Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part I:
Organizational

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Winter term 2025

Organizational

Lectures: Fri, 10:15–11:45 (ÜR I)

English/German, depending on audience.

hands-on, with tasks/questions for audience

Tutorials: Tue, 10:15–11:45 (SE I), starting Oct. 21, 2025.

discussing old solutions and solving new tasks

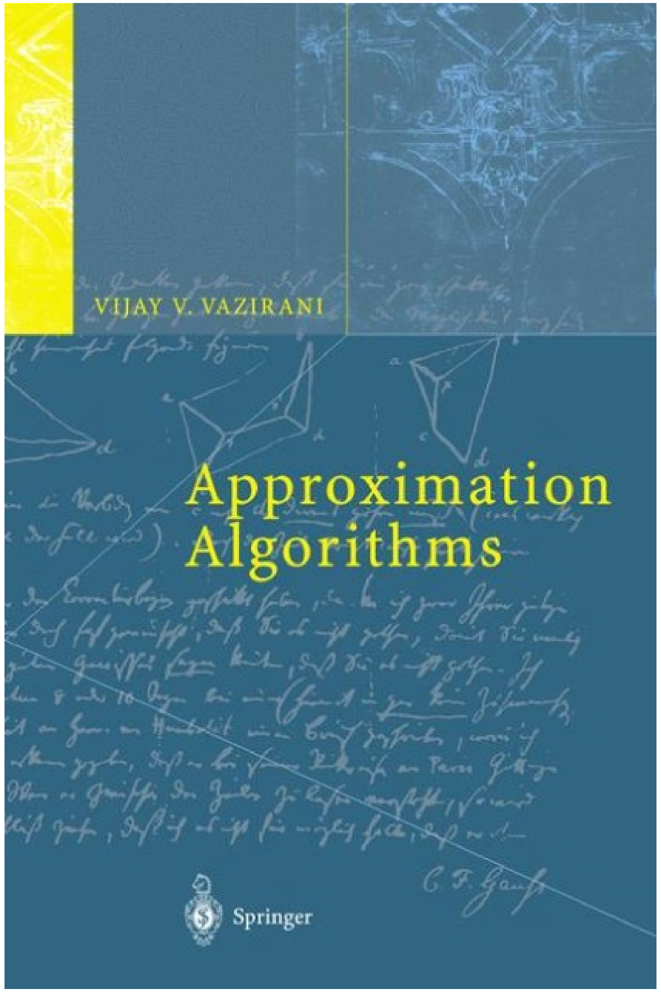
roughly one exercise sheet per lecture

bonus (+0.3 on final grade) for $\geq$ 50% points

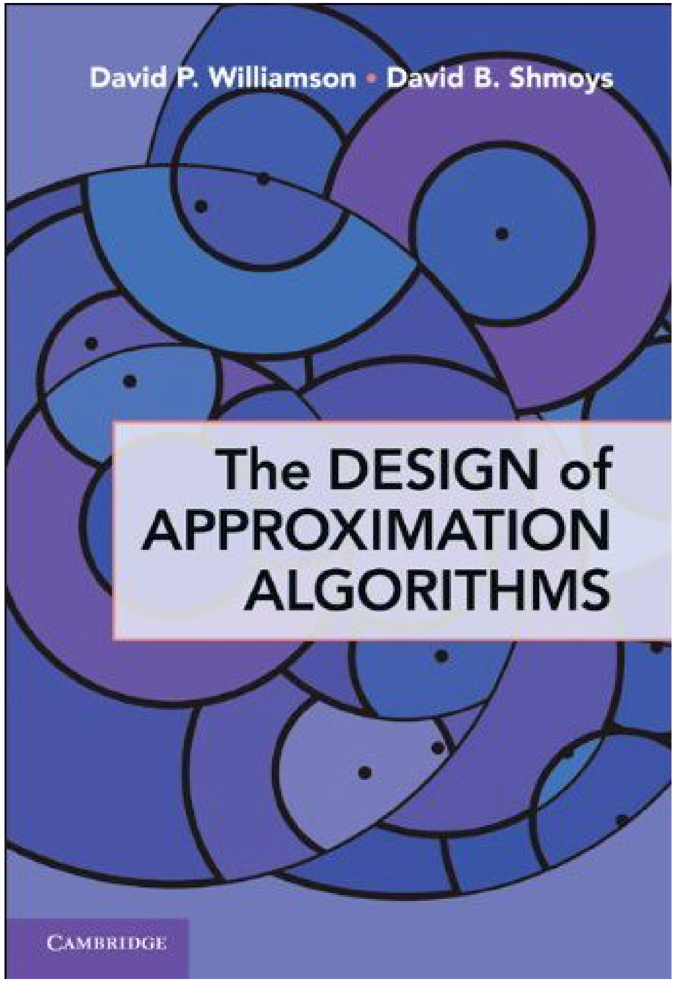
Up to two students can hand in solutions together.
Make sure to write both names!

Most slides are due to Joachim Spoerhase,
polishing & colors are due to Philipp Kindermann – thanks!

Textbooks



Vijay V. Vazirani:
Approximation Algorithms
Springer-Verlag, 2003.



D. P. Williamson & D. B. Shmoys:
The Design of Approximation Algorithms
Cambridge-Verlag, 2011.

http://www.designofapproxalgs.com/

Approximation Algorithms

„All exact science is dominated by the idea of approximation.“

Bertrand Russell
(1872 – 1970)

Approximation Algorithms

- Many optimization problems are NP-hard!
  (For example, the traveling salesperson problem.)
- $\leadsto$ an optimal solution cannot be efficiently computed unless P=NP.
- However, good approximate solutions can often be found efficiently!
- **Techniques** for the design and analysis of approximation algorithms arise from studying specific optimization problems.

Overview

Combinatorial algorithms

- Introduction (Vertex Cover)
- Set Cover via Greedy
- Shortest Superstring via reduction to SC
- Steiner Tree via MST
- Multiway Cut via Greedy
- k-Center via Parametrized Pruning
- Min-Degree Spanning Tree and local search
- Knapsack via DP and Scaling
- Euclidean TSP via Quadtrees

LP-based algorithms

- Introduction to LP-Duality
- Set Cover via LP Rounding
- Set Cover via Primal–Dual Schema
- Maximum Satisfiability
- Scheduling und Extreme Point Solutions
- Steiner Forest via Primal–Dual

Approximation Algorithms

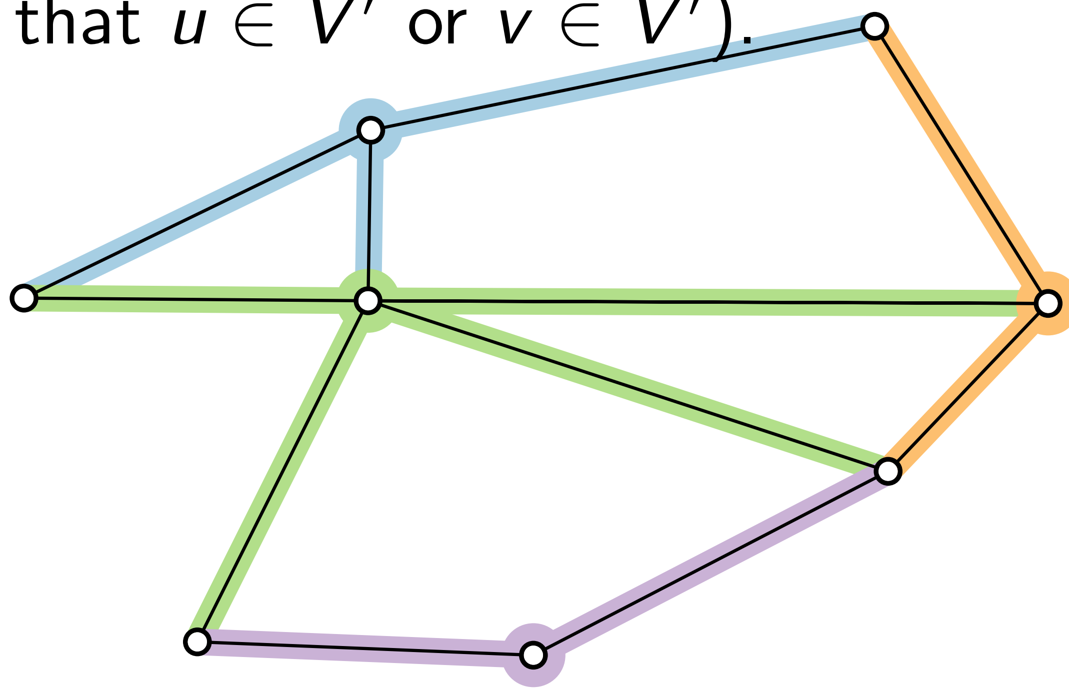
Lecture 1:
Introduction and Vertex Cover

Part II:
(Cardinality) Vertex Cover

VertexCover (card.)

Input: graph G

Output: a minimum **vertex cover**, that is,
a minimum-cardinality vertex set $V' \subseteq V(G)$ s. t.
every edge is **covered** (i.e., for every $uv \in E(G)$, it holds that $u \in V'$ or $v \in V'$).



Optimum (OPT $= 4$) – but in general NP-hard to find :-(

Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part III:
NP-Optimization Problem

NP-Optimization Problem

An **NP-optimization problem** Π is given by:

- A set D_Π of **instances**.
  We denote the size of an instance $I \in D_\Pi$ by $|I|$.
- For each instance $I \in D_\Pi$,
  a set $S_\Pi(I) \neq \emptyset$ of **feasible solutions** for I such that:
  - for each solution $s \in S_\Pi(I)$,
    its size $|s|$ is polynomially bounded in $|I|$, and
  - there is a polynomial-time algorithm that decides,
    for each pair (s, I), whether $s \in S_\Pi(I)$.
- A polynomial time computable **objective function** obj_Π
  which assigns a positive objective value $\mathrm{obj}_\Pi(I, s) \geq 0$ to
  any given pair (s, I) with $s \in S_\Pi(I)$.
- Π is either a minimization or maximization problem.

VertexCover: NP-Optimization Problem

Task: Fill in the gaps for $\Pi = \textsc{Vertex Cover}$.

$D_\Pi =$ set of all graphs

For $I \in D_\Pi$: $\quad |I| =$ number of vertices of G

graph G $\quad S_\Pi(I) =$ set of all vertex covers of G

- Why is $|s| \in \text{poly}(|I|)$ for every $s \in S_\Pi(I)$?

  $s \subseteq V \Rightarrow |s| \leq |V(G)| = |I|$

- For a given pair (s, I), how can we efficiently decide whether $s \in S_\Pi(I)$? Test whether all edges are covered.

$\text{obj}_\Pi(I, s) = |s|$

Π is a minimization problem.

Optimum and Optimal Objective Value

maximization problem

Let Π be a minimization problem and $I \in D_{\Pi}$ an instance of Π.

A feasible solution $s^* \in S_{\Pi}(I)$ is **optimal** if

maximum

$\mathrm{obj}_{\Pi}(I, s^*)$ is minimum among the objective values attained by the feasible solutions of I.

The optimal value $\mathrm{obj}_{\Pi}(I, s^*)$ of the objective function is denoted by $\mathrm{OPT}_{\Pi}(I)$ or simply by OPT in context.

Approximation Algorithms

maximization problem $\quad \alpha\colon \mathbb{N} \to \mathbb{Q}$

Let Π be a minimization problem and $\cancel{\alpha \in \mathbb{Q}^+}$.

A factor-α approximation algorithm for Π is an efficient algorithm that provides, for **any** instance $I \in D_\Pi$, a feasible solution $s \in S_\Pi(I)$ such that

$$\frac{\mathrm{obj}_\Pi(I, s)}{\mathrm{OPT}_\Pi(I)} \ \overset{\geq}{\leq}\ \cancel{\alpha}. \quad \alpha(|I|)$$

Approximation Algorithms

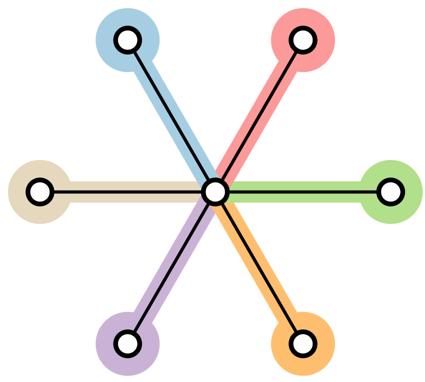
Lecture 1: Introduction and Vertex Cover

Part IV: Approximation Algorithm for $\textsc{VertexCover}$

Approximation Alg. for VertexCover

Ideas?

- Edge-Greedy
- Vertex-Greedy



Quality?

Problem: How can we estimate $\mathrm{obj}_{\Pi}(I, s)/\mathrm{OPT}$ – if it is hard to compute OPT?

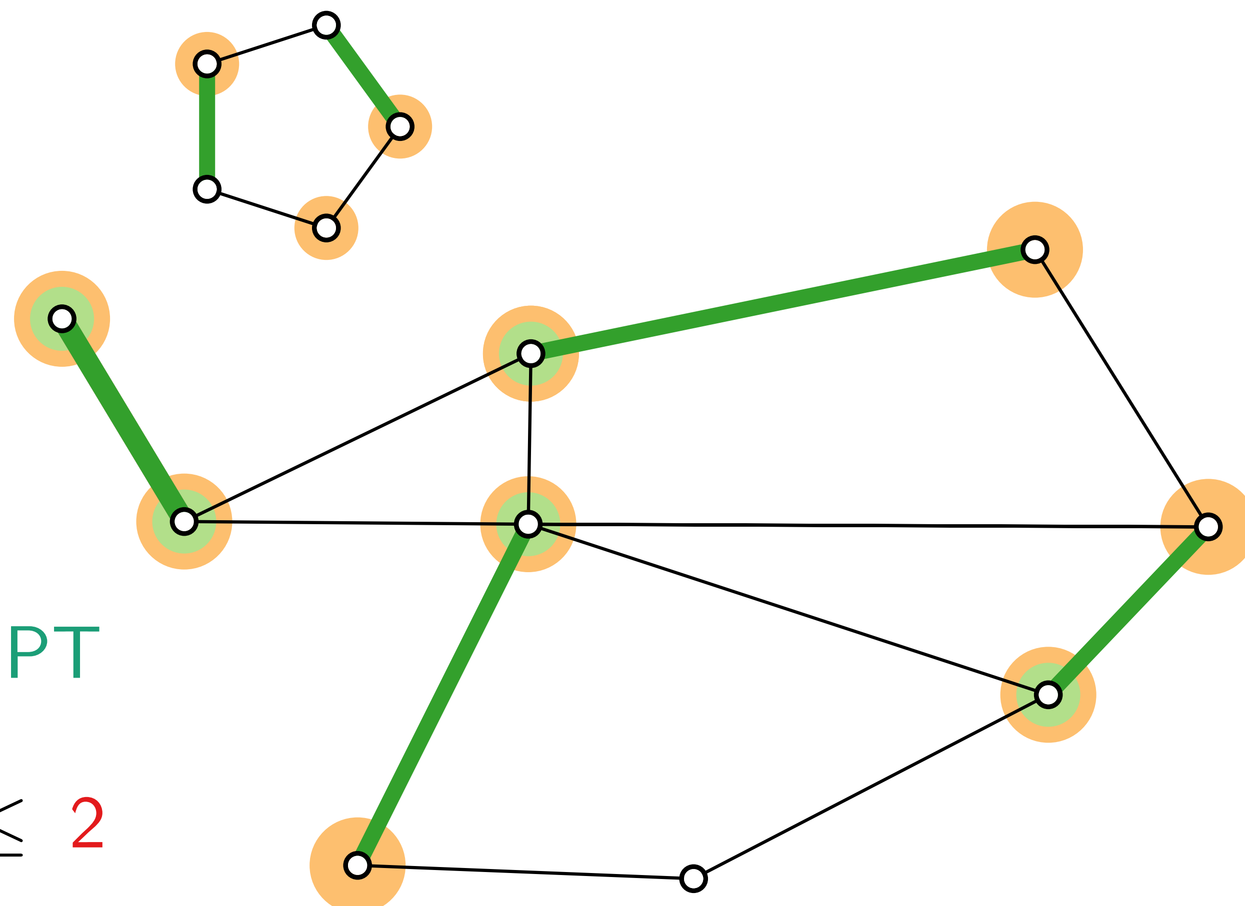
Idea: Find a "good" lower bound $L \leq \mathrm{OPT}$ for OPT and compare it to our approximate solution.

$$\frac{\mathrm{obj}_{\Pi}(I, s)}{\mathrm{OPT}} \leq \frac{\mathrm{obj}_{\Pi}(I, s)}{L}$$

Lower Bound by Matchings

Given a graph G, a set M of edges of G is a **matching** if no two edges of M are adjacent (i.e., share an end vertex).

M is **maximal** if there is no matching M' with $M' \supsetneq M$.



$\mathrm{OPT} \geq |M|$

$\cancel{\mathrm{OPT} = |M|}$?

Vertex cover of M

Vertex cover of $E(G)$

$\mathrm{ALG} = 2 \cdot |M| \leq 2 \cdot \mathrm{OPT}$

$$\Rightarrow \frac{\mathrm{obj}_{\Pi}(I, s)}{\mathrm{OPT}} = \frac{\mathrm{ALG}}{\mathrm{OPT}} \leq 2$$

Approximation Alg. for $\textsc{VertexCover}$

Algorithm VertexCover(G)

$M \leftarrow \emptyset$

foreach $e \in E(G)$ **do**

  if e is not adjacent to any edge in M **then**

    $M \leftarrow M \cup \{e\}$

return $\{\, u, v \mid uv \in M \,\}$

Theorem. The above algorithm is a factor-2 approximation algorithm for $\textsc{VertexCover}$.

Proof. $\mathrm{ALG} = 2 \cdot |M| \leq 2 \cdot \mathrm{OPT}$ $\square$

Approximability of Vertex Cover

The best known approximation factor for VertexCover is

$$2 - \Theta(1/\sqrt{\log n}).$$

If P $\neq$ NP, VertexCover cannot be approximated within a factor of 1.3606.

VertexCover cannot be approximated within a factor of $2 - \Theta(1)$ – if the *Unique Games Conjecture* holds.

Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part V:
An LP-based Algorithm for VertexCover

Task

Write an integer linear program (ILP) for VertexCover:

Using integer (and/or real) variables, express the problem using

- linear constraints and
- a linear objective function.

You can iterate over the vertices / edges of the given graph G.

Variables: for each vertex v of G, we introduce $x_v \in \{0, 1\}$.

v not in the solution
v in the solution

Objective: minimize $\sum_{v \in V(G)} x_v$

Constraints: for each edge uv of G, we require that

$$x_u + x_v \geq 1.$$

Standard ILP Format

LP relaxation

minimize	$\sum_{v \in V(G)} x_v$	
subject to	$x_u + x_v \geq 1$	for each $uv \in E(G)$
	$x_v \geq 0$ $\cancel{x_v \in \{0,1\}}$	for each $v \in V(G)$

Problem: It's NP-hard to solve ILPs in general.

But: LPs can be solved efficiently (in $O(L \cdot n^{3.5})$ time),

where $n = \#$ variables and $L =$ total bit complexity of coefficients.

Problem$'$: Now we can get fractional solutions, i.e., $x_v \in (0, 1)$.

Task: Find a graph G with $\mathrm{OPT_{LP}} \neq \mathrm{OPT_{ILP}}$!

Solution? Round the LP solution to get an integral solution!

Rounding the LP Solution

$$\begin{array}{lll} \text{minimize} & \sum_{v \in V(G)} x_v & \\ \text{subject to} & x_u + x_v \geq 1 & \text{for each } uv \in E(G) \\ & x_v \geq 0 & \text{for each } v \in V(G) \end{array}$$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

Need to check: Is $(x'_v)_{v \in V(G)}$ a feasible solution?

In other words: Is $\{v \in V(G) \colon x'_v = 1\}$ a vertex cover of G?

Need to make sure that every edge uv of G is covered.

Is $x'_u = 0 = x'_v$ possible? But then $x_u < 0.5$ and $x_v < 0.5$.

This contradicts $x_u + x_v \geq 1$. $\Rightarrow$ $x'_u = 1$ or $x'_v = 1$ $\Rightarrow$ (x'_v) feasible!

Cost of the Solution

minimize	$\sum_{v \in V(G)} x_v$	
subject to	$x_u + x_v \geq 1$	for each $uv \in E(G)$
	$x_v \geq 0$	for each $v \in V(G)$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

$$\mathrm{ALG} = \sum_{v \in V(G)} x'_v \leq 2 \cdot \sum_{v \in V(G)} x_v = 2 \cdot \mathrm{OPT}_{\mathrm{LP}} \leq 2 \cdot \mathrm{OPT}_{\mathrm{ILP}}$$

Theorem. The LP rounding algorithm is a factor-2 approximation algorithm for VertexCover.

Cost of the Solution

$$\begin{array}{lll} \text{minimize} & \sum_{v \in V(G)} x_v \cdot w(v) & \\ \text{subject to} & x_u + x_v \geq 1 & \text{for each } uv \in E(G) \\ & x_v \geq 0 & \text{for each } v \in V(G) \end{array}$$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

$$\text{ALG} = \sum_{v \in V(G)} x'_v \cdot w(v) \leq 2 \cdot \sum_{v \in V(G)} x_v \cdot w(v) = 2 \cdot \text{OPT}_{\text{LP}} \leq 2 \cdot \text{OPT}_{\text{ILP}}$$

Theorem. The LP rounding algorithm is a factor-2 approximation algorithm for WEIGHTEDVERTEXCOVER.

Approximation Algorithms

Lecture 1:
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Part VI:
The Vertex Cover Polytope

The Vertex Cover Polytope

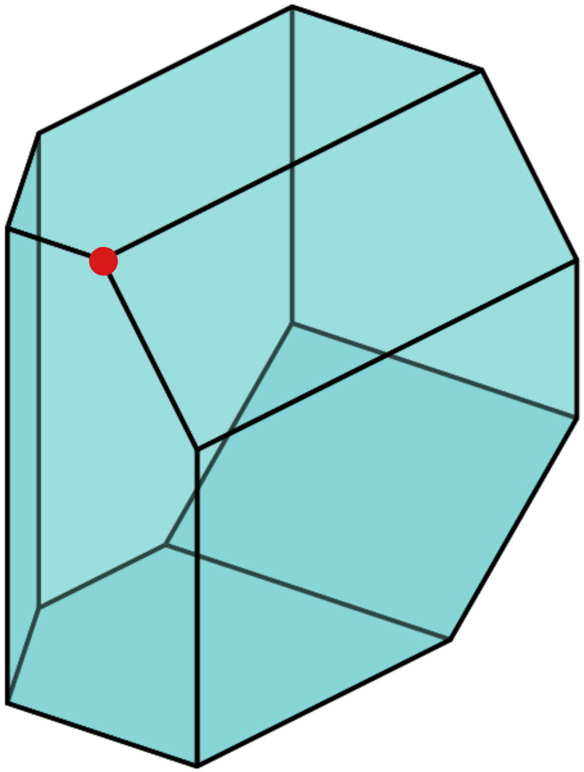
minimize	$\sum_{v \in V(G)} x_v$	
subject to	$x_u + x_v \geq 1$	for each $uv \in E(G)$
	$x_v \geq 0$	for each $v \in V(G)$

The solution space of an LP is a *convex* polytope in $\mathbb{R}^n$, where n is the number of variables.

The number of facets corresponds to the number of "relevant" constraints.

An *extreme point* of the polytope is a point that is extreme in some direction.

Most LP solvers return extreme-point solutions.



Half-Integrality

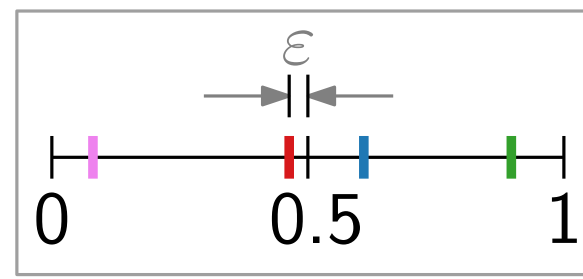
Thm. The extreme points of the vertex cover polytope are half-integral, that is, their coordinates are in $\{0, 0.5, 1\}$.

Proof. Let x be an LP solution that is not half-integral.

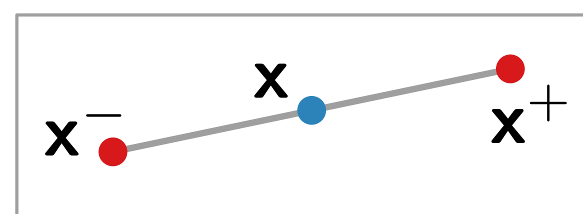
$\Rightarrow \exists v \in V(G)\colon x_v \in (0, 0.5) \cup (0.5, 1)$.

Let $\varepsilon = \min_{u \in V(G)} \{x_u, 1 - x_u, |x_u - 0.5|\} \setminus \{0\}$.

For every $v \in V(G)$, let



$$\left.\begin{aligned} x_v^{+} &= \begin{cases} x_v + \varepsilon & \text{if } x_v \in (0.5, 1) \\ x_v - \varepsilon & \text{if } x_v \in (0, 0.5) \\ x_v & \text{else.} \end{cases} \\ x_v^{-} &= \begin{cases} x_v - \varepsilon & \text{if } x_v \in (0.5, 1) \\ x_v + \varepsilon & \text{if } x_v \in (0, 0.5) \\ x_v & \text{else.} \end{cases} \end{aligned}\right\} \Rightarrow \frac{\mathbf{x}^{+} + \mathbf{x}^{-}}{2} = \mathbf{x}$$



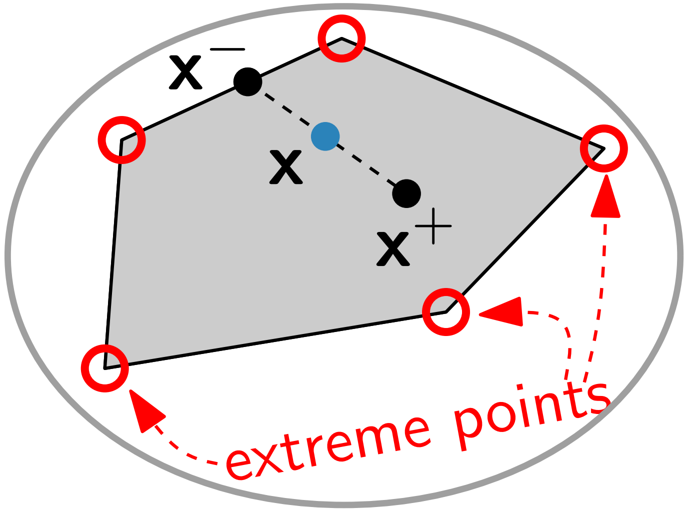
Extreme Points

Claim: $\mathbf{x}^+$ is feasible.

Need to show: For every edge uv, we have $x_u^+ + x_v^+ \geq 1$.

Assume $x_u^+ < 0.5 \Rightarrow x_v^+ > 0.5 \Rightarrow x_u^+ + x_v^+ = x_u + x_v \geq 1$ ✓

Symmetrically: $\mathbf{x}^-$ is feasible.



We have shown:

If a solution is not half-integral, it cannot be extreme.

In other words:

All extreme-point solutions are half-integral.

For VC, standard LP solvers return half-integral solutions. :-)