Approximation Algorithms

Lecture 2: SetCover and ShortestSuperString

Part I: SetCover

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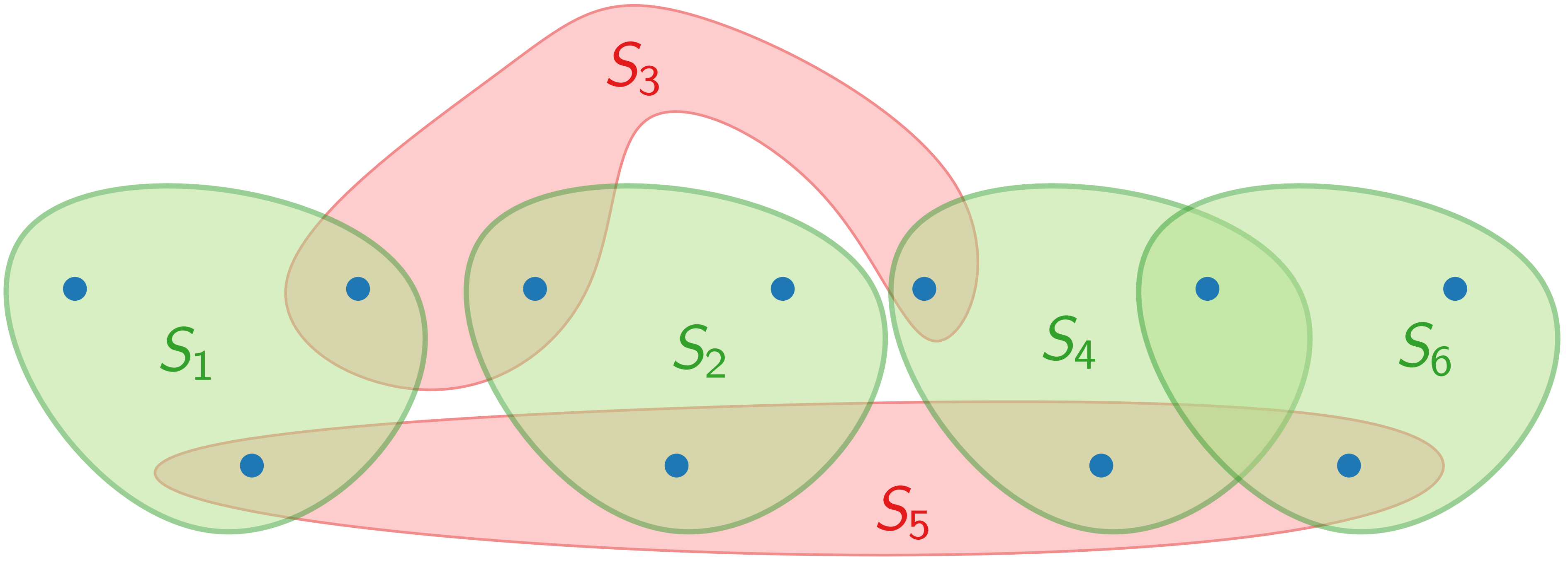
SetCover (card.)

Let U be some **ground set** (universe),
and let $\mathcal{S}$ be a family of **subsets** of U with $\bigcup \mathcal{S} = U$.

Find a **cover** $\mathcal{S}' \subseteq \mathcal{S}$ of U (i.e., with $\bigcup \mathcal{S}' = U$) of minimum cardinality.

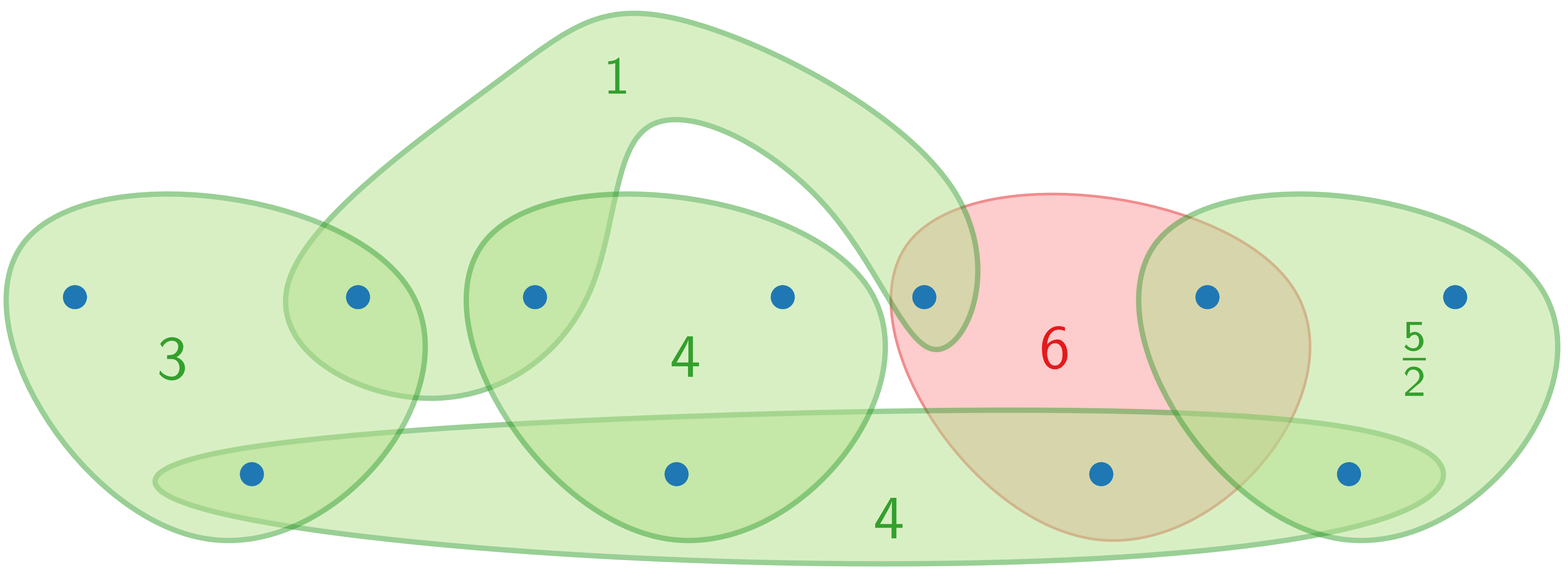


SetCover (general)

Let U be some **ground set** (universe),
and let $\mathcal{S}$ be a family of **subsets** of U with $\bigcup \mathcal{S} = U$.

Each $S \in \mathcal{S}$ has cost $c(S) > 0$.

Find a **cover** $\mathcal{S}' \subseteq \mathcal{S}$ of U (i.e., with $\bigcup \mathcal{S}' = U$) of minimum ~~cardinality.~~ total cost $c(\mathcal{S}') := \sum_{S \in \mathcal{S}'} c(S)$.



Approximation Algorithms

Lecture 2:
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Part II:
Greedy for SetCover

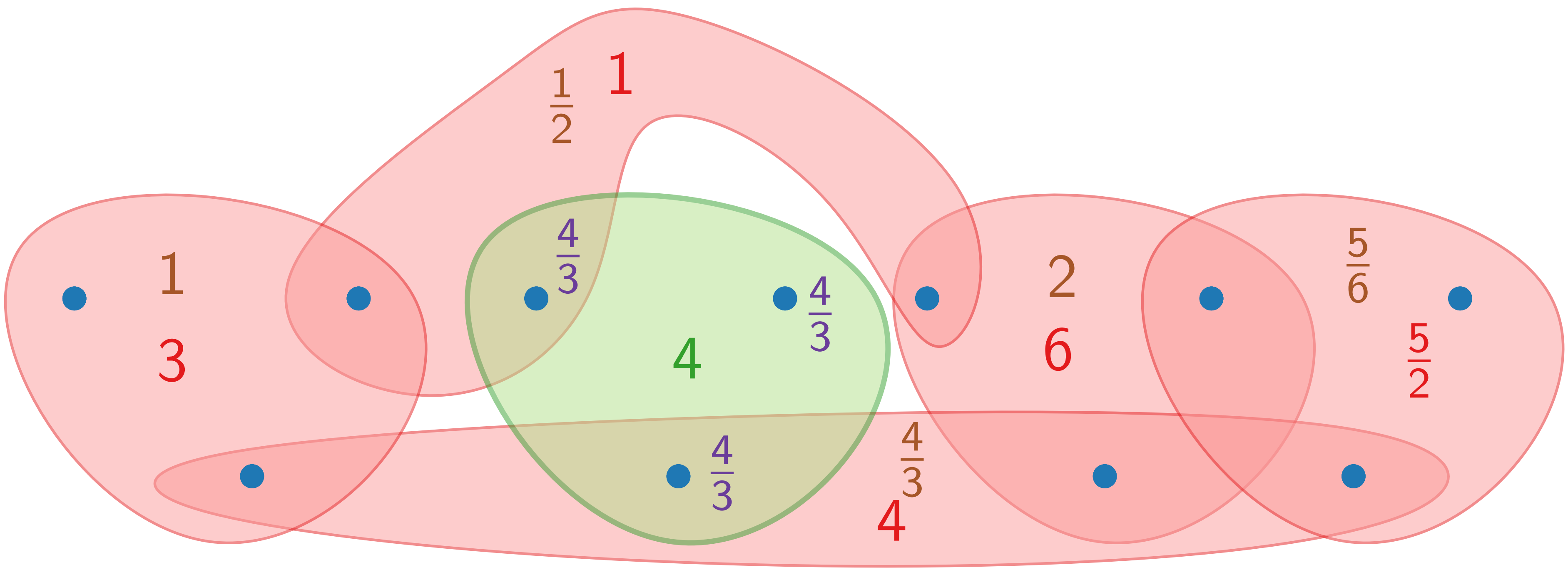
"Buying" Elements Iteratively

What is the real cost of picking a set?

Set with k elements and cost c has per-element cost c/k.

What happens if we "buy" a set?

Fix price of elements bought and recompute per-element cost.



"Buying" Elements Iteratively

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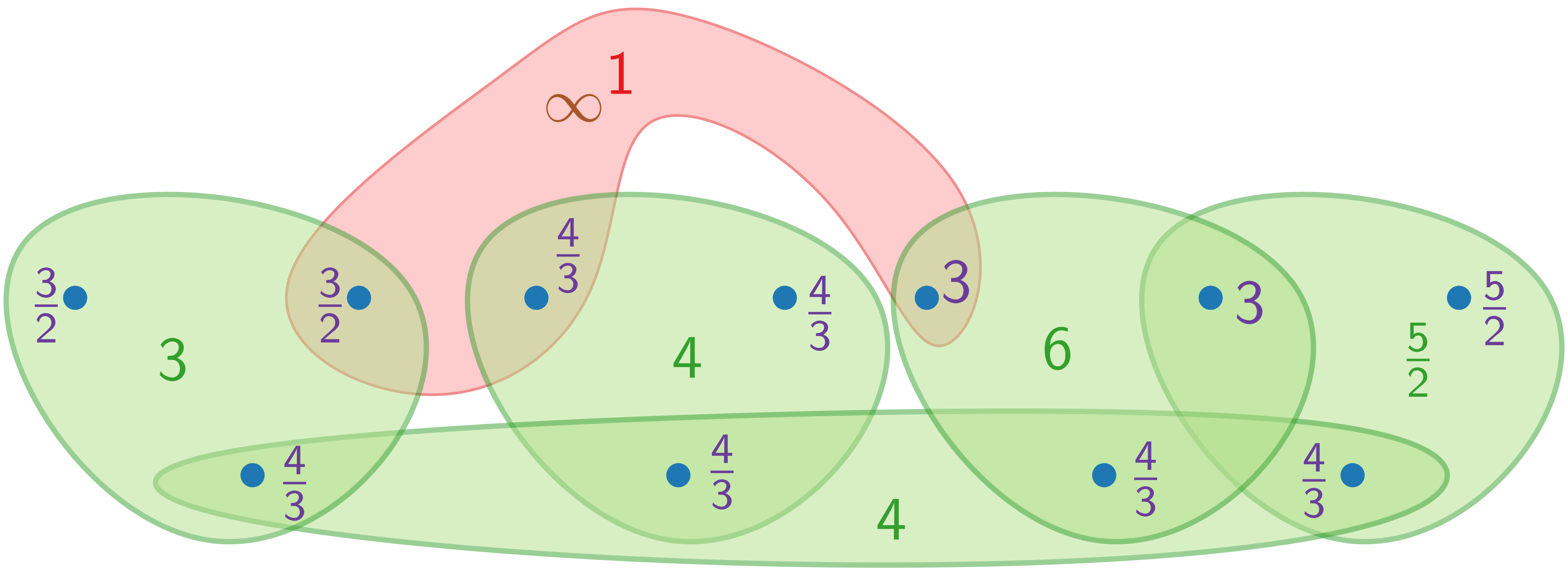
total cost: $\sum_{u \in U} \text{price}(u)$

Greedy: Always choose the set with minimum per-element cost.



Greedy for SetCover

GreedySetCover($U, \mathcal{S}, c$)

$C \leftarrow \emptyset$
$\mathcal{S}' \leftarrow \emptyset$
while $C \neq U$ **do**
  $S \leftarrow$ set in $\mathcal{S}$ that minimizes $\frac{c(S)}{|S \setminus C|}$
  foreach $u \in S \setminus C$ **do**
    $\text{price}(u) \leftarrow \frac{c(S)}{|S \setminus C|}$
  $C \leftarrow C \cup S$
  $\mathcal{S}' \leftarrow \mathcal{S}' \cup \{S\}$
return $\mathcal{S}'$ `// Cover of` U

Approximation Algorithms

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Part III:
Analysis

Analysis

Theorem. GreedySetCover is a factor-$\mathcal{H}_k$ approximation algorithm for $\textsc{SetCover}$, where k is the cardinality of the largest set in $\mathcal{S}$ and $\mathcal{H}_k := 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} \to 0.5 + \ln k$.

Lemma. Let $S \in \mathcal{S}$, and let $u_1, \ldots, u_\ell$ be the elements of S in the order in which they are covered ("bought") by GreedySetCover. Then, for every $j \in \{1, \ldots, \ell\}$: $\mathrm{price}(u_j) \leq c(S)/(\ell - j + 1)$.

Proof. Consider the iteration when the algorithm buys u_j:

- At most $j - 1$ elements of S already bought.
- At least $\ell - j + 1$ elements of S not yet bought.
- Per-element cost for S: at most $c(S)/(\ell - j + 1)$
- Price by alg. no larger due to greedy choice.

Analysis

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Lemma. $\mathrm{price}(S) := \sum_{i=1}^{\ell} \mathrm{price}(u_i) \le c(S) \cdot \mathcal{H}_\ell$.

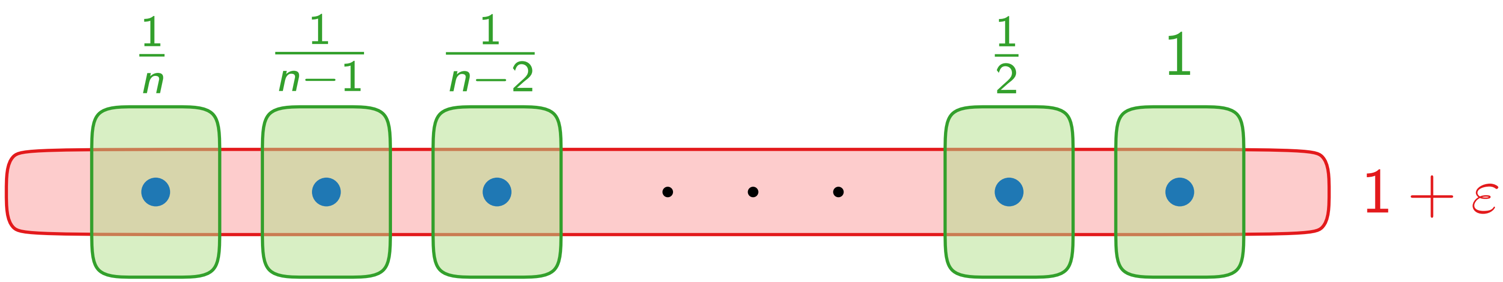
Proof. Let $\{S_1, \ldots, S_m\}$ be an opt. sol. $\mathrm{OPT} = \sum_{i=1}^{m} c(S_i)$.

$$\mathrm{ALG} = \sum_{u \in U} \mathrm{price}(u) \overset{!!}{\le} \sum_{i=1}^{m} \mathrm{price}(S_i) \quad \text{(since } U = \textstyle\bigcup_i S_i\text{)}$$

$$\le \sum_{i=1}^{m} c(S_i) \cdot \mathcal{H}_{|S_i|} \le \mathrm{OPT} \cdot \mathcal{H}_k \qquad \square$$

Analysis tight?

Theorem. GreedySetCover is a factor-$\mathcal{H}_k$ approximation algorithm for $\textsc{SetCover}$, where k is the cardinality of the largest set in $\mathcal{S}$ and $\mathcal{H}_k := 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} \leq 1 + \ln k$.



$\mathrm{ALG} = \mathrm{price}(U) = \mathcal{H}_n$ $\qquad$ $\mathrm{OPT} = 1 + \varepsilon$ $\qquad$ $\frac{\mathrm{ALG}}{\mathrm{OPT}} \xrightarrow[\varepsilon \to 0]{} \mathcal{H}_n$

Can we do better?

No – for any $\varepsilon > 0$, it is NP-hard to approximate $\textsc{SetCover}$ with factor $(1 - \varepsilon) \cdot \ln n$.

[Feige, JACM 1998]
[Dinur, Steurer, STOC 2014]

Approximation Algorithms

Lecture 2:
SetCover and ShortestSuperString

Part IV:
ShortestSuperString

ShortestSuperString (SSS)

Given a set $\{s_1, \ldots, s_n\} \subseteq \Sigma^+$ of strings over a finite alphabet Σ.

Find a **shortest string** s (*superstring*) such that, for each $i \in \{1, \ldots, n\}$, the string s_i is a *substring* of s.

Example. $U := \{cbaa, abc, bcb\} \rightarrow cbaabcb$?

W.l.o.g.: No string s_i is a substring of any other string s_j.

$abcbaa$ "covers" all strings in U

abc

bcb

$cbaa$

SSS as a SetCover Problem

SetCover Instance: ground set U, set family $\mathcal{S}$, costs c.

Ground set $U := \{s_1, \ldots, s_n\}$.

Let be σ_{ijk} be the unique string with prefix s_i and suffix s_j where s_i and s_j *overlap* on k characters (for suitable i, j, k)

s_i: cabab s_j: ababc

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cabab            cabab
  ababc            ababc
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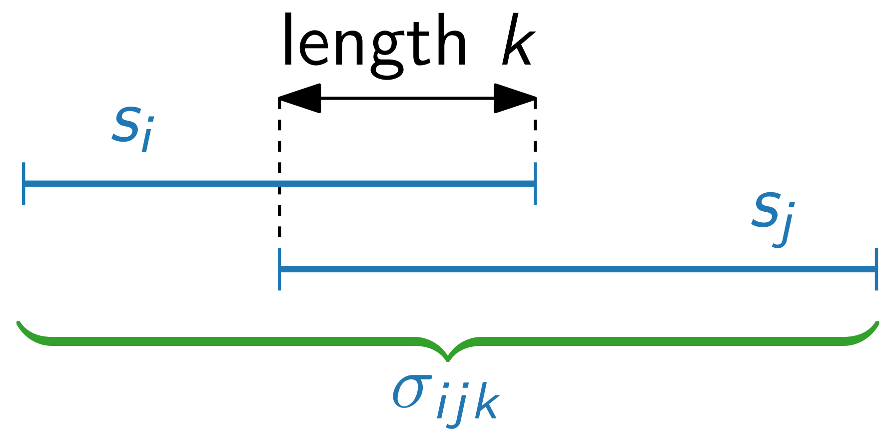
σ_{ij2}: cabababc σ_{ij4}: cababc



$S(\sigma_{ijk}) = \{s \in U \mid s \text{ substring of } \sigma_{ijk}\}$ – contains the elements of the ground set covered by σ_{ijk}.

$c\,(S(\sigma_{ijk})) = |\sigma_{ijk}|$ (number of characters in σ_{ijk})

$\mathcal{S} = \{S(\sigma_{ijk}) \mid 1 \leq i, j \leq n, \text{suitable } k \geq 0\}$

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Part V:
Solving ShortestSuperString via SetCover

Relating SSS and SetCover

Lemma. Let $\mathrm{OPT}_{\mathrm{SSS}}$ be the length of a shortest superstring of U, and let $\mathrm{OPT}_{\mathrm{SC}}$ be the minimum cost of the corresponding SetCover instance. Then

$$\mathrm{OPT}_{\mathrm{SSS}} \leq \mathrm{OPT}_{\mathrm{SC}}.$$

Proof.

Consider an optimal set cover $\{S(\pi_1), \ldots, S(\pi_k)\}$ of U.

Then $s := \pi_1 \circ \cdots \circ \pi_k$ is a superstring of U of length $\sum_{i=1}^{k} |\pi_i| = \sum_{i=1}^{k} c(S(\pi_i)) = \mathrm{OPT}_{\mathrm{SC}}$.

Thus, $\mathrm{OPT}_{\mathrm{SSS}} \leq |s| = \mathrm{OPT}_{\mathrm{SC}}$.

Relating SSS and SetCover

Lemma. $\mathrm{OPT}_{\mathrm{SC}} \leq 2 \cdot \mathrm{OPT}_{\mathrm{SSS}}$.

Proof. Consider an optimal superstring s.
Construct a set cover of cost $\leq 2|s| = 2 \cdot \mathrm{OPT}_{\mathrm{SSS}}$.

s

s_{b_1}

Leftmost occurence of a string $s_{b_1} \in U$.

Relating SSS and SetCover

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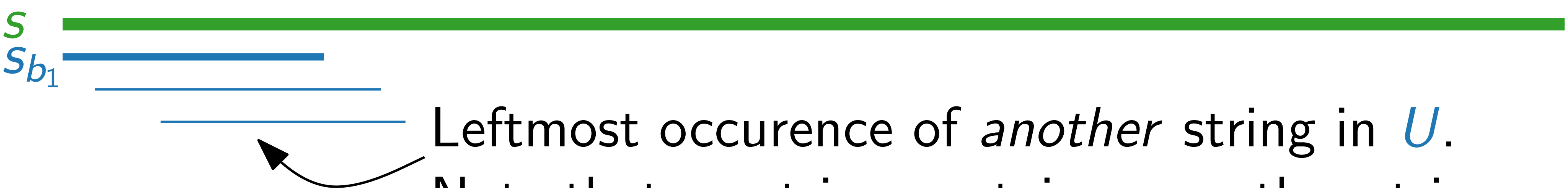

Leftmost occurence of *another* string in U.
Note that no string contains any other string.
$\Rightarrow$ Right endpoints are ordered like left endpoints.

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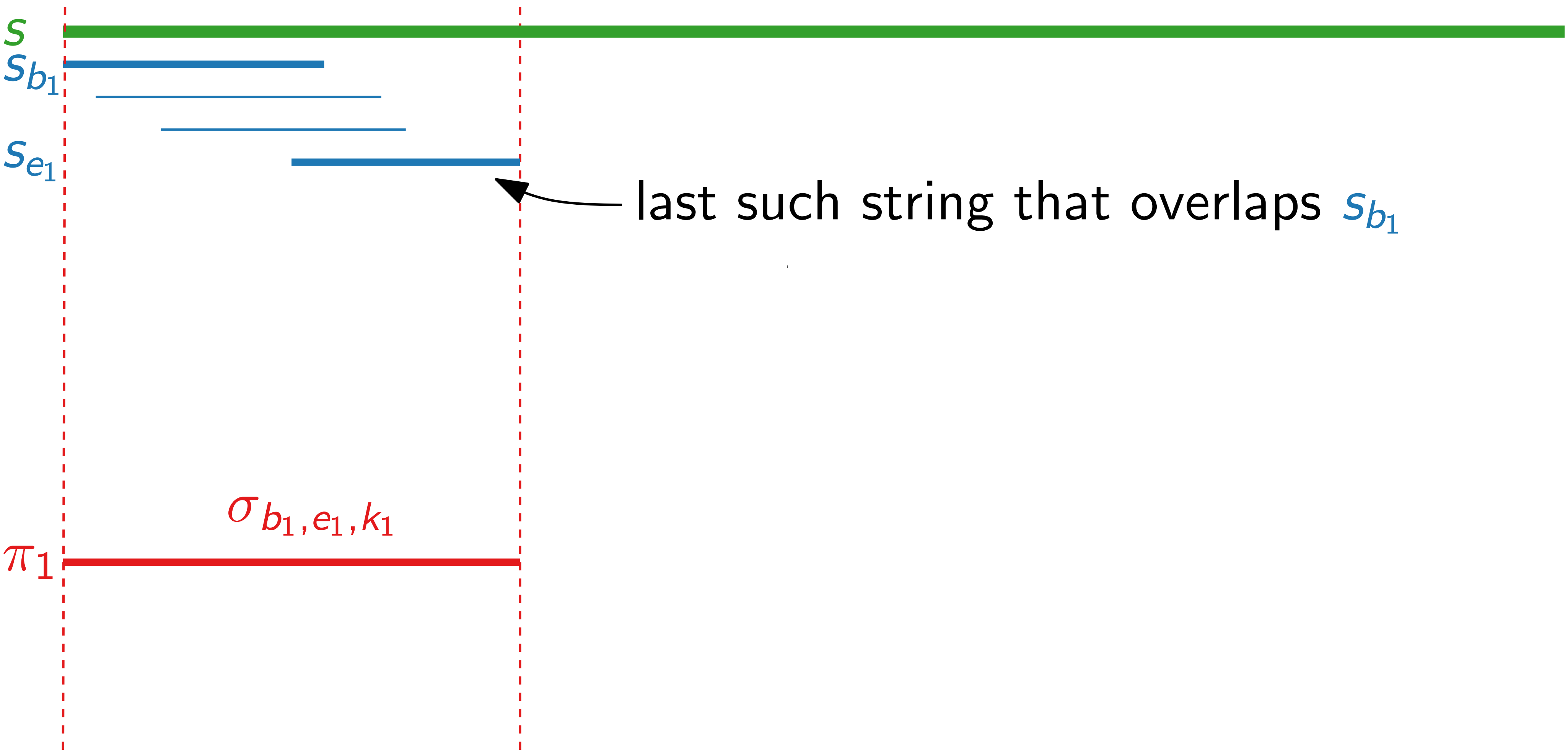

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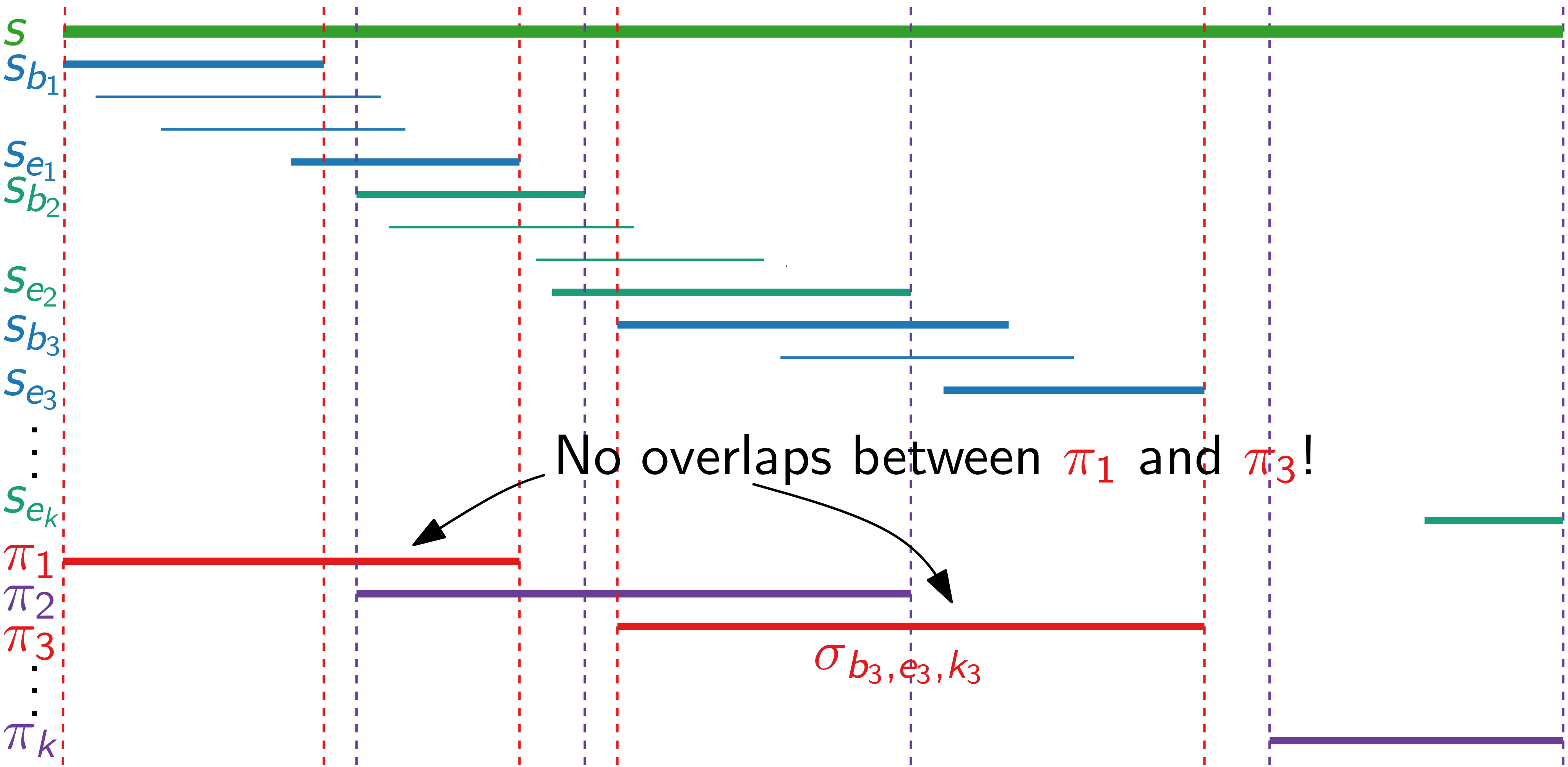

Relating SSS and SETCOVER

Lemma. $\mathrm{OPT}_{\mathrm{SC}} \leq 2 \cdot \mathrm{OPT}_{\mathrm{SSS}}$.

Proof.

Each string $s_i \in U$ is a substring of some π_j.

$\{S(\pi_1), \ldots, S(\pi_k)\}$ is **a** solution for the SETCOVER instance with cost $\sum_i |\pi_i|$.

For $j \in \{1, \ldots, k-2\}$, substrings π_j, π_{j+2} do **not** overlap.

Each character of the optimal superstring s lies in at most **two** (subsequent) substrings, say, π_j and π_{j+1}.

$\mathrm{OPT}_{\mathrm{SC}} \leq \sum_i |\pi_i| \leq 2|s| = 2 \cdot \mathrm{OPT}_{\mathrm{SSS}}$ $\square$

Algorithm for SSS

1. Construct SetCover instance $\langle U, \mathcal{S}, c \rangle$.
2. Compute a set cover $\{S(\pi_1), \ldots, S(\pi_k)\}$ with the algorithm GreedySetCover.
3. Return $\pi_1 \circ \cdots \circ \pi_k$ as the superstring.

What is n?

$$n = \max_{S \in \mathcal{S}} |S| = \max |S(\sigma_{ijk})| = \max_{u \in U} |u|$$

Theorem. This algorithm is a factor-$2\mathcal{H}_n$ approximation algorithm for ShortestSuperString.

Lemma. $\mathrm{OPT}_{\mathrm{SC}} \leq 2 \cdot \mathrm{OPT}_{\mathrm{SSS}}$.

Theorem. GreedySetCover is a factor-$\mathcal{H}_k$ approximation algorithm for SetCover, where k is the cardinality of the largest set in $\mathcal{S}$ and $\mathcal{H}_k := 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} \leq 1 + \ln k$.

Can we do better?

- The best known approximation factor for $\textsc{ShortestSuperString}$ is $(\sqrt{67}+14)/9 \approx 2.466$.

  [Englert, Matsakis, Veselý: STOC 2022, ISAAC 2023]

- $\textsc{ShortestSuperString}$ cannot be approximated within factor $\frac{333}{332} \approx 1.003$ (unless $\mathsf{P} = \mathsf{NP}$).

  [Karpinski & Schmied: CATS 2013]