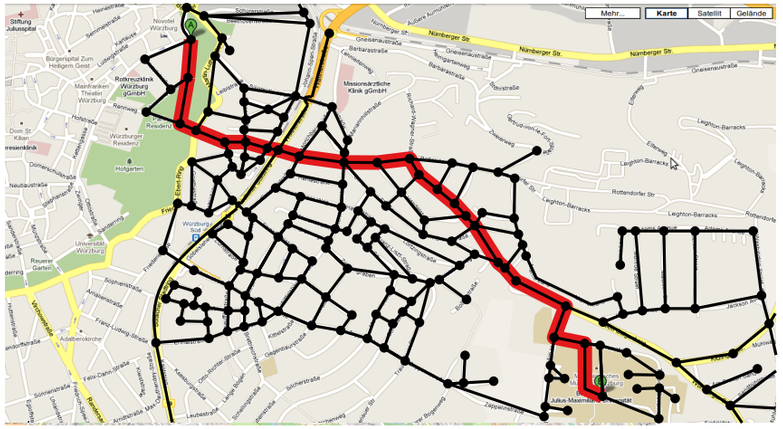
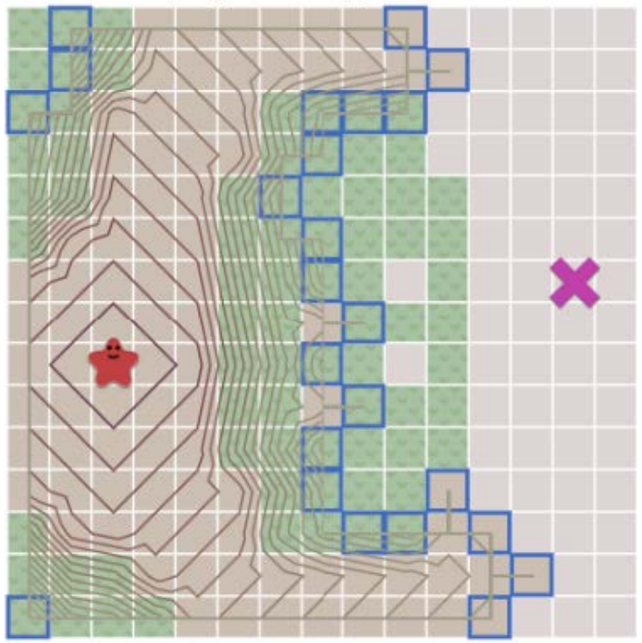
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Algorithmen und Datenstrukturen



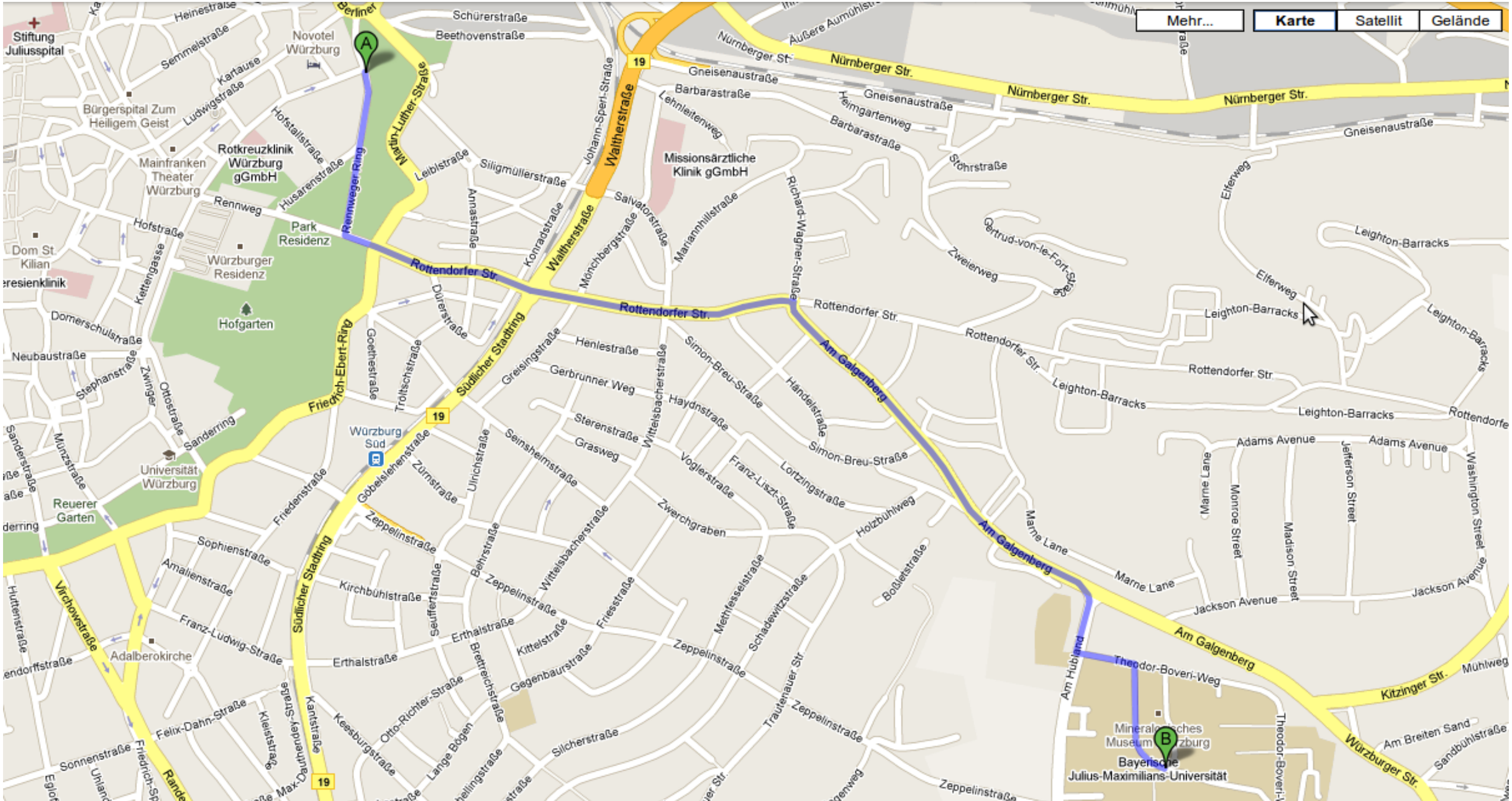
Vorlesung 19:
Kürzeste Wege und Dijkstras Algorithmus



Alexander Wolff

Wintersemester 2025

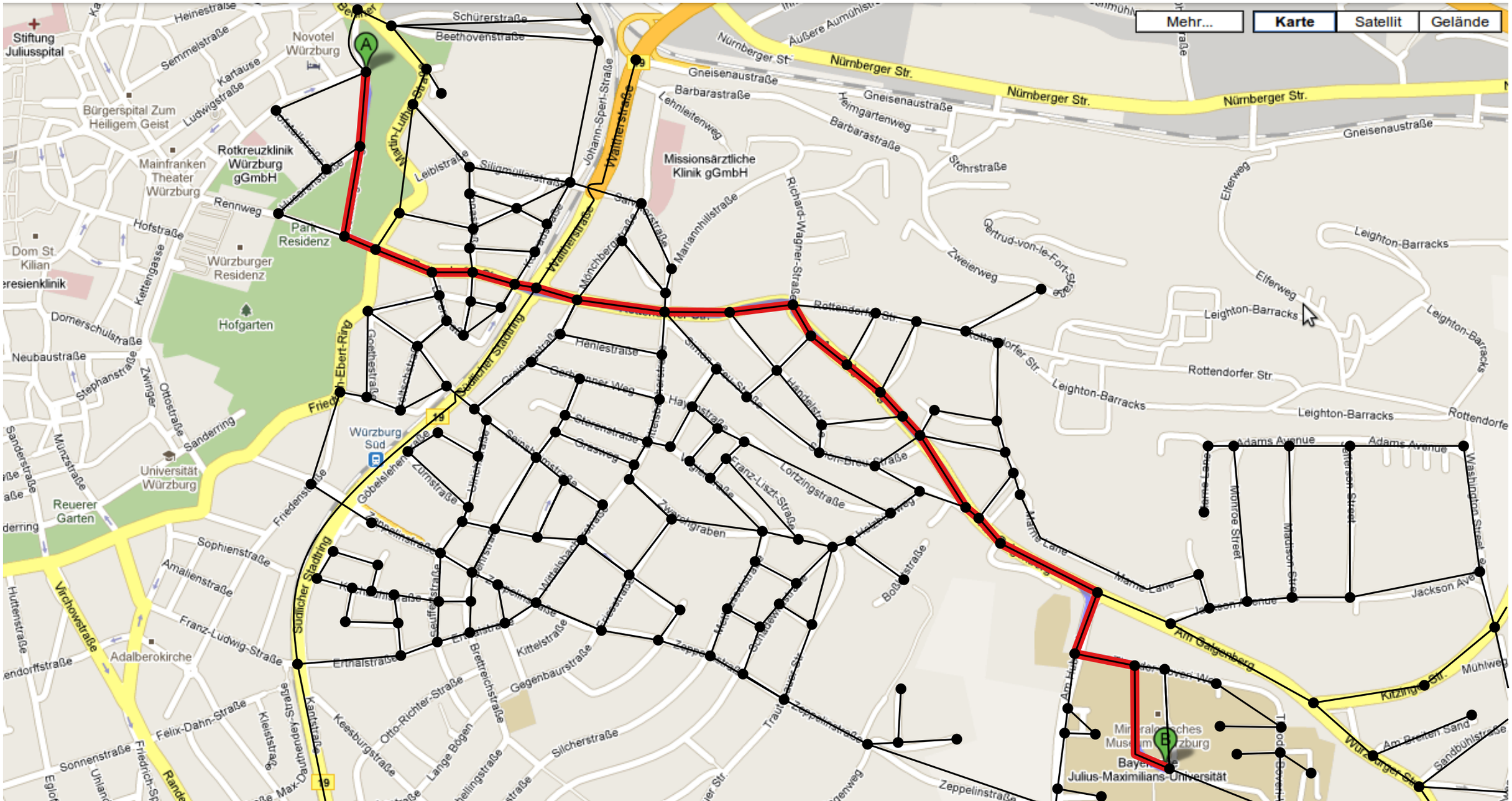
Routenplanung



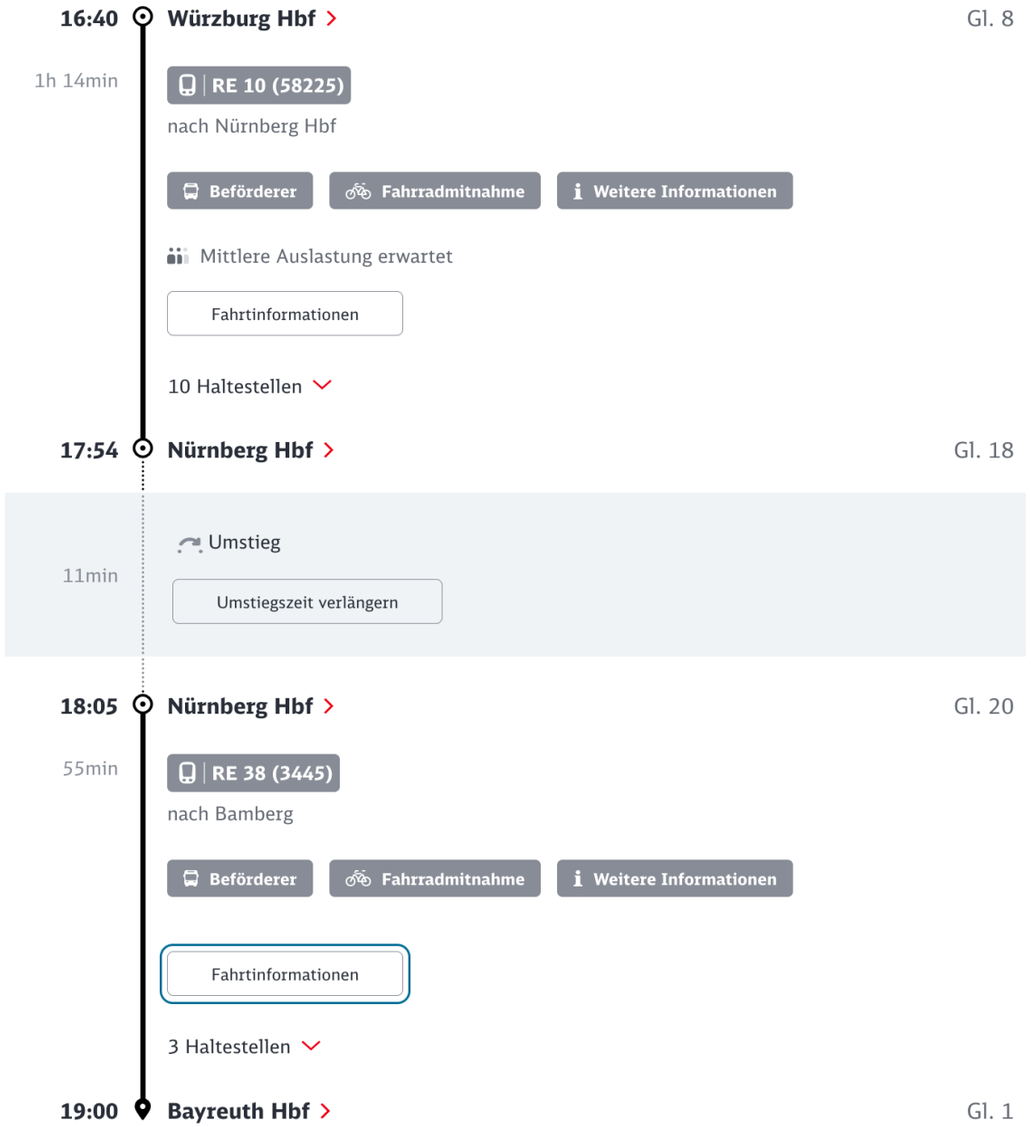
Modellierung des Problems Routenplanung

Straßenkreuzung	➡	Knoten
Straßenabschnitt	➡	zwei entgegengerichtete Kanten
Einbahnstraßenabschnitt	➡	in Fahrtrichtung gerichtete Kante
Fahrtzeit für Abschnitt e	➡	Kantengewicht $w(e) \geq 0$
Straßennetz	➡	gerichteter, gewichteter und zusammenhängender Graph G
Start	➡	Knoten $s \in V(G)$
Ziel	➡	Knoten $t \in V(G)$
Start-Ziel-Route	➡	s-t-Weg: Folge von Kanten $(s, v_1), (v_1, v_2), \ldots, (v_k, t)$ in G

Routenplanung



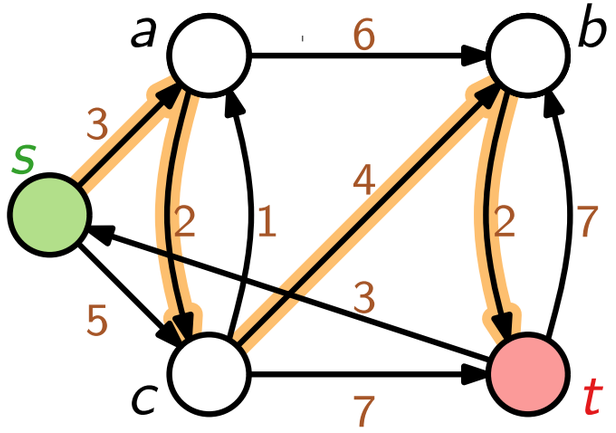
Routenplanung mit Zeitkomponente



Was ist das Problem?

Eingabe:

- gerichteter, zusammenhängender Graph G
  mit nicht-negativen **Kantengewichten** $w\colon E(G) \to \mathbb{Q}_0^+$,
- Knoten s und t



Ausgabe:

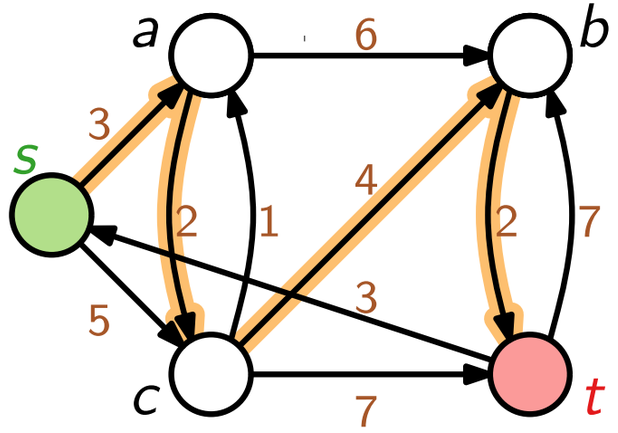
- **kürzester s-t-Weg** W in G, d.h. $\sum_{e \in W} w(e)$ minimal.

  Darstellung durch Vorgänger-Zeiger π:
  für jeden Knoten v sei $\pi(v) \in V(G) \cup \{nil\}$ Vorgänger
  von v auf kürzestem s-v-Weg.

Was ist das Problem?

Eingabe:

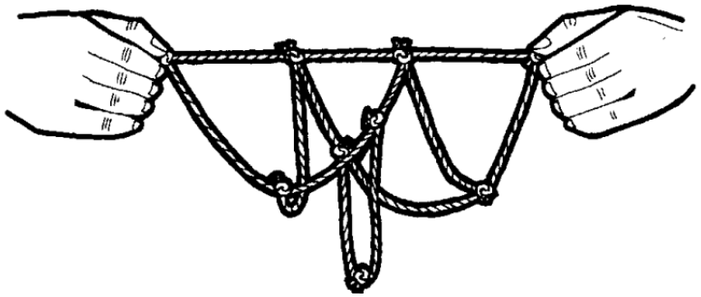
- gerichteter, zusammenhängender Graph G
  mit nicht-negativen **Kantengewichten** $w\colon E(G) \to \mathbb{Q}_0^+$,
- Knoten s ~~und t~~



Ausgabe:

- **kürzester** s-t-**Wege** W_t (für jedes $t \in V(G)$) in G, d.h. $\sum_{e \in W} w(e)$ minimal.

  Darstellung durch Vorgänger-Zeiger π:
  für jeden Knoten v sei $\pi(v) \in V(G) \cup \{nil\}$ Vorgänger von v auf kürzestem s-v-Weg.

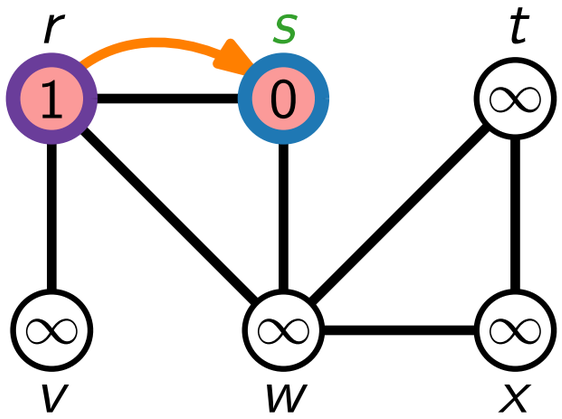


Nebenbemerkung: **Analoge** Berechnungsverfahren?
(nur für ungerichtete Graphen!)

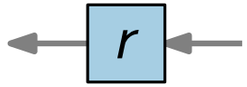
Abbildung aus:
Solving the Shortest Path Problem Using an Analog Network
Linkai Bu & Tzi-Dar Chiueh

Wiederholung Breitensuche

```
BFS(Graph G, Vertex s)
    Initialize(G, s)
    Q = new Queue()
    Q.Enqueue(s)
    while not Q.Empty() do
        u = Q.Dequeue()
        foreach v ∈ Adj[u] do
            if v.color == white then
                v.color = red
                v.d = u.d + 1
                v.π = u
                Q.Enqueue(v)
        u.color = blue
```



Q:

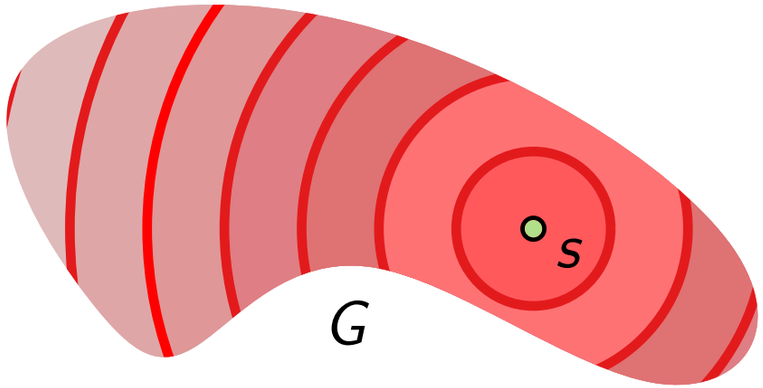


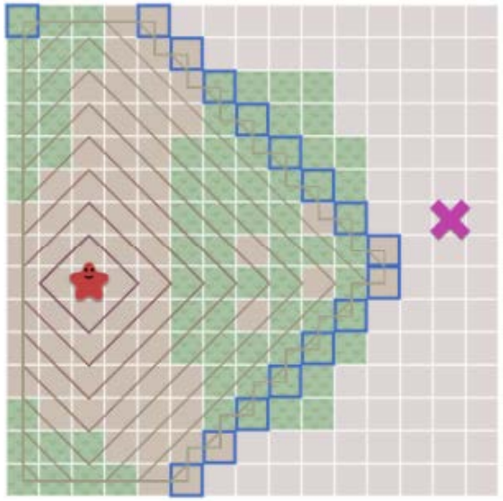

```
Initialize(Graph G, Vertex s)
    foreach u ∈ V(G) do
        u.color = white
        u.d = ∞
        u.π = nil
    s.color = red
    s.d = 0
```

Ausbreitung

Breitensuche (breadth-first search, BFS)

Alle Kanten gleich schwer

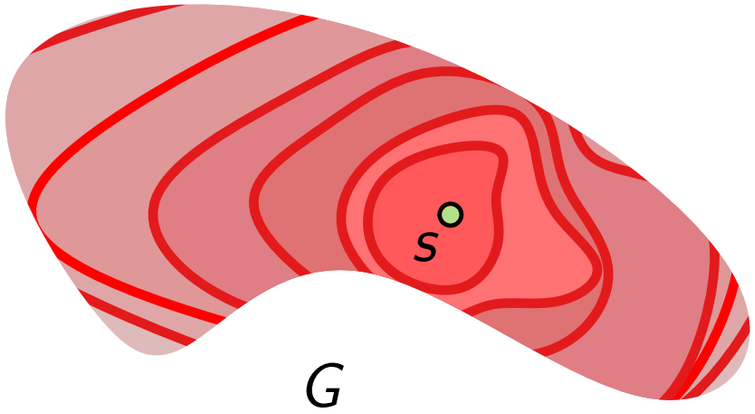


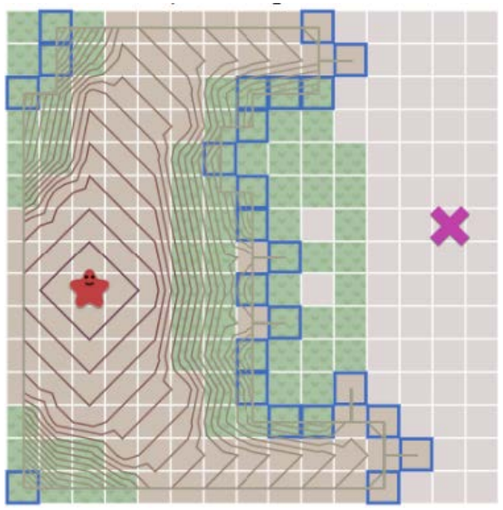


Amit Patel, "Introduction to the A^* Algorithm", Red Blob Games, 2014, https://www.redblobgames.com/pathfinding/a-star/introduction.html

Dijkstra

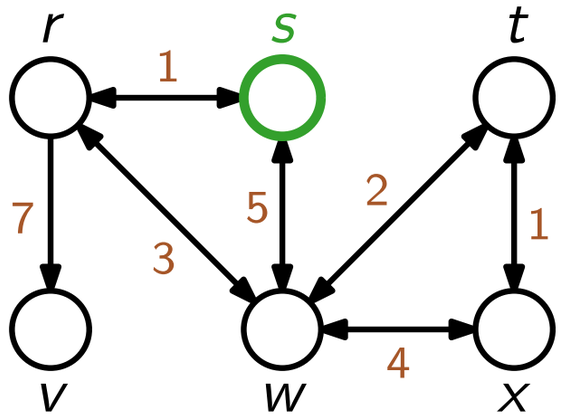
Kanten unterschiedlich schwer





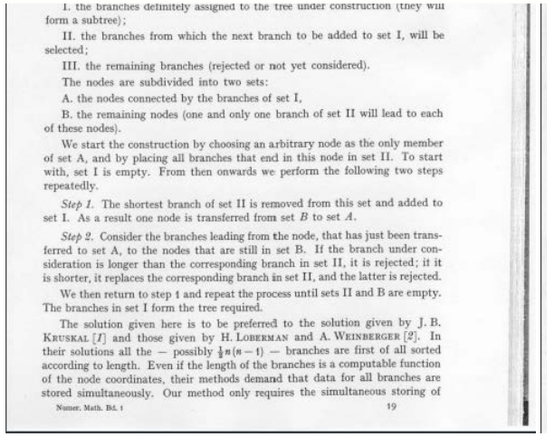
Dijkstra

Edsger W. Dijkstra
(Rotterdam 1930–2002 Nuenen)



A note on two problems in connexion with graphs.
In: Numerische Mathematik, Band 1, 1959

The solution given above is to be preferred to the solution by L. R. Ford [3] as described by C. Berge [4], for, irrespective of the number of branches, we need not store the data for all branches simultaneously but only those for the branches in sets I and II, and this number is always less than n. Furthermore, the amount of work to be done seems to be considerably less.



I. the branches definitely assigned to the tree under construction (they will form a subtree);

II. the branches from which the next branch to be added to set I, will be selected;

III. the remaining branches (rejected or not yet considered).

The nodes are subdivided into two sets:

A. the nodes connected by the branches of set I,

B. the remaining nodes (one and only one branch of set II will lead to each of these nodes).

We start the construction by choosing an arbitrary node as the only member of set A, and by placing all branches that end in this node in set II. To start with, set I is empty. From then onwards we perform the following two steps repeatedly.

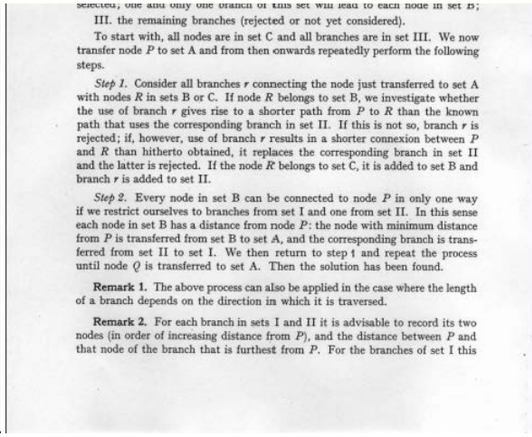
Step 1. The shortest branch of set II is removed from this set and added to set I. As a result one node is transferred from set B to set A.

Step 2. Consider the branches leading from the node, that has just been transferred to set A, to the nodes that are still in set B. If the branch under consideration is longer than the corresponding branch in set II, it is rejected; if it is shorter, it replaces the corresponding branch in set II, and the latter is rejected.

We then return to step 1 and repeat the process until sets II and B are empty. The branches in set I form the tree required.

The solution given here is to be preferred to the solution given by J. B. Kruskal [1] and those given by H. Loberman and A. Weinberger [2]. In their solutions all the — possibly $\frac{1}{2}n(n-1)$ — branches are first of all sorted according to length. Even if the length of the branches is a computable function of the node coordinates, their methods demand that data for all branches are stored simultaneously. Our method only requires the simultaneous storing of

Numer. Math. Bd. 1 19



selected; one and only one branch of this set will lead to each node in set B;

III. the remaining branches (rejected or not yet considered).

To start with, all nodes are in set C and all branches are in set III. We now transfer node P to set A and from then onwards repeatedly perform the following steps.

Step 1. Consider all branches r connecting the node just transferred to set A with nodes R in sets B or C. If node R belongs to set B, we investigate whether the use of branch r gives rise to a shorter path from P to R than the known path that uses the corresponding branch in set II. If this is not so, branch r is rejected; if, however, use of branch r results in a shorter connexion between P and R than hitherto obtained, it replaces the corresponding branch in set II and the latter is rejected. If the node R belongs to set C, it is added to set B and branch r is added to set II.

Step 2. Every node in set B can be connected to node P in only one way if we restrict ourselves to branches from set I and one from set II. In this sense each node in set B has a distance from node P: the node with minimum distance from P is transferred from set B to set A, and the corresponding branch is transferred from set II to set I. We then return to step 1 and repeat the process until node Q is transferred to set A. Then the solution has been found.

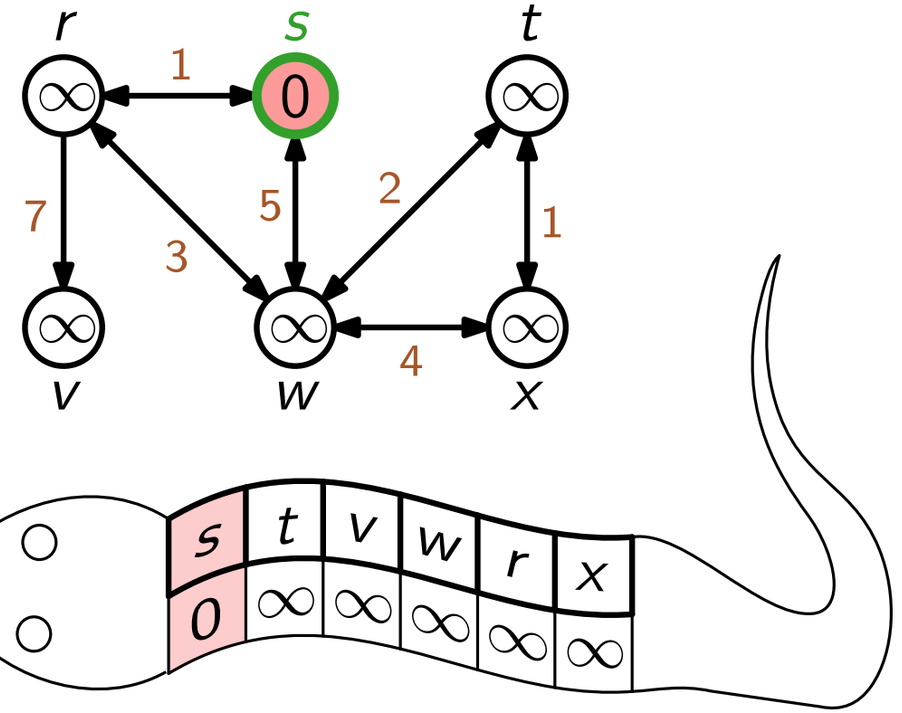
Remark 1. The above process can also be applied in the case where the length of a branch depends on the direction in which it is traversed.

Remark 2. For each branch in sets I and II it is advisable to record its two nodes (in order of increasing distance from P), and the distance between P and that node of the branch that is furthest from P. For the branches of set I this

(Received June 11, 1959)

19*

Dijkstra




```
Dijkstra(Graph G, Edge weights w, Vertex s)
    Initialize(G, s)
    Q = new PriorityQueue(V(G), d)

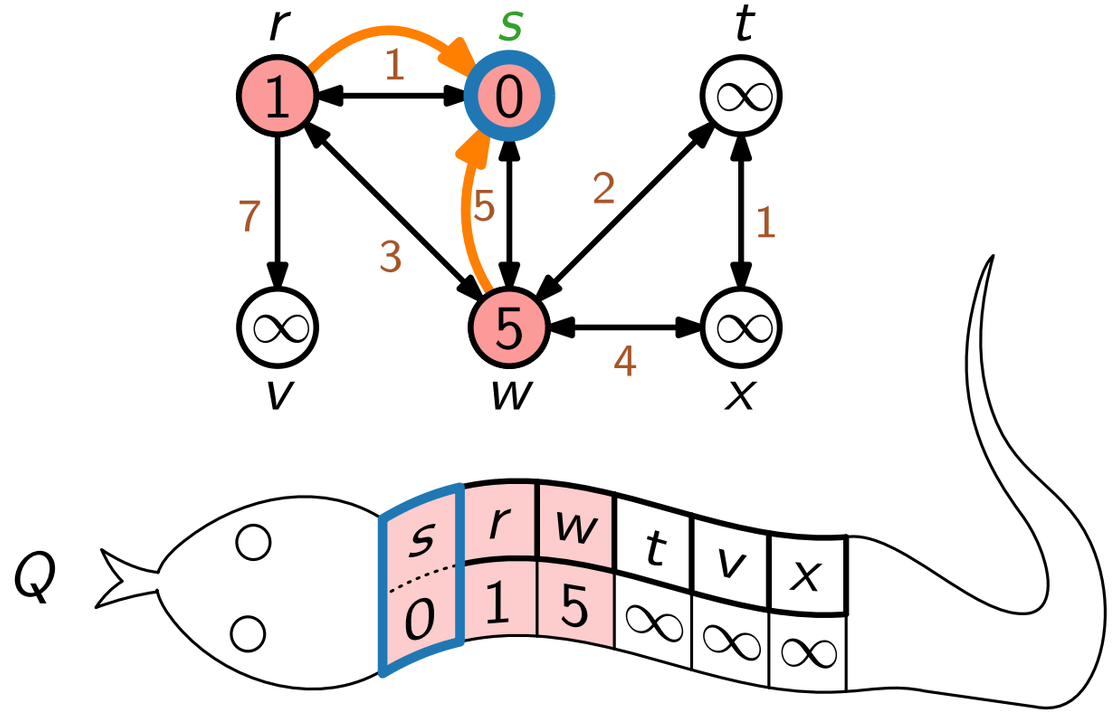
    while not Q.Empty() do
        u = Q.Dequeue()
        foreach v ∈ Adj[u] do
            if v.color == white then
                v.color = red
                v.d = u.d + 1
                v.π = u
                Q.Enqueue(v)
        u.color = blue
```

```
Initialize(Graph G, Vertex s)
    foreach u ∈ V do
        u.color = white
        u.d = ∞
        u.π = nil
    s.color = red
    s.d = 0
```

Dijkstra

```
Dijkstra(Graph G, Edge weights w, Vertex s)
    Initialize(G, s)
    Q = new PriorityQueue(V(G), d)

    while not Q.Empty() do
        u = Q.ExtractMin()
        foreach v ∈ Adj[u] do
            if v.color == white then
                v.color = red
                v.d = u.d + w(u, v)
                v.π = u
                Q.DecreaseKey(v, v.d)
        u.color = blue
```

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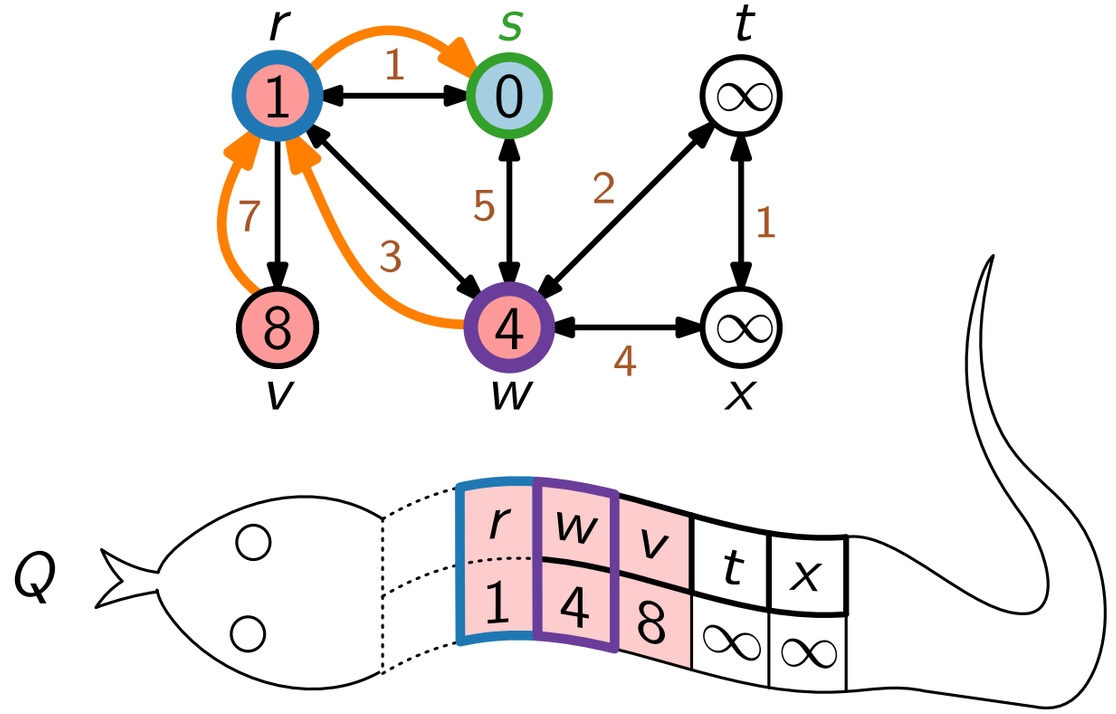


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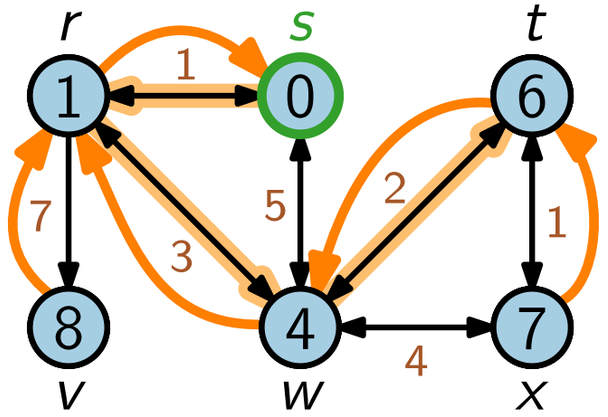
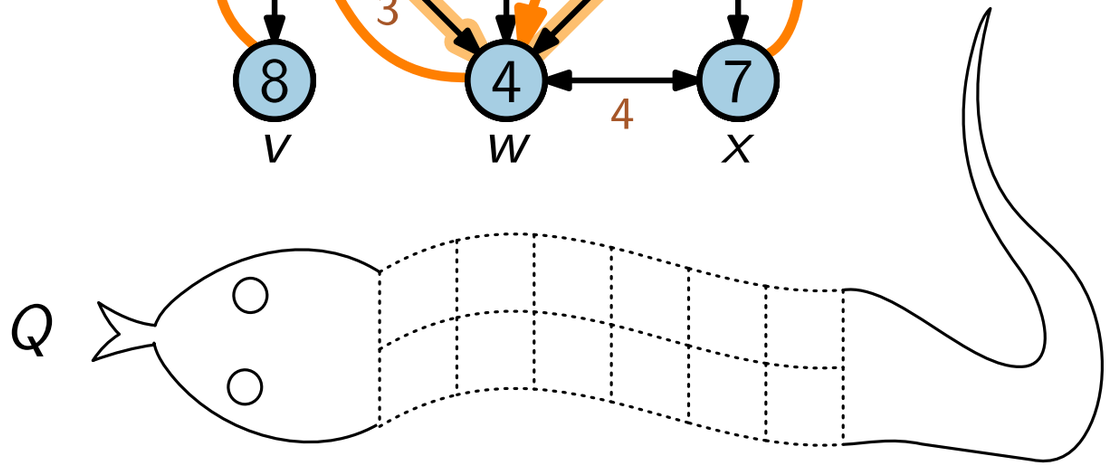
Dijkstra



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    s.color = red
    s.d = 0
```

Dijkstra – die Laufzeit

```
Dijkstra(WeightedGraph G, Vertex s)
    Initialize(G, s)
    Q = new PriorityQueue(V(G), d)
    while not Q.Empty() do
        u = Q.ExtractMin()
        foreach v ∈ Adj[u] do
            if v.d > u.d + w(u, v) then
                v.color = red
                v.d = u.d + w(u, v)
                v.π = u
                Q.DecreaseKey(v, v.d)
        u.color = blue
```

Abk. für $O(|V(G)|)$

$\mathcal{O}(V)$ Zeit

genau $|V(G)|$ mal

Wie oft wird der Schleifeninhalt aufgerufen?

Für jeden Knoten u von G genau $|\text{Adj}[u]| = \deg^{(\text{out-})}(u)$ mal,

also insg. $\Theta(E)$ mal.

Also wird DecreaseKey $\mathcal{O}(E)$ mal aufgerufen.

Dijkstra – die Laufzeit

Satz. Gegeben ein Graph $G=(V,E)$, läuft Dijkstras Alg. in $\Theta(|V|\cdot T_{\text{ExtractMin}}(|V|) + |E|\cdot T_{\text{DecreaseKey}}(|V|))$ Zeit.

Implementierung einer PriorityQueue	$T_{\text{ExtractMin}}(n)$	$T_{\text{DecreaseKey}}(n)$	$T_{\text{Dijkstra}}(\lvert V\rvert,\lvert E\rvert)$
als unsortiertes Feld	$\mathcal{O}(n)$	$\mathcal{O}(1)^{\star}$	$\mathcal{O}(V^2+E)$
als Heap	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)^{\star\star}$	$\mathcal{O}((E+V)\log V)$
als FibonacciHeap	$\mathcal{O}(\log n)$ amortisiert	$\mathcal{O}(1)$ amortisiert	$\mathcal{O}(E+V\log V)$ im Worst-Case!

*) Das geht, weil wir bei ExtractMin Lücken im Feld lassen; daher bleiben die Schlüssel an ihrem Platz ($\rightarrow$ Direktzugriff)
**) Das geht, obwohl wir im Heap nicht suchen können (!). Wir merken uns ständig für jeden Knoten, wo er im Heap steht.

Korollar. In einem Graphen $G=(V,E;w)$ mit $w\colon E\to\mathbb{Q}_{\geq 0}$ kann man in $\mathcal{O}(E+V\log V)$ Zeit die kürzesten Wege von einem zu allen Knoten berechnen (SSSP-Problem).

Dijkstra – die Korrektheit

Definition. Für einen (un)gerichteten Graphen G und $u, v \in V(G)$ sei
$\delta(u, v) :=$ Länge eines kürzesten u-v-Wegs
(falls v von u erreichbar; sonst $\delta(u, v) := \infty$).

Annahme: Es gibt eine Nummerierung $v_1, \ldots, v_n$ der Knotenmenge von G, so dass
$\delta(s, v_1) < \delta(s, v_2) < \ldots < \delta(s, v_n)$

0. Schleifeninvariante

Zu Beginn des k-ten Durchlaufs der **while**-Schleife

I) gilt $v_i.d = \delta(s, v_i)$ für alle $1 \leq i \leq k$

II) enthält Q genau die Knoten $v_k, \ldots, v_n$

III) gilt $v_k = Q.\textsc{FindMin}()$

1. Initialisierung ($k = 1$) ✓

Offensichtlich ist $v_1 = s$ und $\delta(s, s) = 0$.

In $\textsc{Initialize}(G, s)$ wird $s.d = 0$ gesetzt $\Rightarrow$ **I)**

Alle Knoten sind in Q $\Rightarrow$ **II)**

Für jeden Knoten $v \neq s$ wird $v.d = \infty$ gesetzt $\Rightarrow$ **III)**

```
Dijkstra(WeightedGraph G, Vertex s)
  Initialize(G, s)
  Q = new PriorityQueue(V, d)
  while not Q.Empty() do
      u = Q.ExtractMin()
      foreach v ∈ Adj[u] do
          if v.d > u.d + w(u, v) then
              v.d = u.d + w(u, v)
              v.π = u
              Q.DecreaseKey(v, v.d)
```

Dijkstra – die Korrektheit

0. Schleifeninvariante

Zu Beginn des k-ten Durchlaufs der **while**-Schleife

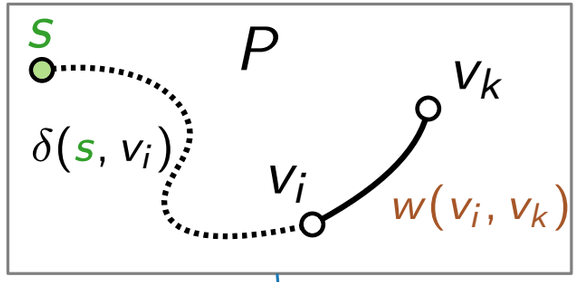
I) gilt $v_i.d = \delta(s, v_i)$ für alle $1 \leq i \leq k$ ✓

II) enthält Q genau die Knoten $v_k, \ldots, v_n$

III) gilt $v_k = Q.\text{FindMin}()$

1. Initialisierung ✓

2. Aufrechterhaltung



Betrachte kürzesten s-v_k-Pfad P.

Sei v_i der Vorgänger von v_k auf P.

Dann ist $i < k$. **I)** $\Rightarrow$ $v_i.d = \delta(s, v_i)$

Betrachte i-ten Schleifendurchlauf mit $u = v_i$, $v = v_k$

$$\Rightarrow v_k.d = v_i.d + w(v_i, v_k) = \delta(s, v_i) + w(v_i, v_k) = \delta(s, v_k) \Rightarrow \textbf{I)}$$

```
Dijkstra(WeightedGraph G, Vertex s)
  Initialize(G, s)
  Q = new PriorityQueue(V, d)
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      u = Q.ExtractMin()
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```

Dijkstra – die Korrektheit

0. Schleifeninvariante

Zu Beginn des k-ten Durchlaufs der **while**-Schleife

I) gilt $v_i.d = \delta(s, v_i)$ für alle $1 \leq i \leq k$ ✓

II) enthält Q genau die Knoten $v_k, \ldots, v_n$ ✓

III) gilt $v_k = Q.\text{FindMin}()$

1. Initialisierung ✓

2. Aufrechterhaltung

Betrachte $(k-1)$-ten Schleifendurchlauf

II) $\Rightarrow$ Q enthielt genau die Knoten $v_{k-1}, \ldots, v_n$

III) $\Rightarrow$ es wurde genau v_{k-1} aus Q entfernt $\Rightarrow$ **II)**

```
Dijkstra(WeightedGraph G, Vertex s)
  Initialize(G, s)
  Q = new PriorityQueue(V, d)
  while not Q.Empty() do
      u = Q.ExtractMin()
      foreach v ∈ Adj[u] do
          if v.d > u.d + w(u, v) then
              v.d = u.d + w(u, v)
              v.π = u
              Q.DecreaseKey(v, v.d)
```

Dijkstra – die Korrektheit

0. Schleifeninvariante

Zu Beginn des k-ten Durchlaufs der **while**-Schleife

- **I)** gilt $v_i.d = \delta(s, v_i)$ für alle $1 \leq i \leq k$ ✓
- **II)** enthält Q genau die Knoten $v_k, \ldots, v_n$ ✓
- **III)** gilt $v_k = Q.\text{FindMin}()$ ✓

1. Initialisierung ✓

2. Aufrechterhaltung ✓

Betrachte Knoten v_ℓ mit $\ell > k$.

Dann gilt $v_\ell.d \geq \delta(s, v_\ell) > \delta(s, v_k) = v_k.d$ (I)

$\delta(s, v_1) < \delta(s, v_2) < \ldots < \delta(s, v_n)$

```
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III) gilt $v_k = Q.\text{FindMin}()$

1. Initialisierung ✓

2. Aufrechterhaltung ✓

3. Terminierung ✓

I) $\Rightarrow$ $v_i.d = \delta(s, v_i)$ für alle $1 \leq i \leq n$

```
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Kürzeste Wege nach Dijkstra

Eingabe	Algorithmus	Laufzeit	
ungewichteter Graph	Breitensuche	$\mathcal{O}(E + V)$	Vorlesung 18
nicht-neg. Kantengew.	Dijkstra	$\mathcal{O}(E + V \log V)$	heute
azyklischer Graph	Topol. Sortieren	$\mathcal{O}(E + V)$	Vorlesung 20
negative Kantengew.	Bellman-Ford	$\mathcal{O}(EV)$	
für alle Knotenpaare	$\lvert V\rvert \times$ Dijkstra	$\mathcal{O}(V(E + V \log V))$	heute
+ negative Kantengew.	Floyd-Warshall	$\mathcal{O}(V^3)$	
	Johnson	$\mathcal{O}(V(E + V \log V))$	
k kürzeste s-t-Wege	Eppstein	$\mathcal{O}(k + E + V \log V)$	