Approximation Algorithms

Lecture 3: SteinerTree and MultiwayCut

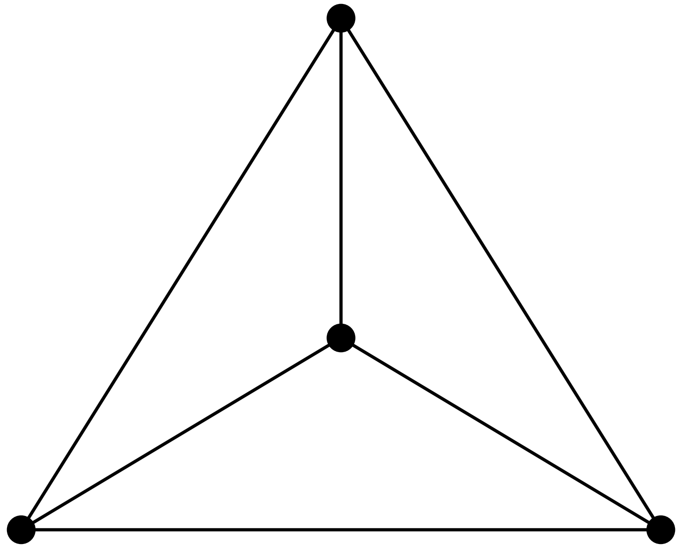
Part I: SteinerTree

Alexander Wolff

Winter term 2025

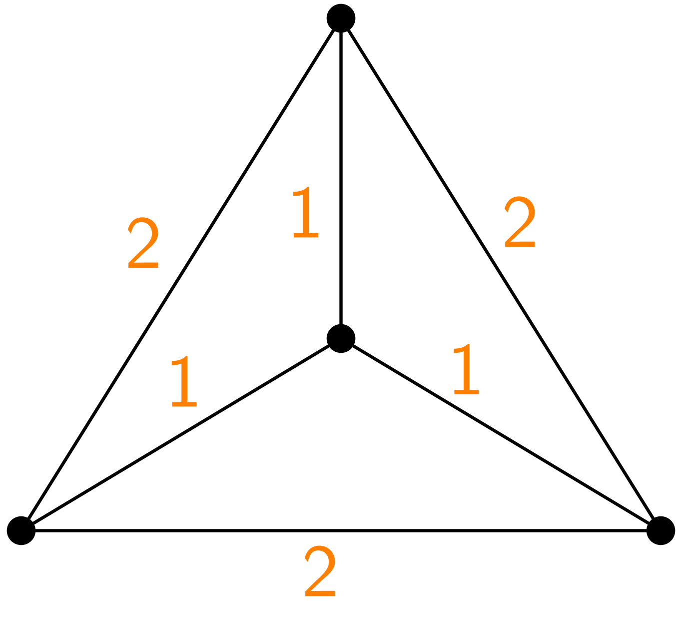
SteinerTree

Given: A graph G



SteinerTree

Given: A graph G with edge weights $c\colon E(G) \to \mathbb{Q}^+$



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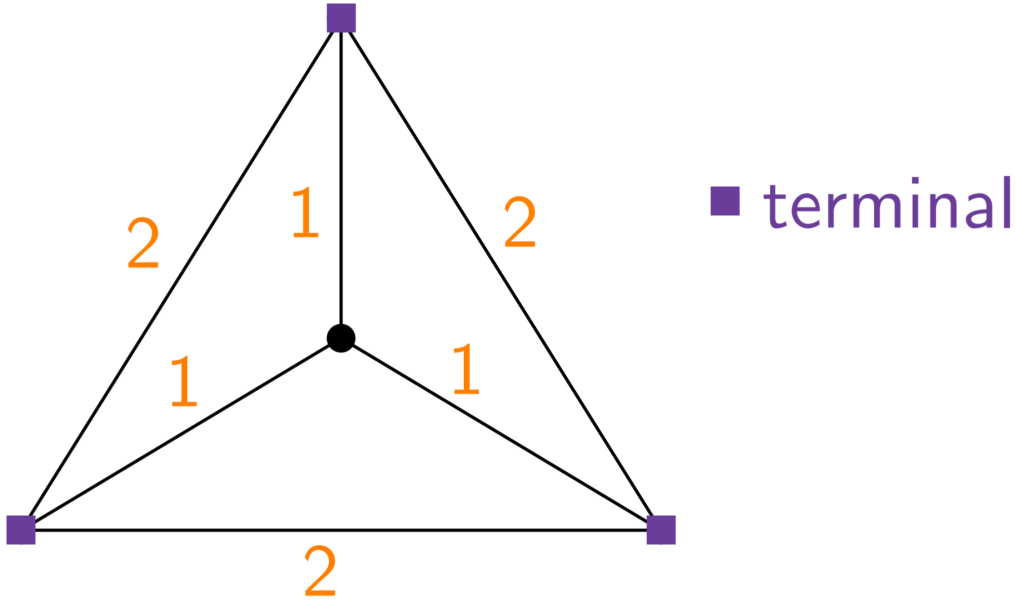

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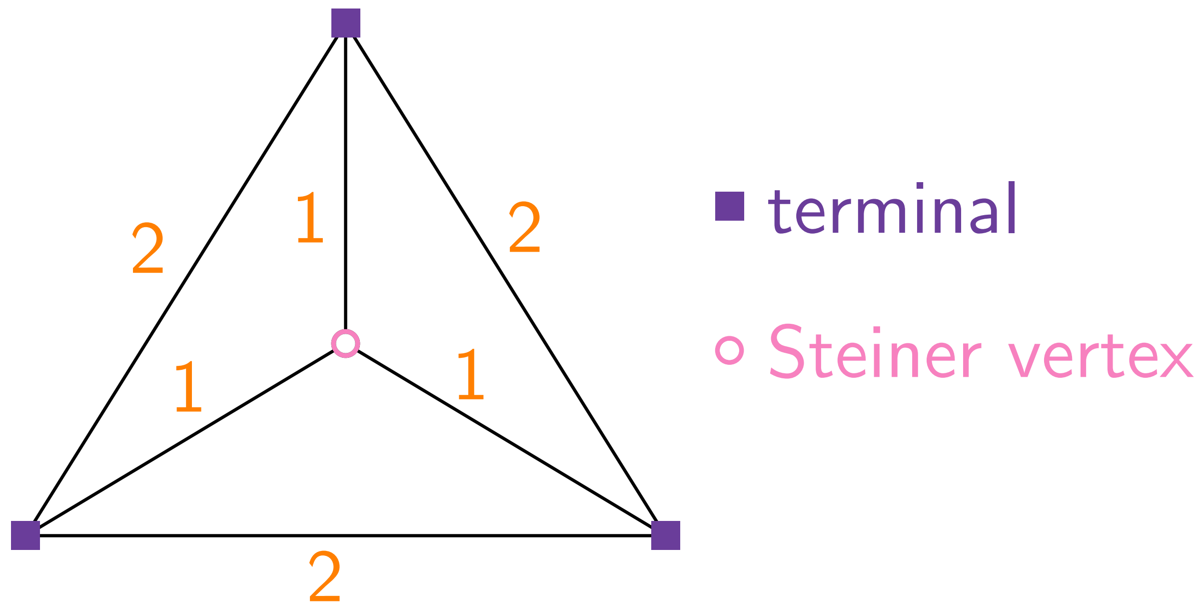

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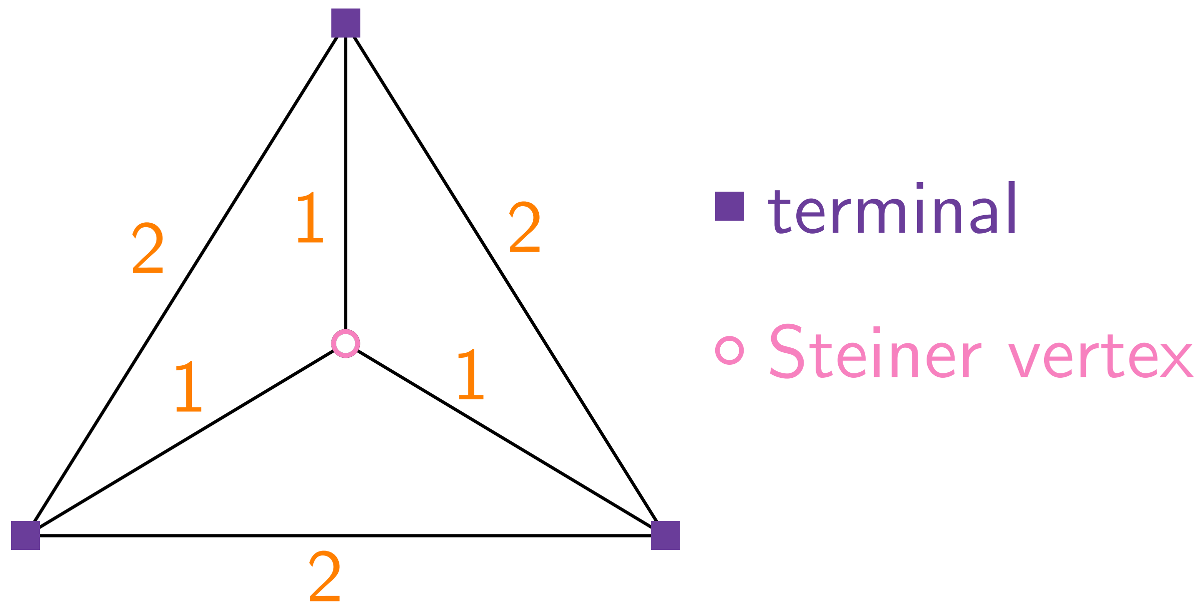

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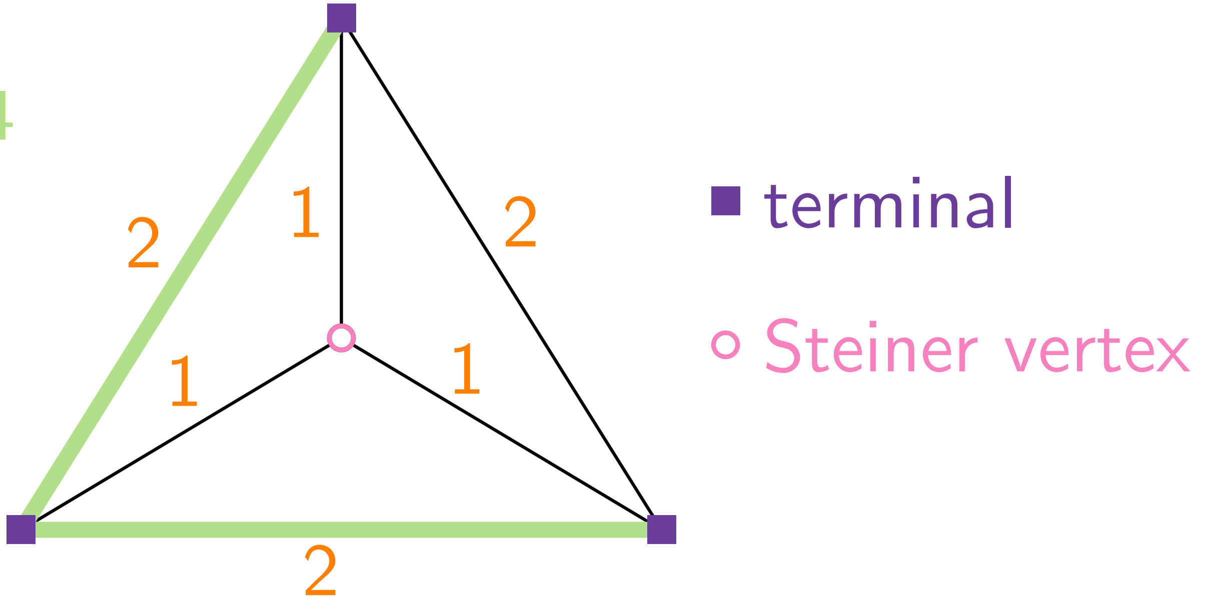

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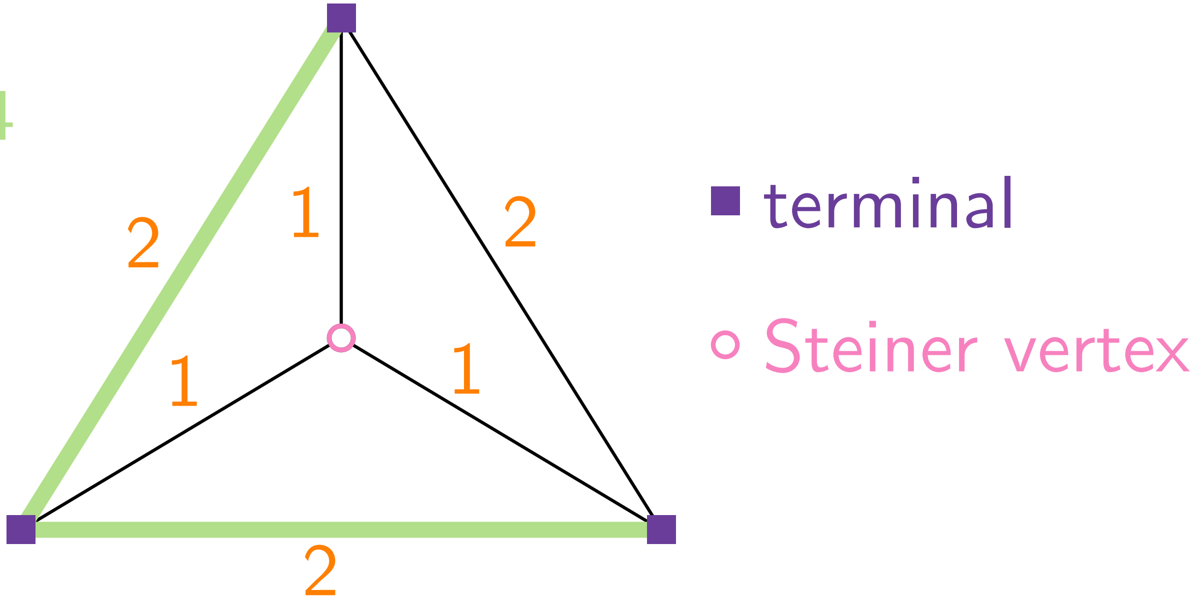

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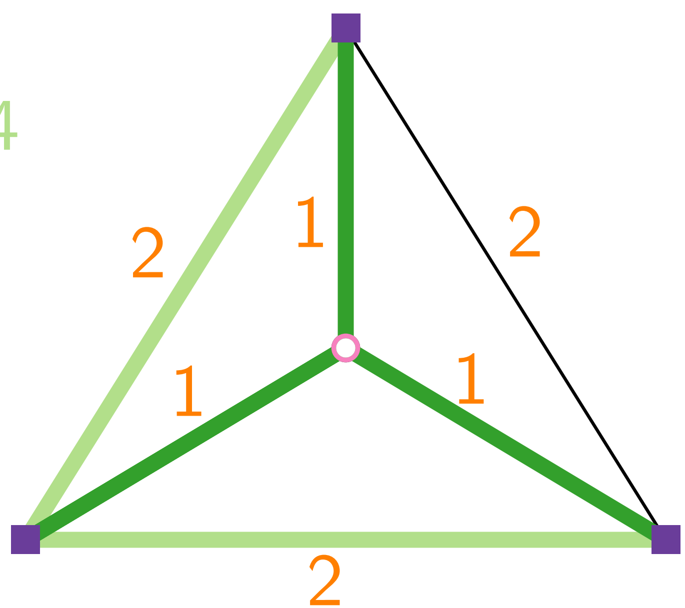

■ terminal

○ Steiner vertex

MetricSteinerTree

Restriction of SteinerTree where the graph G is complete and the cost function is **metric**

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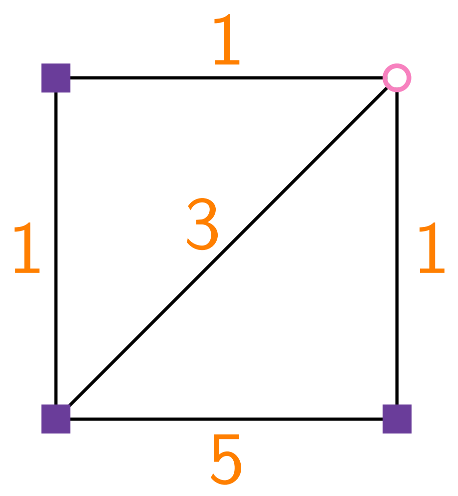

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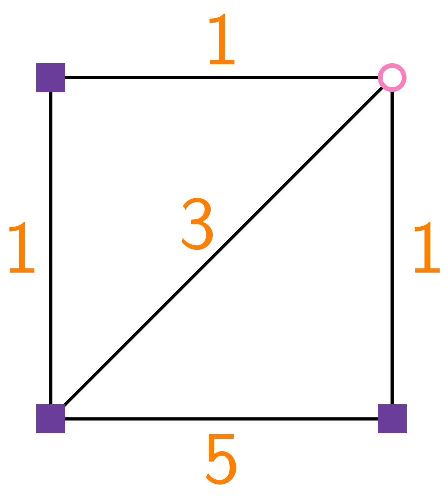

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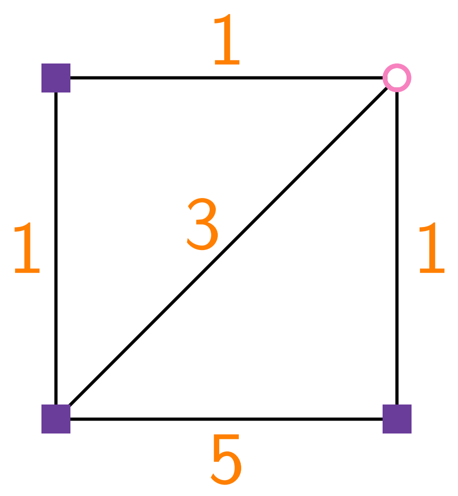

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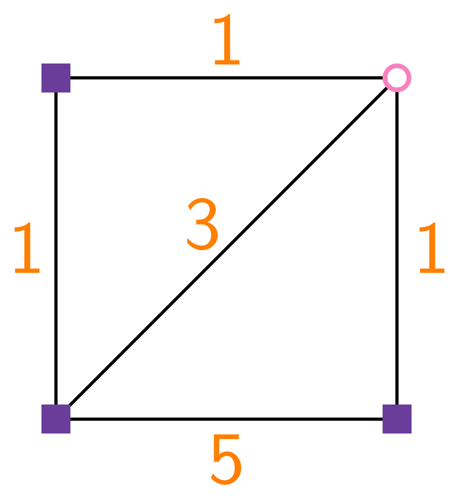

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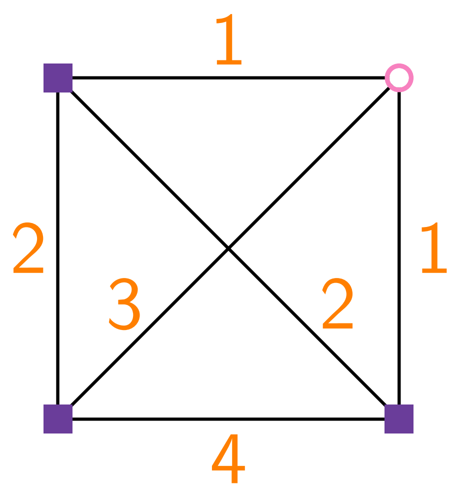

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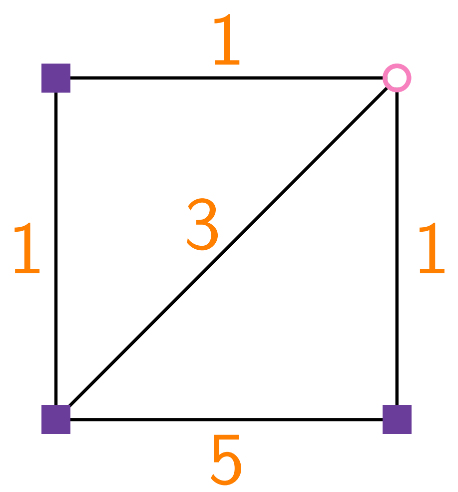

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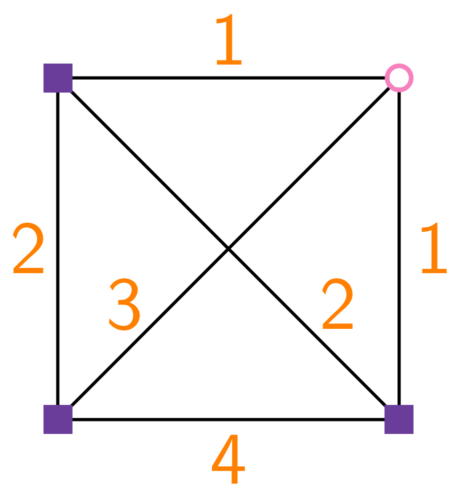

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Approximation Algorithms

Lecture 3: SteinerTree and MultiwayCut

Part II: Approximation-Preserving Reduction

Approximation-Preserving Reduction

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problems Π_1 Π_2

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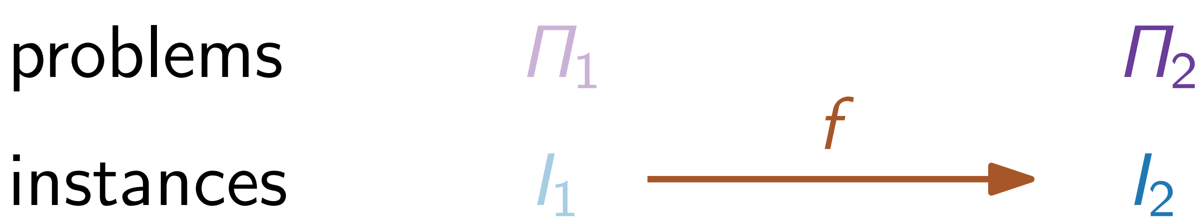

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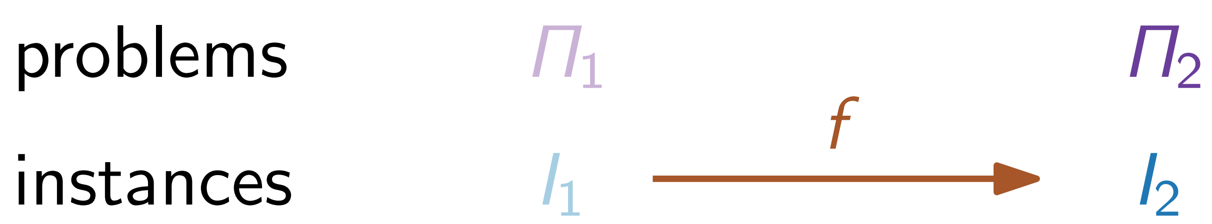

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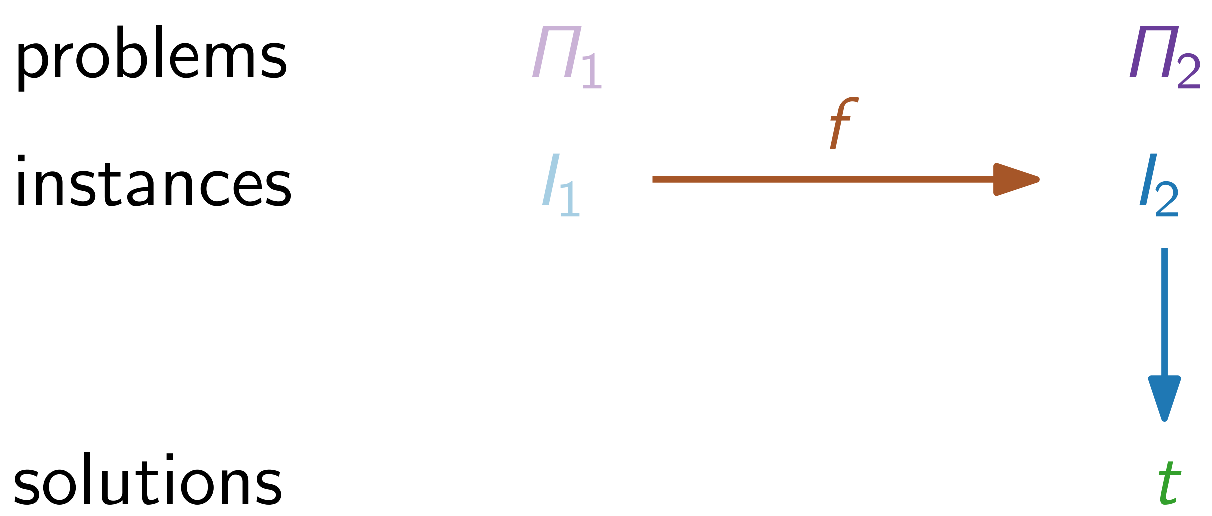

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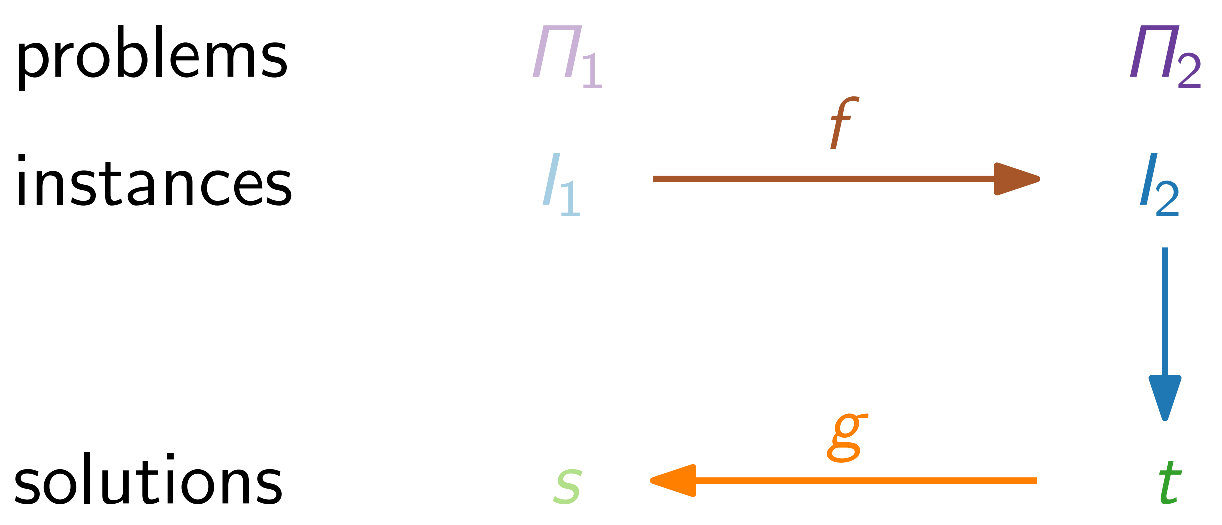

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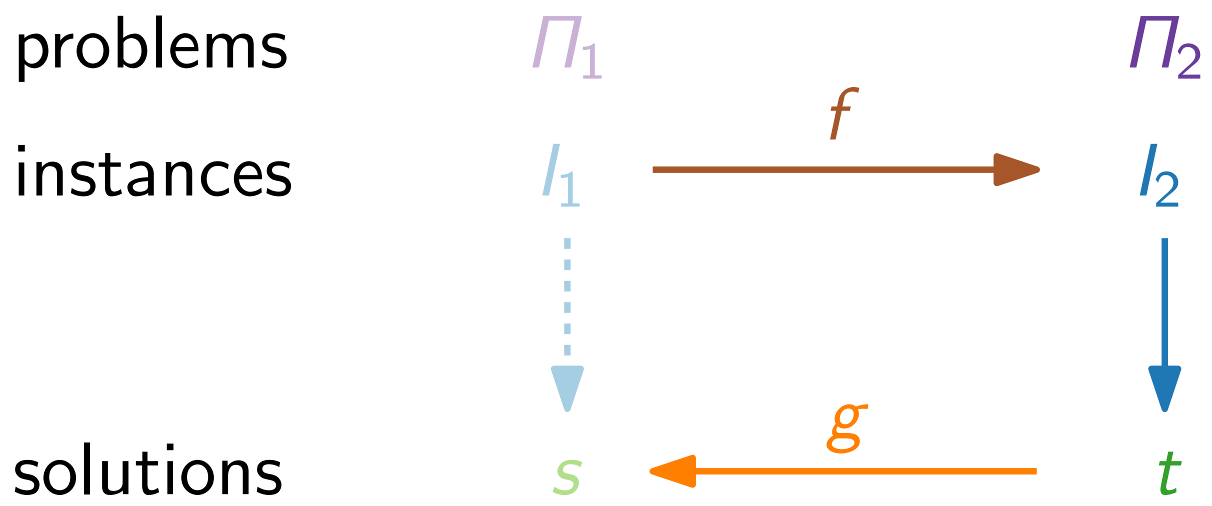

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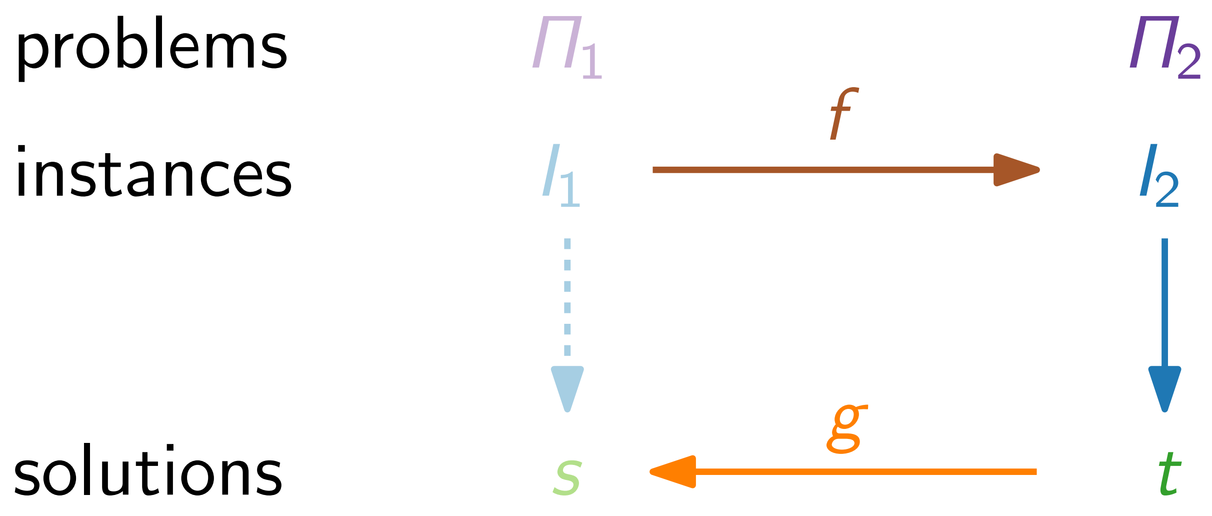

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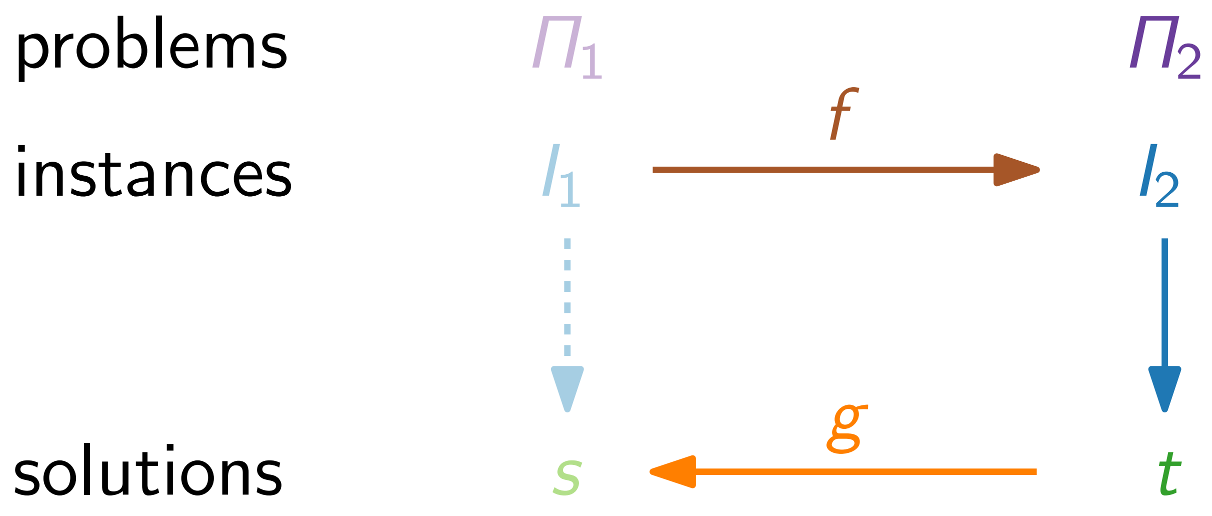
Theorem. Let Π_1, Π_2 be minimization problems with an approximation-preserving reduction (f, g) from Π_1 to Π_2. If there is a factor-α approximation algorithm for Π_2, then there is a factor-? approximation algorithm for Π_1.



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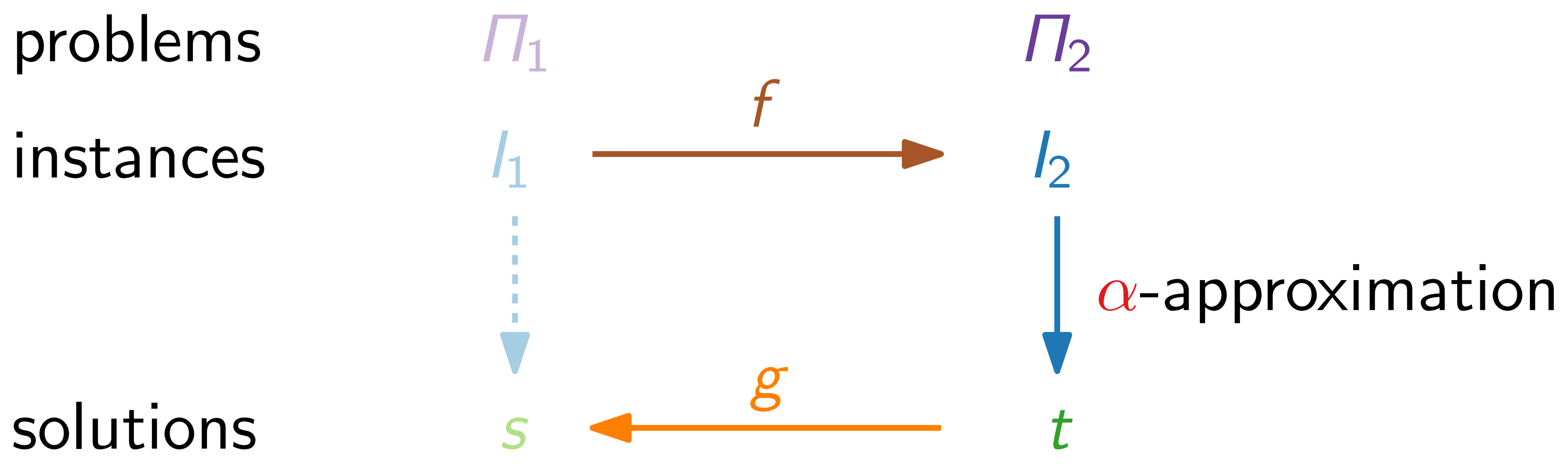

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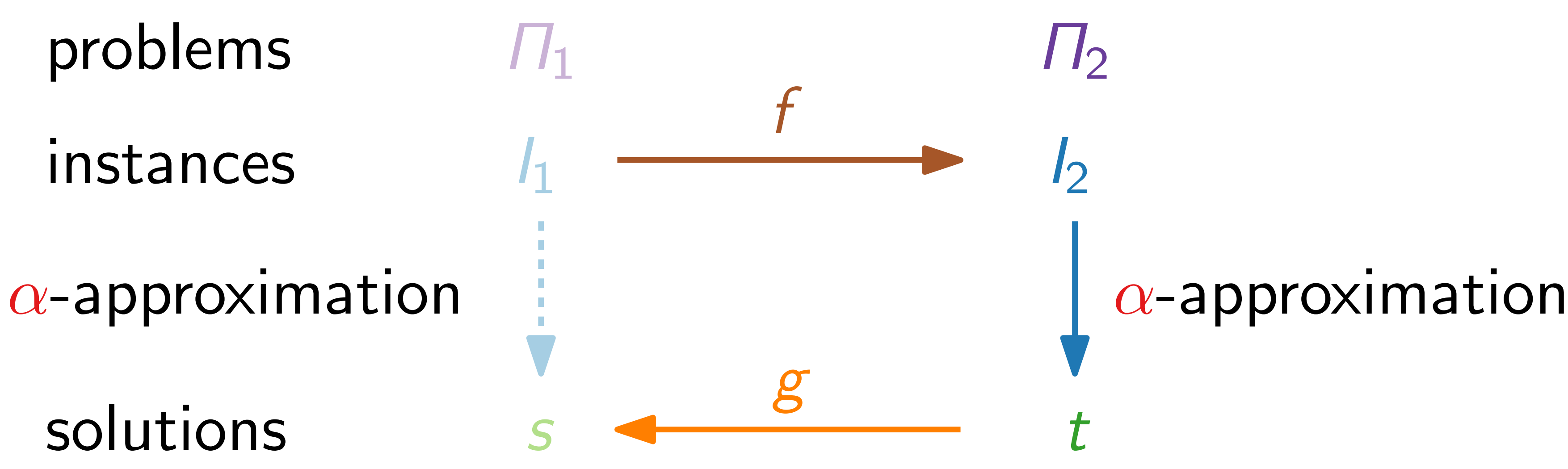

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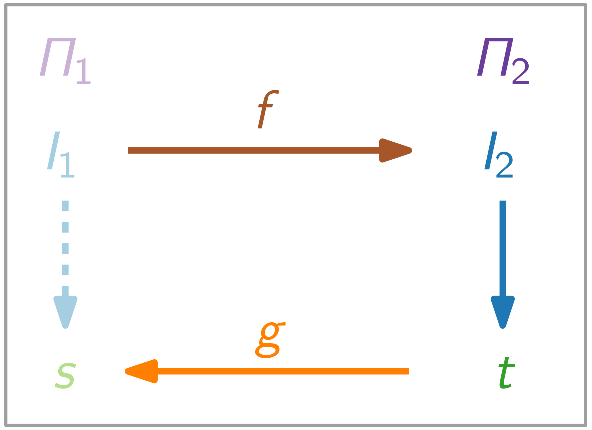

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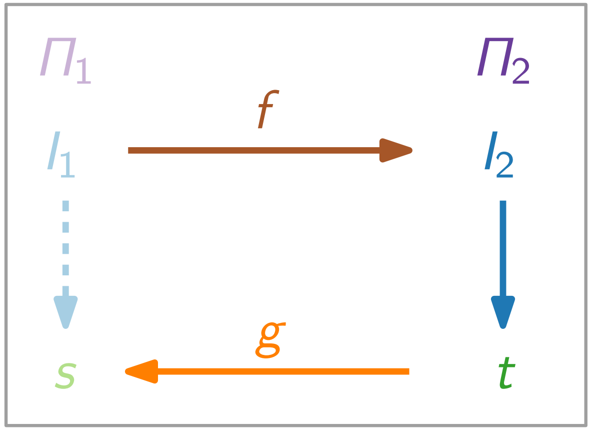

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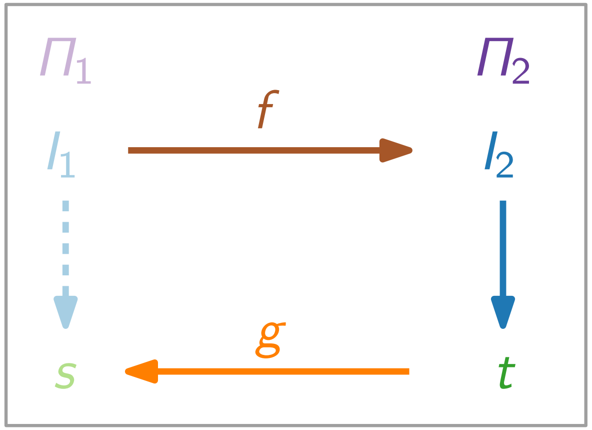

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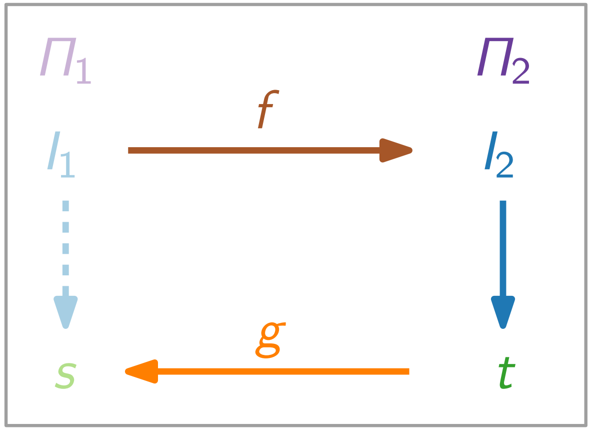

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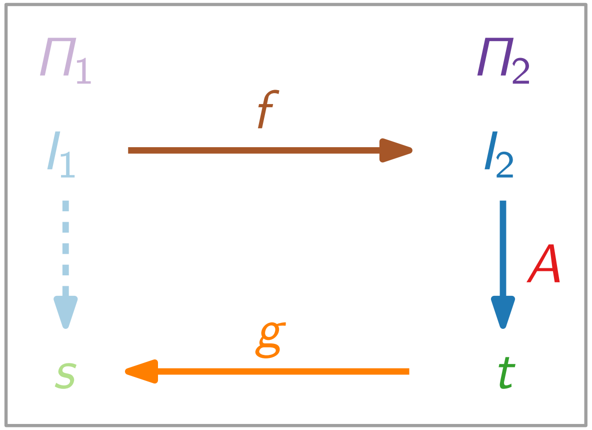

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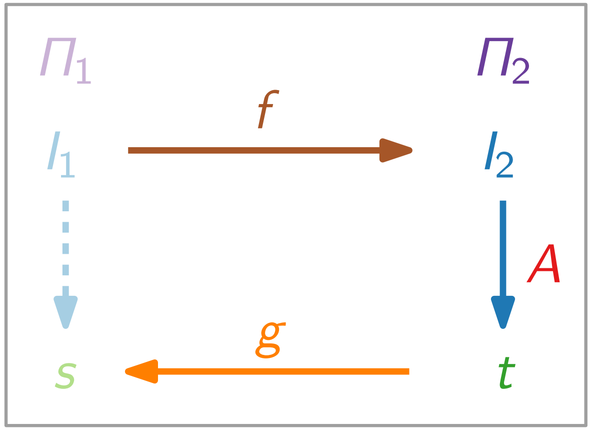

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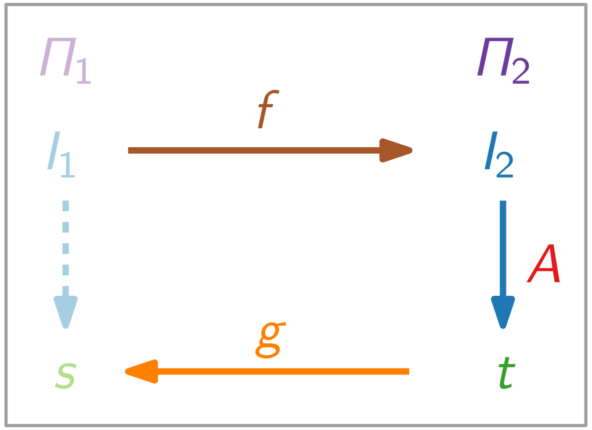

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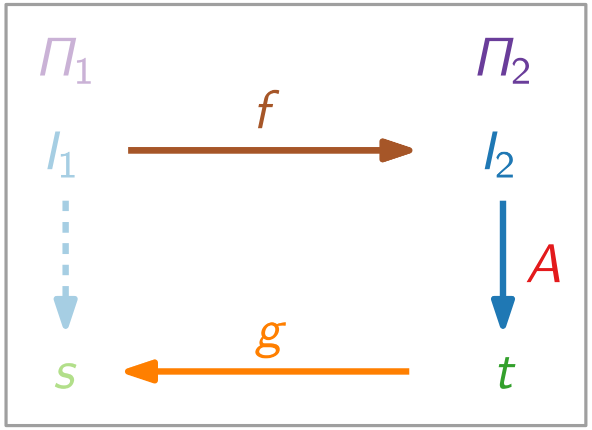

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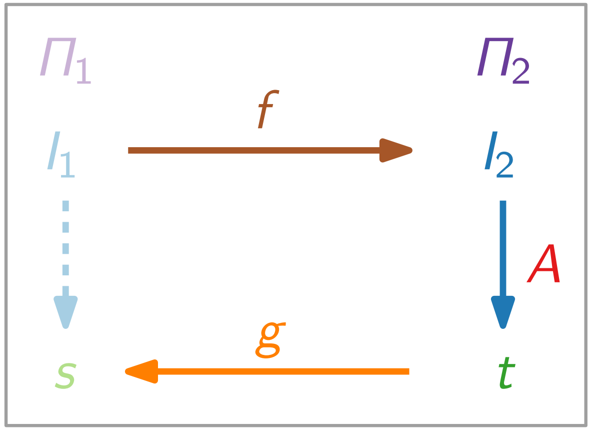

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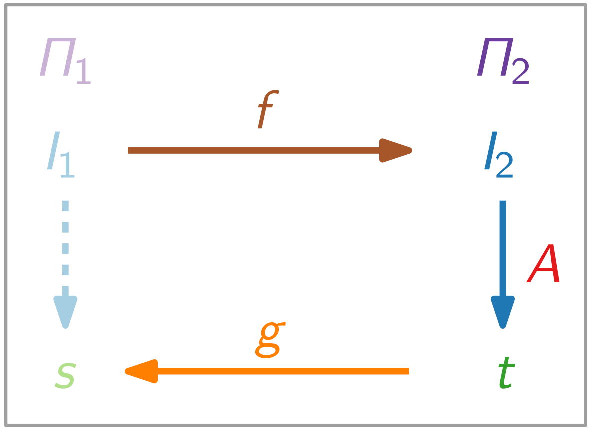

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$\square$

Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part III:
Reduction to MetricSteinerTree

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Graph $G_1 = (V, E_1)$, edge weights c_1, partition $V = T \dot\cup S$

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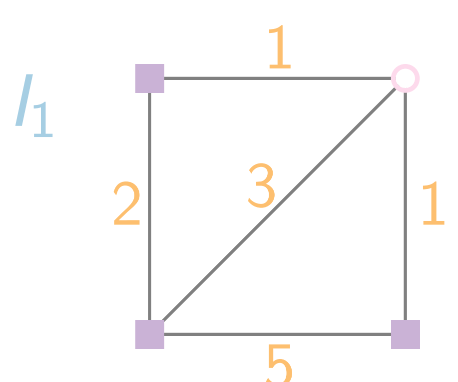

MetricSteinerTree

Theorem. There is an approximation-preserving reduction from SteinerTree to MetricSteinerTree.

Proof. (1) Mapping f

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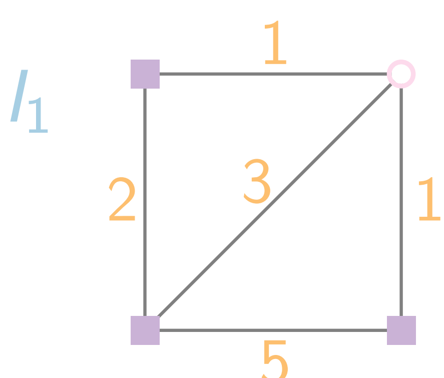

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Proof. (1) Mapping f $\quad I_1 \xrightarrow{f} I_2$

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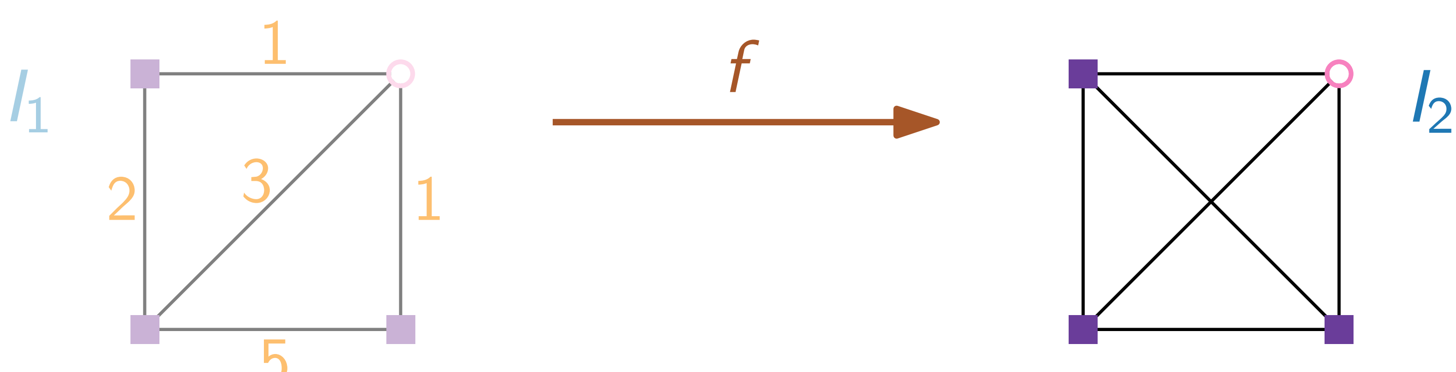

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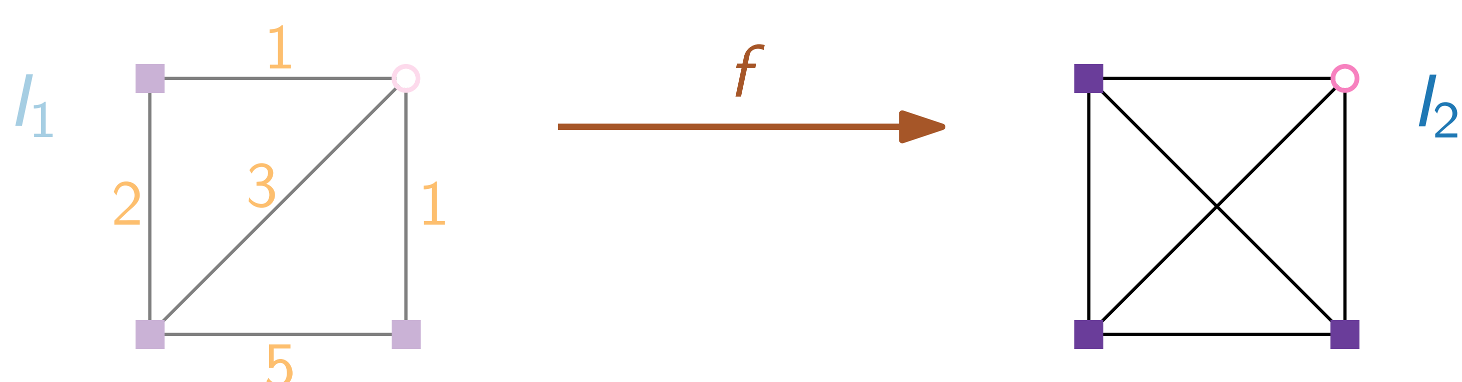

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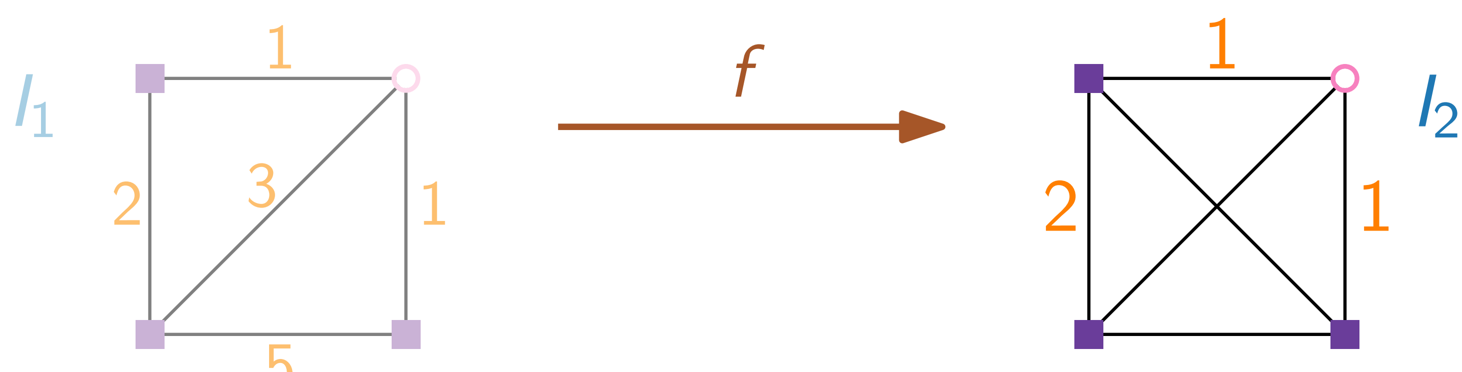

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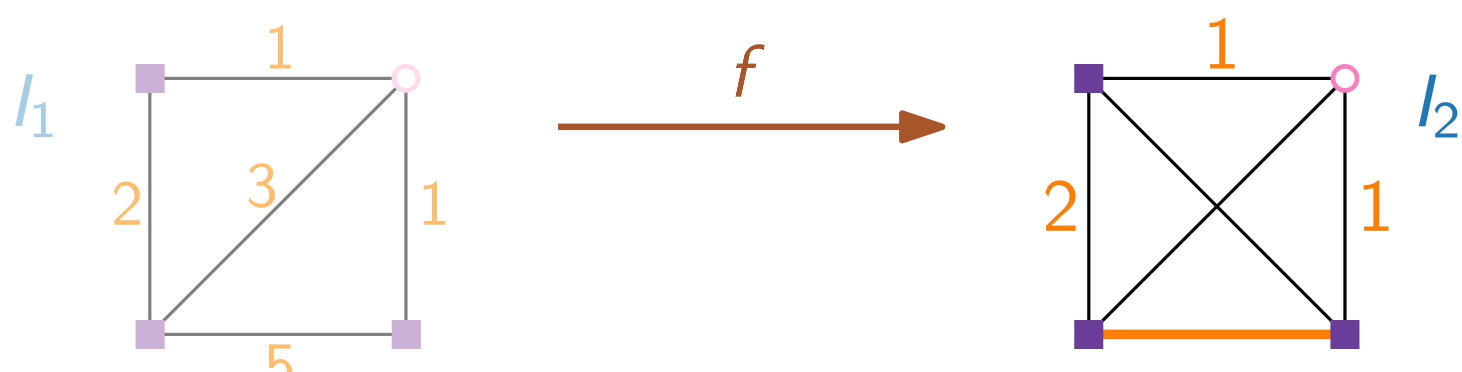

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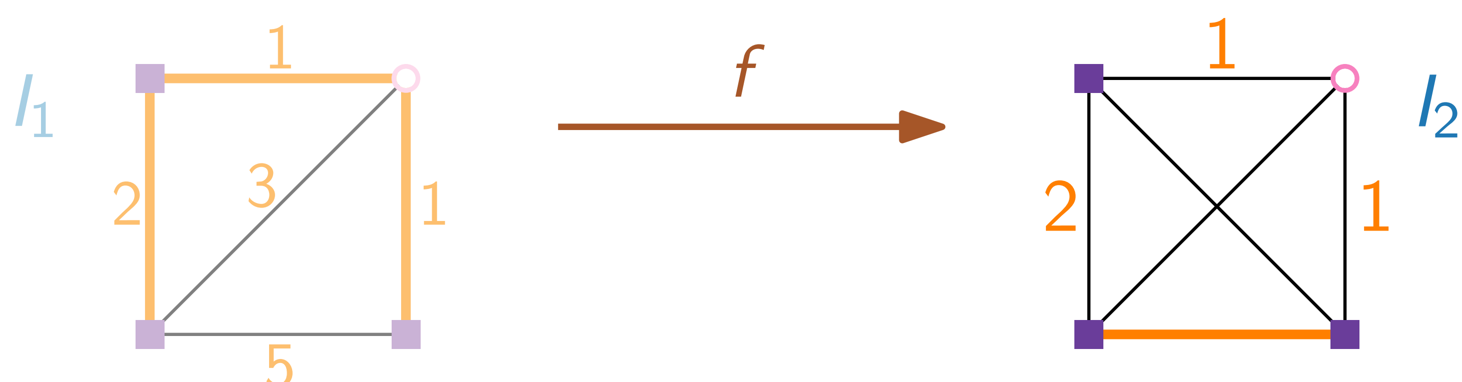

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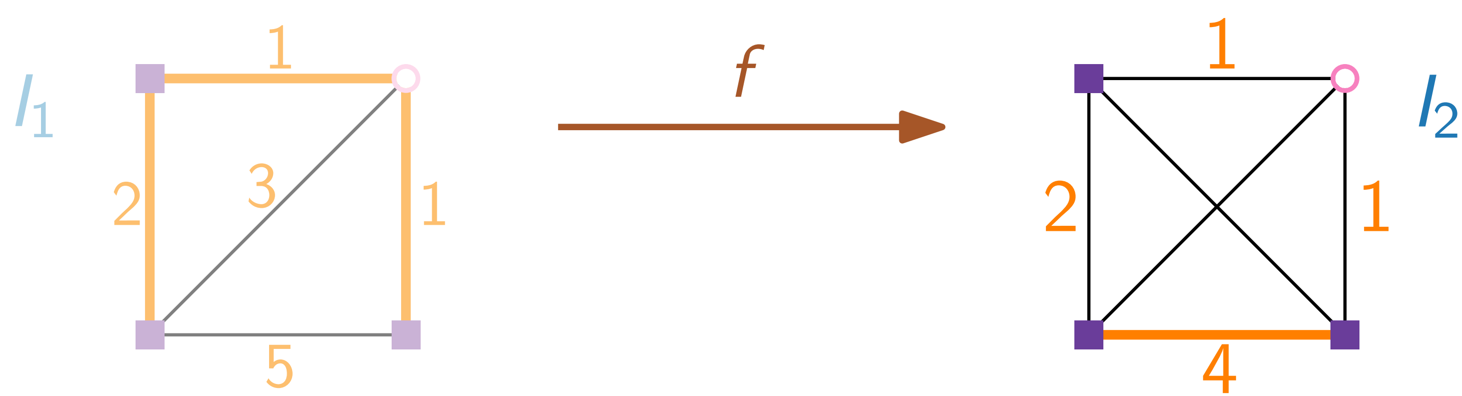

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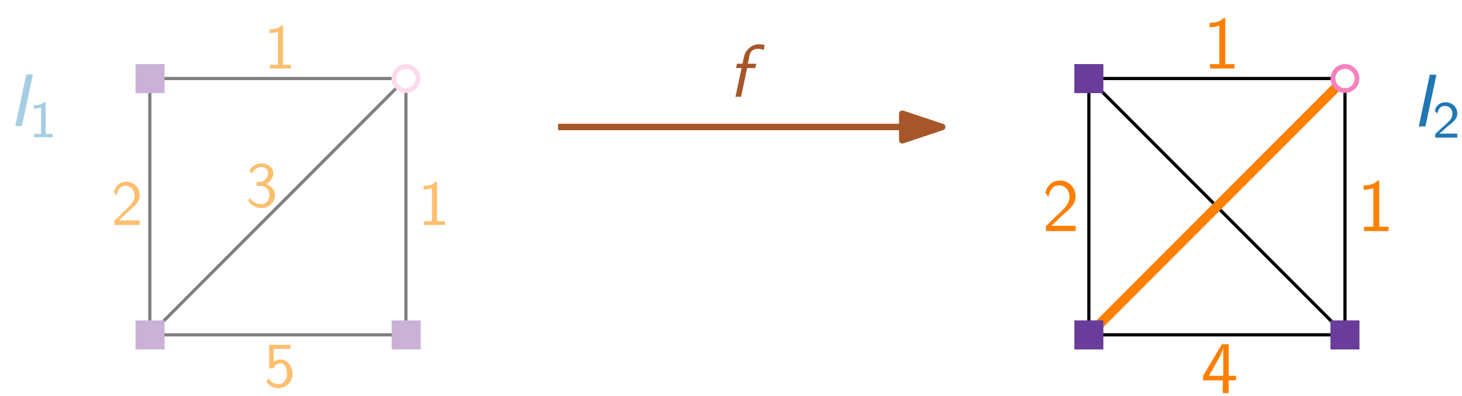

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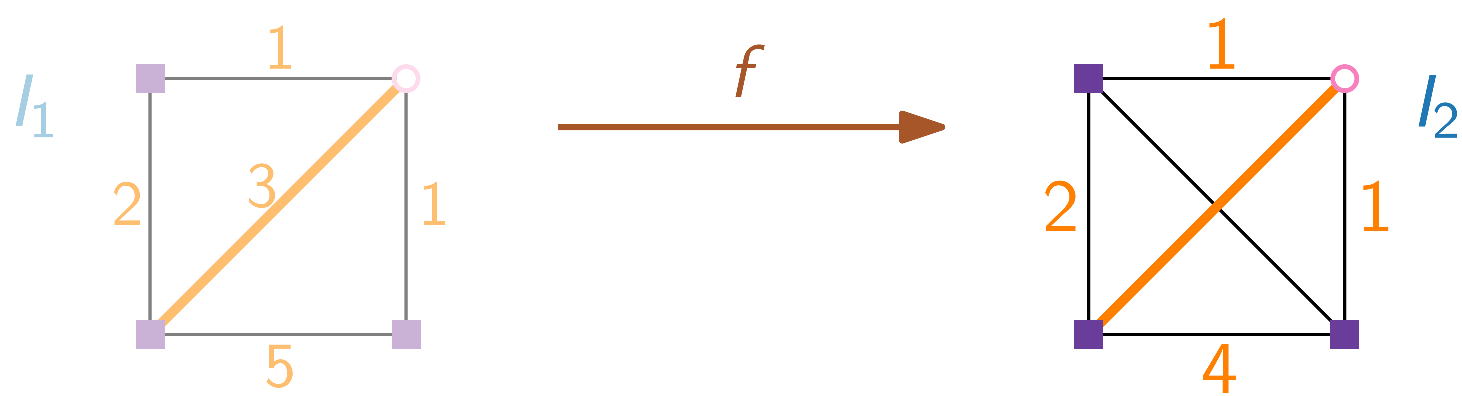

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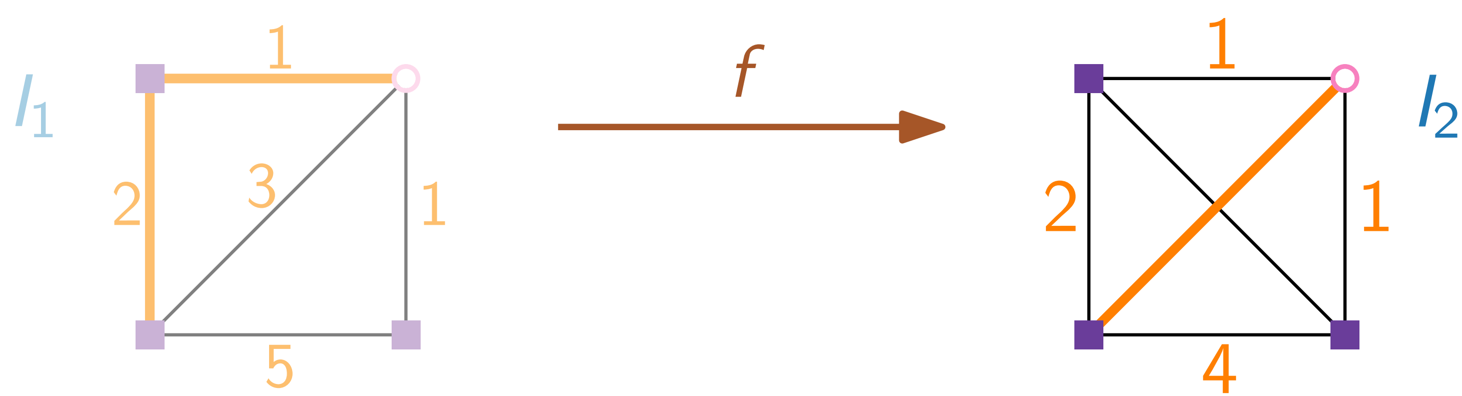

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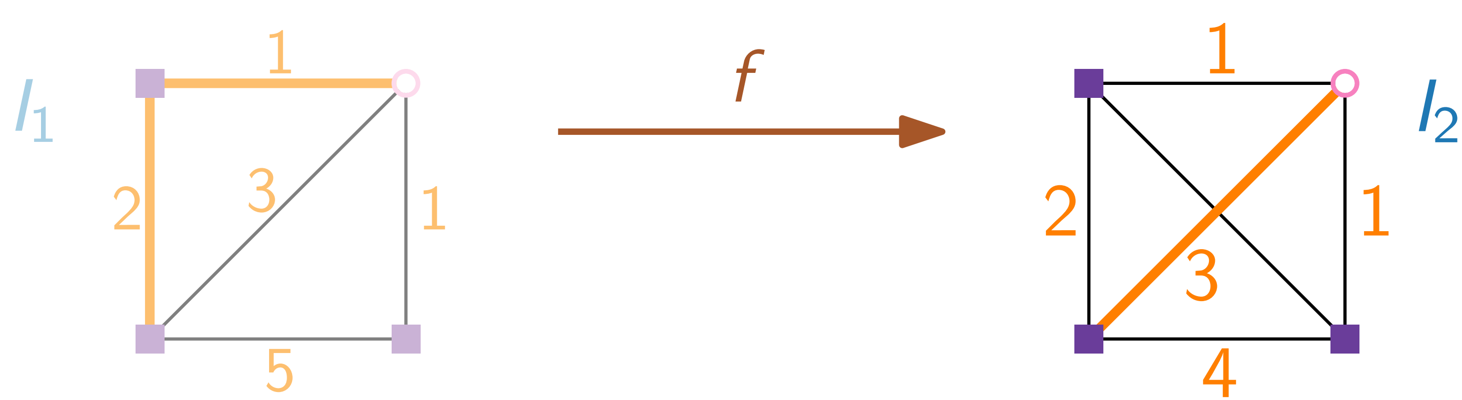

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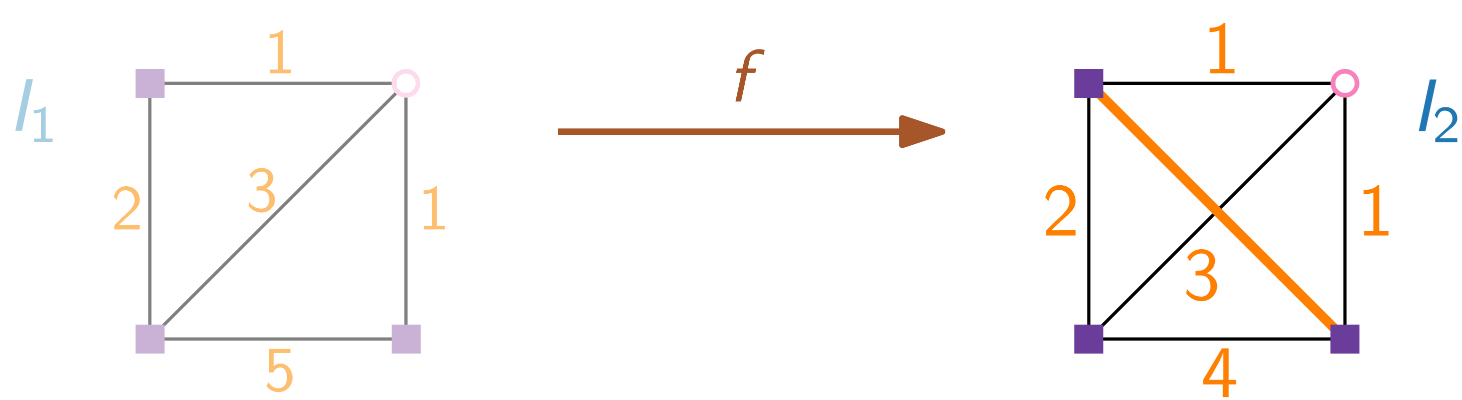

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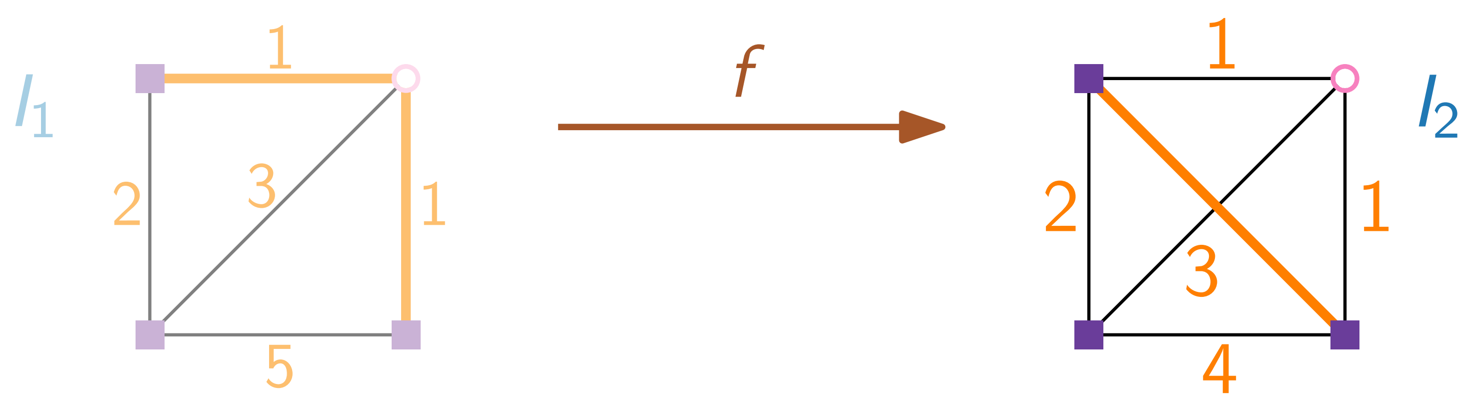

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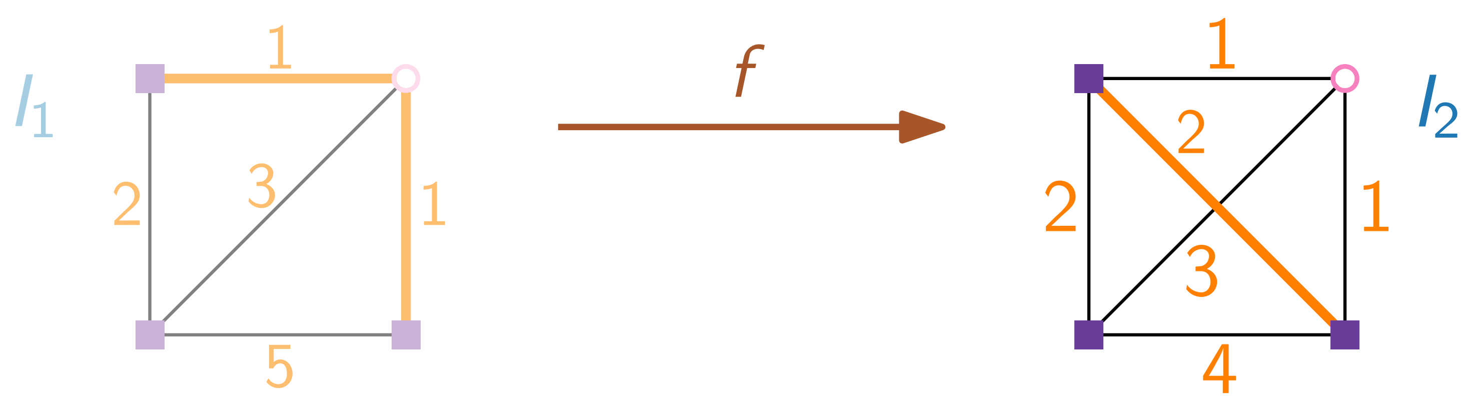

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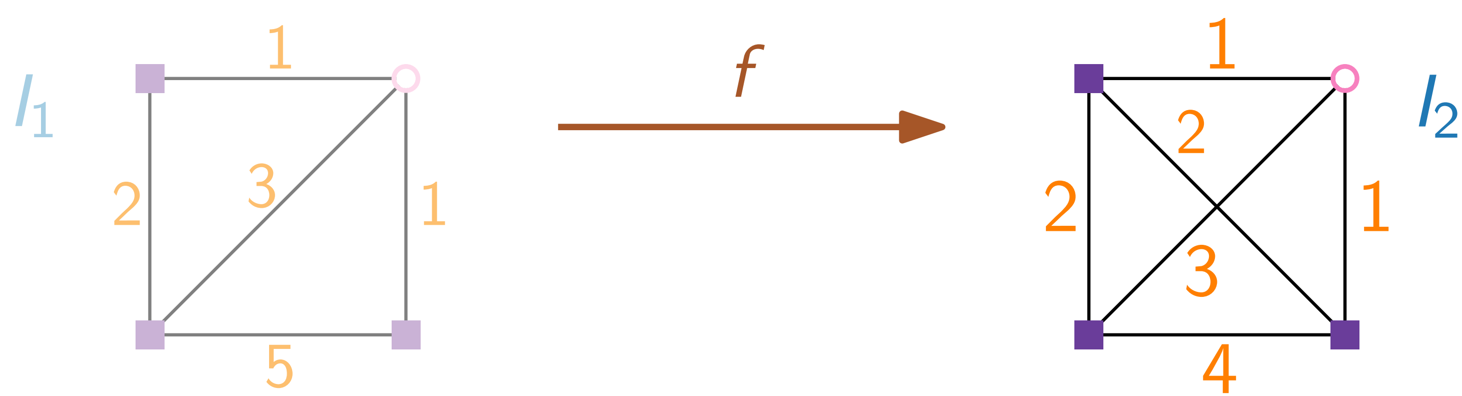

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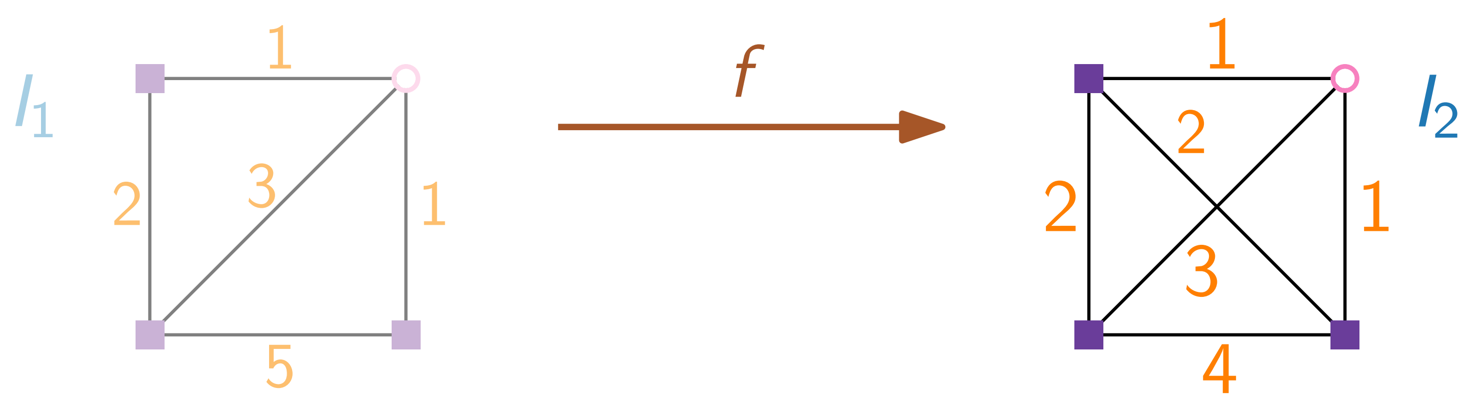

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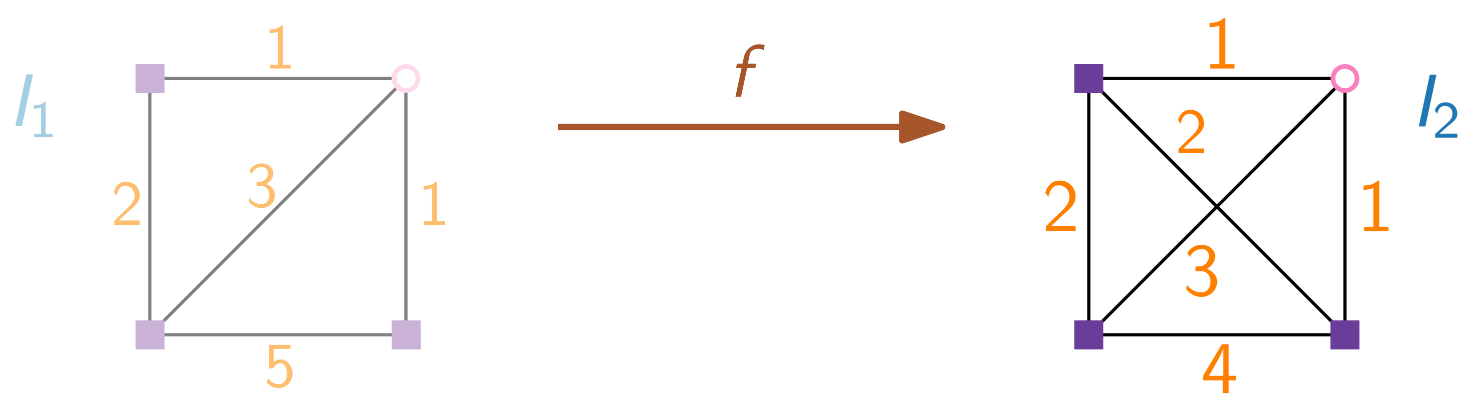
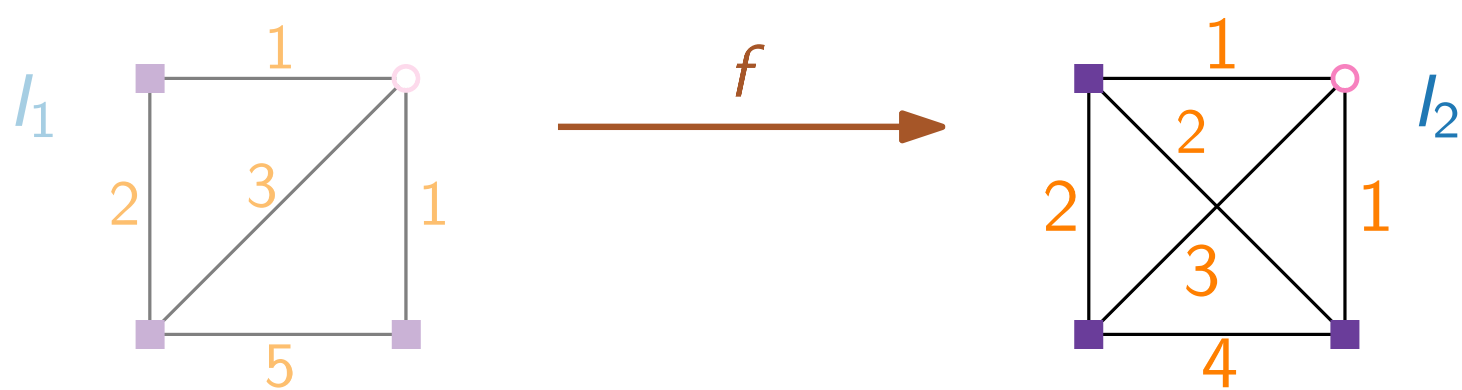
Note: For every edge $(u, v) \in E_1$, $c_2(u, v) \leq c_1(u, v)$.



MetricSteinerTree

Theorem. There is an approximation-preserving reduction from SteinerTree to MetricSteinerTree.

Proof. (2) $\mathrm{OPT}(I_2) \leq \mathrm{OPT}(I_1)$

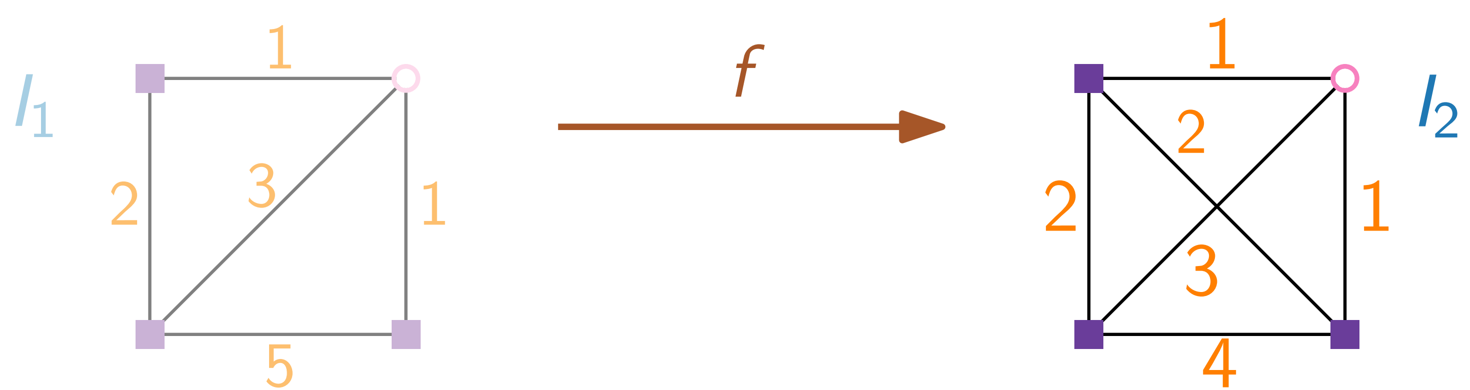


MetricSteinerTree

Theorem. There is an approximation-preserving reduction from SteinerTree to MetricSteinerTree.

Proof. (2) $\mathrm{OPT}(I_2) \leq \mathrm{OPT}(I_1)$

Let B^* be an optimal Steiner tree for I_1.



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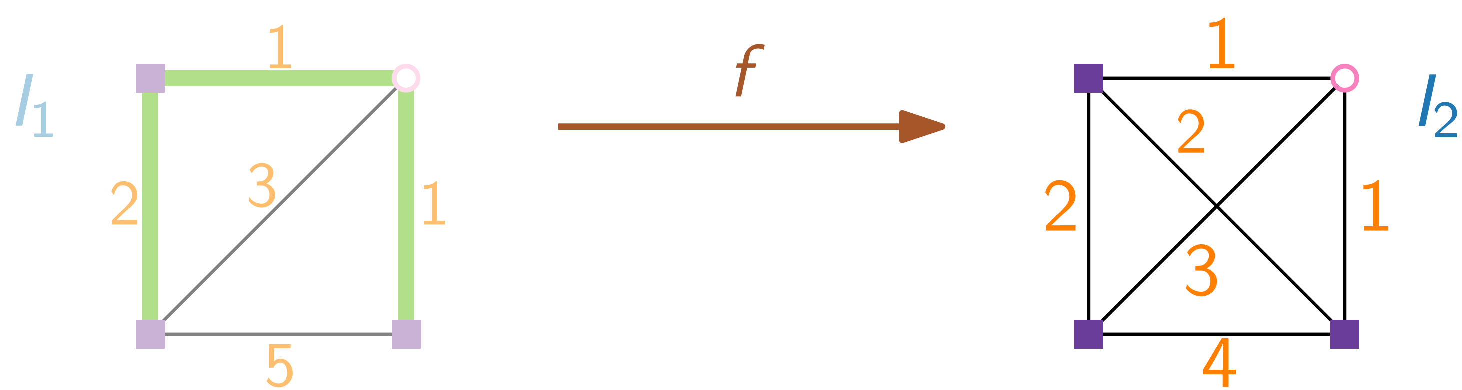

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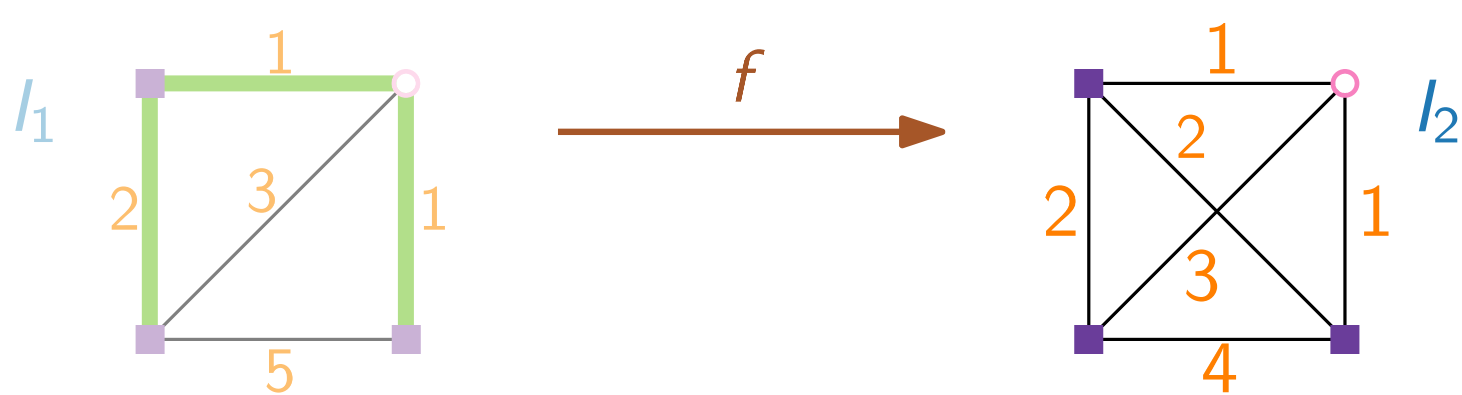

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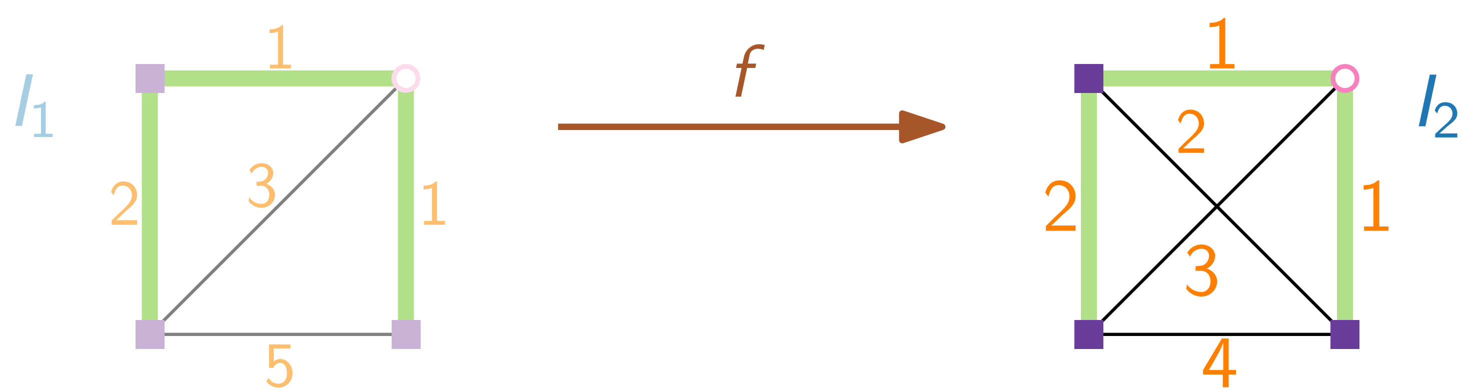

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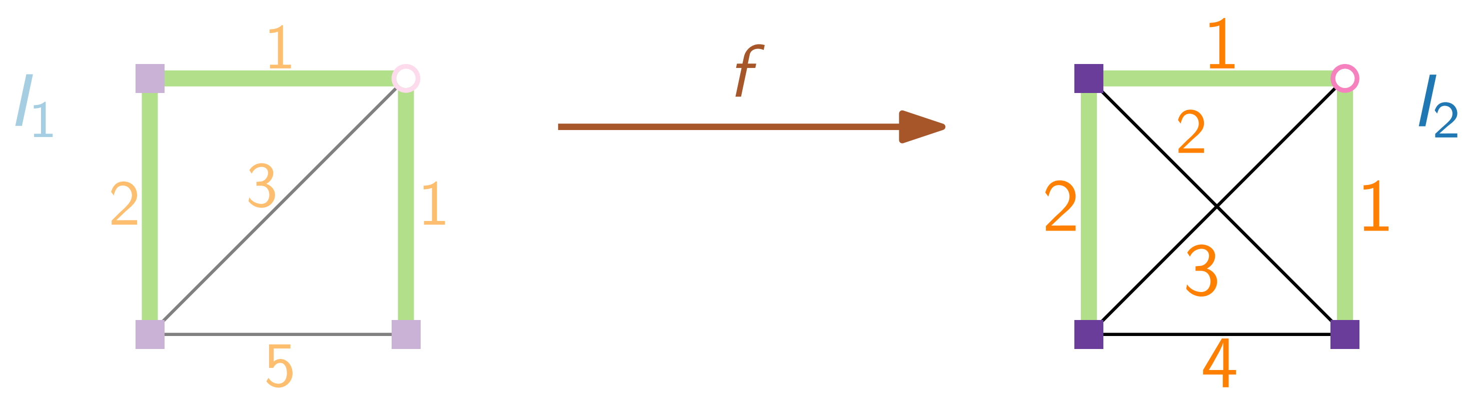
$\mathrm{OPT}(I_2)$



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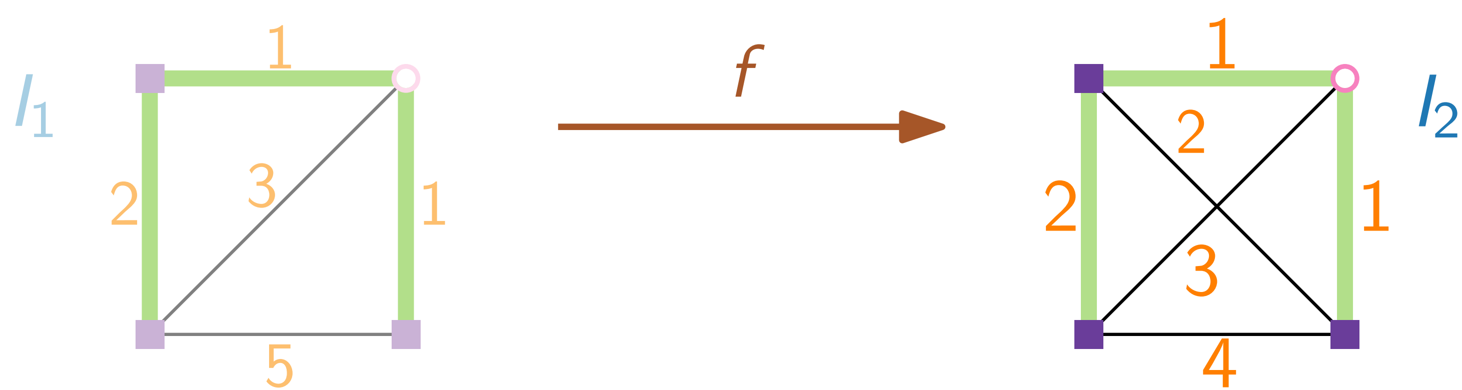
$\mathrm{OPT}(I_2) \leq c_2(B^*)$



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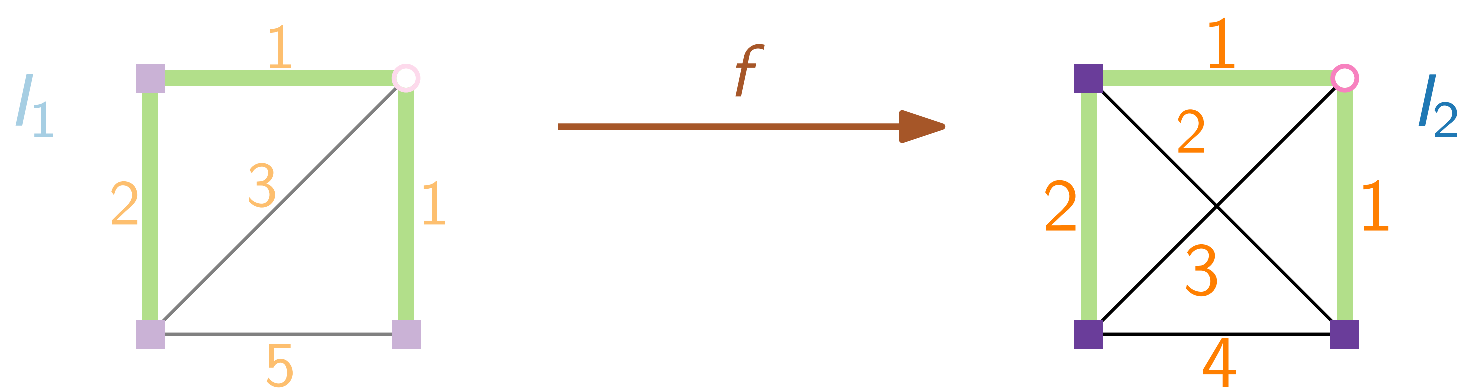

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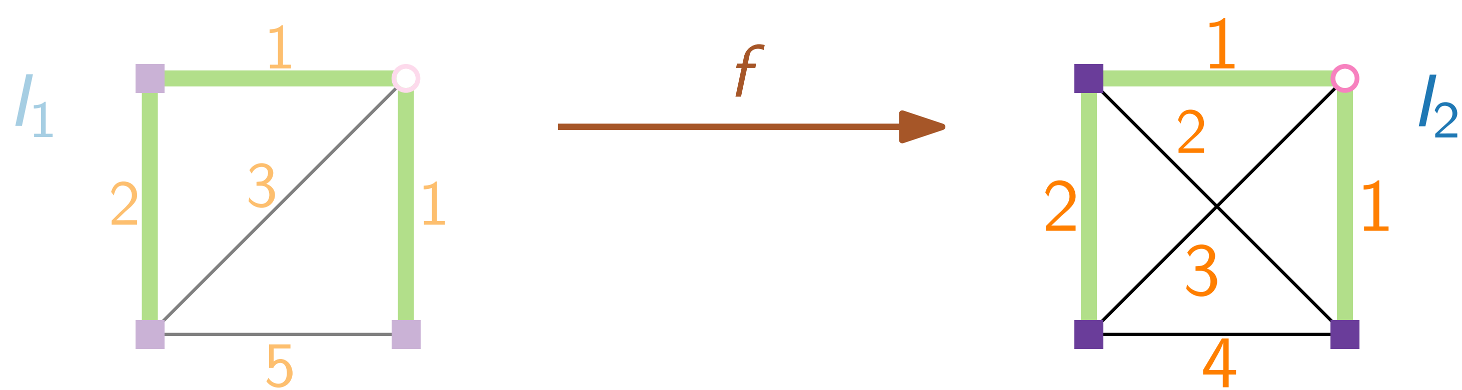

MetricSteinerTree

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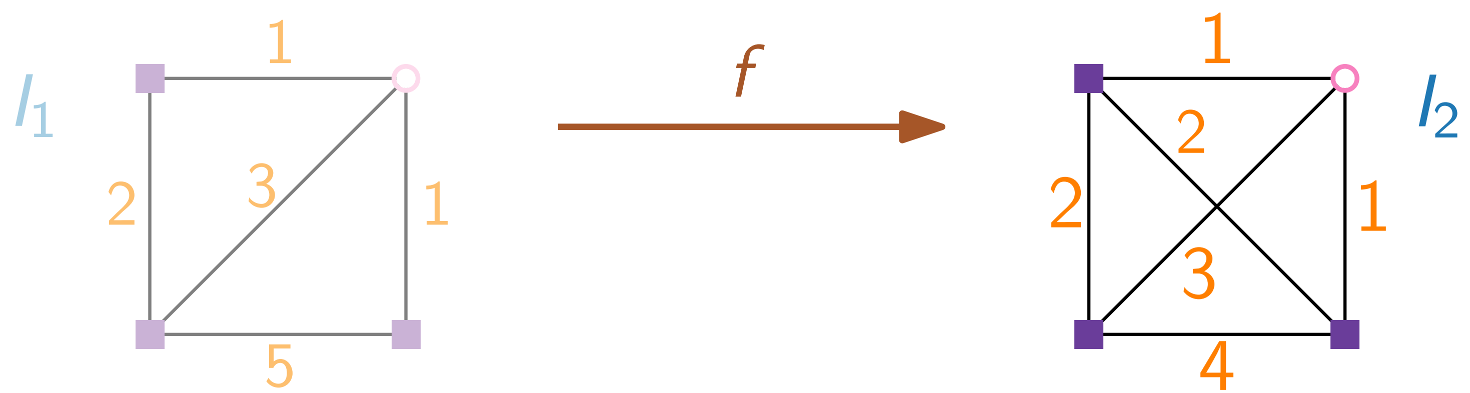
Proof. (3) Mapping g $s \xleftarrow{g} t$



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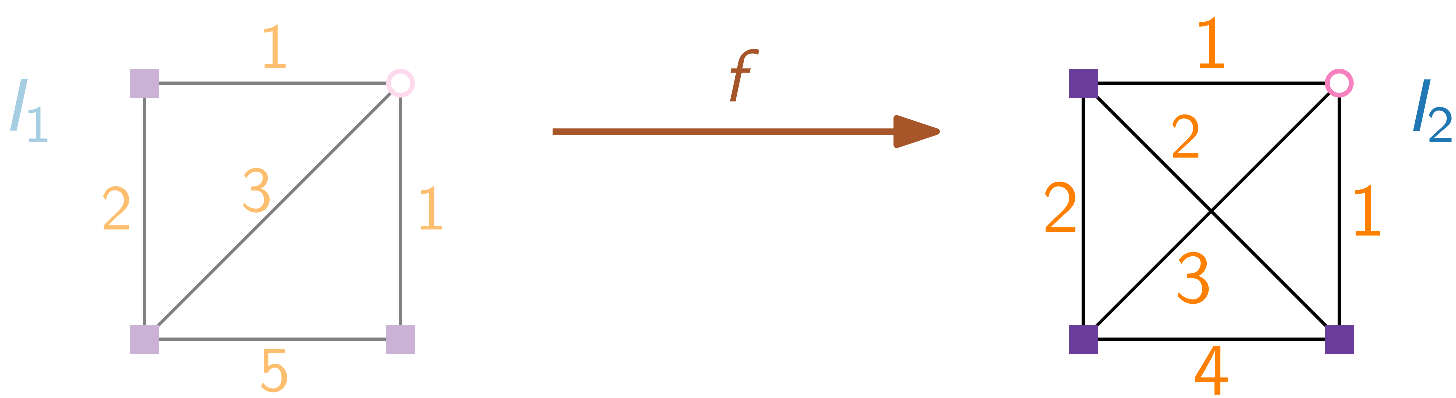

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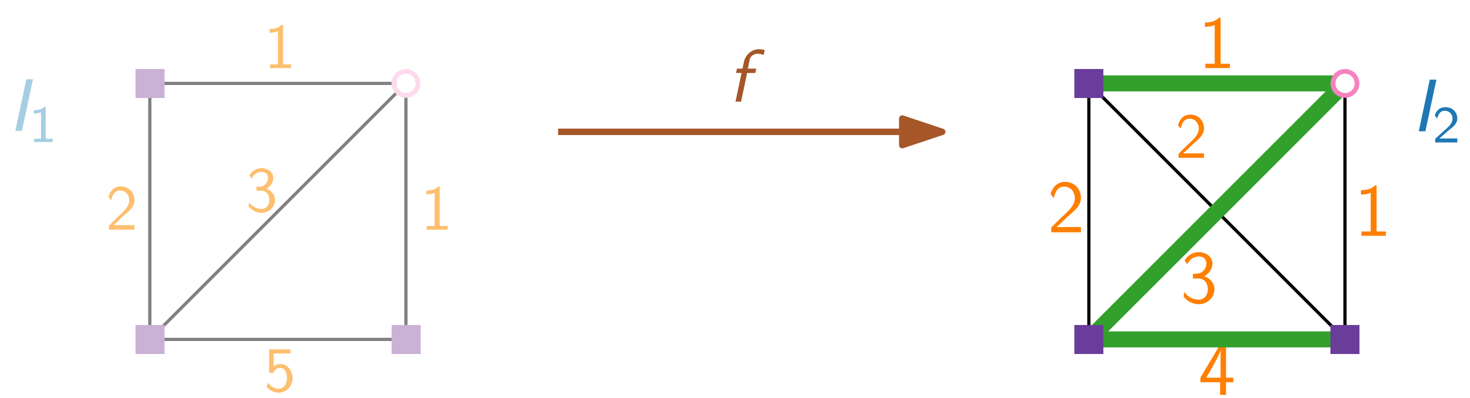

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Proof. (3) Mapping g $\quad s \xleftarrow{g} t$

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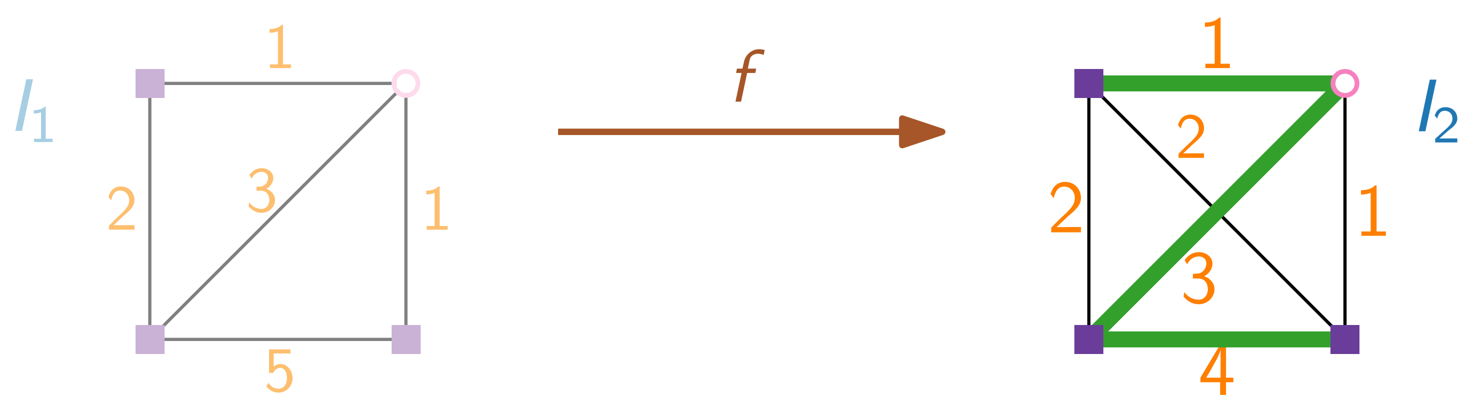

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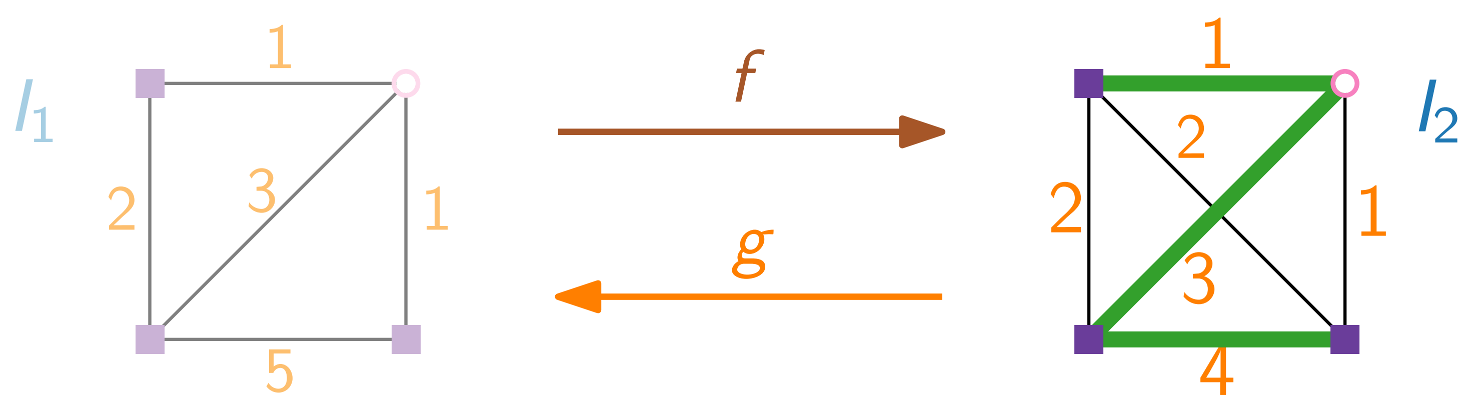

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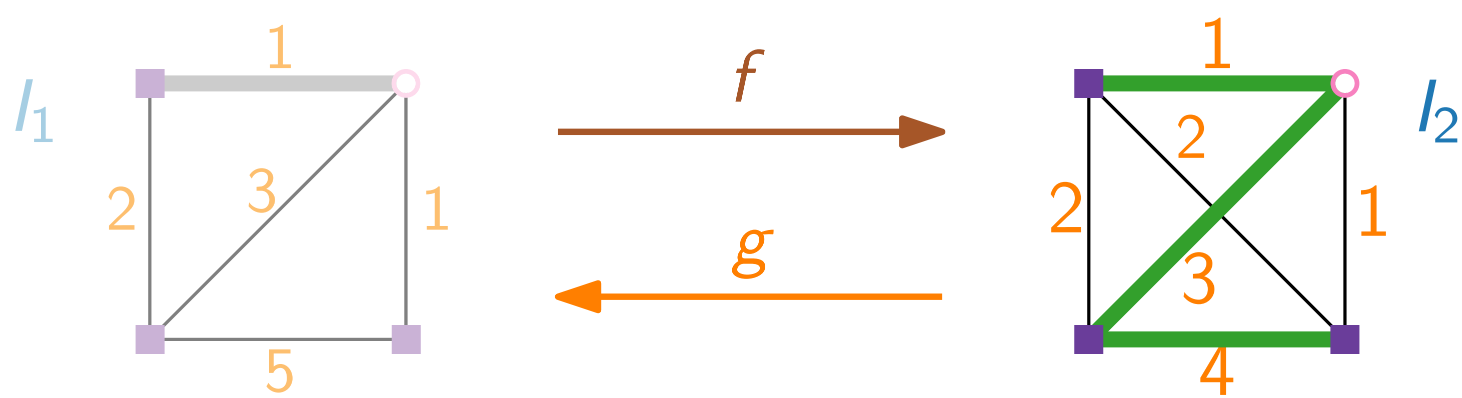

METRICSTEINERTREE

Theorem. There is an approximation-preserving reduction from STEINERTREE to METRICSTEINERTREE.

Proof. (3) Mapping g 

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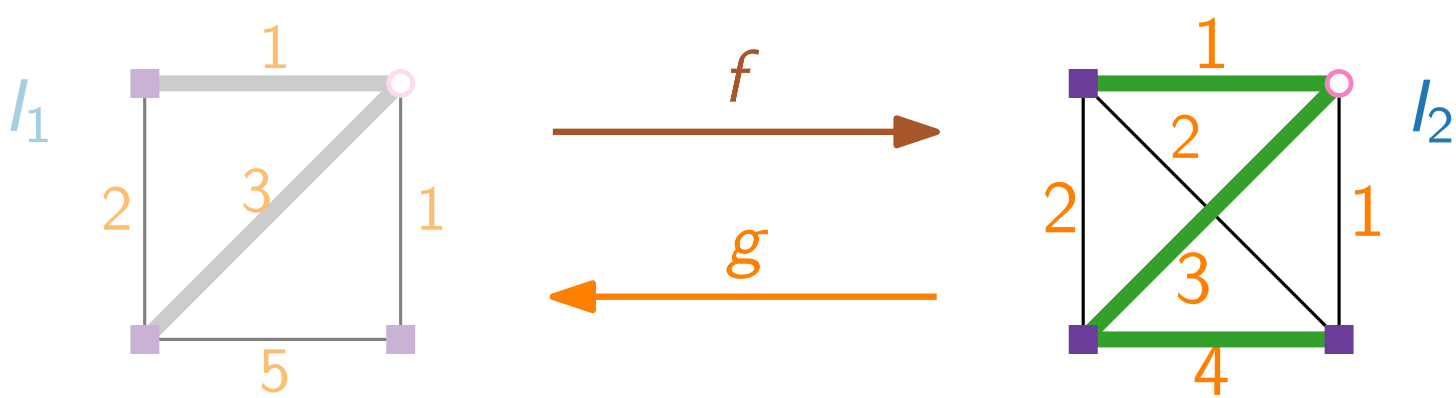

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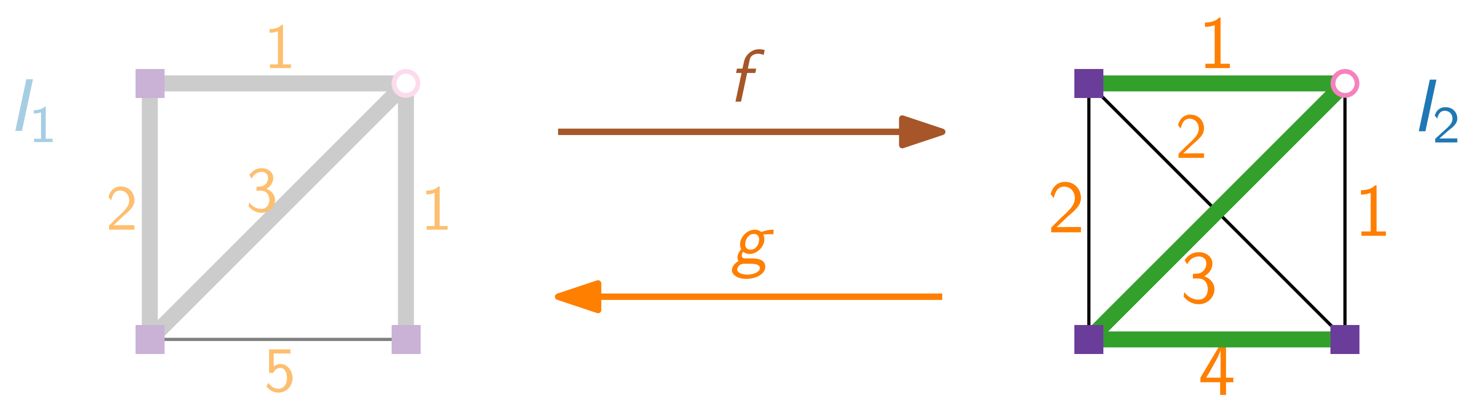

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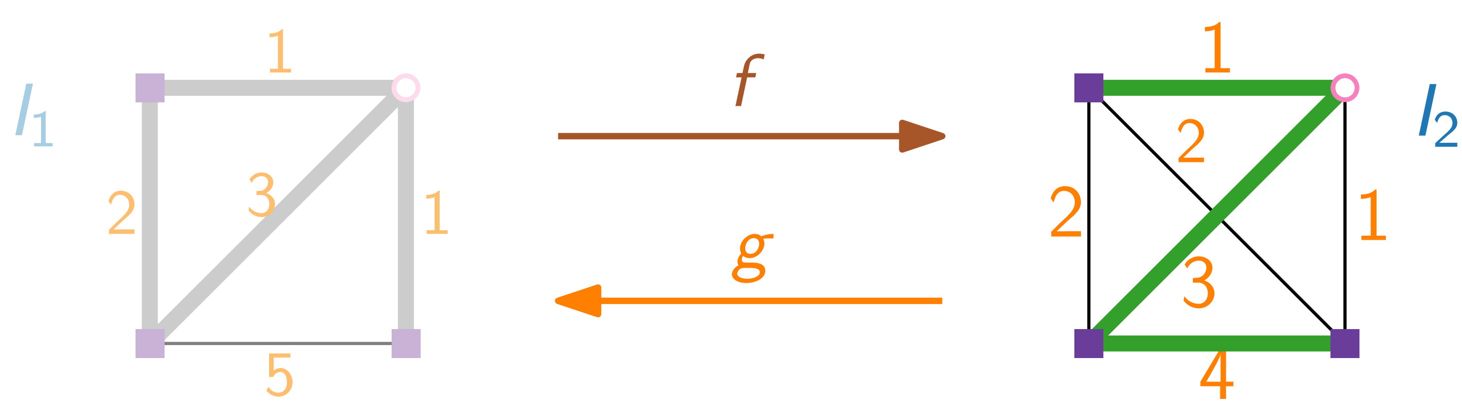

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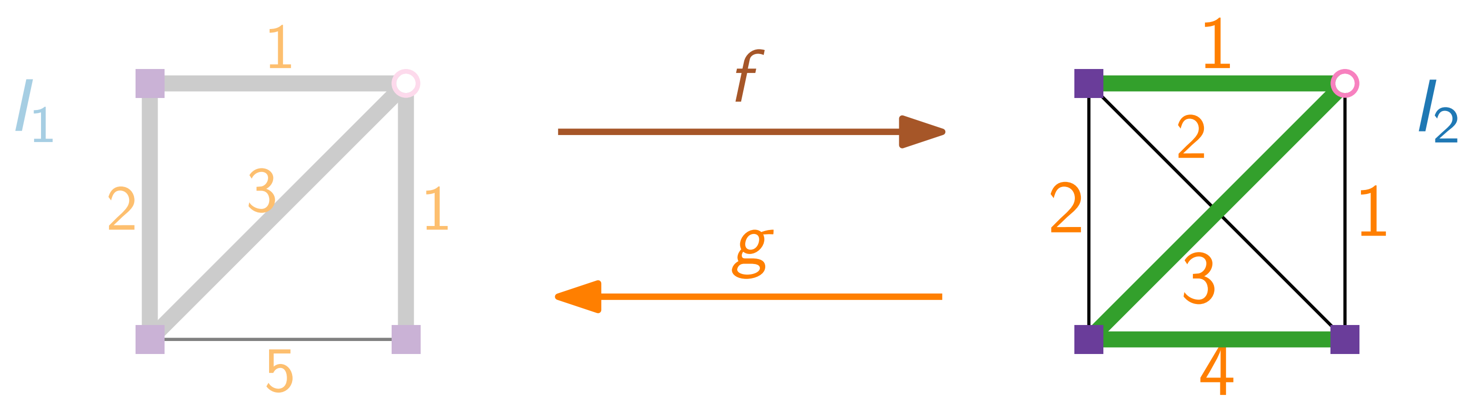

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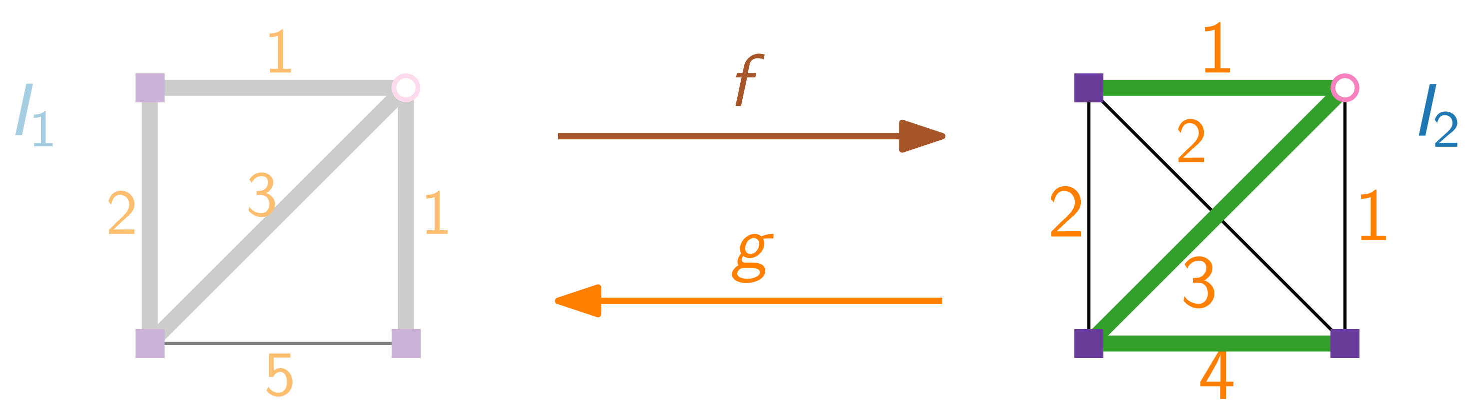

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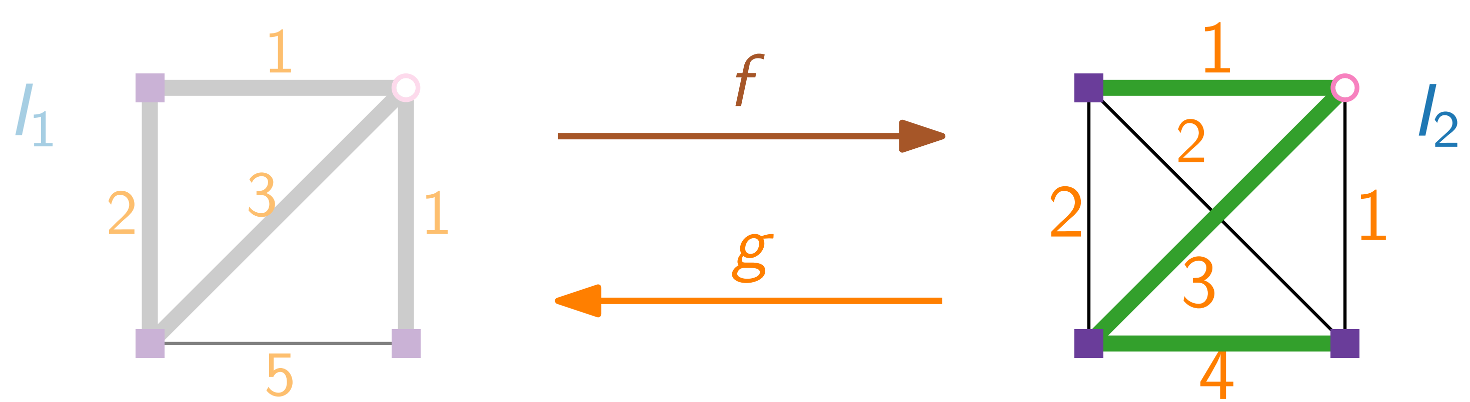

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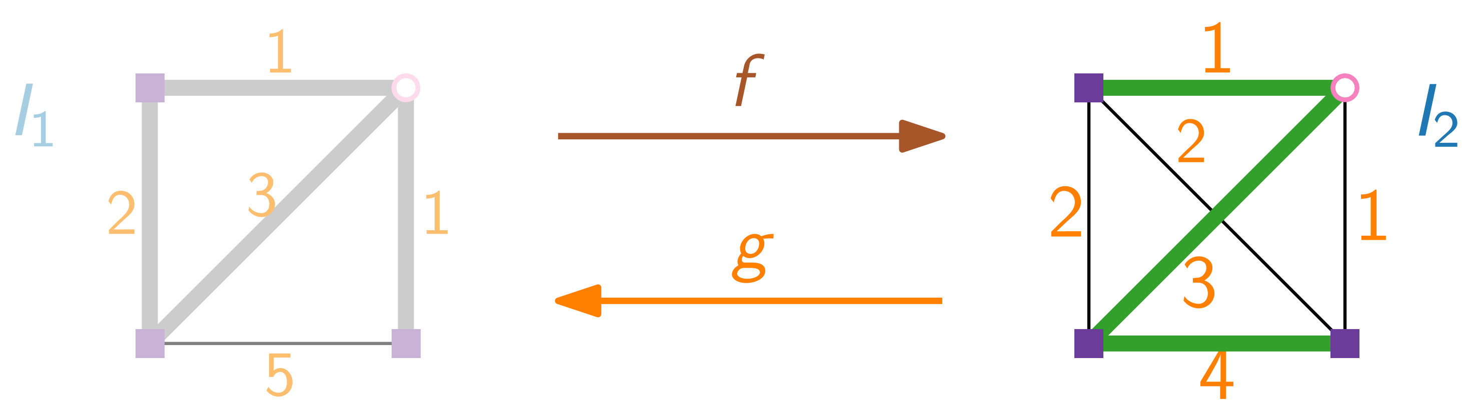

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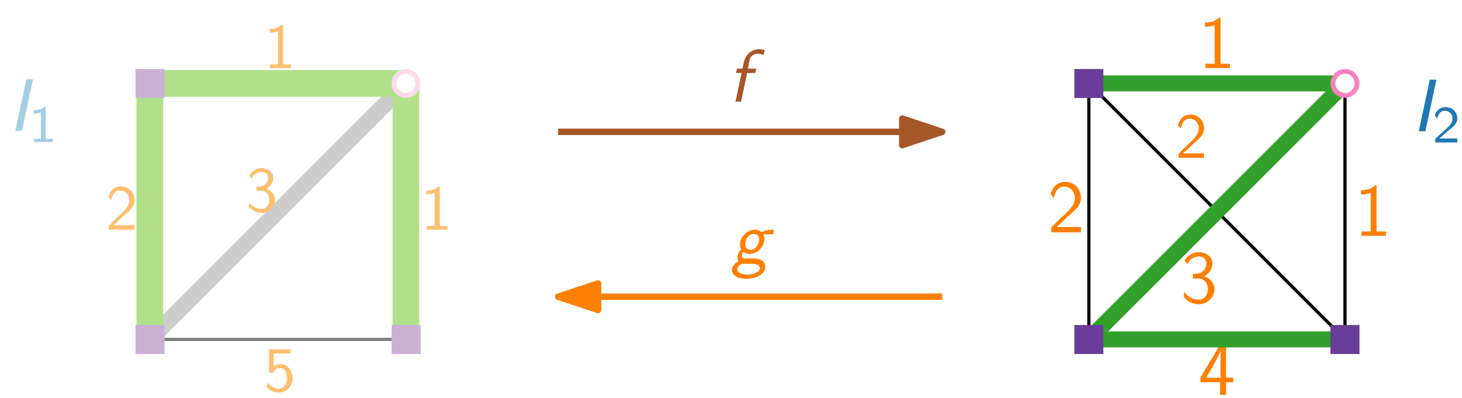

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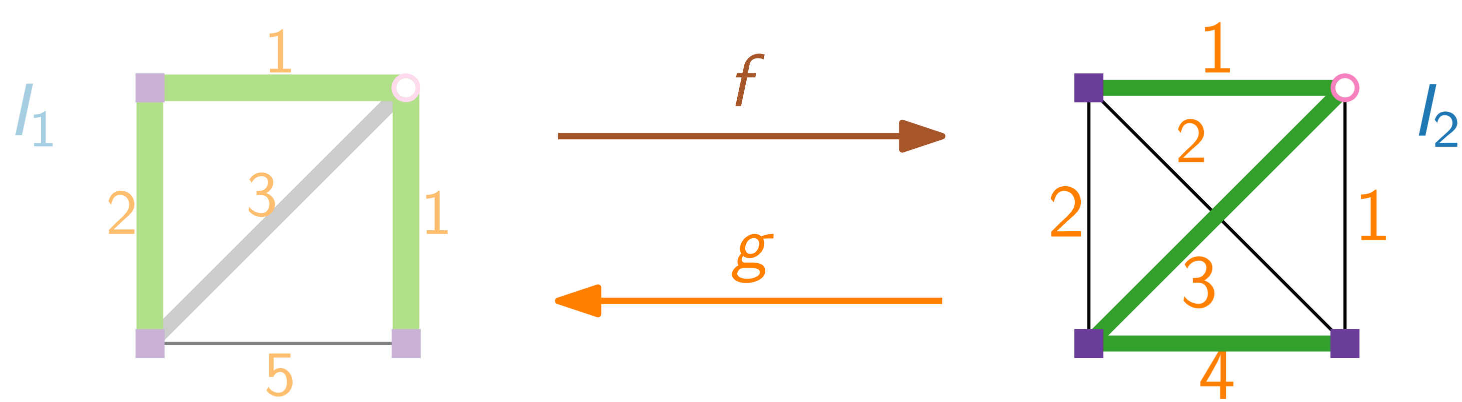

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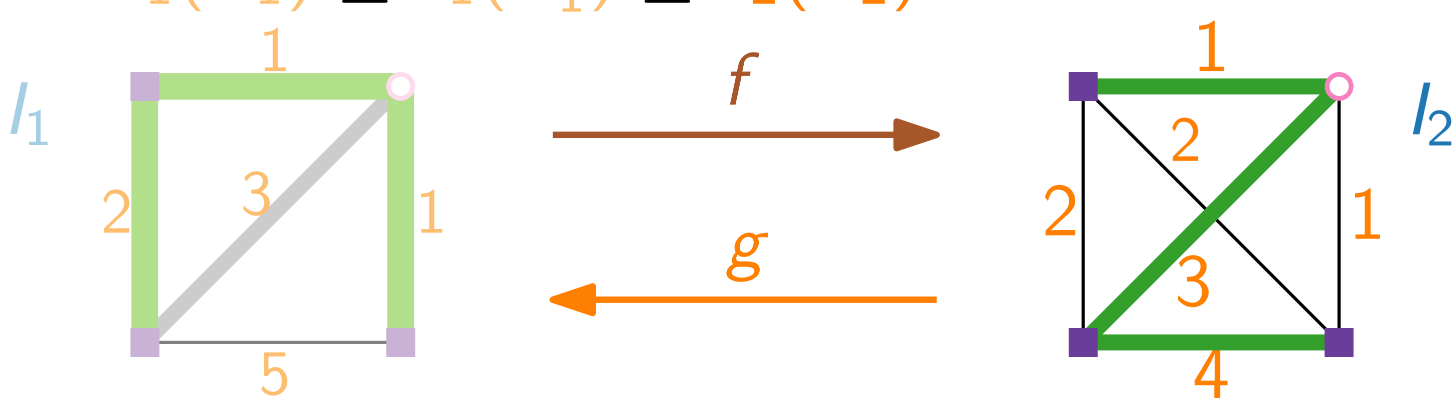
Note that $c_1(B_1) \leq c_1(G_1') \leq c_2(B_2)$.



Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part IV:
2-Approximation for MetricSteinerTree

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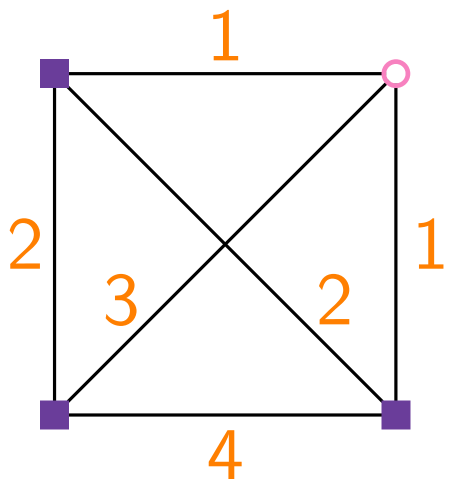

G

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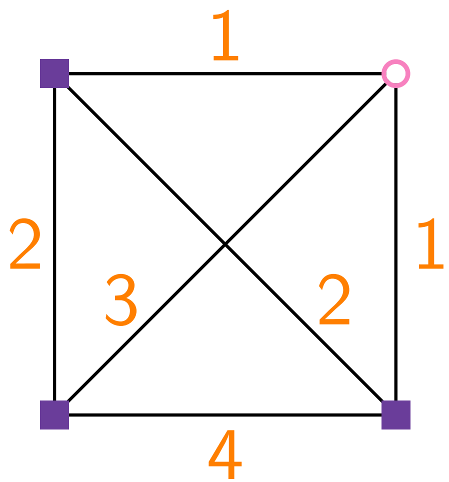

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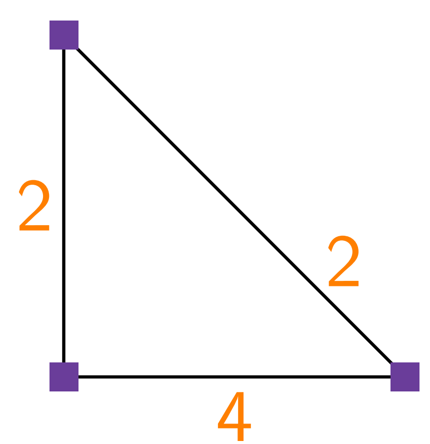

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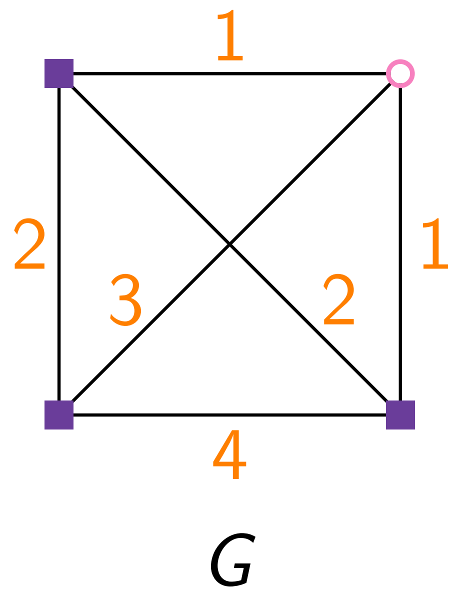

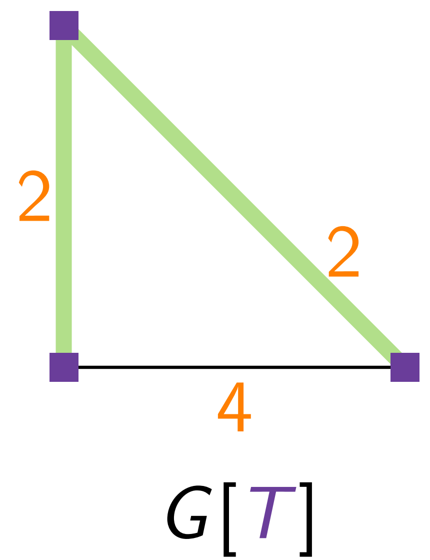

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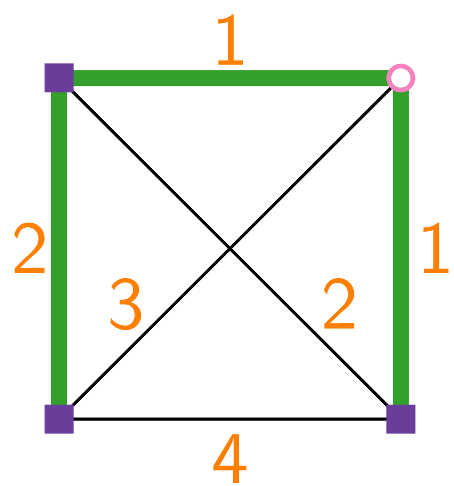

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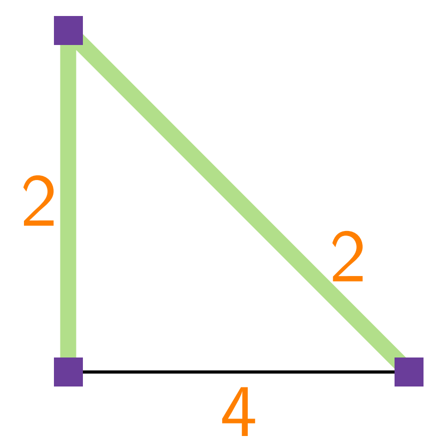

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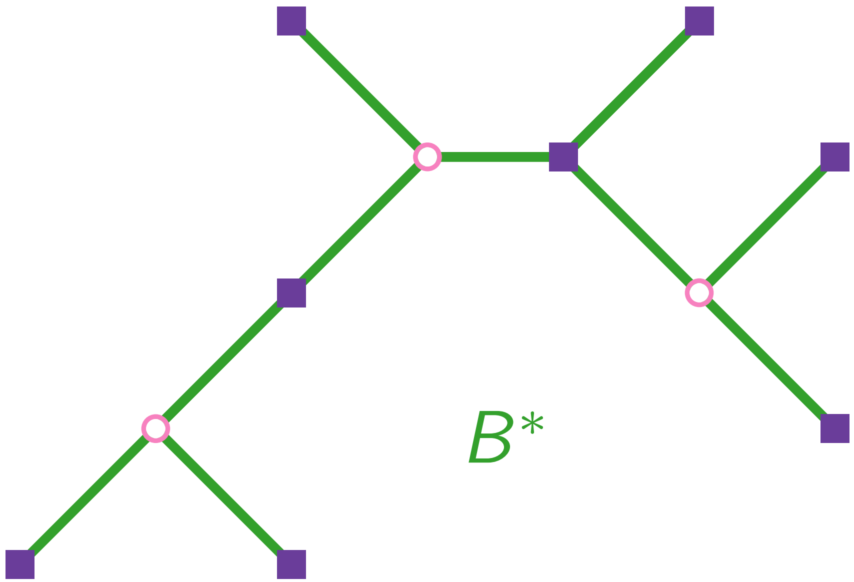

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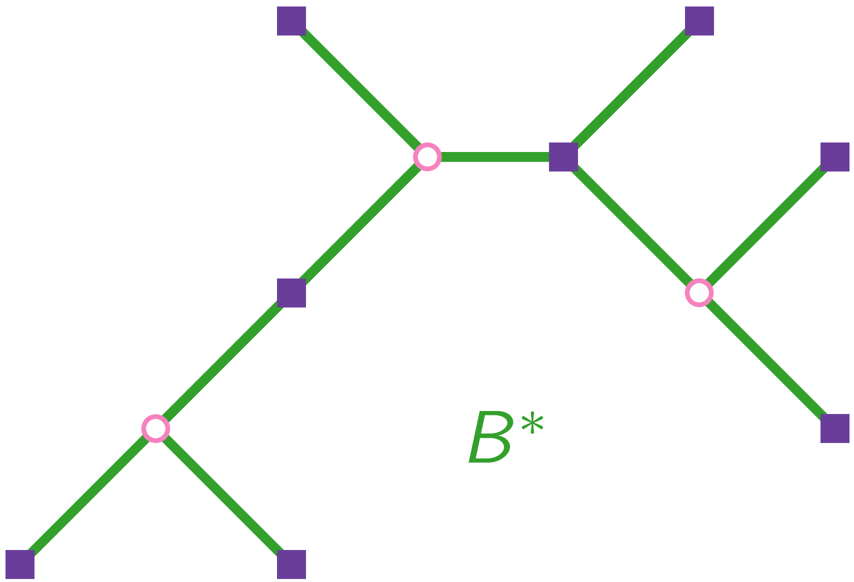

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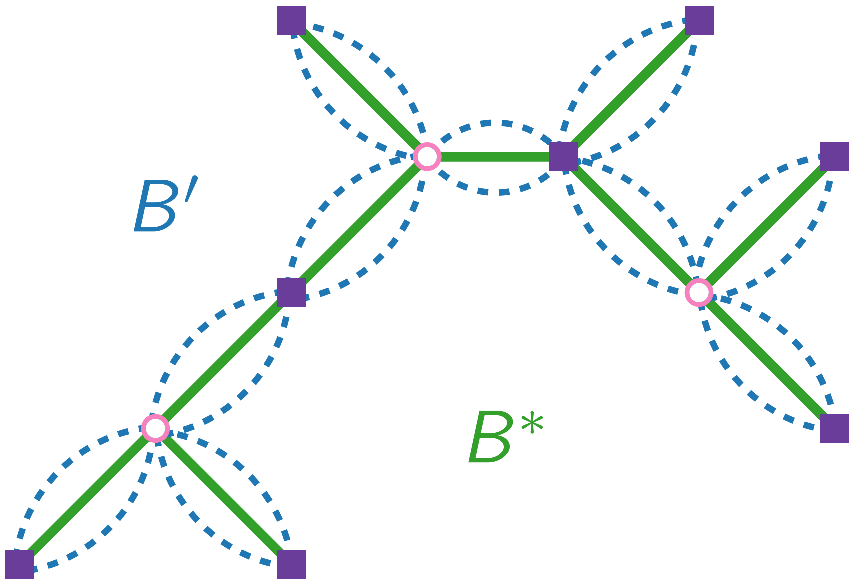

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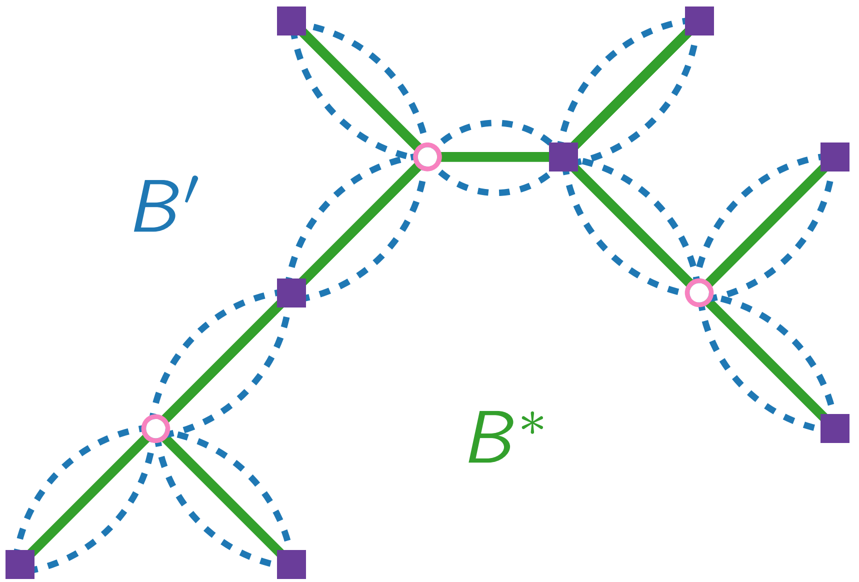

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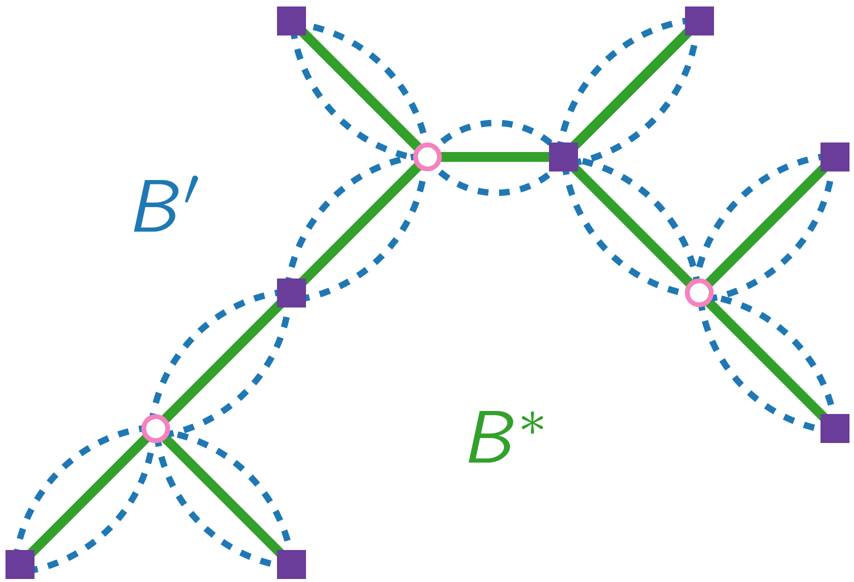

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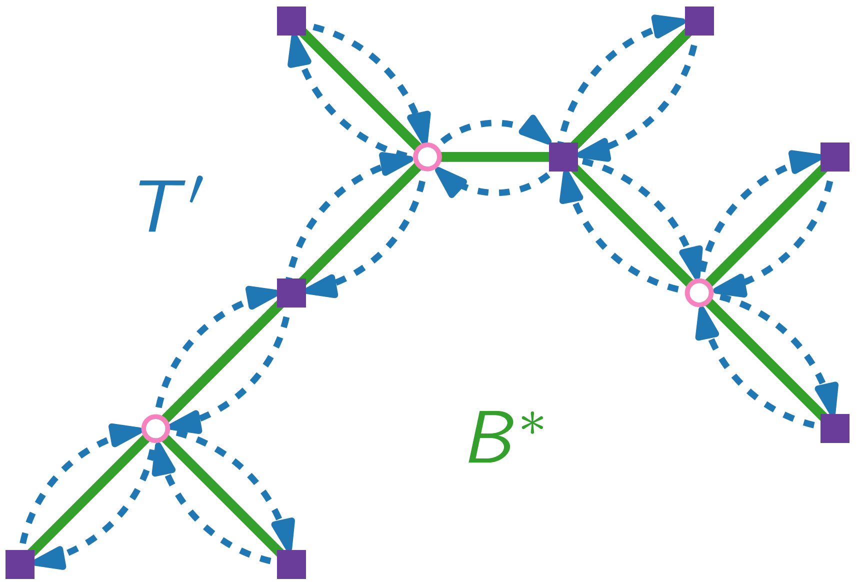

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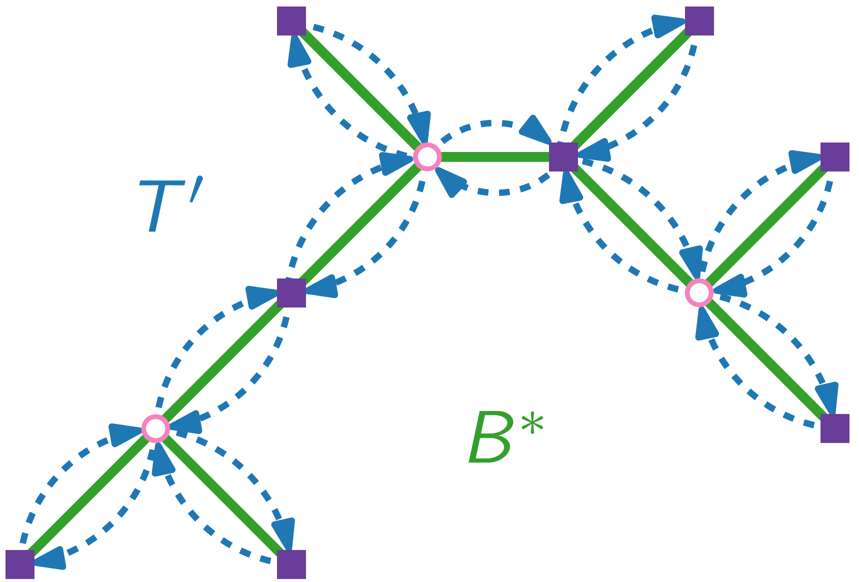

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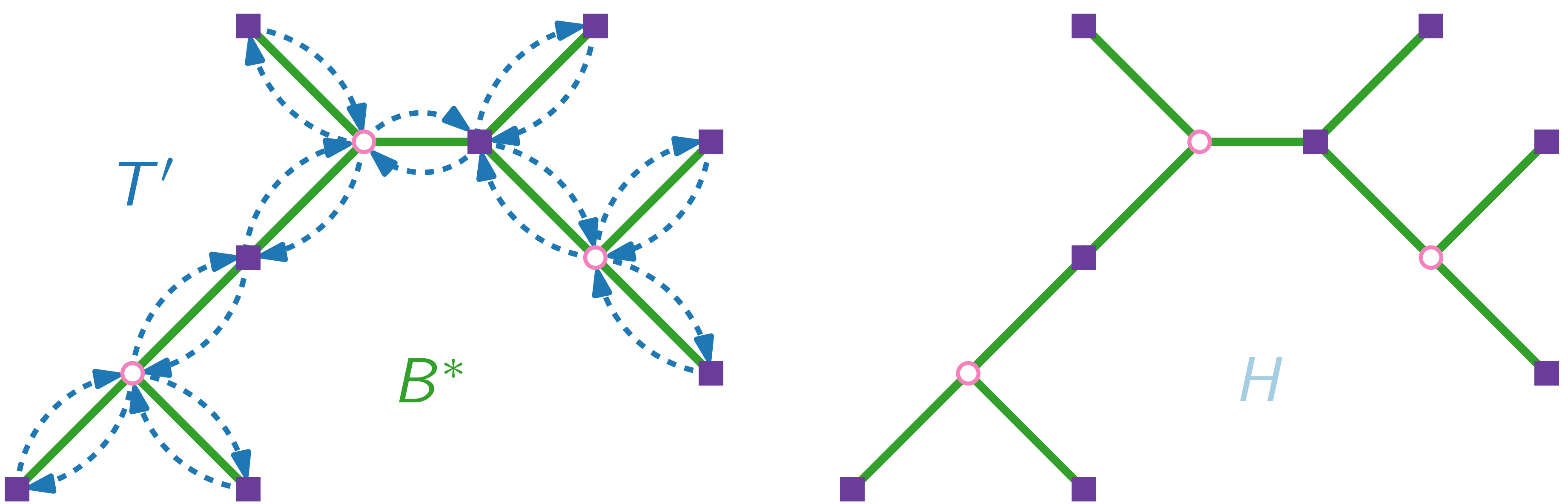

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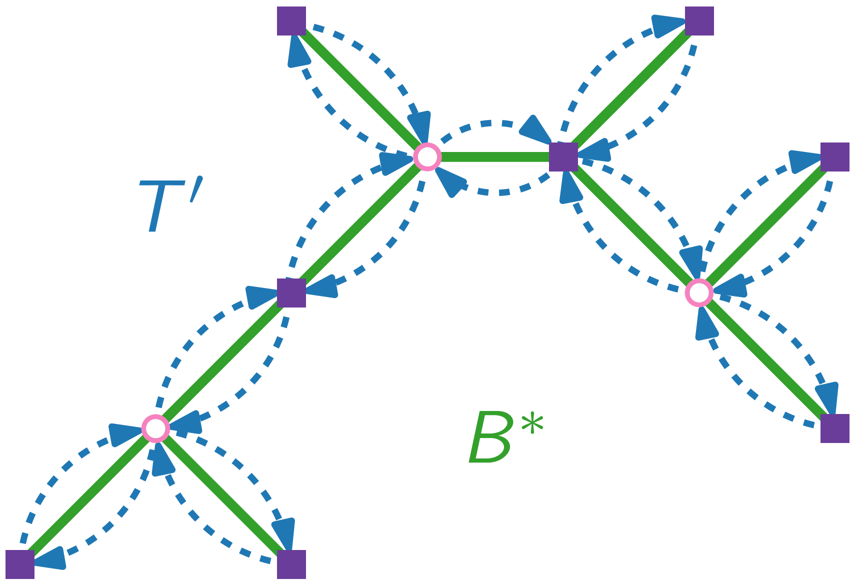

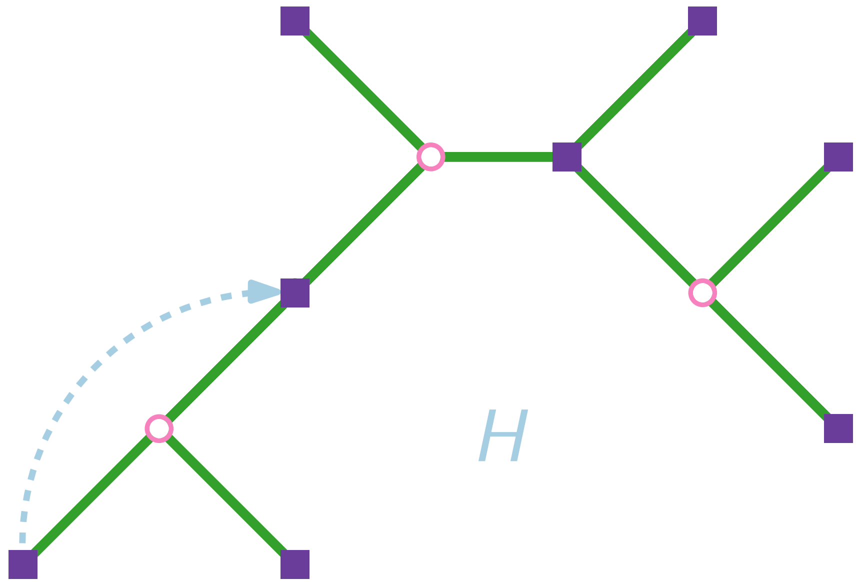

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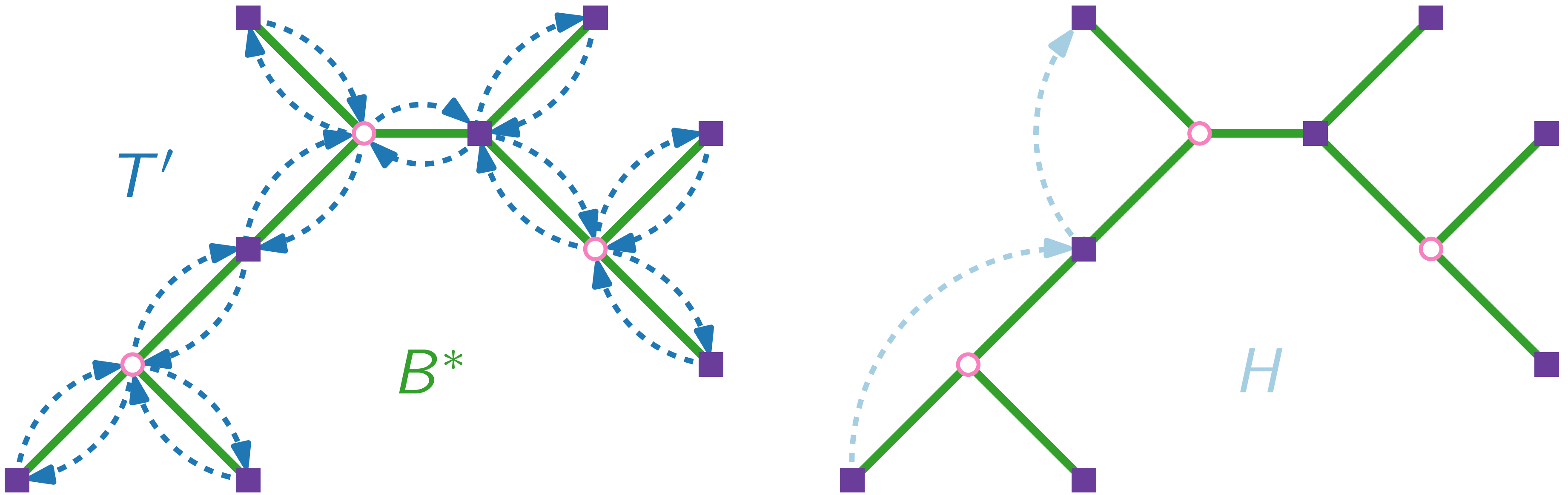

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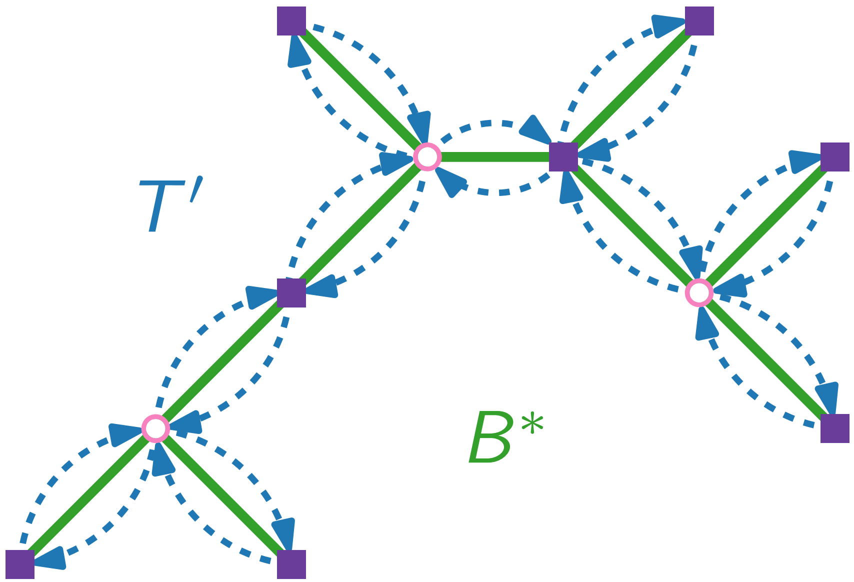

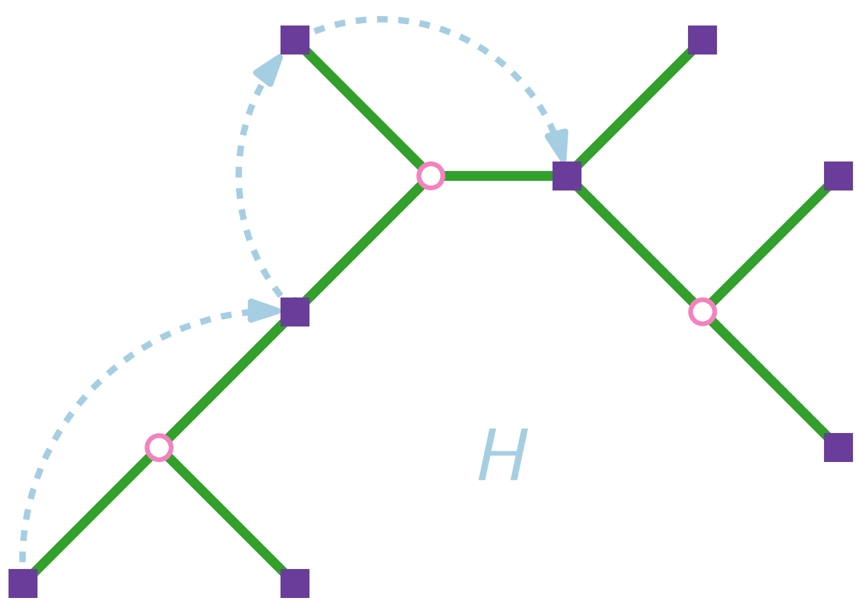

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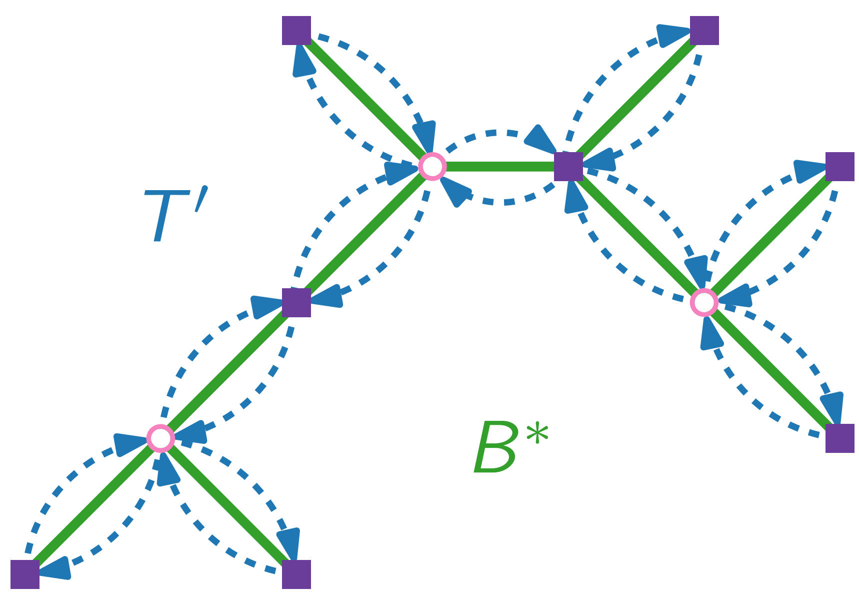

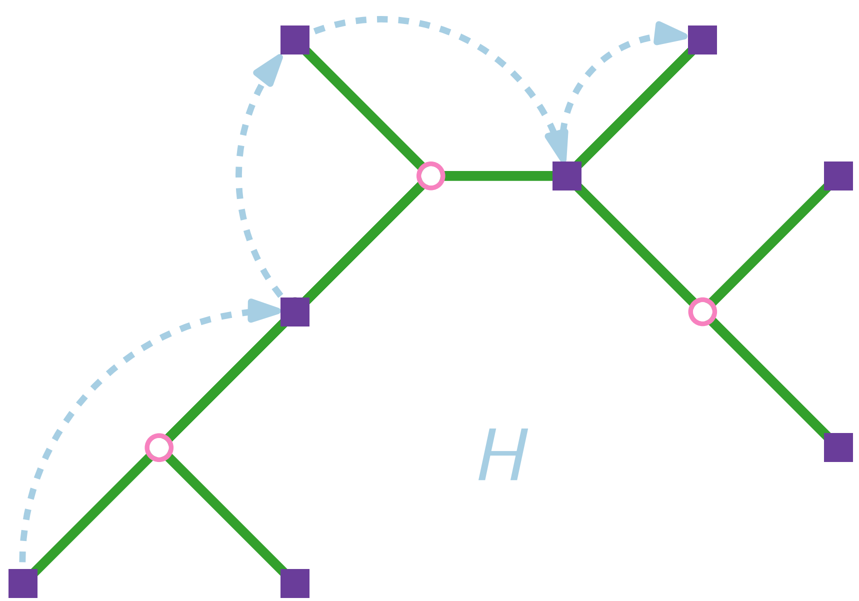

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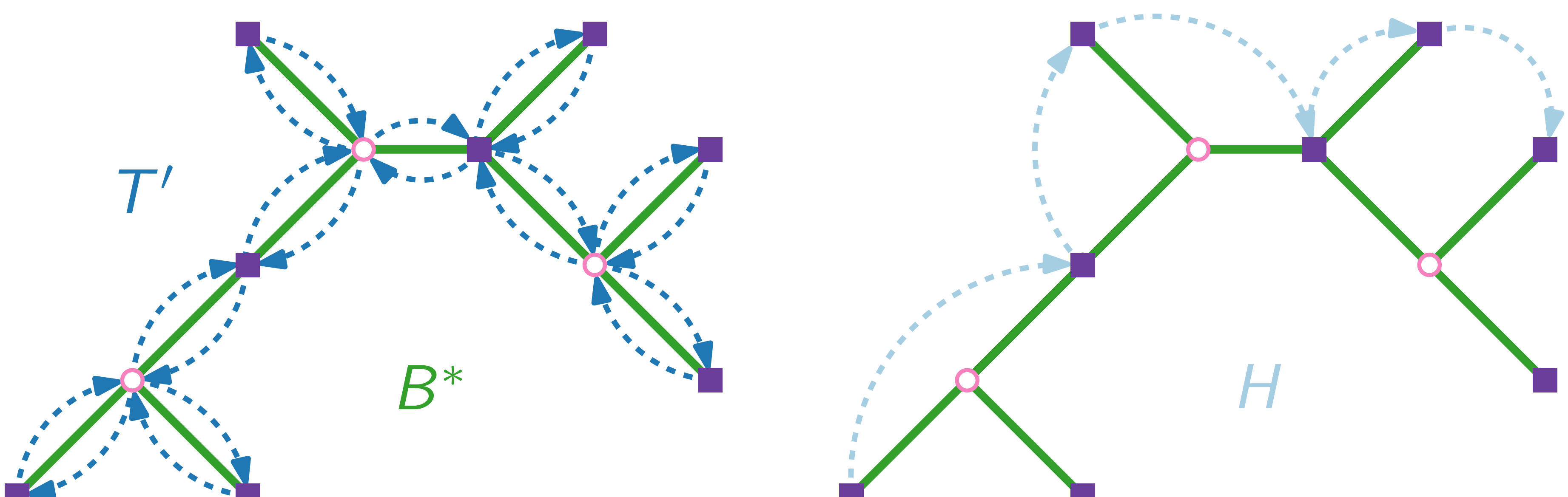

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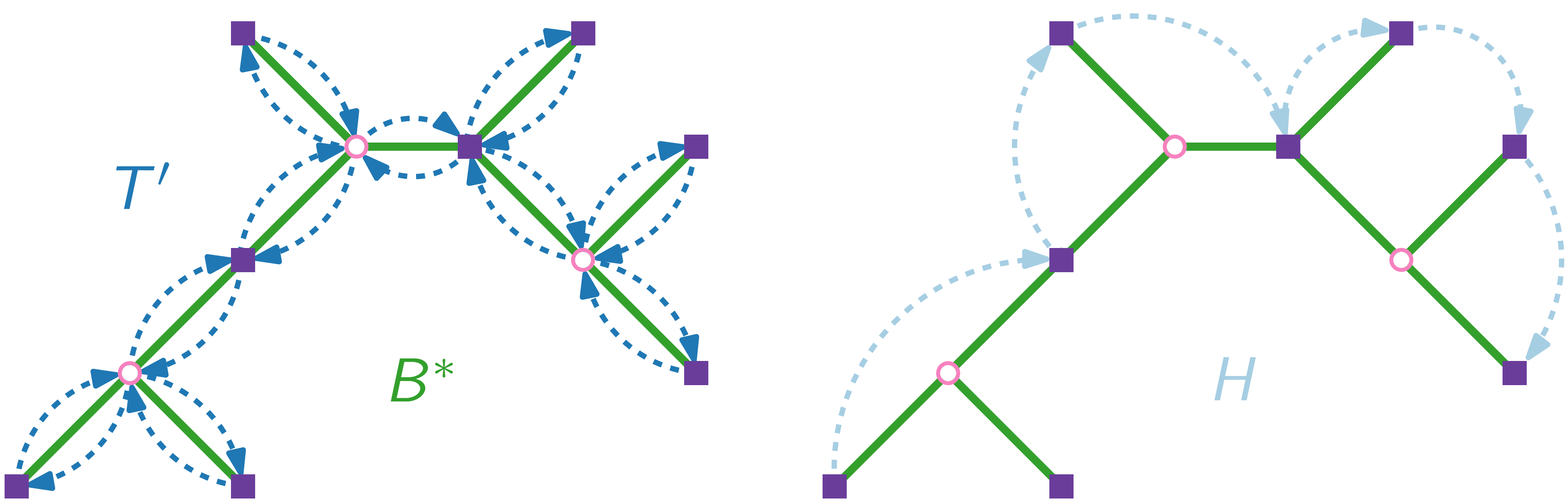

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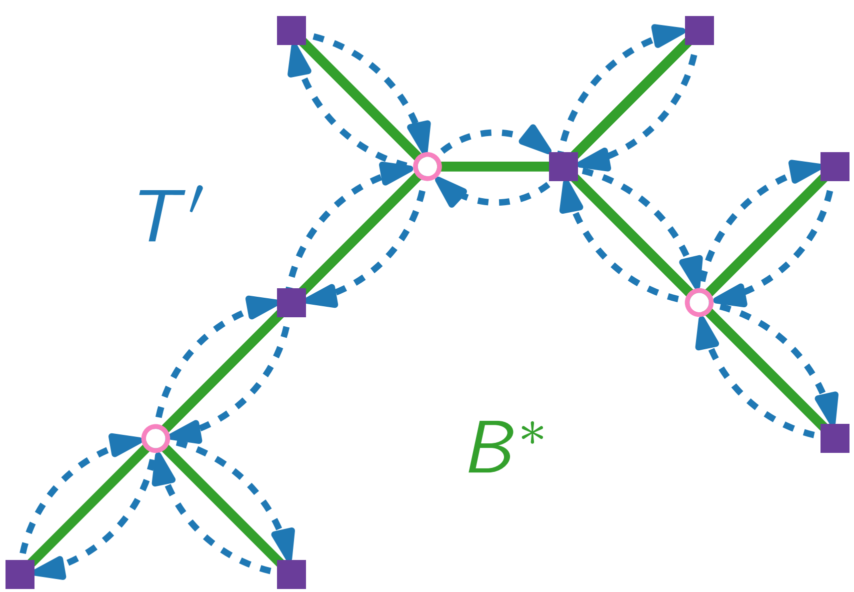

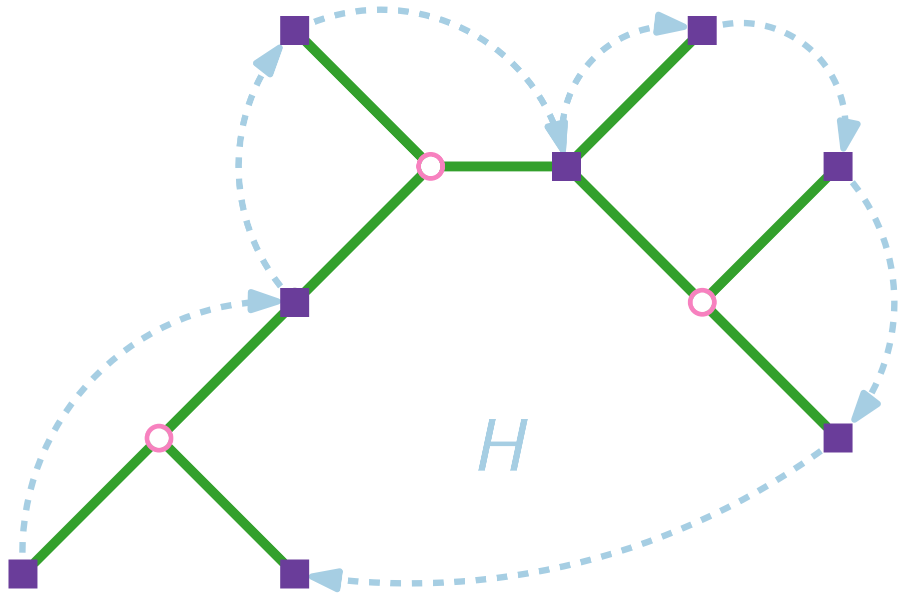

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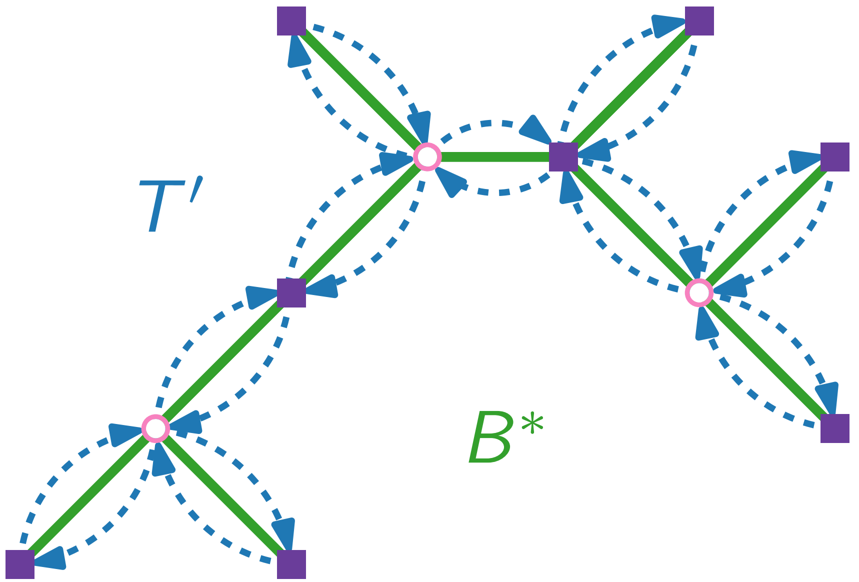

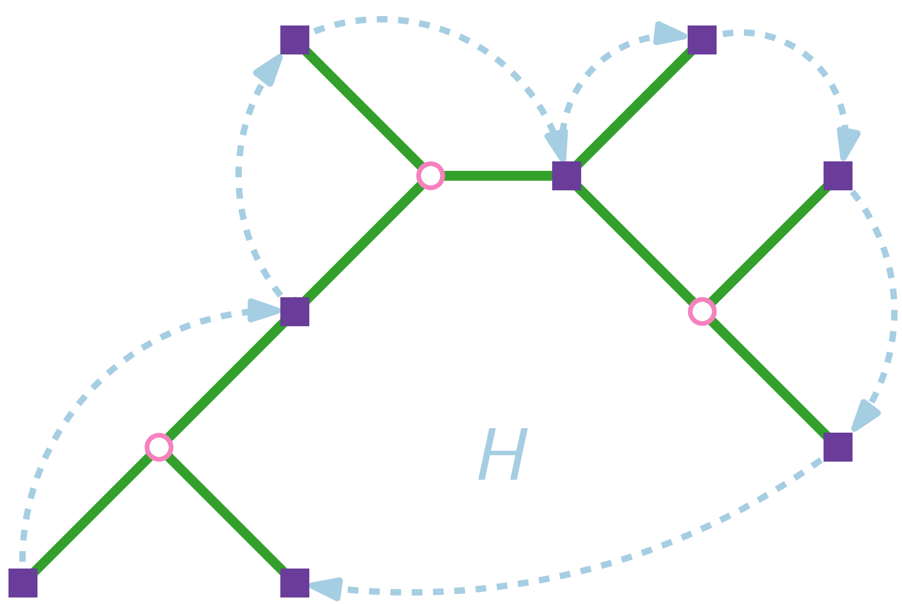

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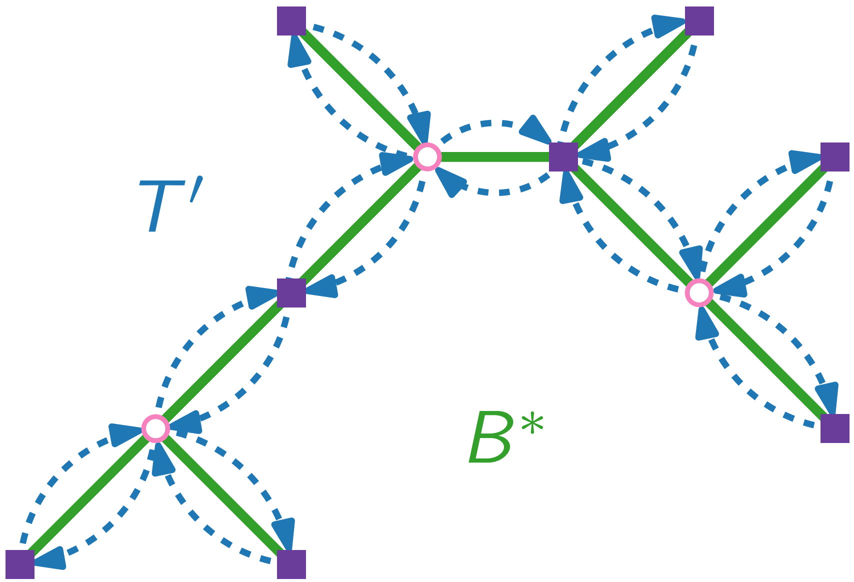

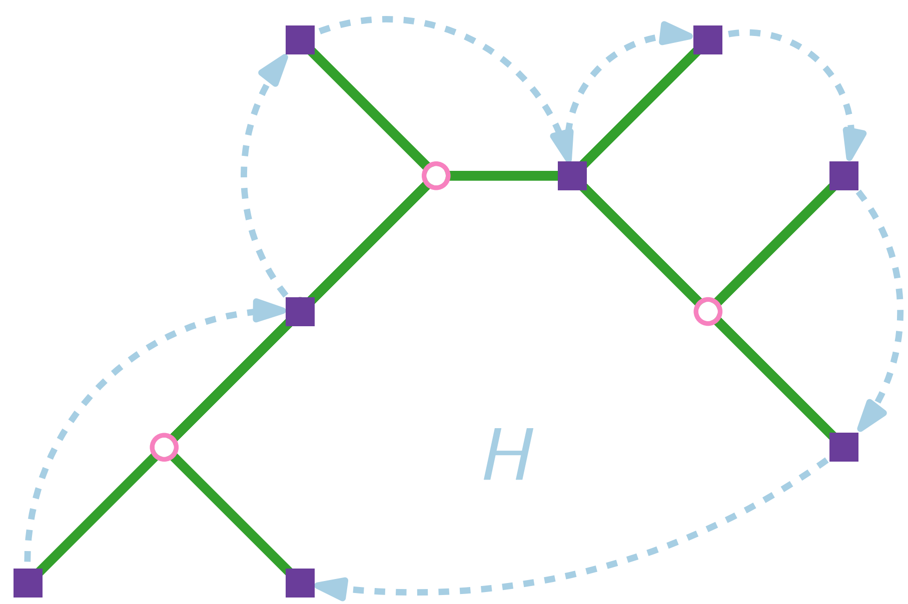

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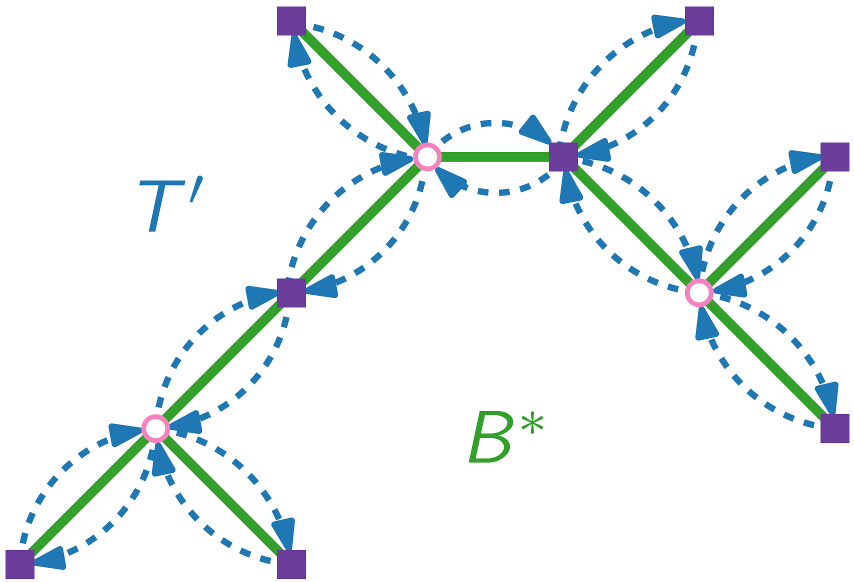

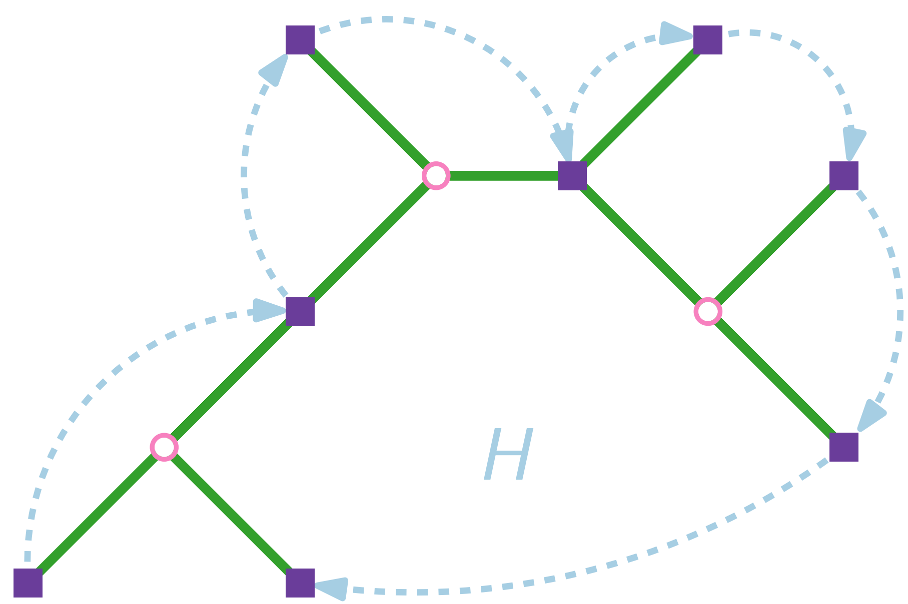

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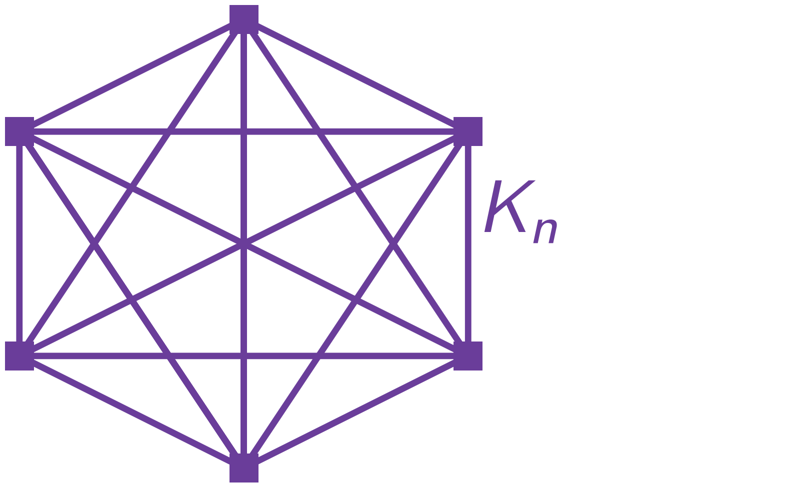

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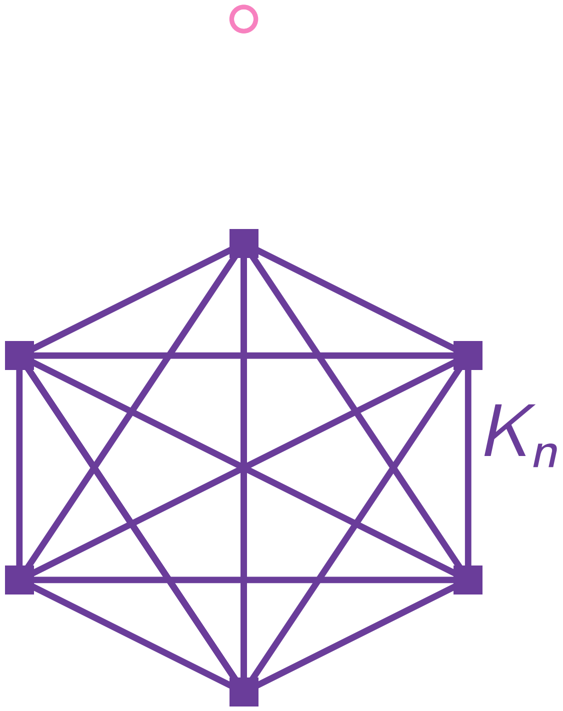
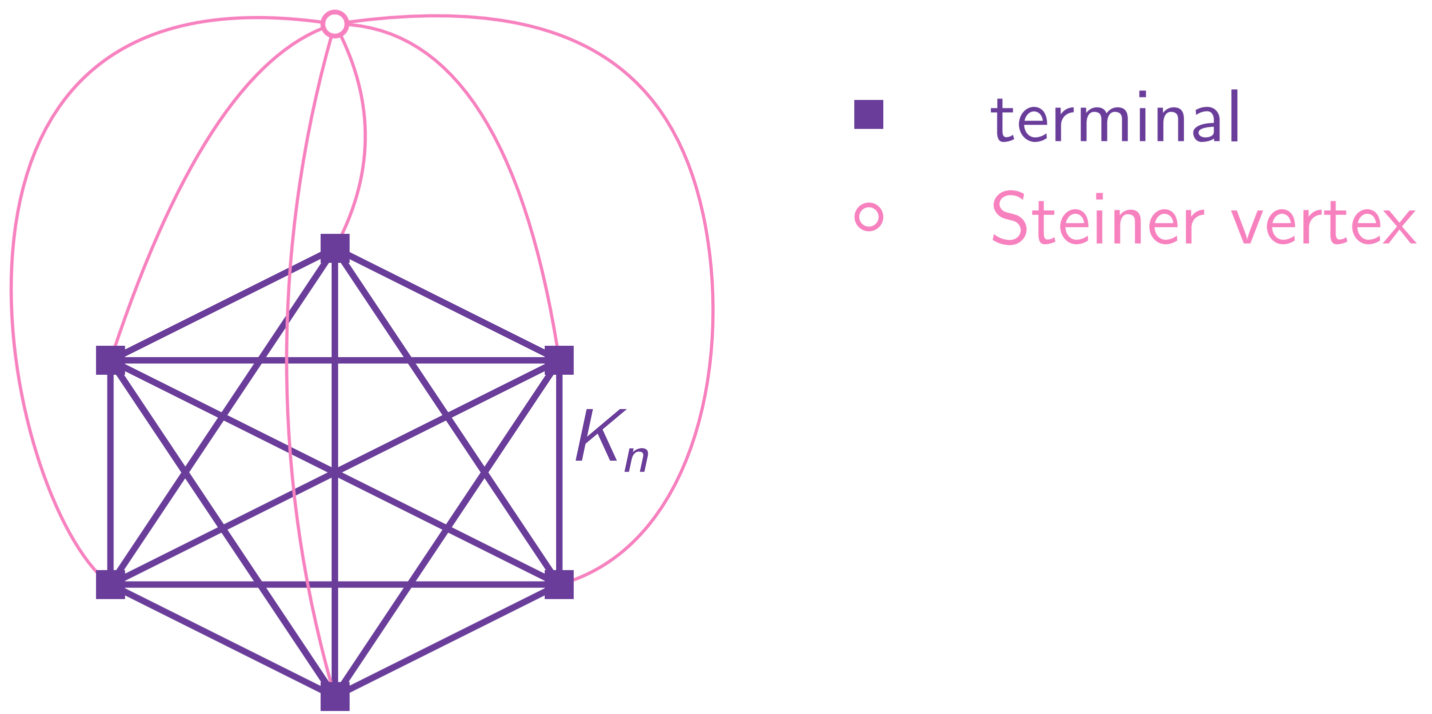


■ terminal

○ Steiner vertex

Analysis Tight?



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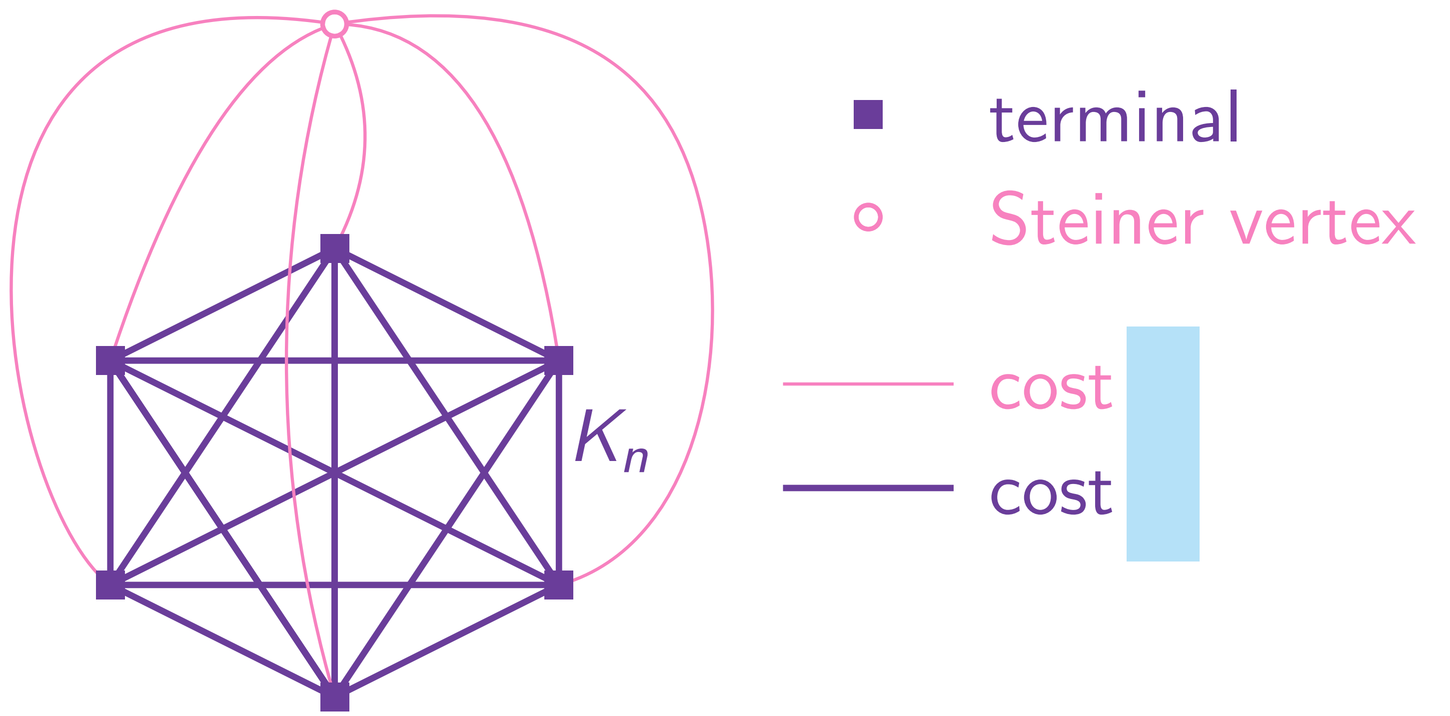

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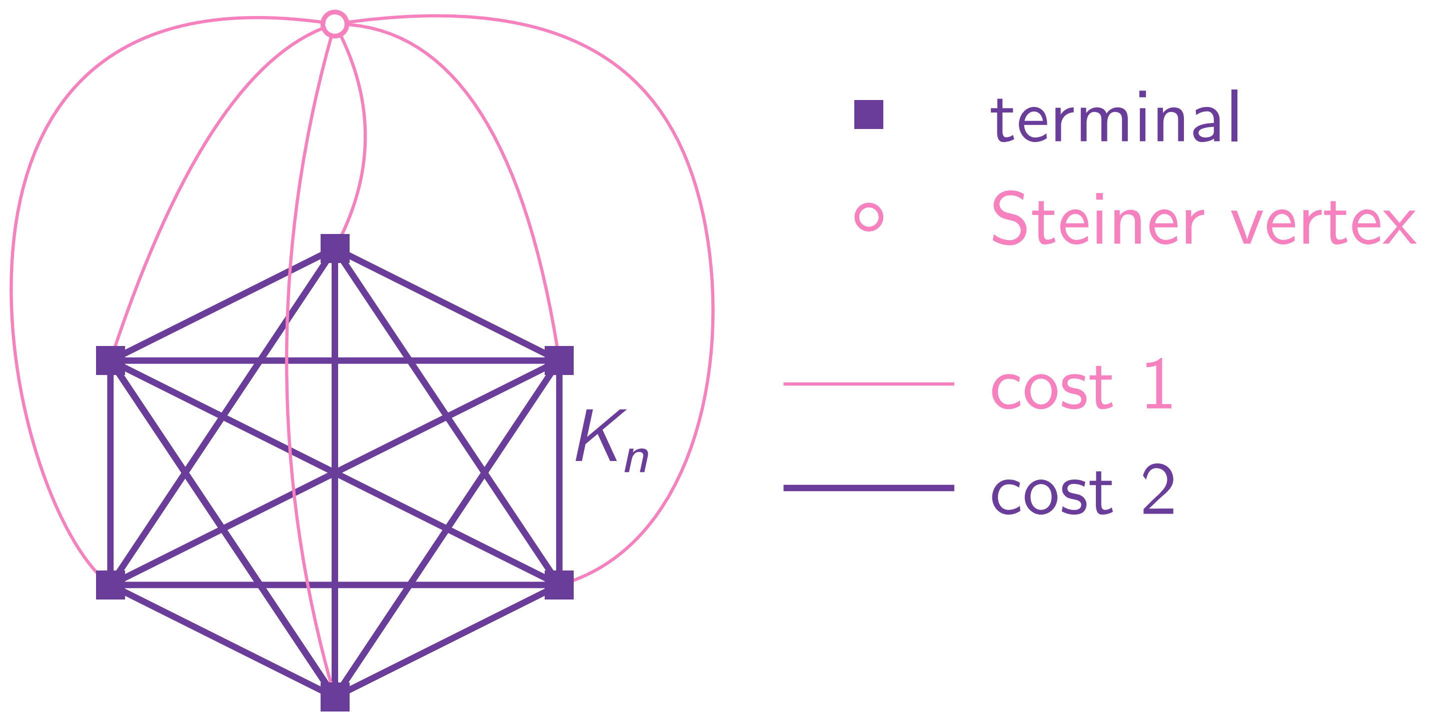
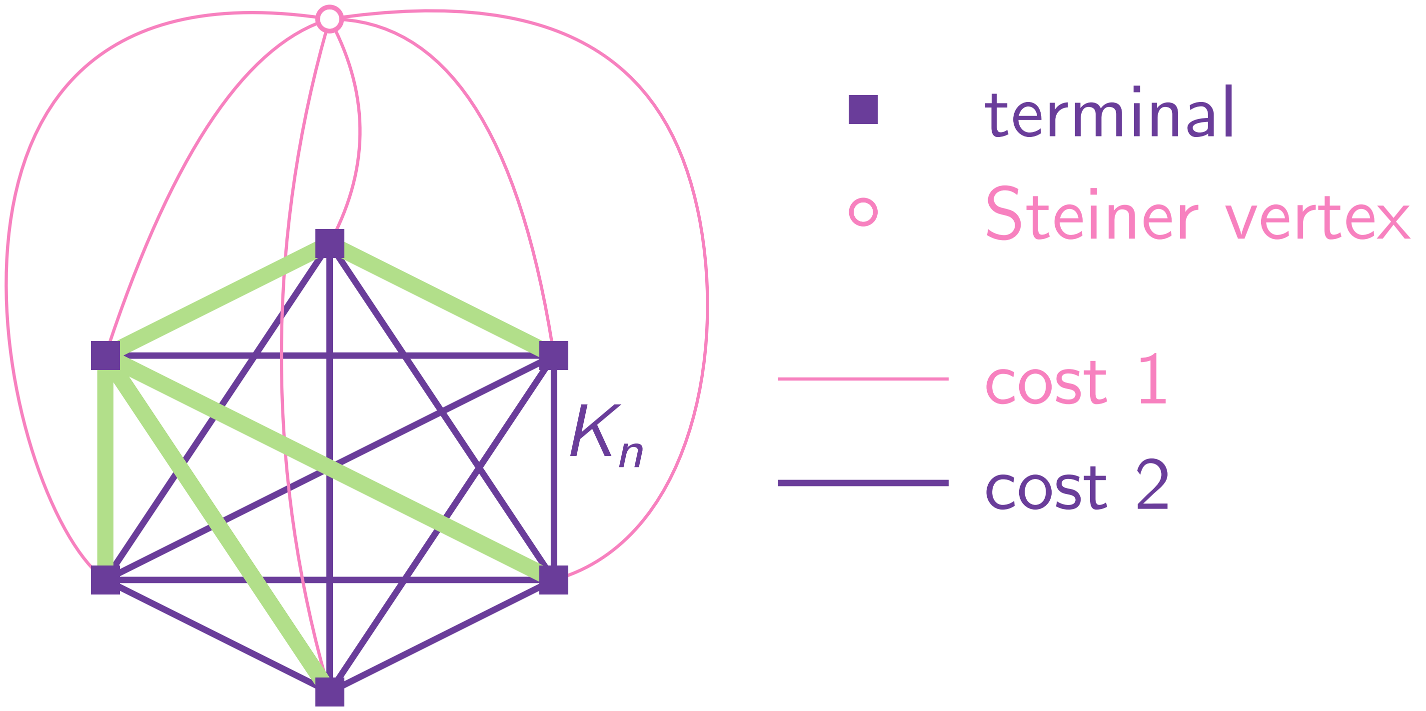


Analysis Tight?

An MST of $G[T]$ has cost



Analysis Tight?

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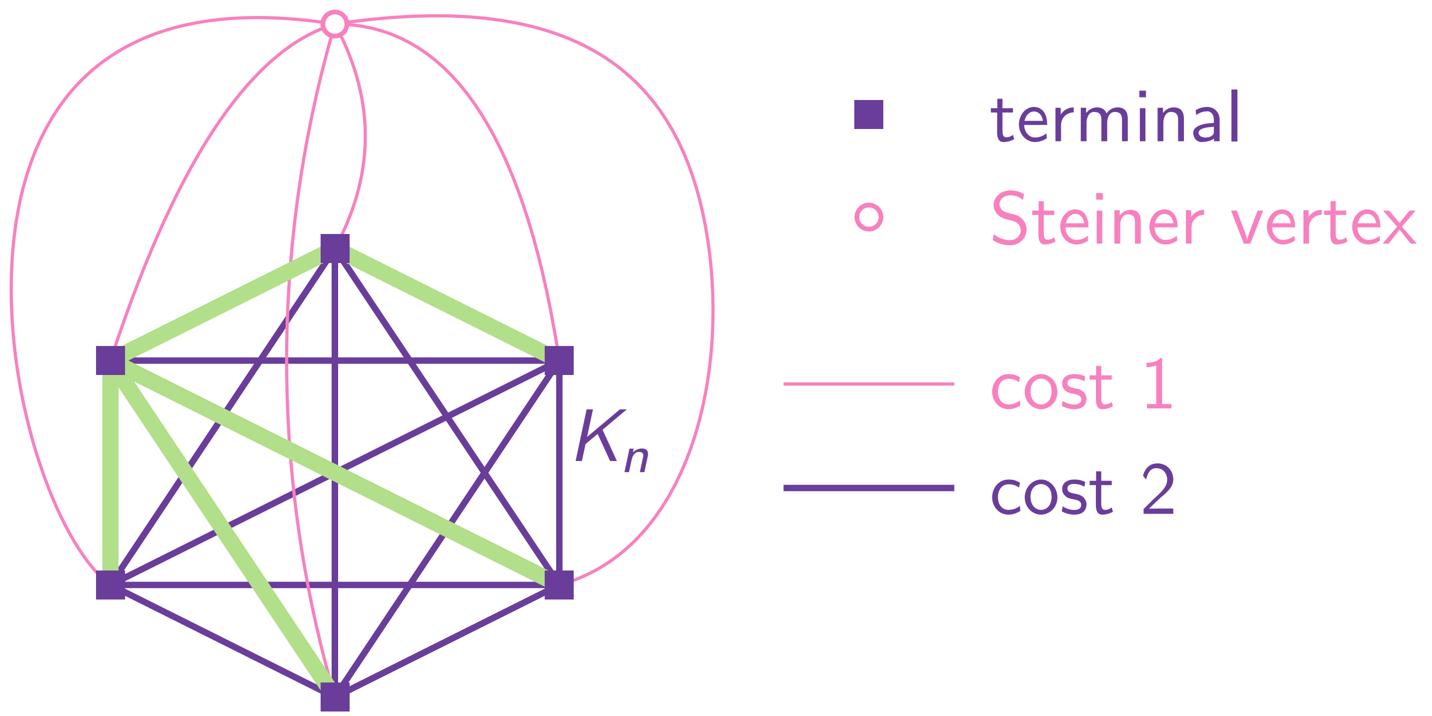

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An MST of $G[T]$ has cost $2(n-1)$.
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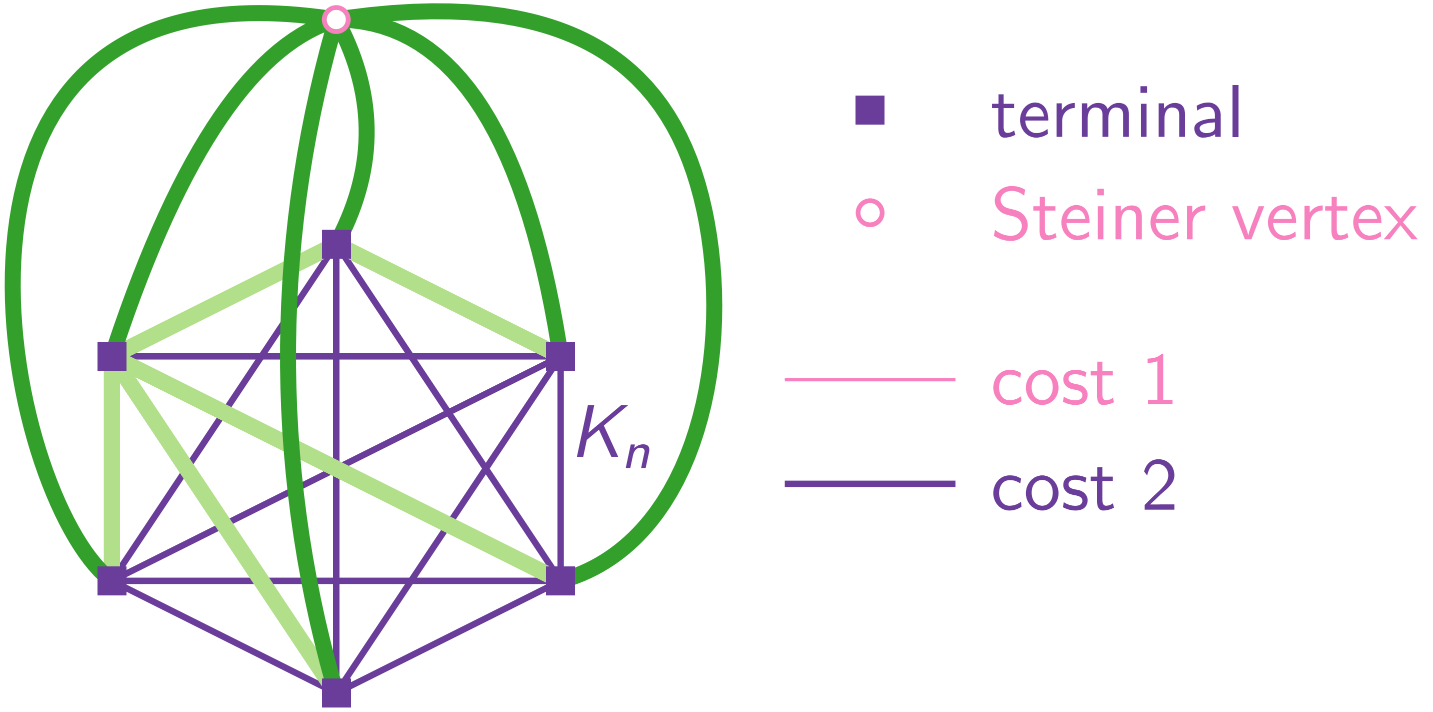

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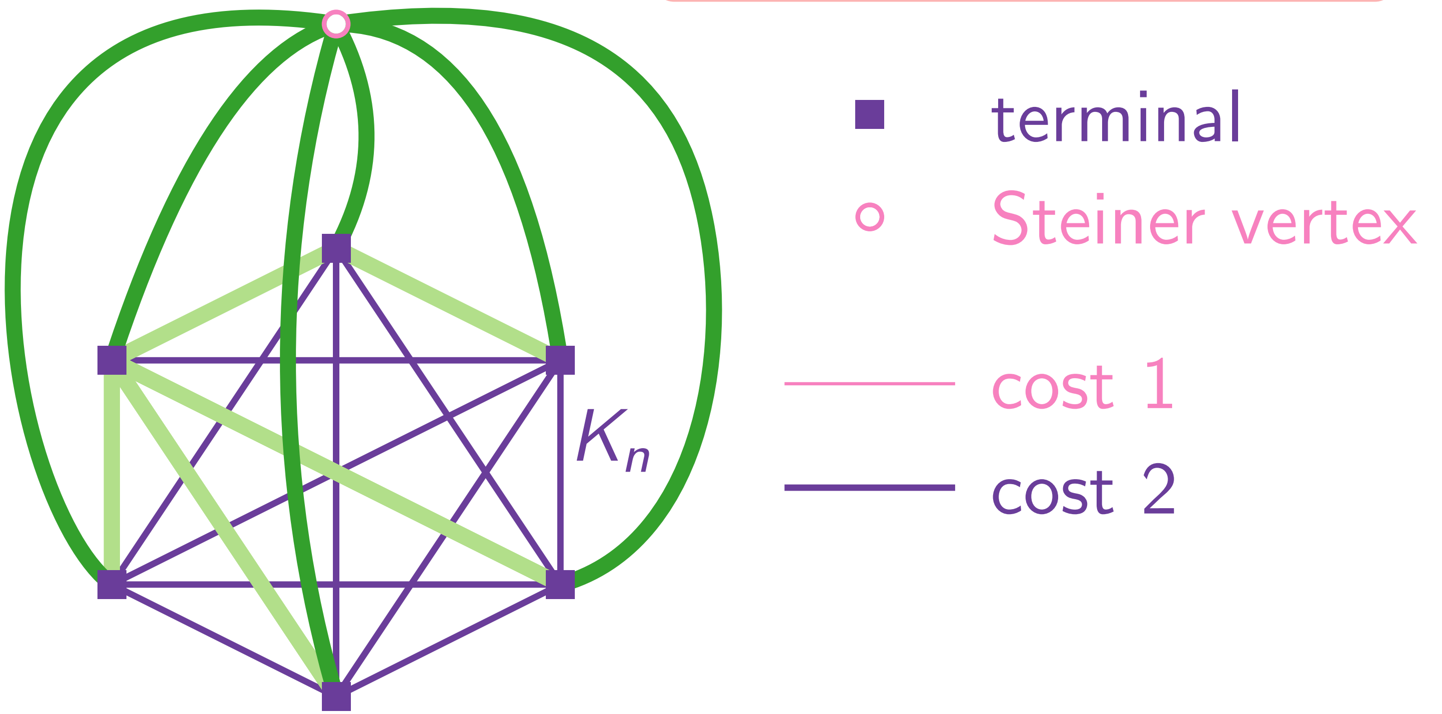

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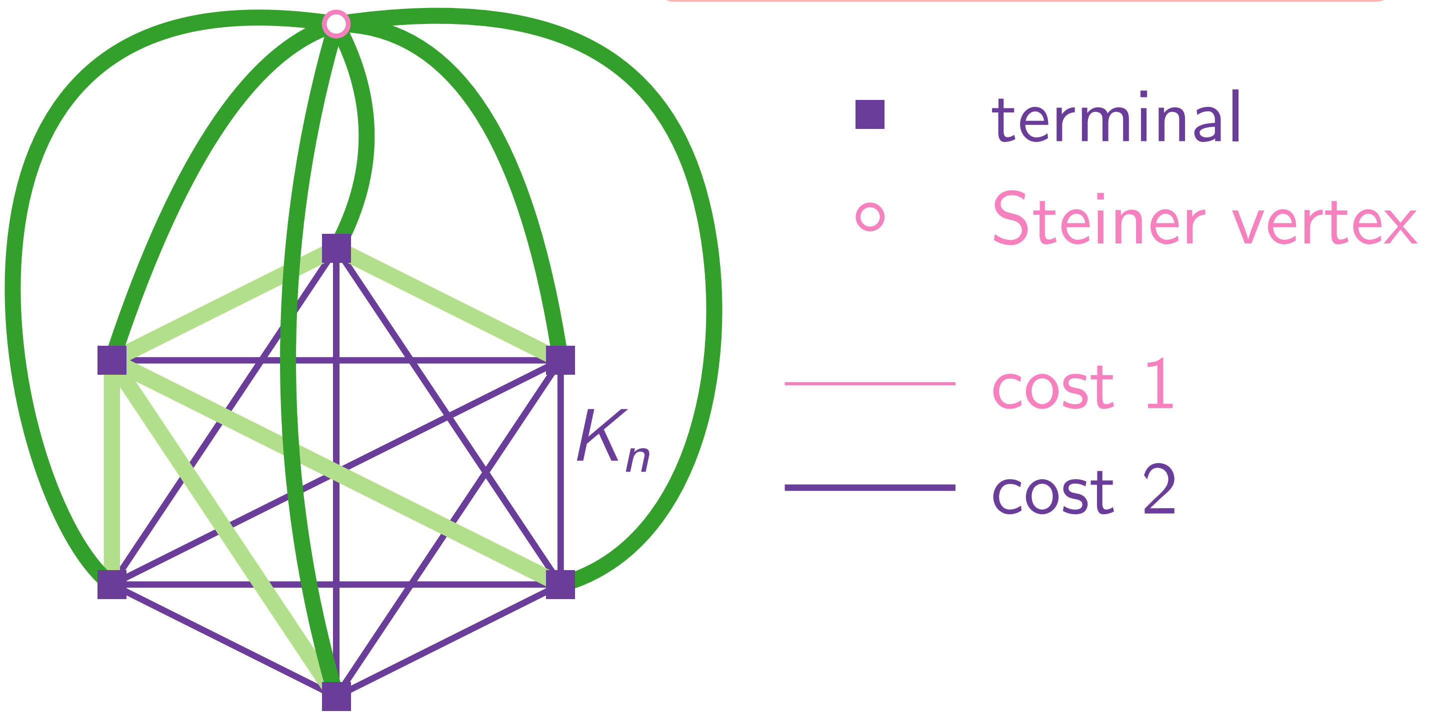

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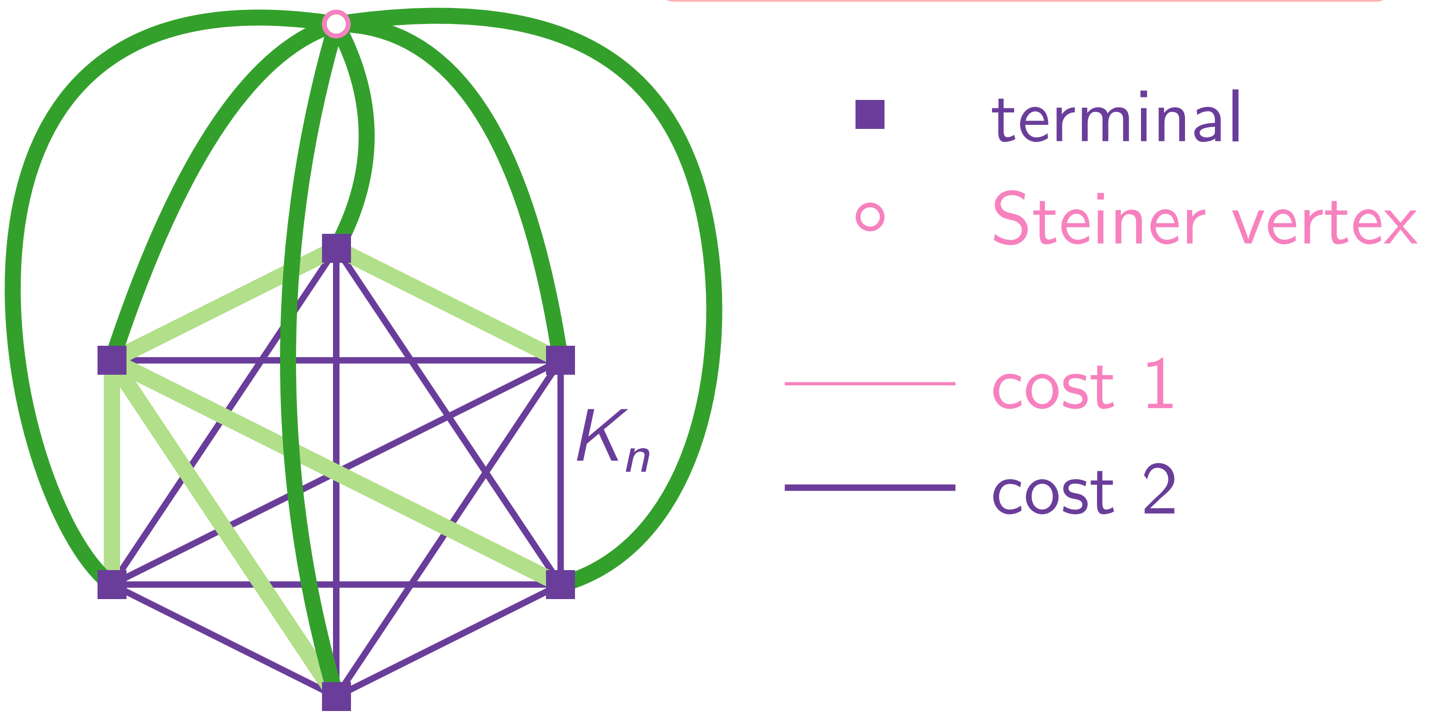

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[Byrka, Grandoni, Rothvoß & Sanità, J. ACM'13]

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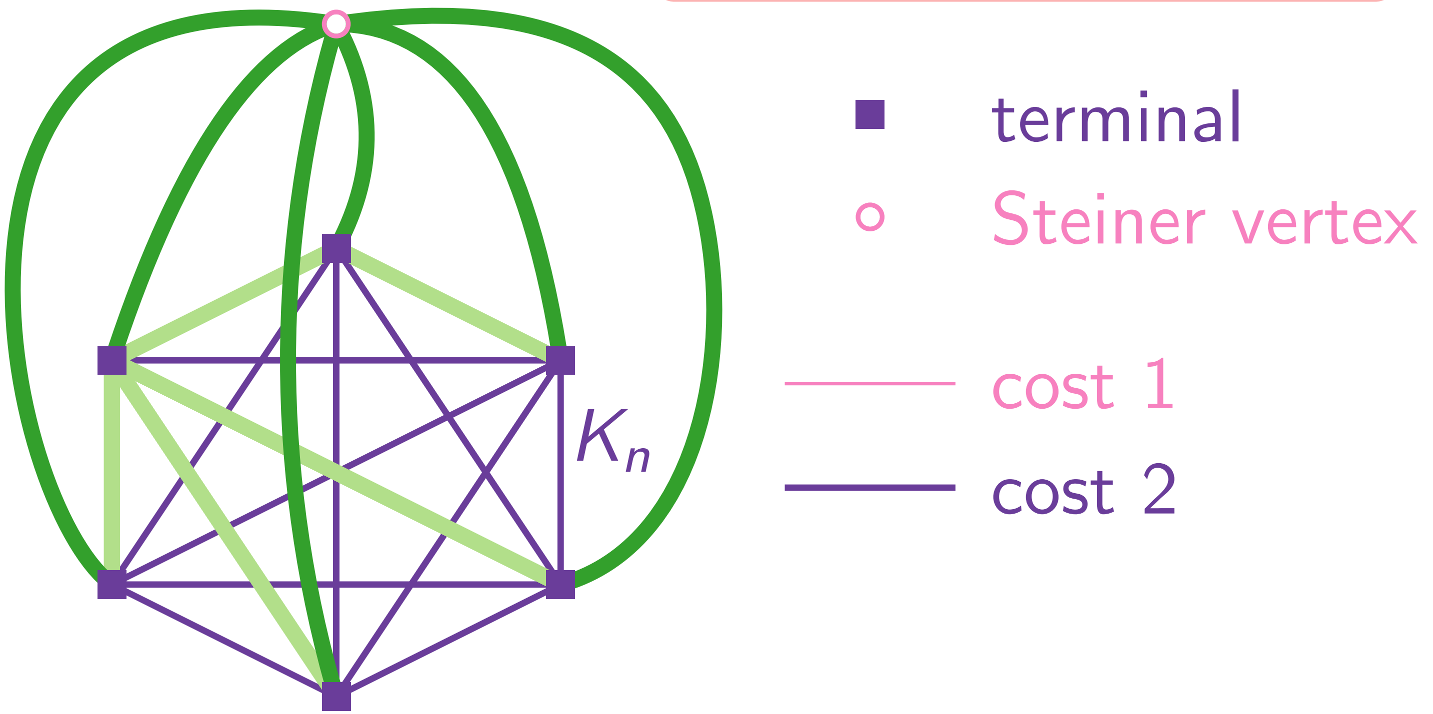

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SteinerTree cannot be approximated within factor $\frac{96}{95} \approx 1.0105$ (unless P = NP).

[Chlebík & Chlebíková, TCS'08]

Approximation Algorithms

Lecture 3: SteinerTree and MultiwayCut

Part V: MultiwayCut

MultiwayCut

Given: A connected graph G

MultiwayCut

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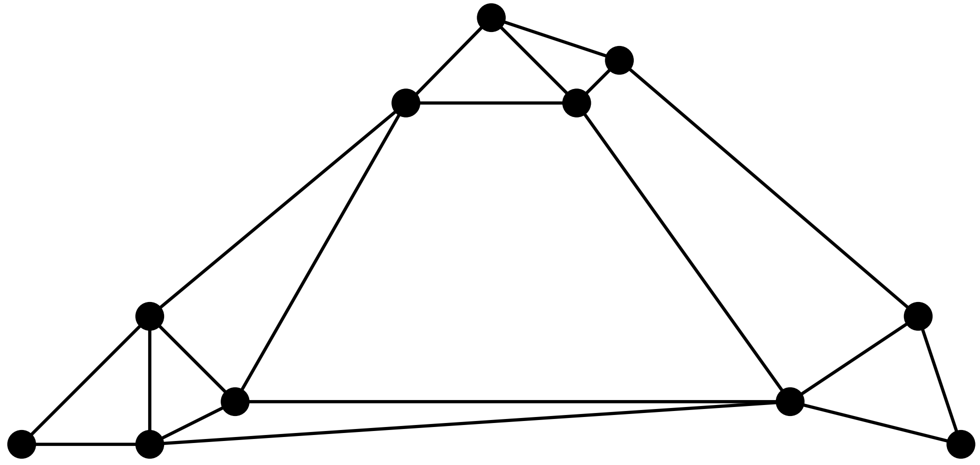
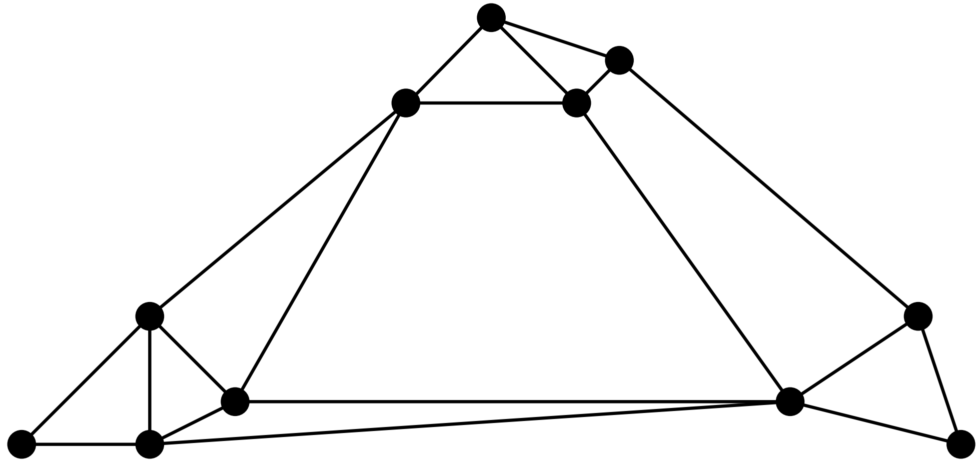


MultiwayCut

Given: A connected graph G with edge costs $c\colon E(G) \to \mathbb{Q}^+$



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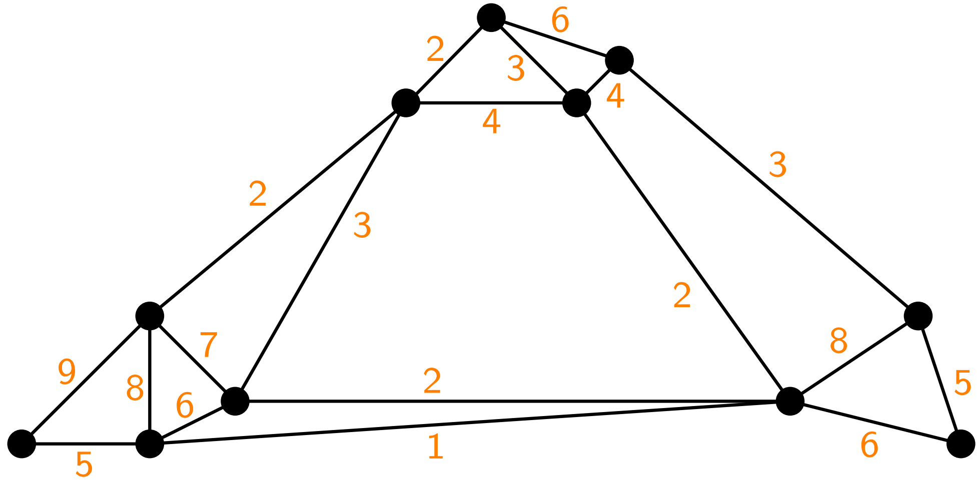

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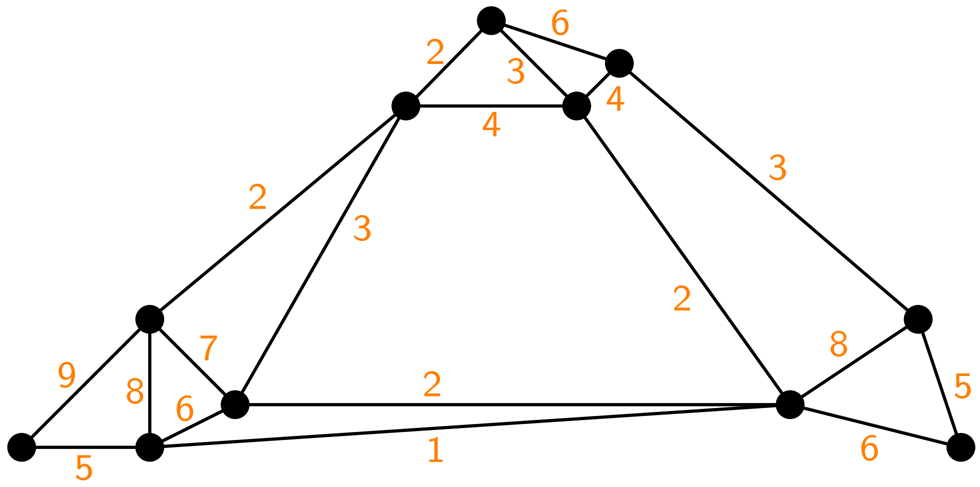

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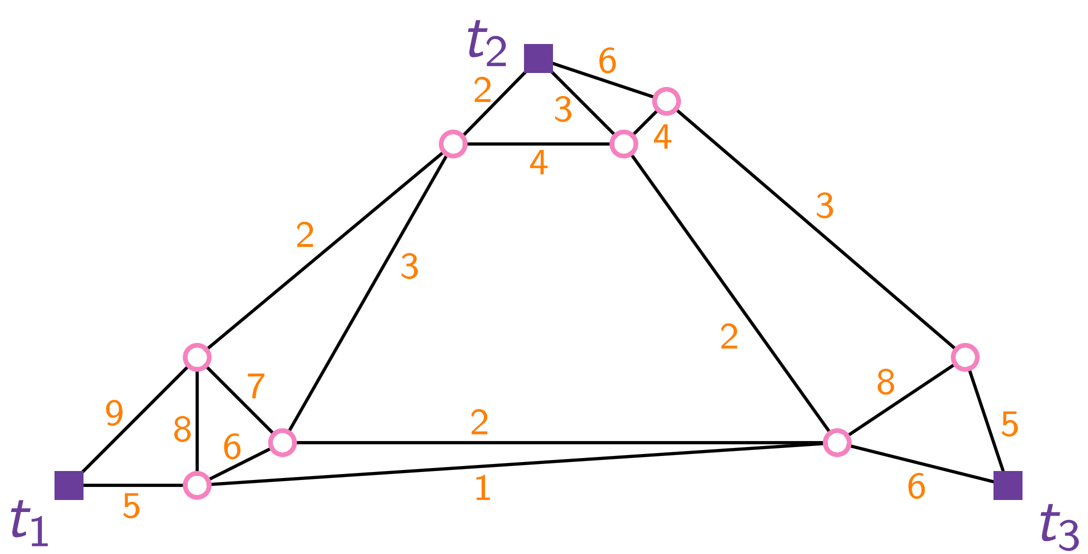

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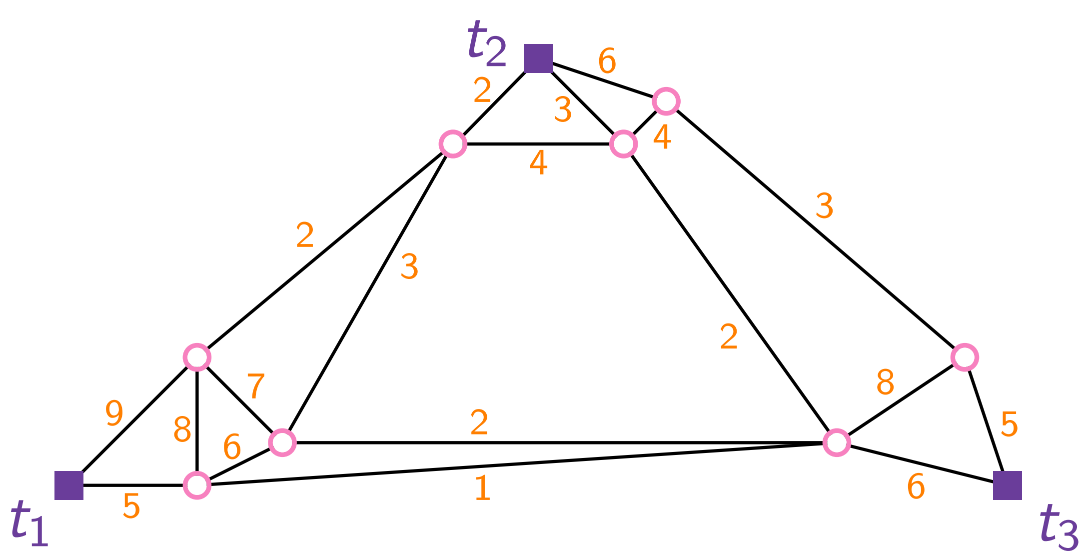

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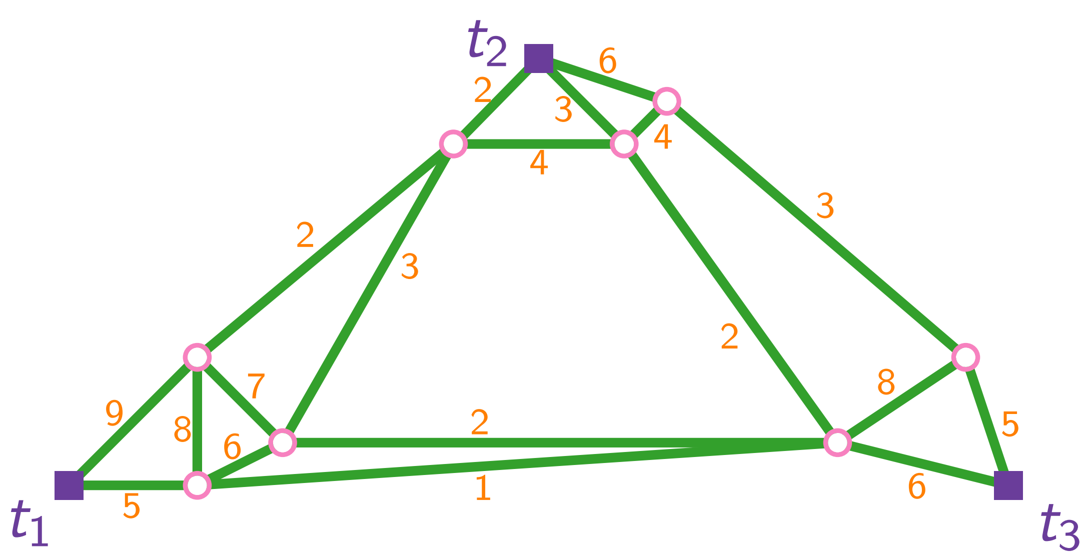

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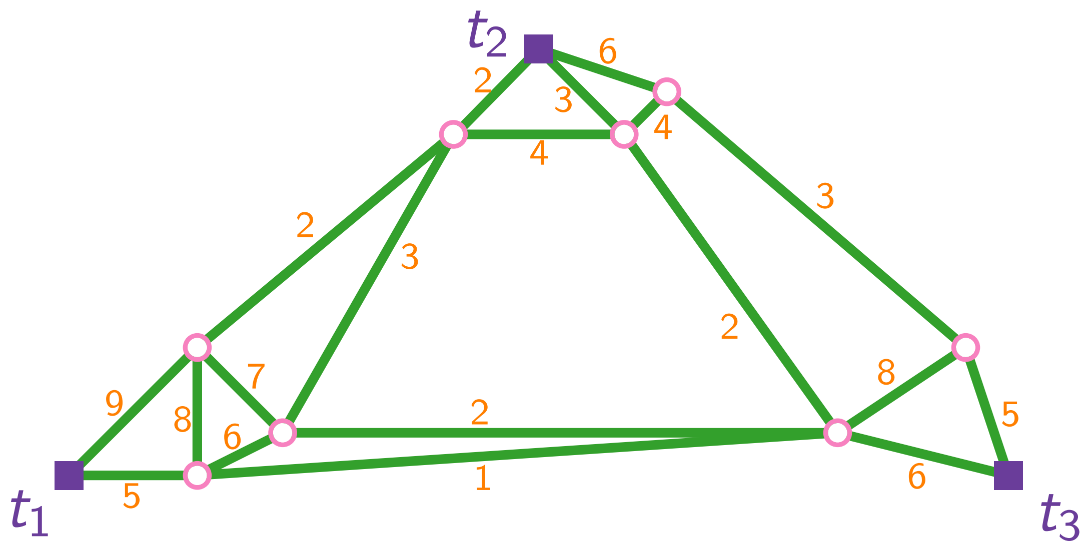

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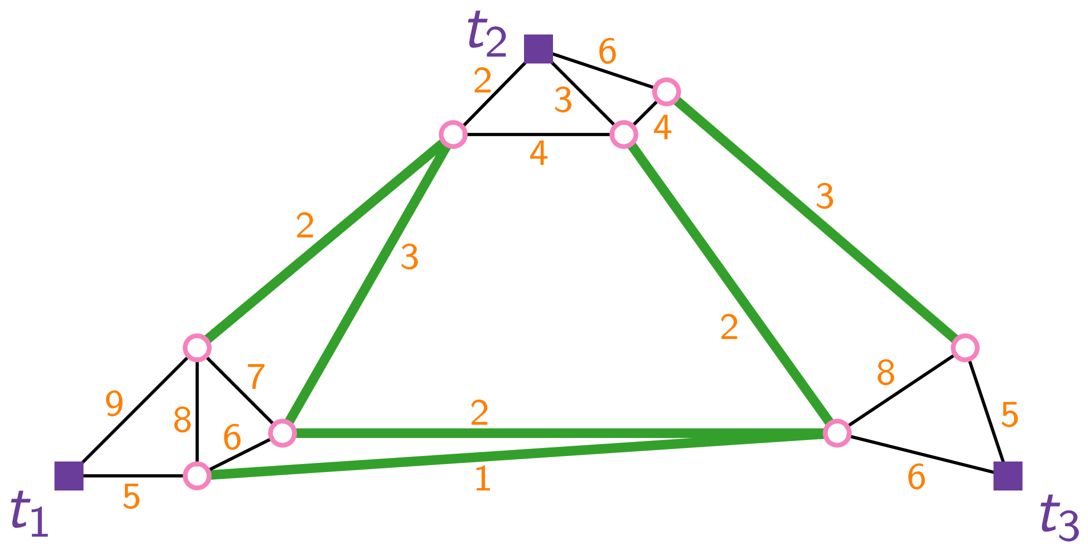

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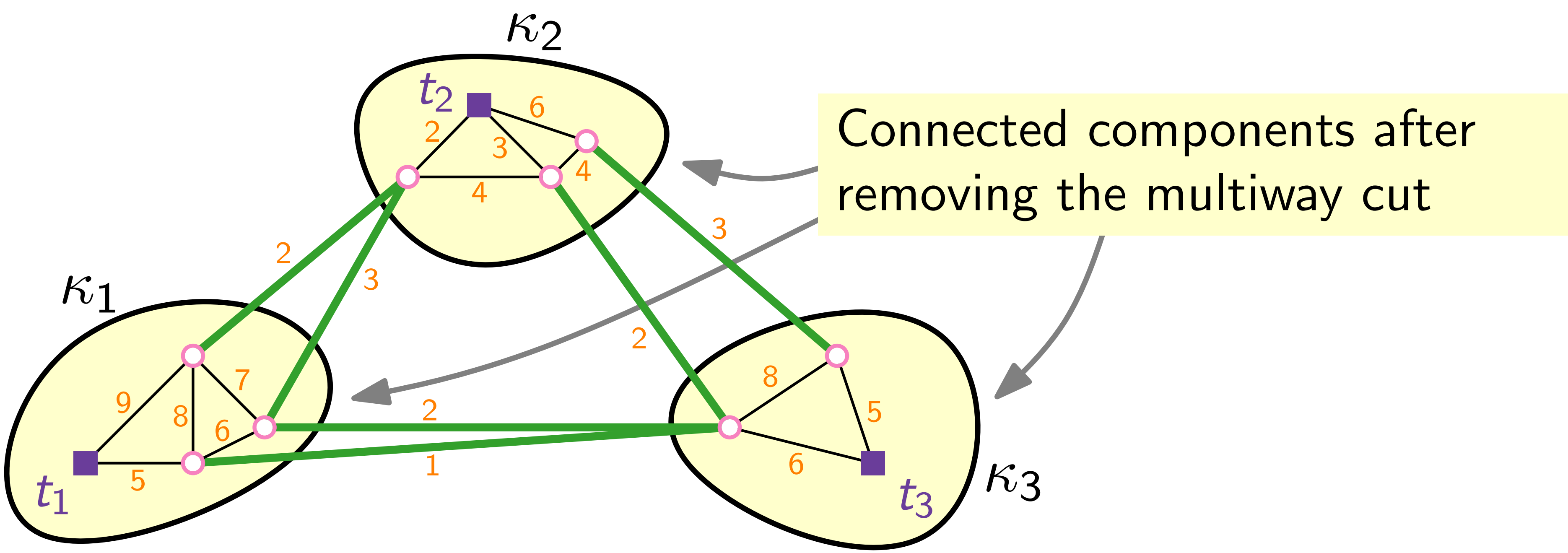

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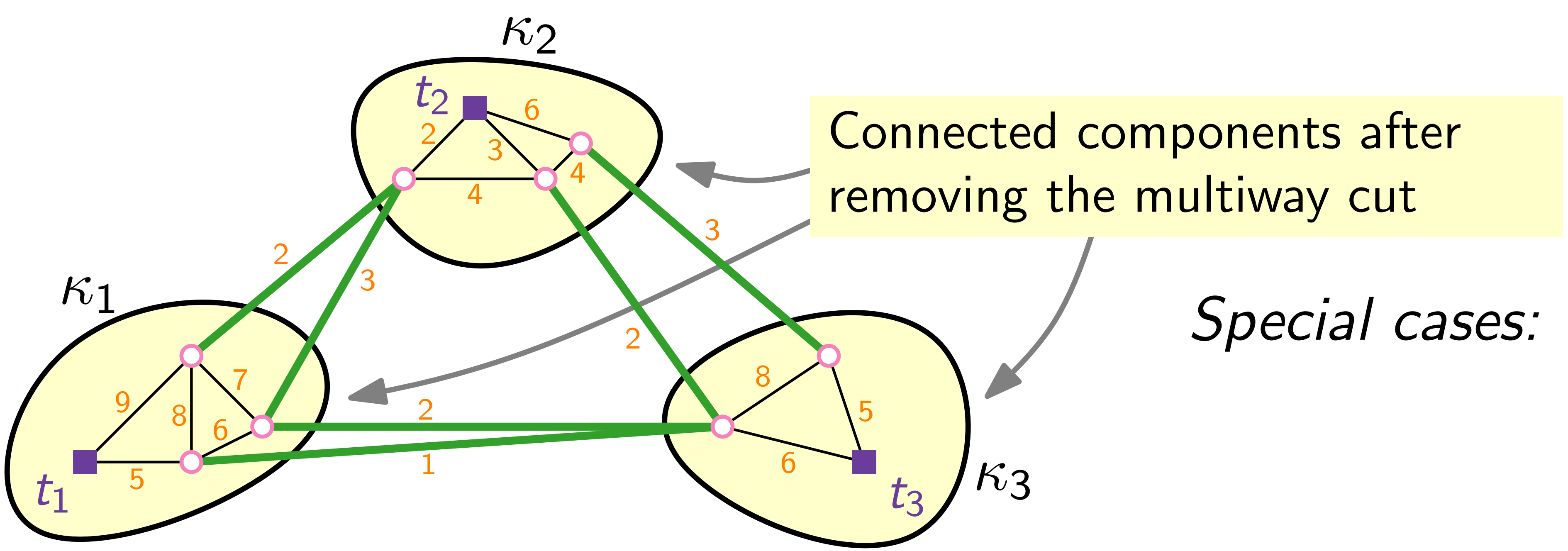

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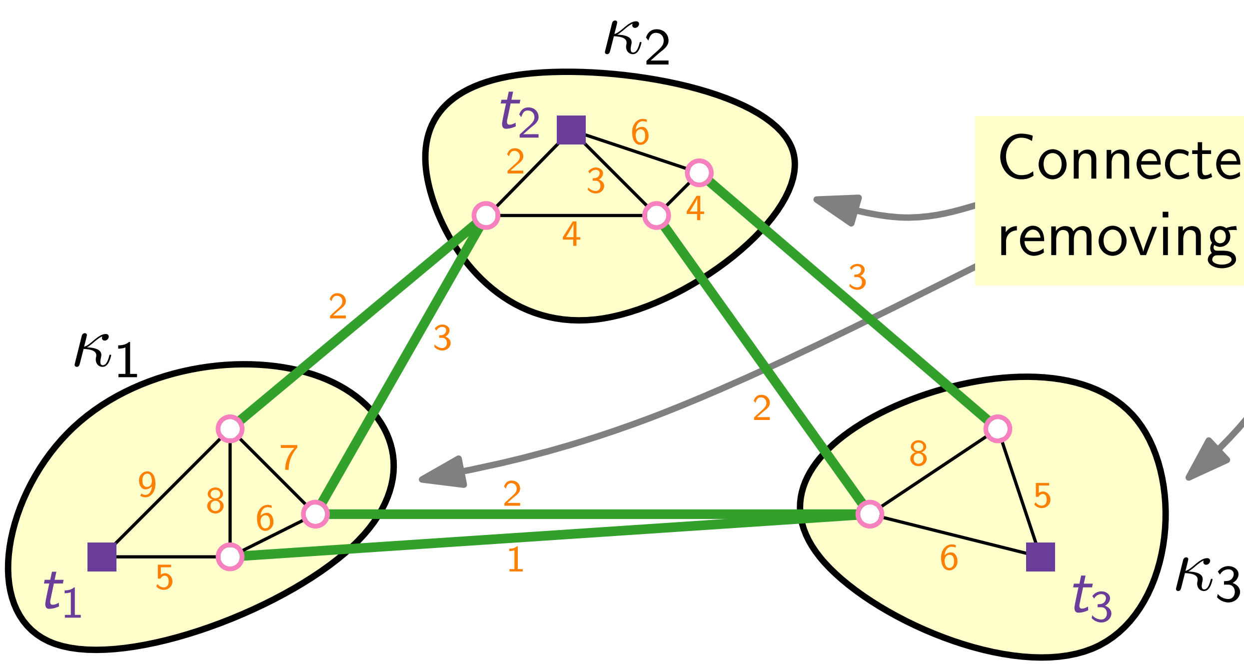

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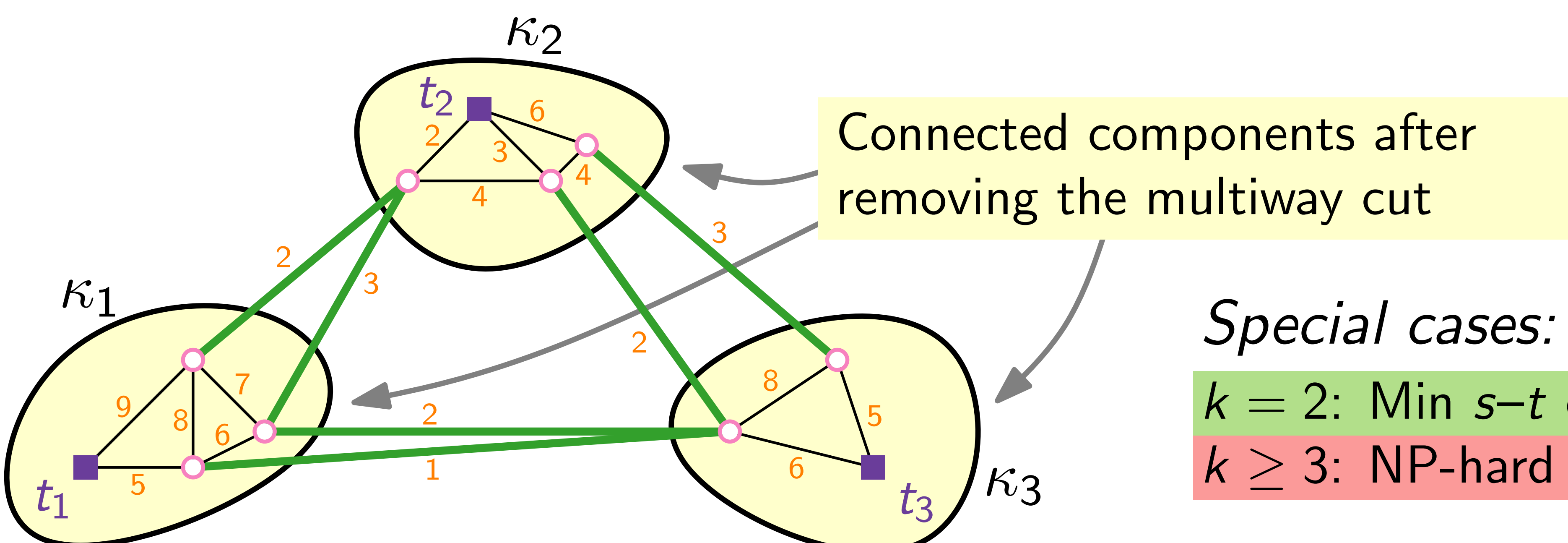

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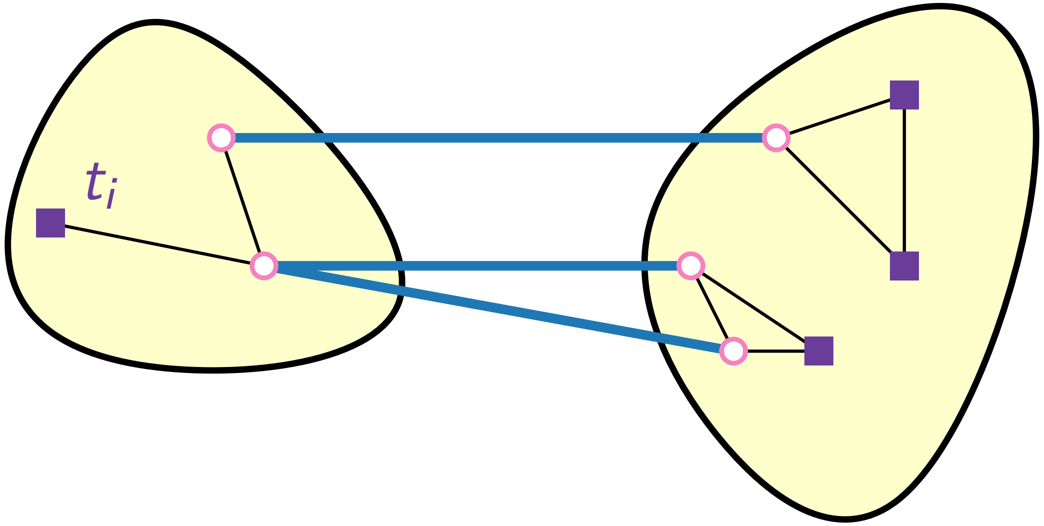

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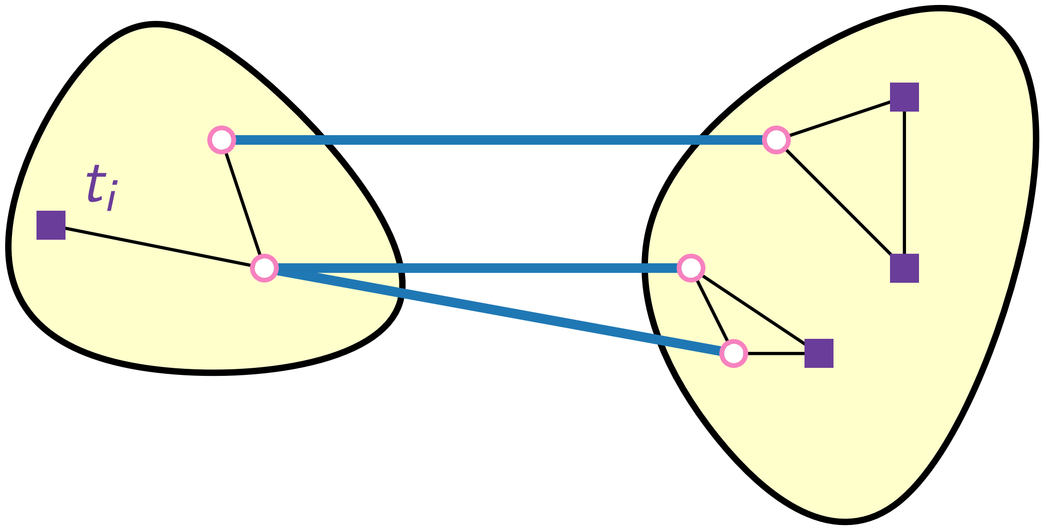

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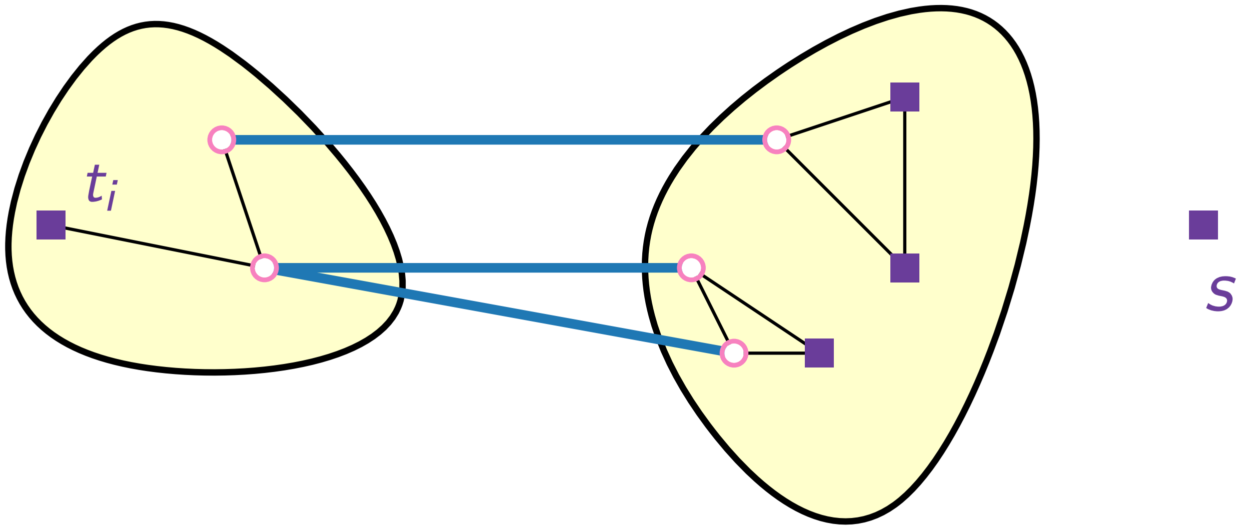

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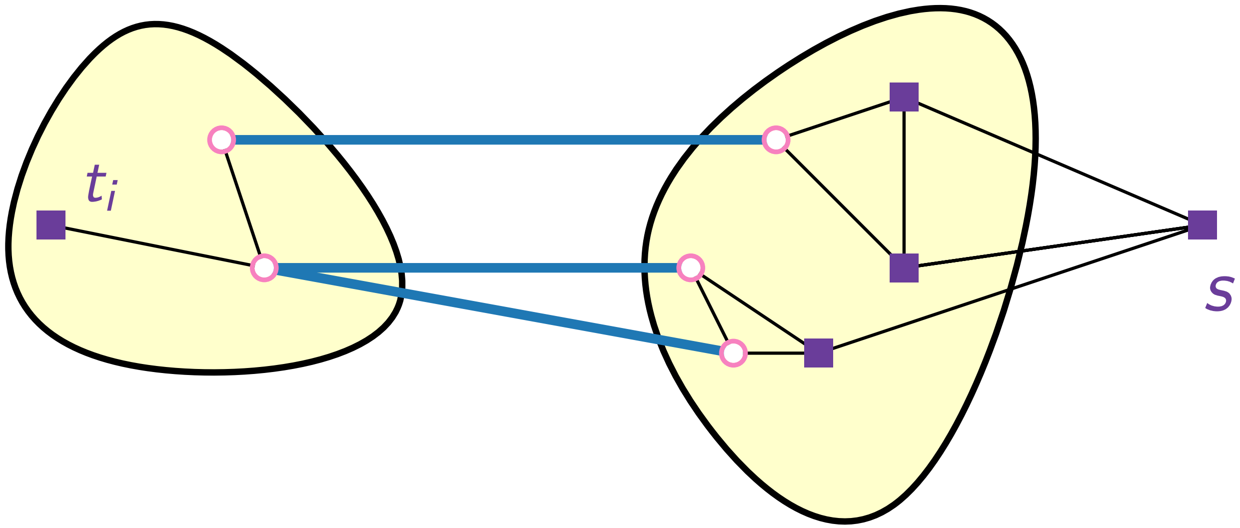

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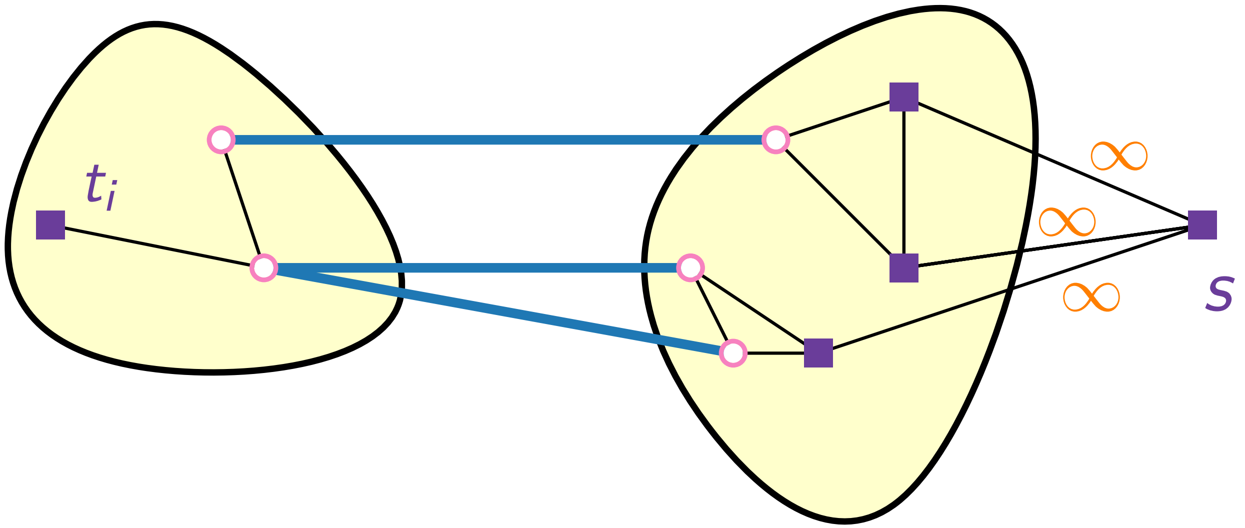

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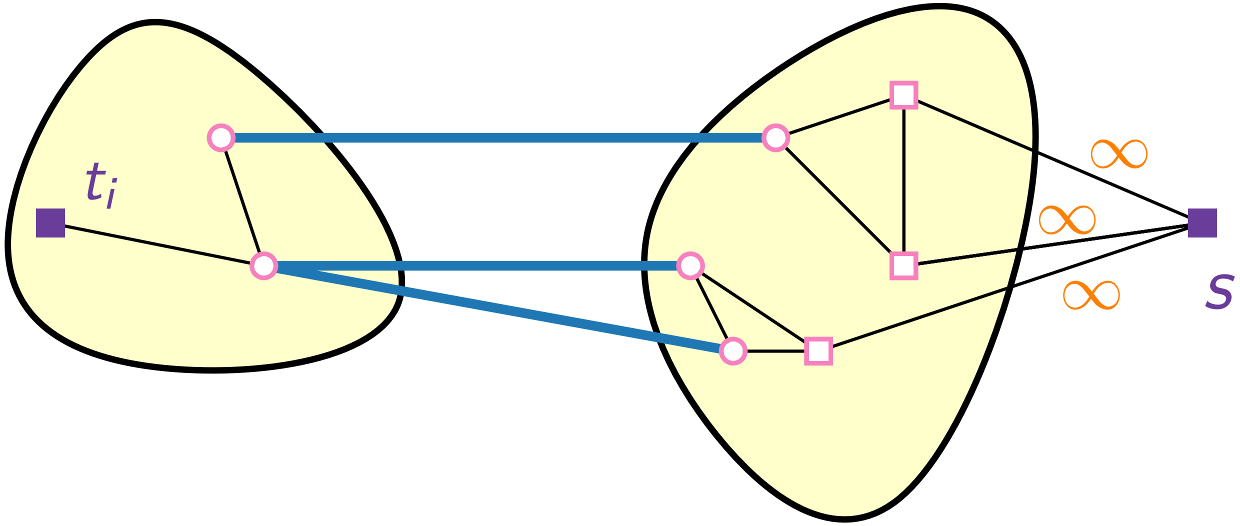

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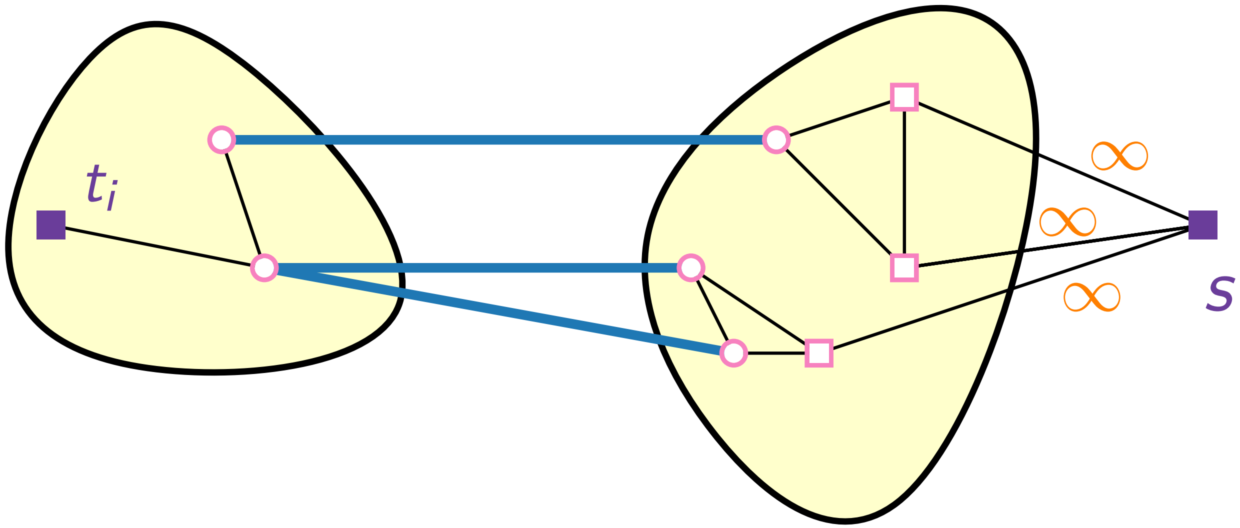

Add dummy terminal s and find a minimum-cost s–t_i cut.

Approximation Algorithms

Lecture 3: SteinerTree and MultiwayCut

Part VI: Algorithm for MultiwayCut

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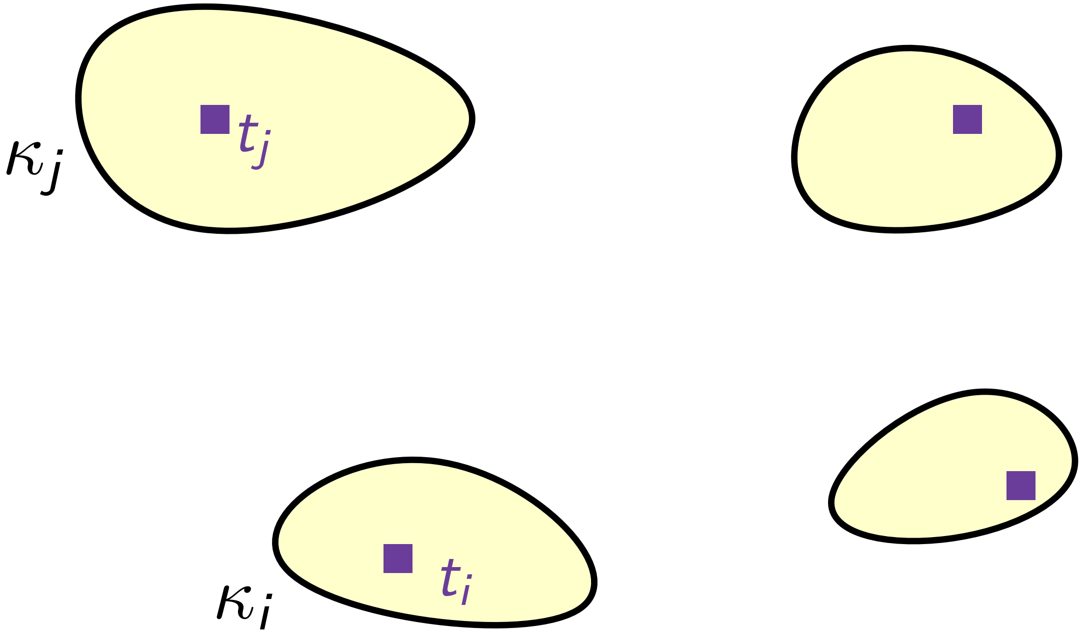

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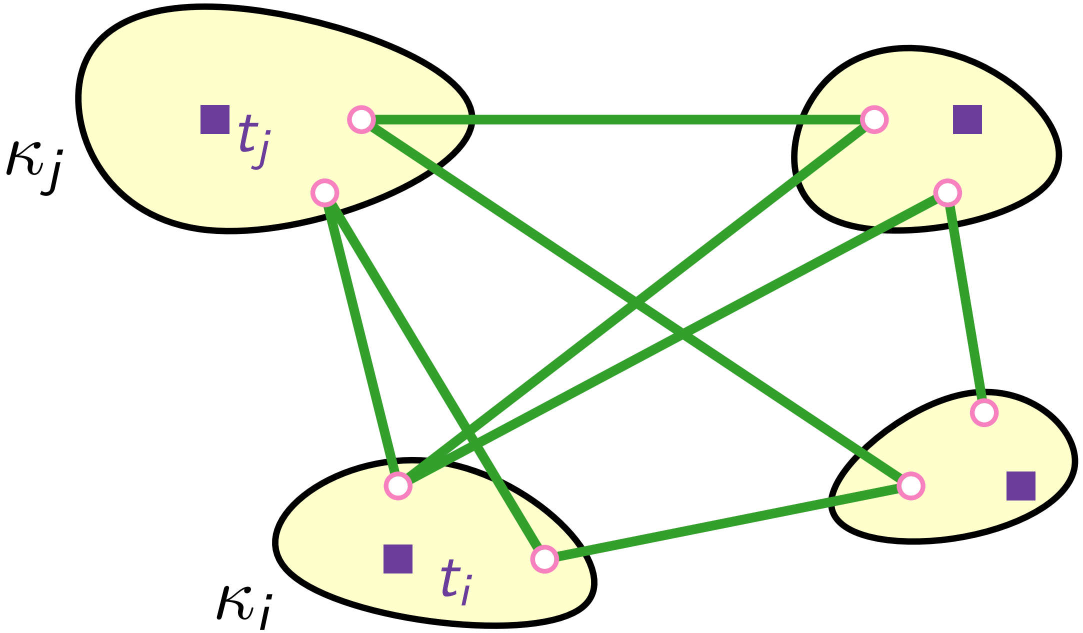

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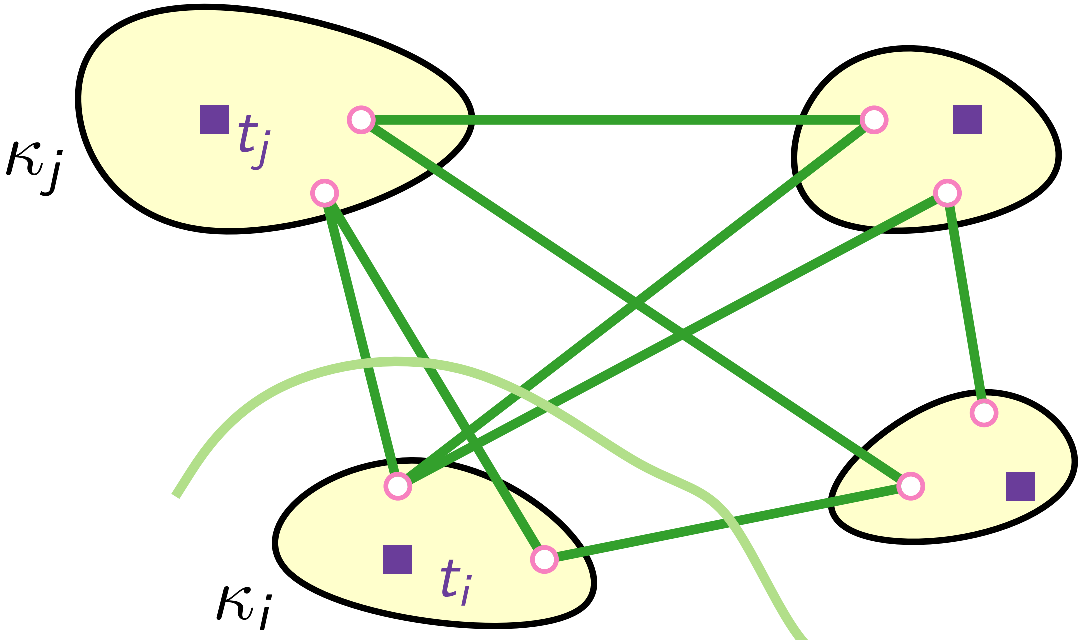

$A_i = \{uv \in \mathcal{A} \colon u \in \kappa_i, v \notin \kappa_i\}$

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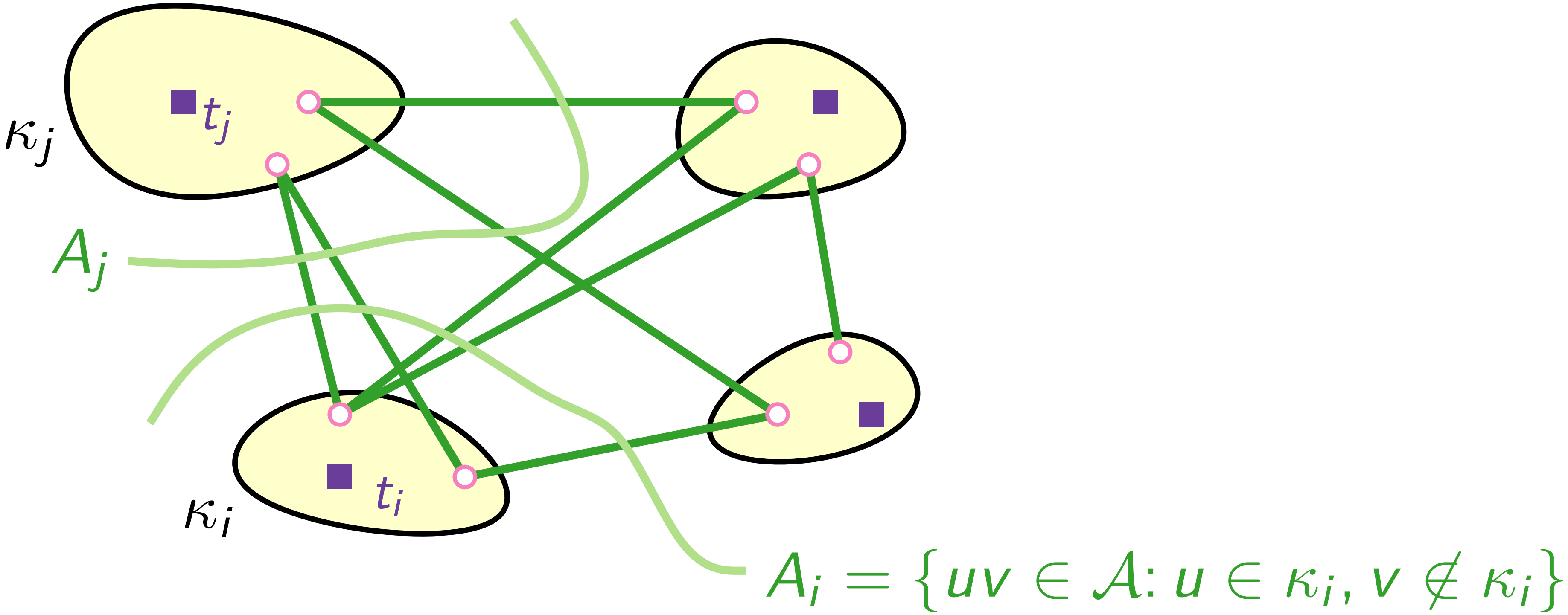

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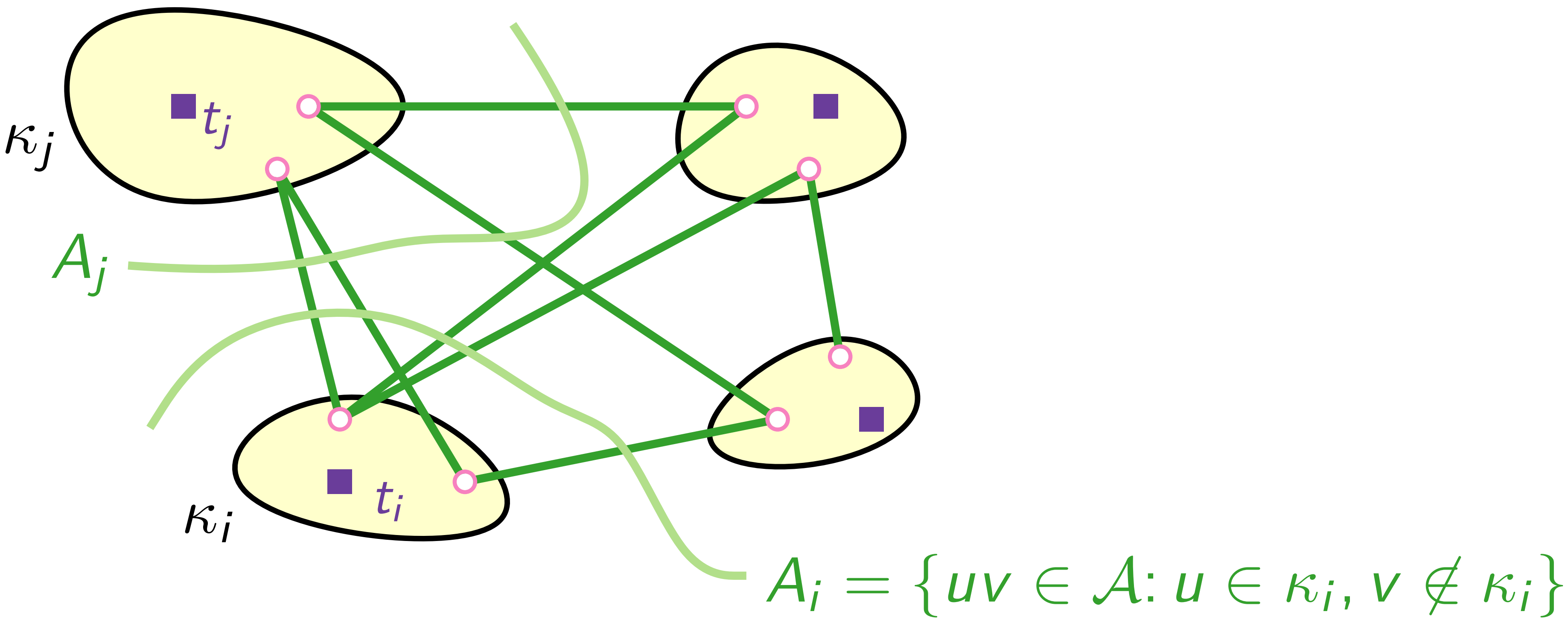

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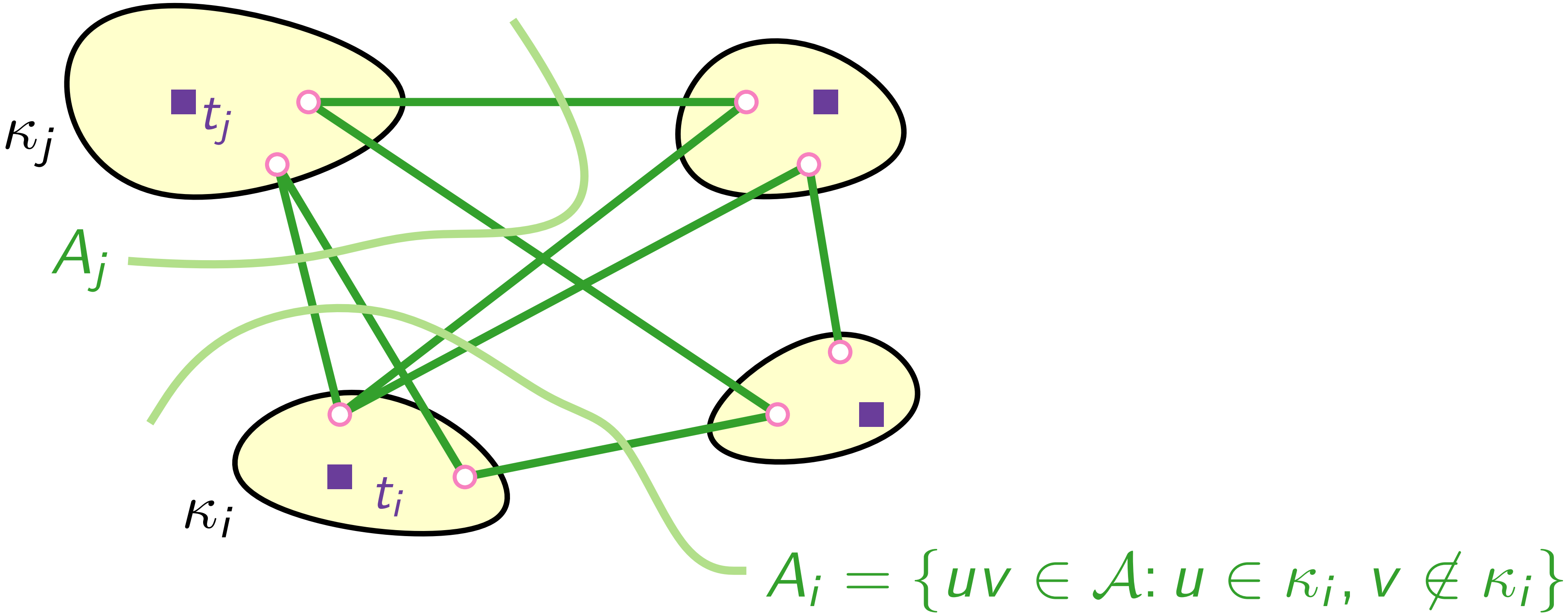

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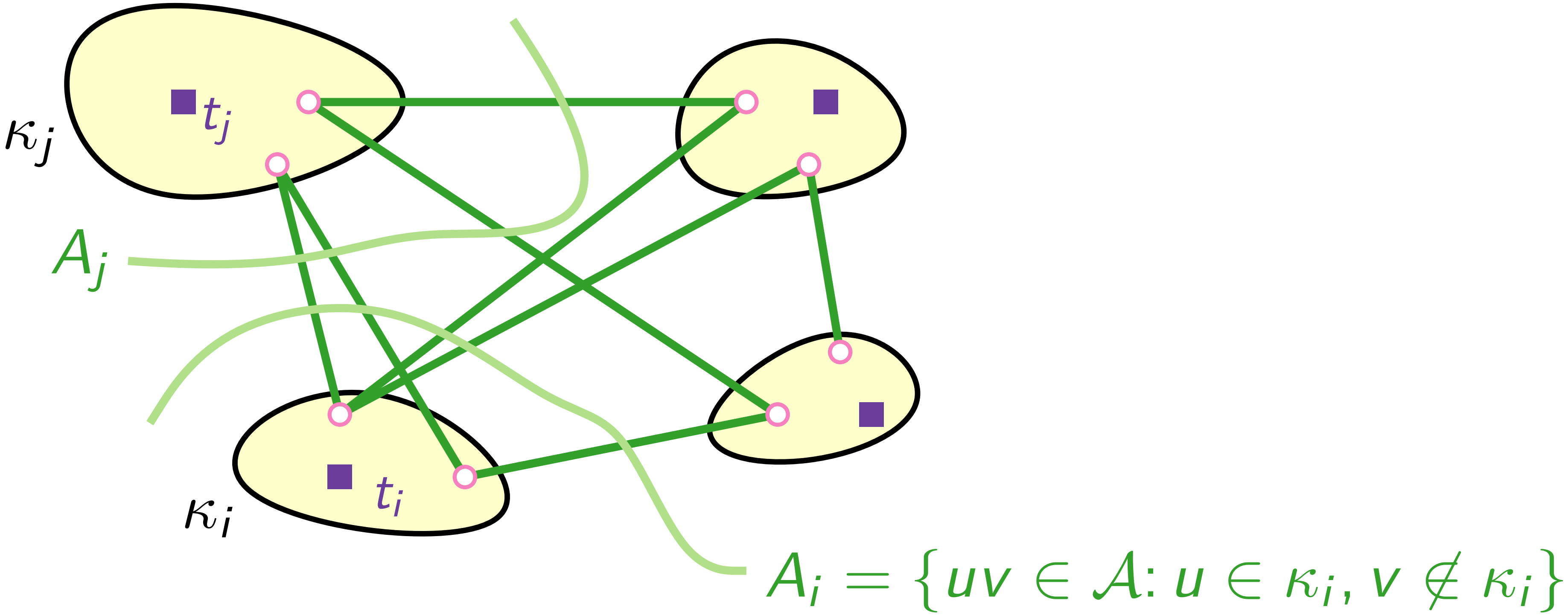

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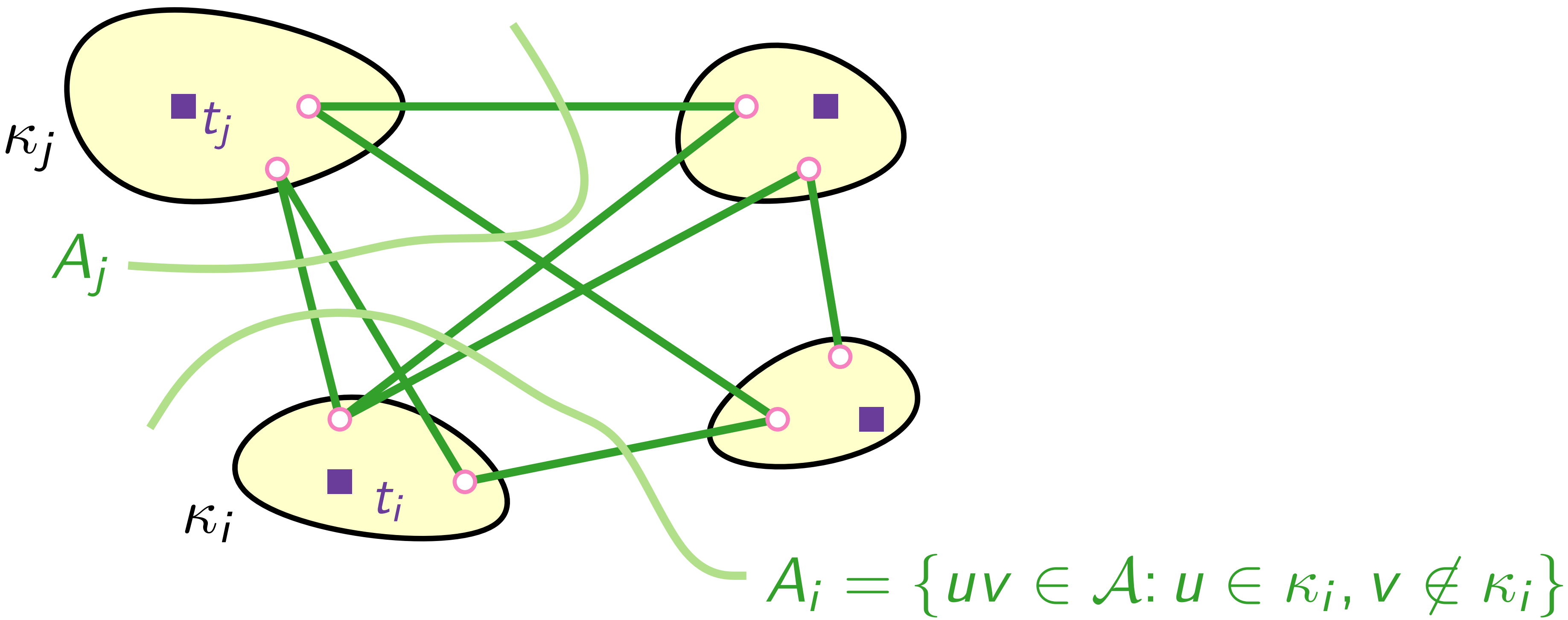

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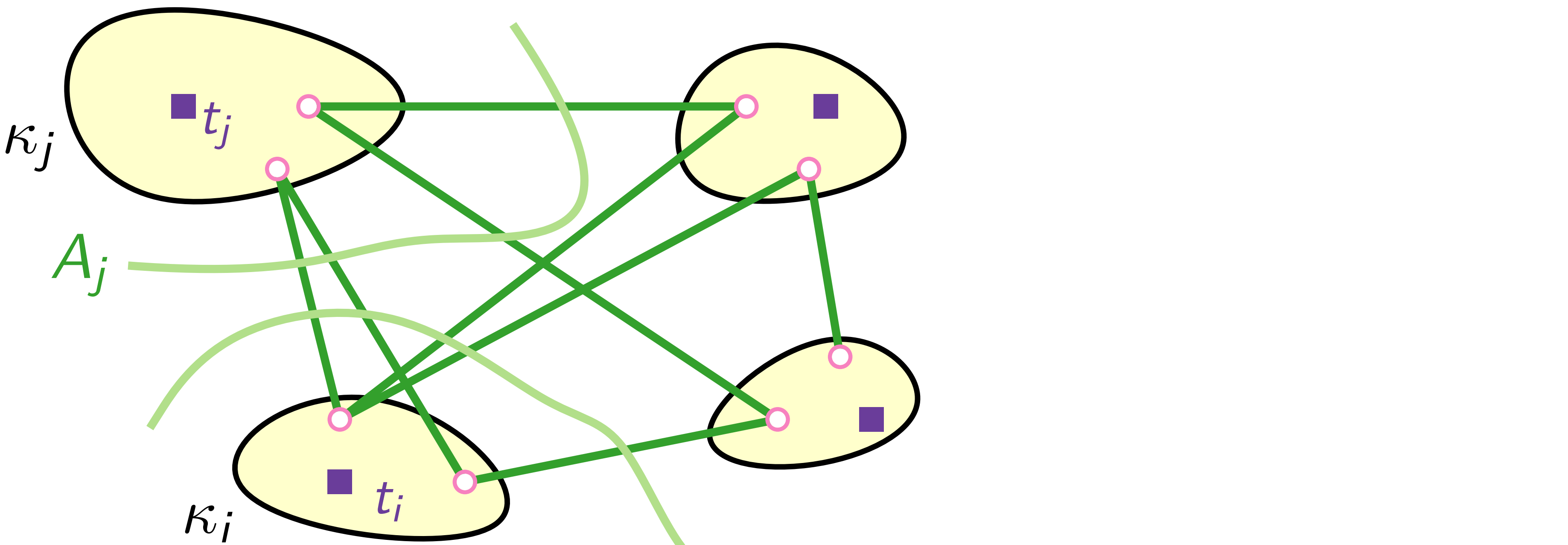

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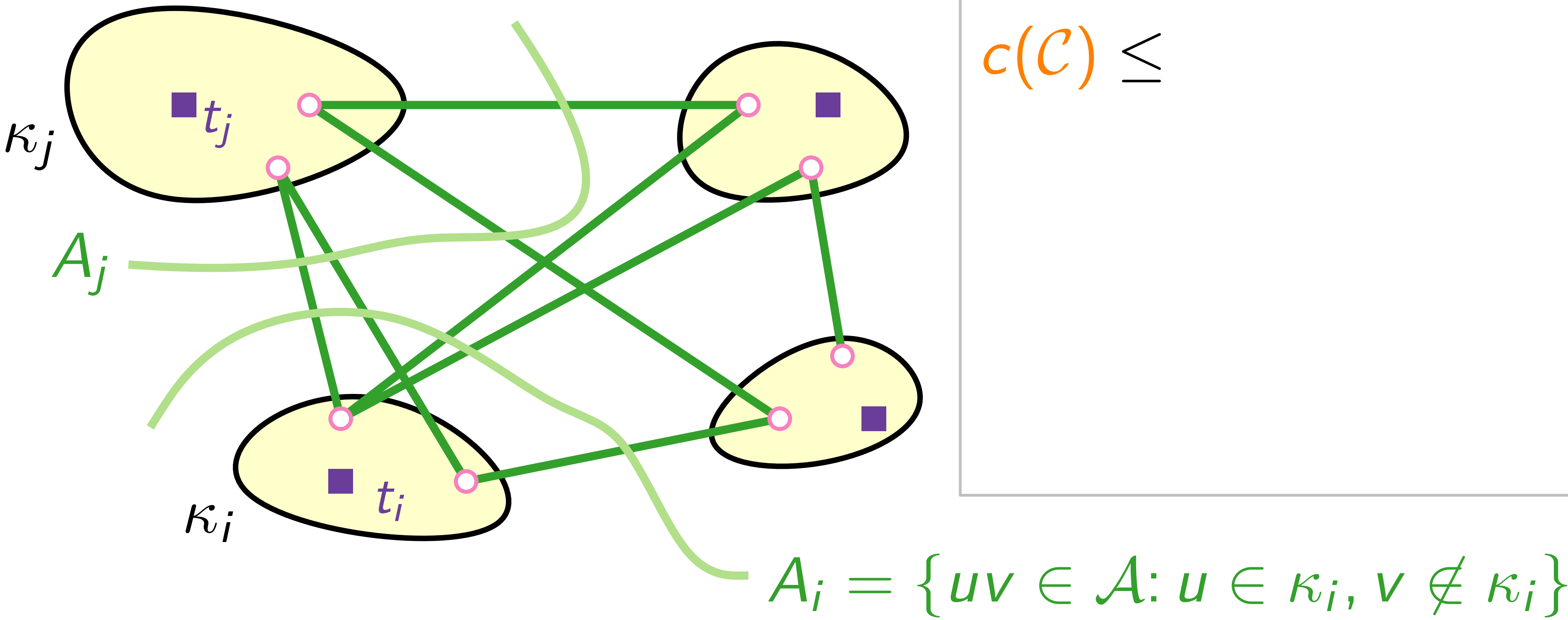

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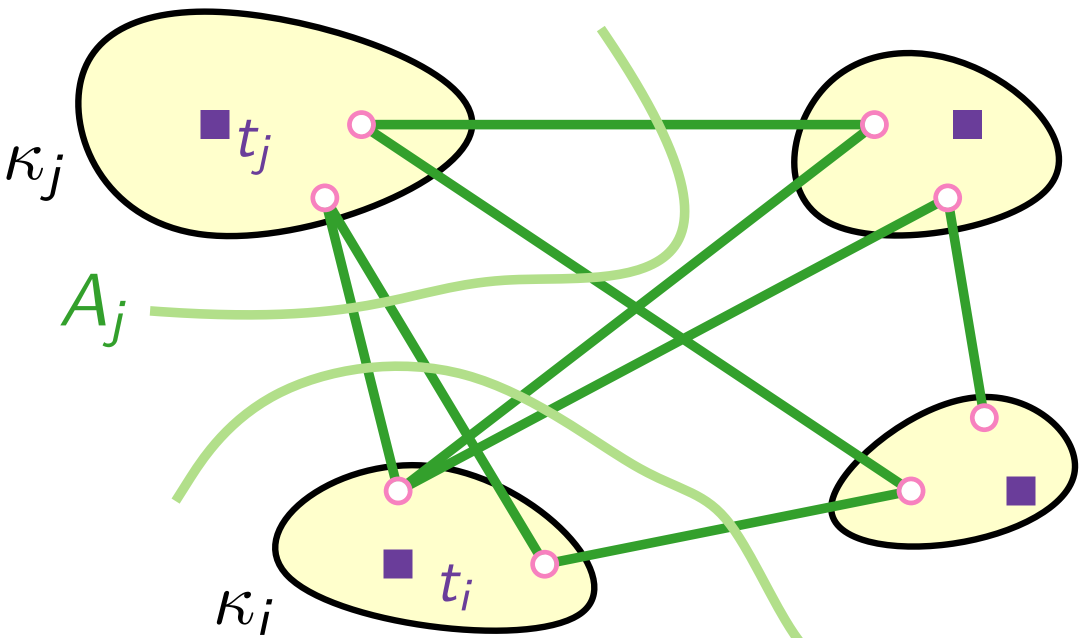

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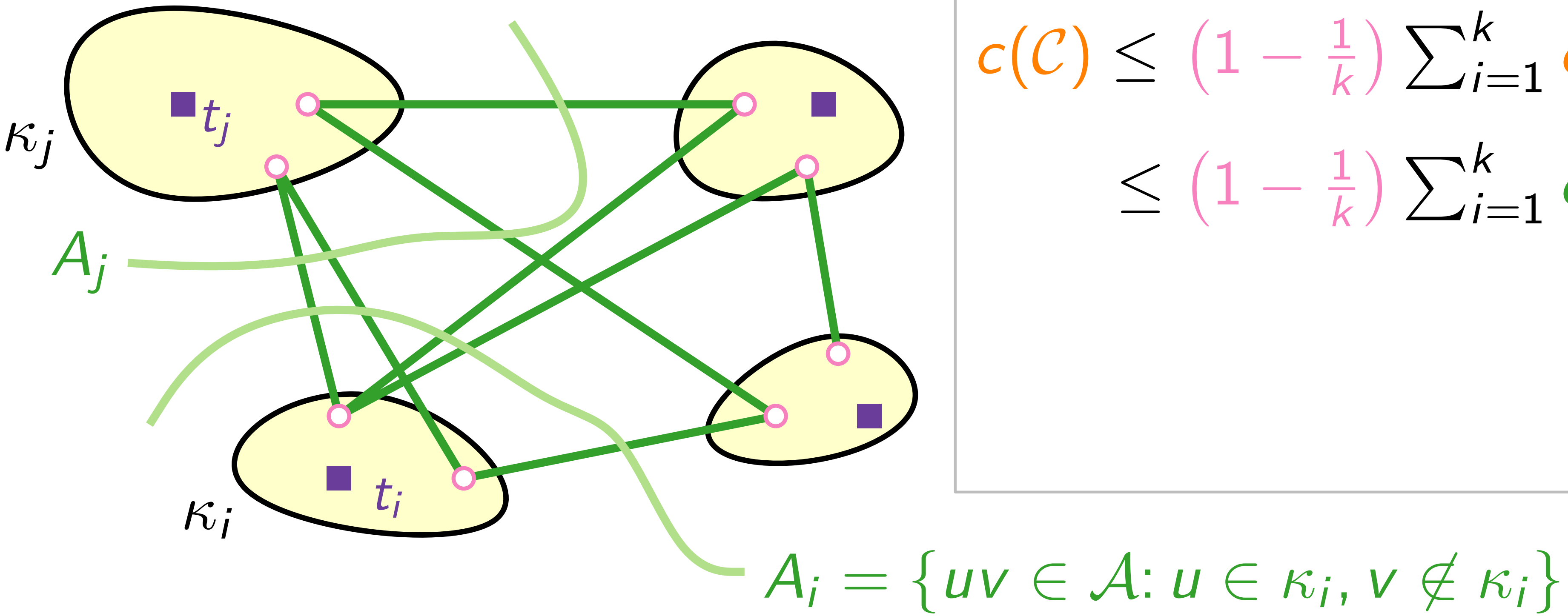

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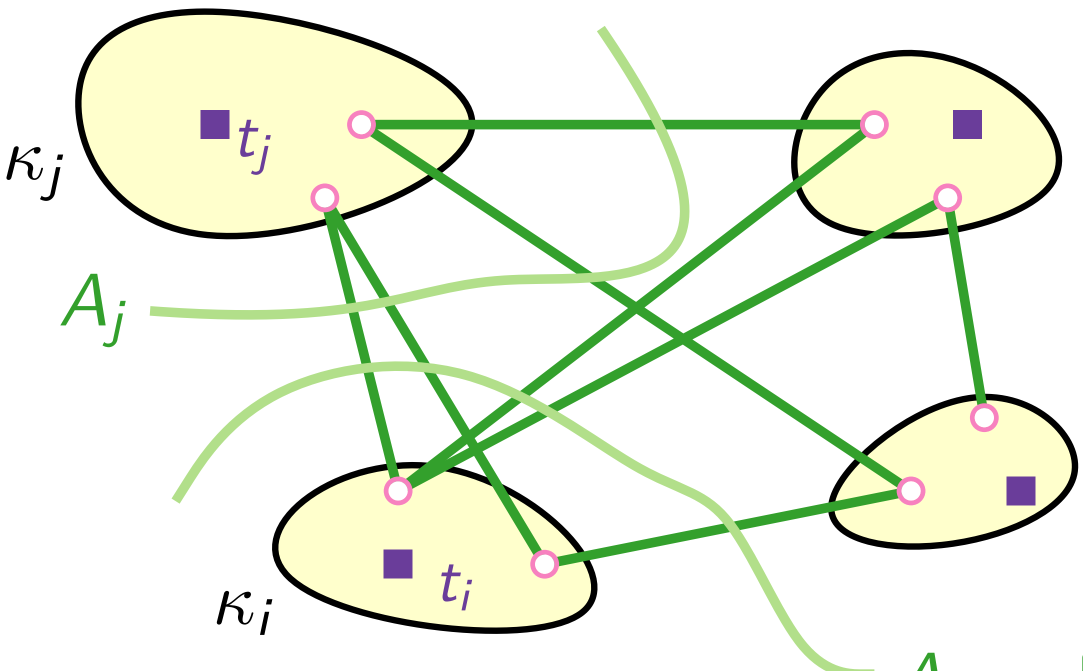

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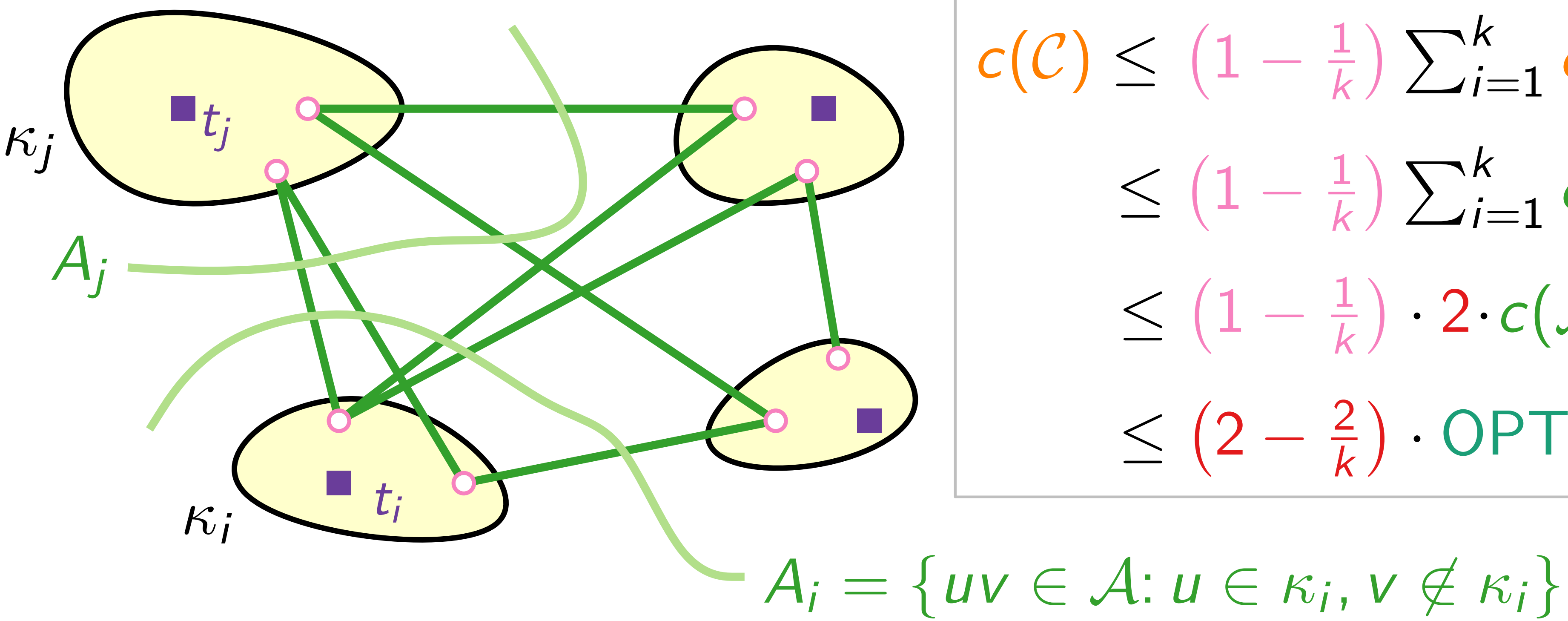

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Analysis Tight?

K_k

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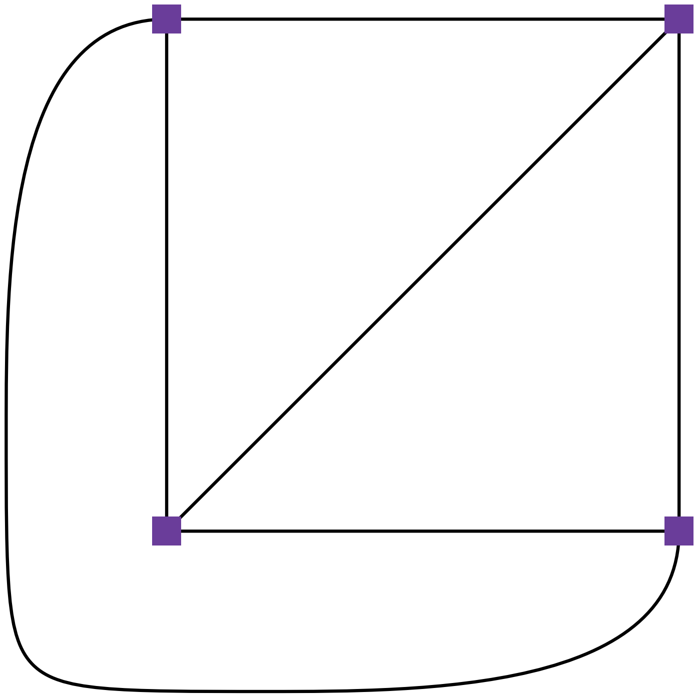

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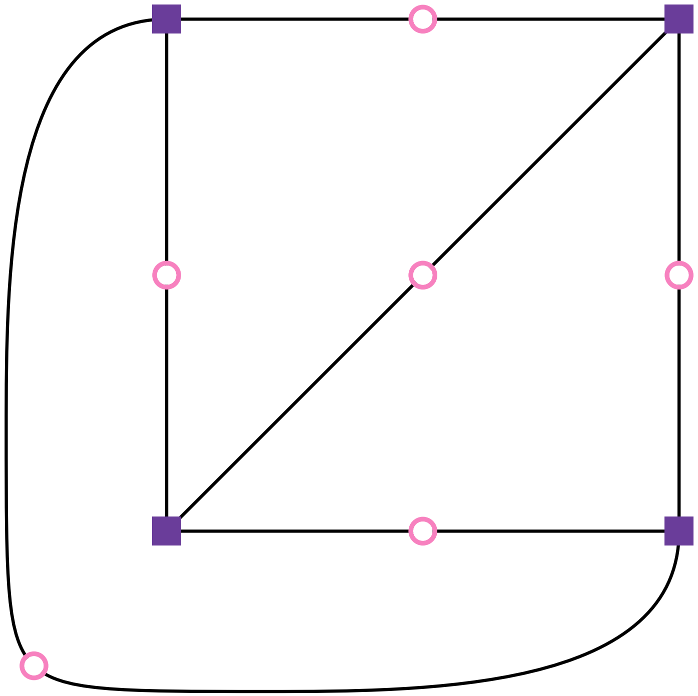

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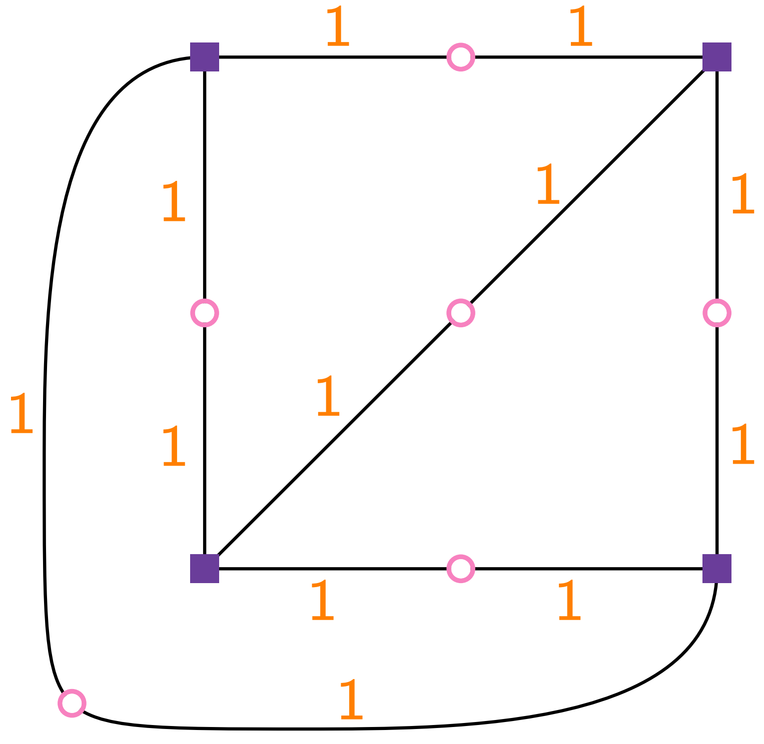

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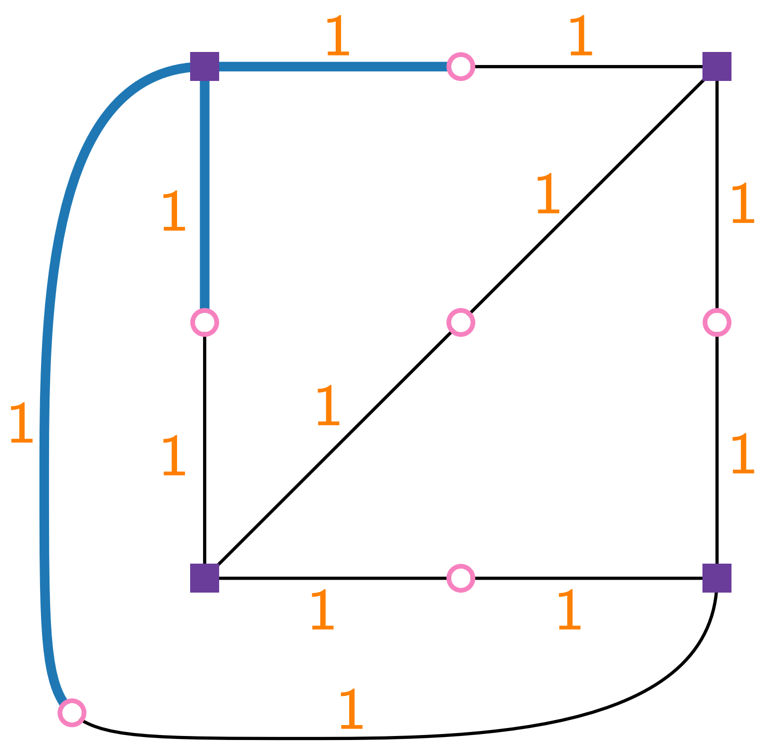

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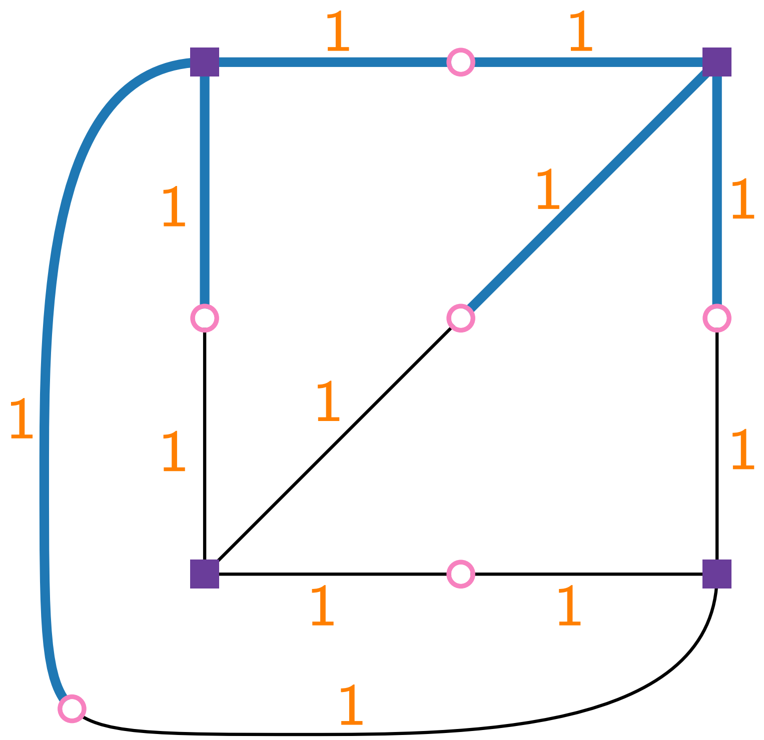

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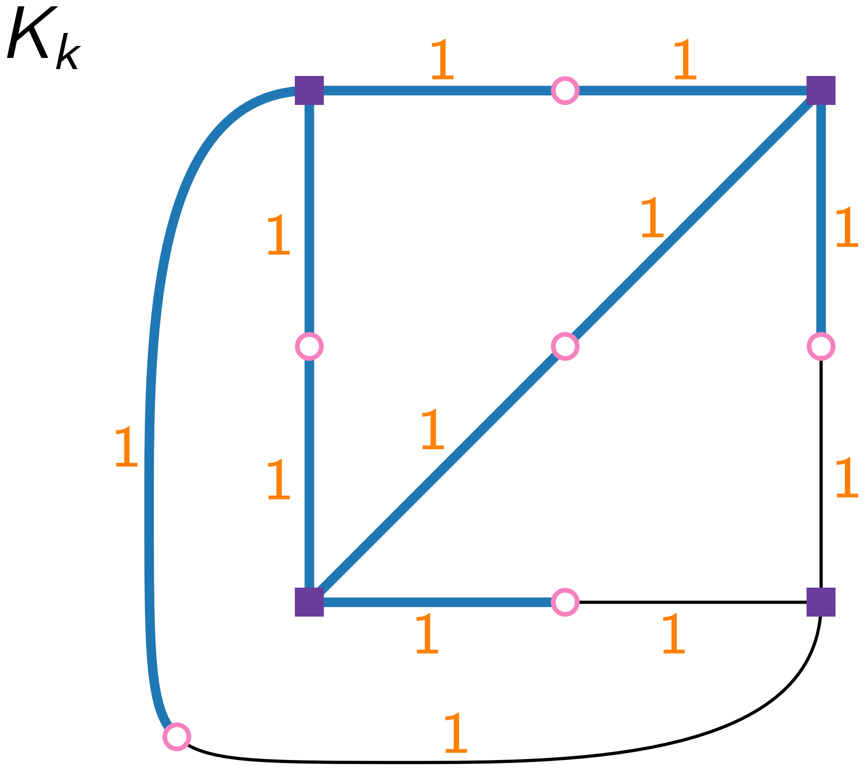

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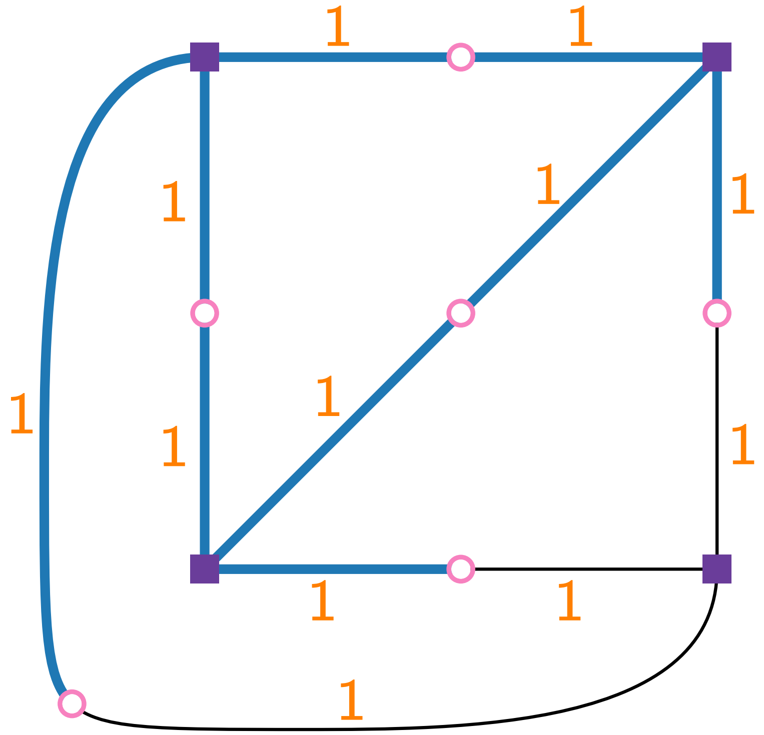

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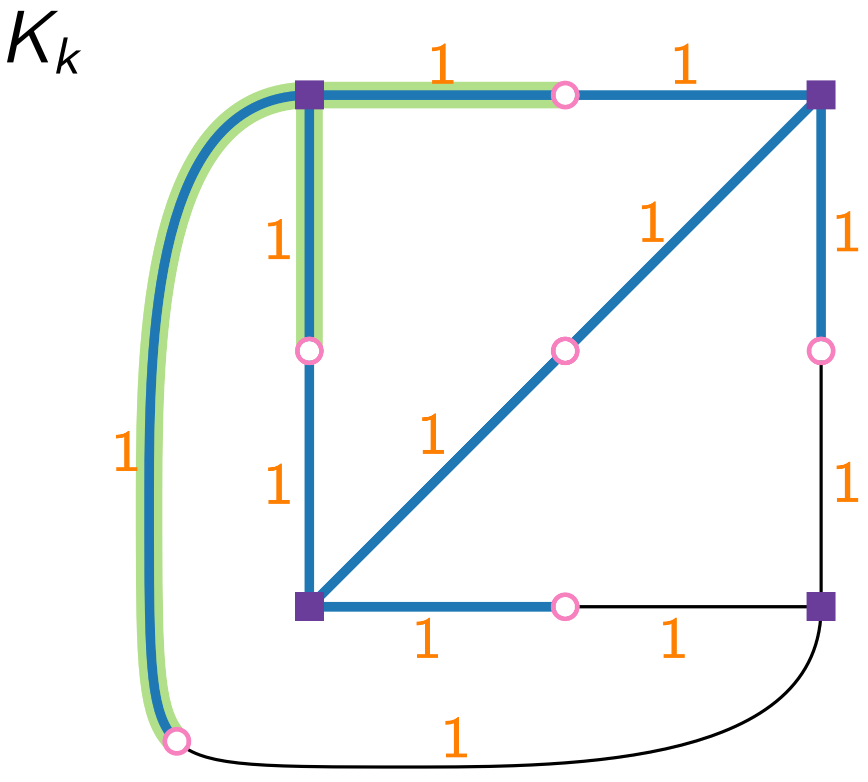

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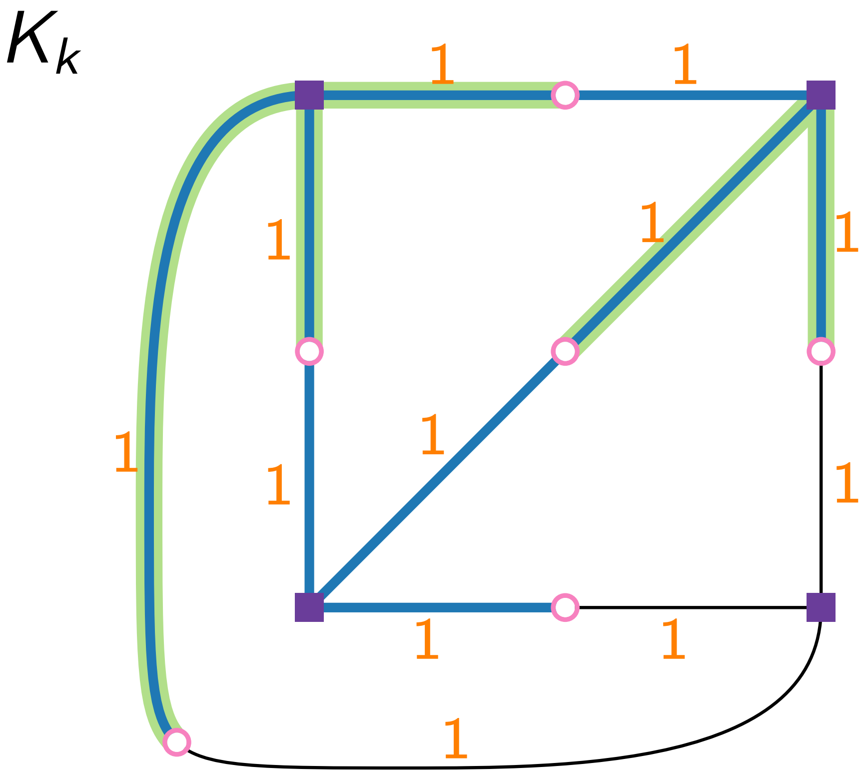

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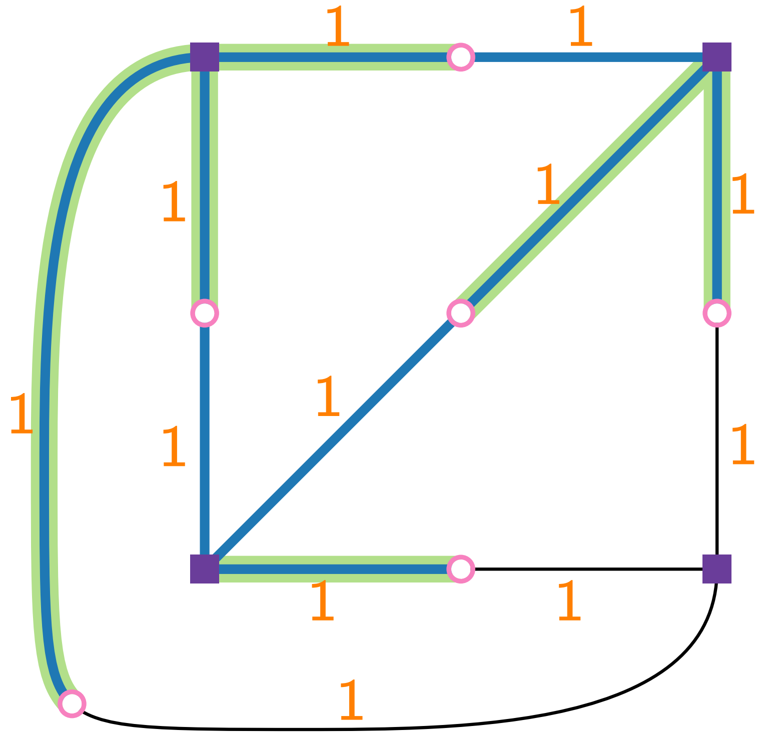

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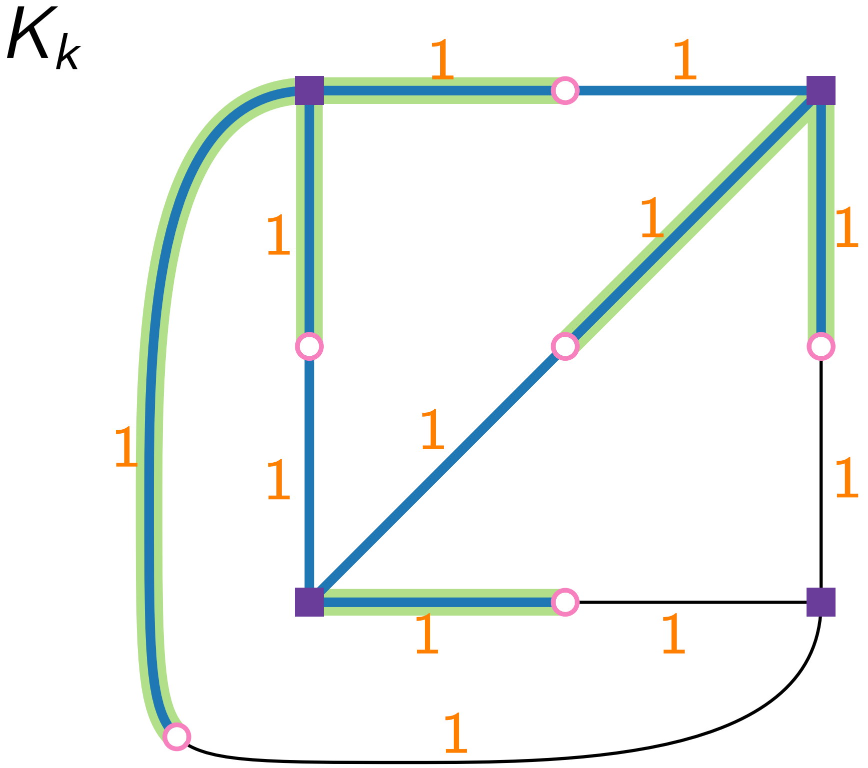

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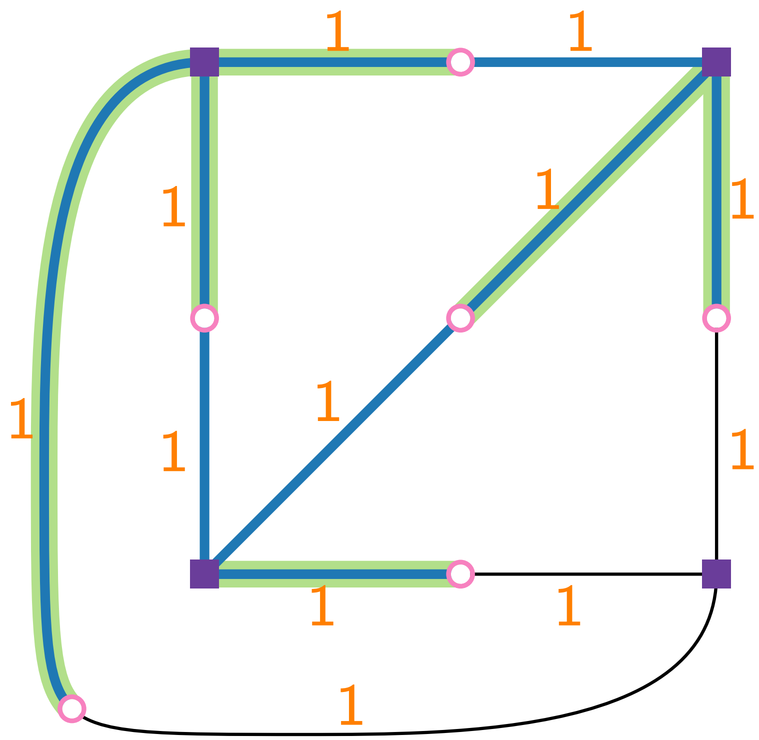

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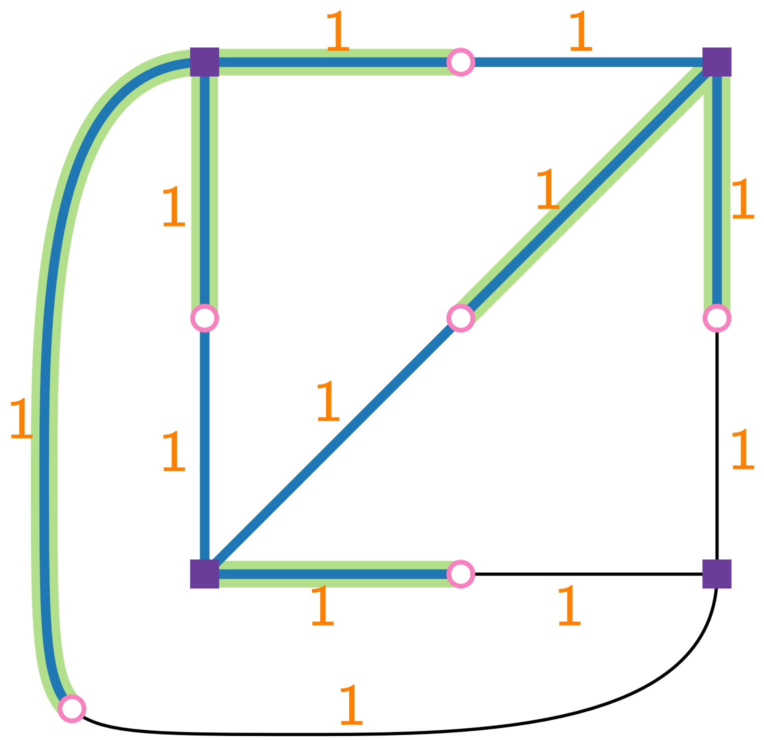

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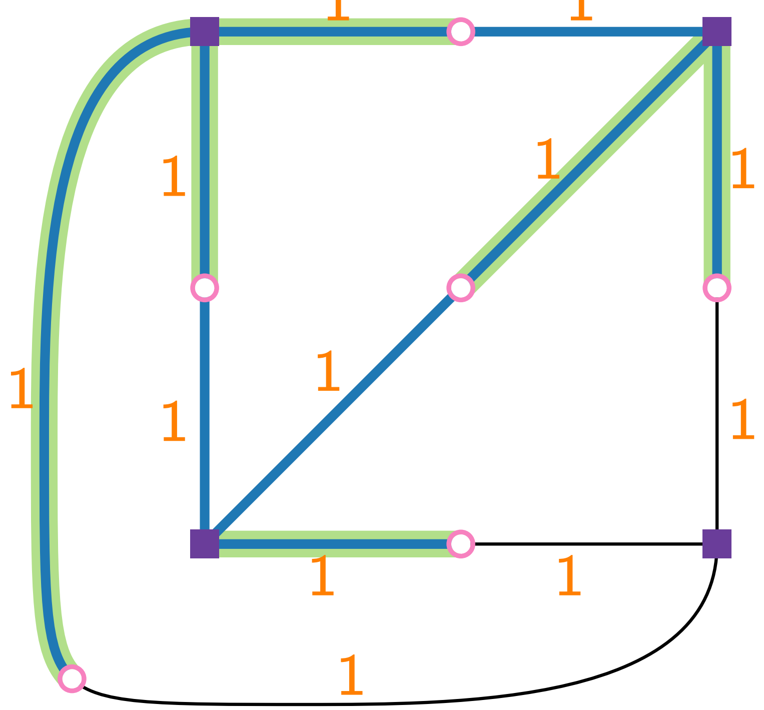

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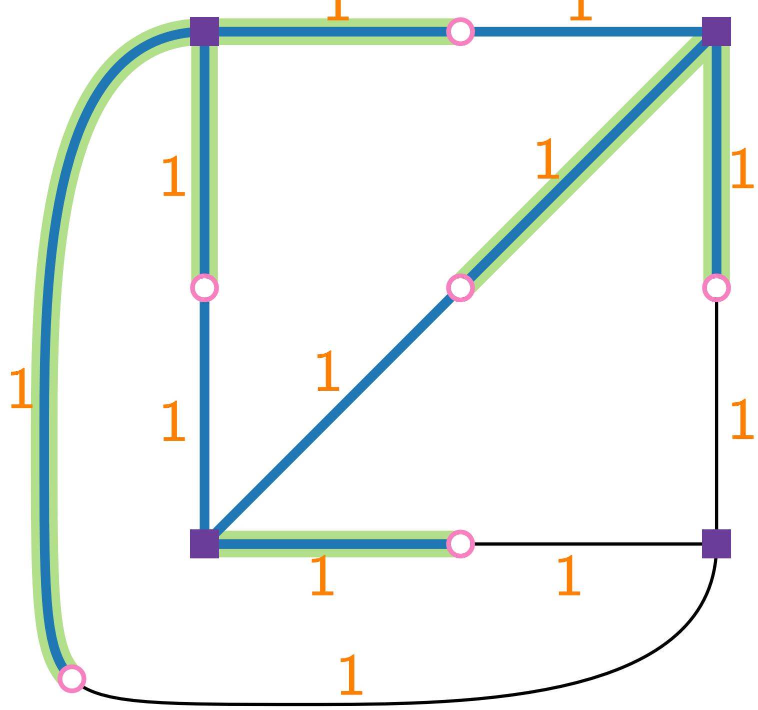

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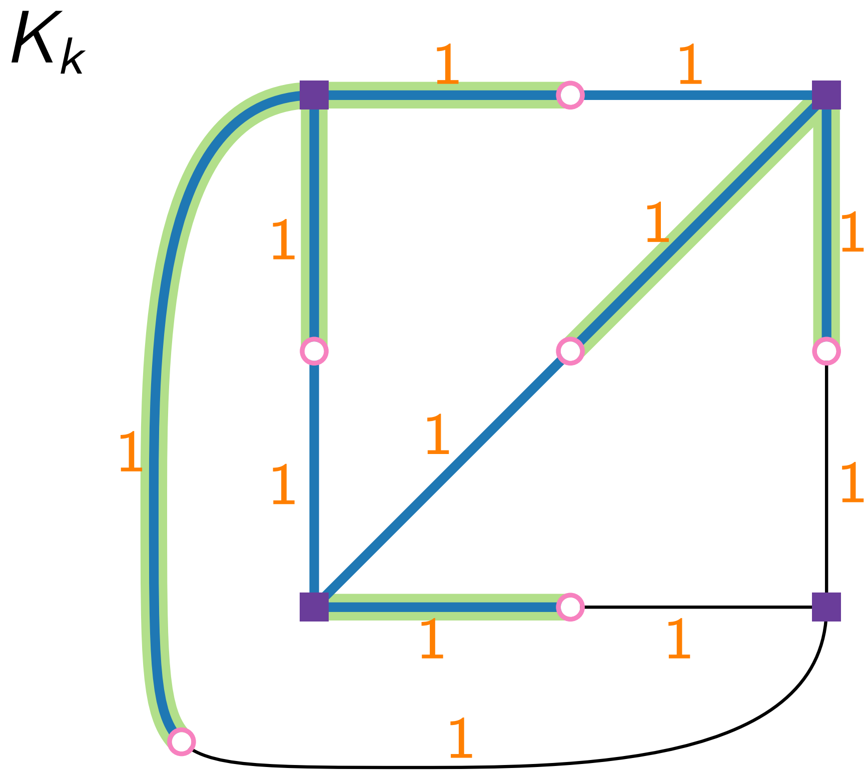

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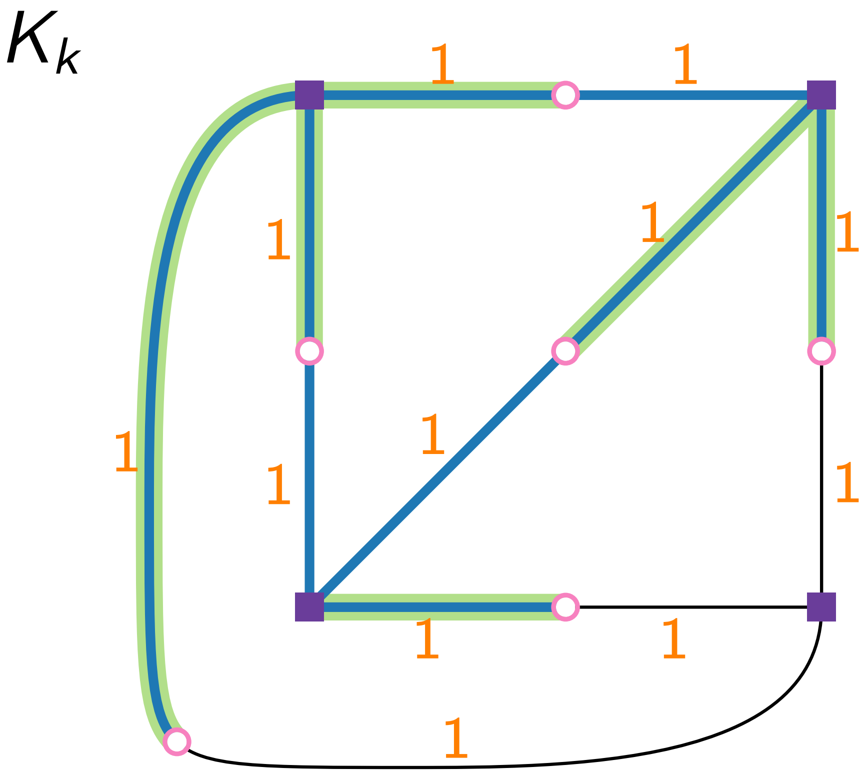

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[Sharma & Vondrák, STOC'14]

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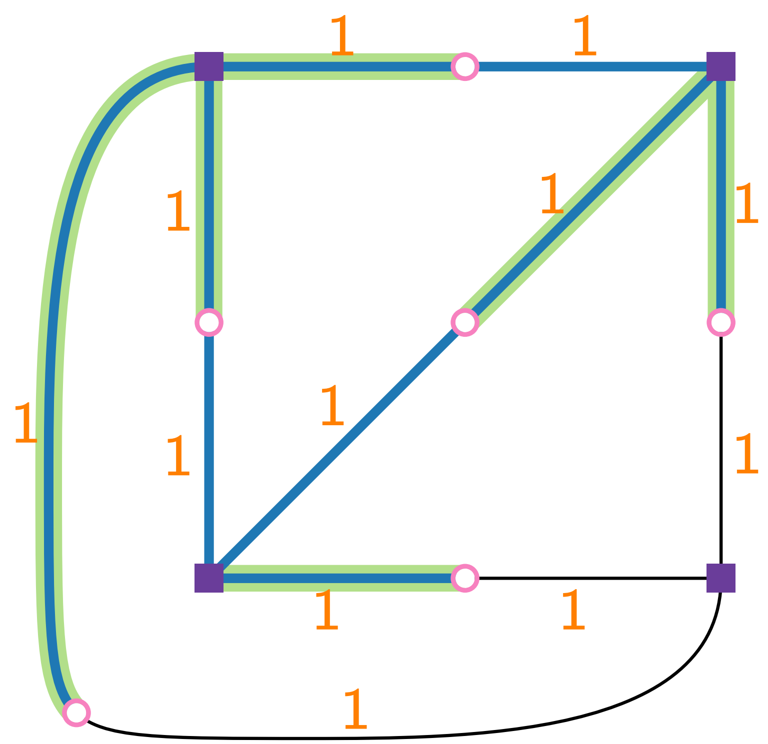

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MultiwayCut cannot be approximated within factor $1.20016 - O(1/k)$ (unless $\text{P} = \text{NP}$).

[Bérczi, Chandrasekaran, Király & Madan, MP'18]