Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part I:
Organizational

Alexander Wolff

Winter term 2025

Organizational

Lectures: Fri, 10:15–11:45 (ÜR I)

English/German, depending on audience.

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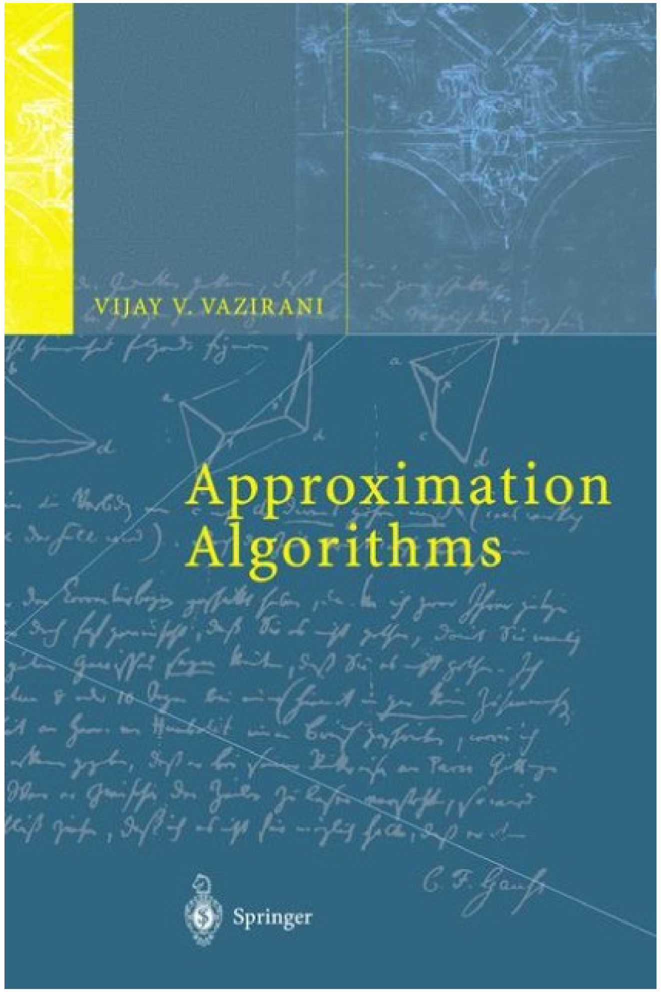
Most slides are due to Joachim Spoerhase,
polishing & colors are due to Philipp Kindermann – thanks!

Textbooks



Vijay V. Vazirani:
Approximation Algorithms
Springer-Verlag, 2003.

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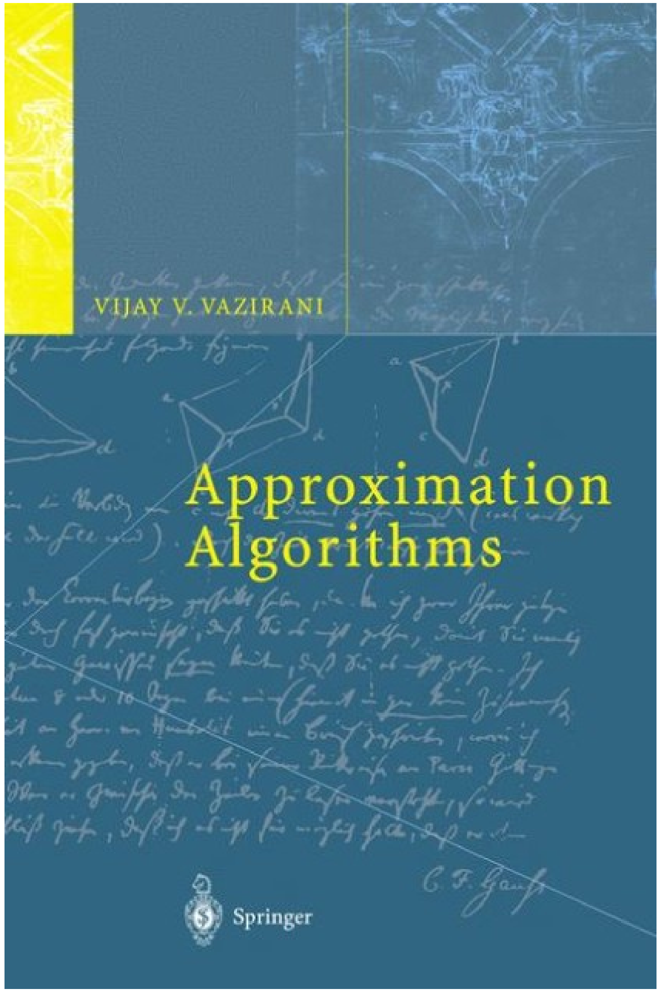

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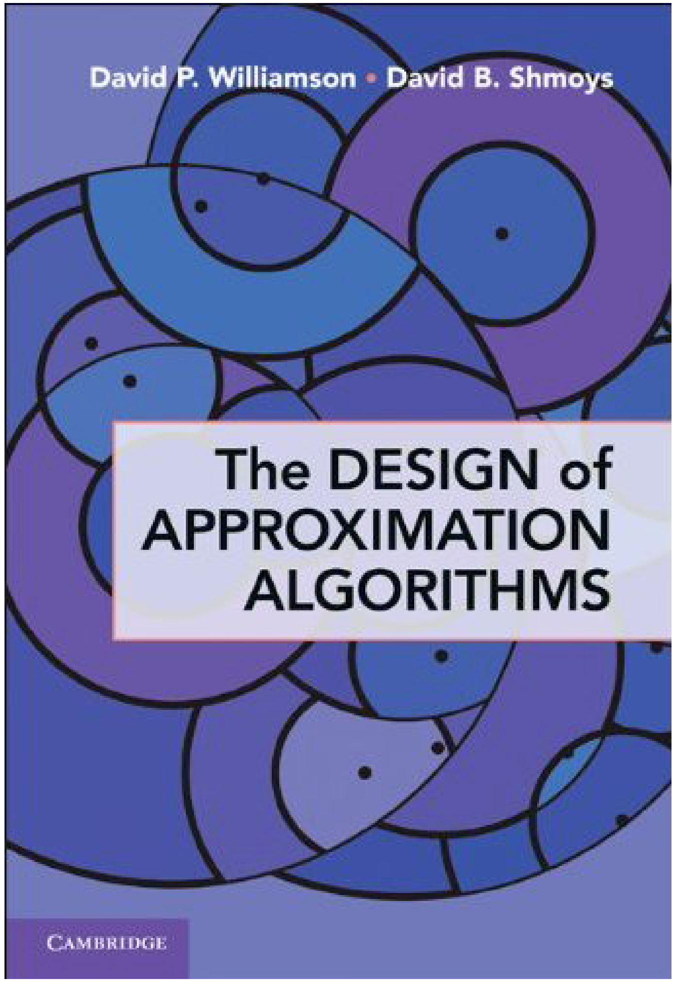

D. P. Williamson & D. B. Shmoys:
The Design of Approximation Algorithms
Cambridge-Verlag, 2011.

http://www.designofapproxalgs.com/

Approximation Algorithms

„All exact science is dominated by the idea of approximation.“

Bertrand Russell
(1872 – 1970)

Approximation Algorithms

- Many optimization problems are NP-hard!
  (For example, the traveling salesperson problem.)

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- Many optimization problems are NP-hard!
  (For example, the traveling salesperson problem.)
- $\rightsquigarrow$ an optimal solution cannot be efficiently computed unless P=NP.
- However, good approximate solutions can often be found efficiently!
- **Techniques** for the design and analysis of approximation algorithms arise from studying specific optimization problems.

Overview

Combinatorial algorithms

- Introduction (Vertex Cover)
- Set Cover via Greedy
- Shortest Superstring
  via reduction to SC
- Steiner Tree via MST
- Multiway Cut via Greedy
- k-Center via Parametrized Pruning
- Min-Degree Spanning Tree
  and local search
- Knapsack via DP and Scaling
- Euclidean TSP via Quadtrees

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LP-based algorithms

- Introduction to LP-Duality
- Set Cover via LP Rounding
- Set Cover via Primal–Dual Schema
- Maximum Satisfiability
- Scheduling und Extreme Point Solutions
- Steiner Forest via Primal–Dual

Approximation Algorithms

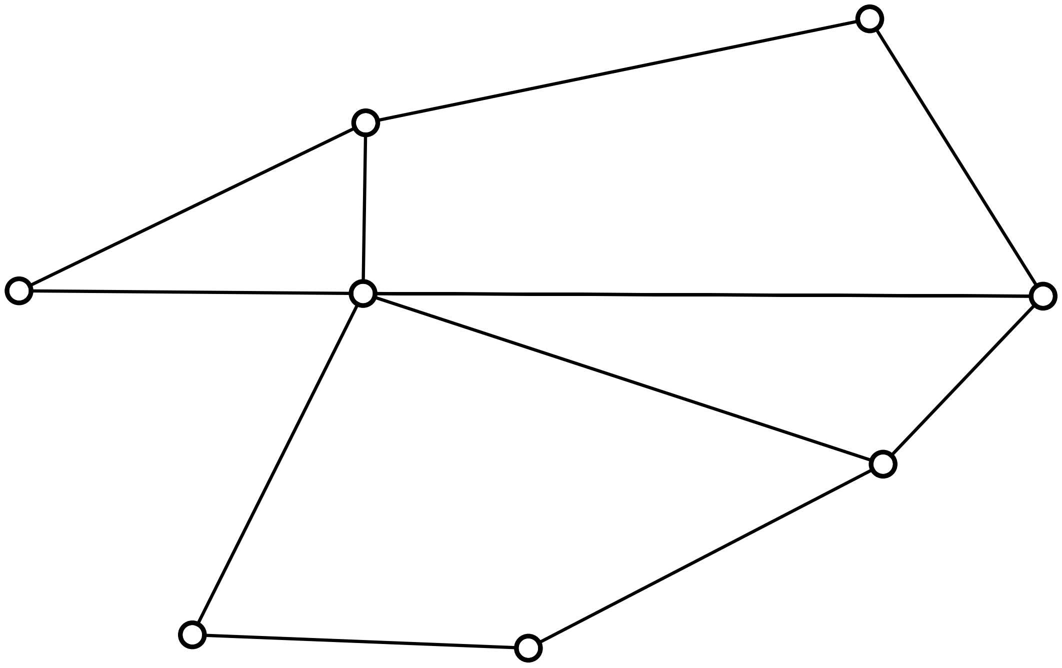
Lecture 1:
Introduction and Vertex Cover

Part II:
(Cardinality) Vertex Cover

VertexCover (card.)

Input: graph G

Output:



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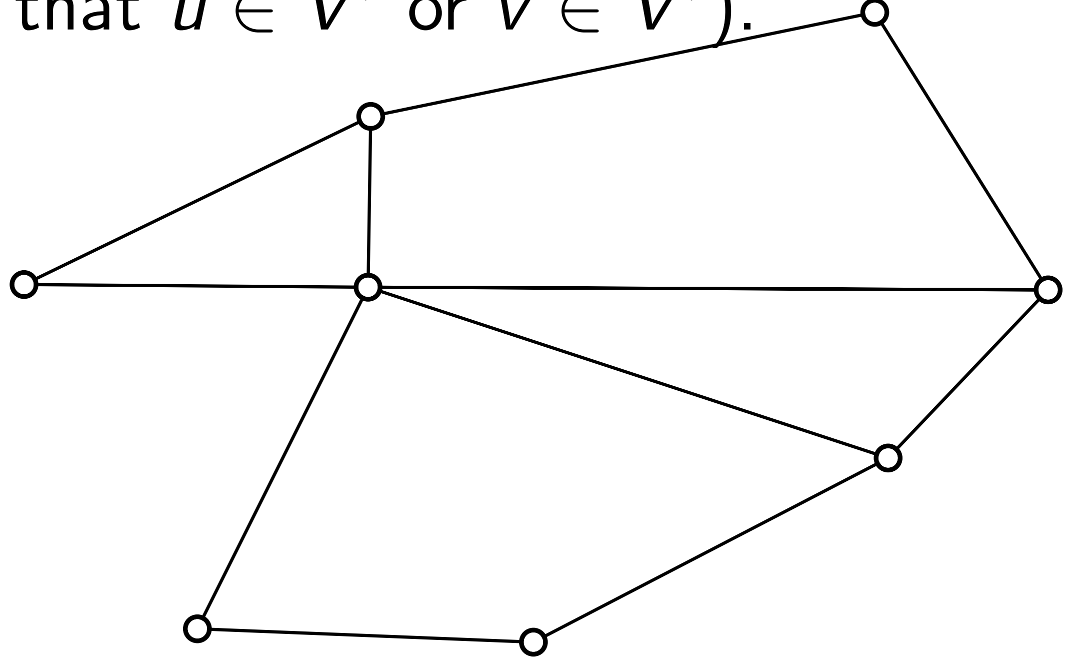

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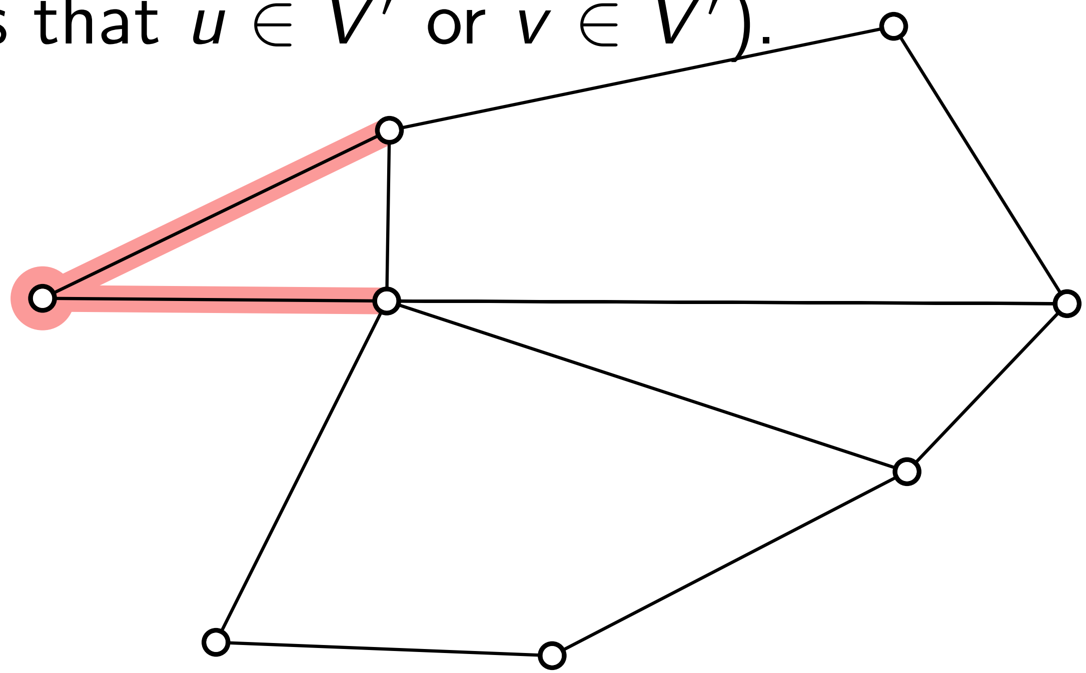

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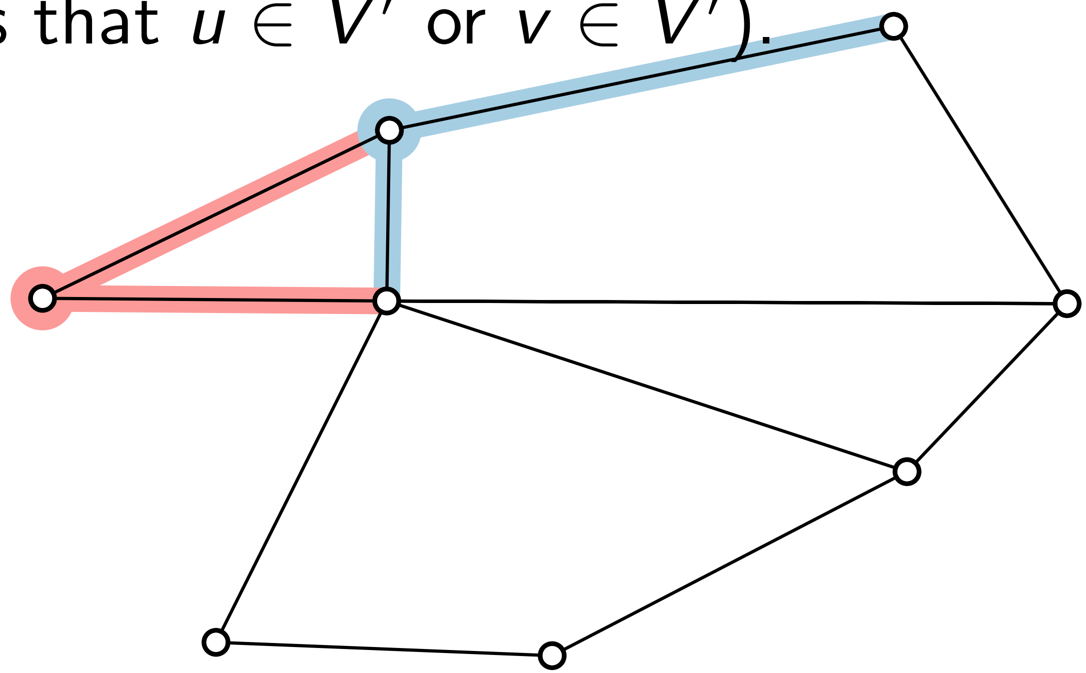

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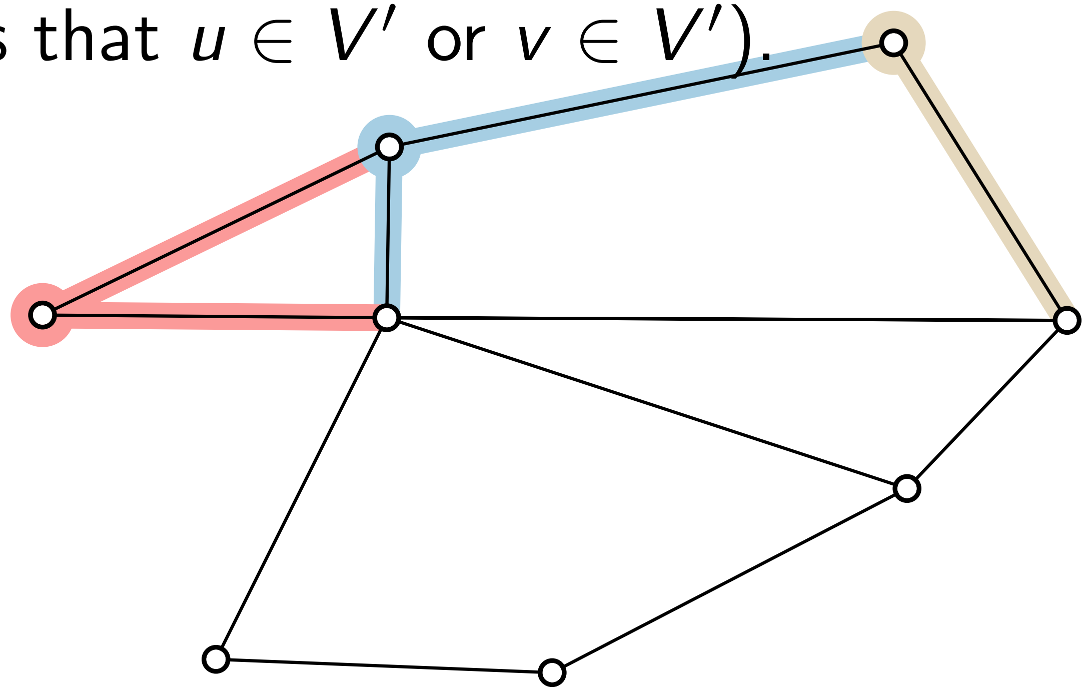

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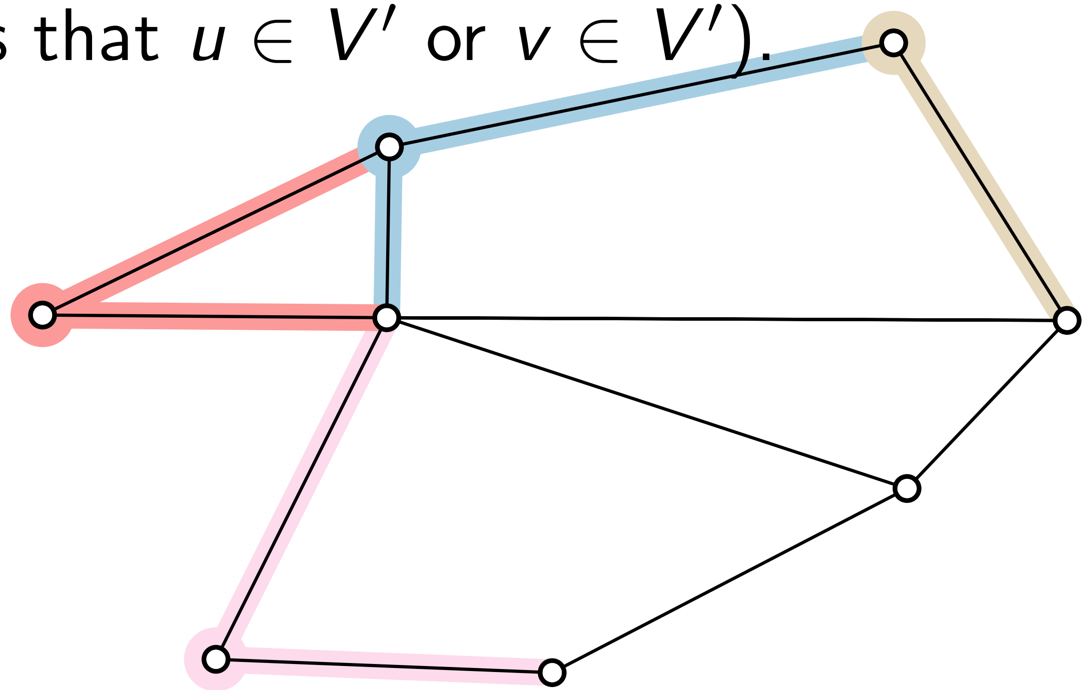

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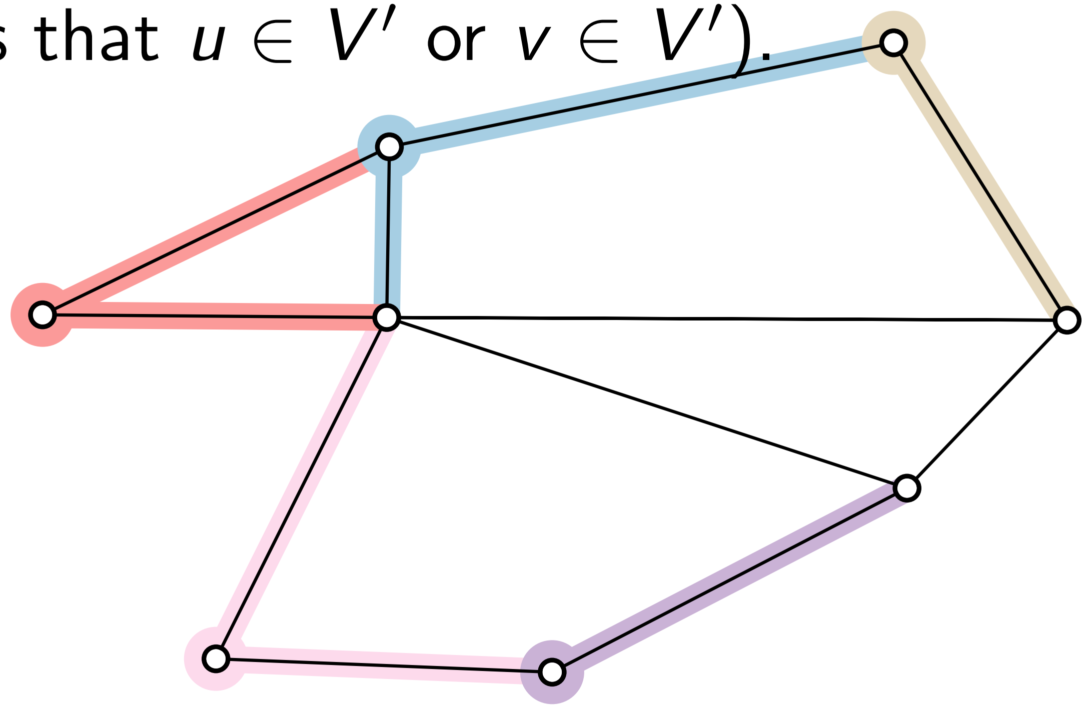

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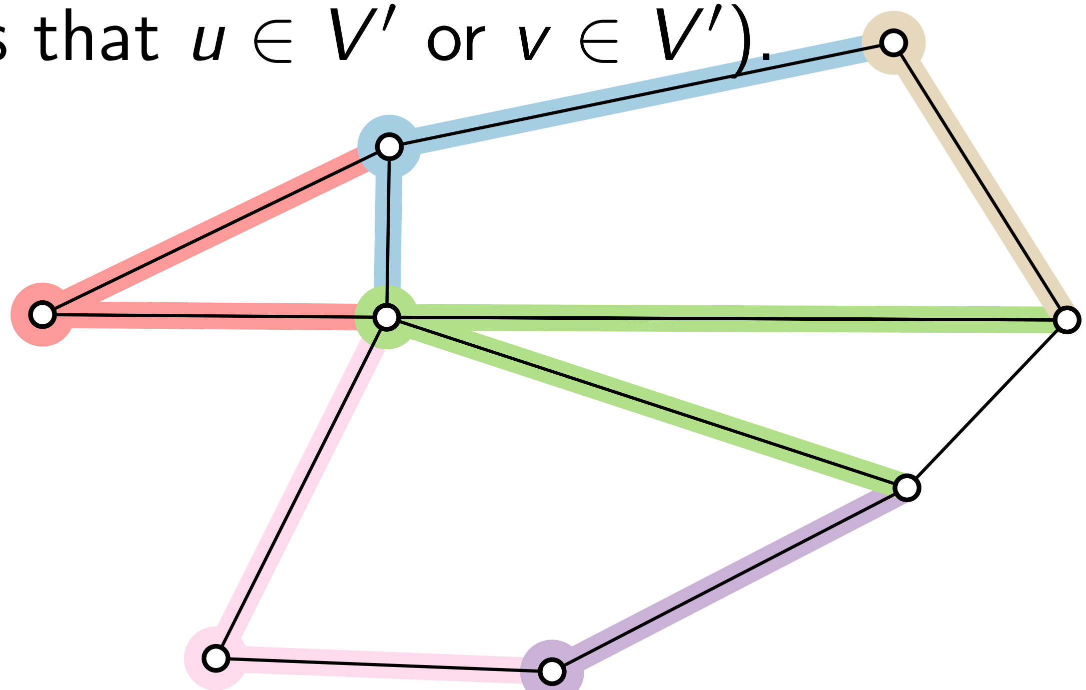

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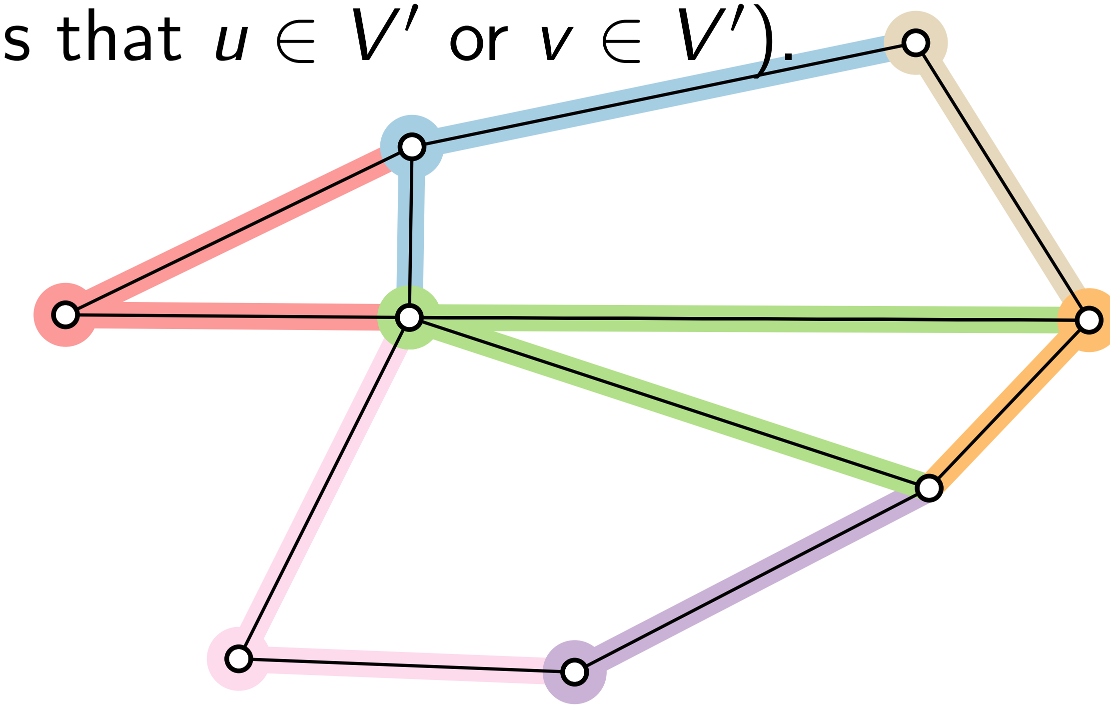

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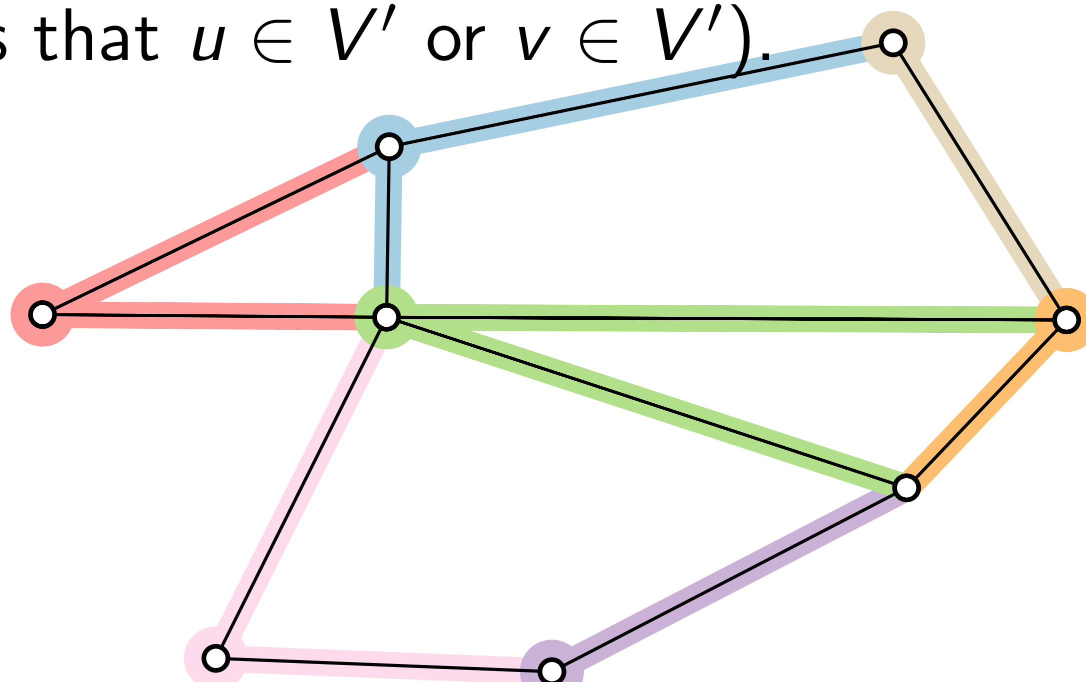

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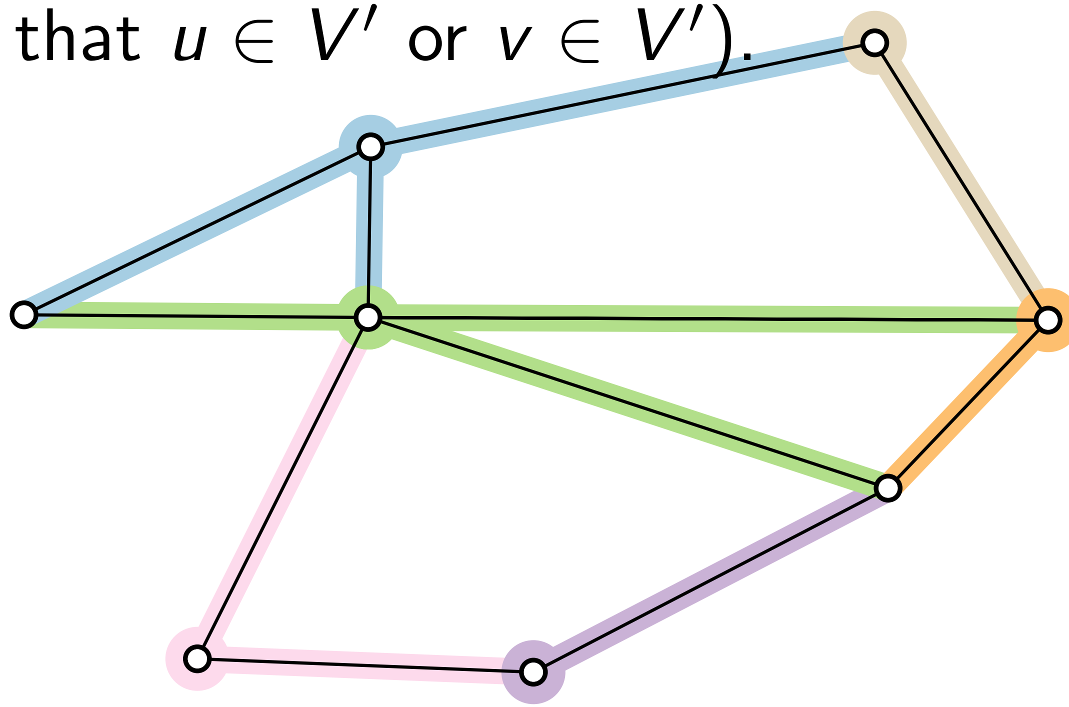

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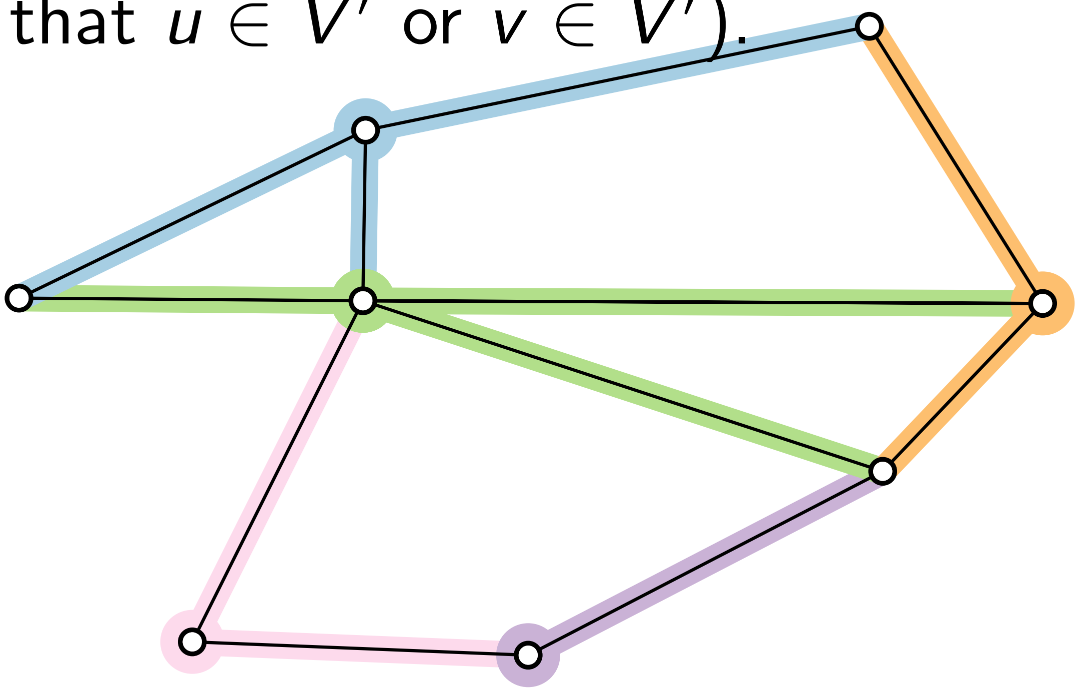

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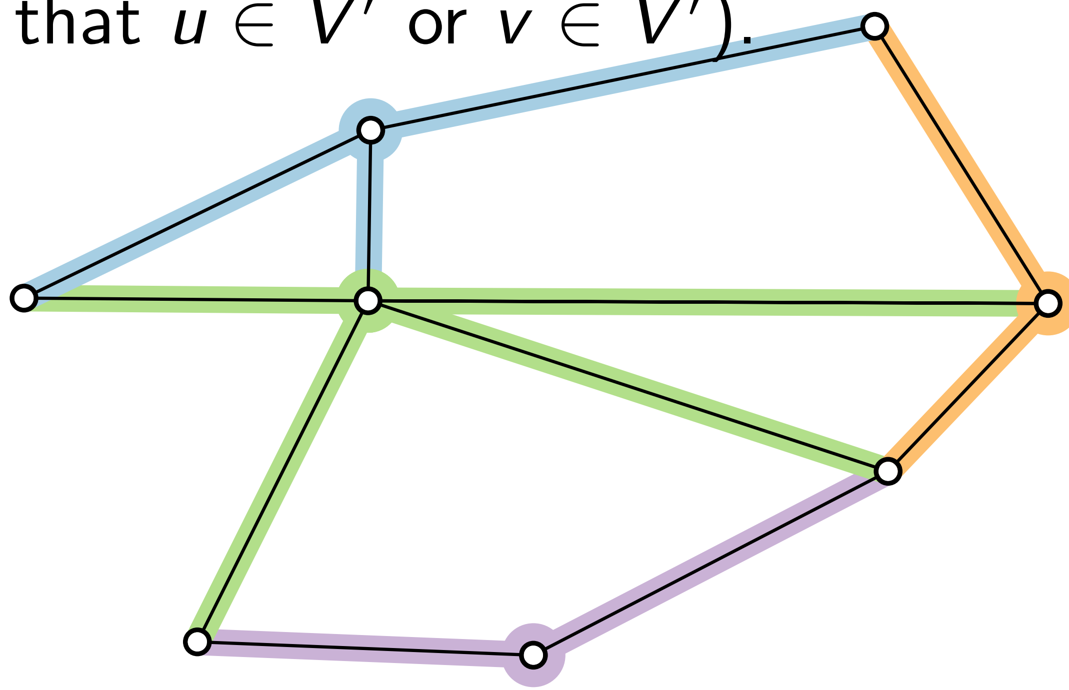

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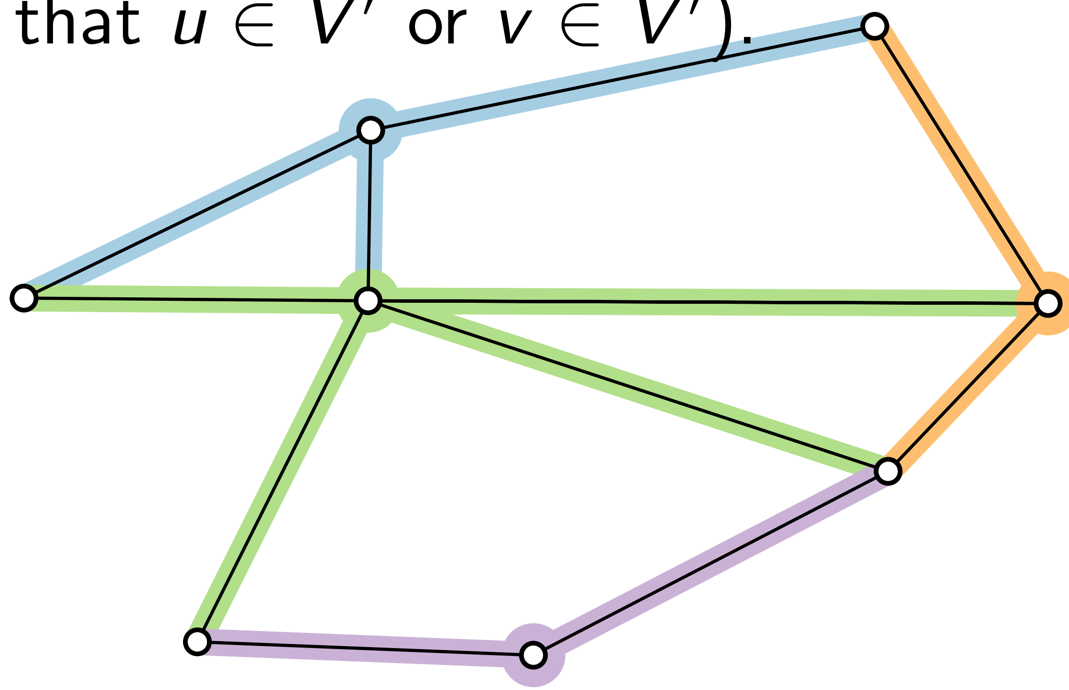

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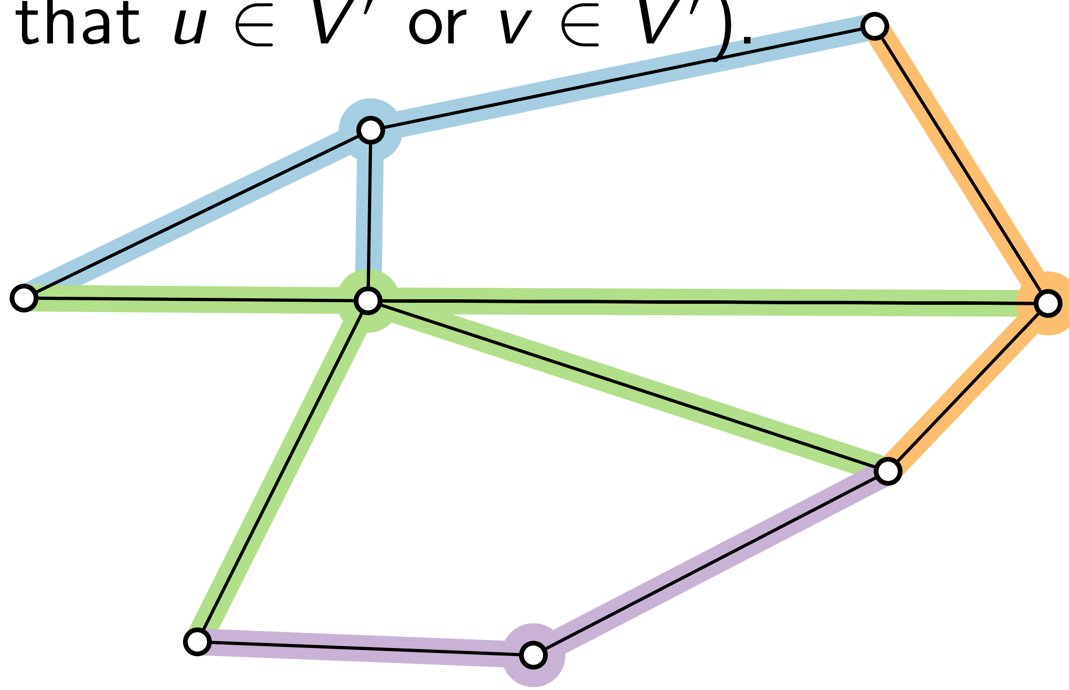

Optimum (OPT $= 4$) – but in general NP-hard to find :-(

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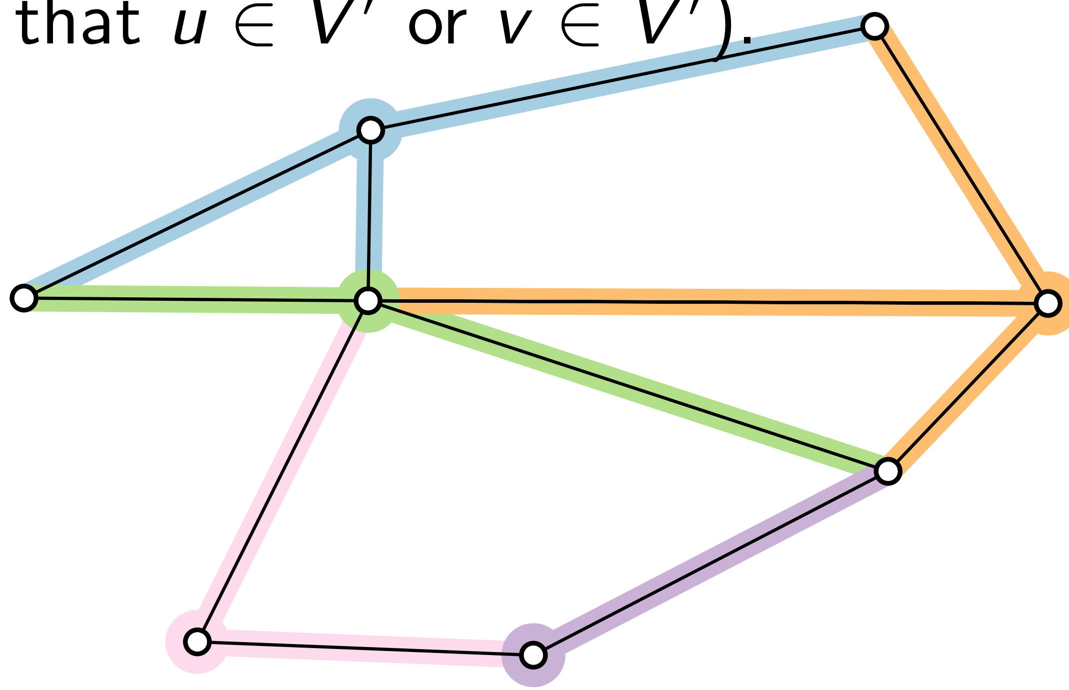

"good" (5/4-) approximate solution

Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part III:
NP-Optimization Problem

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  which assigns a positive objective value $\mathrm{obj}_\Pi(I, s) \geq 0$ to
  any given pair (s, I) with $s \in S_\Pi(I)$.
- Π is either a minimization or maximization problem.

$\textsc{VertexCover}$: NP-Optimization Problem

Task: Fill in the gaps for $\Pi = \textsc{Vertex Cover}$.

$D_\Pi =$

For $I \in D_\Pi$: $|I| =$

$S_\Pi(I) =$

- Why is $|s| \in \text{poly}(|I|)$ for every $s \in S_\Pi(I)$?

- For a given pair (s, I), how can we efficiently decide whether $s \in S_\Pi(I)$?

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Π is a minimization problem.

Optimum and Optimal Objective Value

Let Π be a minimization problem and $I \in D_\Pi$ an instance of Π.

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A feasible solution $s^* \in S_\Pi(I)$ is **optimal** if

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The optimal value $\mathrm{obj}_\Pi(I, s^*)$ of the objective function is denoted by $\mathrm{OPT}_\Pi(I)$ or simply by OPT in context.

Approximation Algorithms

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$$\frac{\mathrm{obj}_\Pi(I, s)}{\mathrm{OPT}_\Pi(I)} \leq \alpha.$$

Approximation Algorithms

$\alpha \colon \mathbb{N} \to \mathbb{Q}$

Let $\mathit{\Pi}$ be a minimization problem and $\cancel{\alpha \in \mathbb{Q}^+}$.

A factor-α approximation algorithm for $\mathit{\Pi}$ is an efficient algorithm that provides, for **any** instance $I \in D_{\mathit{\Pi}}$, a feasible solution $s \in S_{\mathit{\Pi}}(I)$ such that

$$\frac{\mathrm{obj}_{\mathit{\Pi}}(I, s)}{\mathrm{OPT}_{\mathit{\Pi}}(I)} \leq \cancel{\alpha}\ \alpha(|I|).$$

Approximation Algorithms

maximization problem $\alpha\colon \mathbb{N} \to \mathbb{Q}$

Let Π be a minimization problem and $\cancel{\alpha \in \mathbb{Q}^+}$.

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Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part IV:
Approximation Algorithm for VertexCover

Approximation Alg. for VertexCover

Ideas?

Approximation Alg. for VertexCover

Ideas?

- Edge-Greedy

Approximation Alg. for VertexCover

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- Edge-Greedy
- Vertex-Greedy

Approximation Alg. for VertexCover

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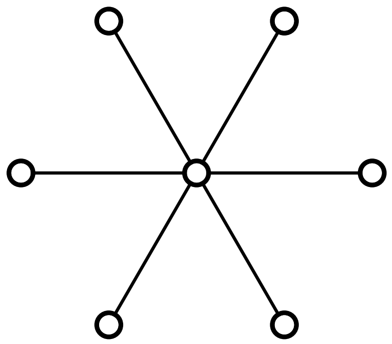

Approximation Alg. for VertexCover

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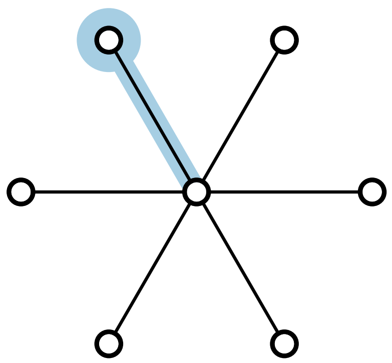

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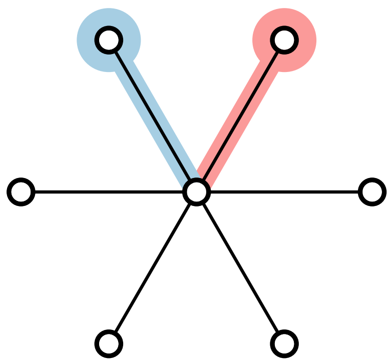

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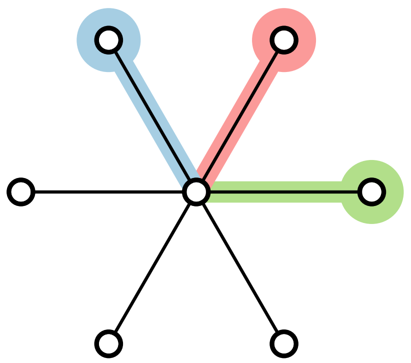

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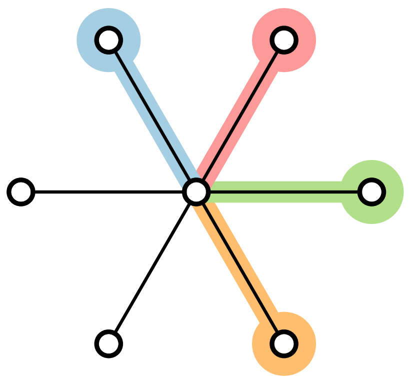

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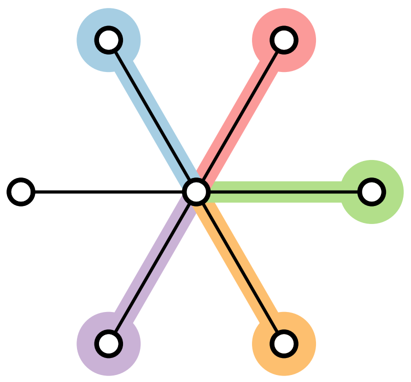

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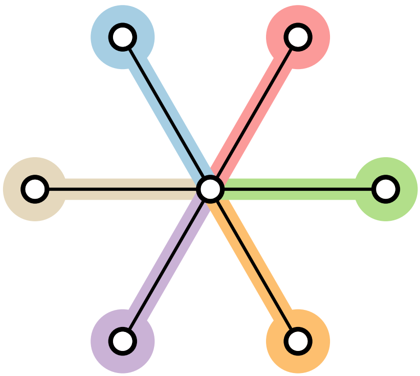

Approximation Alg. for VERTEXCOVER

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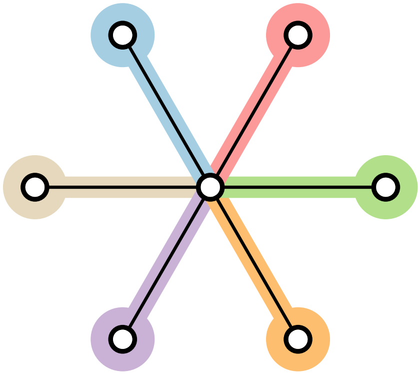

Quality?

Approximation Alg. for VertexCover

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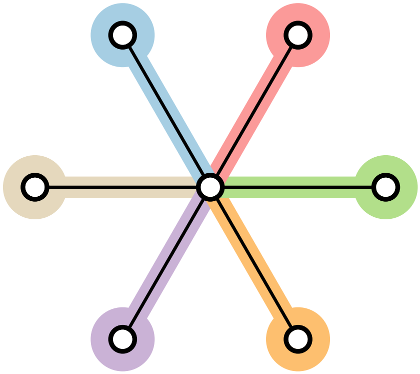

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Problem: How can we estimate $\mathrm{obj}_{\Pi}(I, s)/\mathrm{OPT}$ – if it is hard to compute OPT?

Approximation Alg. for VertexCover

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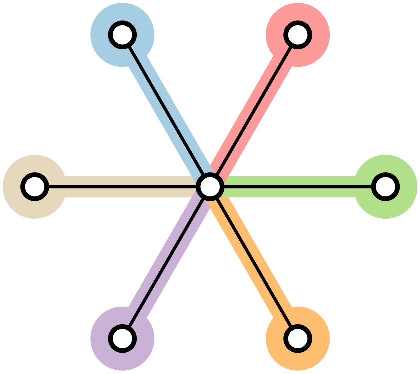

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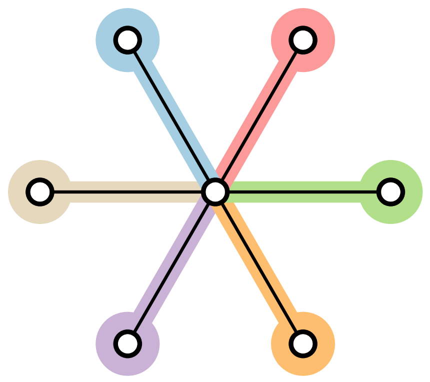

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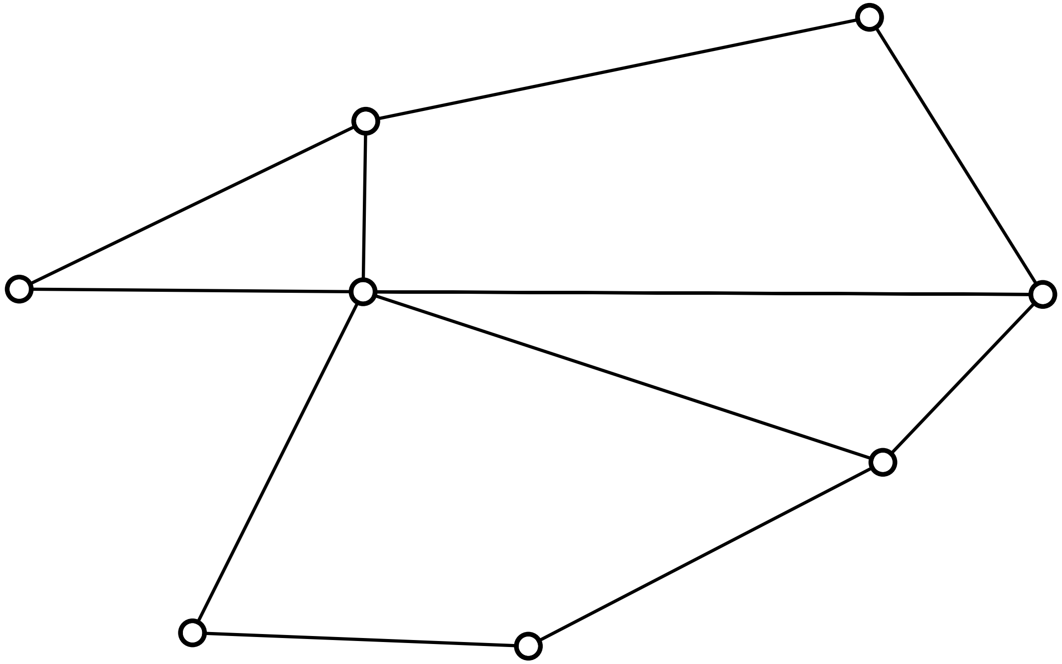
$$\frac{\mathrm{obj}_\Pi(I, s)}{\mathrm{OPT}} \leq \frac{\mathrm{obj}_\Pi(I, s)}{L}$$

Lower Bound



Lower Bound by Matchings

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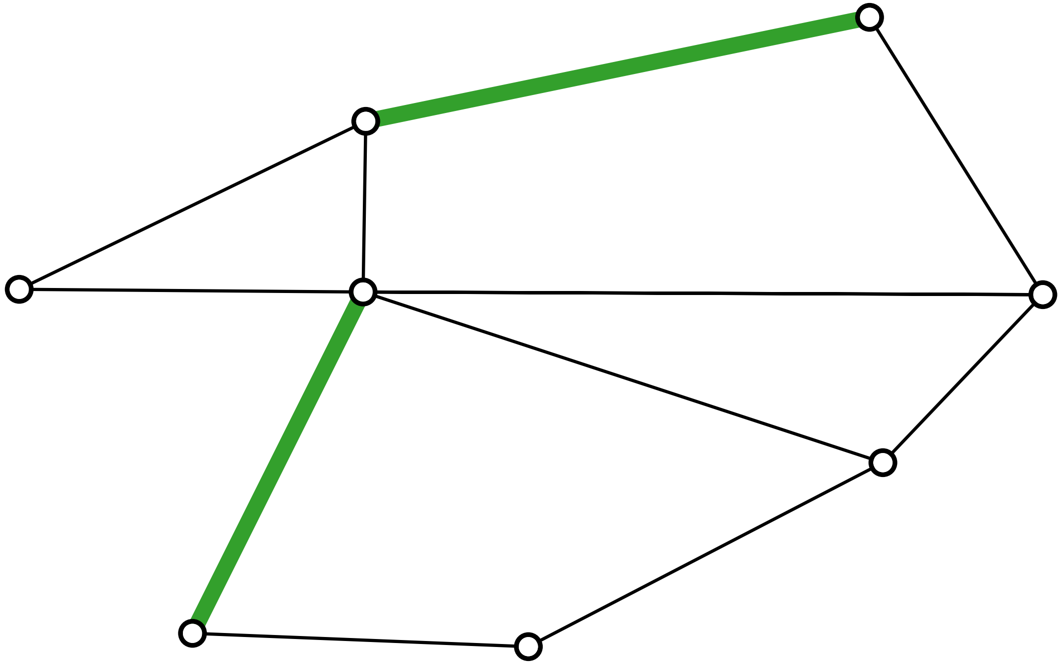

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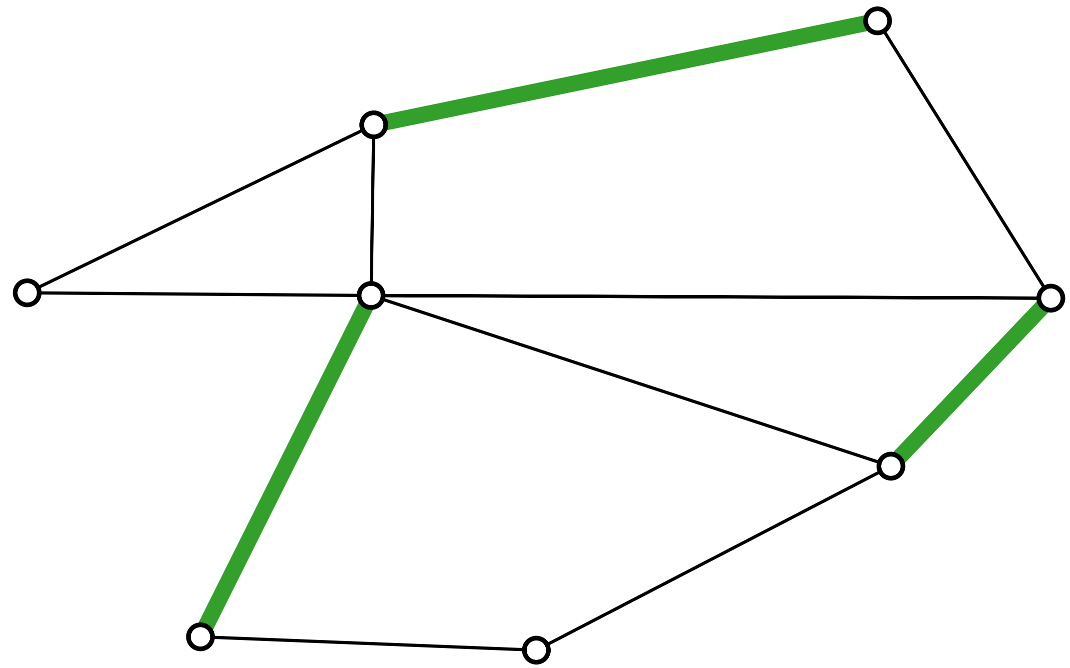

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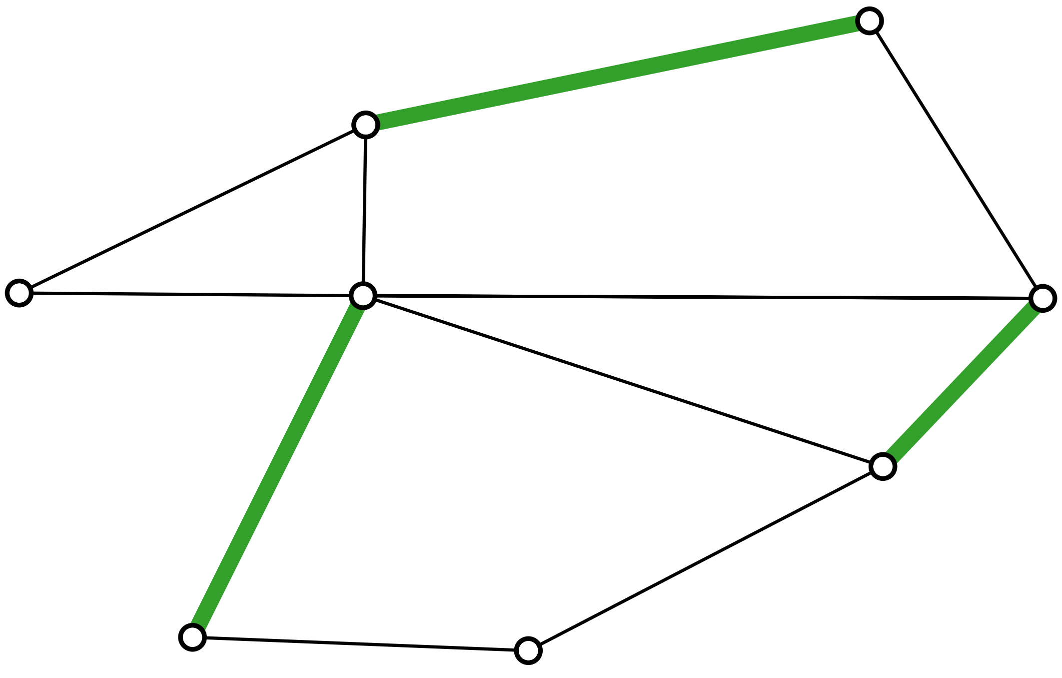
OPT $\geq$



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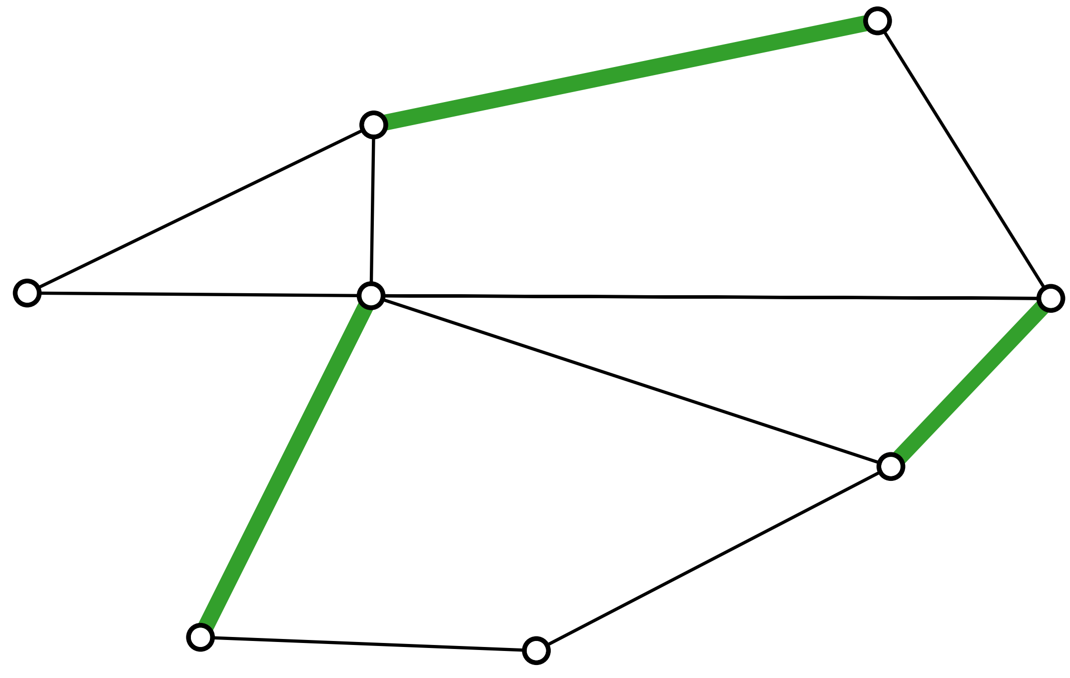

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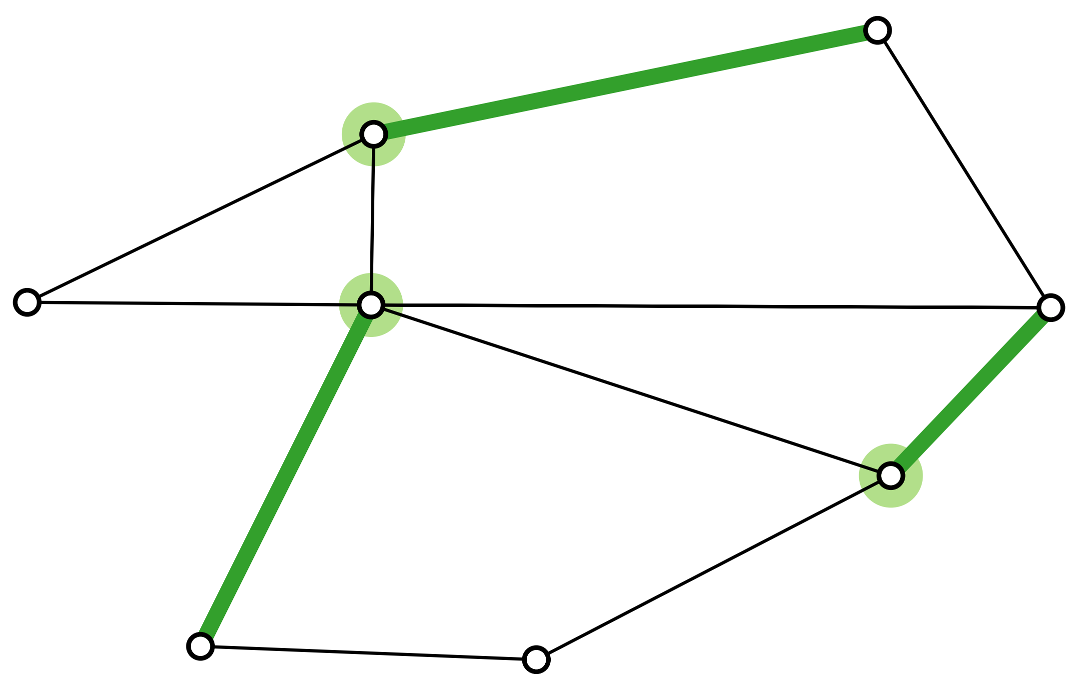
Vertex cover of M



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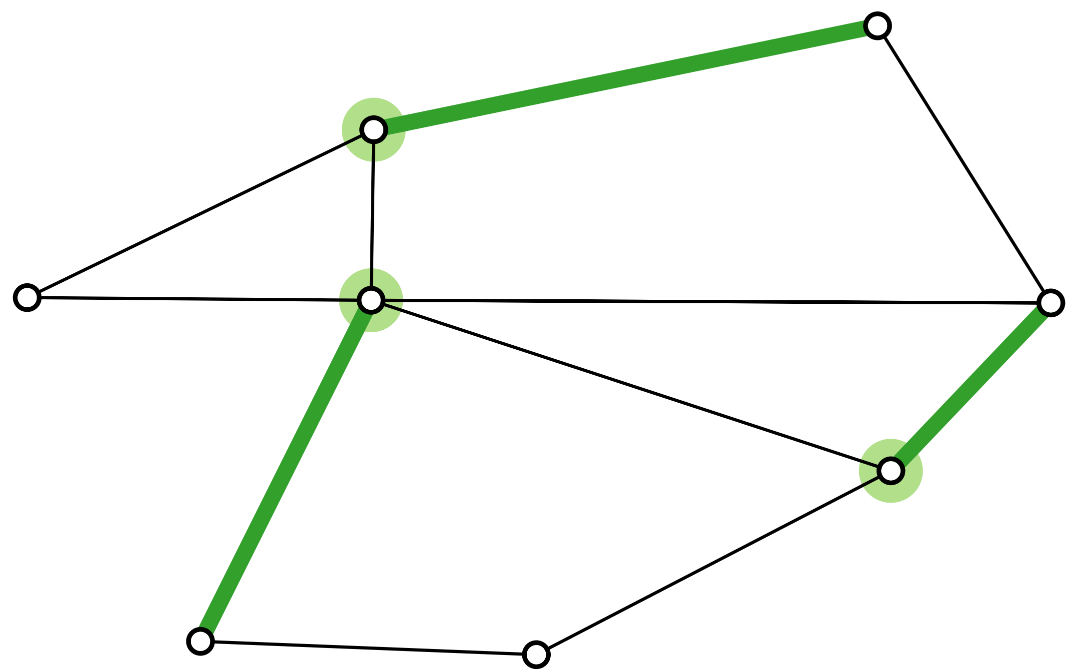

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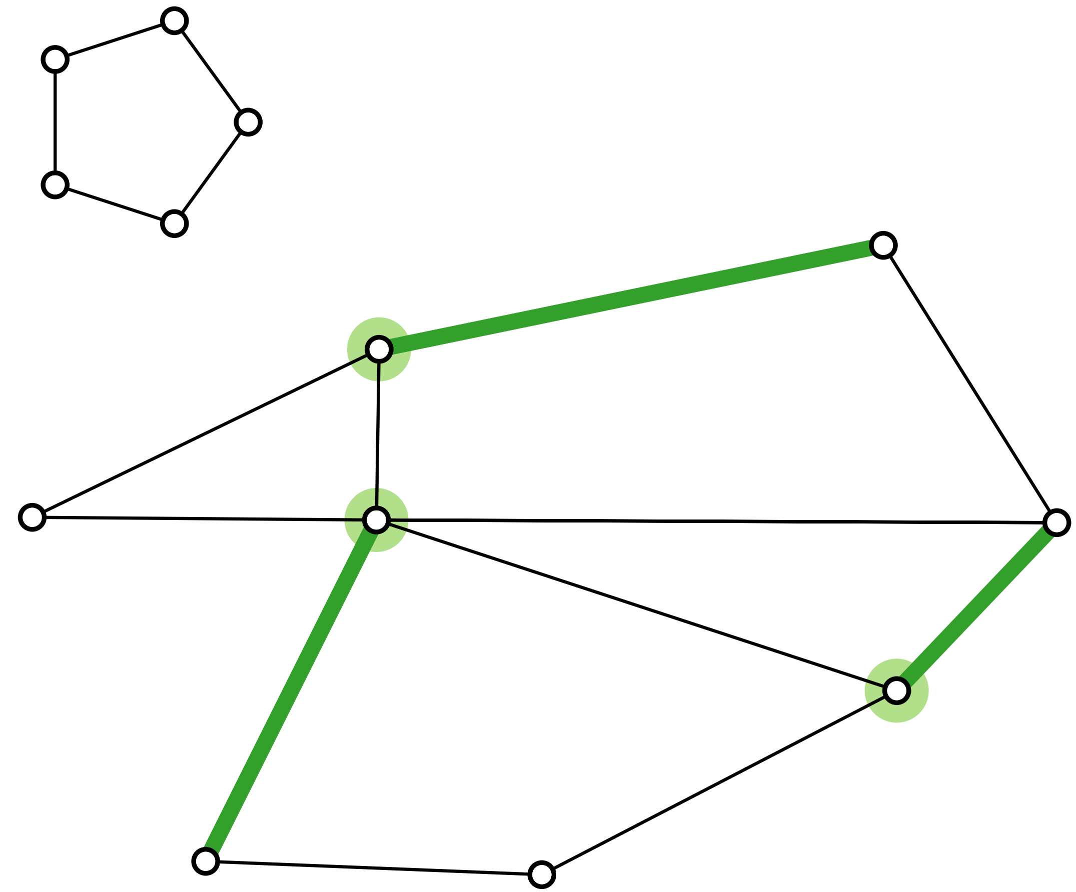

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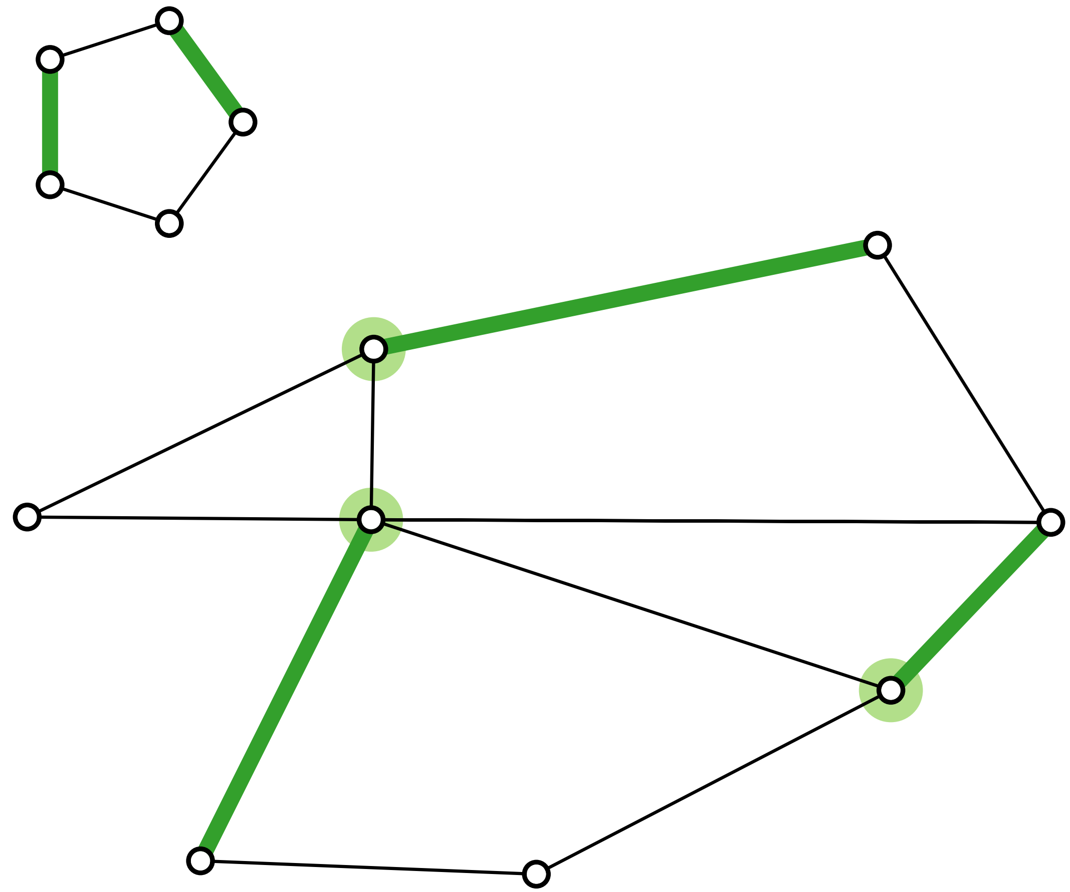

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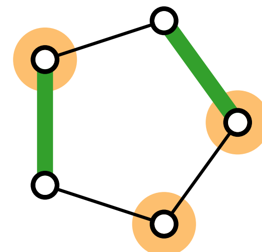

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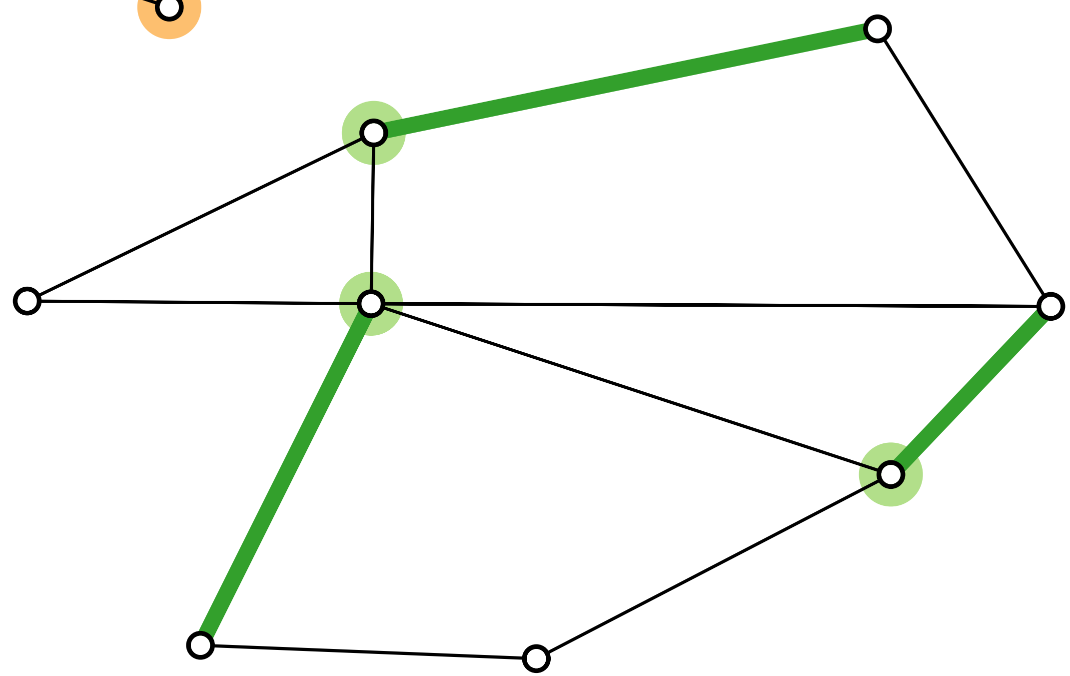

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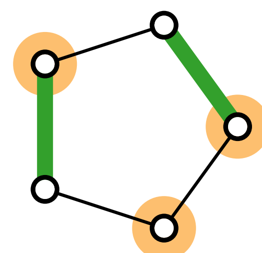

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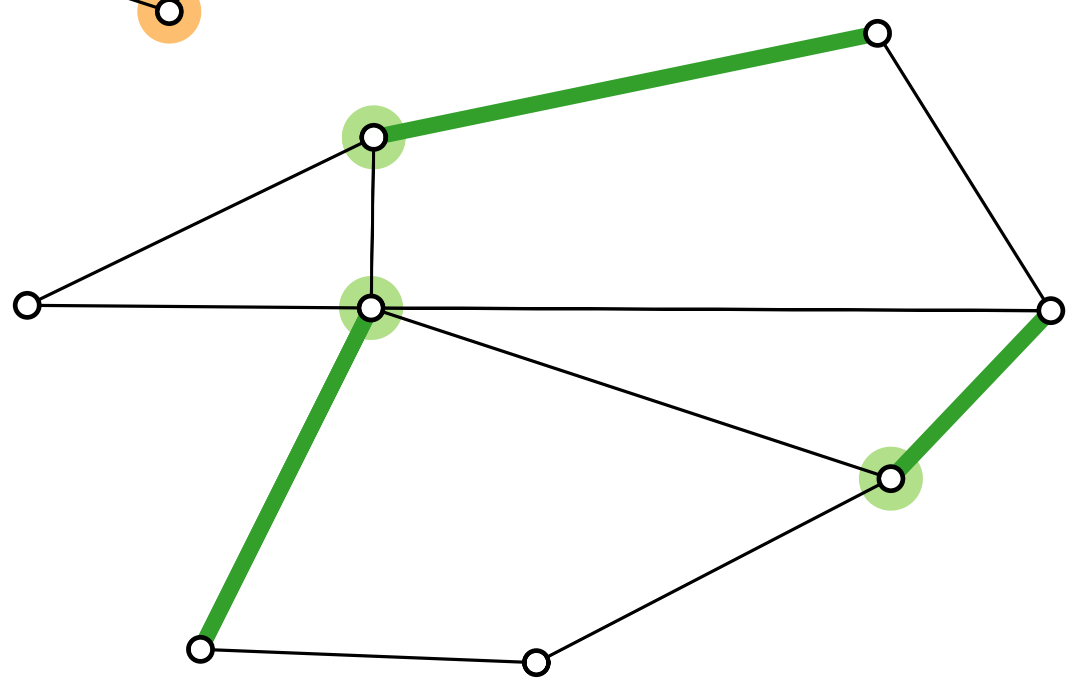

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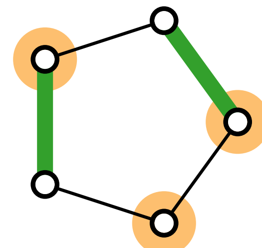

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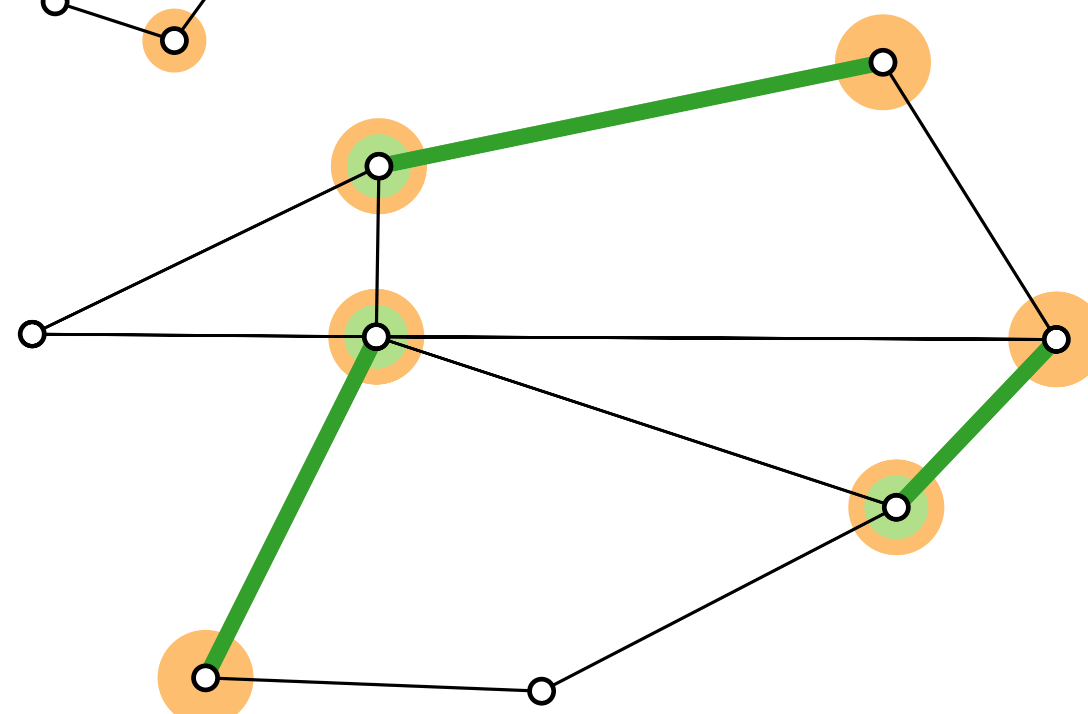
Vertex cover of $E(G)$



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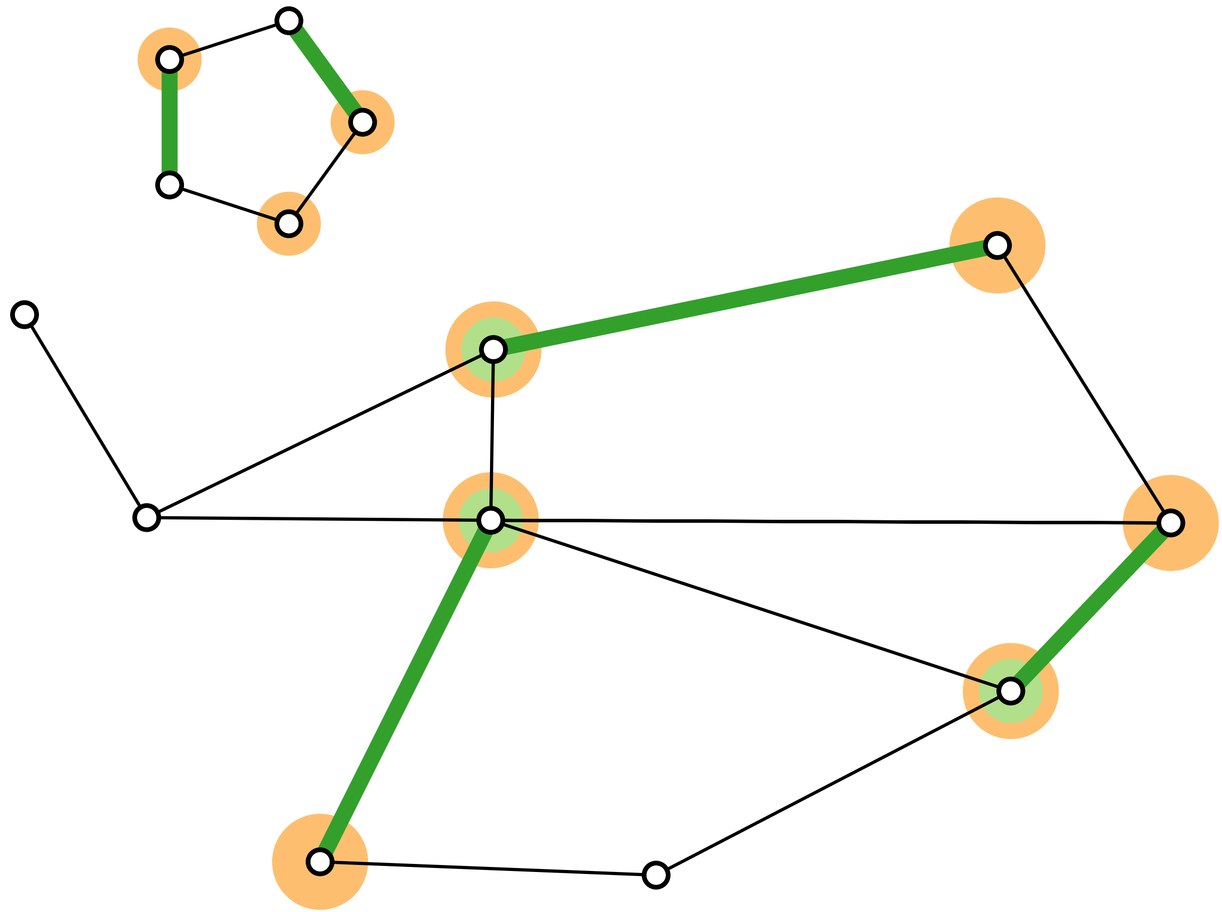

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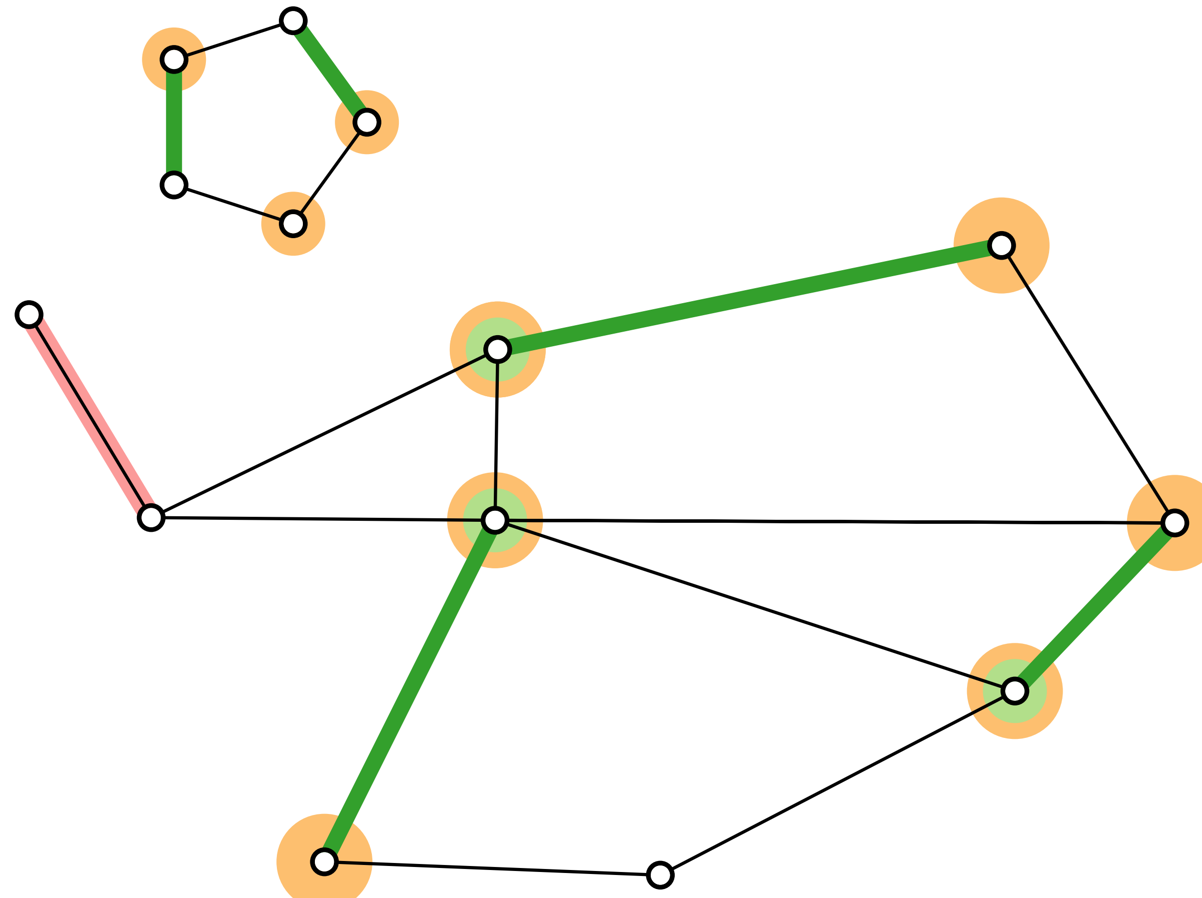

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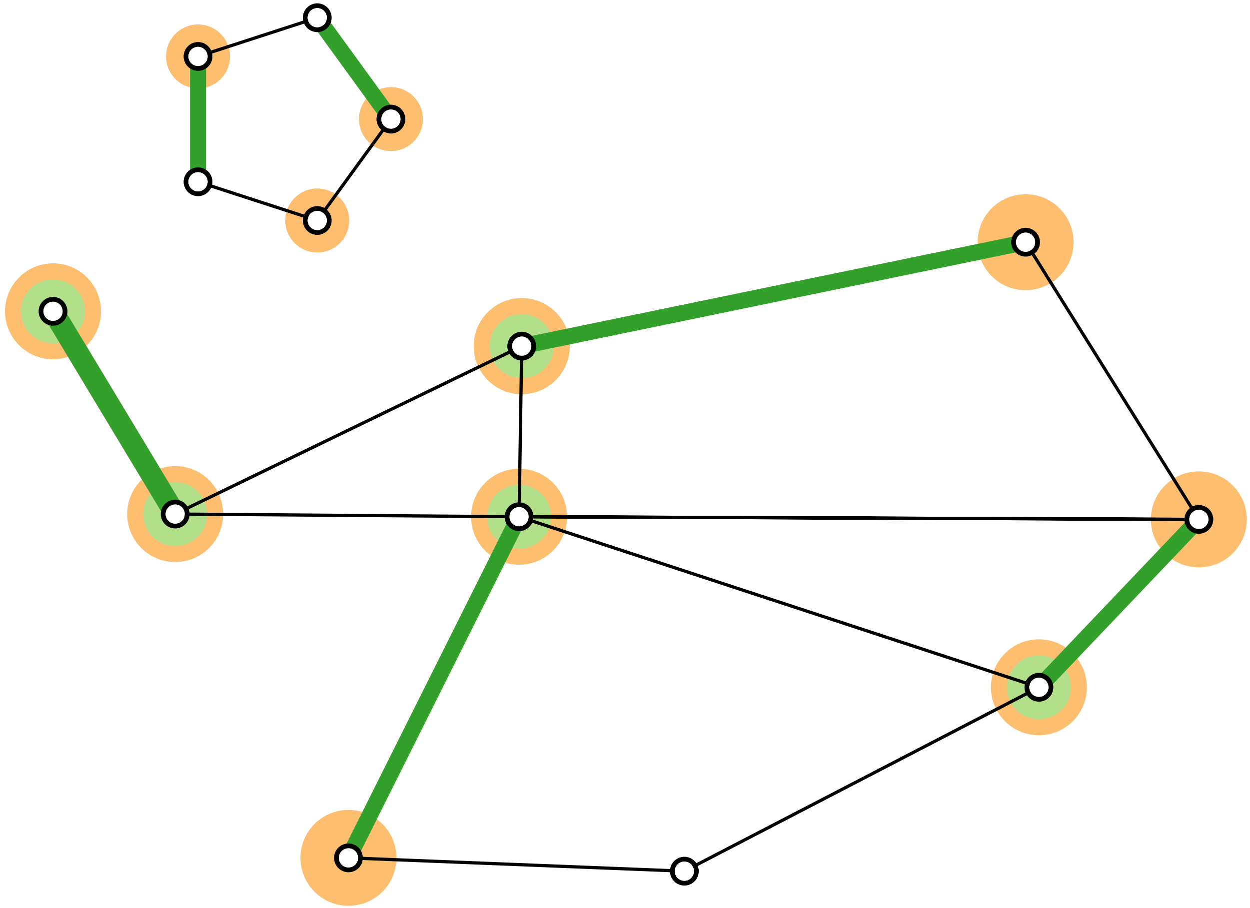

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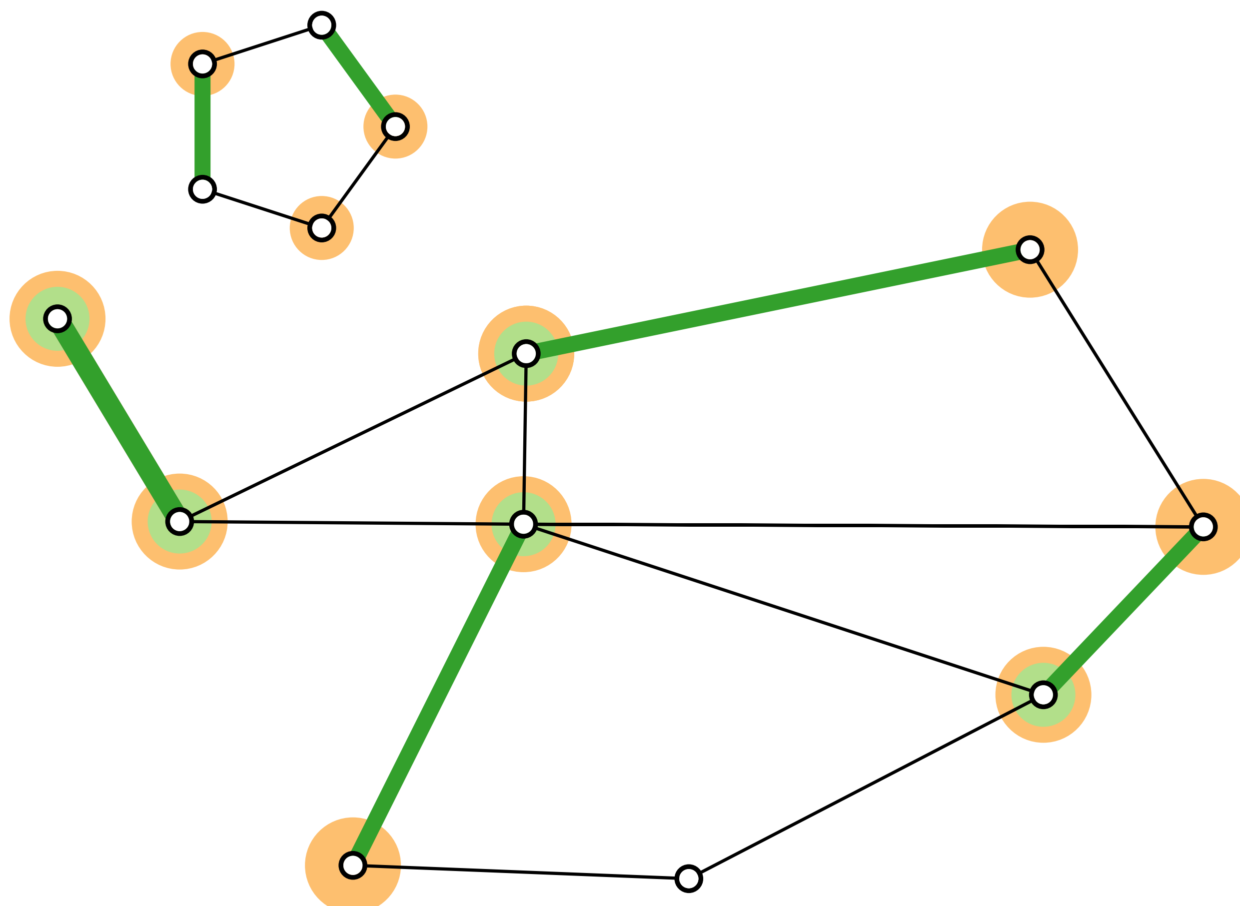

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$\text{ALG} = 2 \cdot |M| \leq$

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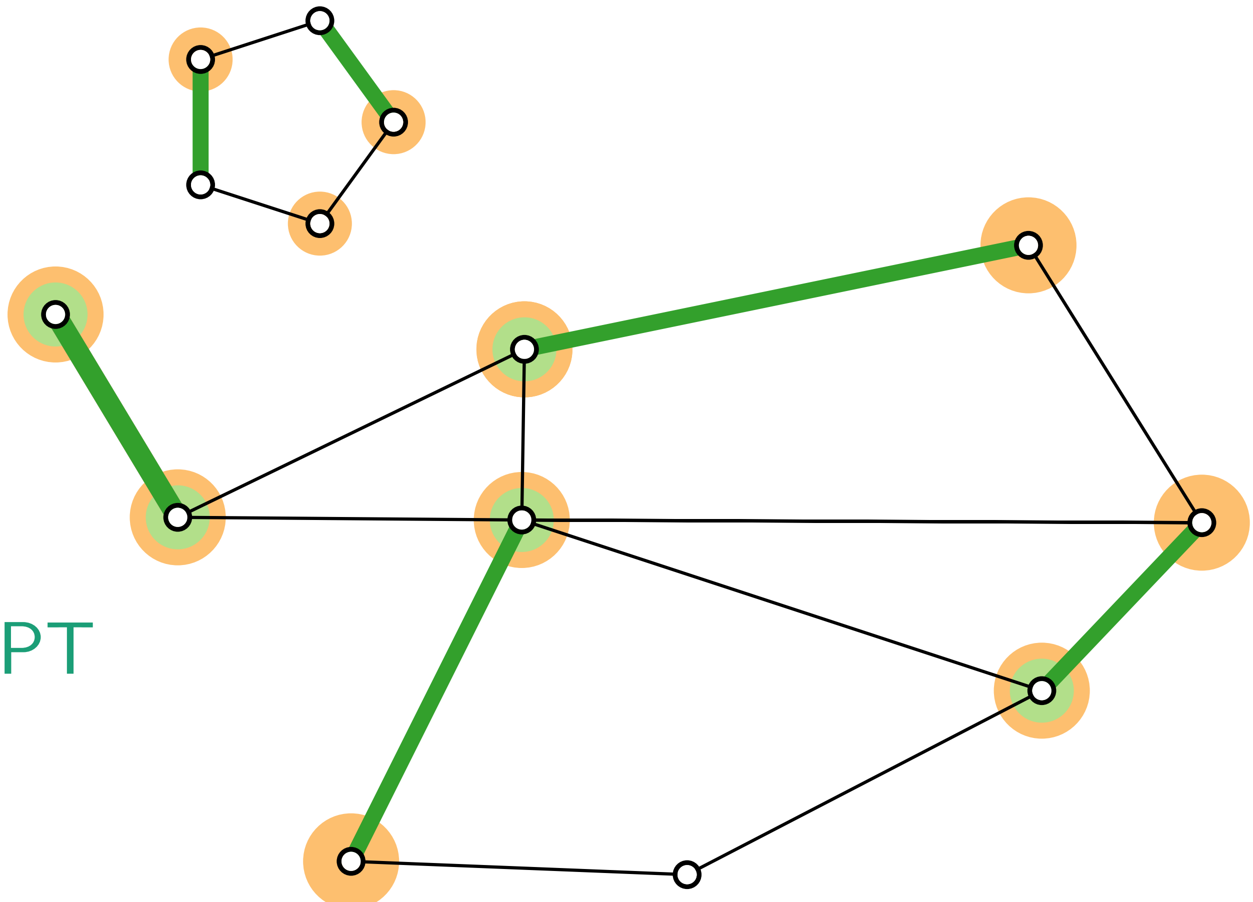

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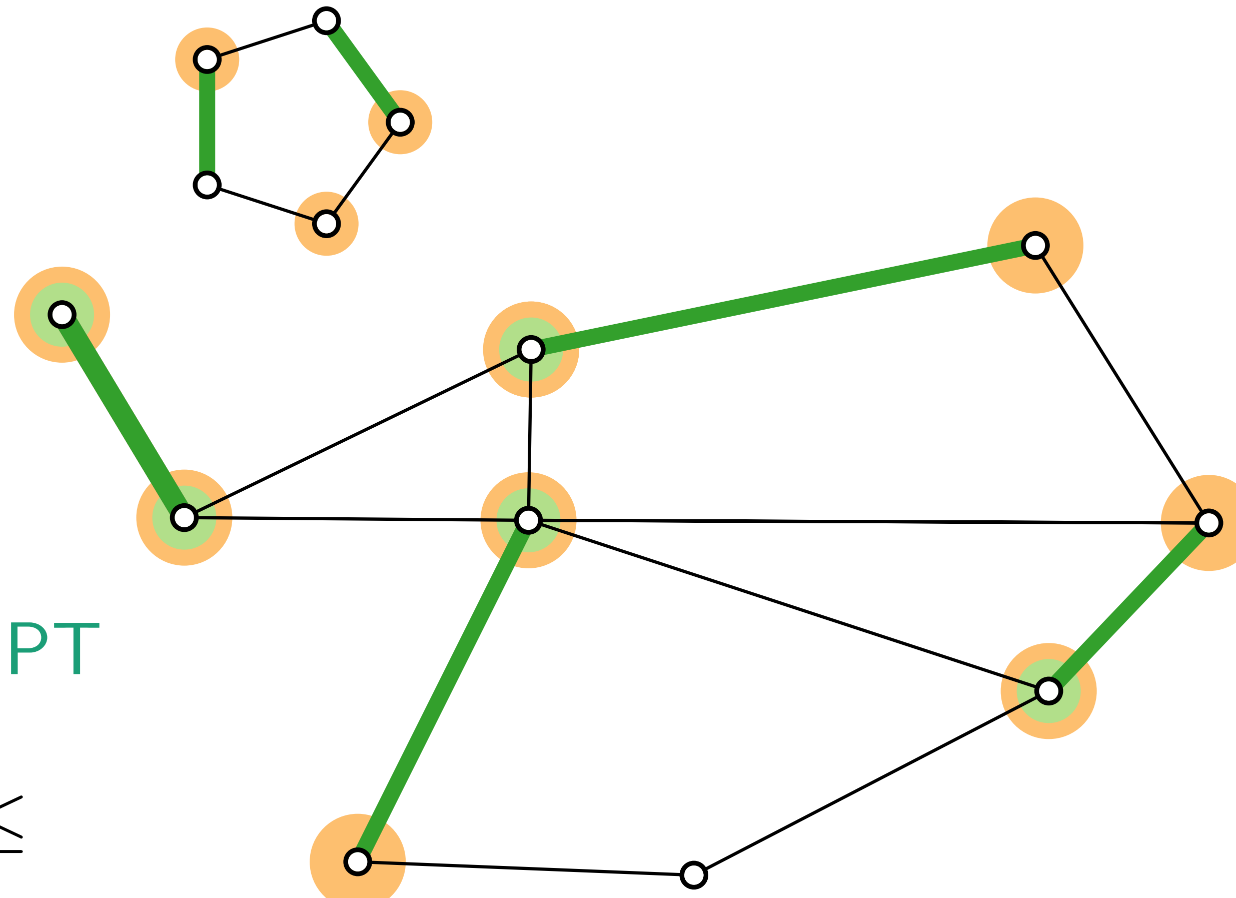

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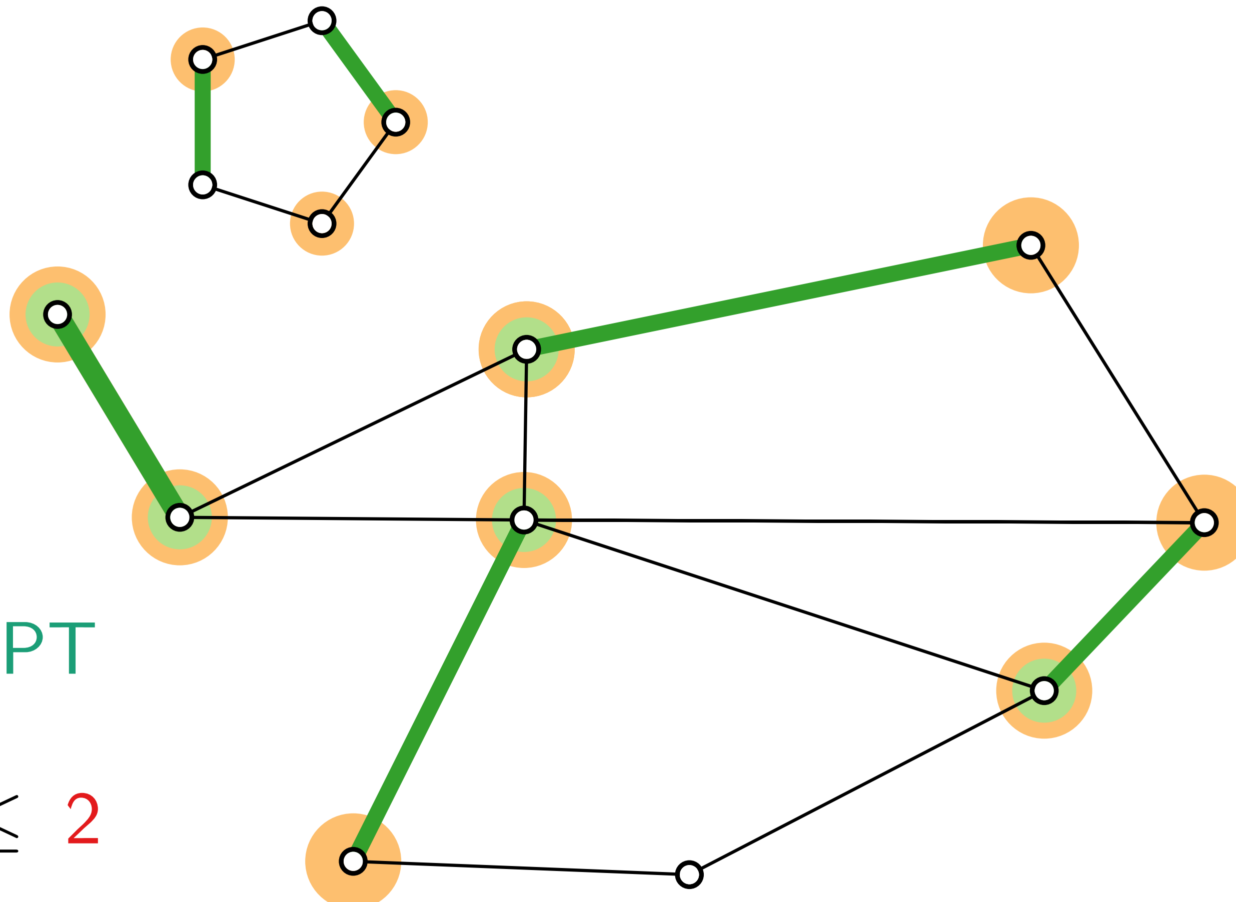

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$$\Rightarrow \frac{\mathrm{obj}_{\Pi}(I, s)}{\mathrm{OPT}} = \frac{\mathrm{ALG}}{\mathrm{OPT}} \leq 2$$

Approximation Alg. for VertexCover

Algorithm VertexCover(G)

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Approximation Alg. for $\textsc{VertexCover}$

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Approximation Alg. for VERTEXCOVER

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Theorem. The above algorithm is a factor-2 approximation algorithm for $\textsc{VertexCover}$.

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Approximability of Vertex Cover

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VertexCover cannot be approximated within a factor of $2 - \Theta(1)$ – if the *Unique Games Conjecture* holds.

Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part V:
An LP-based Algorithm for VertexCover

Task

Write an integer linear program (ILP) for VERTEXCOVER:

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Using integer (and/or real) variables, express the problem using

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- a linear objective function.

You can iterate over the vertices / edges of the given graph G.

Variables: for each vertex v of G, we introduce $x_v \in \{0, 1\}$.

v not in the solution
v in the solution

Objective: minimize $\sum_{v \in V(G)} x_v$

Constraints: for each edge uv of G, we require that

$$x_u + x_v \geq 1.$$

Standard ILP Format

$$\begin{array}{lll} \text{minimize} & \sum_{v \in V(G)} x_v & \\ \text{subject to} & x_u + x_v \geq 1 & \text{for each } uv \in E(G) \\ & x_v \in \{0, 1\} & \text{for each } v \in V(G) \end{array}$$

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\end{array}
$$

Problem:

Standard ILP Format

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Problem: It's NP-hard to solve ILPs in general.

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Problem: It's NP-hard to solve ILPs in general.

But: LPs can be solved efficiently (in $O(L \cdot n^{3.5})$ time),

Standard ILP Format

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Problem: It's NP-hard to solve ILPs in general.

But: LPs can be solved efficiently (in $O(L \cdot n^{3.5})$ time),
where $n = \#$ variables and $L =$ total bit complexity of coefficients.

Standard ILP Format

LP relaxation

minimize	$\sum_{v \in V(G)} x_v$	
subject to	$x_u + x_v \geq 1$	for each $uv \in E(G)$
	$x_v \geq 0$ $\quad \cancel{x_v \in \{0,1\}}$	for each $v \in V(G)$

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Problem′: Now we can get fractional solutions, i.e., $x_v \in (0, 1)$.

Standard ILP Format

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subject to	$x_u + x_v \geq 1$	for each $uv \in E(G)$
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Task: Find a graph G with $\mathrm{OPT_{LP}} \neq \mathrm{OPT_{ILP}}$!

Standard ILP Format

LP relaxation

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\begin{array}{lll}
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Solution?

Standard ILP Format

LP relaxation

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Problem′: Now we can get fractional solutions, i.e., $x_v \in (0,1)$.

Task: Find a graph G with $\mathrm{OPT_{LP}} \neq \mathrm{OPT_{ILP}}$!

Solution? Round the LP solution to get an integral solution!

Rounding the LP Solution

$$\begin{array}{lrl} \text{minimize} & \sum_{v \in V(G)} x_v & \\ \text{subject to} & x_u + x_v \geq 1 & \text{for each } uv \in E(G) \\ & x_v \geq 0 & \text{for each } v \in V(G) \end{array}$$

For each $v \in V(G)$:

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For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

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Need to check:

Rounding the LP Solution

$$\begin{aligned} &\text{minimize} \quad \textstyle\sum_{v \in V(G)} x_v \\ &\text{subject to} \quad x_u + x_v \geq 1 \qquad \text{for each } uv \in E(G) \\ &\qquad\qquad\qquad\quad {\color{red} x_v \geq 0} \qquad \text{for each } v \in V(G) \end{aligned}$$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

Need to check: Is $(x'_v)_{v \in V(G)}$ a feasible solution?

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In other words: Is $\{v \in V(G) \colon x'_v = 1\}$ a vertex cover of G?

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Need to make sure that every edge uv of G is covered.

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Is $x'_u = 0 = x'_v$ possible?

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This contradicts $x_u + x_v \geq 1$.

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This contradicts $x_u + x_v \geq 1. \Rightarrow\ x'_u = 1$ or $x'_v = 1$

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Is $x'_u = 0 = x'_v$ possible? But then $x_u < 0.5$ and $x_v < 0.5$.

This contradicts $x_u + x_v \geq 1. \Rightarrow$ $x'_u = 1$ or $x'_v = 1$ $\Rightarrow$ (x'_v) feasible!

Cost of the Solution

$$\begin{array}{lll} \text{minimize} & \sum_{v \in V(G)} x_v & \\ \text{subject to} & x_u + x_v \geq 1 & \text{for each } uv \in E(G) \\ & x_v \geq 0 & \text{for each } v \in V(G) \end{array}$$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

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\end{array}
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Theorem. The LP rounding algorithm is a factor-2 approximation algorithm for VertexCover.

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$x_v \geq 0$ for each $v \in V(G)$

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Theorem. The LP rounding algorithm is a factor-2 approximation algorithm for WEIGHTEDVERTEXCOVER.

Cost of the Solution

$$\begin{array}{lll} \text{minimize} & \sum_{v \in V(G)} x_v \cdot w(v) & \\ \text{subject to} & x_u + x_v \geq 1 & \text{for each } uv \in E(G) \\ & x_v \geq 0 & \text{for each } v \in V(G) \end{array}$$

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Theorem. The LP rounding algorithm is a factor-2 approximation algorithm for WeightedVertexCover.

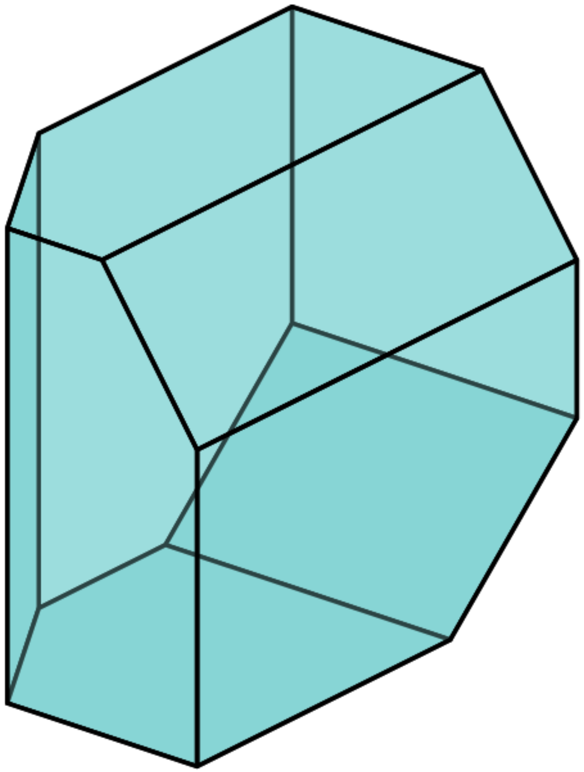
Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part VI:
The Vertex Cover Polytope

The Vertex Cover Polytope

minimize	$\sum_{v \in V(G)} x_v$	
subject to	$x_u + x_v \geq 1$	for each $uv \in E(G)$
	$x_v \geq 0$	for each $v \in V(G)$



The Vertex Cover Polytope

minimize	$\sum_{v \in V(G)} x_v$	
subject to	$x_u + x_v \geq 1$	for each $uv \in E(G)$
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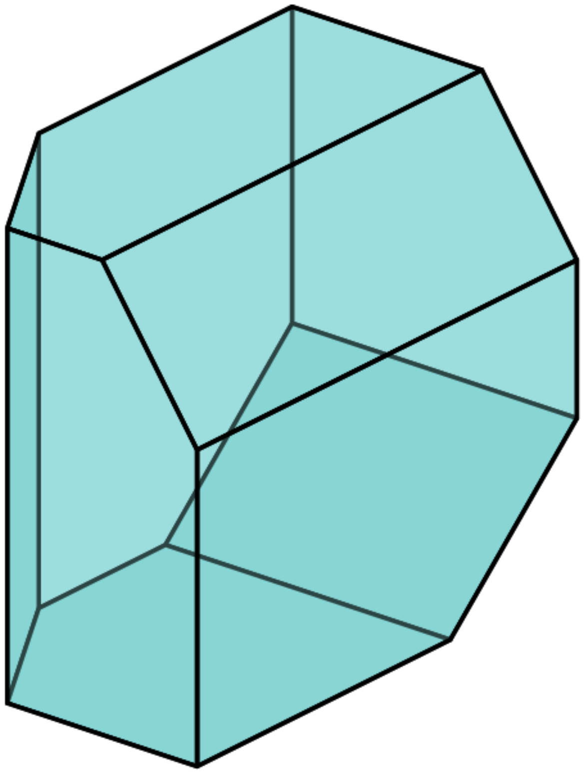
The solution space of an LP is a *convex* polytope in $\mathbb{R}^n$, where n is the number of variables.



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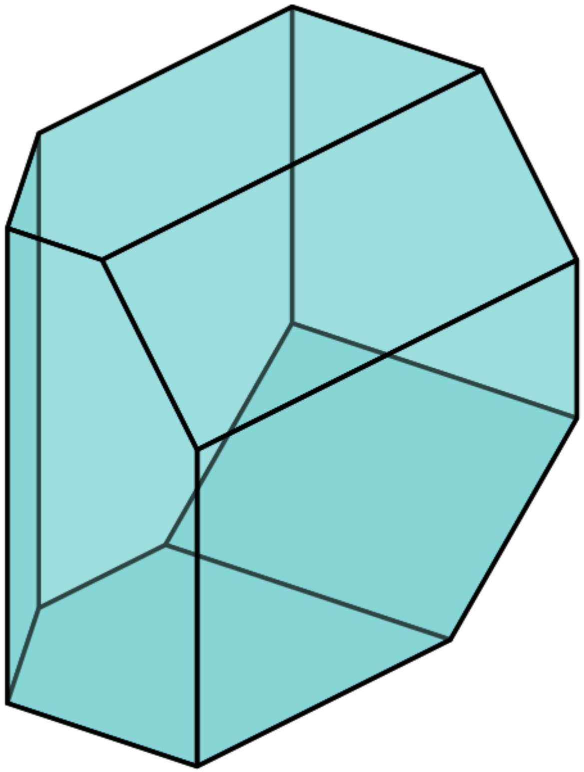
The number of facets corresponds to the number of "relevant" constraints.



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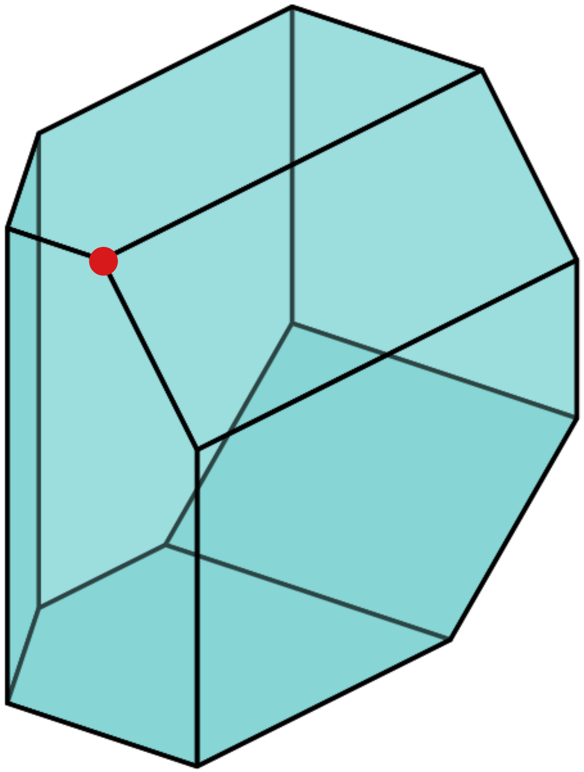
An *extreme point* of the polytope is a point that is extreme in some direction.



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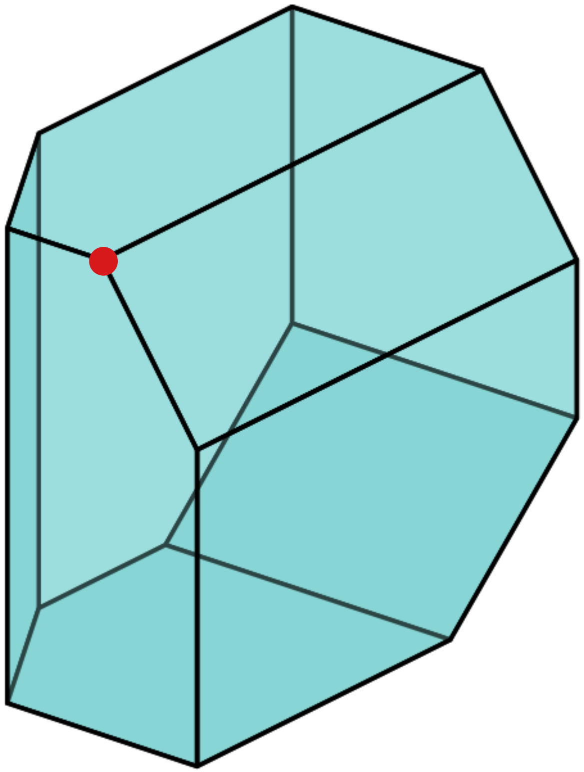
An *extreme point* of the polytope is a point that is extreme in some direction.

Most LP solvers return extreme-point solutions.



Half-Integrality

Thm. The extreme points of the vertex cover polytope are half-integral, that is, their coordinates are in $\{0, 0.5, 1\}$.

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Proof. Let x be an LP solution that is not half-integral.

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$\Rightarrow \exists v \in V(G)\colon x_v \in (0, 0.5) \cup (0.5, 1)$.

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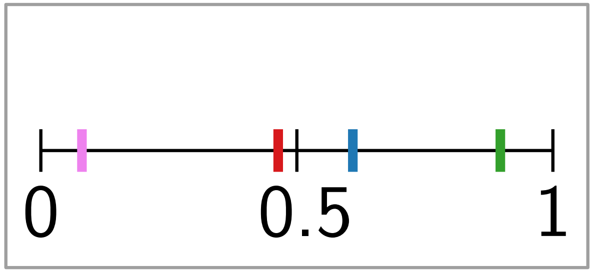

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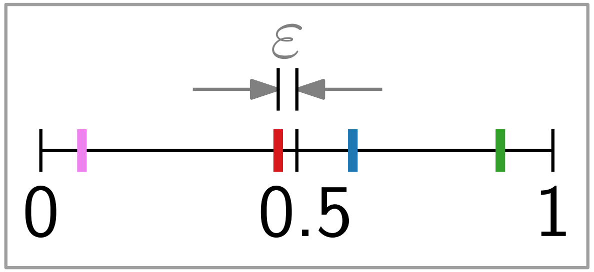

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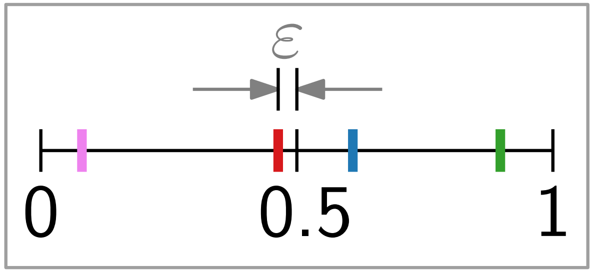

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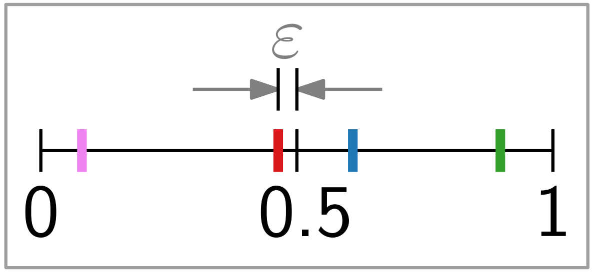

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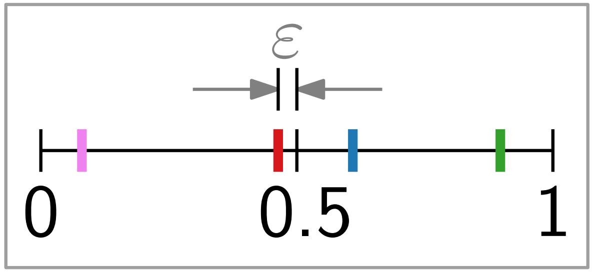

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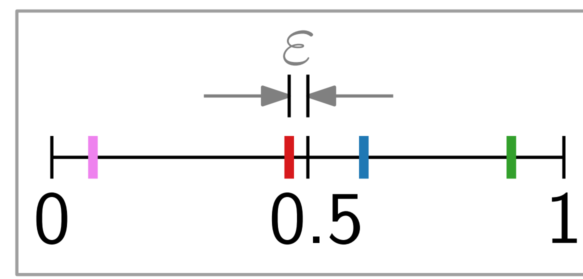

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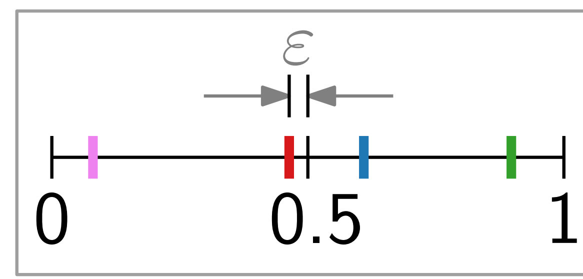

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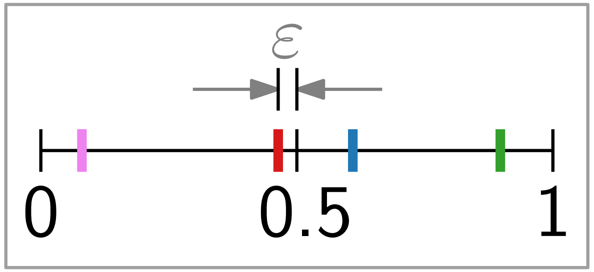

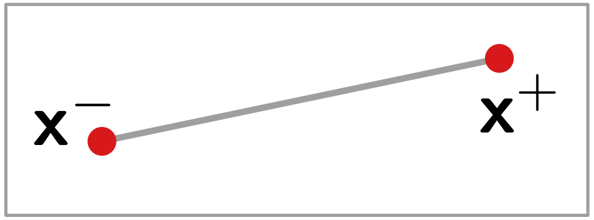

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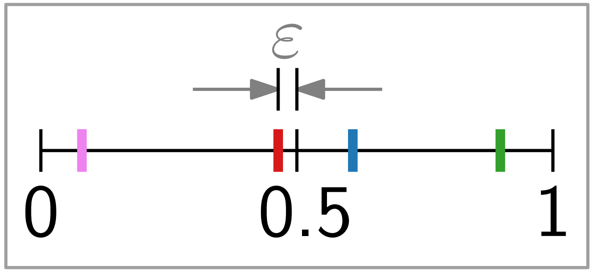

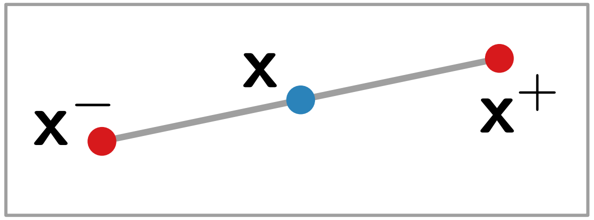

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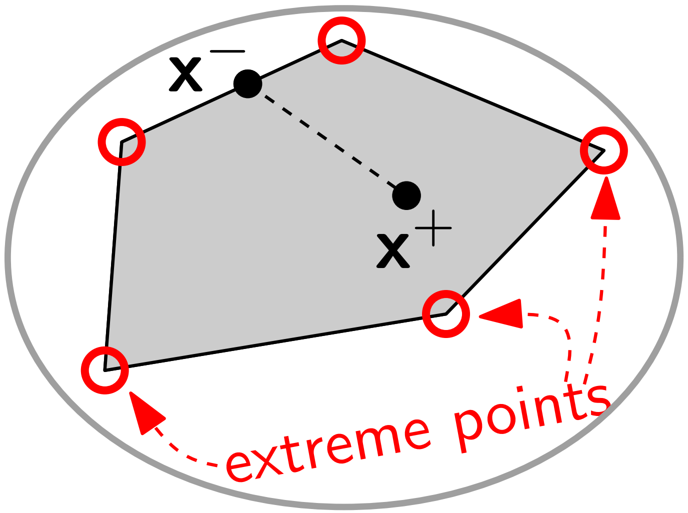

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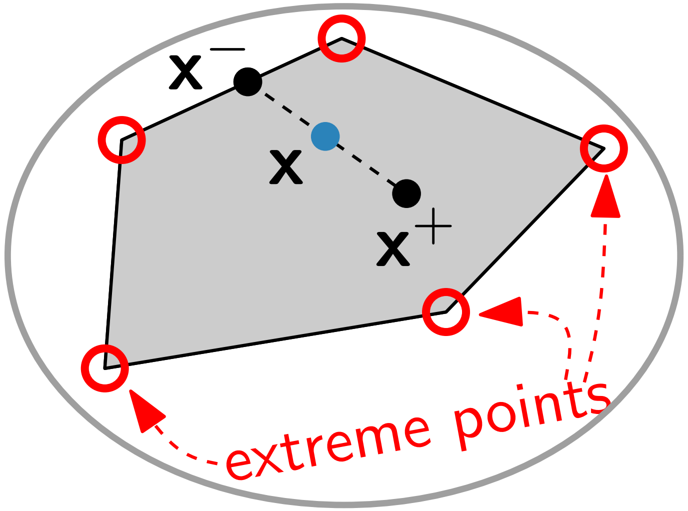

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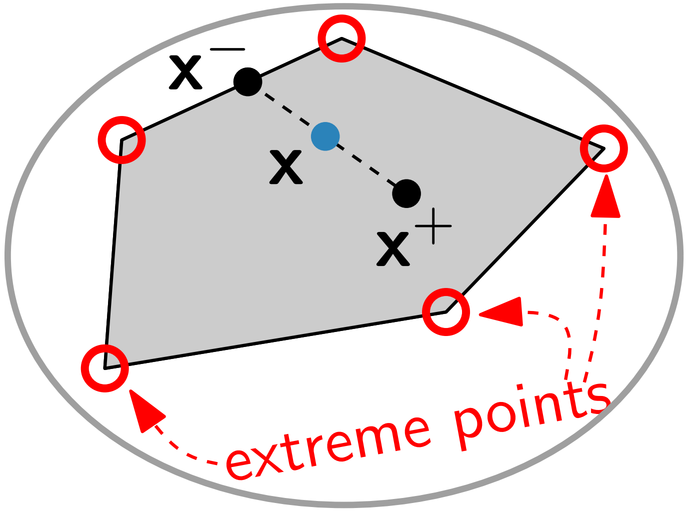

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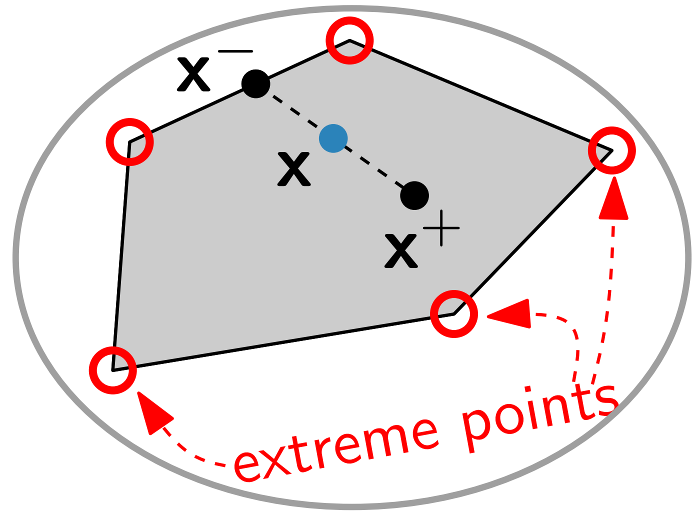

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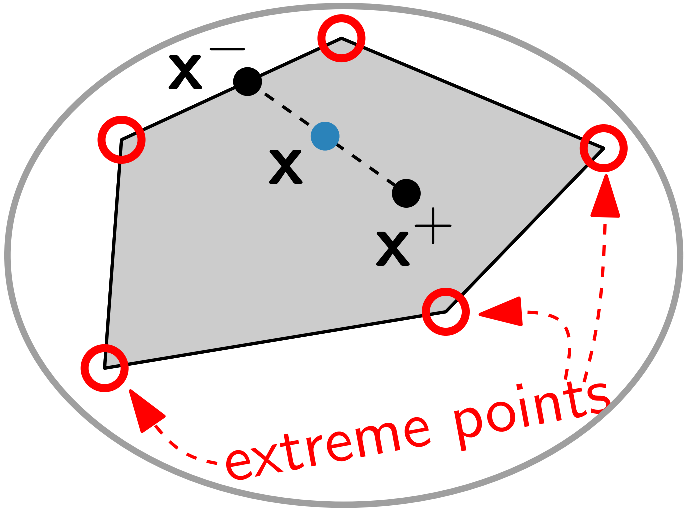

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For VC, standard LP solvers return half-integral solutions. :-)