\Pi_1}(I_1). \qquad \square$$

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Part III:
Reduction to MetricSteinerTree

MetricSteinerTree

Theorem. There is an approximation-preserving reduction from SteinerTree to MetricSteinerTree.

Proof. (1) Mapping f $\quad I_1 \xrightarrow{f} I_2$

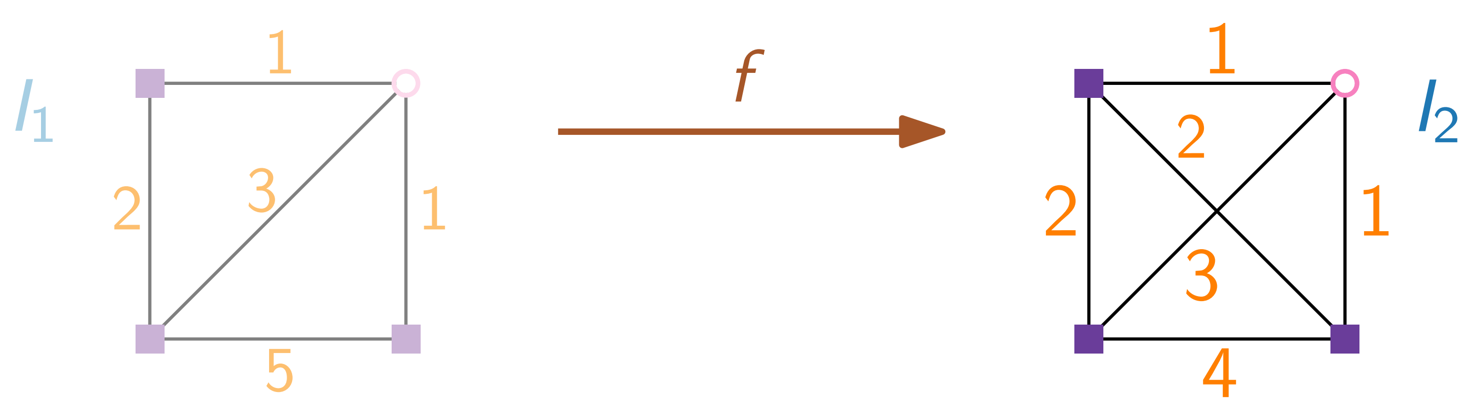
Instance I_1 of SteinerTree:

Graph $G_1 = (V, E_1)$, edge weights c_1, partition $V = T \mathbin{\dot\cup} S$

Metric instance $I_2 := f(I_1)$:

Complete graph $G_2 = (V, E_2)$, partition T, S as in I_1

$c_2(u, v) :=$ length of a shortest u–v path in G_1.



MetricSteinerTree

Theorem. There is an approximation-preserving reduction from SteinerTree to MetricSteinerTree.

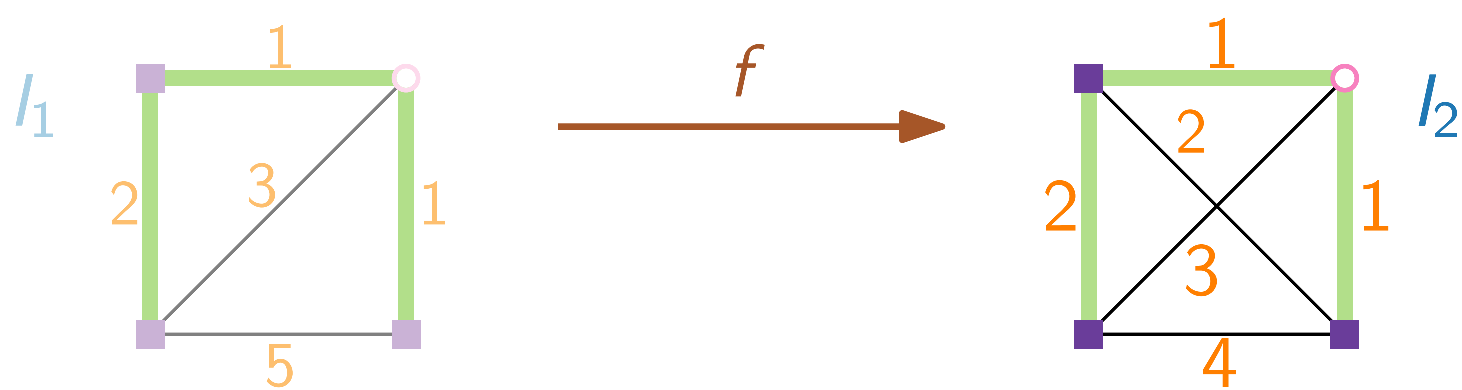
Proof. (2) $\mathrm{OPT}(I_2) \leq \mathrm{OPT}(I_1)$

Let B^* be an optimal Steiner tree for I_1.

Note that B^* is also a feasible solution for I_2:

$E_1 \subseteq E_2$ and the vertex sets V, T, S are the same.

$\mathrm{OPT}(I_2) \leq c_2(B^*) \leq c_1(B^*) = \mathrm{OPT}(I_1)$



MetricSteinerTree

Theorem. There is an approximation-preserving reduction from SteinerTree to MetricSteinerTree.

Proof. (3) Mapping g $\quad s \xleftarrow{g} t$

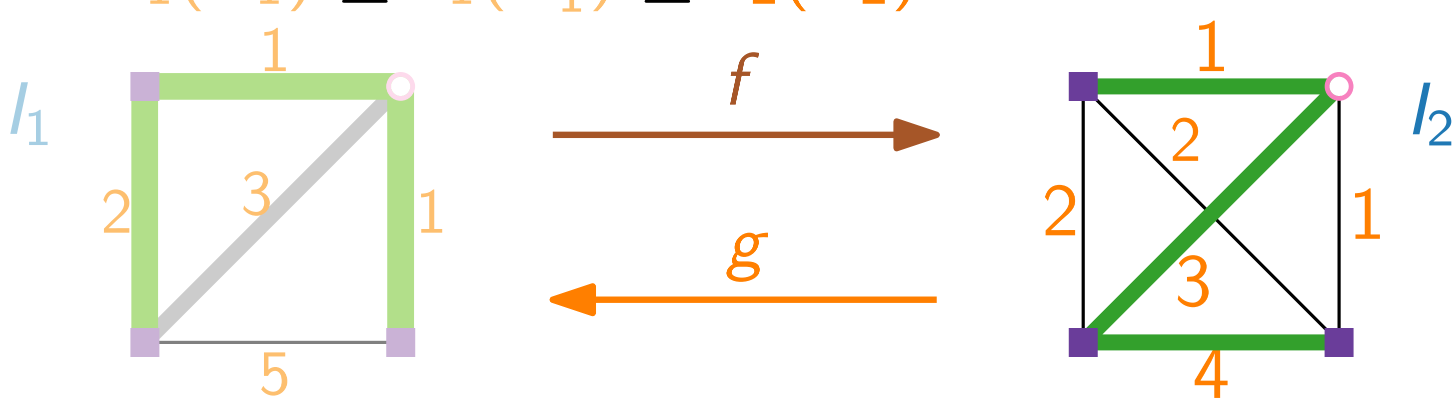
Let B_2 be a Steiner tree of G_2.

Construct $G_1' \subseteq G_1$ from B_2 by replacing each edge (u, v) of B_2 by a shortest u–v path in G_1. Keep ≤ 1 copy per edge.

$c_1(G_1') \leq c_2(B_2)$; G_1' connects all terminals ; maybe not a tree.

Consider spanning tree B_1 of G_1' $\rightsquigarrow$ Steiner tree B_1 of G_1

Note that $c_1(B_1) \leq c_1(G_1') \leq c_2(B_2)$.



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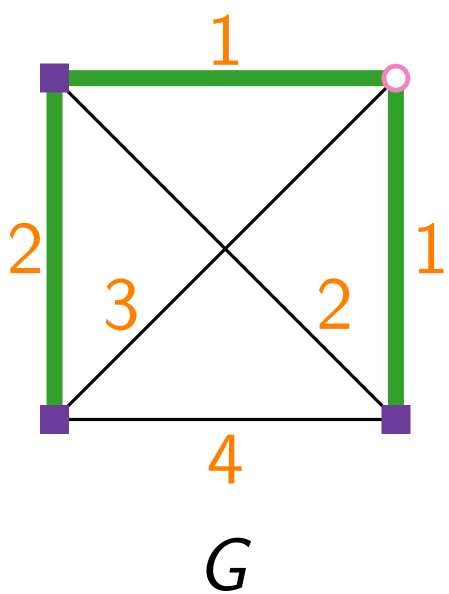
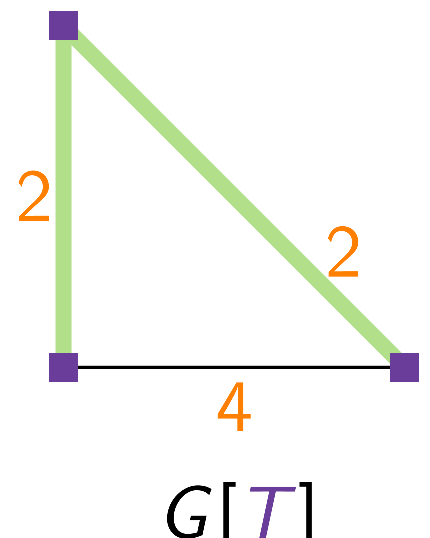
Part IV:
2-Approximation for MetricSteinerTree

2-Approximation for MetricSteinerTree

Theorem. For an instance of MetricSteinerTree, let B be a minimum spanning tree (MST) of the subgraph $G[T]$ induced by the terminal set T. Then $c(B) \leq 2 \cdot \mathrm{OPT}$.



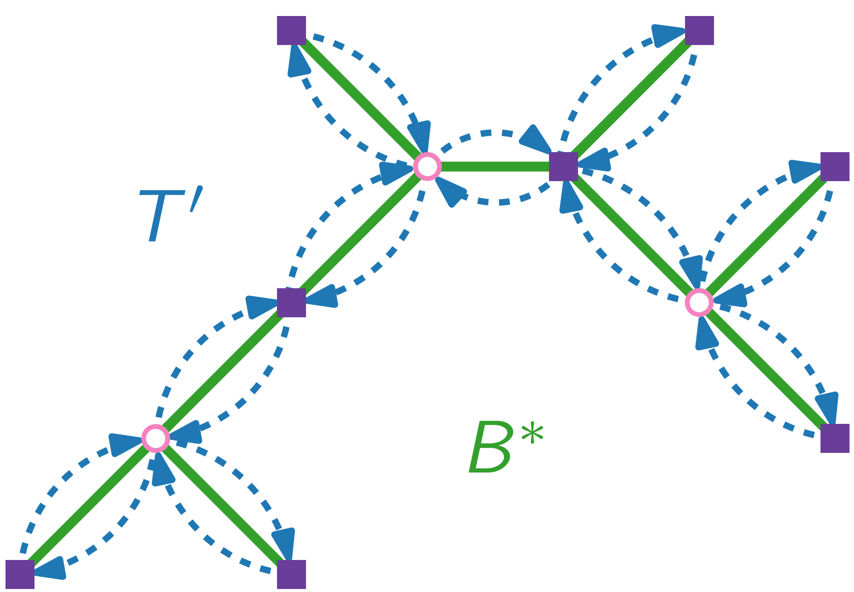


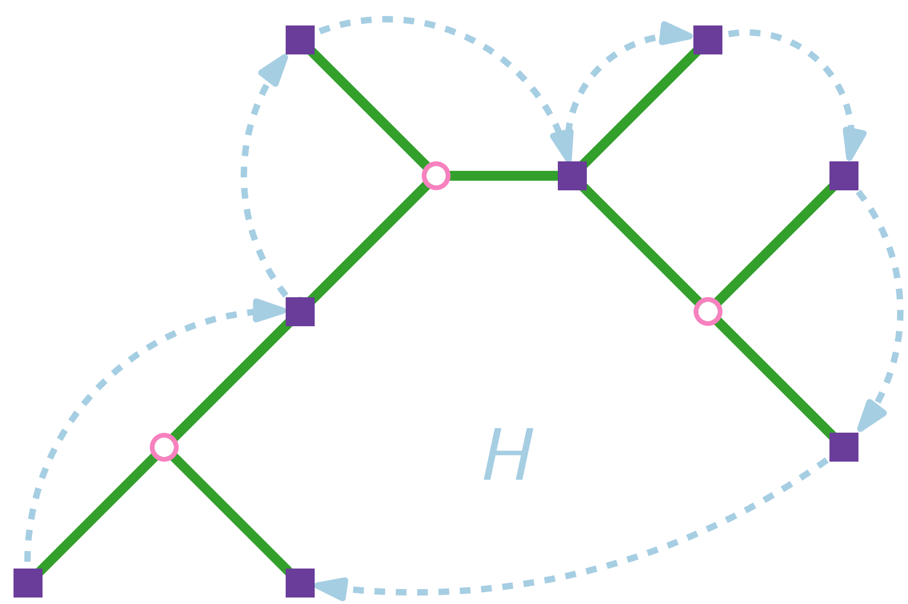
Proof of the Approximation Factor

Consider an optimal Steiner tree B^*.
Duplicate all edges of B^*.
$\Rightarrow$ Eulerian (multi-)graph B' with cost $c(B') = 2 \cdot \mathrm{OPT}$.

Find a Eulerian tour T' in B' $\Rightarrow c(T') = c(B') = 2 \cdot \mathrm{OPT}$
Find a Hamiltonian path H in $G[T]$ by "short-cutting" Steiner vertices and previously visited terminals.
$\Rightarrow c(H) \le c(T') = 2 \cdot \mathrm{OPT}$ since G is metric.

MST B of $G[T]$ costs $c(B) \le c(H) \le 2 \cdot \mathrm{OPT}$
since H is a spanning tree of $G[T]$.

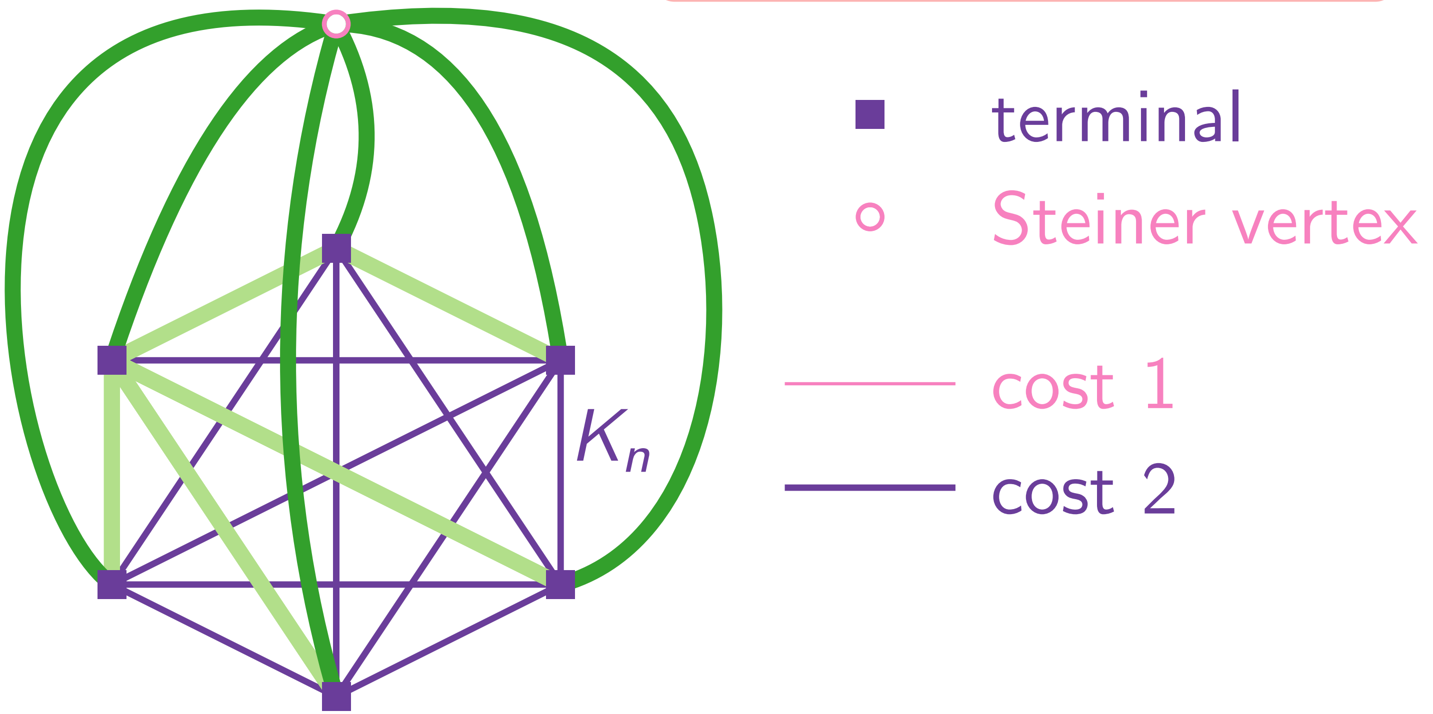




Analysis Tight?

An MST of $G[T]$ has cost $2(n-1)$.
The optimal solution has cost n.

$$\frac{\text{ALG}}{\text{OPT}} = \frac{2(n-1)}{n} \to 2$$



Can we do better?

The best known approximation factor for SteinerTree is $\ln(4) + \varepsilon \approx 1.39$.

[Byrka, Grandoni, Rothvoß & Sanità, J. ACM'13]

SteinerTree cannot be approximated within factor $\frac{96}{95} \approx 1.0105$ (unless $\mathrm{P} = \mathrm{NP}$).

[Chlebík & Chlebíková, TCS'08]

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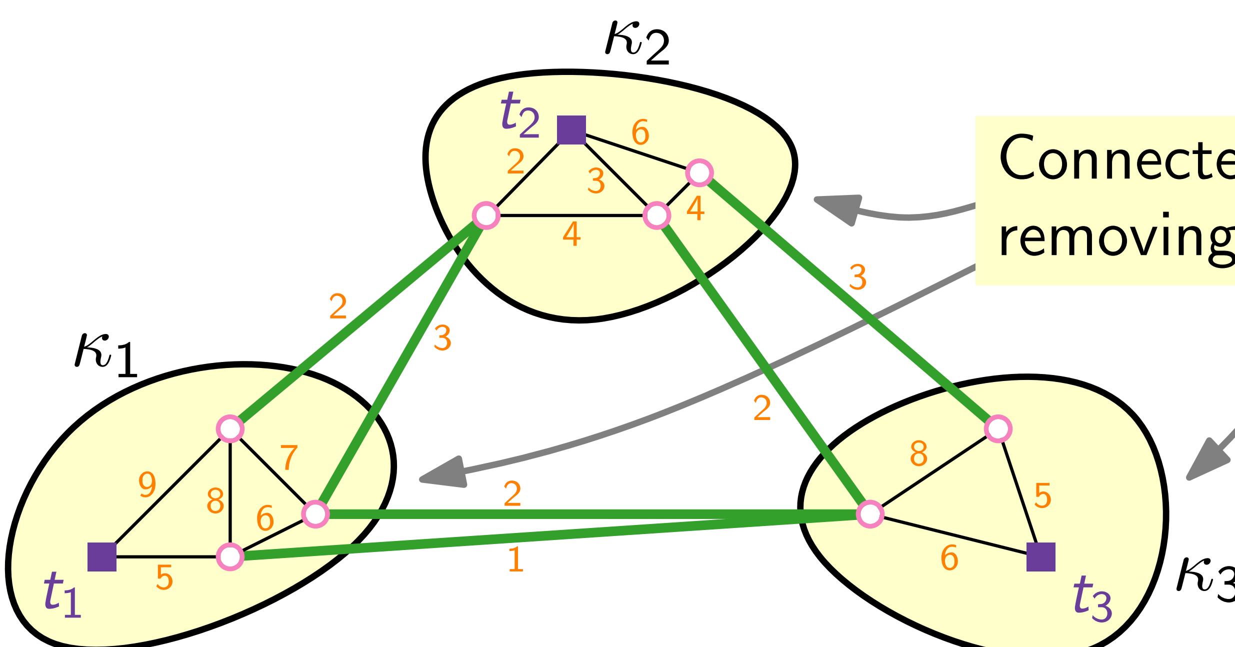
Part V: MultiwayCut

MultiwayCut

Given: A connected graph G with edge costs $c\colon E(G) \to \mathbb{Q}^+$ and a set $T = \{t_1, \ldots, t_k\} \subseteq V(G)$ of **terminals**.

A **multiway cut** of T is a subset E' of edges such that no two terminals in the graph $(V(G), E(G) - E')$ are connected.

Find: A minimum-cost multiway cut of T.



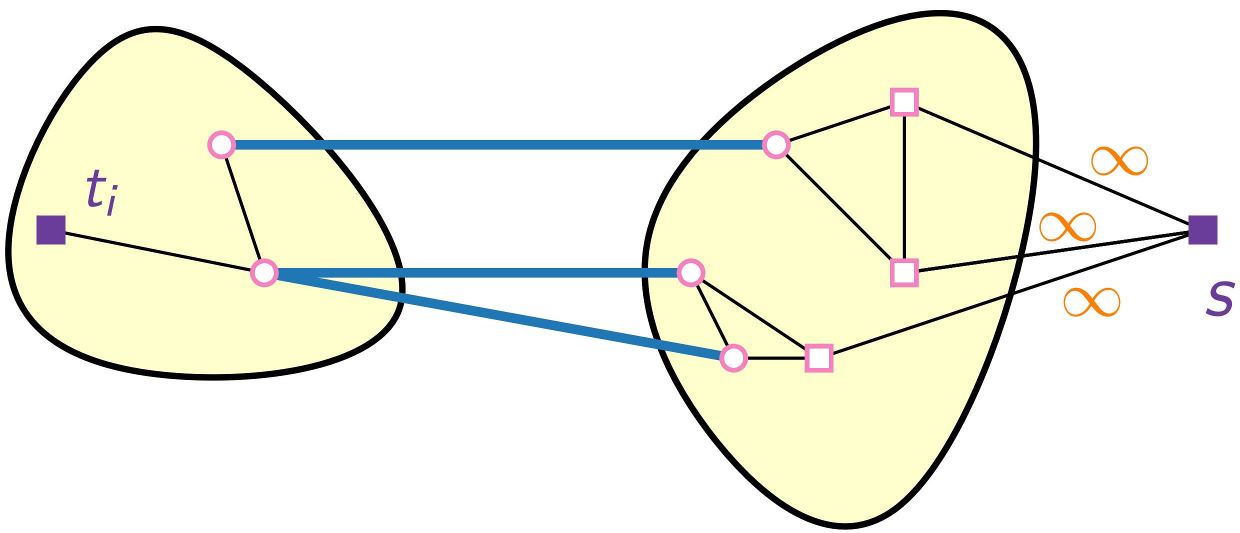
Special cases:

$k = 2$:	Min s–t cut
$k \geq 3$:	NP-hard

Isolating Cuts

An **isolating cut** for a terminal t_i is a set of edges that disconnects t_i from all other terminals.

A minimum-cost isolating cut for t_i can be computed efficiently:



Add dummy terminal s and find a minimum-cost s–t_i cut.

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Part VI: Algorithm for MultiwayCut

Algorithm MultiwayCut

For $i = 1, \ldots, k$:

- Compute a minimum-cost isolating cut C_i for t_i.
- Return the union $\mathcal{C}$ of the $k-1$ cheapest such isolating cuts.

In other words:
Ignore the most expensive one of the isolating cuts $C_1, \ldots, C_k$.

$$\Rightarrow c(\mathcal{C}) \le \left(1 - \frac{1}{k}\right) \sum_{i=1}^{k} c(C_i) \text{ because:}$$

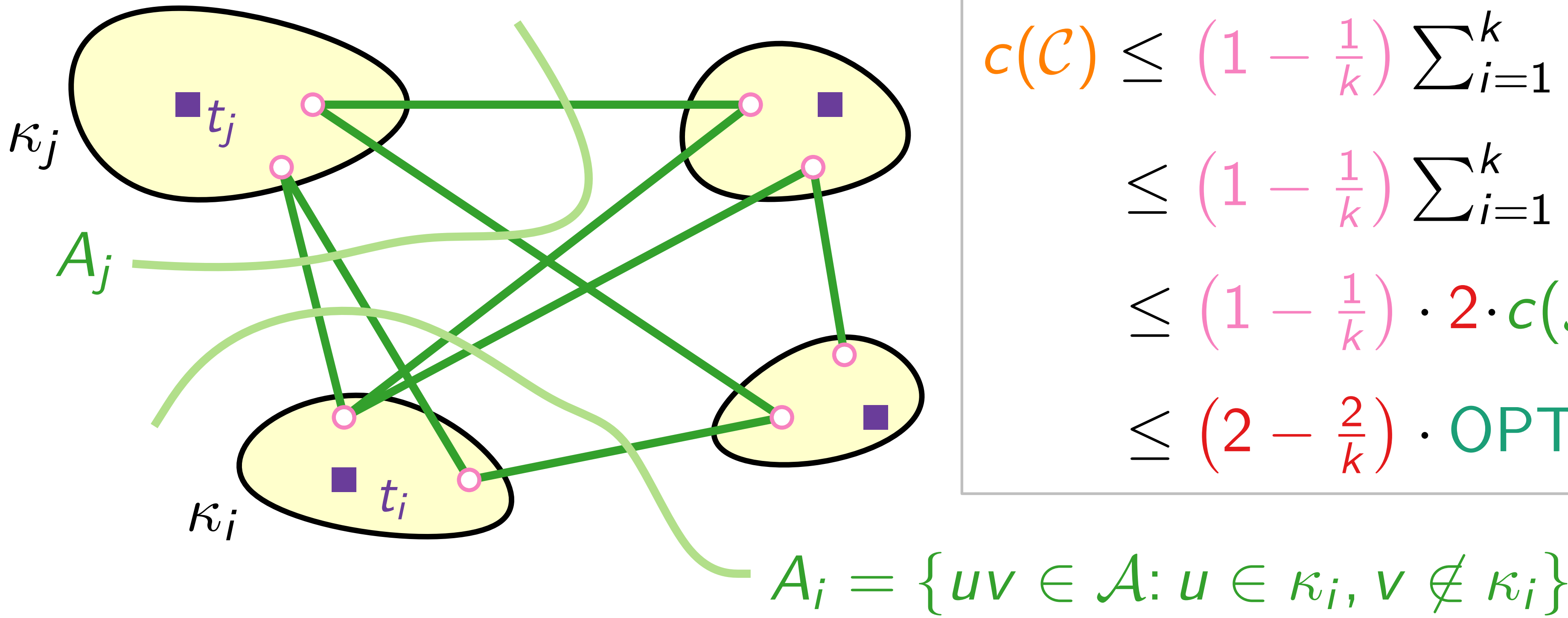
for the most expensive cut of $C_1, \ldots, C_k$, say C_1, we have

$$c(C_1) \ge \frac{1}{k} \sum_{i=1}^{k} c(C_i) \text{ by the pidgeon-hole principle.}$$

Approximation Factor

Theorem. This algorithm is a factor-$(2 - 2/k)$ approximation algorithm for MultiwayCut.

Proof. Consider an opt. multiway cut $\mathcal{A}$:



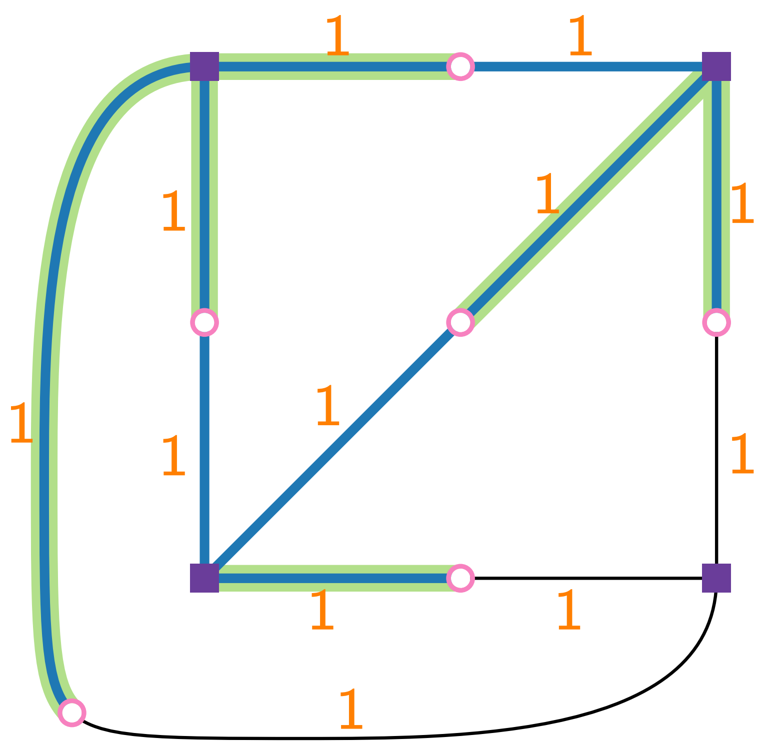
Consider the alg.'s solution $\mathcal{C}$:

$$\begin{aligned} c(\mathcal{C}) &\le \left(1 - \tfrac{1}{k}\right) \textstyle\sum_{i=1}^k c(C_i) \\ &\le \left(1 - \tfrac{1}{k}\right) \textstyle\sum_{i=1}^k c(A_i) \\ &\le \left(1 - \tfrac{1}{k}\right) \cdot 2 \cdot c(\mathcal{A}) \\ &\le \left(2 - \tfrac{2}{k}\right) \cdot \mathrm{OPT} \qquad \square \end{aligned}$$

Observation. $\mathcal{A} = \bigcup_{i=1}^{k} A_i$ and $\sum_{i=1}^k c(A_i) = 2 \cdot c(\mathcal{A}) = 2 \cdot \mathrm{OPT}$.

Analysis Tight?

K_k



$\text{ALG} = (k-1)(k-1)$

$\text{OPT} = \sum_{i=1}^{k-1} i = \frac{k\cdot(k-1)}{2}$

$\text{ALG}/\text{OPT} = \frac{2(k-1)}{k} = 2 - \frac{2}{k}$

Can we do better?

The best known approximation factor for MultiwayCut is $1.2965 - \frac{1}{k}$.
[Sharma & Vondrák, STOC'14]

MultiwayCut cannot be approximated within factor $1.20016 - O(1/k)$ (unless $\text{P} = \text{NP}$). [Bérczi, Chandrasekaran, Király & Madan, MP'18]