Approximation Algorithms

Lecture 12:
SteinerForest via Primal–Dual

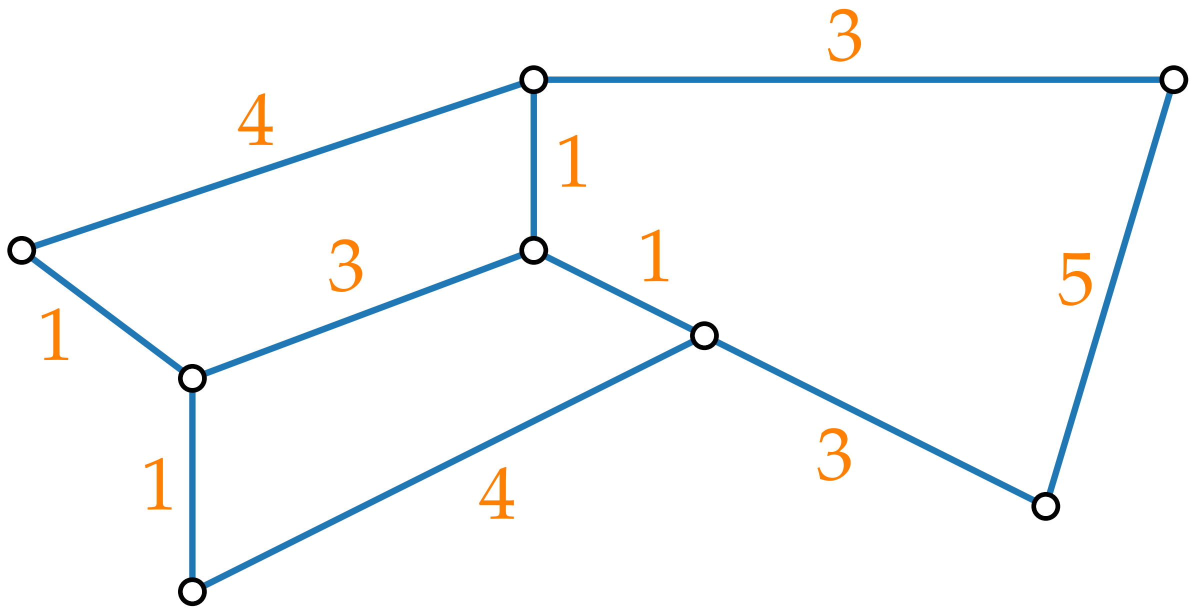
Part I:
SteinerForest

Alexander Wolff

Winter 2023/24

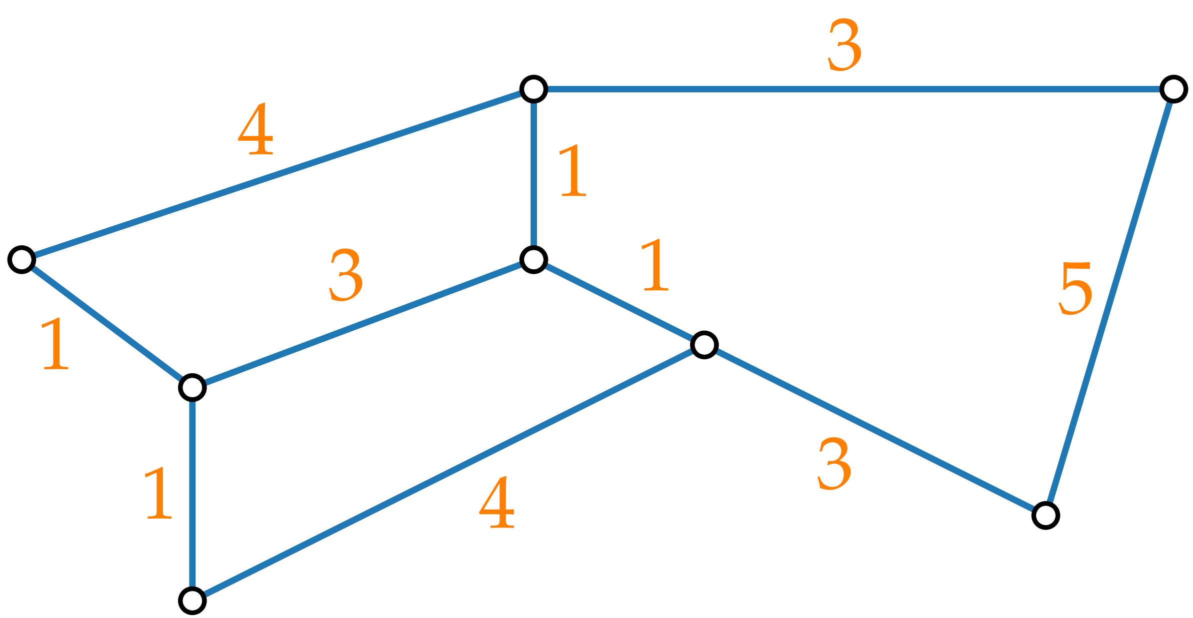
SteinerForest

Given: A graph $G = (V, E)$ with edge costs $c\colon E \to \mathbb{N}$ and



SteinerForest

Given: A graph $G = (V, E)$ with edge costs $c\colon E \to \mathbb{N}$ and a set $R = \{(s_1, t_1), \ldots, (s_k, t_k)\}$ of k vertex pairs.



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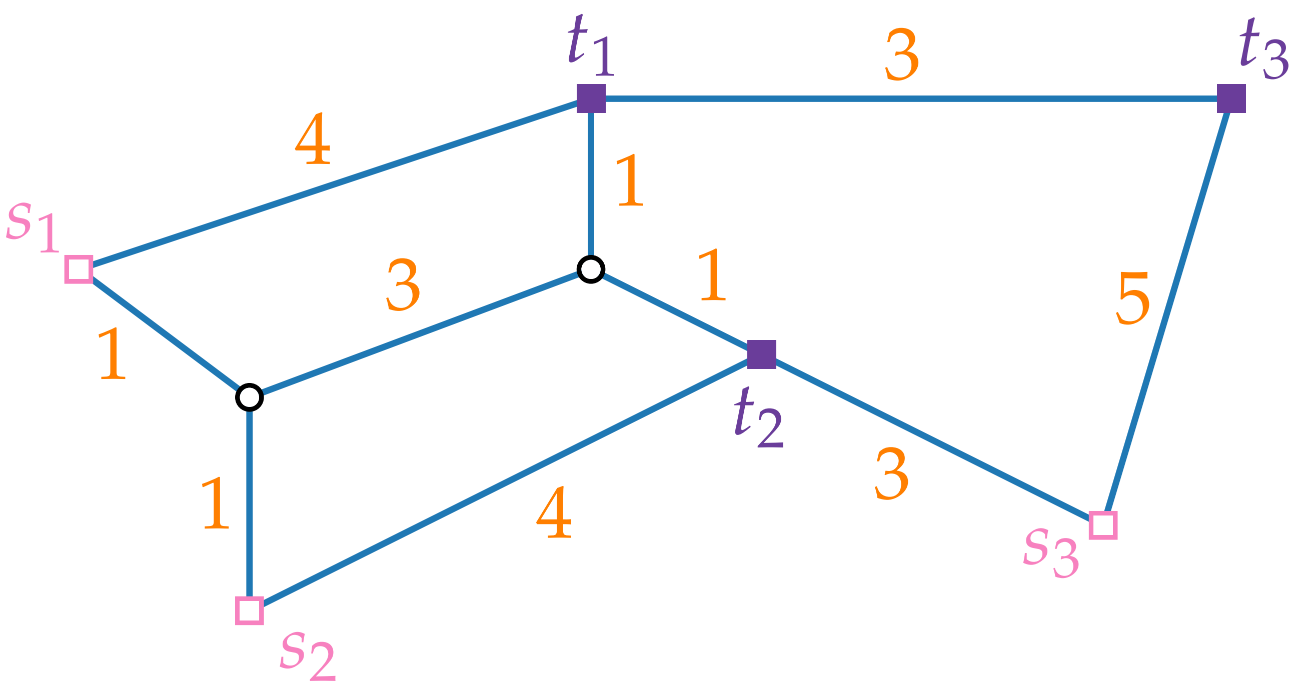

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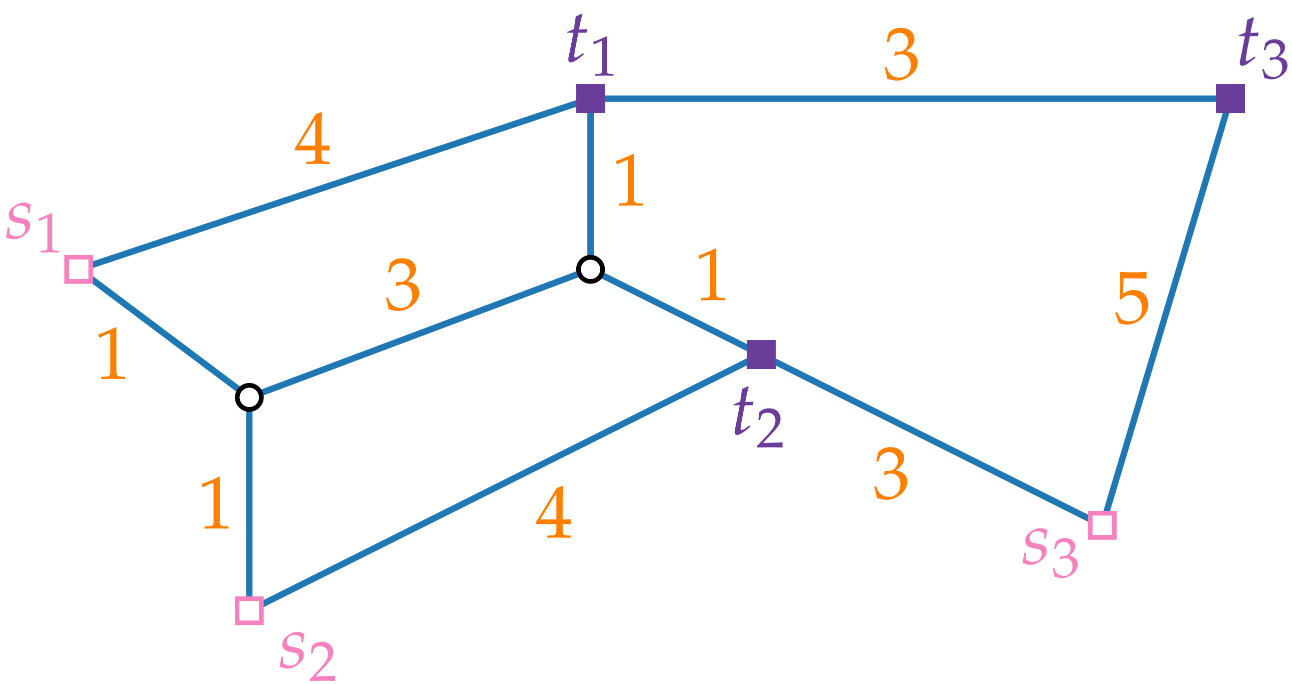
Task: Find an edge set $F \subseteq E$ of minimum total cost $c(F)$



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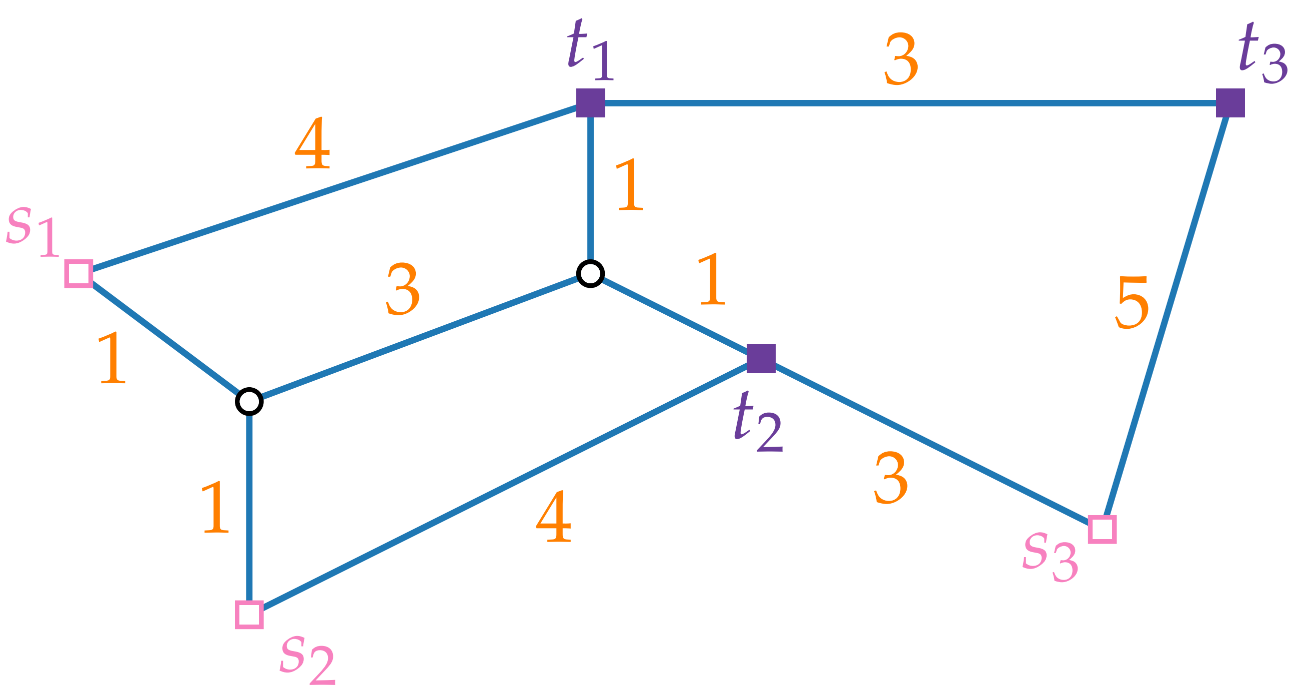

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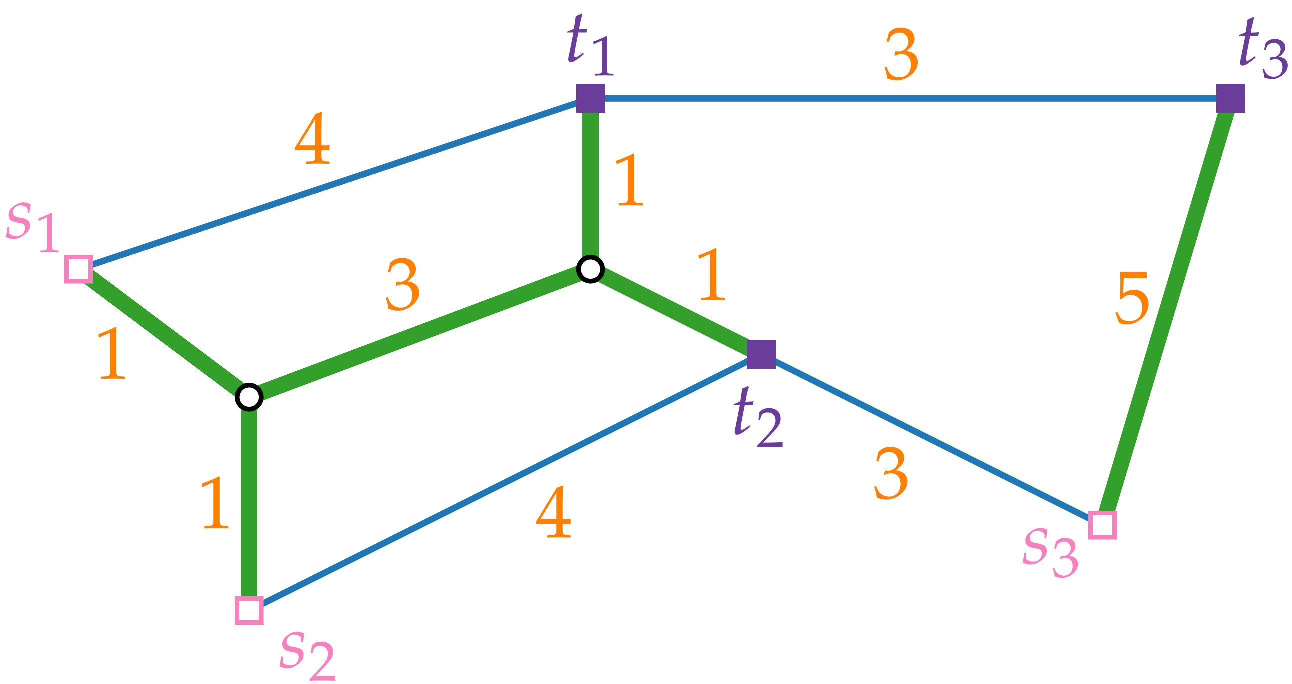

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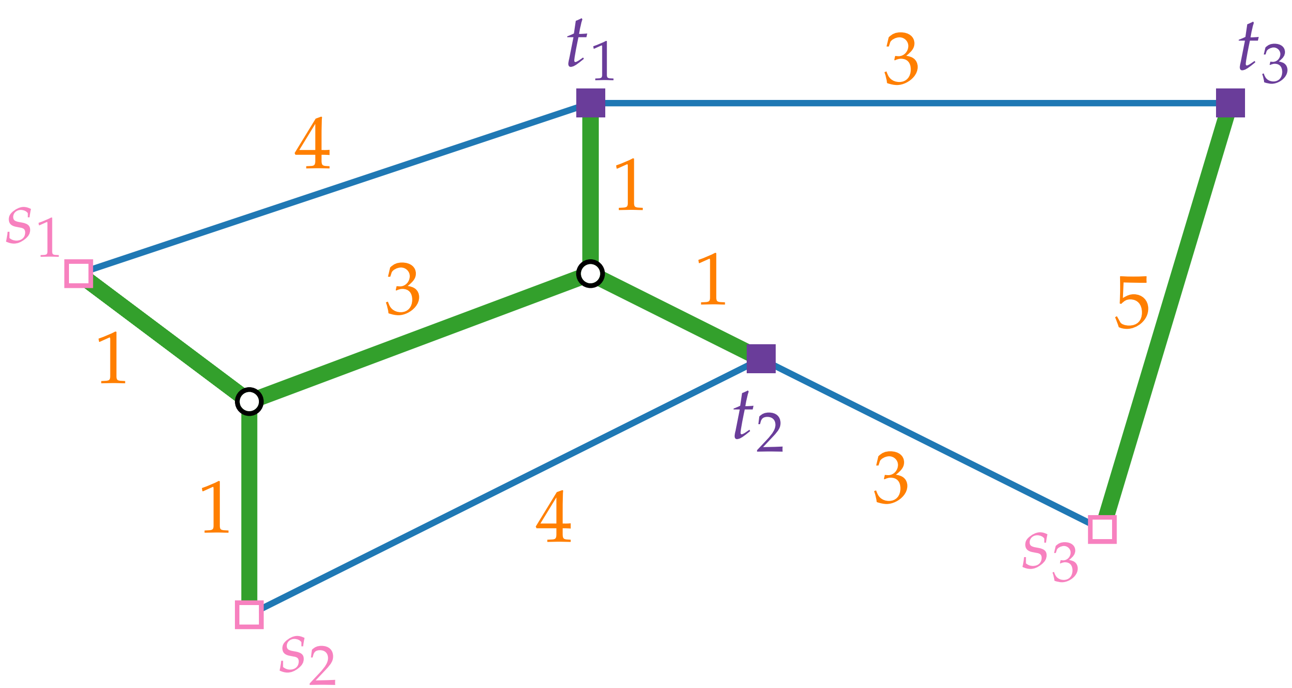

Special cases?

SteinerForest

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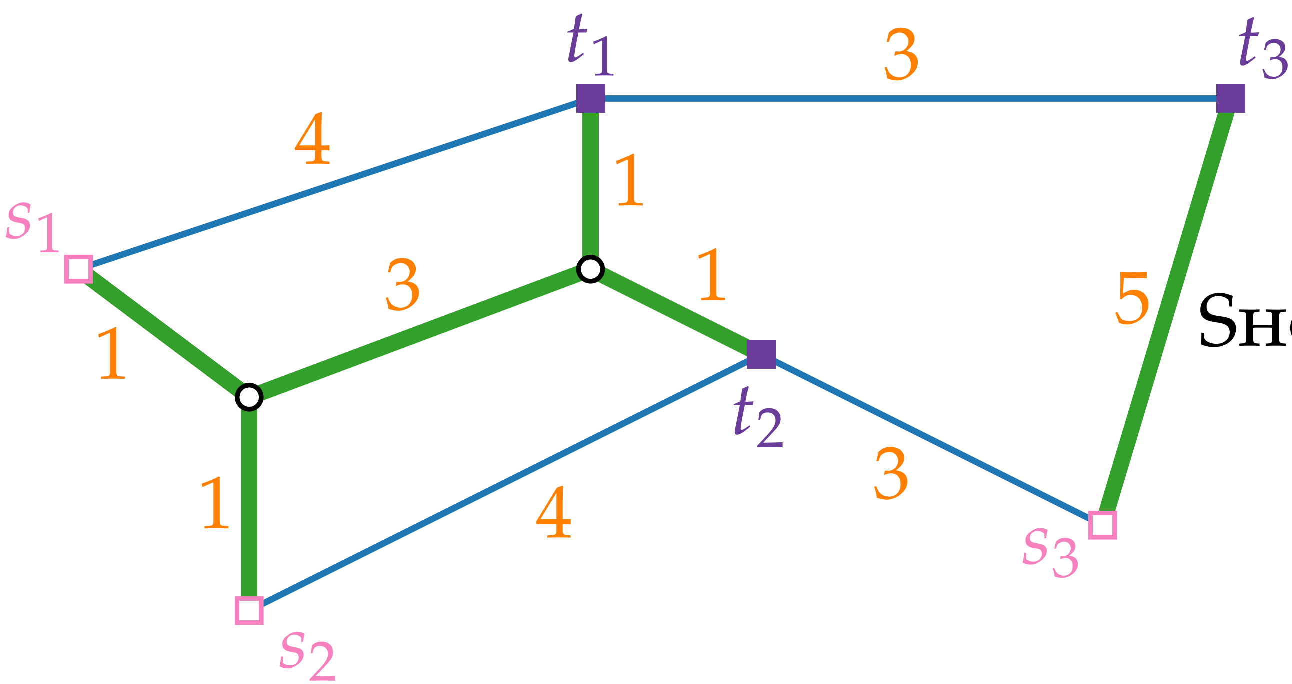

Special cases?

ShortestPath $(R = \{s, t\})$

SteinerForest

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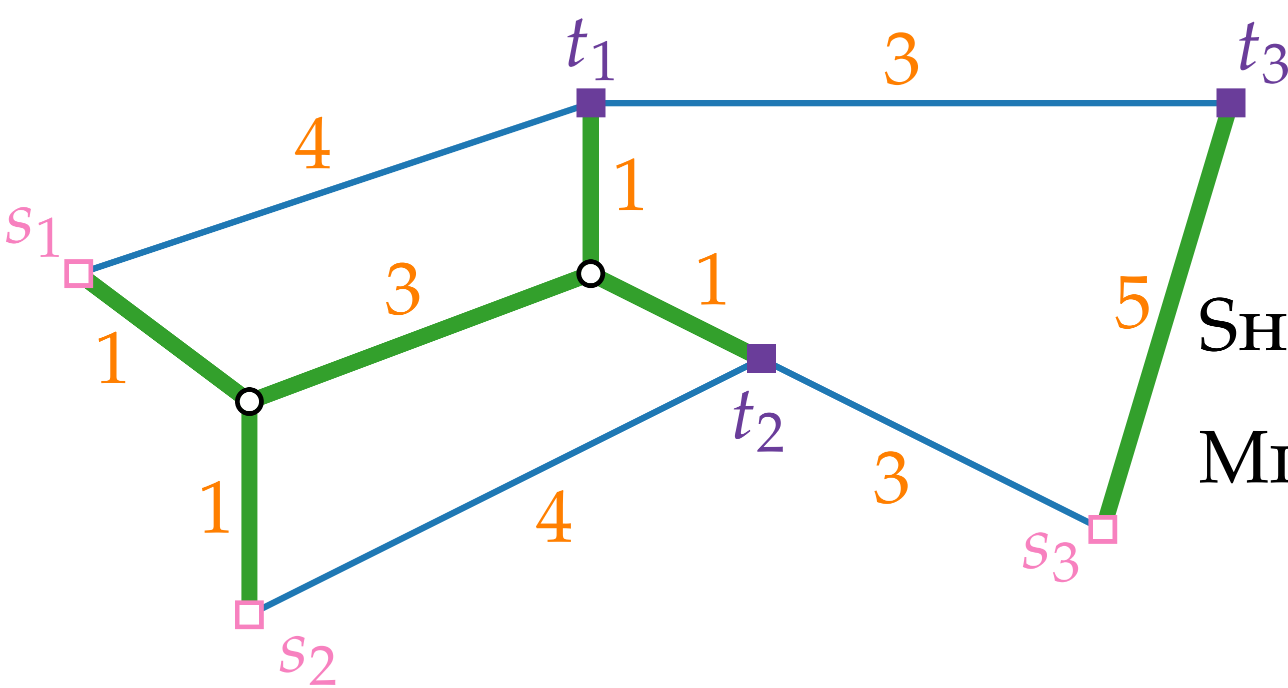

Special cases?

ShortestPath ($R = \{s, t\}$)

MinSpanningTree ($R = E$)

SteinerForest

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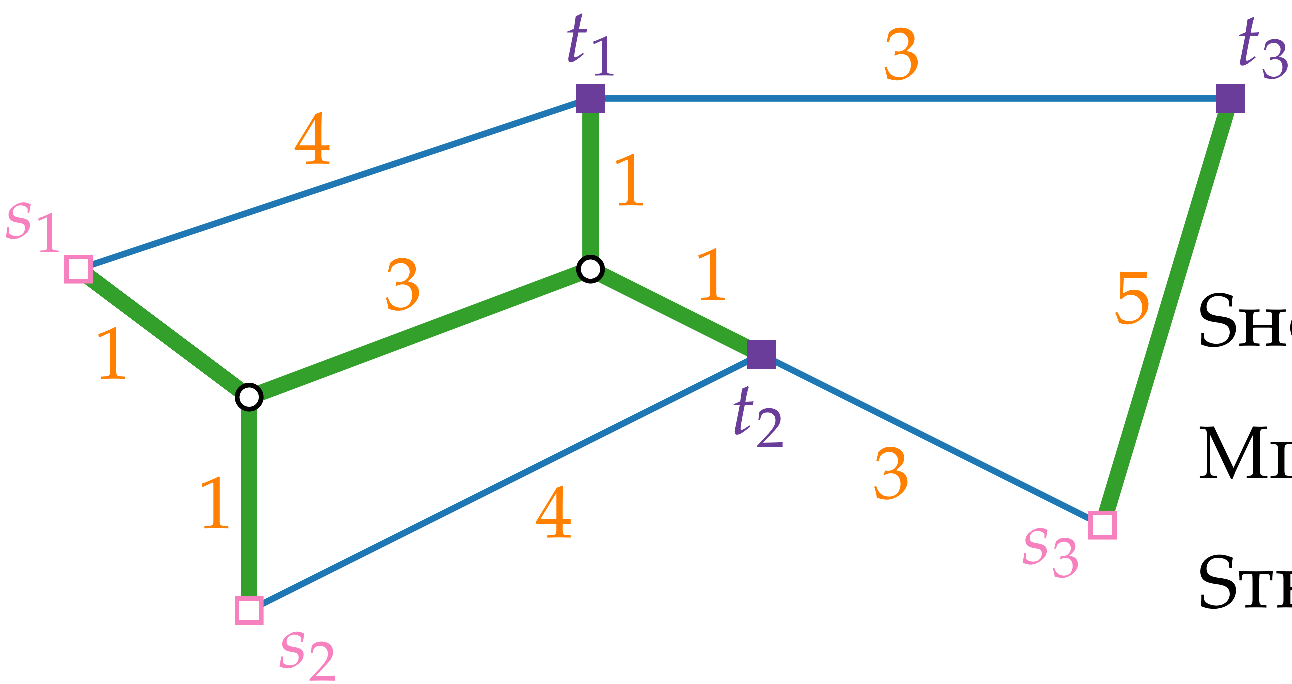


Special cases?

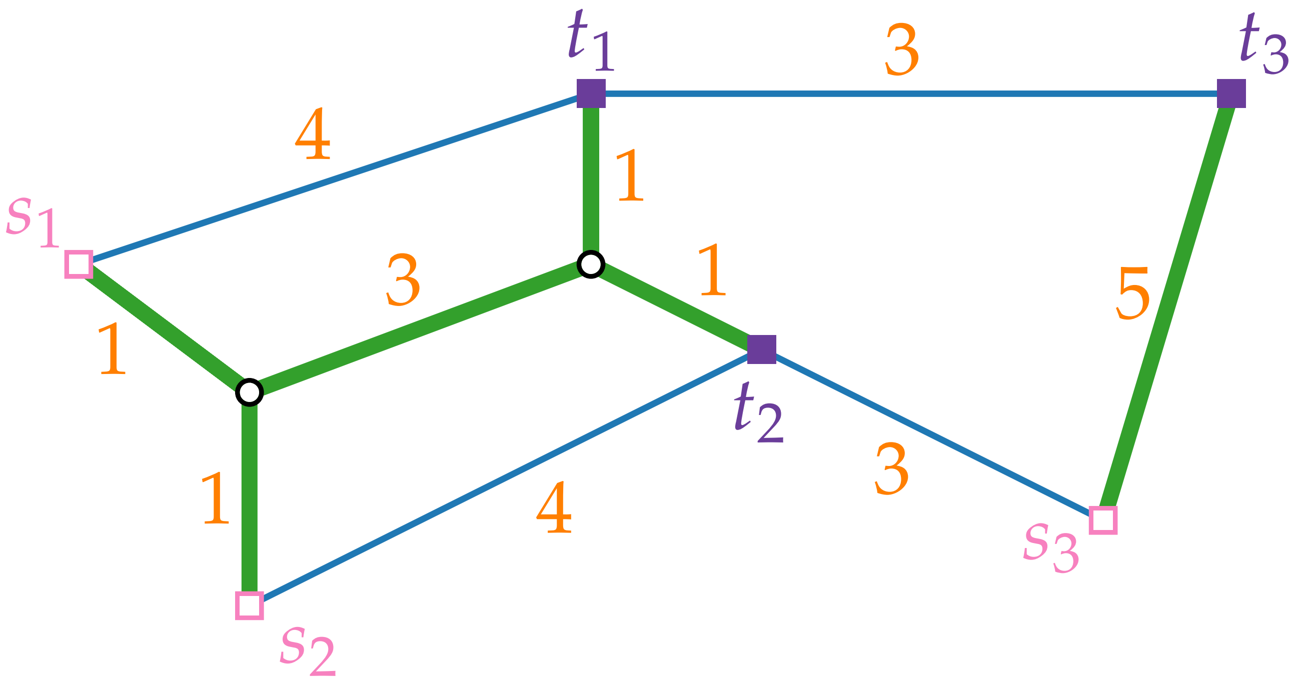
ShortestPath ($R = \{s, t\}$)

MinSpanningTree ($R = E$)

SteinerTree ($R = T \times T$)

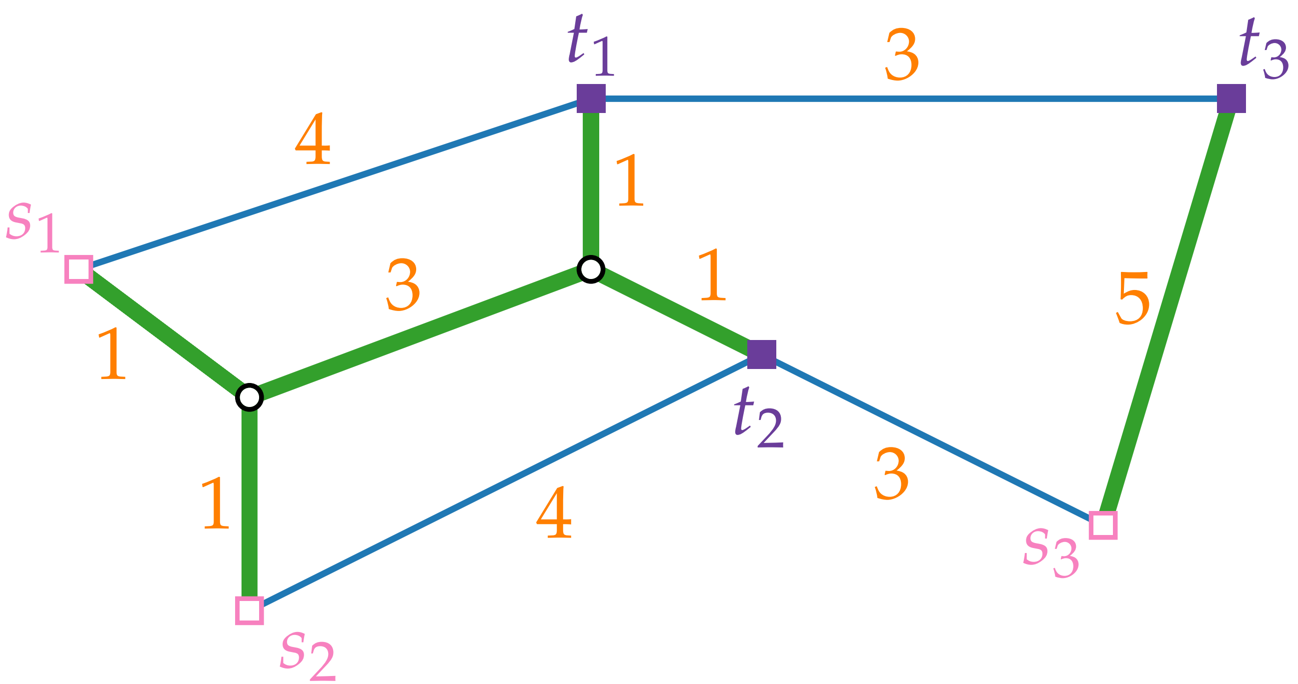
Approaches?

- Merge k shortest s_i–t_i paths



Approaches?

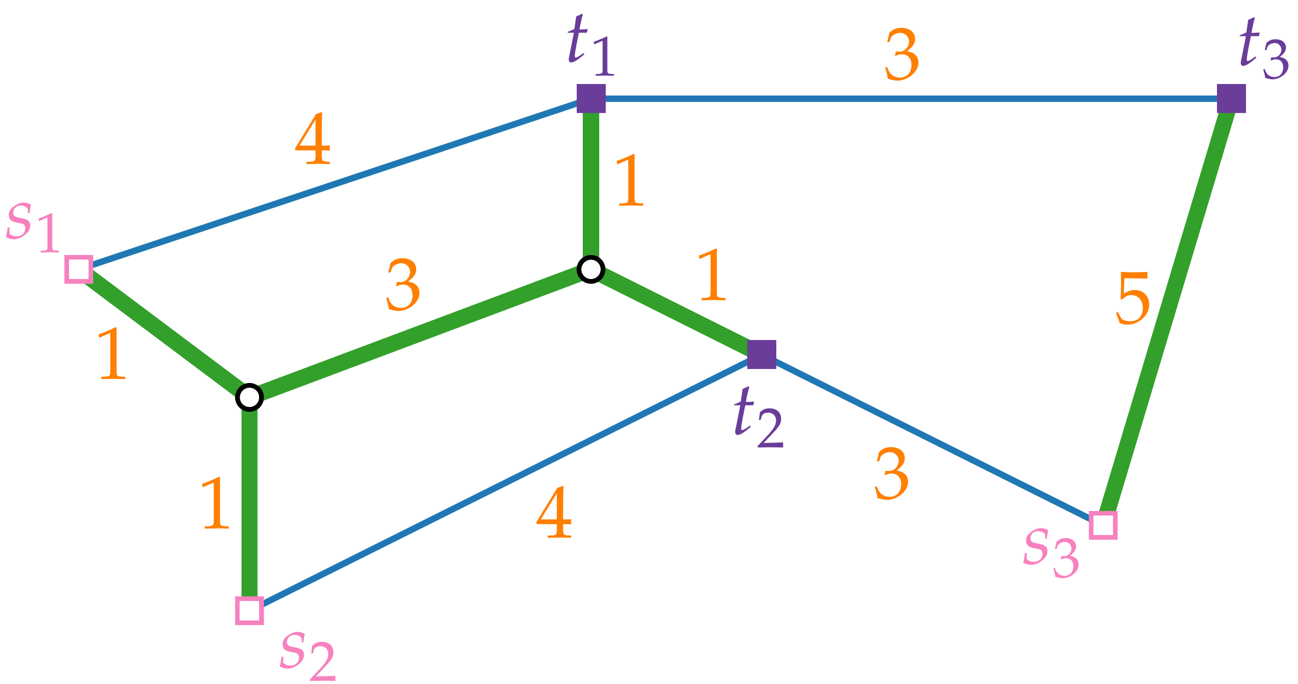
- Merge k shortest s_i–t_i paths
- SteinerTree on the set of terminals



Approaches?

- Merge k shortest s_i–t_i paths
- SteinerTree on the set of terminals

Homework: Both above approaches perform poorly :-(

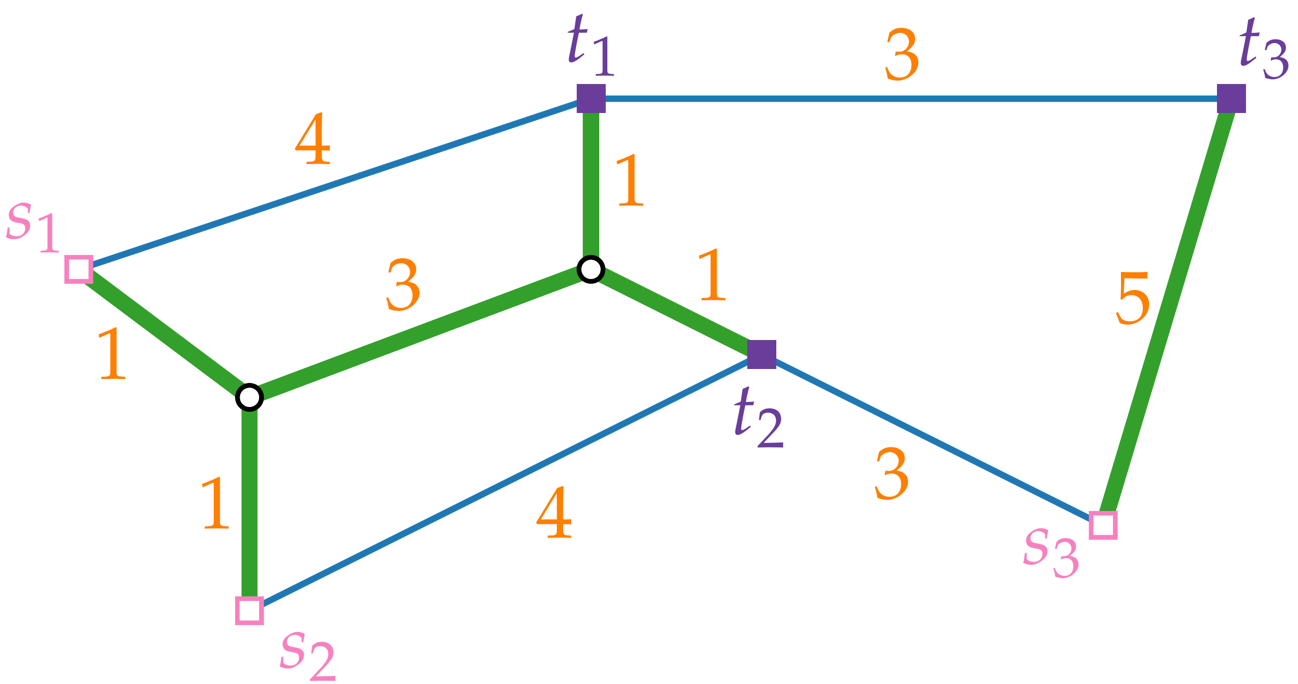


Approaches?

- Merge k shortest s_i–t_i paths
- SteinerTree on the set of terminals

Homework: Both above approaches perform poorly :-(

Difficulty:
Which terminals belong to the same tree of the forest?



Approximation Algorithms

Lecture 12:
SteinerForest via Primal–Dual

Part II:
Primal and Dual LP

An ILP

minimize

subject to

An ILP

minimize

subject to

$$x_e \in \{0,1\} \qquad e \in E$$

An ILP

minimize $\sum_{e \in E} c_e x_e$

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t_i

s_i

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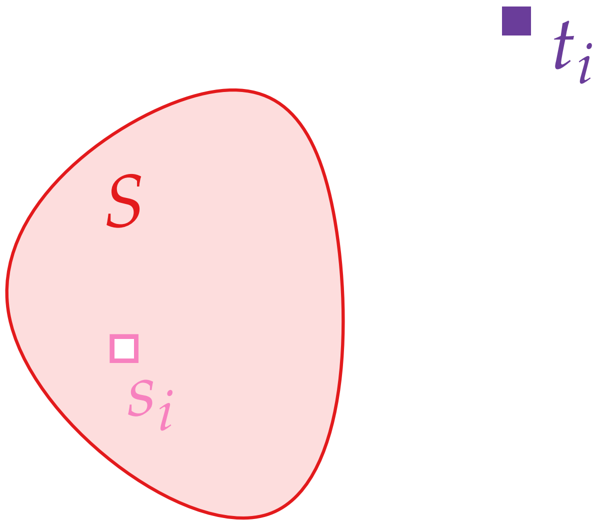

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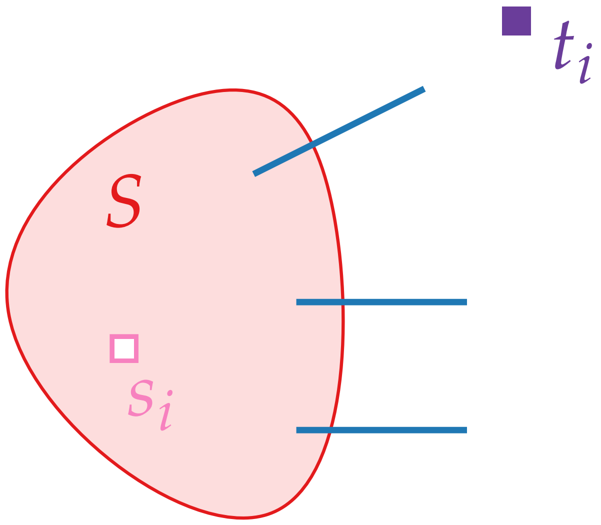

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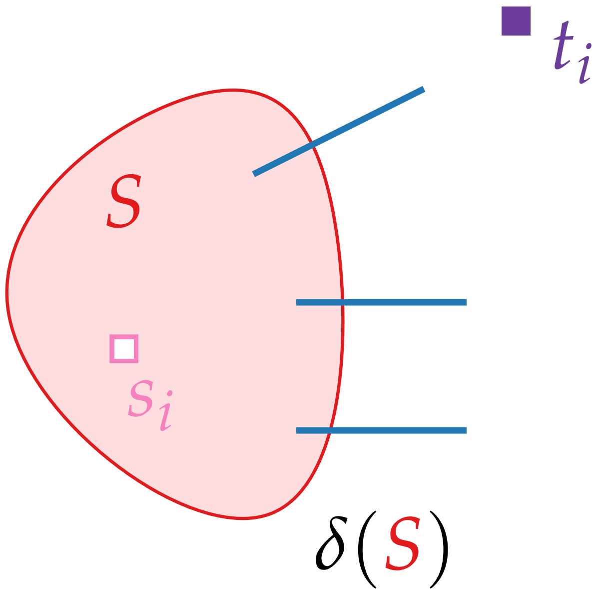

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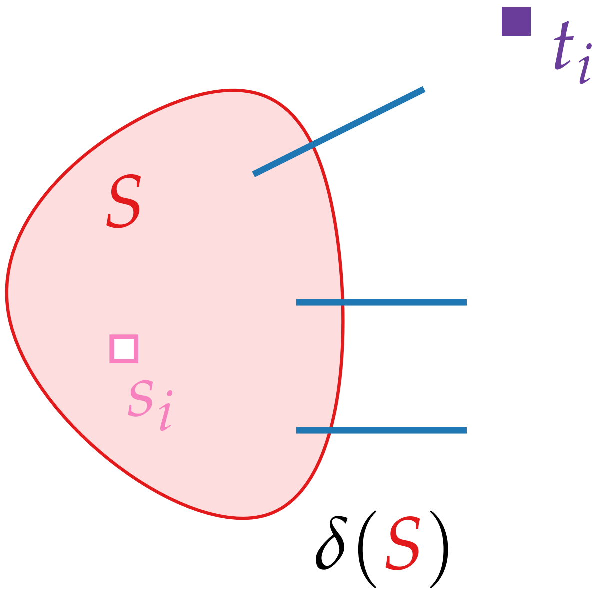
$\delta(S) := \{(u,v) \in E : u \in S \text{ and } v \notin S\}$



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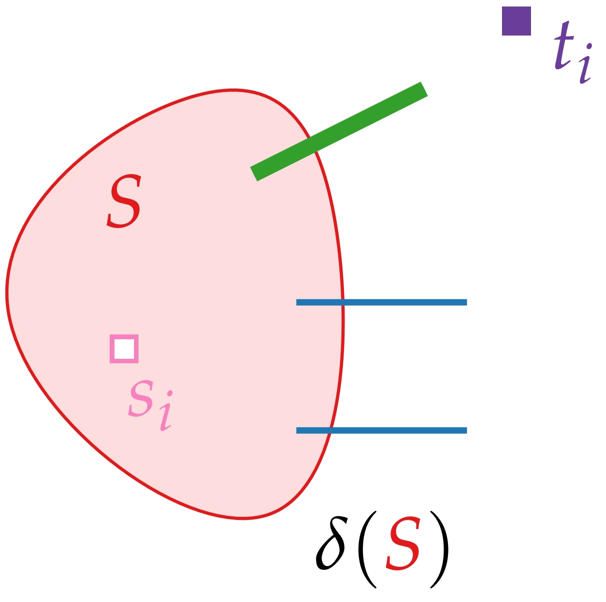

An ILP

minimize	$\sum_{e \in E} c_e x_e$
subject to	$\sum_{e \in \delta(S)} x_e \geq 1$
	$x_e \in \{0,1\} \qquad e \in E$

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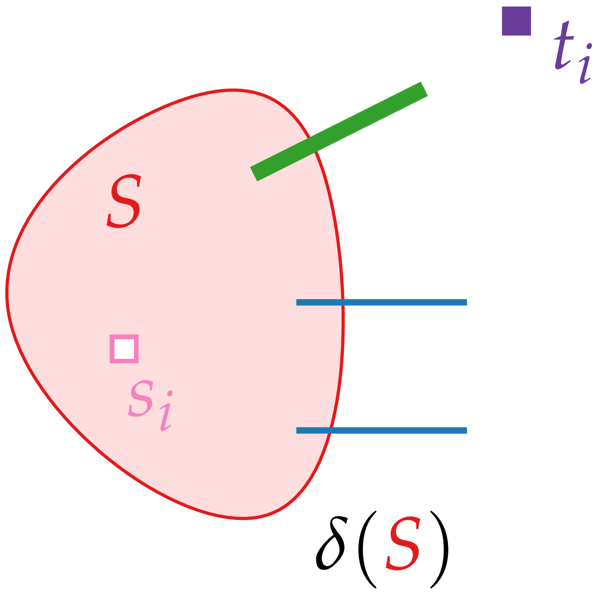

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$$\begin{array}{llll} \textbf{minimize} & \sum_{e \in E} c_e x_e & & \\ \textbf{subject to} & \sum_{e \in \delta(S)} x_e \geq 1 & & S \in \mathcal{S}_i,\ i \in \{1, \ldots, k\} \\ & x_e \in \{0, 1\} & & e \in E \end{array}$$

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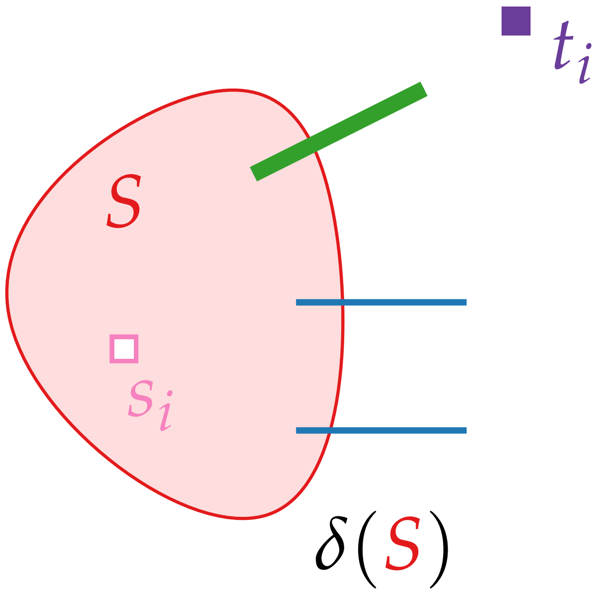

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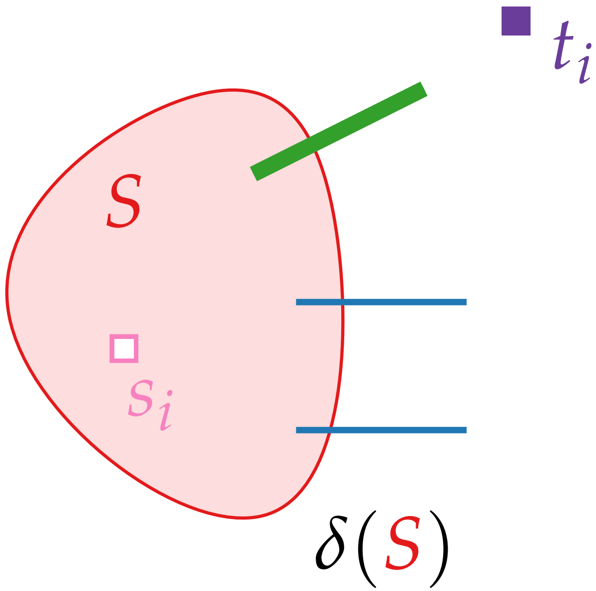

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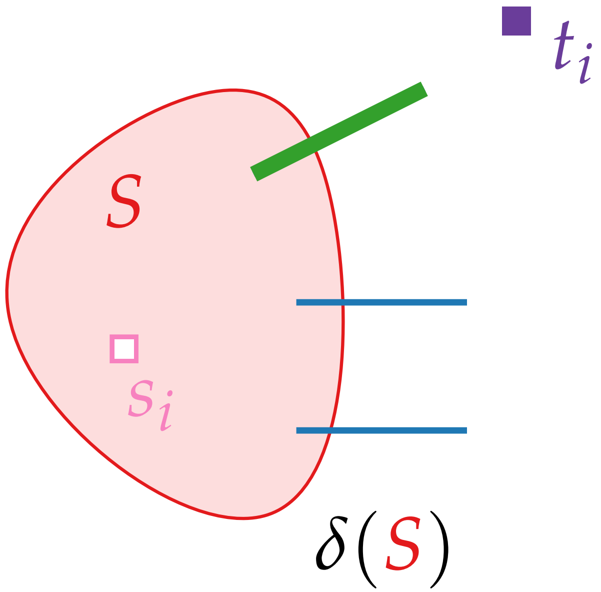
$\rightsquigarrow$ exponentially many constraints!



LP-Relaxation and Dual LP

$$\begin{array}{lll} \textbf{minimize} & \sum_{e \in E} c_e x_e & \\ \textbf{subject to} & \sum_{e \in \delta(S)} x_e \geq 1 & S \in \mathcal{S}_i,\ i \in \{1, \ldots, k\} \\ & x_e \geq 0 & e \in E \end{array}$$

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$x_e \geq 0 \qquad e \in E$

maximize

subject to

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Intuition for the Dual

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The graph is a network of **bridges**, spanning the **moats**.

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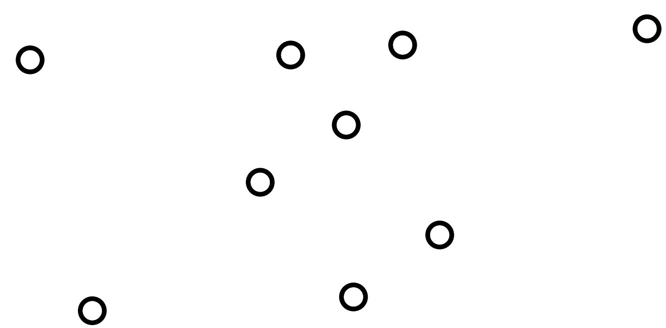

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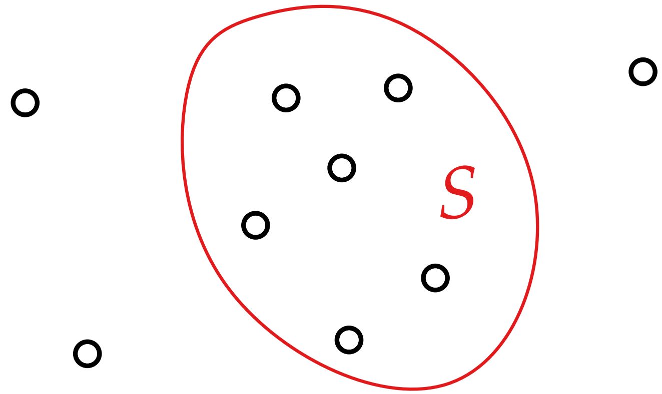

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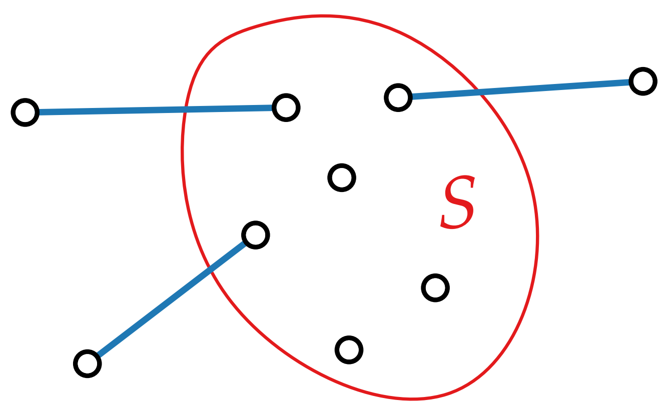

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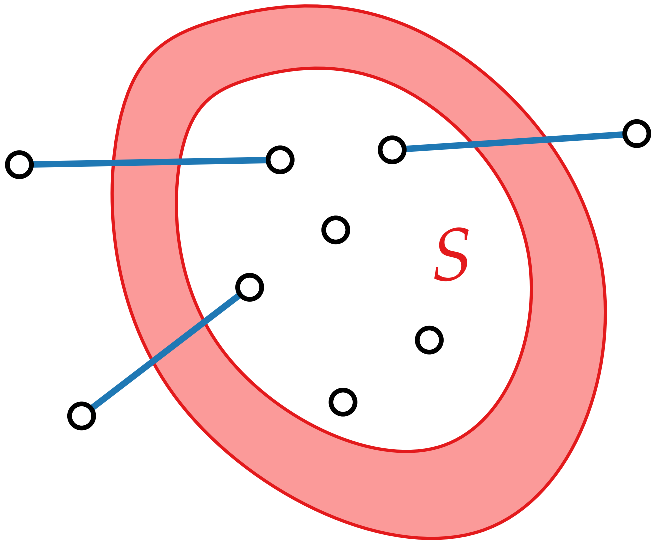

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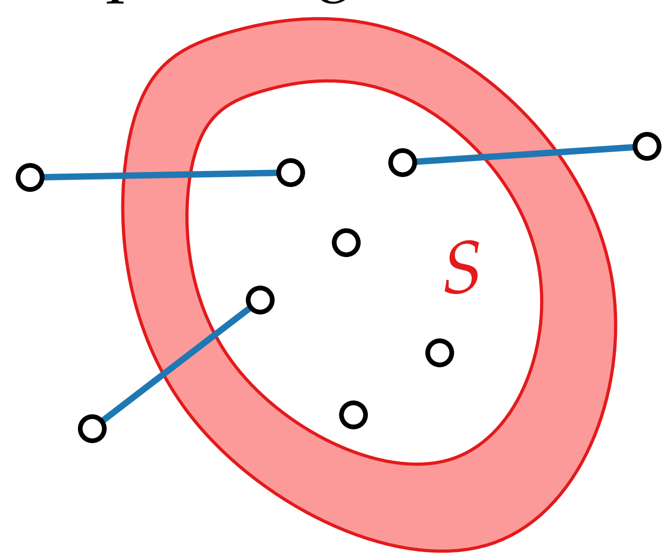

$\delta(S)$ = set of edges / bridges over the moat around S

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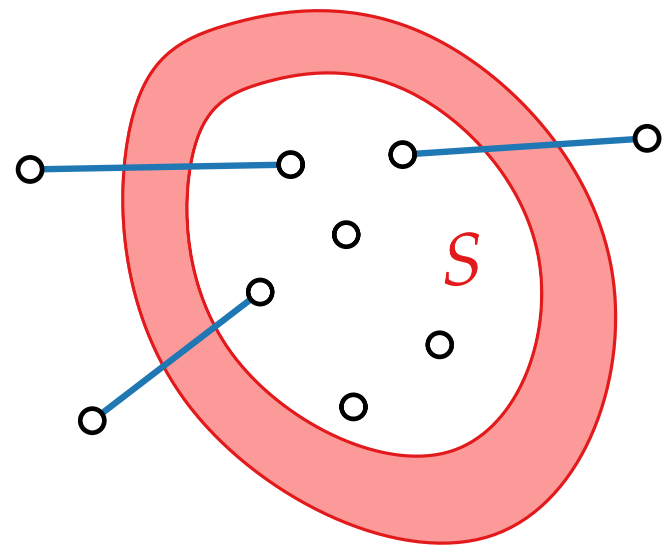

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y_S = width of the **moat** around S

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$$\text{maximize} \quad \sum_{\substack{S \in \mathcal{S}_i \\ i \in \{1,\ldots,k\}}} y_S$$

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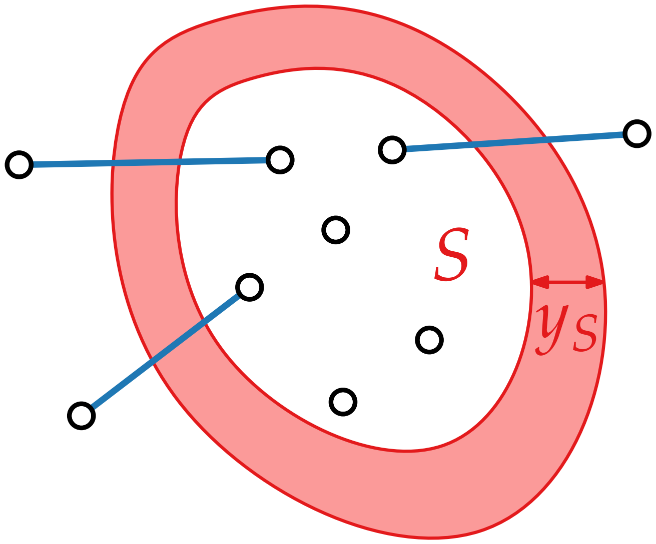

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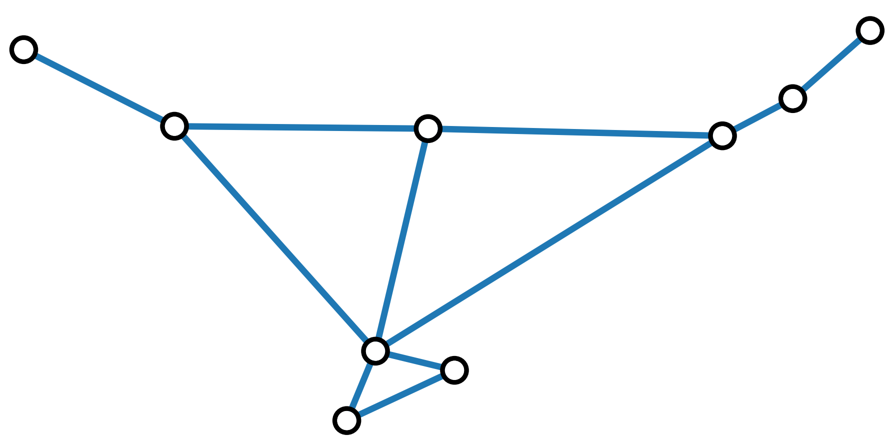

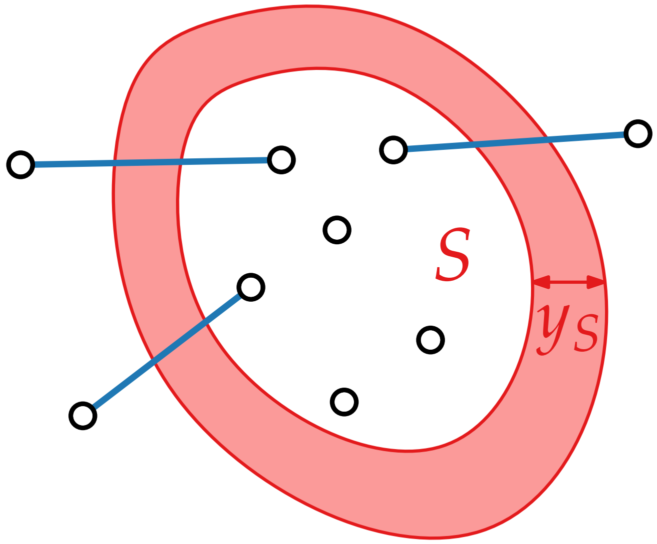

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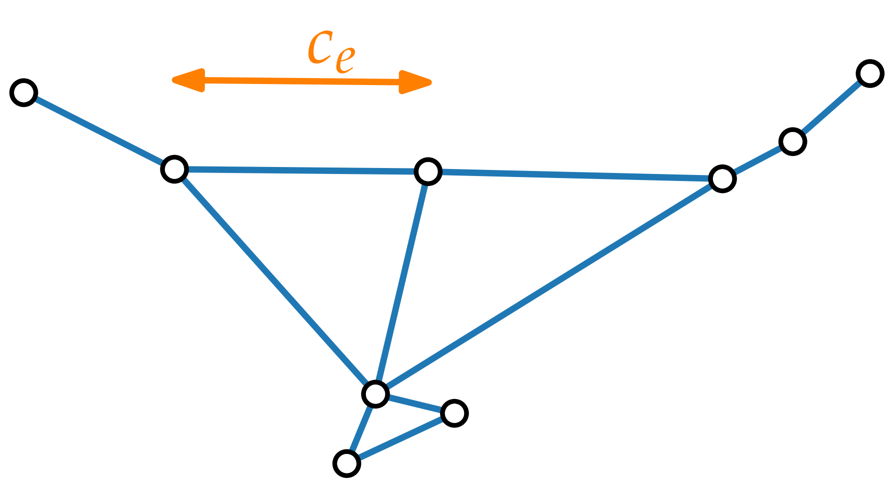
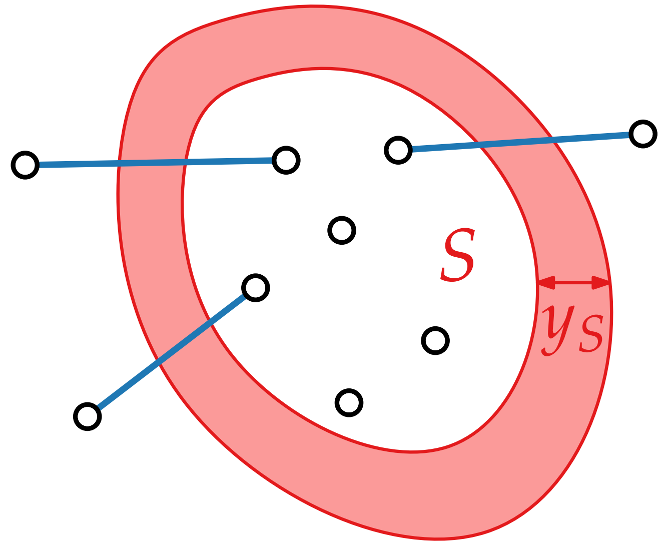
The graph is a network of **bridges**, spanning the **moats**.





$\delta(S)$ = set of edges / bridges over the moat around S

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Intuition for the Dual

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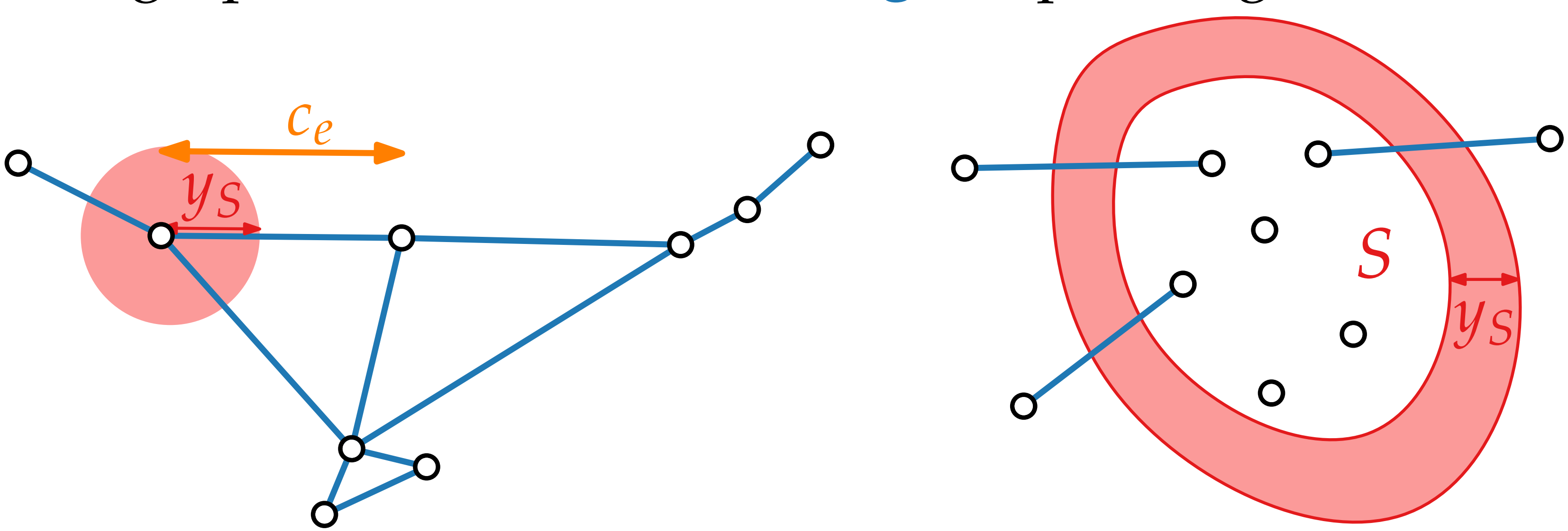

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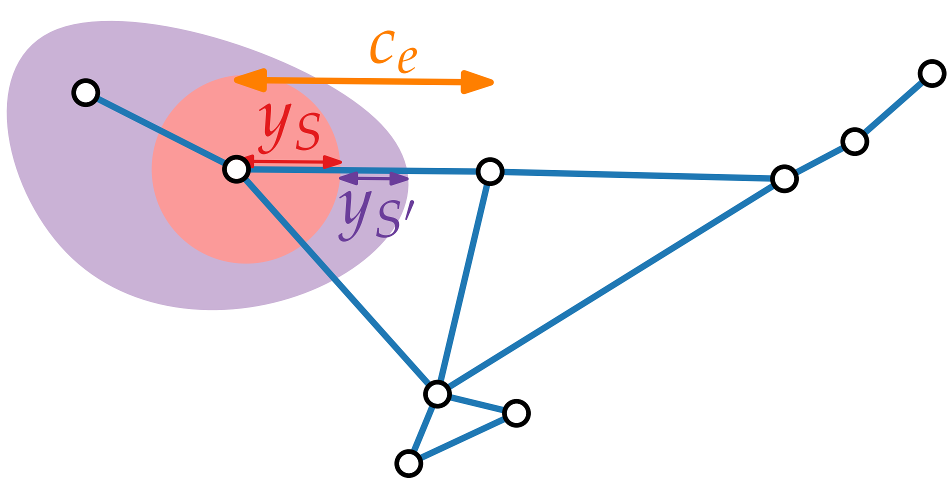

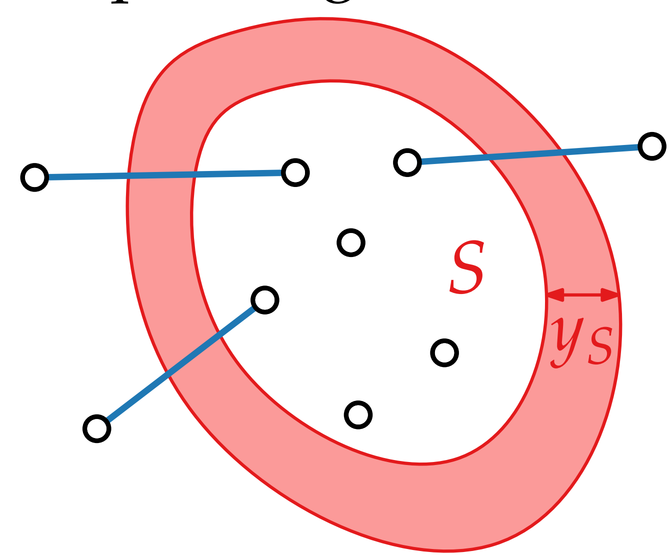

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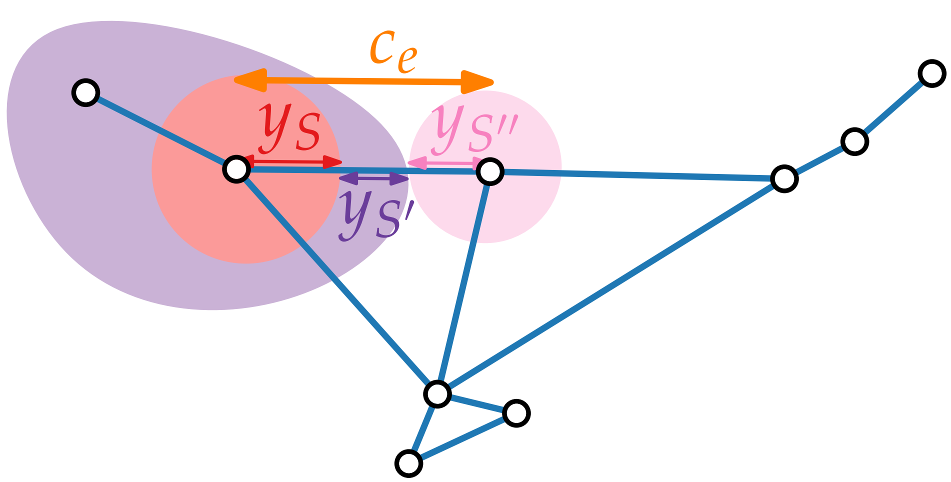

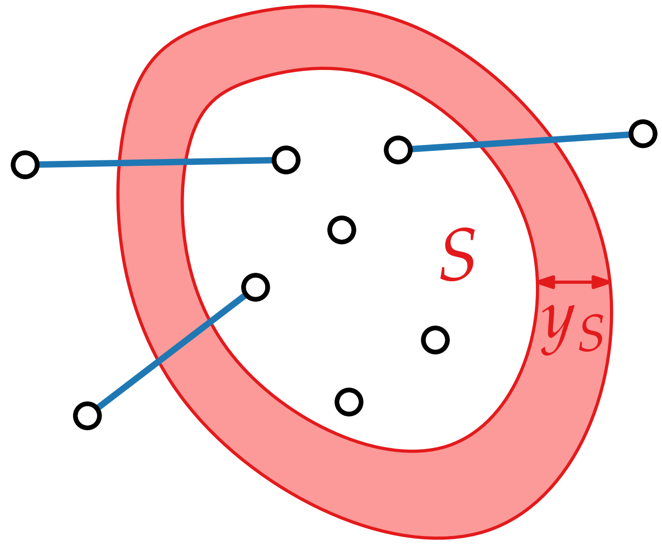

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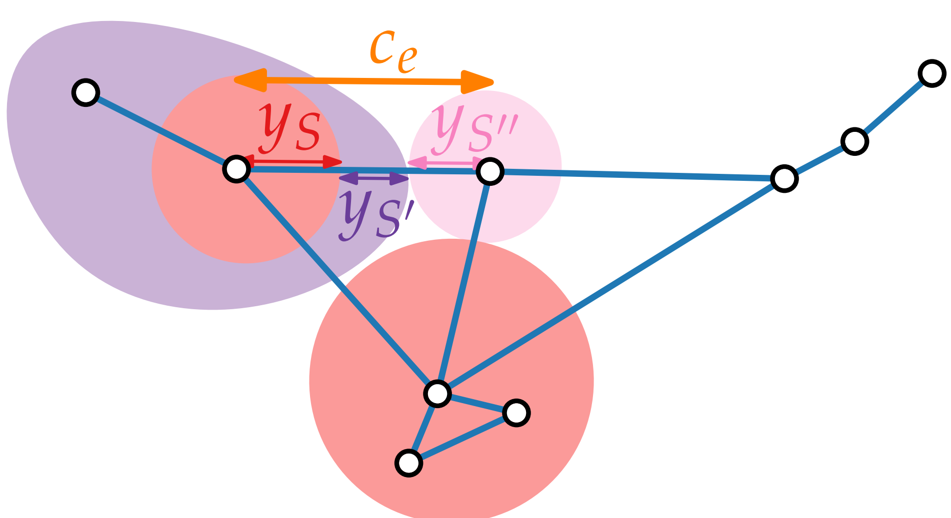

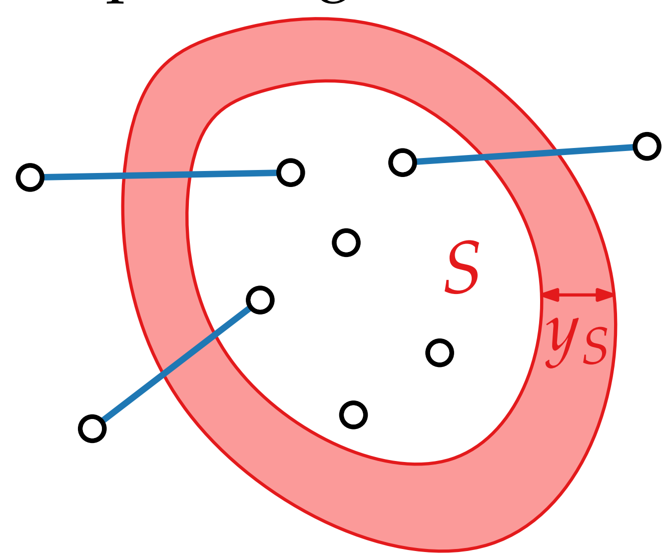

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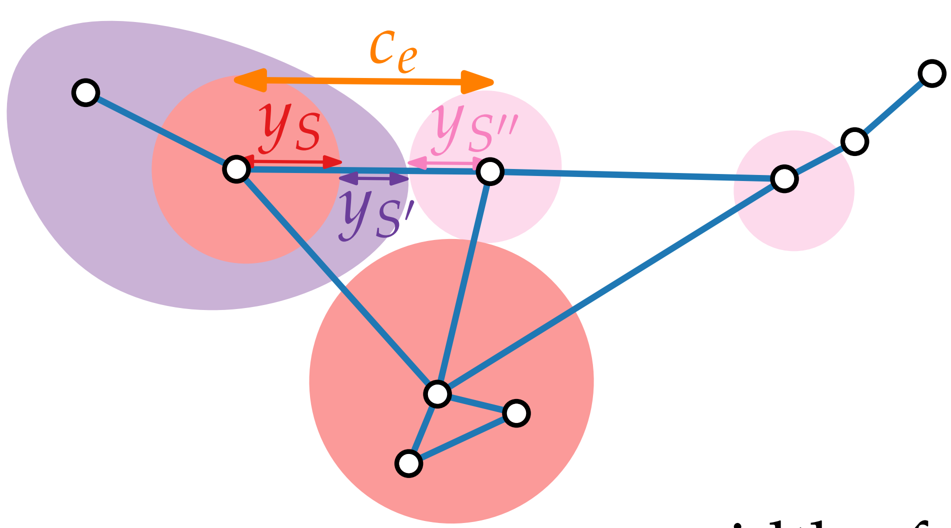

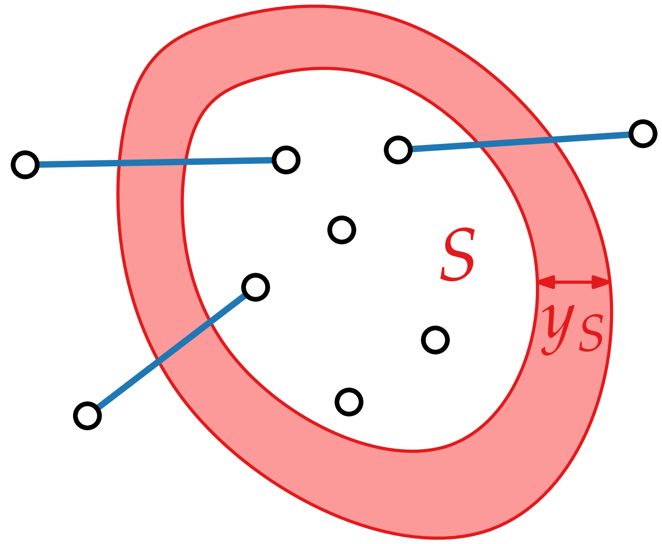

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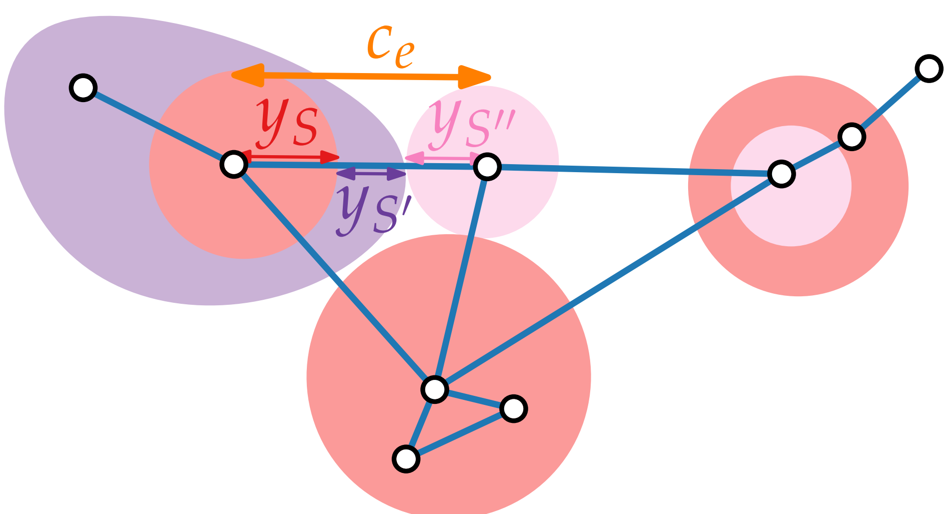

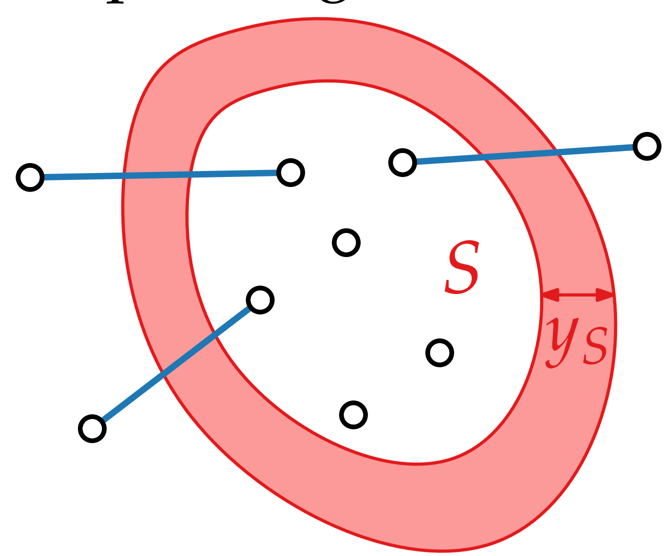

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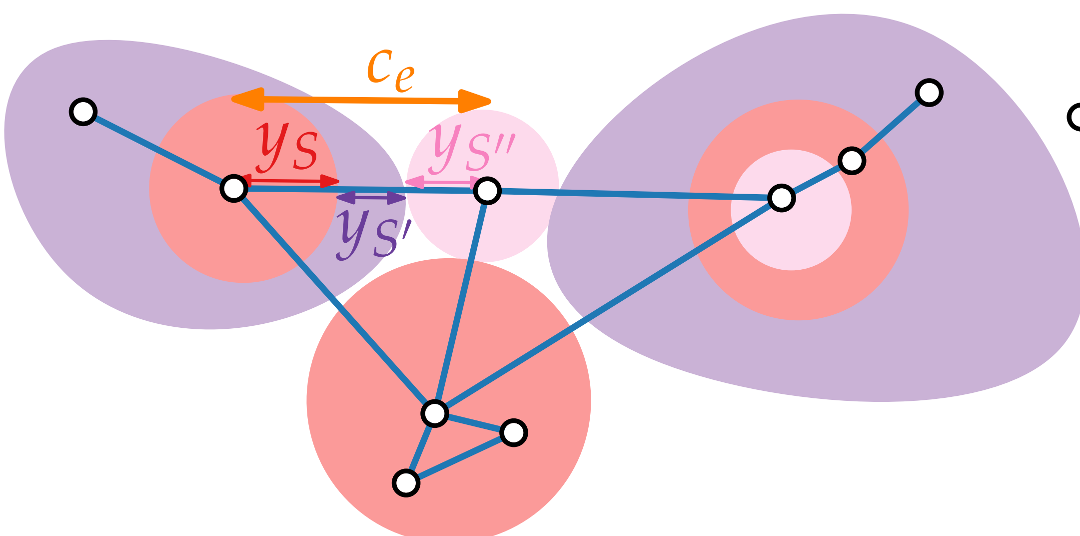

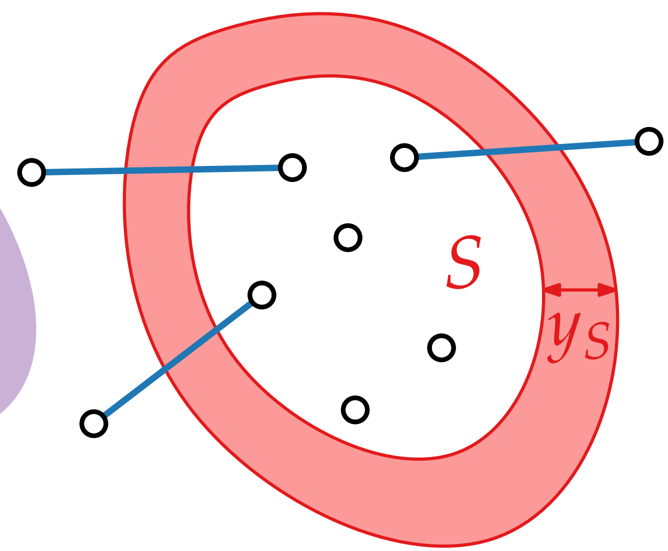

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Approximation Algorithms

Lecture 12:
SteinerForest via Primal–Dual

Part III:
A First Primal–Dual Approach

Complementary Slackness (Reminder)

$$\begin{array}{llll} \textbf{minimize} & c^\top x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{llll} \textbf{maximize} & b^\top y & & \\ \textbf{subject to} & A^\top y & \leq & c \\ & y & \geq & 0 \end{array}$$

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Theorem. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program (resp.). Then x and y are optimal if and only if the following conditions are met:

Primal CS:
For each $j = 1, \ldots, n$: either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
For each $i = 1, \ldots, m$: either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

A First Primal–Dual Approach

Complementary slackness: $x_e > 0 \;\; \Rightarrow$

A First Primal–Dual Approach

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Idea: iteratively build a feasible integral primal solution.

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How to find a violated primal constraint? $\left(\sum_{e \in \delta(S)} x_e < 1\right)$

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How do we iteratively improve the dual solution?

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- Consider related connected component C!

How do we iteratively improve the dual solution?

- Increase y_C (until some edge in $\delta(C)$ becomes critical)!

A First Primal–Dual Approach

PrimalDualSteinerForestNaive(graph G, costs c, pairs R)

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Trick: Handle all y_S with $y_S = 0$ implicitly.

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$$\sum_{e \in F} c_e \stackrel{\text{CS}}{=} \sum_{e \in F} \sum_{S \,:\, e \in \delta(S)} y_S = \sum_{S} |\delta(S) \cap F| \cdot y_S.$$

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Homework!

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But: Average degree of "active components" is less than 2.

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Homework!

But: Average degree of "active components" is less than 2.

$\Rightarrow$ Increase y_C for all active components C simultaneously!

Approximation Algorithms

Lecture 12:
SteinerForest via Primal–Dual

Part IV:
Primal–Dual with Synchronized Increases

Primal–Dual with Synchronized Increases

PrimalDualSteinerForest(graph G, edge costs c, pairs R)

$y \leftarrow 0, F \leftarrow \emptyset, \ell \leftarrow 0$
while some $(s_i, t_i) \in R$ not connected in (V, F) **do**
  $\ell \leftarrow \ell + 1$
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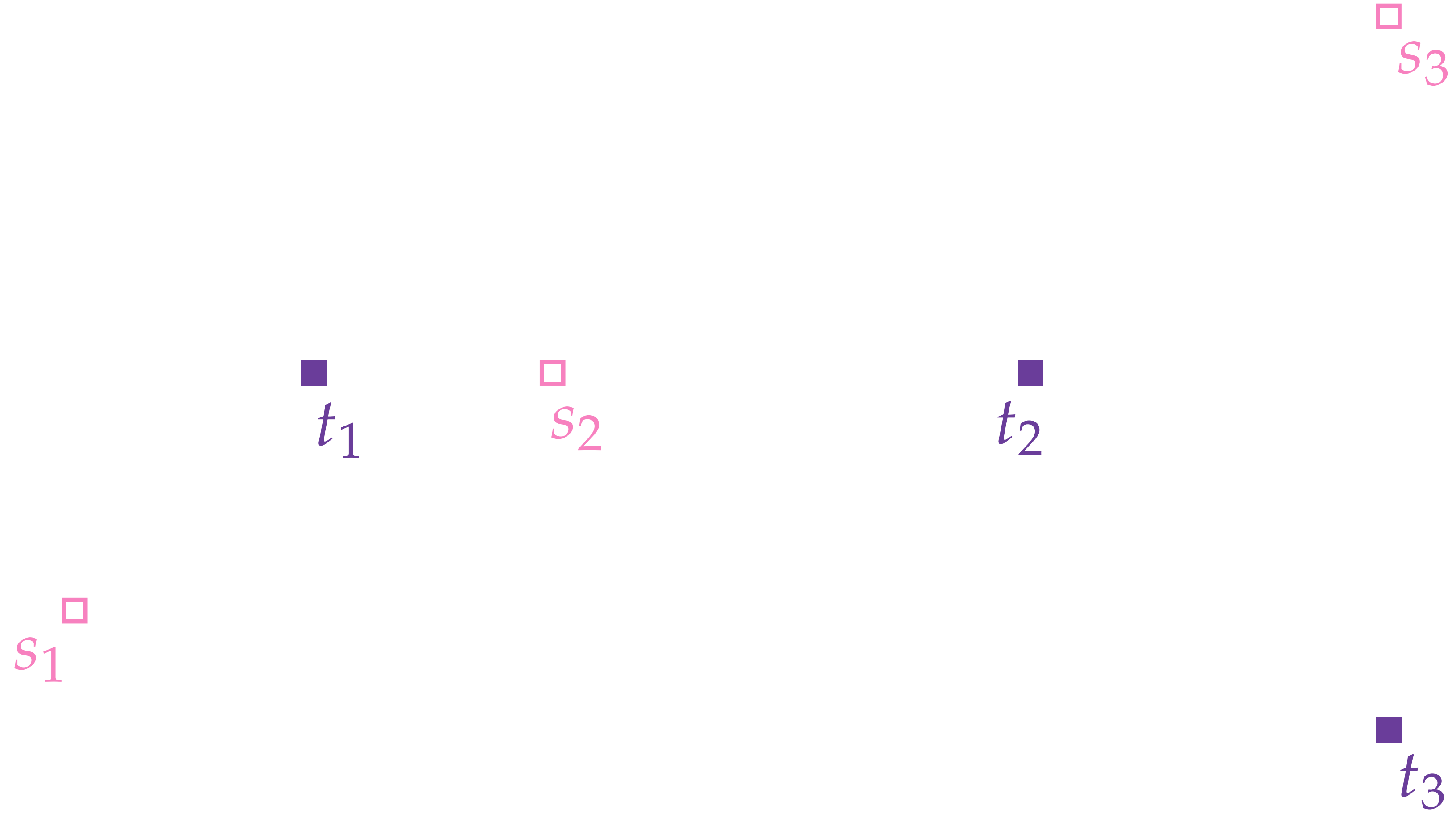
Illustration

$G = K_6$ with Euclidean edge costs



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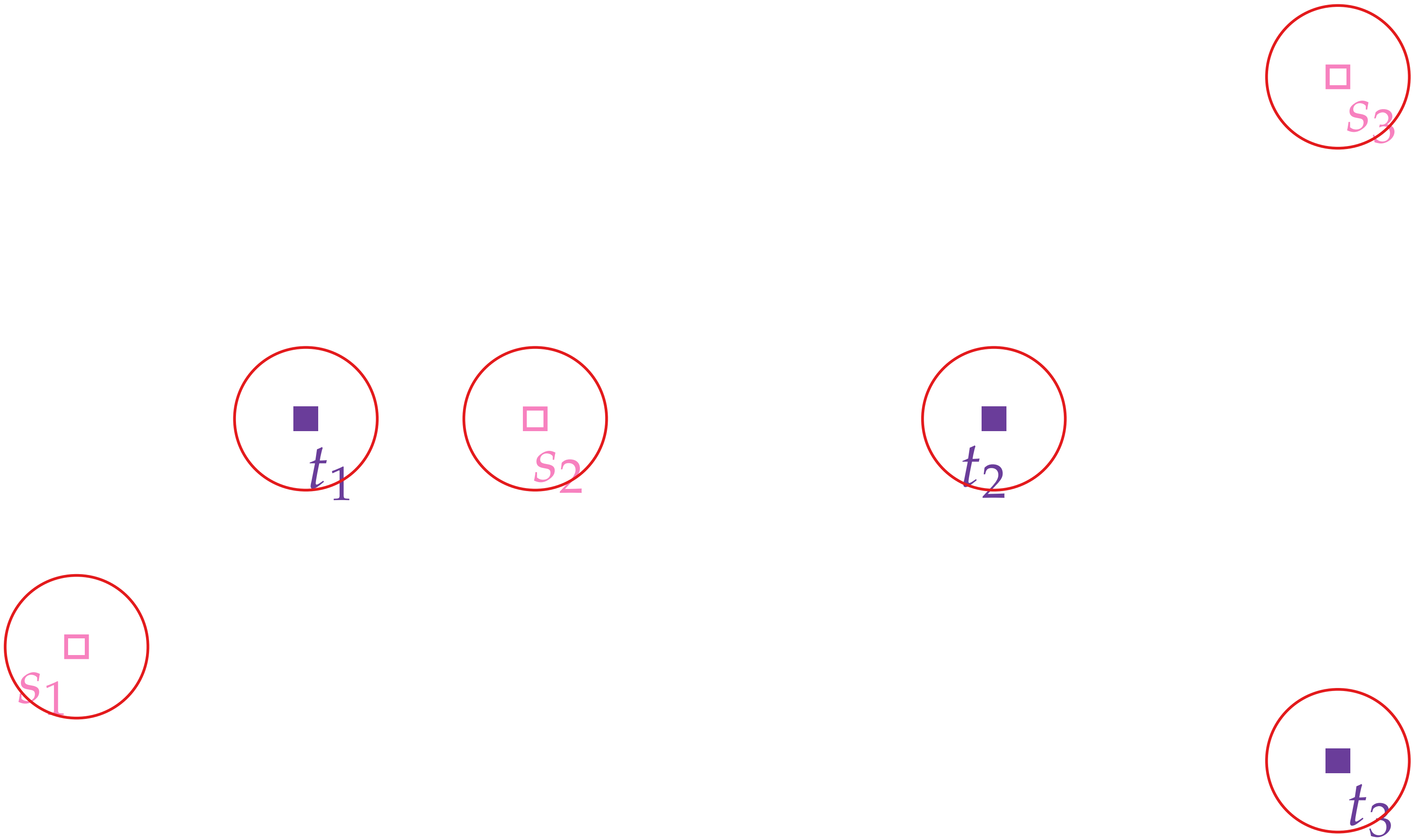

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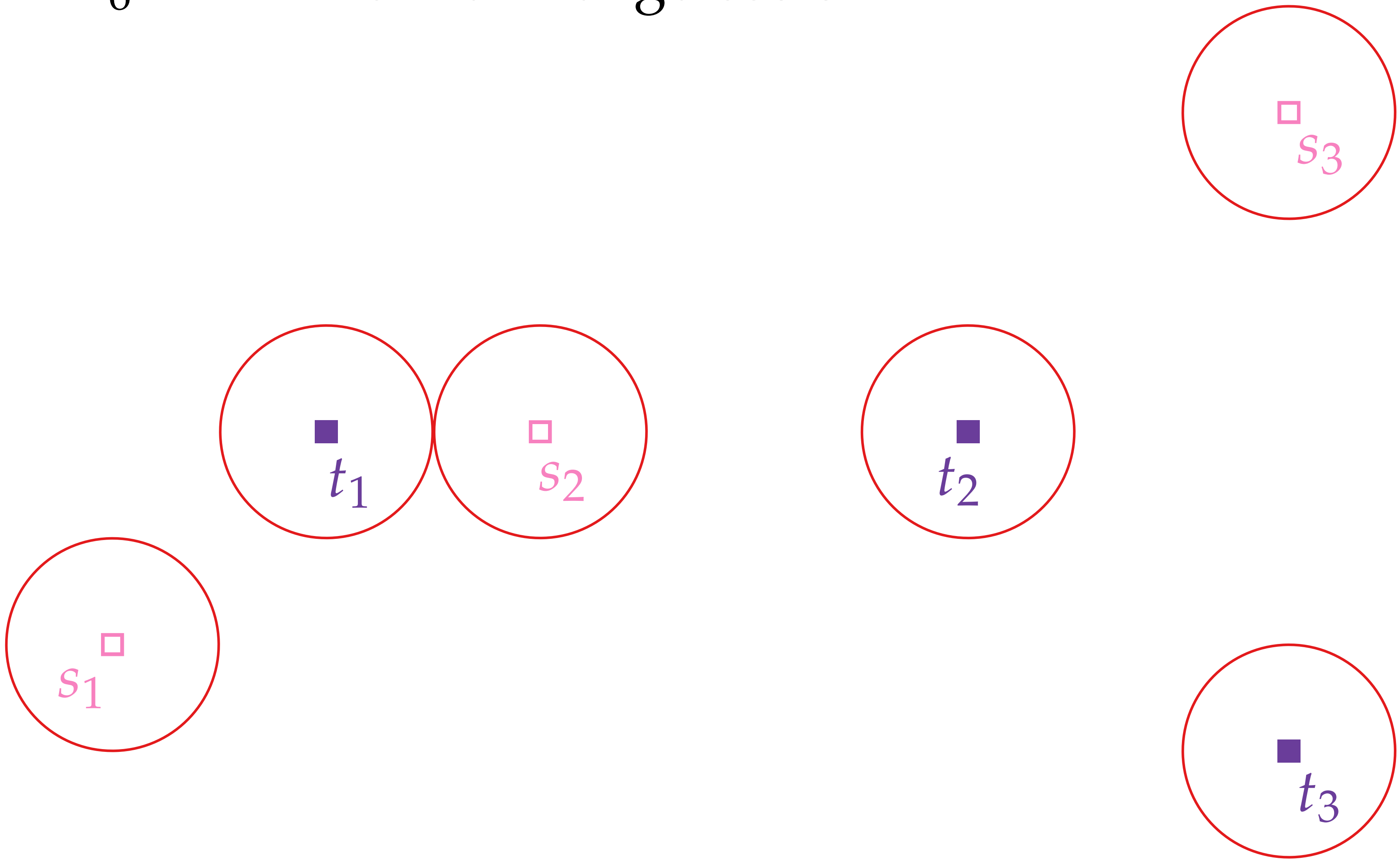

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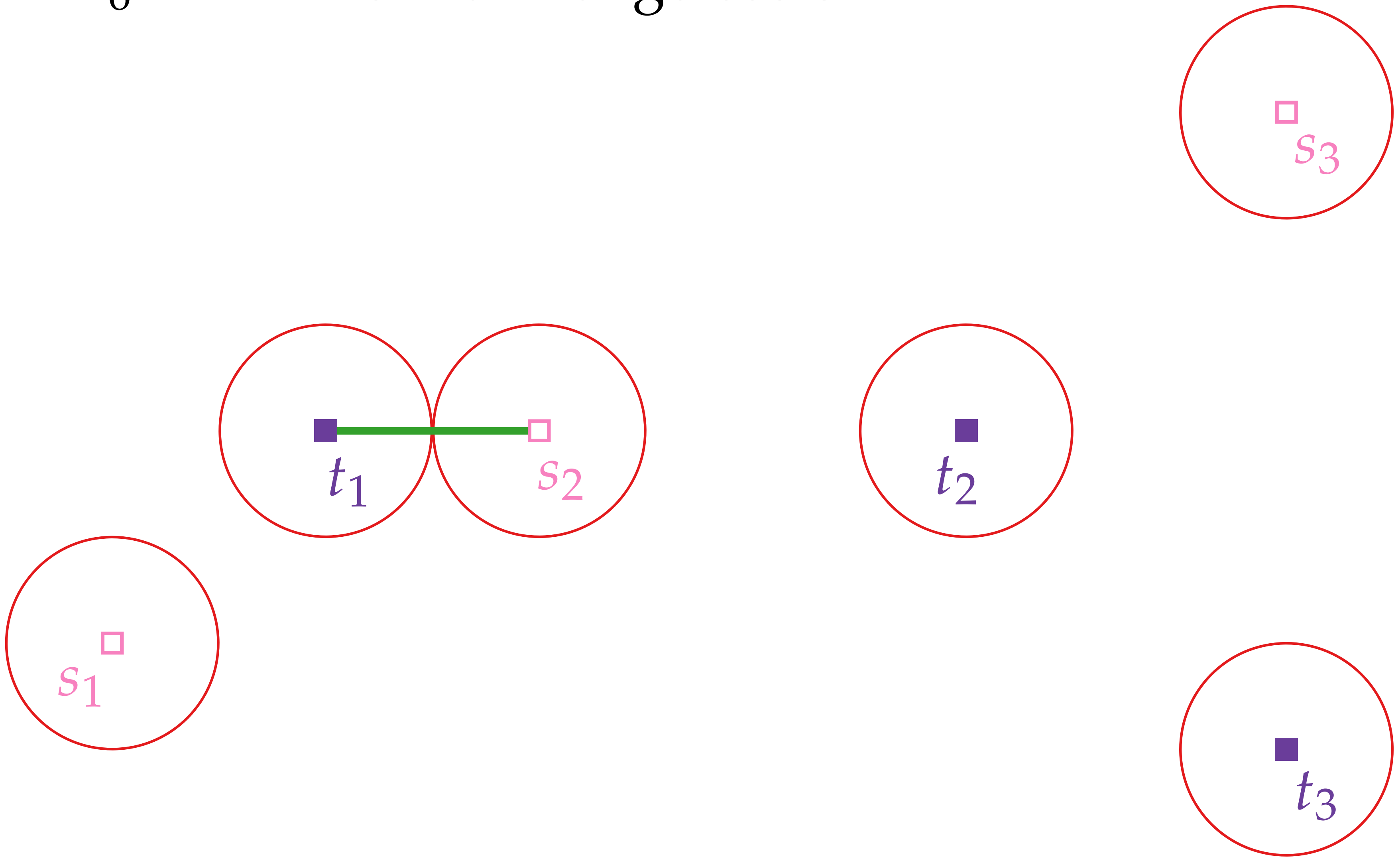

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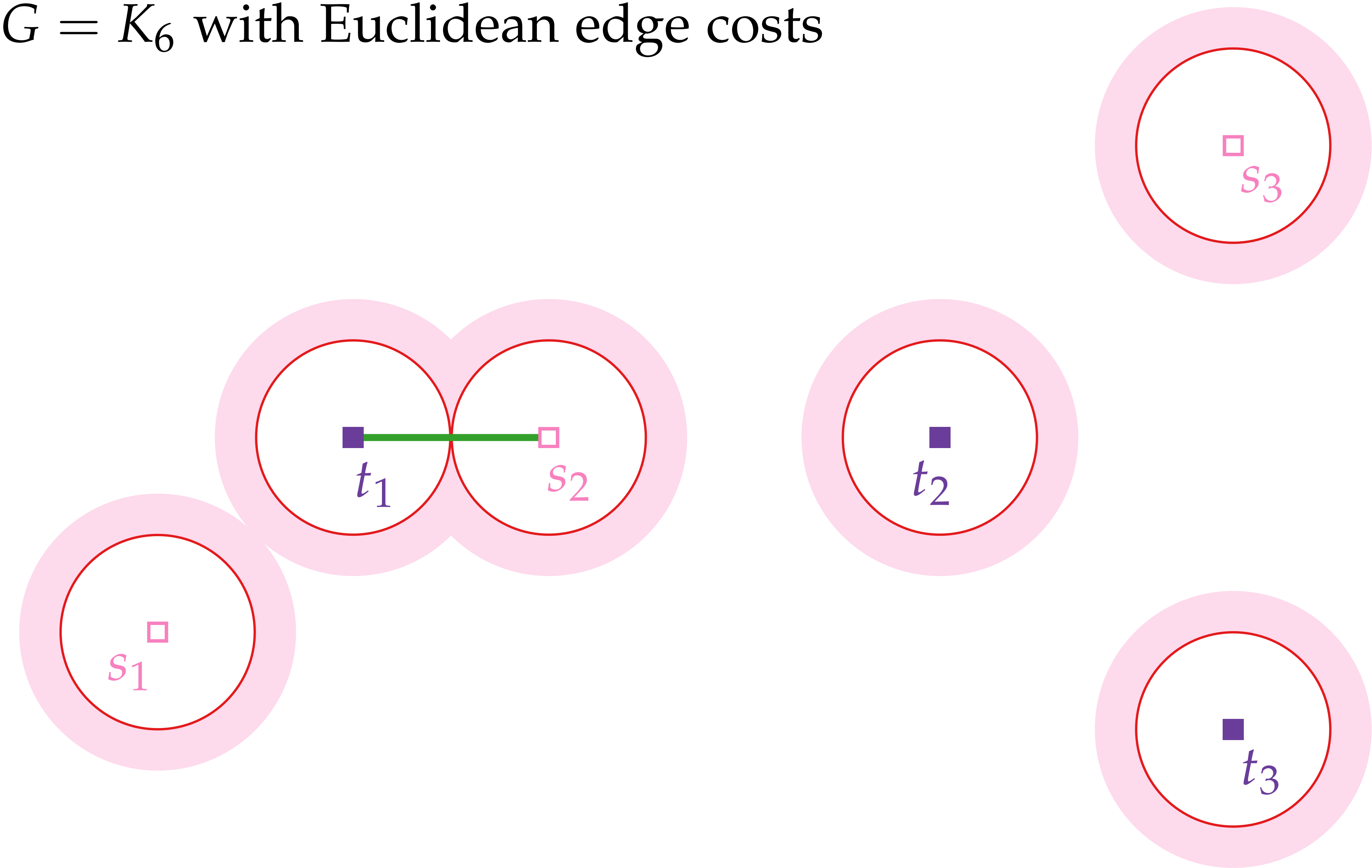

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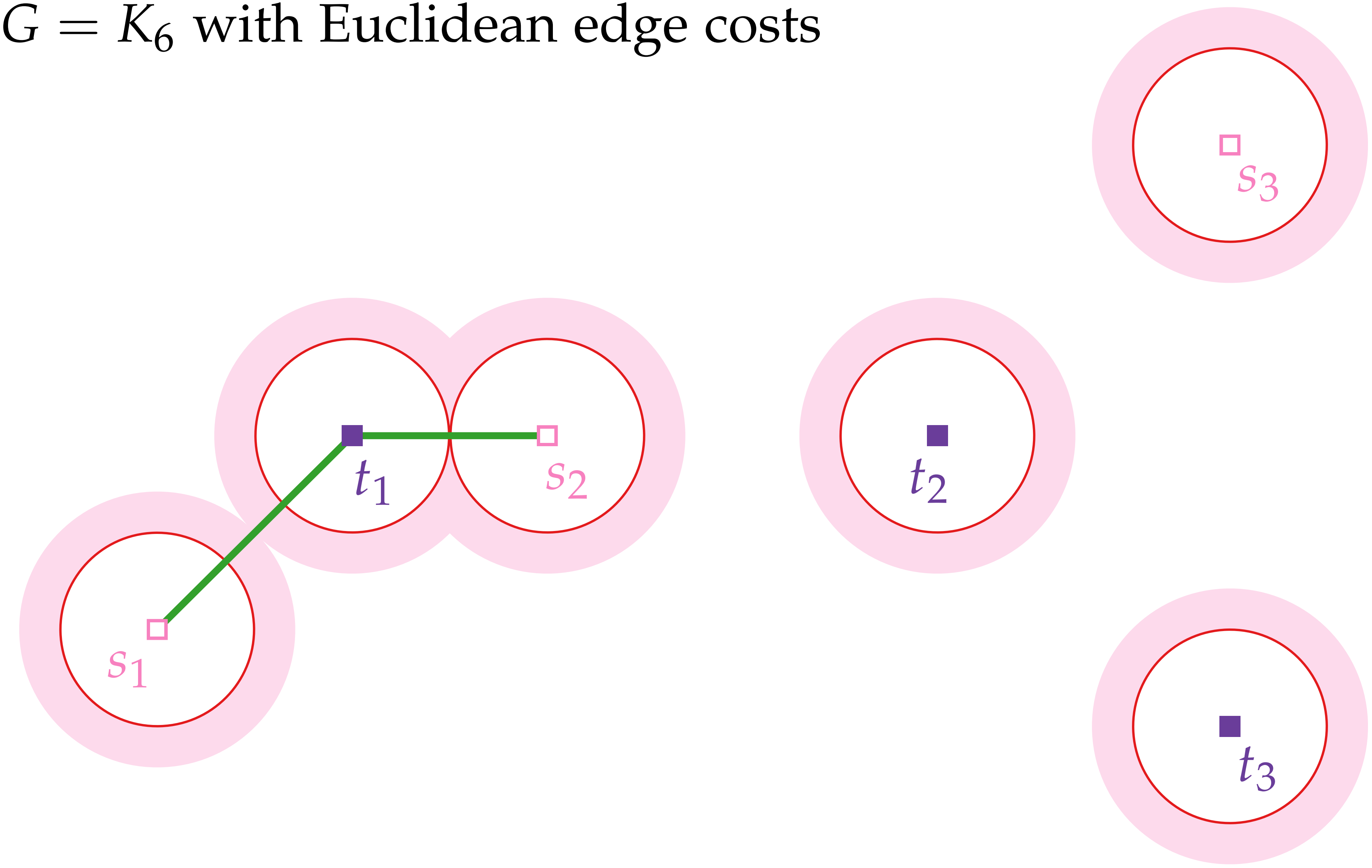

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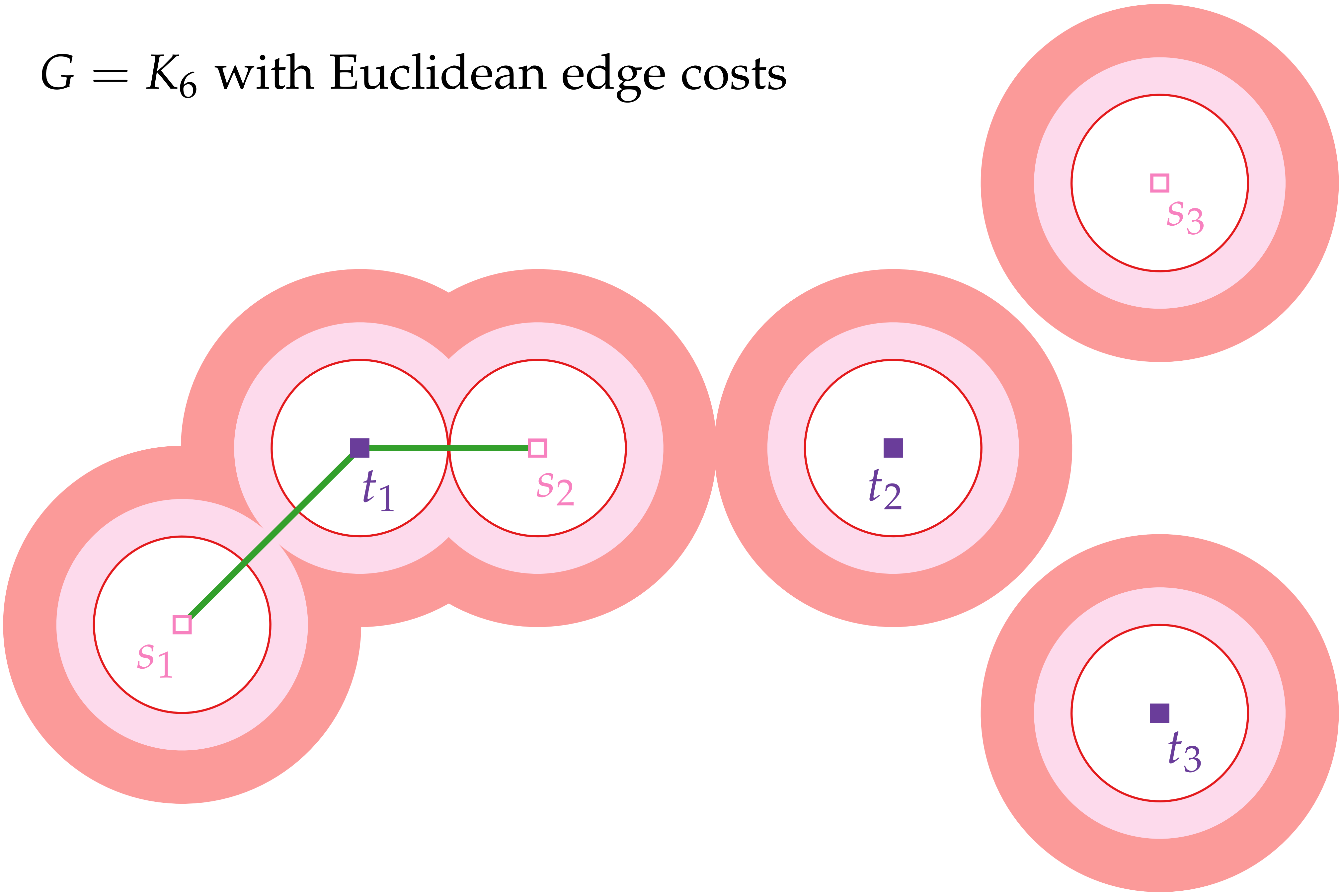

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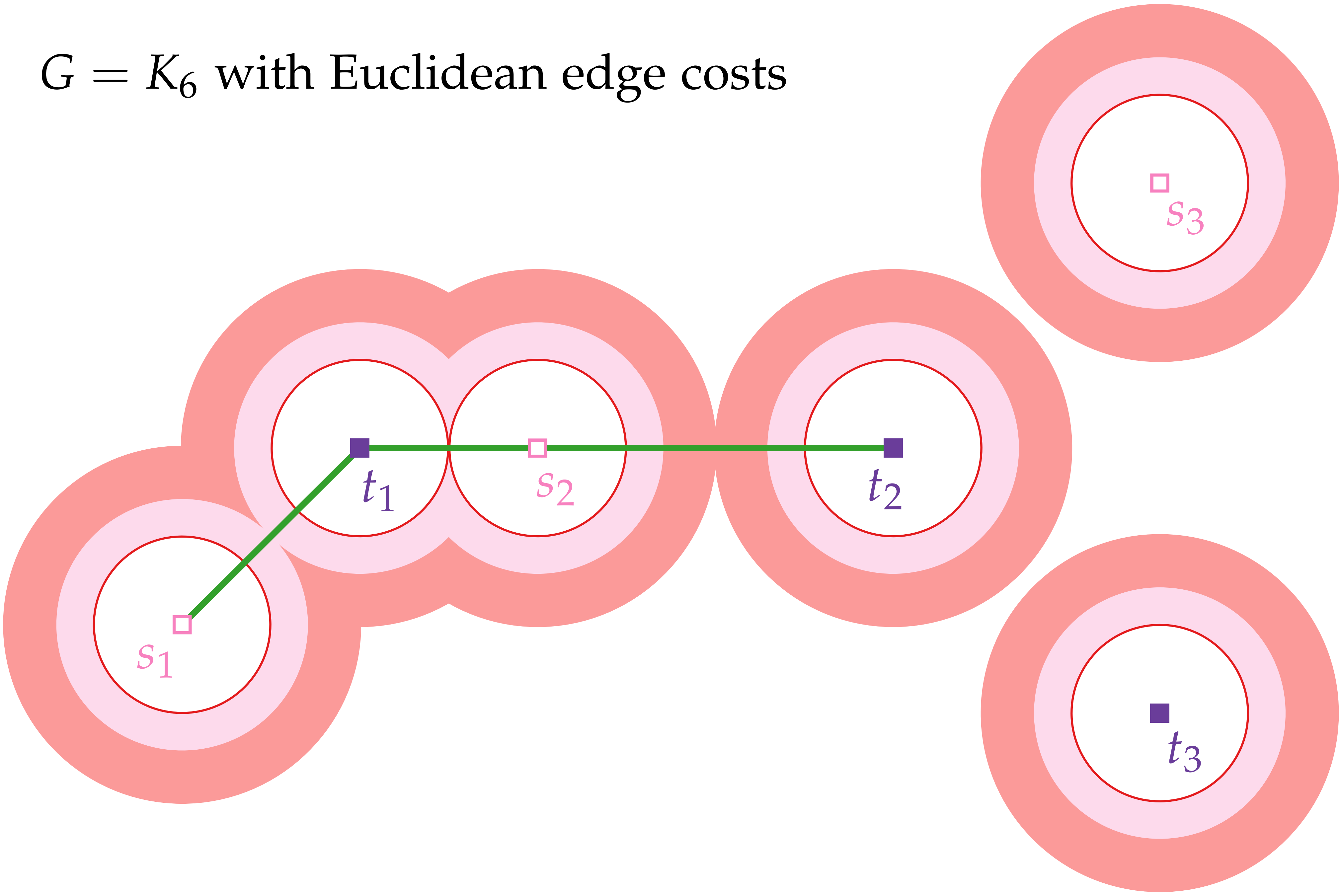

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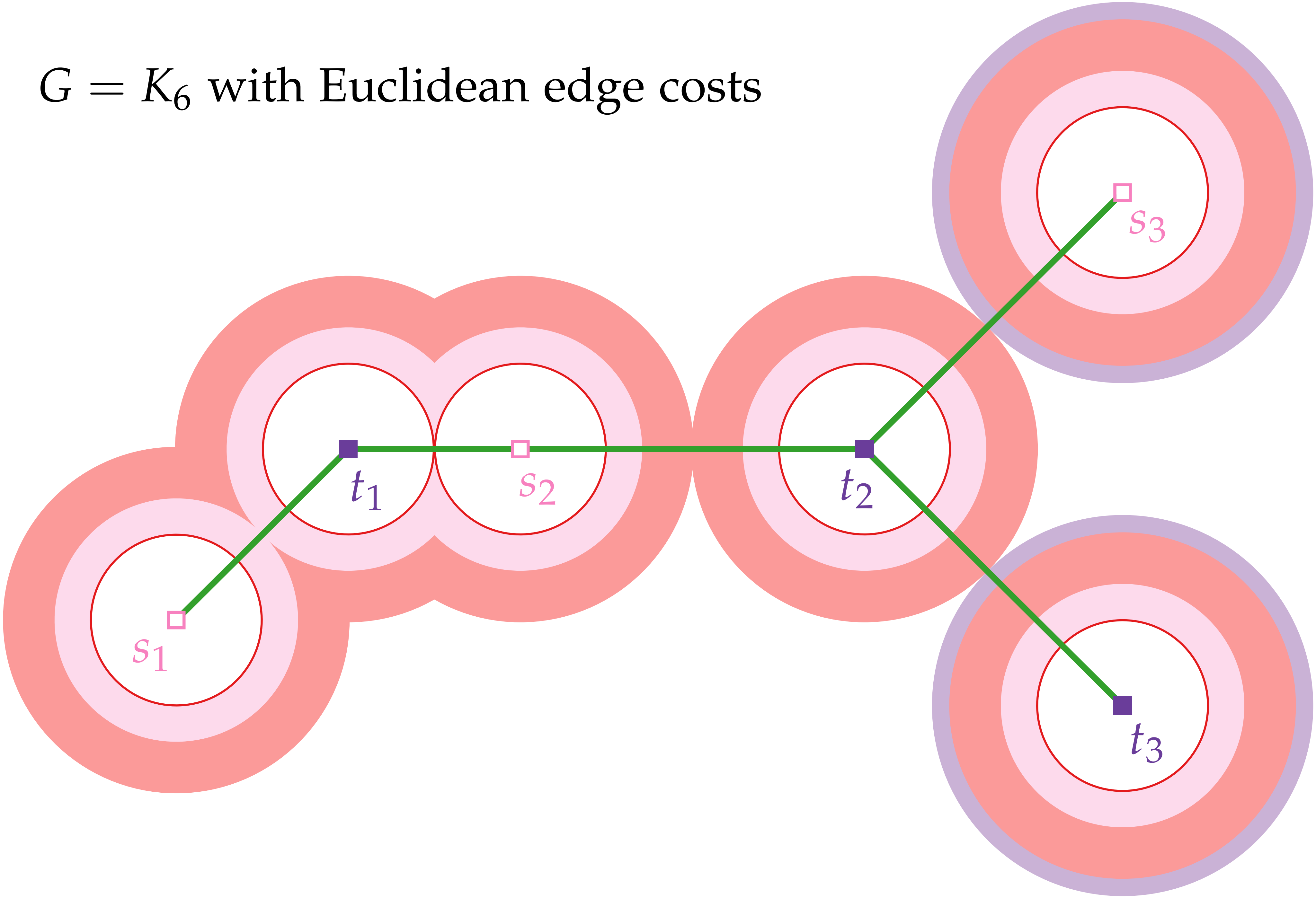

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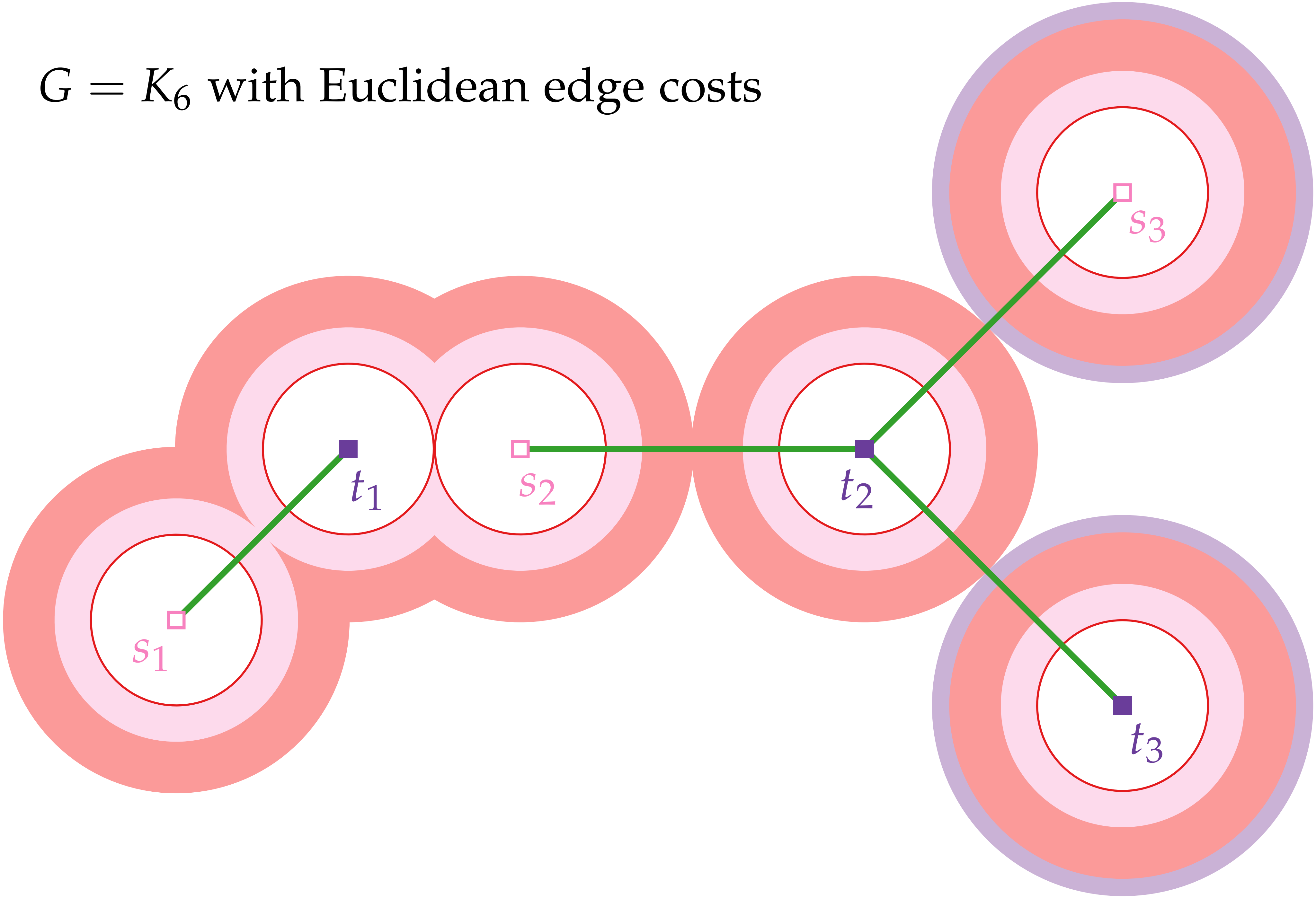

Approximation Algorithms

Lecture 12:
SteinerForest via Primal–Dual

Part V:
Structure Lemma

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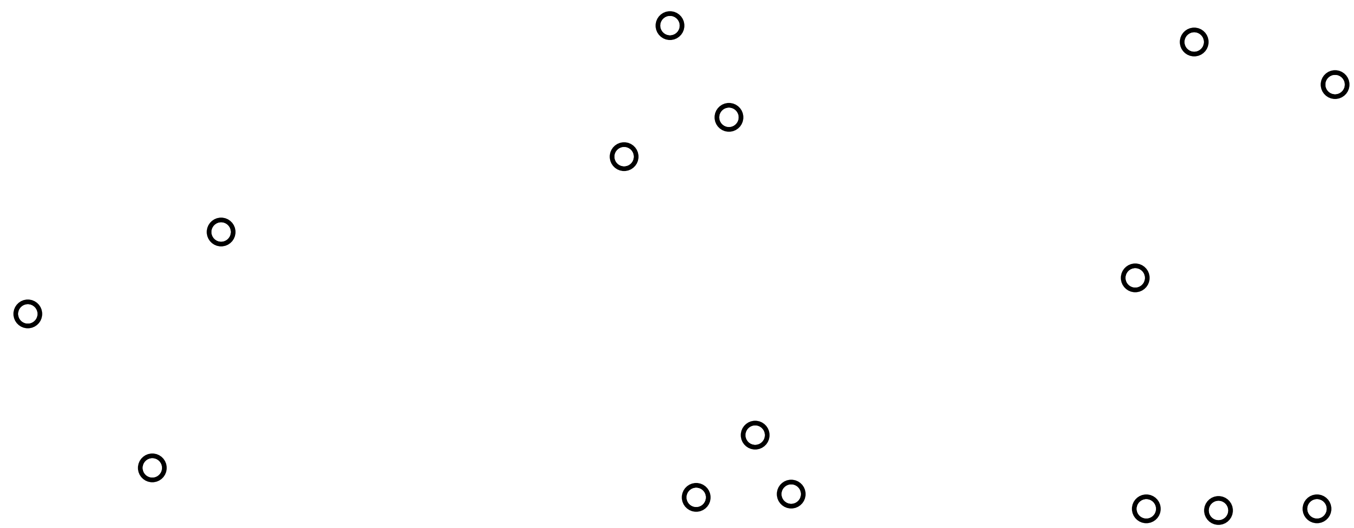

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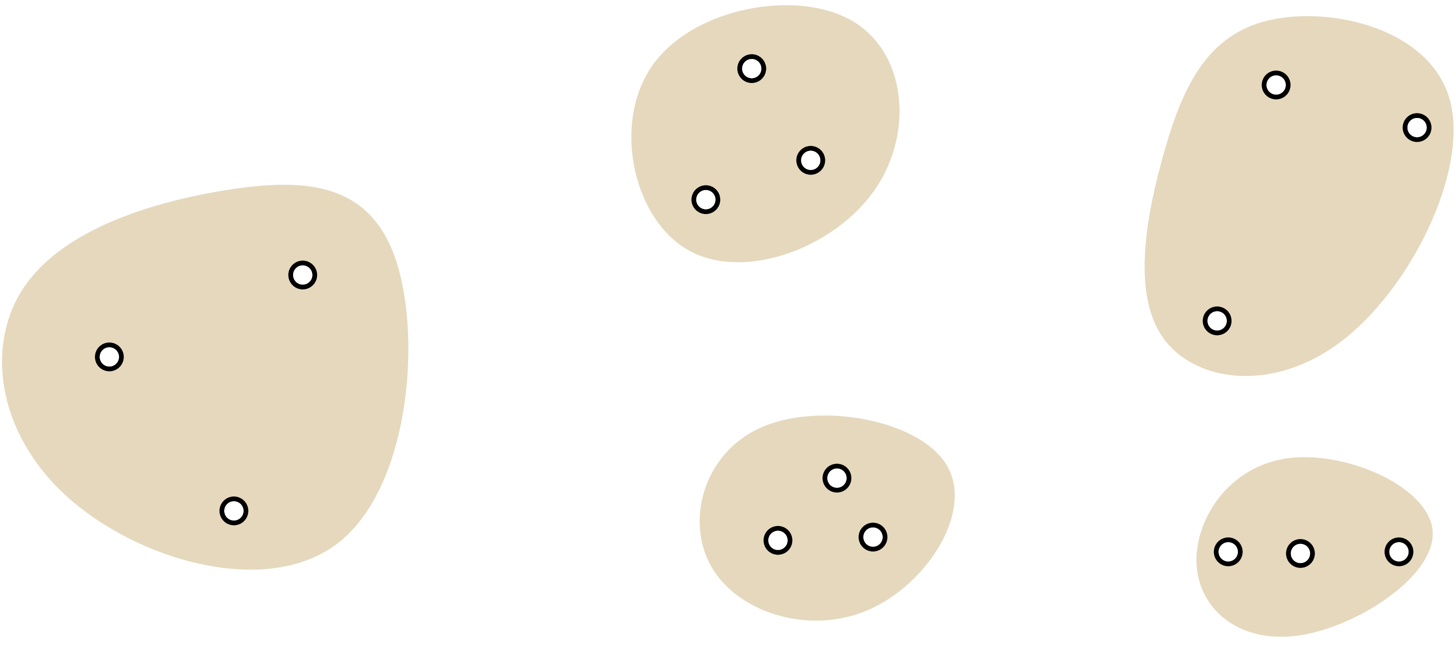

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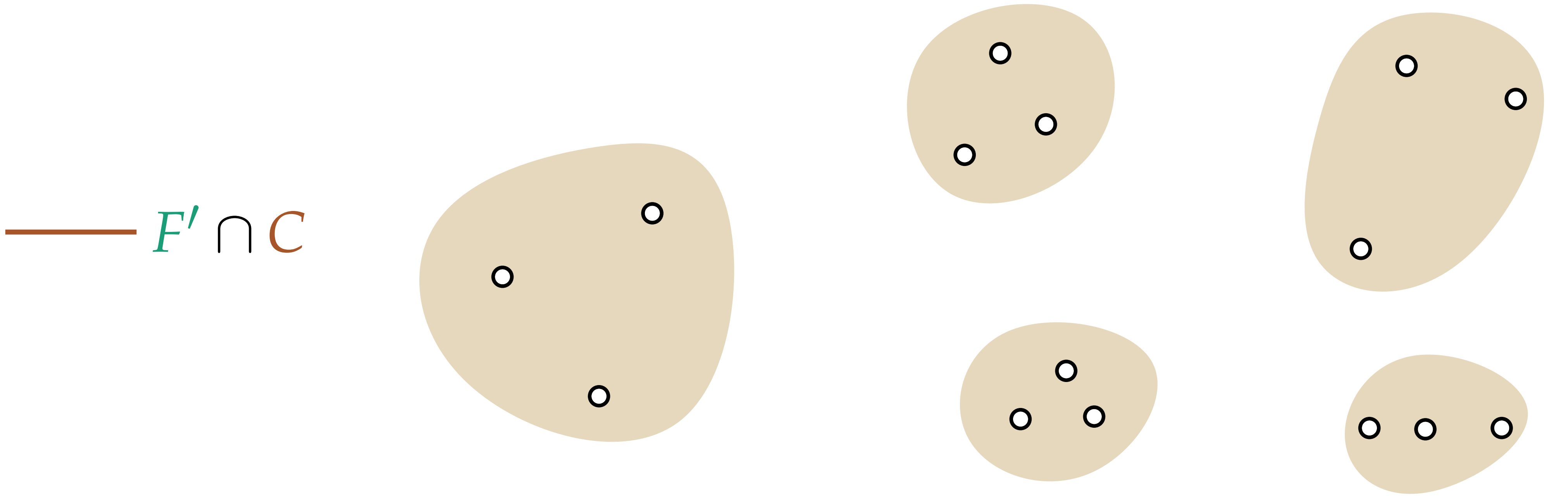

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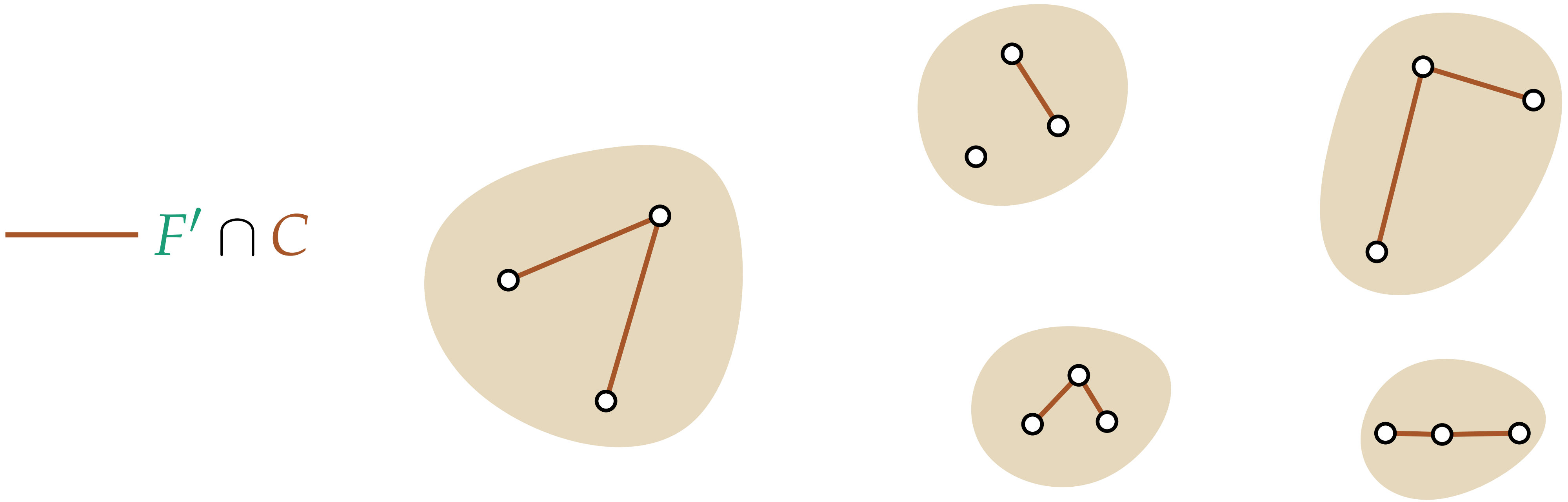

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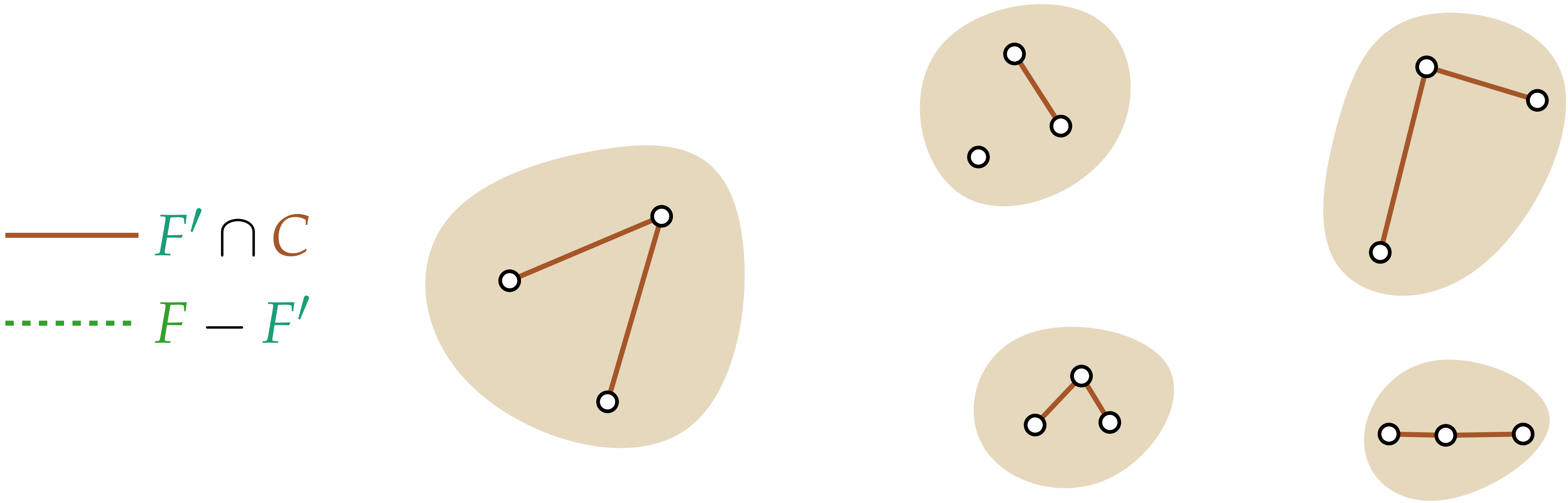

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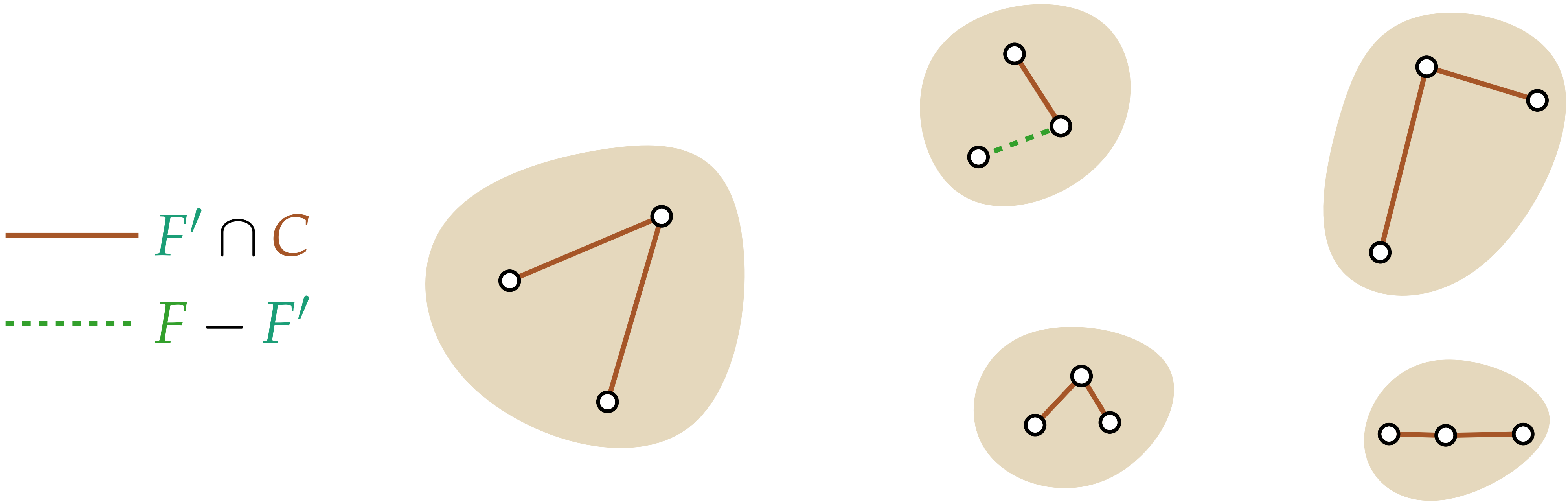

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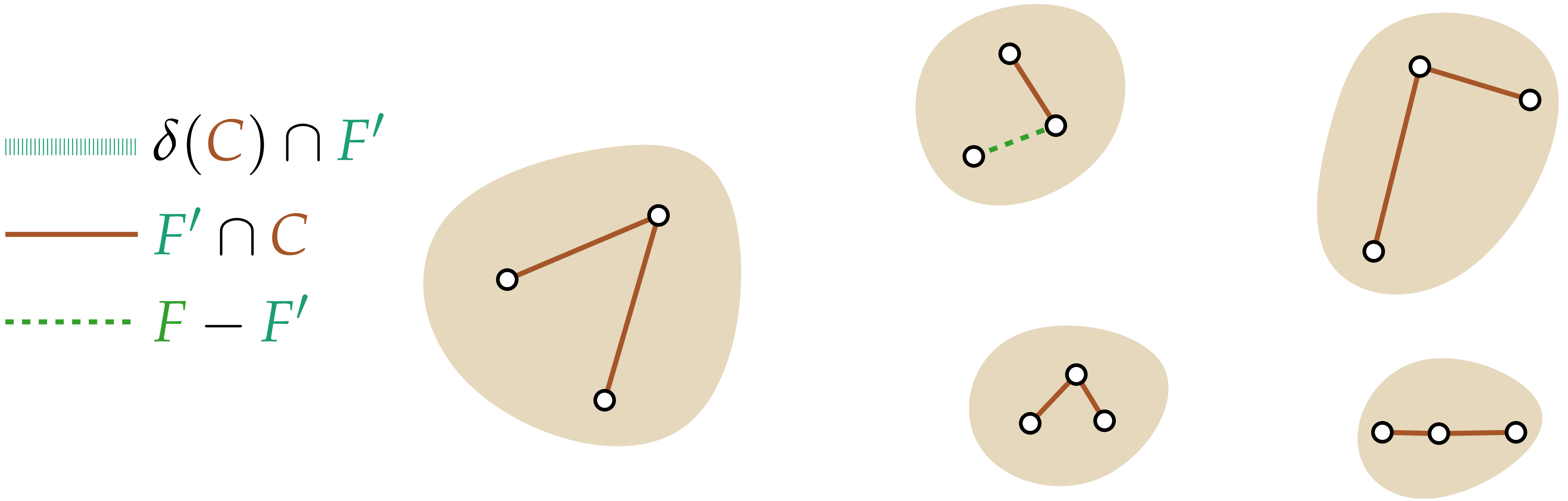

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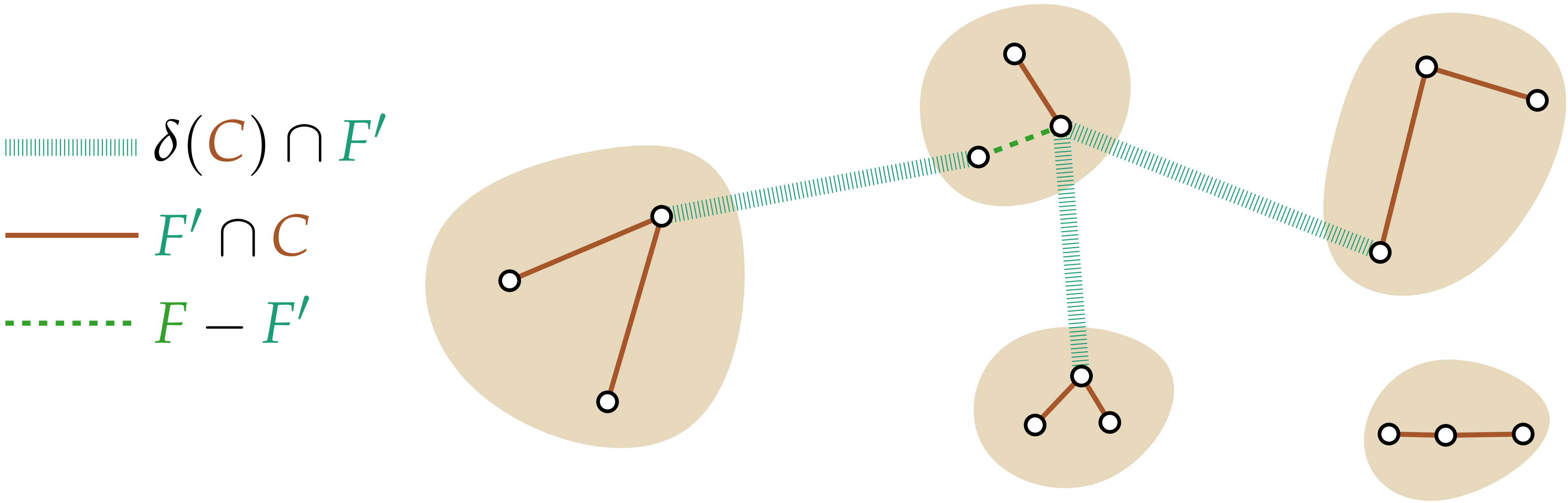

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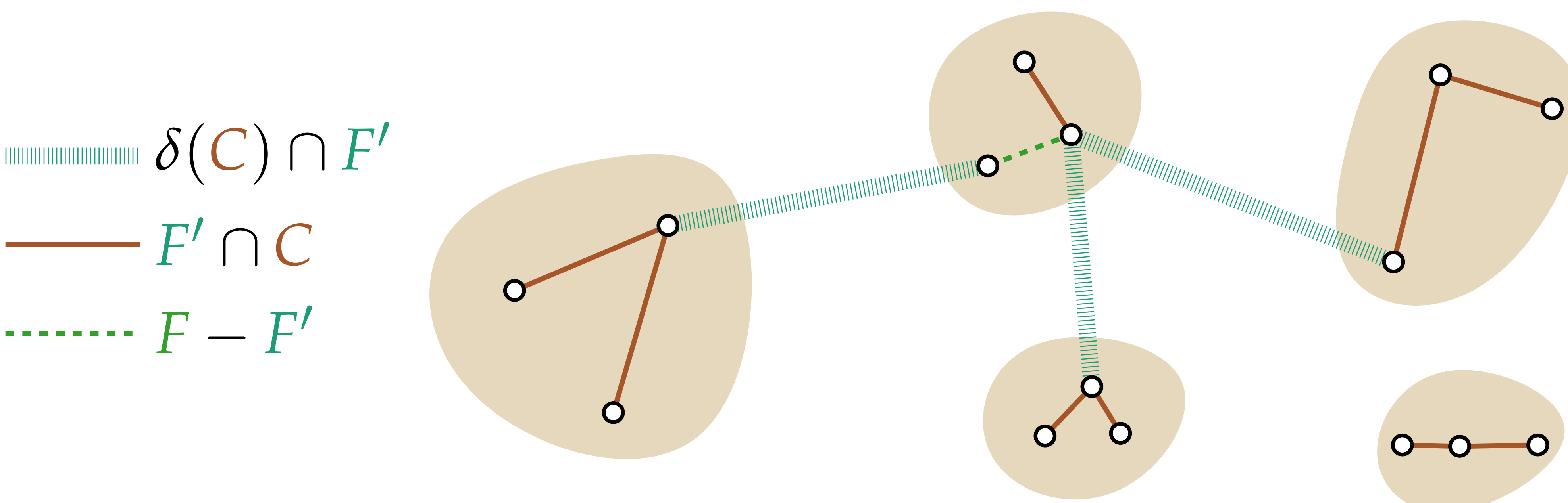

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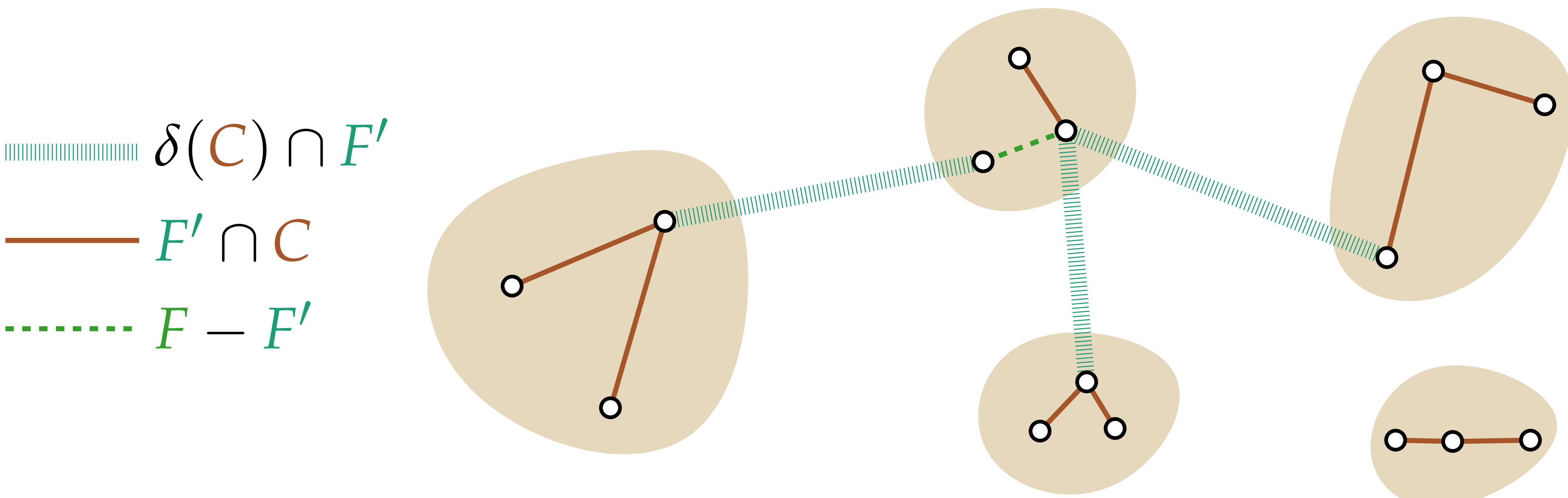

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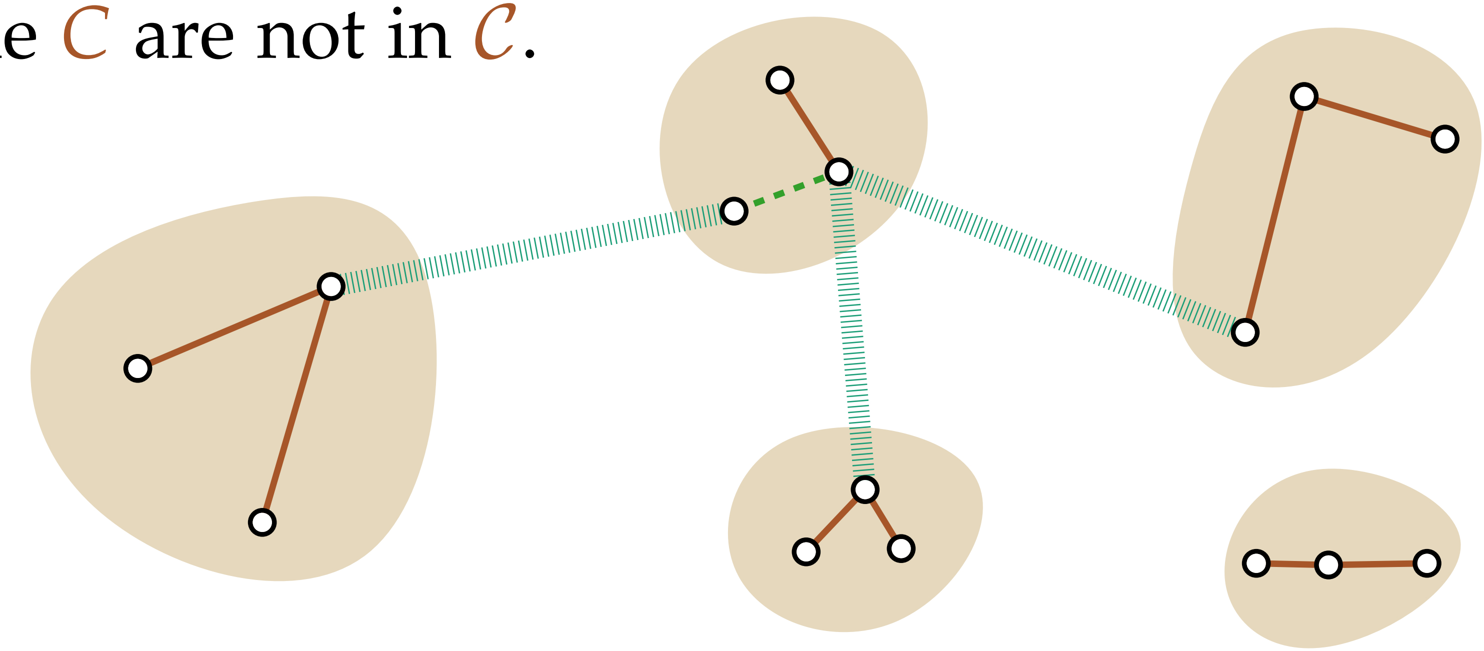
Difficulty: Some C are not in $\mathcal{C}$.



- $\delta(C) \cap F'$
- $F' \cap C$
- $F - F'$

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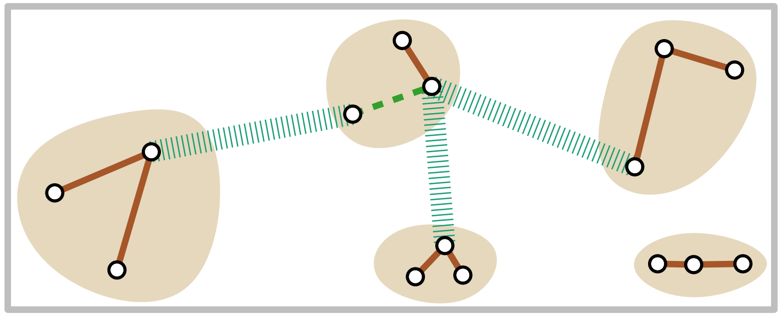

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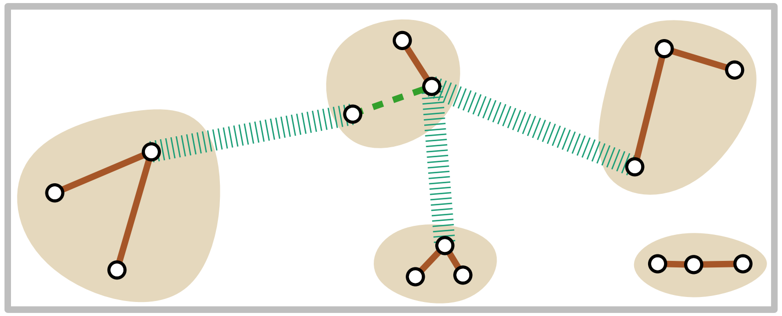

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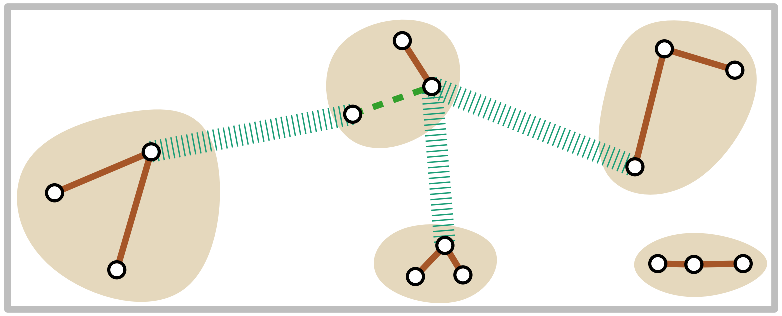

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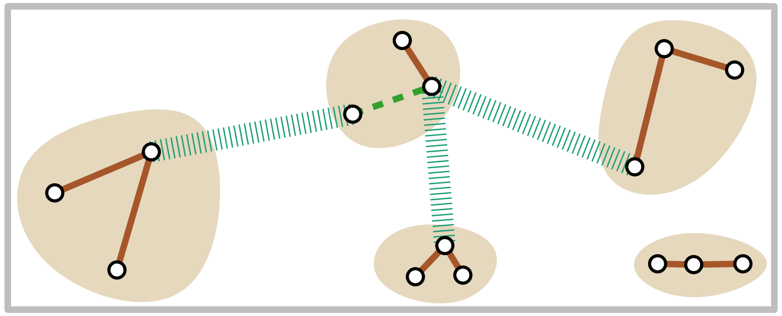

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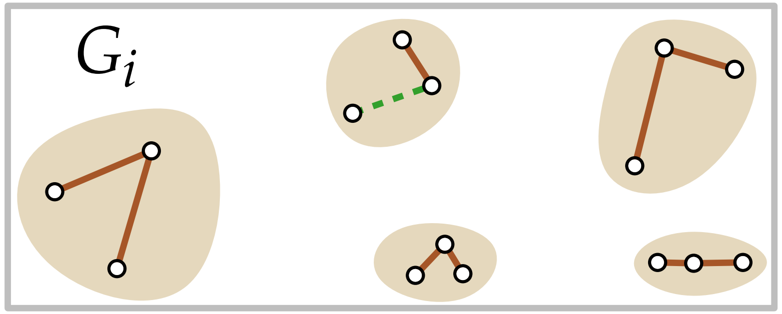

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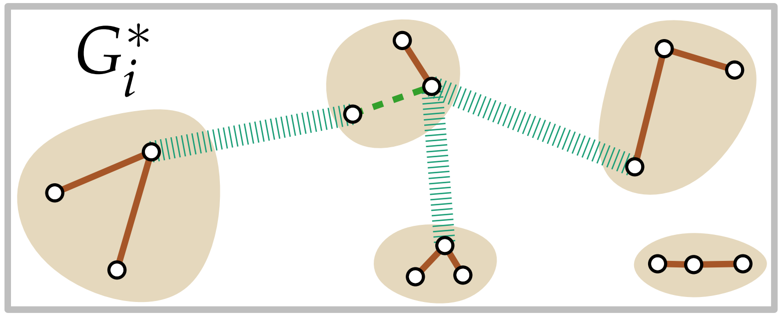

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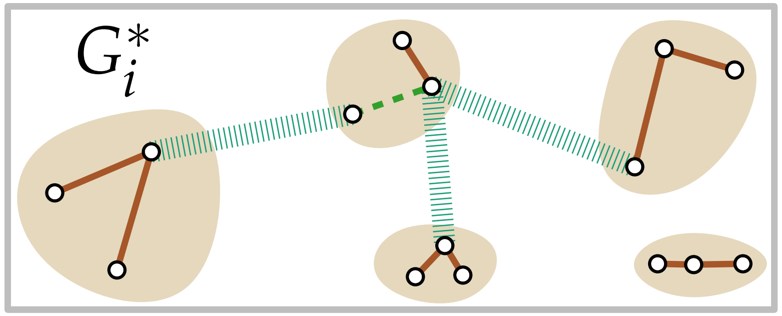

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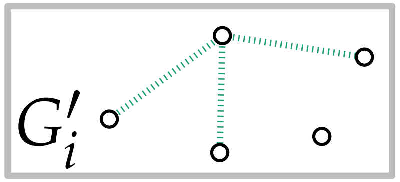

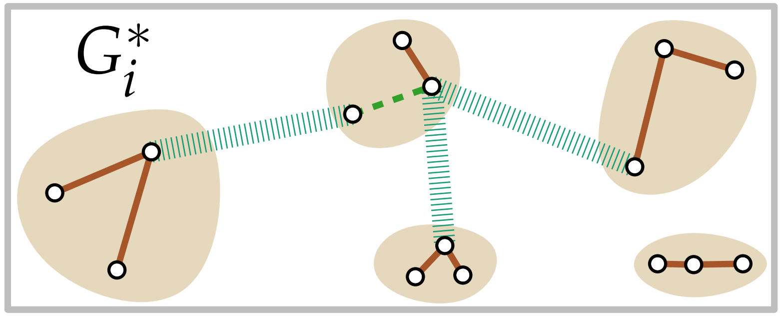

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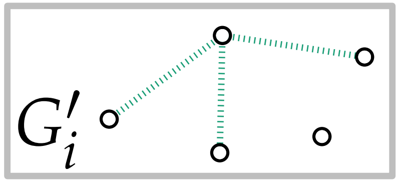

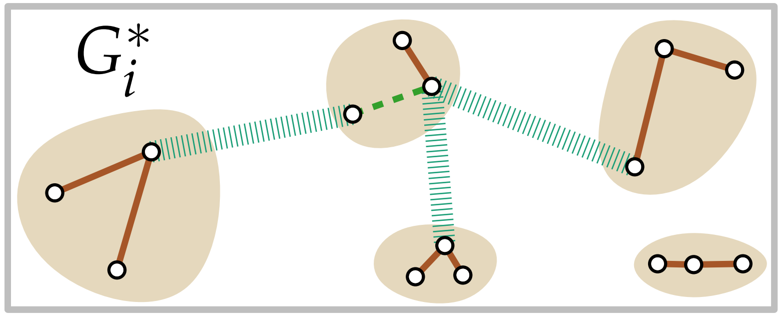

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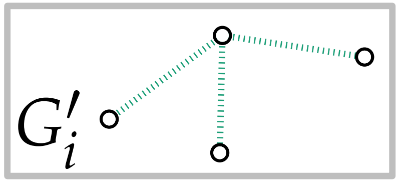

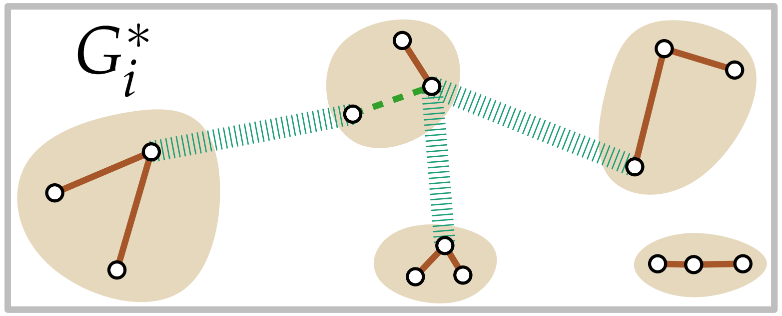

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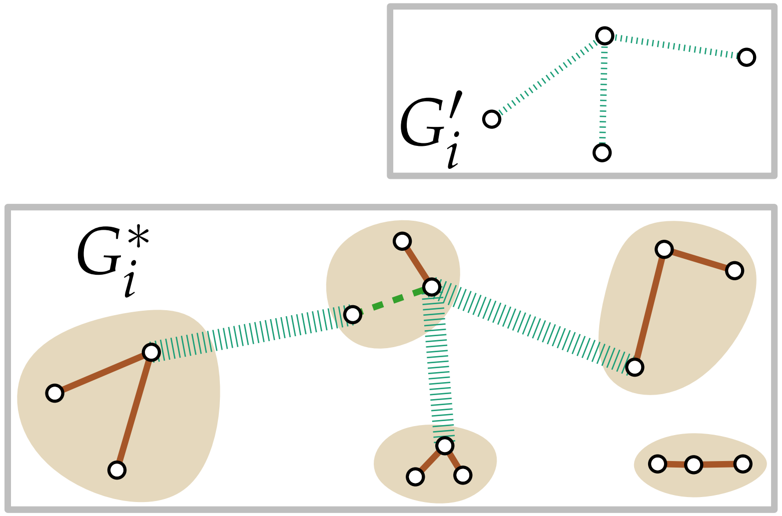

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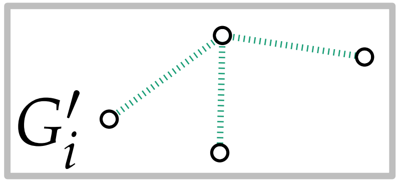

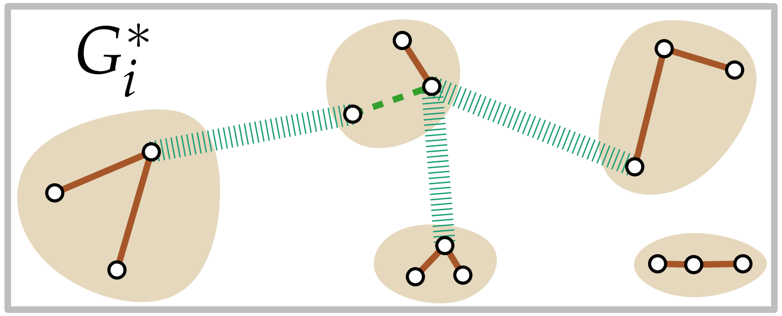

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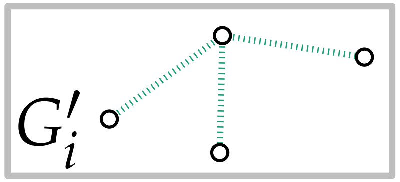

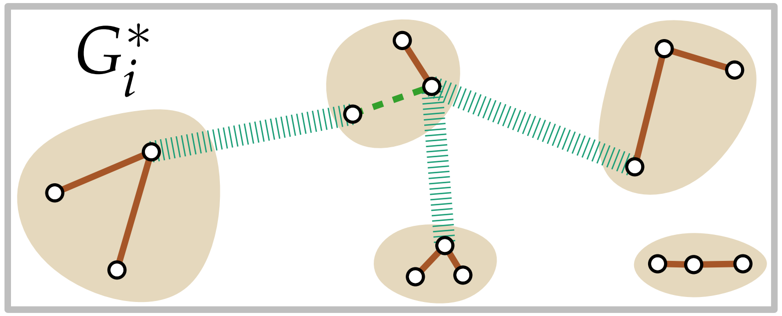

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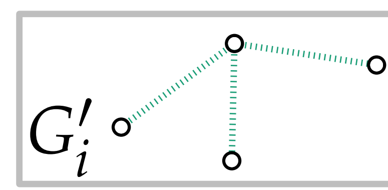

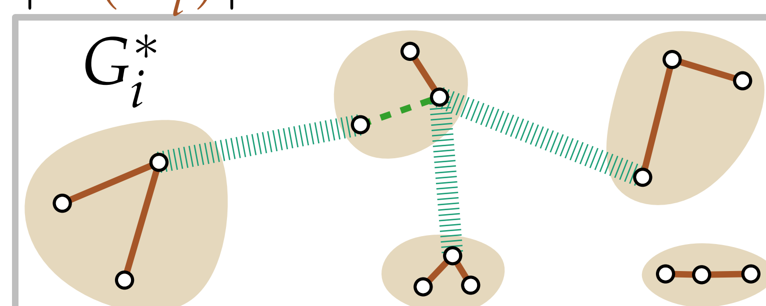

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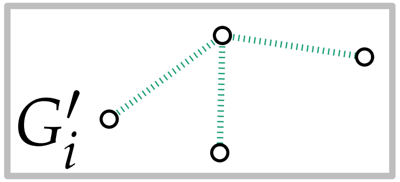

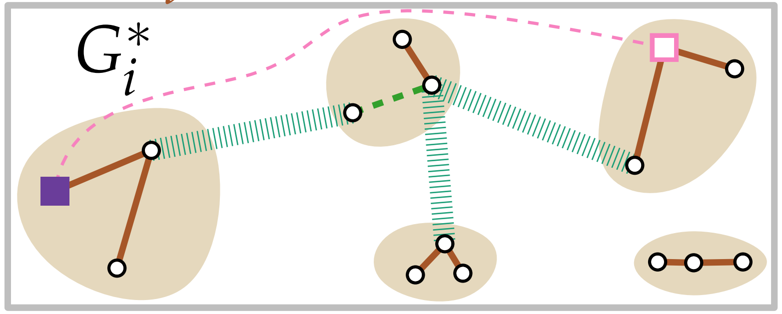

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Lemma. For the set $\mathcal{C}$ in any iteration of the algorithm:

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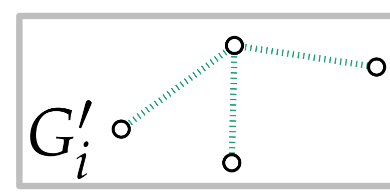

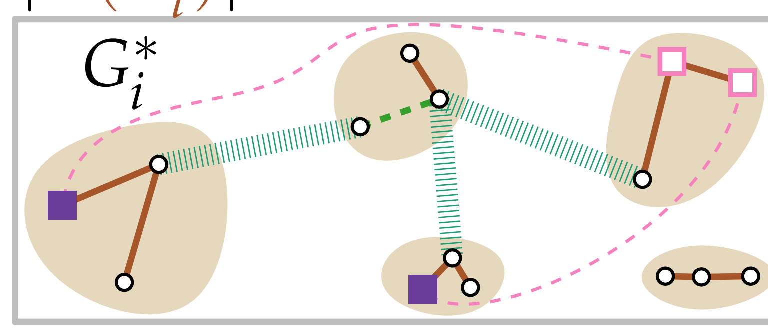

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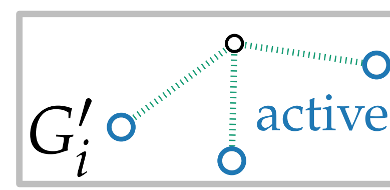

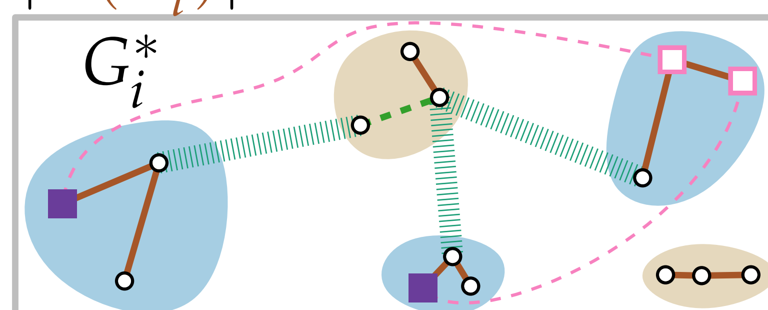

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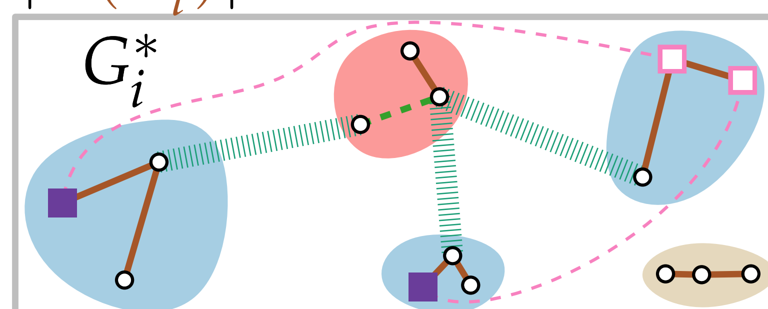

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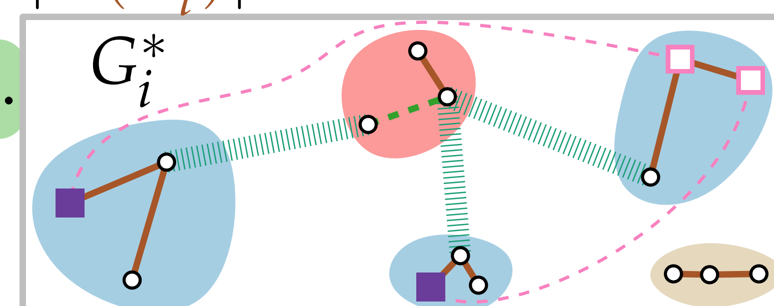

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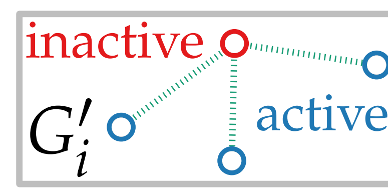

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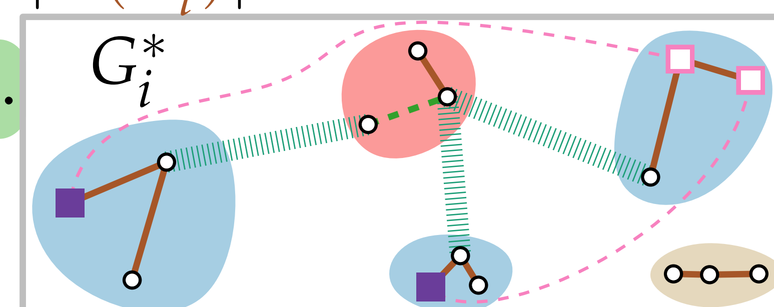

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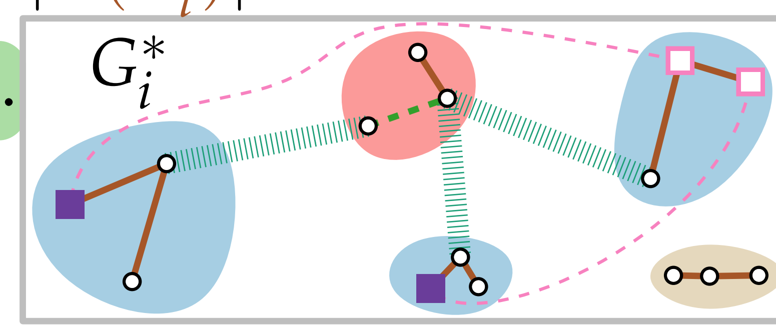

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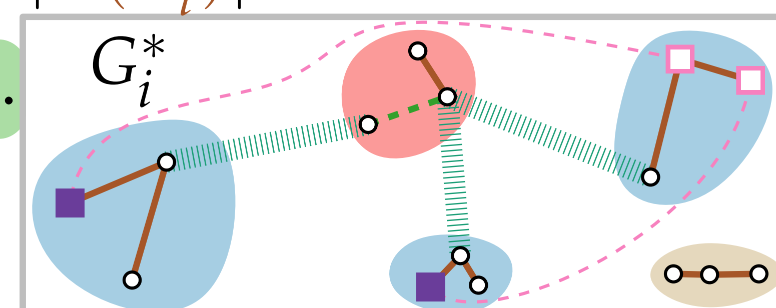

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Approximation Algorithms

Lecture 12:
SteinerForest via Primal–Dual

Part VI:
Analysis

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Theorem. The Primal–Dual algorithm with synchronized increases yields a 2-approximation for SteinerForest.

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From that, the claim of the theorem follows.

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Structure lemma $\Rightarrow (*)$ also holds after the current iteration. □

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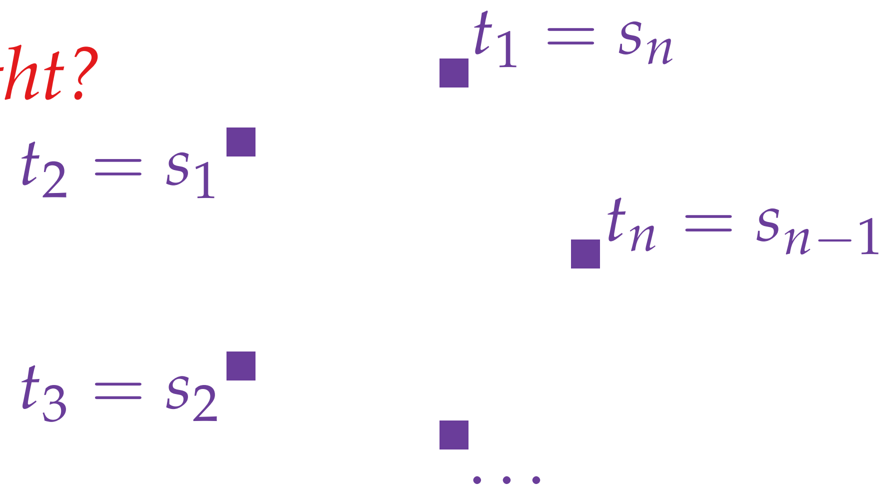

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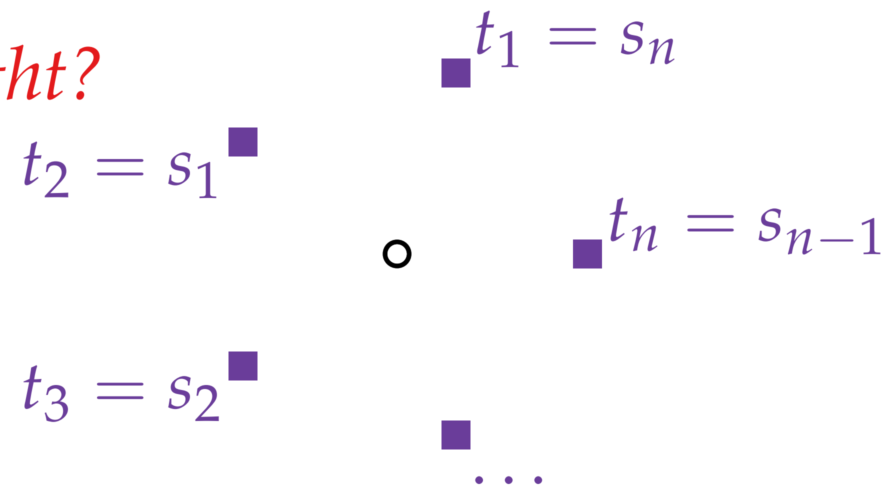

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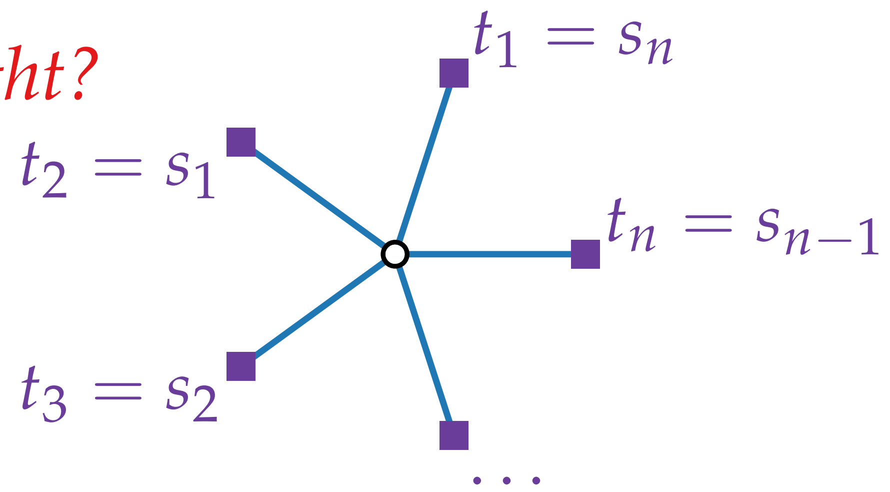

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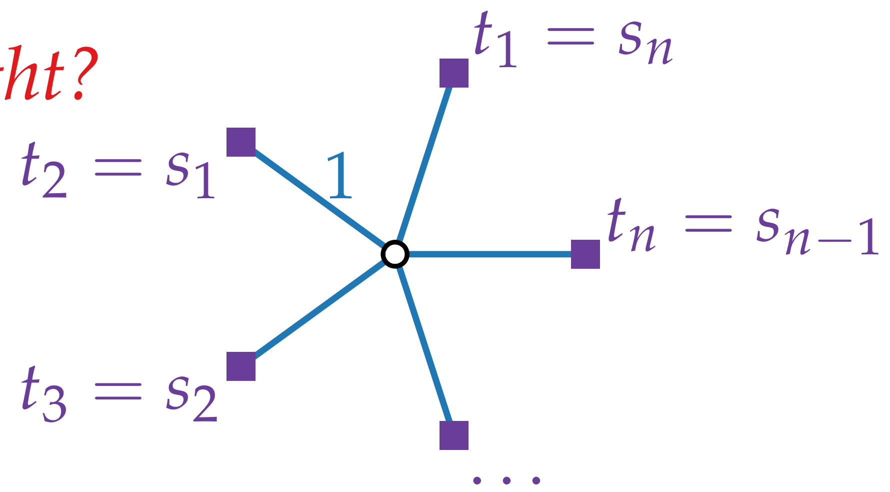

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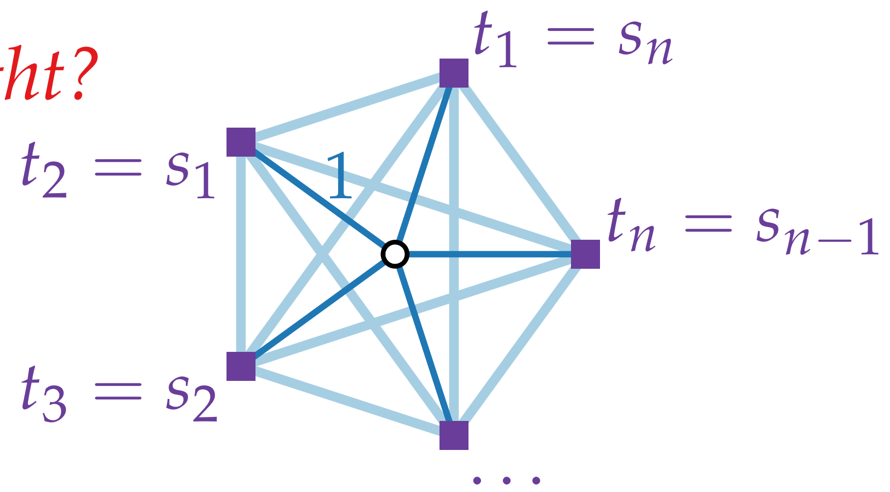

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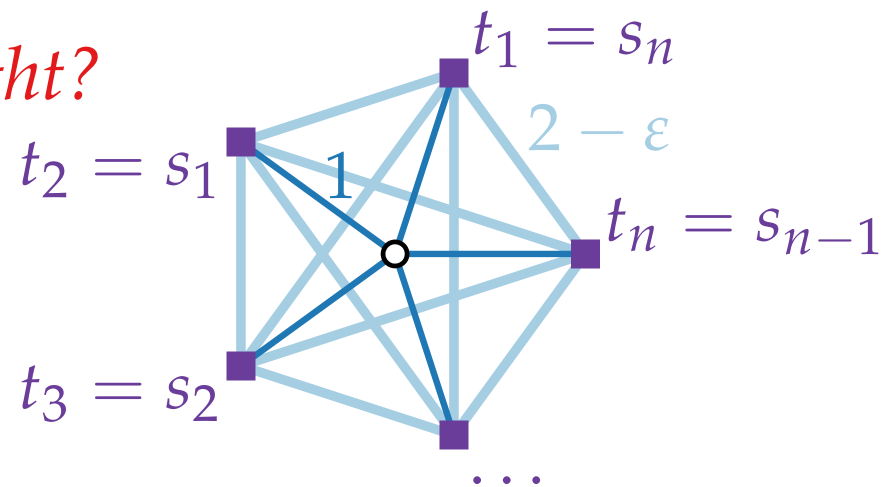

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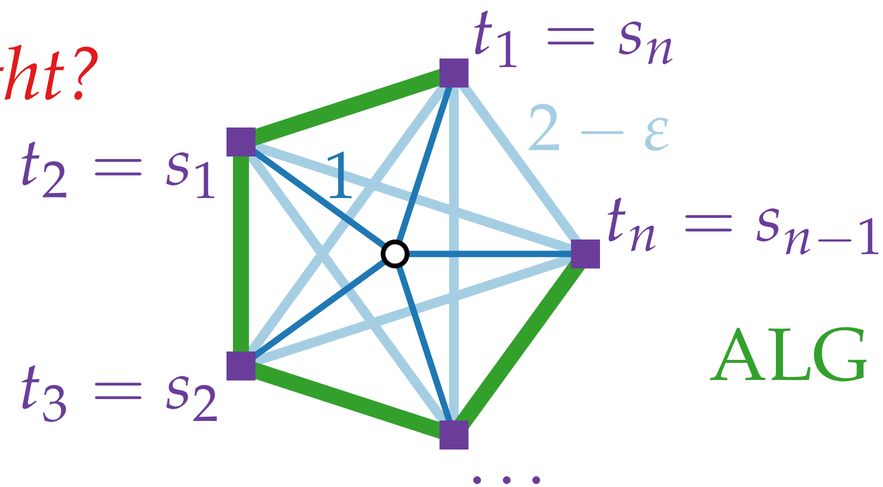

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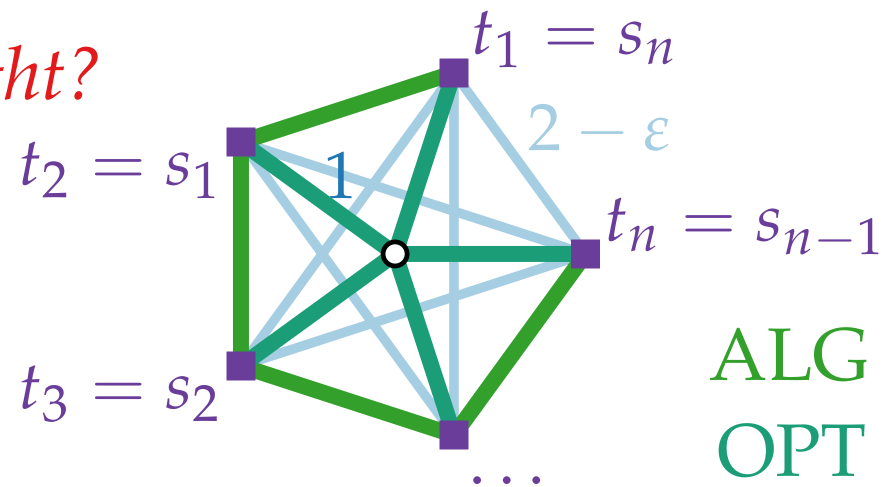

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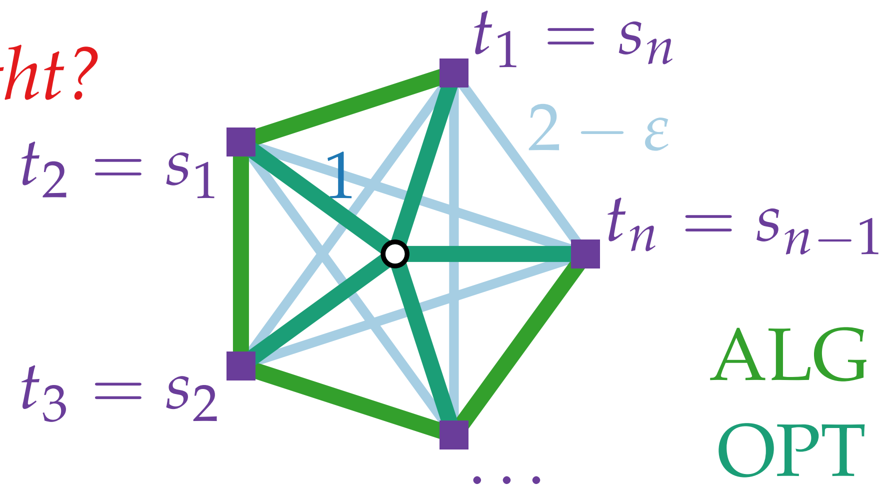

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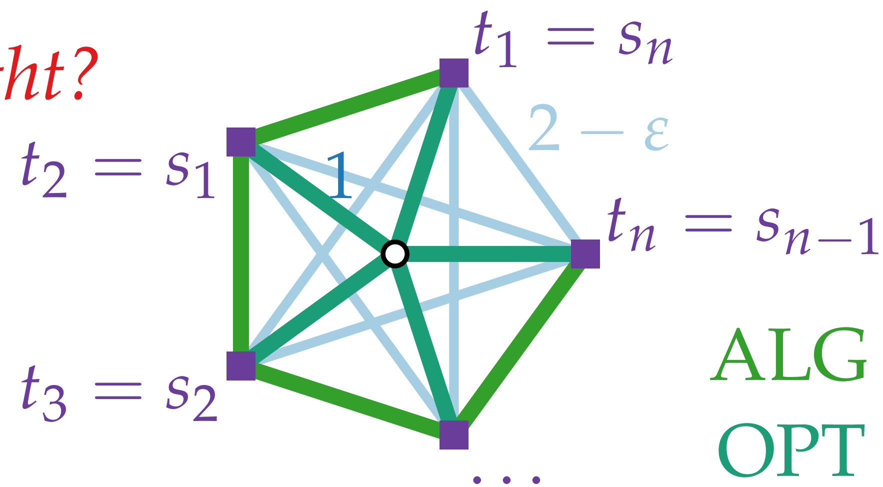

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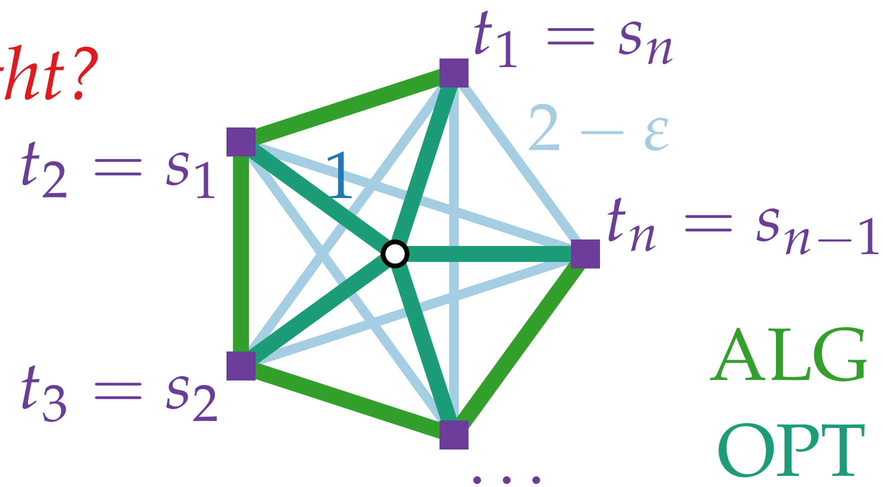

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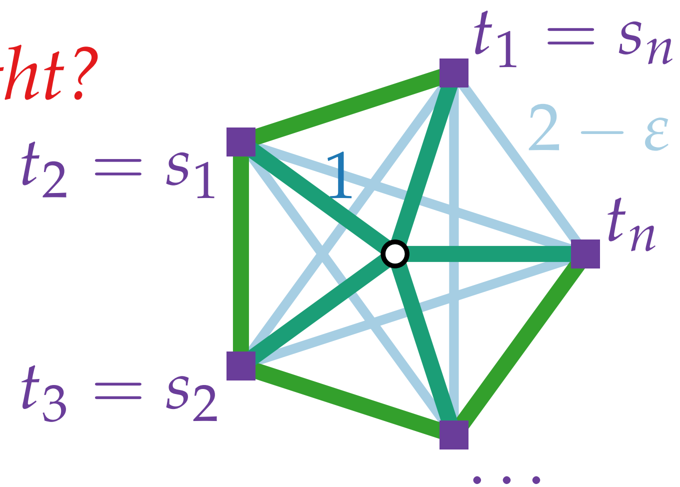

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