Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part I:
Maximum Satisfiability (MaxSat)

Alexander Wolff

Winter 2024/25

Maximum Satisfiability (MaxSat)

Given: Boolean variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$ with weights $w_1, \ldots, w_m$.

Task: Find an assignment of the variables $x_1, \ldots, x_n$ such that the total weight of the *satisfied* clauses is **maximized**.

Literal: Variable or negated variable – e.g., x_1, $\overline{x_1}$.

Clause: Disjunction of literals – e.g., $x_1 \vee \overline{x_2} \vee x_3$.

Length of a clause = number of literals.

Problem is NP-hard since Satisfiability (Sat) is NP-hard:
Is a given formula in conjunctive normal form satisfiable?
E.g., $(x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee \overline{x_4})$.

Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part II:
A Simple Randomized Algorithm

A Simple Randomized Algorithm

Theorem. Independently setting each variable to 1 (`true`) with probability 1/2 provides an expected 1/2-approximation for MaxSat.

Proof.

Let $Y_j \in \{0,1\}$ be a random variable for the truth value of clause C_j.

Let W be a random variable for the total weight of the satisfied clauses.

$$\mathbf{E}[W] = \mathbf{E}\left[\sum_{j=1}^{m} w_j Y_j\right] = \sum_{j=1}^{m} w_j \mathbf{E}[Y_j] = \sum_{j=1}^{m} w_j \mathbf{Pr}[C_j \text{ satisfied}]$$

$l_j := \text{length}(C_j) \;\Rightarrow \mathbf{Pr}[C_j \text{ satisfied}] = 1 - (1/2)^{l_j} \geq 1/2.$

Thus, $\mathbf{E}[W] \geq \frac{1}{2}\sum_{j=1}^{m} w_j \geq \text{OPT}/2.$ □

Approximation Algorithms

Lecture 11: MaxSat via Randomized Rounding

Part III: Derandomization by Conditional Expectation

Derandomization by Conditional Expectation

Theorem. The previous algorithm can be derandomized, i.e., there is a deterministic $1/2$-approximation algorithm for $\textsc{MaxSat}$.

Proof.

We set x_1 deterministically, but $x_2, \dots, x_n$ randomly.

Namely: set $x_1 = 1 \Leftrightarrow \mathbf{E}[W \mid x_1 = 1] \geq \mathbf{E}[W \mid x_1 = 0]$.

$\mathbf{E}[W] = \big(\mathbf{E}[W \mid x_1 = 0] + \mathbf{E}[W \mid x_1 = 1]\big)/2.$ [because of the original random choice of x_1]

If x_1 was set to $b_1 \in \{0, 1\}$,
then $\mathbf{E}[W \mid x_1 = b_1] \geq \mathbf{E}[W] \geq \mathrm{OPT}/2$.

Derandomization by Conditional Expectation

Assume (by induction) that we have set $x_1, \ldots, x_i$ to $b_1, \ldots, b_i$ such that

$$\mathbf{E}[W \mid x_1 = b_1, \ldots, x_i = b_i] \geq \mathrm{OPT}/2$$

Then (similar to the base case):

$$\begin{aligned}
&\big(\mathbf{E}[W \mid x_1 = b_1, \ldots, x_i = b_i, x_{i+1} = 0] \\
&+\mathbf{E}[W \mid x_1 = b_1, \ldots, x_i = b_i, x_{i+1} = 1]\big)/2 \\
&= \mathbf{E}[W \mid x_1 = b_1, \ldots, x_i = b_i] \geq \mathrm{OPT}/2
\end{aligned}$$

So we set $x_{i+1} = 1 \Leftrightarrow$

$$\begin{aligned}
&\mathbf{E}[W \mid x_1 = b_1, \ldots, x_i = b_i, x_{i+1} = 1] \\
&\geq \mathbf{E}[W \mid x_1 = b_1, \ldots, x_i = b_i, x_{i+1} = 0]
\end{aligned}$$

Derandomization by Conditional Expectation

Thus, the algorithm can be derandomized –
if the conditional expectation can be computed efficiently!

Consider a partial assignment $x_1 = b_1, \ldots, x_i = b_i$
and a clause C_j.

If C_j is already satisfied, then it contributes exactly w_j to
$\mathbf{E}[W \mid x_1 = b_1, \ldots, x_i = b_i]$.

If C_j is not yet satisfied and contains k unassigned variables,
then it contributes exactly $w_j(1 - (1/2)^k)$ to
$\mathbf{E}[W \mid x_1 = b_1, \ldots, x_i = b_i]$.

The conditional expectation is simply the sum of the
contributions from each clause. □

Summary

Using *Conditional Expectation* is a standard procedure with which many randomized algorithms can be derandomized.

Requirement: respective conditional probabilities can be appropriately estimated for each random decision.

The algorithm simply chooses the best option at each step.

Quality of the obtained solution is then at least as high as the expected value.

The algorithm iteratively sets the variables and greedily decides for the locally best assignment.

Global optimization?

Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part IV:
Randomized Rounding

An ILP . . . and Its Relaxation

Let $\quad C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x_i} \quad$ for $j = 1, \dots, m$.

$$\textbf{maximize} \quad \sum_{j=1}^{m} w_j z_j$$

$$\textbf{subject to} \quad \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j \quad \text{for } j = 1, \dots, m$$

$$\sout{y_i \in \{0,1\}},\ 0 \leq y_i \leq 1 \quad \text{for } i = 1, \dots, n$$

$$\sout{z_j \in \{0,1\}},\ 0 \leq z_j \leq 1 \quad \text{for } j = 1, \dots, m$$

Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation. Independently setting each variable x_i to 1 with probability y_i^* provides a $0.63 \approx (1 - 1/e)$-approximation for MaxSat.

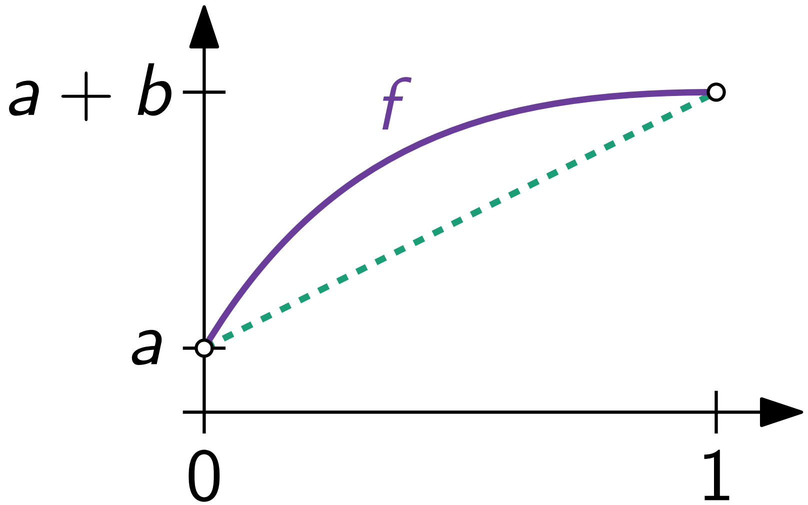
Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part V:
Randomized Rounding – Proof

Mathematical Toolkit

Let f be a function that is concave on $[0, 1]$ (i.e. $f''(x) \leq 0$ on $[0, 1]$) with $f(0) = a$ and $f(1) = a + b$



$\Rightarrow f(x) \geq bx + a$ for $x \in [0, 1]$.

Arithmetic–Geometric Mean Inequality (AGMI):

For all non-negative numbers $a_1, \ldots, a_k$:

$$\left(\prod_{i=1}^{k} a_i \right)^{1/k} \leq \frac{1}{k} \left(\sum_{i=1}^{k} a_i \right)$$

Randomized Rounding (Proof)

Consider a fixed clause C_j of length l_j. Then we have:

$$\mathbf{Pr}[C_j \text{ not sat.}] = \prod_{i \in P_j} (1 - y_i^*) \prod_{i \in N_j} y_i^*$$

$$\left(\prod_{i=1}^{k} a_i \right)^{1/k} \leq \frac{1}{k} \left(\sum_{i=1}^{k} a_i \right)$$

AGMI

$$\leq \left[\frac{1}{l_j} \left(\sum_{i \in P_j} (1 - y_i^*) + \sum_{i \in N_j} y_i^* \right) \right]^{l_j}$$

$$= \left[1 - \frac{1}{l_j} \underbrace{\left(\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1 - y_i^*) \right)}_{\geq z_j^* \text{ by LP constraints}} \right]^{l_j}$$

$$\leq \left(1 - \frac{z_j^*}{l_j} \right)^{l_j}$$

Randomized Rounding (Proof)

The function $f(z_j^*) = 1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$ is concave on $[0, 1]$.

Thus

$$\begin{aligned}
\mathbf{Pr}[C_j \text{ satisfied}] &\geq f(z_j^*) \geq f(1) \cdot z_j^* + f(0) \\
&\geq \left[1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right] z_j^* \\
&\geq \left(1 - \frac{1}{e}\right) z_j^*
\end{aligned}$$

$$1 + x \leq e^x$$

$$x = -\tfrac{1}{l_j} \Rightarrow \quad 1 - \tfrac{1}{l_j} \leq e^{-1/l_j}$$

Randomized Rounding (Proof)

Therefore

$$\begin{aligned}
\mathbf{E}[W] &= \sum_{j=1}^{m} \mathbf{Pr}[C_j \text{ satisfied}] \cdot w_j \\
&\geq \left(1 - \frac{1}{e}\right) \boxed{\sum_{j=1}^{m} w_j z_j^*} \quad \text{LP objective function} \\
&= \left(1 - \frac{1}{e}\right) \mathrm{OPT}_{\mathrm{LP}} \\
&\geq \left(1 - \frac{1}{e}\right) \mathrm{OPT} \qquad \square
\end{aligned}$$

Theorem. The previous algorithm can be derandomized by the method of conditional expectation.

Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part VI:
Combining the Algorithms

Take the better of the two solutions!

Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a $3/4$-approximation for MaxSat.

Proof.

We use another probabilistic argument.
With probability $1/2$, choose the solution of the first algorithm; otherwise the solution of the second algorithm.

The better solution is at least as good as the expectation of the above randomized algorithm.

Take the better of the two solutions!

The probability that clause C_j is satisfied is at least:

$$\frac{1}{2}\left[\underbrace{\left(1-\left(1-\frac{1}{l_j}\right)^{l_j}\right)}_{\text{LP-rounding}} + \underbrace{(1-2^{-l_j})}_{\text{rand. alg.}}\right]\cdot z_j^* \underbrace{\geq \frac{3}{4} z_j^*}_{\text{we claim!}}.$$

(The rest follows similarly as in the proofs of the previous two theorems by linearity of expectation.)

For $l_j \in \{1,2\}$, a simple calculation yields exactly $\frac{3}{4} z_j^*$.

For $l_j \geq 3$, $\quad 1-(1-1/l_j)^{l_j} \geq (1-1/e)$ and $1-2^{-l_j} \geq \frac{7}{8}$.

Thus, we have at least:

$$\frac{1}{2}\left[\left(1-\frac{1}{e}\right)+\frac{7}{8}\right] z_j^* \approx 0.753 z_j^* \geq \frac{3}{4} z_j^* \qquad \square$$

Visualization and Derandomization

- Randomized alg. is better for large values of l_j.
- Randomized LP-rounding is better for small values of l_j

$\Rightarrow$ higher probability of satisfying clause C_j.

The mean of the two solutions is at least $3/4$ for *integer* l_j.

The maximum is at least as large as the mean.

This algorithm, too, can be derandomized by conditional expectation.